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MODIFIED DAI-YUAN ITERATIVE SCHEME FOR NONLINEAR SYSTEMS AND ITS APPLICATION

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Abstract. By exploiting the idea employed in the spectral Dai-Yuan method by Xue et al. [IEICE Trans. Inf. Syst. 101 (12)2984-2990 (2018)] and the approach applied in the modified Hager-Zhang scheme for nonsmooth optimization [PLos ONE 11(10): e0164289 (2016)], we develop a Dai-Yuan type iterative scheme for convex constrained nonlinear monotone system. The scheme's algorithm is obtained by combining its search direction with the projection method. One of the new scheme's attribute is that it is derivative-free, which makes it ideal for solving non-smooth problems. Furthermore, we apply the method to recover obscure or blurry images in compressed sensing. By employing mild assumptions, global convergence of the scheme is determined and results of some numerical experiments show the method to be favorable compared to some recent iterative methods.

1. **Introduction.** In this paper, we consider the system of nonlinear equations given by

$$F(x) = 0, \quad x \in \Omega \subset \mathbf{R}^n, \tag{1}$$

where $F : \mathbf{R}^n \longrightarrow \mathbf{R}^n$ is a nonlinear, monotone mapping, which may not be differentiable and $x = (x_1, x_2, ..., x_n)$. Also, Ω in (1) is a nonempty closed convex set, while monotonicity of F implies that $\forall x, y \in \mathbf{R}^n$, the following holds:

$$(F(x) - F(y))^T(x - y) \geq 0. \tag{2}$$

Finding solutions of nonlinear systems of equations is a popular trend in the field of optimization. This is due to real life applications of the concept in science, engineering, economics and other areas of human endeavor. In radiactive transfer and transport theory [25], the Chandrasekhar integral equation [11] is discretized and expressed as system of nonlinear equations. The economic equilibrium problems that arise in [13, 35], are remodeled as systems of nonlinear equations. Also, some ℓ_1-*norm* regularized optimization problems in signal and image processing [30, 53] are obtained by reformulating systems of monotone nonlinear equations. For more applications of the concept, see [43, 59].

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The conjugate gradient (CG) method stands out as one of the most ideal iterative scheme for solving large-scale unconstrained optimization problems of the form

$$\min_{x \in \mathbf{R}^n} f(x), \tag{3}$$

where $f : \mathbf{R}^n \longrightarrow \mathbf{R}$ is a smooth nonlinear function whose gradient $\nabla f(x)$ is obtainable. The scheme is implemented using the iterative formula

$$x_{k+1} = x_k + s_k, \quad s_k = \alpha_k d_k, \quad k = 0, 1, ..., \tag{4}$$

where x_k represents the current iterate, α_k is a steplength obtained using some line search technique, and d_k is the CG search direction defined by

$$d_k = -g_k + \beta_k d_{k-1}, \quad d_0 = -g_0, \tag{5}$$

where $g_k = g(x_k) = \nabla f(x_k)$ and β_k represents the CG update parameter, which is crucial in the scheme.

The attributes of the CG scheme includes less storage requirement and strong global convergence properties [4]. The most essential CG methods include the Fletcher-Reeves (FR) method [17], Conjugate gradient (CD) method [18], Dai-Yuan (DY) method [12], Hestenes-Stiefel (HS) method [24], Polak-Ribi‘ere-Polyak (PRP) method [38, 39], and the Liu-Storey (LS) method [33]. The HS, PRP and LS methods are numerically effective due to their ability to initiate a restart whenever a bad direction is encountered. However, they fail to generate descent search directions, which is vital for global convergence. On the other hand, the FR, DY and CD methods posses the ability to generate descent directions under some suitable line search rule, but they are not numerically efficient. The reason for this stems from the fact that these methods are prone to jamming phenomenon, which results when small steps are made without making significant progress to the minimum.

Quite a number of CG methods have been developed for solving systems of nonlinear equations as extensions of the classical CG methods described above. By combining the PRP method [38, 39] with the hyperplane projection scheme [44], Cheng [9] developed an effective PRP-type scheme for solving nonlinear systems of equations. Li and Li [27] equally proposed a set of PRP-type algorithms for solving system of nonlinear monotone equations by employing the projection strategy [44]. Liu et al. [29] developed an LS-type scheme for constrained system of nonlinear monotone equations by implementing Powell's idea on the classical Liu-Storey scheme [33]. The method converges globally under basic assumptions. Inspired by the self-adaptive method proposed in [23], Wang et al. [46] proposed a self-adaptive three-term CG method for monotone nonlinear equations with convex constraints. By extending the modified FR method developed by Zhang et al. [58] for unconstrained optimization, Li and Wang [26] proposed an FR-type iterative scheme for symmetric nonlinear equations . The method was shown to converge globally under mild assumptions. Abubakar et al. [1, 2] recently proposed efficient FR-type methods for convex constrained monotone systems with their applications to signal and image processing problems. Also, the DY method [12] has gained the attention of researchers recently. By combining the DY scheme [12] with the spectral gradient method [6] and the projection technique [44], Liu and Li [31] developed a spectral DY-type projection scheme for solving system of nonlinear equations. Global convergence of the scheme was established without the differentiability assumption on the system of equations. Liu and Li [32] combined the DY method [12] with the multivariate spectral gradient scheme [56] to propose a multivariate spectral DY method for system of monotone nonlinear equations with convex constraint. The scheme is derivative-free, which makes it suitable for non-smooth equations. Global convergence of the method was also proved under mild conditions. Inspired by the work in [27] and [31], Liu and Feng [28] proposed a DY-type scheme for solving system of nonlinear monotone equations with convex constraint. An attribute of the scheme is its ability to solve large-scale non-smooth problems, which stems from the fact that it is derivative-free and requires low storage to implement. Global convergence of the proposed method is proved under the Lipschitz continuity assumption. Recently, motivated by the work in [28], Sani et al. [3] proposed a DY-type method for convex constrained system of monotone equations. Search direction of the scheme is taken as a convex combination of the DY update parameter and the classical CD parameter. Under mild conditions, global convergence of the scheme is proven. For further recent progress on addition methods, we refer the reader to [20, 47, 48, 49, 50, 40, 41, 51, 21, 22] and the references therein.

The aim of this article is to develop an efficient DY-type projection scheme for solving constrained systems of nonlinear equations with its application to image deblurring. We are inspired by the work of Xue et al. [54] and Yuan et al. [57], where in [54] the authors took advantage of the nice attribute of DY scheme to develop an algorithm with a spectral conjugate parameter and a search direction with the descent property vital for global convergence. Also, in [57], the authors developed an update parameter that switches between the efficient $CG\_DESCENT$ scheme [19] and a scheme that guarantees sufficient descent. We are also motivated by the fact that the DY method for unconstrained as well as constrained systems of nonlinear equations are rare in the literature as most of the research conducted on the scheme are focused on solving the minimization problem (8).

Apart from developing a new scheme with global convergence properties, another contribution of this research is its application in image restoration problems in compressed sensing.

The article is prepared as follows: Preliminaries leading to derivation of the new scheme is presented in the next section. The third section deals with derivation as well as convergence analysis of the scheme. Results of numerical experiments carried out on the scheme and three other methods are discussed in the fourth section. In section five, we present application of the proposed scheme to highlight its effectiveness. concluding remarks are presented in section six.

2. **Preliminaries and the Proposed Method.** Throughout the article, $\|x\| = \sqrt{x^T x}$, represents the $\ell_2\ norm$, $F_k = F(x_k)$, $F_{k-1} = F(x_{k-1})$, and $s_{k-1} = x_k - x_{k-1}$. Now, we give two assumptions, that will be required later in the article:

(i). A solution $\bar{x} \in \Omega$ exists such that $F(\bar{x}) = 0$.

(ii). The mapping F satisfies the Lipschitz continuity property; i.e, there exists a positive constant L such that for all $x, y \in \mathbf{R}^n$, the following holds:

$$\|F(x) - F(y)\| \leq L\|x - y\|. \tag{6}$$

Next, we discuss the projection scheme [44], in which a sequence $\{\bar{z}_k\}$ is generated satisfying

$$\bar{z}_k = x_k + \alpha_k d_k, \tag{7}$$

and $\alpha_k > 0$ is a steplength obtained in the direction d_k via suitable line search technique so that

$$F(\bar{z}_k)^T (x_k - \bar{z}_k) > 0. \tag{8}$$

For a solution $\bar{x}$ of (1), and using the monotonicity of F, we obtain

$$F(\bar{z}_k)^T(\bar{x}-\bar{z}_k) = F(\bar{z}_k) - F(\bar{x})^T(\bar{x}-\bar{z}_k) \leq 0. \tag{9}$$

Hence, by (8) and (9) it is clear that

$$H_k = \{x \in \mathbf{R}^n | F(\bar{z}_k)^T(x-\bar{z}_k) = 0\}, \tag{10}$$

represents the hyperplane, which separates x_k strictly from the solution $\bar{x}$ of (1). Solodov and Svaiter [44] employed this approach, where they suggested the use of the projection of x_k onto the hyperplane H_k to be the new iterate, namely

$$x_{k+1} = x_k - \frac{F(\bar{z}_k)^T(x_k-\bar{z}_k)}{\|F(\bar{z}_k)\|^2}F(\bar{z}_k). \tag{11}$$

