Задача 3

$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ спектр A?

σ_ε : z - с.з. $A+\delta A$ с $\|\delta A\|_2 < \epsilon$

$i \Leftrightarrow iv$ - Th-то:

i: z - с.з. для $A+\delta A$ с $\|\delta A\|_2 < \epsilon$ $\quad \cancel{\|A\|_2 = \sigma_{max} = \max_k |\lambda_k|}$

iv: $\|(zI - A)^{-1}\|_2 \geq \epsilon^{-1}$ $\quad \|A\|_2 = \sqrt{\lambda_{max}(A^* A)}$

$z = x+iy \quad z^* = x-iy$

$$B = z\hat{\mathbb{1}} - A = \begin{bmatrix} z & -1 \\ 0 & z \end{bmatrix} = B \quad B^{-1} = \frac{1}{z^2}\begin{bmatrix} z & +1 \\ 0 & z \end{bmatrix} = \begin{bmatrix} \frac{1}{z} & +1/z^2 \\ 0 & \frac{1}{z} \end{bmatrix} \quad zz^* = |z|^2$$

$$B^{-1*} = \begin{bmatrix} \frac{1}{z^*} & 0 \\ \frac{1}{z^{2*}} & \frac{1}{z^*} \end{bmatrix}$$

$$B^{-1*}B^{-1} = \begin{bmatrix} \frac{1}{z^*} & 0 \\ \frac{1}{z^{2*}} & \frac{1}{z^*} \end{bmatrix} \begin{bmatrix} \frac{1}{z} & \frac{1}{z^2} \\ 0 & \frac{1}{z} \end{bmatrix} = \frac{1}{|z|^2}$$

$z = x+iy$

$z^2 = x^2 - y^2 + 2xiy$

$z^{2*} = x^2 - y^2 - 2xiy = (x-iy)^2$

$z^{2*} = z^{*2}$

$$= \begin{bmatrix} \frac{1}{|z|^2} & \frac{1}{|z|^2 z} \\ \frac{1}{|z|^2 z^*} & \frac{1}{|z|^4} + \frac{1}{|z|^2} \end{bmatrix} = \begin{bmatrix} \frac{1}{|z|^2} & \frac{x-iy}{|z|^4} \\ \frac{x+iy}{|z|^4} & \frac{1}{|z|^4} + \frac{1}{|z|^2} \end{bmatrix}$$

$\frac{1}{|z|^2} = \frac{1}{x^2+y^2} = \alpha$

$$\begin{bmatrix} \alpha & (x-iy)\alpha^2 \\ \alpha^2(x+iy) & \alpha^2+\alpha \end{bmatrix} = B^{-1*}B^{-1} \quad \det(B^{-1*}B^{-1} - \lambda I) =$$

$$= \det\begin{bmatrix} \alpha-\lambda & (x-iy)\alpha \\ \alpha(x+iy) & \alpha^2+\alpha-\lambda \end{bmatrix} = 0 \Leftrightarrow$$

$\Leftrightarrow (\alpha-\lambda)(\alpha^2+\alpha-\lambda) - \frac{1}{\alpha}\alpha^4 = 0$

$\cancel{\alpha^3} + \alpha^2 - \lambda\alpha - \lambda\alpha^2 - \lambda\alpha + \lambda^2 - \cancel{\alpha^3} = 0$

$\lambda^2 - \lambda(2\alpha + \alpha^2) + \alpha^2 = 0$

$\lambda^2 - \lambda\alpha(2+\alpha) + \alpha^2 = 0$

$$\lambda_{max} = \frac{1}{2}\left(\alpha(2+\alpha) + \sqrt{\alpha^2(2+\alpha)^2 - 4\alpha^2}\right)$$

$$\lambda_{max} = \alpha + \frac{\alpha^2}{2} + \frac{1}{2}\alpha^2\sqrt{1+\frac{4}{\alpha}}$$

$\|(zI - A)^{-1}\|_2 \geq \frac{1}{\epsilon}$

$\sigma_{0,1}$: $\sqrt{\alpha + \frac{\alpha^2}{2} + \frac{1}{2}\alpha^2\sqrt{1+\frac{4}{\alpha}}} \geq 10$

$\sigma_{0,01}$: $\sqrt{\alpha + \frac{\alpha^2}{2} + \frac{1}{2}\alpha^2\sqrt{1+\frac{4}{\alpha}}} \geq 100$