№4

$$f(A) = A^{-1}X(X^T A^{-1} X)^{-1} \quad f(\Delta) = \Delta^{-1}X(X^T \Delta^{-1} X)^{-1}$$

$$\Delta f(\Delta) X^T \Delta^{-1} X = X$$

$$\Delta f(\Delta) X^T \Delta^{-1} = \hat{\mathbb{1}}$$

$$\Delta^{-1} \Delta f(\Delta) X^T \Delta^{-1} \Delta = \Delta^{-1} \Delta$$

$$f(\Delta) X^T = \hat{\mathbb{1}}$$

$$f(\Delta) = (X^T)^{-1}.$$

Таким образом $f(A) = (X^T)^{-1} \ \forall A.$

$$A = X\Omega X^T + \Delta$$

$$(X^T)^{-1} = f(\Delta) = f(X\Omega X^T + \Delta)$$

Задача 1

а) $\|x\|_2 \le \sqrt{m}\,\|x\|_\infty$

$$\sqrt{\sum_i |x_i|^2} \le \sqrt{\sum_i (\max x_i)^2} \le \sqrt{m(\max x_i)^2} \le \sqrt{m}\,\|x\|_\infty$$

$$\|x\|_2 \le \sqrt{m}\,\|x\|_\infty$$

нер-во достиг. при $\forall_i\ x_i = a$

$$\sqrt{\sum_i |x_i|^2} \le \sqrt{m}\,\max_i x_i$$

$$\sqrt{ma^2} \le \sqrt{m}\cdot a \quad \sqrt{m}\,a = \sqrt{m}\,a$$

б) $\|A\|_\infty \le \sqrt{n}\,\|A\|_2 \quad A \in mat(n \times n)$

$$\|A\|_\infty = \max_i \sum_j (a_{ij} x_j), \text{ где } A_{ij} = (a_{ij})$$

$i = \overline{1,\dots n}$

$j = \overline{1,\dots,n}$

$$\|A\|_2 = \sup_{(x,x)=1} \sqrt{(Ax,Ax)} \qquad (Ax)_i = \sum_j a_{ij} x_j$$

$$(Ax,Ax) = \sum_i \left(\sum_j a_{ij} x_j\right)^2 \qquad (x,x) = \sum_i x_i^2$$

$$\|A\|_2 = \sup_{(x,x)=1} \sqrt{\sum_i \left(\sum_j a_{ij} x_j\right)^2}$$

$$\cancel{\|A\|_2 = \sup_{(x,x)=1} \sqrt{\sum_i \left(\sum_{k,j} a_{kj} x_j x_k\right)}}$$

$$\|A\|_2 = \sup_{(x,x)=1} \sqrt{\sum_i \left(\sum_j a_{ij} x_j\right)^2}$$

$$\cancel{\|A\|_2 \le \sup_{(x,x)=1} \sqrt{\max_i \left(\sum_j a_{ij} x_j\right)^2}}$$

$$\|A\|_2 \le \sup_{(x,x)=1} \sqrt{n\sum_i \left(\sum_j a_{ij} x_j\right)^2},$$

$$\|A\|_2 \le \sqrt{n}\,\|A_2\|$$

$$\|A\|_\infty = \max_i \sum_j (a_{ij} x_j) \le \sup_{(x,x)=1} \sqrt{\sum_i \left(\sum_j a_{ij} x_j\right)^2} = \|A\|_2$$

$$\|A\|_\infty \le \sqrt{n}\,\|A\|_2$$

Задача 2.

(a) $\begin{bmatrix} 3 & 0 \\ 0 & -2 \end{bmatrix} = A$ $\quad \cancel{\begin{vmatrix} 3-\lambda & 0 \\ 0 & -2-\lambda \end{vmatrix} = 0 \quad (3-\lambda)(-2-\lambda) = 0 \quad \lambda = 3 \quad \lambda = -2}$

$A^T = \begin{bmatrix} 3 & 0 \\ 0 & -2 \end{bmatrix}$ $\quad AA^T = \begin{bmatrix} 9 & 0 \\ 0 & 4 \end{bmatrix}$ $\quad \lambda_1 = 9 \quad \lambda = 4$ $\quad \sigma_1 = \sqrt{\lambda_1} = 3 \quad \sigma_2 = \sqrt{\lambda_2} = 2$

$A = U \Sigma V^T$ $\quad \begin{pmatrix} 3 & 0 \\ 0 & -2 \end{pmatrix} \begin{pmatrix} V_{11} & V_{12} \\ V_{21} & V_{22} \end{pmatrix} = \begin{pmatrix} 3 & 0 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{pmatrix}$

$\begin{pmatrix} 3V_{11} & 3V_{12} \\ -2V_{21} & -2V_{22} \end{pmatrix} = \begin{pmatrix} 3u_{11} & 3u_{12} \\ 2u_{21} & 2u_{22} \end{pmatrix}$

$A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 3 & 0 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}^T$

б) $\underset{n \times m}{A} = \begin{pmatrix} 0 & 2 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} = \underset{n\times n}{U} \underset{n\times m}{\Sigma} \underset{m\times m}{V^T}$ $\quad A^T = \begin{pmatrix} 0 & 0 & 0 \\ 2 & 0 & 0 \end{pmatrix}$

$AA^T = \begin{pmatrix} 0 & 2 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ 2 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 4 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ $\quad \lambda_1 = 4 \quad \lambda_2 = 0 \quad \lambda_3 = 0$

$AV = U\Sigma$ $\quad \begin{pmatrix} 0 & 2 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} V_{11} & V_{12} \\ V_{21} & V_{22} \end{pmatrix} = \begin{pmatrix} u_{11} & u_{12} & u_{13} \\ u_{21} & u_{22} & u_{23} \\ u_{31} & u_{32} & u_{33} \end{pmatrix} \begin{pmatrix} 2 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}$ $\quad \sigma = 2$

$\begin{pmatrix} 2V_{21} & 2V_{22} \\ 0 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 2u_{11} & 0 \\ 2u_{21} & 0 \\ 2u_{31} & 0 \end{pmatrix}$

$V = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$

$V^T = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$

$A = \begin{pmatrix} 0 & 2 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 2 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 2 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}$

$|A - \lambda|^2 - 1 = 0$

6| $A = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$

$$A^T A = \begin{pmatrix} 2 & 2 \\ 2 & 2 \end{pmatrix} \qquad \lambda_1 = 4 \quad \sigma_1 = 2$$

$$\lambda_2 = 0 \quad \sigma_2 = 0$$

$$A = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix}^T$$