Next we introduce the projection operator. Let $\Omega \subset \mathbf{R}^n$ be a nonempty, closed convex set. Then for each vector $x \in \mathbf{R}^n$, its projection onto Ω is given by

$$P_\Omega(x) = \arg\min\|x-y\| : y \in \Omega. \tag{12}$$

$P_\Omega : \mathbf{R}^n \to \Omega$ is refereed to as the projection operator, with the nonexpansive property given by,

$$\|P_\Omega(x) - P_\Omega(y)\| \leq \|x-y\|, \quad \forall x, y \in \mathbf{R}^n, \tag{13}$$

for which, we can write

$$\|P_\Omega(x) - y\| \leq \|x-y\|, \quad \forall y \in \Omega. \tag{14}$$

3. **Modified DY Method and Global Convergence.** This section deals with derivation of the scheme, its algorithm and convergence analysis. Firstly, we recall that a vital factor in global convergence of the CG method is the sufficient descent condition given by

$$g_k^T d_k \leq -\Gamma\|g_k\|^2, \quad \forall k \geq 0. \tag{15}$$

In the scheme to solve (8) by Hager and Zhang [19], namely

$$\beta_k^{HZ} = \frac{g_k^T y_{k-1}}{d_{k-1}^T y_{k-1}} - 2\frac{\|y_{k-1}\|^2}{d_{k-1}^T y_{k-1}}\frac{g_k^T d_{k-1}}{d_{k-1}^T y_{k-1}}, \quad y_{k-1} = g_k - g_{k-1}, \tag{16}$$

the sufficient descent condition (15) holds for $\Gamma = \frac{7}{8}$. In an attempt to develop a scheme with descent direction, Yuan et al. [57] gave the following modification of (16):

$$\beta_k^{GN} = \left(y_{k-1} - 2\frac{\|y_{k-1}\|^2}{T_k}d_{k-1}\right)^T \frac{g_k}{T_k}, \tag{17}$$

where

$$T_k = \max\left\{\nu\|d_{k-1}\|\|y_{k-1}\|, d_{k-1}^T y_{k-1}, \frac{2\|y_{k-1}\|^2 d_{k-1}^T g_k}{y_{k-1}^T g_k}\right\}, \tag{18}$$

with $\nu \in (0,1)$. Clearly, if $T_k = d_{k-1}^T y_{k-1}$, then (17) reduces to (16). Inspired by [19], Yu et al. [55] proposed a descent modification of the classical DY method with search direction defined by

$$d_k^{DDY} = -g_k + \beta_k^{DDY} d_{k-1}, \tag{19}$$

where

$$\beta_k^{DDY} = \beta_k^{DY} - \frac{C\|g_k\|^2}{(d_{k-1}^T y_{k-1})^2} d_{k-1}^T g_k, \tag{20}$$

with the update parameter β_k^{DY} given by

$$\beta_k^{DY} = \frac{g_k^T g_k}{d_{k-1}^T y_{k-1}}. \tag{21}$$

The scheme satisfies condition (15) for $\Gamma = 1 - \frac{1}{4C}$, where $C > \frac{1}{4}$. Recently, motivated by (20), and its improvement, Xue et al. [54] presented a modified DY method, which has application in image denoising. Search direction of the scheme is given by

$$\beta_k^{SDDY} = \beta_k^{SDY} - \min\left\{\beta_k^{SDY}, \frac{C\|g_k\|^2}{\delta_k (d_{k-1}^T y_{k-1})^2} d_{k-1}^T g_k\right\}, \tag{22}$$

where

$$\beta_k^{SDY} = \frac{\beta_k^{DY}}{\delta_k}, \quad \delta_k = \frac{s_{k-1}^T y_{k-1}}{\|s_{k-1}\|^2}. \tag{23}$$

By exploiting the descent PRP method by Cheng [8], the authors in [54] defined search direction of the scheme as

$$d_k = \begin{cases} -g_k + \beta_k^{SDDY}\left(I - \frac{g_k g_k^T}{g_k^T g_k}\right) d_{k-1} & k \geq 1; \\ -g_k, & \text{if} \quad k = 0. \end{cases} \tag{24}$$

Careful observation reveals that (24) satisfies the condition (15). Here, with motivation from the work of Xue et al. [54] as described by (22) and the approach employed by Yuan et al. [57] as presented in (17) and (18), we propose the following modified DY search direction

$$d_k = \begin{cases} -F_k + \beta_k^{MDDYM} s_{k-1}, & k \geq 1; \\ -F_k, & \text{if} \quad k = 0, \end{cases} \tag{25}$$

with

$$\beta_k^{MDDYM} = \beta_k^{MDY} - \min\left\{\beta_k^{MDY}, \frac{\mu\|F_k\|^2}{\Phi_k^2} F_k^T s_{k-1}\right\}, \quad \beta_k^{MDY} = \frac{F_k^T F_k}{\Phi_k}, \tag{26}$$

and

$$\Phi_k = \max\left\{\theta\|F_k\|\|s_{k-1}\|, s_{k-1}^T \bar{y}_{k-1}, \frac{\mu F_k^T F_k}{F_k^T y_{k-1}}\right\}, \quad \theta \in (0,1), \quad \mu \geq \frac{1}{4}. \tag{27}$$

Here, $y_{k-1} = F_k - F_{k-1}$, and $\bar{y}_{k-1}$ is defined as

$$\bar{y}_{k-1} = y_{k-1} + \bar{m}\frac{\|F_k\|}{\|s_{k-1}\|} s_{k-1}, \quad \bar{m} > 0. \tag{28}$$

From (28) and the monotonicity of F we obtain

$$s_{k-1}^T \bar{y}_{k-1} = s_{k-1}^T y_{k-1} + \bar{m}\|F_k\|\|s_{k-1}\| \geq \bar{m}\|F_k\|\|s_{k-1}\| > 0. \tag{29}$$

Considering (3), it is easy to deduce that

$$\Phi_k = \max\left\{\theta\|F_k\|\|d_{k-1}\|, s_{k-1}^T \bar{y}_{k-1}, \frac{\mu F_k^T F_k}{F_k^T y_{k-1}}\right\} \geq s_{k-1}^T \bar{y}_{k-1} > 0, \tag{30}$$

and

$$\beta_k^{MDDYM} \geq 0, \quad \forall k.$$

Next, we describe algorithm of the proposed method as follows:

Algorithm 1 Modified Descent Dai-Yuan Method (MDDYM)

Step 0: Choose a tolerance $\epsilon > 0$, initial guess $x_0 \in \Omega$, parameters $\beta \in (0,1)$,

$\rho \in (0,1)$, $\eta > 0$, $\theta \in (0,1)$, $\mu \geq \frac{1}{4}$ and $\bar{m} > 0$. Set $k = 0$ and $d_0 = -F_0$.

Step 1: Obtain $F(x_k)$. If $\|F(x_k)\| \leq \epsilon$, stop. If not, proceed to **Step** 2.

Step 2: Find $\bar{z}_k = x_k + \alpha_k d_k$, where $\alpha_k = \max\{\beta\rho^\iota : \iota = 0, 1, 2, ...\}$, for which

$$-F(x_k + \alpha_k d_k)^T d_k \geq \eta\alpha_k \|F(x_k + \alpha_k d_k)\| \|d_k\|^2, \tag{31}$$

holds.

Step 3: If $\bar{z_k} \in \Omega$ and $\|F(\bar{z_k})\| \leq \epsilon$, stop, else determine

$$x_{k+1} = P_\Omega \left[x_k - \bar{\mu}_k F(\bar{z_k})\right],$$

where

$$\bar{\mu}_k = \frac{(\bar{z}_k)^T (x_k - \bar{z}_k)}{\|F(\bar{z}_k)\|^2}.$$

Step 4: Generate direction d_{k+1} by (25),(26),(27).

Step 5: Set $k = k + 1$. Goto **Step** 1.

Global convergence of the MDDYM scheme is discussed in the remainder of this section.

Lemma 3.1. *Consider the sequence $\{d_k\}$ generated by (25),(26),(27). Then $\forall k \geq 0$, the following holds:*

$$F_k^T d_k \leq -\psi\|F_k\|^2 \quad \psi > 0, \quad \forall k \in \boldsymbol{N}. \tag{32}$$

Proof. Clearly by (25), if $k = 0$, then $F_0^T d_0 = -\|F_0\|^2$. For $k \geq 1$, using (25), (26), and (27), we have

$$\begin{aligned} F_k^T d_k &= -\|F_k\|^2 + \beta_k^{MDDYM} s_{k-1}^T F_k \\ &= -\|F_k\|^2 + \left(\beta_k^{MDY} - \min\left\{\beta_k^{MDY}, \frac{\mu\|F_k\|^2}{\Phi_k^2} s_{k-1}^T F_k\right\}\right) s_{k-1}^T F_k \\ &= -\|F_k\|^2 + \left(\frac{\|F_k\|^2}{\Phi_k} - \min\left\{\frac{\|F_k\|^2}{\Phi_k}, \frac{\mu\|F_k\|^2}{\Phi_k^2} s_{k-1}^T F_k\right\}\right) s_{k-1}^T F_k \end{aligned} \tag{33}$$

Now, two cases need to be analyzed.

1. If $\beta_k^{MDY} < \frac{\mu\|F_k\|^2}{\Phi_k^2} s_{k-1}^T F_k$, then by (33), $F_k^T d_k = -\|F_k\|^2$.

2. If $\beta_k^{MDY} \geq \frac{\mu\|F_k\|^2}{\Phi_k^2} s_{k-1}^T F_k$, then (33) can be expressed as

$$\begin{aligned}
F_k^T d_k &= -\|F_k\|^2 + \frac{\|F_k\|^2}{\Phi_k} s_{k-1}^T F_k - \frac{\mu\|F_k\|^2}{\Phi_k^2} \left(s_{k-1}^T F_k\right)^2 \\
&= \frac{\|F_k\|^2 \Phi_k s_{k-1}^T F_k - \Phi_k^2 \|F_k\|^2 - \mu \|F_k\|^2 \left(s_{k-1}^T F_k\right)^2}{\Phi_k^2} \\
&= \frac{\frac{\Phi_k F_k^T}{\sqrt{2\mu}} (s_{k-1}^T F_k) F_k \sqrt{2\mu} - \Phi_k^2 \|F_k\|^2 - \mu \|F_k\|^2 \left(s_{k-1}^T F_k\right)^2}{\Phi_k^2} \\
&\leq \frac{\frac{1}{2}\left(\frac{\Phi_k^2\|F_k\|^2}{2\mu} + 2\mu\|F_k\|^2 (s_{k-1}^T F_k)^2\right) - \Phi_k^2\|F_k\|^2 - \mu\|F_k\|^2 (s_{k-1}^T F_k)^2}{\Phi_k^2} \\
&= \frac{\frac{\Phi_k^2\|F_k\|^2}{4\mu} - \Phi_k^2\|F_k\|^2}{\Phi_k^2} \\
&= -\left(1 - \frac{1}{4\mu}\right)\|F_k\|^2.
\end{aligned} \tag{34}$$

The fourth inequality resulted from the relation $a^T b = \frac{1}{2}(\|a\|^2 + \|b\|^2)$, where

$$a = \frac{\Phi_k}{\sqrt{2\mu}} F_k, \quad b = s_{k-1}^T F_k \sqrt{2\mu} F_k.$$

Setting $\psi = \min\{1, 1 - \frac{1}{4\mu}\}$, we obtain the result. □

In the following Lemma, it is indicated that when solution of (1) is not attained, there exists always a step-size α_k such that the line search condition (31) is satisfied.

Lemma 3.2. *Let Assumptions (i) and (ii) hold. Then a step-size α_k exists satisfying the line search (31) $\forall k \geq 0$.*

Proof. For the sake of contradiction, suppose there exists $k_0 \geq 0$ such that for any non-negative integer i (31) does not hold at the k_0^{th} iterate, i.e,

$$-F(x_{k_0} + \beta\rho^i d_{k_0})^T d_{k_0} < \eta\beta\rho^i \|F(x_{k_0} + \beta\rho^i d_{k_0})\| \|d_{k_0}\|^2. \tag{35}$$

Applying continuity of F, (31), with the fact that $\rho \in (0,1)$, allowing i to grow to infinity, i.e $i \to \infty$, we obtain

$$-F(x_{k_0})^T d_{k_0} \leq 0. \tag{36}$$

From (32), we have

$$-F(x_{k_0})^T d_{k_0} = \|F(x_{k_0})\|^2 > 0, \tag{37}$$

which clearly contradicts (36). □

Lemma 3.3. *Let Assumptions (i), and (ii) hold and the sequences $\{x_k\}$ and $\{\bar{z}_k\}$ be generated by* ***Algorithm*** *1. Then a positive constant α_k exists satisfying*

$$\alpha_k \geq \min\left\{\beta, \frac{\rho\psi\|F_k\|^2}{(L + \eta\|F(x_k + \bar{\alpha}_k d_k)\|)\|d_k\|^2}\right\}. \tag{38}$$

Proof. By (31), if $\alpha_k = \beta$, then (38) holds. On the other hand, if $\alpha_k \neq \beta$ then $\bar{\alpha}_k = \alpha_k \rho^{-1}$ will not satisfy (38), which implies that,

$$-F(x_k + \bar{\alpha}_k d_k)^T d_k < \eta\bar{\alpha}_k \|F(x_k + \bar{\alpha}_k d_k)\| \|d_k\|^2, \tag{39}$$

By Assumption (ii) and (32), we can write

$$\begin{aligned}\psi\|F_k\|^2 &= -F_k^T d_k \\ &= (F(x_k+\bar{\alpha}_k d_k - F_k)^T d_k - F(x_k+\bar{\alpha}_k d_k)^T d_k \\ &\leq \alpha_k \rho^{-1}(L+\eta\|F(x_k+\bar{\alpha}_k d_k)\|)\|d_k\|^2.\end{aligned} \tag{40}$$

So,

$$\alpha_k \geq \frac{\rho\psi\|F_k\|^2}{(L+\eta\|F(x_k+\bar{\alpha}_k d_k)\|)\|d_k\|^2}. \tag{41}$$

Hence (38) holds, which completes the proof. □

Lemma 3.4. *Let $\{x_k\}$ be the sequence generated by* ***Algorithm*** 1*, then*

$$\lim_{k\to\infty} \alpha_k\|d_k\| = 0. \tag{42}$$

Proof. First, boundedness of the sequences $\{x_k\}$ and $\{\bar{z}_k\}$ needs to be shown. Given a solution $\bar{x}$ of (1), applying monotonicity of F we can write

$$F(\bar{z}_k)^T(x_k-\bar{x}) \geq F(\bar{z}_k)^T(x_k-\bar{z}_k). \tag{43}$$

From the line search condition (31) and (7), we obtain

$$F(\bar{z}_k)^T(x_k-\bar{z}_k) \geq \eta\alpha_k^2\|d_k\|^2 > 0. \tag{44}$$

Also, by using (13) and (3) we get

$$\begin{aligned}\|x_{k+1}-\bar{x}\|^2 &= \|P_\Omega(x_k-\bar{\mu}_k F(\bar{z}_k)) - (\bar{x})\|^2 \\ &\leq \|x_k-\bar{\mu}_k F(\bar{z}_k)-\bar{x}\|^2 \\ &= \|x_k-\bar{x}\|^2 - 2\bar{\mu}_k F(\bar{z}_k)^T(x_k-\bar{x}) + \bar{\mu}_k^2\|F(\bar{z}_k)\|^2.\end{aligned} \tag{45}$$

Since F is monotone, we can write

$$\begin{aligned}F(\bar{z}_k)^T(x_k-\bar{x}) &= F(\bar{z}_k)^T(x_k-\bar{z}_k) + F(\bar{z}_k)^T(\bar{z}_k-\bar{x}) \\ &\geq F(\bar{z}_k)^T(x_k-\bar{z}_k) + F(\bar{x})^T(\bar{z}_k-\bar{x}) \\ &= F(\bar{z}_k)^T(x_k-\bar{z}_k).\end{aligned} \tag{46}$$

From (44), (45), (46), and Cauchy Schwartz inequality, we have

$$\begin{aligned}\|x_{k+1}-\bar{x}\|^2 &\leq \|x_k-\bar{x}\|^2 - 2\bar{\mu}_k F(\bar{z}_k)^T(x_k-\bar{z}_k) + \bar{\mu}_k^2\|F(\bar{z}_k)\|^2 \\ &= \|x_k-\bar{x}\|^2 - \frac{F(\bar{z}_k)^T(x_k-\bar{z}_k)}{\|F(\bar{z}_k)\|^2} \\ &\leq \|x_k-\bar{x}\|^2,\end{aligned} \tag{47}$$

for which we obtain

$$\|x_{k+1}-\bar{x}\| \leq \|x_k-\bar{x}\|, \quad \forall k \geq 0. \tag{48}$$

So, $\{\|x_k-\bar{x}\|\}$ clearly is a sequence that is decreasing, and therefore, bounded. Furthermore, utilizing Assumption **(ii)**, (6) and (48) we have

$$\|F(x_{k+1})\| = \|F(x_{k+1}) - F(\bar{x})\| \leq L\|x_{k+1}-\bar{x}\| \leq L\|x_0-\bar{x}\|. \tag{49}$$

Setting $L\|x_0-\bar{x}\| = \pi$, the boundedness of $\{F_k\}$ is established.
By using (7), (44), Cauchy inequality, and the fact that F is monotone, we have

$$\eta\|x_k-\bar{z}_k\| = \frac{\eta\|\alpha_k d_k\|^2}{\|x_k-\bar{z}_k\|} \leq \frac{F(\bar{z}_k)^T(x_k-\bar{z}_k)}{\|x_k-\bar{z}_k\|} \leq \frac{F_k^T(x_k-\bar{z}_k)}{\|x_k-\bar{z}_k\|} \leq \|F_k\|. \tag{50}$$

So, using the boundedness of $\{x_k\}$, $\{F_k\}$, and utilizing (50), we deduce that $\{\bar{z}_k\}$ is bounded.

Now, boundedness of $\{\bar{z}_k\}$ implies that $\{\|\bar{z}_k - \bar{x}\|\}$ is bounded, namely, a constant $\bar{\varphi} > 0$ exists for any $\bar{x} \in \Omega$, such that

$$\|\bar{z}_k - \bar{x}\| \leq \bar{\varphi}. \tag{51}$$

From (6) and (51), we have

$$\|F(\bar{z}_k)\| = \|F(\bar{z}_k) - F(\bar{x})\| \leq L\|\bar{z}_k - \bar{x}\| \leq L\bar{\varphi}. \tag{52}$$

Now, from (42), we deduce that there exists a constant $\bar{\kappa} > 0$ satisfying $t_k\|d_k\| \leq \bar{\kappa}$. So, using $\bar{z}_k = x_k + \alpha_k\rho^{-1}d_k$ and employing similar argument as above, we can write

$$\|F(\bar{z}_k)\| = \|F(\bar{z}_k) - F(\bar{x})\| \leq L\|\bar{z}_k - \bar{x}\| \leq L(\|x_0 - \bar{x}\| + \rho^{-1}\bar{\kappa}) = \zeta. \tag{53}$$

Also, from (47), (50), (52), we obtain

$$\frac{\eta^2}{(L\bar{\varphi})^2}\sum_{k=0}^{\infty}\|x_k - \bar{z}_k\|^4 \leq \sum_{k=0}^{\infty}\frac{F(\bar{z}_k)^T(x - \bar{z}_k)^2}{\|F(\bar{z}_k)\|^2} \leq \sum_{k=0}^{\infty}(\|x_k - \bar{x}\|^2 - \|x_{k+1} - \bar{x}\|^2) < \infty. \tag{54}$$

Hence, taking limit as k approaches infinity of the convergent series in (54) and applying definition of $\bar{z}_k$, we obtain

$$\lim_{k \to \infty}\|x_k - \bar{z}_k\| = \lim_{k \to \infty}\alpha_k\|d_k\| = 0. \tag{55}$$

□

Lemma 3.5. *Let Assumptions **(i)** and **(ii)** hold and let $\{d_k\}$ be the sequence of directions generated by **Algorithm** 1. Then $\{d_k\}$ is bounded, namely, a constant $\vartheta > 0$ exists such that*

$$\|d_k\| \leq \vartheta, \quad \forall k \quad \textit{positive}. \tag{56}$$

Proof. From (42) we have that $\|\alpha_k d_k\| = \|s_k\|$ is convergent, hence bounded; i.e, there exists a constant $\bar{\varpi}$ such that

$$\|s_k\| \leq \bar{\varpi}, \quad \forall k. \tag{57}$$

By utilizing (25), (26), and (30), we obtain

$$\begin{aligned} \|d_k\| &= \| - F_k + \beta_k^{MDDYM}s_{k-1}\| \\ &= \| - F_k + \left(\beta_k^{MDY} - \min\left\{\beta_k^{MDY}, \frac{\mu\|F_k\|^2}{\Phi_k^2}s_{k-1}^TF_k\right\}\right)s_{k-1}\|. \end{aligned} \tag{58}$$

Here, we consider two possibilities.
1. If $\beta_k^{MDY} < \frac{\mu\|F_k\|^2}{\Phi_k^2}s_{k-1}^TF_k$, using (49), then (58) becomes

$$\|d_k\| = \|F_k\| < \pi = \vartheta_1. \tag{59}$$

2. If $\beta_k^{MDY} \geq \frac{\mu\|F_k\|^2}{\Phi_k^2}s_{k-1}^TF_k$, then from (27) we observe three cases or possibilities for Φ_k.

CASE I. $\Phi_k = \theta\|s_{k-1}\|\|F_k\|$. Using (30), (49), (57), and the Cauchy Schwarz inequality, then (58) becomes

$$\begin{aligned}
\|d_k\| &= \|F_k + \beta_k^{MDY} s_{k-1} - \frac{\mu\|F_k\|^2}{\Phi_k}\frac{s_{k-1}^T F_k}{\Phi_k} s_{k-1}\| \\
&= \|F_k + \frac{\|F_k\|^2}{\Phi_k} s_{k-1} - \frac{\mu\|F_k\|^2}{\Phi_k}\frac{s_{k-1}^T F_k}{\Phi_k} s_{k-1}\| \\
&\leq \|F_k\| + \frac{\|F_k\|^2\|s_{k-1}\|}{\theta\|s_{k-1}\|\|F_k\|} + \frac{\mu\|F_k\|^3\|s_{k-1}\|^2}{\theta^2\|s_{k-1}\|^2\|F_k\|^2} \\
&= \|F_k\| + \frac{\|F_k\|}{\theta} + \frac{\mu\|F_k\|}{\theta^2} \\
&\leq \pi + \frac{\pi}{\theta} + \frac{\mu\pi}{\theta^2} \\
&= \left(1 + \frac{1}{\theta} + \frac{\mu}{\theta^2}\right)\pi = \vartheta_2.
\end{aligned} \tag{60}$$

CASE II. $\Phi_k = s_{k-1}^T\bar{y}_{k-1}$. Using (3), (49), (57), the Cauchy Schwarz inequality, then (58) becomes

$$\begin{aligned}
\|d_k\| &= \|F_k + \beta_k^{MDY} s_{k-1} - \frac{\mu\|F_k\|^2}{\Phi_k}\frac{s_{k-1}^T F_k}{\Phi_k} s_{k-1}\| \\
&= \|F_k + \frac{\|F_k\|^2}{\Phi_k} s_{k-1} - \frac{\mu\|F_k\|^2}{\Phi_k}\frac{s_{k-1}^T F_k}{\Phi_k} s_{k-1}\| \\
&\leq \|F_k\| + \frac{\|F_k\|^2\|s_{k-1}\|}{s_{k-1}^T\bar{y}_{k-1}} + \frac{\mu\|F_k\|^3\|s_{k-1}\|^2}{(s_{k-1}^T\bar{y}_{k-1})^2} \\
&\leq \|F_k\| + \frac{\|F_k\|^2\|s_{k-1}\|}{\bar{m}\|F_k\|\|s_{k-1}\|} + \frac{\mu\|F_k\|^3\|s_{k-1}\|^2}{\bar{m}^2\|F_k\|^2\|s_{k-1}\|^2} \\
&= \|F_k\| + \frac{\|F_k\|}{\bar{m}} + \frac{\mu\|F_k\|}{\bar{m}^2} \\
&= \pi + \frac{\pi}{\bar{m}} + \frac{\mu\pi}{\bar{m}^2} \\
&= \left(1 + \frac{1}{\bar{m}} + \frac{\mu}{\bar{m}^2}\right)\pi = \vartheta_3.
\end{aligned} \tag{61}$$

CASE III. $\Phi_k = \frac{\mu\|F_k\|^2}{F_k^T y_{k-1}}$. Using (30), (49), (57), the Cauchy Schwarz inequality, then (58) becomes

$$\begin{aligned}
\|d_k\| &= \|F_k + \beta_k^{MDY} s_{k-1} - \frac{\mu\|F_k\|^2}{\Phi_k}\frac{s_{k-1}^T F_k}{\Phi_k} s_{k-1}\| \\
&= \|F_k + \frac{\|F_k\|^2}{\Phi_k} s_{k-1} - \frac{\mu\|F_k\|^2}{\Phi_k}\frac{s_{k-1}^T F_k}{\Phi_k} s_{k-1}\| \\
&\leq \|F_k\| + \frac{\|F_k\|^3\|s_{k-1}\|\|y_{k-1}\|}{\mu\|F_k\|^2} + \frac{\mu\|F_k\|^5\|s_{k-1}\|^2\|y_{k-1}\|^2}{\mu^2\|F_k\|^4} \\
&= \|F_k\| + \frac{\|F_k\|\|s_{k-1}\|\|y_{k-1}\|}{\mu} + \frac{\|F_k\|\|s_{k-1}\|^2\|y_{k-1}\|^2}{\mu} \\
&\leq \|F_k\| + \frac{L\|F_k\|\|s_{k-1}\|^2}{\mu} + \frac{L^2\|F_k\|\|s_{k-1}\|^4}{\mu}
\end{aligned} \tag{62}$$

$$
\begin{aligned}
&\leq \quad \pi+\frac{L\pi\bar{\varpi}^2}{\mu}+\frac{L^2\pi\bar{\varpi}^4}{\mu}\\
&= \quad \left(1+\frac{L\bar{\varpi}^2}{\mu}+\frac{L^2\bar{\varpi}^4}{\mu}\right)\pi=\vartheta_4.
\end{aligned}
$$

Setting $\vartheta=\max\{\vartheta_1,\vartheta_2,\vartheta_3,\vartheta_4\}$, the proof is completed. □

The following theorem establishes global convergence of **Algorithm** 1.

Theorem 3.6. *Let Assumptions **(i)** and **(ii)** hold. Consider the sequences* $\{x_k\}$ *and* $\{\bar{z}_k\}$ *generated by **Algorithm** 1. Then*

$$\liminf_{k\to\infty}\|F_k\|=0. \tag{63}$$

Proof. We proceed by contradiction. Let the conclusion (63) be false, then there exists $\epsilon_1>0$ satisfying

$$\|F_k\|\geq\epsilon_1,\quad \forall k\geq 0. \tag{64}$$

From (32) and the Cauchy-Schwarz inequality, we have

$$\psi\|F_k\|^2\leq -F_k^Td_k\leq\|F(x_k)\|\|d_k\|, \tag{65}$$

for which, using (64) we obtain,

$$\|d_k\|\geq\psi\epsilon_1. \tag{66}$$

Applying (38), (53), (56), (64) and (66) we get

$$
\begin{aligned}
\alpha_k\|d_k\|&\geq\min\left\{\beta,\frac{\rho\psi\|F_k\|^2}{(L+\eta\|F(x_k+\bar{\alpha}_kd_k)\|)\|d_k\|^2}\right\}\|d_k\|\\
&\geq\min\left\{\beta\psi\epsilon_1,\frac{\rho\psi\epsilon_1^2}{(L+\eta\zeta)\vartheta}\right\}>0.
\end{aligned} \tag{67}
$$

Taking limits of both sides of the last inequality yields a contradiction with (42), and we conclude that $\liminf_{k\to\infty}\|F_k\|=0$. □

4. **Numerical Experiments and Discussions.** To investigate effectiveness of the $MDDYM$ scheme, its performance is compared with the three recent efficient methods presented in [28, 3, 2]. For simplicity, the three methods are labeled as $PDYM$, MDY, and MFR and the line search parameters are set as the authors applied them in each of the papers. On the other hand, the line search parameters for the $MDDYM$ method was set as $\rho=0.45$, $\beta=0.95$, $\eta=10^{-4}$. In addition, we set $\mu=0.26,\theta=0.1$. Codes for the algorithms were written using Matlab R2015a and run on a PC with configuration (2.30GHZ CPU, 4GB RAM). In the experiments, all four algorithms are set to terminate if $\|F(x_k)\|\leq 10^{-8}$ or $\|F(z_k)\|\leq 10^{-8}$ or iterations exceed 1000.

Furthermore, numerical results of the experiments conducted are reported in Tables $1-8$, where "Nvars" denote dimensions, "IGuess" and "Niter" represent the initial starting point and number of iterations respectively. Also, "Nfev" and "Cpt" indicate total number of function evaluations and processing time recorded. In addition, "$Norm$" stands for residual at stopping point, while the symbol "$**$" indicates failure to obtain a solution after 1000 iterations. Moreover, to support the results of Tables $1-8$, performance profile of Dolan and Moré [14] was utilized as an evaluation tool to approximately assess the performances of the four schemes.

The following test problems were used to test the four methods:

Problem 4.1. Non-smooth function [10].
$F_i(x) = 2x_i - \sin|x_i|, \quad i = 1, 2, \ldots, n,$
where $\Omega = \left\{ x \in \mathbf{R}^n | \sum_{i=1}^{n} x_i \leq n, \quad x_i \geq 0, \quad i = 1, 2, \ldots, n \right\}$.

Problem 4.2. [10]
$F_i(x) = \min(\min(|x_i|, x_i^2), \max(|x_i|, x_i^3)), \quad i = 1, 2 \ldots, n,$
where $\Omega = \mathbf{R}^n_+$.

Problem 4.3. Trigexp function [37].
$F_1(x) = 3x_1^3 + 2x_2 - 5 + \sin(x_1 - x_2)\sin(x_1 + x_2),$
$F_i(x) = -x_{i-1}e^{(x_{i-1} - x_i)} + x_i(4 + 3x_i^2) + 2x_{i+1} + \sin(x_i - x_{i+1})\sin(x_i + x_{i+1}) - 8,$
$F_n(x) = -x_{n-1}e^{(x_{n-1} - x_n)} + 4x_n - 3, \quad i = 2, 3, \ldots, n - 1,$
where $\Omega = \mathbf{R}^n_+$.

Problem 4.4. Strictly convex function [45].
$F_i(x) = e^{x_i} - 1, \quad i = 1, 2, \ldots, n,$
where $\Omega = \mathbf{R}^n_+$.

Problem 4.5. Tridiagonal Exponential Function [30].
$F_1(x) = x_1 - e^{\left(cos\frac{x_1 + x_2}{n+1}\right)},$
$F_i(x) = x_i - e^{\left(cos\frac{x_{i-1} + x_i + x_{i+1}}{n+1}\right)}, \quad i = 2, 3, \ldots, n - 1,$
$F_n(x) = x_n - e^{\left(cos\frac{x_{n-1} + x_n}{n+1}\right)}.$
where $\Omega = \mathbf{R}^n_+$.

Problem 4.6. Non-smooth function [53]
$F_i(x) = x_i - \sin|x_i - 1|, \quad i = 1, 2, \ldots, n,$
where $\Omega = \left\{ x \in \mathbf{R}^n | \sum_{i=1}^{n} x_i \leq n, \quad x_i \geq -1, \quad i = 1, 2, \ldots, n \right\}$.

Problem 4.7. Non-smooth function [53]
$F_i(x) = x_i - 2\sin|x_i - 1|, \quad i = 1, 2, \ldots, n,$
where $\Omega = \mathbf{R}^n_+$.

Problem 4.8. The problem is obtained from [37].
$F_1(x) = -2x_1 - x_2 + e^{x_1} - 1,$
$F_i(x) = -x_{i-1} + 2x_i - x_{i+1} + e^{x_i} - 1, \quad i = 2, 3, \ldots, n - 1,$
$F_n(x) = 2x_n - x_{n-1} + e^{x_n} - 1.$
where $\Omega = \mathbf{R}^n_+$.

For each of the above test problems, 24 numerical experiments were performed with dimensions $5,000$, $10,000$, $50,000$, and the following initial starting points:

$x1 = (\frac{1}{100}, \frac{1}{100}, ..., \frac{1}{100})^T, x2 = (\frac{2}{100}, \frac{2}{100}, ..., \frac{2}{100})^T, x3 = (\frac{1}{10}, \frac{1}{10}, ..., \frac{1}{10})^T,$
$x4 = (0.75, 0.75, ..., 0.75)^T, x5 = (1.25, 1.25, ..., 1.25)^T, x6 = (1.75, 1.75, ..., 1.75)^T,$
$x7 = (2.25, 2.25, ..., 2.25)^T, x8 = (2.5, 2.5, ..., 2.5)^T$. In all, a total of 192 cases were considered for the experiments.

TABLE 1. Test results of the four methods for problems 4.1

Nvars	IGuess	MDDYM				PDYM				MDY				MFR			
		Niter	**Nfev**	**Cpt**	**Norm**	**Niter**	**Nfev**	**Cpt**	**Norm**	**Niter**	**Nfev**	**Cpt**	**Norm**	**Niter**	**Nfev**	**Cpt**	**Norm**
5000	**x1**	6	12	0.27234	2.0475E-09	16	35	0.30469	4.1222E-09	11	25	0.13273	2.4989E-09	26	106	0.36405	0.0000E+00
5000	**x2**	6	12	0.07889	4.0428E-09	16	35	0.05510	8.2456E-09	12	28	0.05936	2.2971E-09	26	106	0.12459	0.0000E+00
5000	**x3**	3	5	0.01436	8.4985E-09	18	39	0.06173	4.5974E-09	12	28	0.07119	5.7248E-09	30	122	0.17739	0.0000E+00
5000	**x4**	5	9	0.02079	1.8997E-09	20	43	0.06536	4.6462E-09	14	34	0.05971	2.5758E-09	34	138	0.16181	0.0000E+00
5000	**x5**	11	23	0.03935	3.9754E-10	20	43	0.07339	8.5913E-09	15	37	0.05510	2.6441E-09	36	146	0.18173	0.0000E+00
5000	**x6**	20	45	0.08341	8.4879E-09	21	45	0.07166	3.4438E-09	15	36	0.06630	5.1003E-09	35	141	0.17659	0.0000E+00
5000	**x7**	12	25	0.05233	5.4026E-10	21	46	0.08474	4.3326E-09	15	39	0.09094	5.0139E-09	35	142	0.16383	0.0000E+00
5000	**x8**	10	21	0.04351	9.5806E-09	21	46	0.07816	5.2005E-09	15	41	0.07043	4.5835E-09	35	142	0.18567	0.0000E+00
10000	**x1**	6	12	0.04366	2.8956E-09	16	35	0.11302	5.8297E-09	10	23	0.05868	8.5342E-09	26	106	0.22574	0.0000E+00
10000	**x2**	6	12	0.04652	5.7173E-09	17	37	0.10115	3.8700E-09	12	28	0.13233	3.1978E-09	26	106	0.25446	0.0000E+00
10000	**x3**	4	6	0.02748	5.9571E-10	18	39	0.10617	6.5017E-09	12	28	0.08322	3.7136E-09	30	122	0.25620	0.0000E+00
10000	**x4**	5	9	0.03605	2.6866E-09	20	43	0.12137	6.5707E-09	14	33	0.10267	4.4531E-09	34	138	0.26421	0.0000E+00
10000	**x5**	11	23	0.09120	5.6221E-10	21	45	0.12382	4.0482E-09	15	37	0.11103	5.1673E-09	36	146	0.31887	0.0000E+00
10000	**x6**	21	48	0.14197	6.1811E-09	20	44	0.15292	5.5936E-09	16	39	0.12905	3.0303E-09	35	141	0.31019	0.0000E+00
10000	**x7**	12	25	0.08928	7.6404E-10	21	46	0.13180	6.1272E-09	16	43	0.12267	3.0885E-09	35	142	0.29184	0.0000E+00
10000	**x8**	11	22	0.07167	3.5449E-10	21	47	0.12698	7.3547E-09	15	41	0.14901	2.9293E-09	35	142	0.29874	0.0000E+00
50000	**x1**	6	12	0.18477	6.4748E-09	17	37	0.43468	4.3261E-09	12	28	0.31329	3.4966E-09	26	106	1.02635	0.0000E+00
50000	**x2**	7	13	0.18950	4.9344E-10	17	37	0.39080	8.6535E-09	12	28	0.32265	5.2020E-09	26	106	0.99223	0.0000E+00
50000	**x3**	4	6	0.09771	1.3320E-09	19	41	0.47699	4.8342E-09	13	31	0.37285	4.4055E-09	30	122	1.16776	0.0000E+00
50000	**x4**	5	9	0.13988	6.0075E-09	21	45	0.48957	4.8925E-09	16	42	0.43939	2.3366E-09	34	138	1.26896	0.0000E+00
50000	**x5**	11	23	0.29815	1.2571E-09	23	53	0.54401	5.3731E-09	18	52	0.53469	2.9724E-09	36	146	1.33839	0.0000E+00
50000	**x6**	22	51	0.55331	7.5655E-09	23	54	0.54909	7.2056E-09	19	55	0.59269	3.2582E-09	35	141	1.28348	0.0000E+00
50000	**x7**	12	25	0.32319	1.7085E-09	23	54	0.58040	4.9765E-09	19	61	0.60155	3.3731E-09	35	142	1.33907	0.0000E+00
50000	**x8**	11	22	0.26948	7.9265E-10	23	54	0.56286	5.4177E-09	18	56	0.56586	2.7351E-09	35	142	1.31569	0.0000E+00

TABLE 2. Test results of the four methods for problems 4.2

Nvars	IGuess	MDDYM				PDYM				MDY				MFR			
		Niter	**Nfev**	**Cpt**	**Norm**	**Niter**	**Nfev**	**Cpt**	**Norm**	**Niter**	**Nfev**	**Cpt**	**Norm**	**Niter**	**Nfev**	**Cpt**	**Norm**
5000	**x1**	1	2	0.01078	0.0000E+00	1	2	0.01097	0.0000E+00	1	2	0.01014	0.0000E+00	1	2	0.01003	0.0000E+00
5000	**x2**	1	2	0.02040	0.0000E+00	1	2	0.00944	0.0000E+00	1	2	0.01005	0.0000E+00	1	2	0.02019	0.0000E+00
5000	**x3**	1	2	0.01013	0.0000E+00	1	2	0.00965	0.0000E+00	1	2	0.00947	0.0000E+00	1	2	0.01038	0.0000E+00
5000	**x4**	1	2	0.00951	0.0000E+00	1	2	0.00966	0.0000E+00	1	2	0.00928	0.0000E+00	**	**	**	**
5000	**x5**	1	2	0.01045	0.0000E+00	1	2	0.00748	0.0000E+00	1	2	0.00649	0.0000E+00	1	2	0.00520	0.0000E+00
5000	**x6**	1	2	0.00940	0.0000E+00	1	2	0.00732	0.0000E+00	1	2	0.00662	0.0000E+00	1	2	0.00842	0.0000E+00
5000	**x7**	1	2	0.00979	0.0000E+00	1	2	0.00660	0.0000E+00	1	2	0.00667	0.0000E+00	1	2	0.00757	0.0000E+00
5000	**x8**	1	2	0.01111	0.0000E+00	1	2	0.00795	0.0000E+00	1	2	0.00687	0.0000E+00	1	2	0.00779	0.0000E+00
10000	**x1**	1	2	0.01478	0.0000E+00	1	2	0.01622	0.0000E+00	1	2	0.01650	0.0000E+00	1	2	0.01449	0.0000E+00
10000	**x2**	1	2	0.01571	0.0000E+00	1	2	0.01633	0.0000E+00	1	2	0.01488	0.0000E+00	1	2	0.01594	0.0000E+00
10000	**x3**	1	2	0.01490	0.0000E+00	1	2	0.02384	0.0000E+00	1	2	0.03357	0.0000E+00	1	2	0.01579	0.0000E+00
10000	**x4**	1	2	0.01739	0.0000E+00	1	2	0.01549	0.0000E+00	1	3	0.02037	0.0000E+00	**	**	**	**
10000	**x5**	1	2	0.01636	0.0000E+00	1	2	0.01000	0.0000E+00	1	2	0.01013	0.0000E+00	1	2	0.00759	0.0000E+00
10000	**x6**	1	2	0.01637	0.0000E+00	1	2	0.00939	0.0000E+00	1	2	0.01043	0.0000E+00	1	2	0.01118	0.0000E+00
10000	**x7**	1	2	0.01602	0.0000E+00	1	2	0.00893	0.0000E+00	1	2	0.00994	0.0000E+00	1	2	0.01012	0.0000E+00
10000	**x8**	1	2	0.01539	0.0000E+00	1	2	0.01026	0.0000E+00	1	2	0.01045	0.0000E+00	1	2	0.01100	0.0000E+00
50000	**x1**	1	2	0.05796	0.0000E+00	1	2	0.06032	0.0000E+00	1	2	0.06280	0.0000E+00	1	2	0.05899	0.0000E+00
50000	**x2**	1	2	0.06203	0.0000E+00	1	2	0.05772	0.0000E+00	1	2	0.06076	0.0000E+00	1	2	0.05922	0.0000E+00
50000	**x3**	1	2	0.05224	0.0000E+00	1	2	0.05857	0.0000E+00	1	2	0.06821	0.0000E+00	1	2	0.06629	0.0000E+00
50000	**x4**	1	2	0.06079	0.0000E+00	1	3	0.07697	0.0000E+00	1	5	0.11996	0.0000E+00	**	**	**	**
50000	**x5**	1	2	0.06053	0.0000E+00	1	2	0.03772	0.0000E+00	1	2	0.03890	0.0000E+00	1	2	0.03026	0.0000E+00
50000	**x6**	1	2	0.05694	0.0000E+00	1	2	0.03597	0.0000E+00	1	2	0.03897	0.0000E+00	1	2	0.03691	0.0000E+00
50000	**x7**	1	2	0.06281	0.0000E+00	1	2	0.03922	0.0000E+00	1	2	0.03959	0.0000E+00	1	2	0.03842	0.0000E+00
50000	**x8**	1	2	0.05664	0.0000E+00	1	2	0.03785	0.0000E+00	1	2	0.03866	0.0000E+00	1	2	0.04029	0.0000E+00

TABLE 3. Test results of the four methods for problems 4.3

Nvars	IGuess	MDDYM				PDYM				MDY				MFR			
		Niter	Nfev	Cpt	Norm	Niter	Nfev	Cpt	Norm	Niter	Nfev	Cpt	Norm	Niter	Nfev	Cpt	Norm
5000	**x1**	44	271	0.62935	3.1455E-09	42	258	0.60676	6.9529E-09	33	161	0.45597	7.0484E-09	**	**	**	**
5000	**x2**	39	243	0.63451	5.8143E-09	42	258	0.63349	6.8440E-09	49	253	0.65486	6.8393E-09	**	**	**	**
5000	**x3**	40	252	0.60480	7.5147E-09	41	252	0.58345	9.1694E-09	31	147	0.42323	8.9989E-09	**	**	**	**
5000	**x4**	37	233	0.57729	3.7953E-09	35	216	0.53482	9.0123E-09	40	194	0.51740	7.4457E-09	**	**	**	**
5000	**x5**	33	202	0.46893	9.9461E-09	35	217	0.53483	8.7841E-09	42	204	0.52178	7.4199E-09	**	**	**	**
5000	**x6**	34	212	0.52856	8.7562E-09	40	247	0.57437	6.6311E-09	64	358	0.89216	7.2966E-09	**	**	**	**
5000	**x7**	43	267	0.70223	9.1513E-09	37	228	0.55987	7.0629E-09	33	160	0.43868	9.8882E-09	**	**	**	**
5000	**x8**	39	244	0.62983	5.5424E-09	38	236	0.56374	9.7994E-09	39	212	0.51684	4.9792E-09	**	**	**	**
10000	**x1**	46	283	1.24261	8.9039E-09	41	252	1.06589	8.7049E-09	43	201	0.99828	6.2921E-09	**	**	**	**
10000	**x2**	43	270	1.20866	6.3134E-09	41	252	1.09452	8.5691E-09	33	125	0.65028	8.5659E-09	**	**	**	**
10000	**x3**	35	213	0.98029	7.5116E-09	41	252	1.11311	7.5636E-09	38	200	0.90475	9.7816E-09	**	**	**	**
10000	**x4**	41	256	1.13918	6.2659E-09	35	216	0.95719	7.4878E-09	**	**	**	**	**	**	**	**
10000	**x5**	38	232	1.09266	4.3520E-09	35	217	0.95539	7.2962E-09	32	165	0.78884	6.7343E-09	**	**	**	**
10000	**x6**	38	232	1.05984	8.2915E-09	39	241	1.02612	8.3641E-09	40	189	0.88094	8.8953E-09	**	**	**	**
10000	**x7**	41	262	1.15504	6.3728E-09	32	200	0.91508	9.2095E-09	42	207	1.01358	9.6843E-09	**	**	**	**
10000	**x8**	44	263	1.22024	4.4229E-09	45	278	1.23987	7.9323E-09	**	**	**	**	**	**	**	**
50000	**x1**	51	316	6.68143	6.4806E-09	42	260	5.45556	7.9101E-09	38	187	4.12196	7.2817E-09	**	**	**	**
50000	**x2**	48	301	6.33527	8.4813E-09	42	260	5.42795	7.8102E-09	40	181	4.31980	5.7514E-09	**	**	**	**
50000	**x3**	36	218	4.66628	7.4951E-09	42	260	5.36750	7.0264E-09	44	190	4.36631	3.5019E-09	**	**	**	**
50000	**x4**	33	206	4.39647	3.1500E-09	34	210	4.33667	7.4626E-09	34	143	3.36253	3.8370E-09	**	**	**	**
50000	**x5**	36	226	4.71243	9.0560E-09	34	211	4.36821	7.2827E-09	32	132	3.13728	9.7006E-09	**	**	**	**
50000	**x6**	38	235	4.97988	9.5093E-09	40	249	5.25738	8.0717E-09	36	164	3.76700	5.7396E-09	**	**	**	**
50000	**x7**	44	282	5.86905	6.3230E-09	37	234	4.87686	6.5499E-09	40	200	4.41362	5.0952E-09	**	**	**	**
50000	**x8**	37	231	4.86337	9.4485E-09	33	210	4.50616	9.9184E-09	36	138	3.29200	6.6412E-09	**	**	**	**

TABLE 4. Test results of the four methods for problems 4.4

Nvars	IGuess	MDDYM				PDYM				MDY				MFR			
		Niter	Nfev	Cpt	Norm	Niter	Nfev	Cpt	Norm	Niter	Nfev	Cpt	Norm	Niter	Nfev	Cpt	Norm
5000	**x1**	4	6	0.01613	1.1694E-09	16	35	0.06067	4.1367E-09	11	25	0.04204	2.7254E-09	13	22	0.05005	2.5649E-09
5000	**x2**	4	6	0.01681	8.3703E-09	16	35	0.06441	8.3035E-09	6	14	0.02475	3.7553E-09	15	25	0.05655	1.3159E-09
5000	**x3**	3	5	0.01304	3.3286E-09	18	39	0.05937	4.7509E-09	12	28	0.04195	4.8432E-09	17	28	0.05855	5.2831E-09
5000	**x4**	15	33	0.05422	1.7766E-09	20	43	0.06423	3.6850E-09	13	30	0.05059	2.4951E-09	21	35	0.09843	3.9220E-09
5000	**x5**	10	23	0.05208	1.0831E-09	16	35	0.05671	5.9886E-09	15	36	0.06009	5.0128E-09	21	37	0.07153	3.9833E-09
5000	**x6**	8	17	0.04271	6.9887E-09	20	44	0.06183	7.2667E-09	16	41	0.06403	4.8801E-09	22	42	0.09015	3.9830E-09
5000	**x7**	20	44	0.06693	1.8089E-09	20	46	0.07138	9.6847E-09	15	43	0.05725	3.8385E-09	22	43	0.08100	3.9830E-09
5000	**x8**	8	16	0.03136	2.2485E-09	20	46	0.06917	6.3318E-09	15	44	0.52479	4.4557E-09	22	44	0.08352	3.9830E-09
10000	**x1**	4	6	0.02494	1.6538E-09	16	35	0.08973	5.8502E-09	11	25	0.06575	2.1388E-09	13	22	0.07581	3.6273E-09
10000	**x2**	5	7	0.02770	1.0276E-10	17	37	0.08415	3.8974E-09	12	28	0.06498	3.1362E-09	15	25	0.12385	1.8610E-09
10000	**x3**	3	5	0.02101	4.7073E-09	18	39	0.08491	6.7188E-09	11	26	0.07826	6.4560E-09	17	28	0.12292	7.4714E-09
10000	**x4**	15	33	0.10815	2.5125E-09	20	43	0.09236	5.2114E-09	15	37	0.09462	2.7562E-09	21	35	0.13054	5.5465E-09
10000	**x5**	10	23	0.06283	1.5317E-09	17	38	0.09884	9.9575E-09	16	40	0.21861	3.1439E-09	21	37	0.13807	5.6333E-09
10000	**x6**	8	17	0.05651	9.8835E-09	21	48	0.11028	7.9275E-09	16	42	0.10622	4.2741E-09	22	42	0.13112	5.6328E-09
10000	**x7**	20	44	0.10776	2.5582E-09	21	49	0.12883	4.5754E-09	16	46	0.08559	5.0752E-09	22	43	0.12979	5.6328E-09
10000	**x8**	8	16	0.05015	3.1798E-09	20	47	0.11975	8.9545E-09	16	47	0.11307	5.1334E-09	22	44	0.12479	5.6328E-09
50000	**x1**	4	6	0.09720	3.6981E-09	17	37	0.36221	4.3415E-09	12	28	0.27787	3.4805E-09	13	22	0.29020	8.1110E-09
50000	**x2**	5	7	0.11296	2.2973E-10	17	37	0.32769	8.7148E-09	12	28	0.27745	5.2196E-09	15	25	0.48431	4.1613E-09
50000	**x3**	4	7	0.09722	5.7871E-09	19	41	0.37649	4.9966E-09	13	31	0.30385	4.6998E-09	19	31	0.40698	2.0700E-09
50000	**x4**	15	33	0.34967	5.6182E-09	22	49	0.42467	3.5662E-09	16	41	0.39497	5.2693E-09	23	38	0.48994	1.5367E-09
50000	**x5**	10	23	0.25464	3.4249E-09	22	52	0.44383	8.0423E-09	18	50	0.44577	4.6654E-09	23	40	0.49968	1.5607E-09
50000	**x6**	9	18	0.19377	6.5941E-10	23	55	0.46324	4.6940E-09	19	54	0.46182	5.3105E-09	24	45	0.53573	1.5606E-09
50000	**x7**	20	44	0.43657	5.7203E-09	23	58	0.53040	9.1580E-09	18	57	0.47142	3.8366E-09	24	46	0.54006	1.5606E-09
50000	**x8**	8	16	0.17111	7.1103E-09	23	58	0.49150	8.6064E-09	18	57	0.48898	3.4965E-09	24	47	0.56776	1.5606E-09

Table 5. Test results of the four methods for problems 4.5

Nvars	**IGuess**	**MDDYM**				**PDYM**				**MDY**				**MFR**			
		Niter	**Nfev**	**Cpt**	**Norm**	**Niter**	**Nfev**	**Cpt**	**Norm**	**Niter**	**Nfev**	**Cpt**	**Norm**	**Niter**	**Nfev**	**Cpt**	**Norm**
5000	**x1**	13	29	0.08409	6.1206E-09	21	45	0.11835	4.5793E-09	13	34	0.07926	8.7285E-09	27	57	0.14177	9.2581E-09
5000	**x2**	13	29	0.08607	6.0981E-09	21	45	0.11653	4.5624E-09	13	34	0.07905	8.6940E-09	27	57	0.14553	9.2256E-09
5000	**x3**	13	29	0.08117	5.9183E-09	21	45	0.09188	4.4271E-09	13	34	0.09186	8.4192E-09	27	57	0.13343	8.9623E-09
5000	**x4**	13	29	0.07941	4.4547E-09	20	43	0.14263	9.9982E-09	12	29	0.09527	8.3184E-09	27	57	0.14216	6.6834E-09
5000	**x5**	13	29	0.07700	3.3263E-09	20	43	0.11080	7.4583E-09	15	38	0.09559	3.4919E-09	27	57	0.12148	4.9324E-09
5000	**x6**	12	27	0.07471	8.0821E-09	20	43	0.12521	4.9185E-09	14	34	0.08074	3.5022E-09	26	55	0.12577	7.3575E-09
5000	**x7**	12	27	0.11041	3.8935E-09	19	41	0.11220	7.1229E-09	13	31	0.07748	4.0165E-09	25	53	0.13185	8.0804E-09
5000	**x8**	11	25	0.06216	4.7068E-09	18	39	0.10240	9.9853E-09	13	31	0.08459	3.3404E-09	24	51	0.51949	8.5601E-09
10000	**x1**	13	30	0.15350	2.3785E-09	23	53	0.19978	5.1107E-09	15	46	0.16562	8.1229E-09	28	58	0.31646	0.0000E+00
10000	**x2**	15	37	0.15897	5.1927E-09	23	53	0.19735	5.0918E-09	15	46	0.17351	8.0936E-09	28	58	0.24591	0.0000E+00
10000	**x3**	19	54	0.23207	2.7569E-09	23	53	0.21008	4.9409E-09	15	46	0.15890	7.8592E-09	28	58	0.23659	0.0000E+00
10000	**x4**	13	32	0.13566	2.2501E-09	21	45	0.18849	4.7069E-09	14	39	0.16351	6.5113E-09	27	57	0.37153	9.3333E-09
10000	**x5**	14	35	0.13436	7.6581E-09	21	45	0.16255	3.5112E-09	13	34	0.12313	6.3921E-09	27	57	0.23985	6.9447E-09
10000	**x6**	12	28	0.12876	7.1509E-09	20	43	0.18314	6.9563E-09	11	25	0.12049	8.3813E-09	27	56	0.25423	0.0000E+00
10000	**x7**	9	18	0.08808	5.6775E-09	20	43	0.21754	3.3642E-09	12	29	0.13141	4.5477E-09	26	54	0.21101	0.0000E+00
10000	**x8**	7	13	0.06663	5.1294E-09	19	41	0.16822	4.6958E-09	13	31	0.14696	4.3632E-09	25	52	0.23674	0.0000E+00
50000	**x1**	8	14	0.31637	9.7835E-10	26	67	1.02269	9.0422E-09	23	107	1.49375	8.1128E-09	24	50	0.92306	0.0000E+00
50000	**x2**	8	14	0.29982	9.7482E-10	26	67	1.06621	9.0088E-09	23	107	1.41950	8.0837E-09	24	50	0.92547	0.0000E+00
50000	**x3**	8	14	0.29072	9.4607E-10	26	67	1.07177	8.7417E-09	23	104	1.38072	6.5311E-09	24	50	0.89133	0.0000E+00
50000	**x4**	8	14	0.29903	7.1236E-10	25	63	1.00707	7.9619E-09	20	80	1.19303	6.3556E-09	24	50	0.94953	0.0000E+00
50000	**x5**	8	14	0.29453	5.3210E-10	23	53	0.87334	6.1957E-09	17	58	0.90448	6.6277E-09	24	50	0.88540	0.0000E+00
50000	**x6**	8	14	0.31612	3.5134E-10	22	49	0.86319	7.0213E-09	14	40	0.63377	7.7993E-09	22	46	0.86539	0.0000E+00
50000	**x7**	8	14	0.28685	1.7015E-10	20	43	0.74746	7.5227E-09	15	38	0.65625	3.5221E-09	22	46	0.85031	0.0000E+00
50000	**x8**	7	13	0.27290	5.4616E-09	20	43	0.74983	3.5066E-09	11	27	0.48111	6.0062E-09	22	46	0.85129	0.0000E+00

TABLE 6. Test results of the four methods for problems 4.6

Nvars	IGuess	MDDYM				PDYM				MDY				MFR			
		Niter	**Nfev**	**Cpt**	**Norm**	**Niter**	**Nfev**	**Cpt**	**Norm**	**Niter**	**Nfev**	**Cpt**	**Norm**	**Niter**	**Nfev**	**Cpt**	**Norm**
5000	**x1**	11	29	0.05830	5.9634E-09	21	66	0.09121	4.9608E-09	12	33	0.05388	5.3440E-09	60	245	0.30692	9.2918E-09
5000	**x2**	11	29	0.05493	4.3213E-09	21	66	0.07995	4.7837E-09	12	33	0.05358	5.8173E-09	60	245	0.29193	9.1558E-09
5000	**x3**	10	27	0.04793	1.1728E-09	20	63	0.08563	9.6409E-09	12	33	0.05520	4.2467E-09	60	245	0.30544	7.9678E-09
5000	**x4**	8	21	0.03816	8.8309E-09	6	15	0.02606	1.5409E-09	10	25	0.04814	5.5576E-09	59	241	0.27576	8.6175E-09
5000	**x5**	17	52	0.08657	6.7905E-09	22	69	0.10643	3.8403E-09	12	32	0.06375	3.6002E-09	61	248	0.29183	8.0471E-09
5000	**x6**	12	31	0.04854	6.6177E-09	22	68	0.11060	4.6406E-09	13	42	0.10346	5.9588E-09	62	250	0.29224	8.1876E-09
5000	**x7**	13	36	0.05865	5.1067E-09	23	72	0.11025	5.6953E-09	11	31	0.05111	2.8649E-09	62	250	0.27411	7.6296E-09
5000	**x8**	14	38	0.06893	9.5514E-10	7	18	0.03452	1.0412E-09	11	29	0.05033	7.4314E-09	62	250	0.31480	8.3682E-09
10000	**x1**	11	29	0.12935	8.4335E-09	21	66	0.14482	7.0156E-09	13	37	0.09583	1.4459E-09	61	249	0.49287	9.5333E-09
10000	**x2**	11	29	0.08446	6.1113E-09	21	66	0.18801	6.7652E-09	13	37	0.10481	1.7840E-09	61	249	0.50182	9.3937E-09
10000	**x3**	10	27	0.08571	1.6586E-09	21	66	0.15276	4.9435E-09	12	33	0.10223	4.7426E-09	61	249	0.48216	8.1748E-09
10000	**x4**	9	23	0.07085	1.8142E-09	6	15	0.05041	2.1791E-09	10	25	0.08289	9.5969E-09	60	245	0.46931	8.8414E-09
10000	**x5**	17	52	0.14664	9.6032E-09	22	69	0.18159	5.4309E-09	11	28	0.08064	8.7915E-09	62	252	0.50543	8.2562E-09
10000	**x6**	12	31	0.08264	9.3589E-09	23	73	0.15916	6.3474E-09	14	46	0.10846	1.7662E-09	63	254	0.51418	8.4004E-09
10000	**x7**	13	36	0.12906	7.2220E-09	23	73	0.25665	8.5323E-09	14	43	0.10606	2.1715E-09	63	254	0.51744	7.8279E-09
10000	**x8**	14	38	0.10402	1.3508E-09	7	18	0.07598	1.4725E-09	14	42	0.09957	2.1096E-09	63	254	0.49114	8.5857E-09
50000	**x1**	12	31	0.33042	9.3840E-10	22	69	0.62427	5.6816E-09	14	41	0.42098	2.3754E-09	64	261	2.32850	8.1397E-09
50000	**x2**	12	31	0.36379	5.6107E-10	22	69	0.62287	5.4788E-09	14	41	0.42743	2.1971E-09	64	261	2.27368	8.0206E-09
50000	**x3**	10	27	0.29782	3.7088E-09	22	69	0.64410	4.0033E-09	13	39	0.41428	8.9237E-09	63	257	2.26461	9.6210E-09
50000	**x4**	9	23	0.26084	4.0568E-09	6	15	0.17896	4.8726E-09	13	38	0.39586	3.2734E-09	63	257	2.21871	7.5490E-09
50000	**x5**	18	54	0.54905	3.5244E-09	24	77	0.69938	4.1819E-09	17	62	0.58483	2.4241E-09	64	260	2.27889	9.7167E-09
50000	**x6**	13	33	0.38900	1.9096E-09	24	77	0.74722	5.1440E-09	15	50	0.45349	5.4303E-09	65	262	2.28170	9.8864E-09
50000	**x7**	14	38	0.38756	1.1869E-09	24	77	0.67827	6.9131E-09	14	39	0.40820	1.4432E-09	65	262	2.33294	9.2126E-09
50000	**x8**	14	38	0.40306	3.0204E-09	24	77	0.69055	7.2256E-09	14	38	0.38383	1.4634E-09	66	266	2.30200	7.3306E-09

TABLE 7. Test results of the four methods for problems 4.7

Nvars	IGuess	MDDYM				PDYM				MDY				MFR			
		Niter	Nfev	Cpt	Norm	Niter	Nfev	Cpt	Norm	Niter	Nfev	Cpt	Norm	Niter	Nfev	Cpt	Norm
5000	**x1**	16	59	0.09432	1.3050E-09	30	124	0.14592	7.0705E-09	10	32	0.04996	7.5384E-09	57	349	0.32182	9.4550E-09
5000	**x2**	14	53	0.07568	4.0067E-09	30	124	0.13148	6.9353E-09	10	30	0.04617	9.1316E-09	57	349	0.35728	9.3912E-09
5000	**x3**	16	60	0.08765	6.4133E-09	30	124	0.13896	5.8827E-09	9	30	0.03916	7.4550E-09	57	349	0.30529	8.7821E-09
5000	**x4**	16	62	0.07775	5.5391E-09	11	37	0.06037	1.8485E-09	9	29	0.04321	4.7455E-09	53	325	0.31769	7.4711E-09
5000	**x5**	15	57	0.07999	9.8573E-09	12	40	0.05478	9.3263E-09	11	37	0.05279	5.2468E-09	57	346	0.31532	7.5276E-09
5000	**x6**	15	58	0.07809	7.8166E-09	13	43	0.06090	2.5833E-09	**	**	**	**	58	349	0.29898	8.0592E-09
5000	**x7**	16	59	0.08286	1.3795E-09	14	44	0.07650	3.3787E-09	**	**	**	**	56	333	0.32301	7.5313E-09
5000	**x8**	12	42	0.06033	2.4066E-10	14	43	0.06044	6.0314E-09	**	**	**	**	54	321	0.29418	8.5637E-09
10000	**x1**	16	59	0.14692	1.8456E-09	30	124	0.26347	9.9993E-09	11	33	0.06356	7.5740E-09	58	355	0.54602	9.3986E-09
10000	**x2**	14	53	0.10996	5.6664E-09	30	124	0.23092	9.8079E-09	11	34	0.07902	4.5397E-11	58	355	0.55024	9.3352E-09
10000	**x3**	16	60	0.13400	9.0698E-09	30	124	0.21597	8.3194E-09	13	43	0.10110	9.3470E-09	58	355	0.53853	8.7297E-09
10000	**x4**	16	62	0.13363	7.8334E-09	11	37	0.30166	2.6142E-09	6	19	0.04657	4.5079E-10	54	331	0.52338	7.4265E-09
10000	**x5**	16	61	0.12671	4.8938E-09	13	43	0.07998	1.9679E-09	11	39	0.08054	2.4455E-09	58	352	0.52326	7.4827E-09
10000	**x6**	16	62	0.12887	5.9649E-09	13	43	0.09948	3.6534E-09	**	**	**	**	59	355	0.61184	8.0112E-09
10000	**x7**	16	59	0.14343	1.9510E-09	14	44	0.07941	4.7783E-09	**	**	**	**	57	339	0.51324	7.4864E-09
10000	**x8**	12	42	0.09692	3.4034E-10	14	43	0.09317	8.5297E-09	167	1820	2.60568	4.7429E-11	55	327	0.52918	8.5127E-09
50000	**x1**	16	59	0.52241	4.1268E-09	32	132	0.95785	4.9313E-09	14	54	0.46701	1.1179E-09	61	373	2.61151	7.2981E-09
50000	**x2**	15	56	0.46085	5.5395E-09	32	132	0.98458	4.8369E-09	16	60	0.51123	4.0239E-10	61	373	2.58425	7.2489E-09
50000	**x3**	17	64	0.56293	7.8990E-09	31	128	0.97264	8.7364E-09	16	58	0.52341	3.8080E-09	60	367	2.53142	9.6440E-09
50000	**x4**	17	65	0.52033	1.7365E-09	11	37	0.33492	5.8455E-09	10	32	0.30108	3.4982E-09	56	343	2.30189	8.2044E-09
50000	**x5**	17	64	0.56208	1.1172E-09	13	43	0.40677	4.4003E-09	87	897	5.90688	5.5572E-09	60	364	2.46086	8.2665E-09
50000	**x6**	17	66	0.57559	4.9142E-09	13	43	0.38142	8.1692E-09	61	577	4.09402	3.9378E-09	61	367	2.51340	8.8502E-09
50000	**x7**	16	59	0.51799	4.3625E-09	17	54	0.49638	1.9299E-09	66	643	4.51278	7.7185E-09	59	351	2.42242	8.2705E-09
50000	**x8**	12	42	0.40991	7.6102E-10	35	137	1.04612	6.9307E-09	185	1959	12.83235	7.1829E-09	57	339	2.30068	9.4043E-09

TABLE 8. Test results of the four methods for problems 4.8

Nvars	IGuess	MDDYM				PDYM				MDY				MFR			
		Niter	Nfev	Cpt	Norm	Niter	Nfev	Cpt	Norm	Niter	Nfev	Cpt	Norm	Niter	Nfev	Cpt	Norm
5000	**x1**	14	105	0.16757	0.0000E+00	**	**	**	**	8	76	0.08422	0.0000E+00	3	19	0.03360	0.0000E+00
5000	**x2**	14	102	0.14631	0.0000E+00	**	**	**	**	8	76	0.10084	0.0000E+00	5	32	0.05756	0.0000E+00
5000	**x3**	14	96	0.13992	0.0000E+00	4	34	0.03438	0.0000E+00	15	141	0.16698	0.0000E+00	4	20	0.06854	0.0000E+00
5000	**x4**	17	126	0.17229	0.0000E+00	5	44	0.06671	0.0000E+00	16	90	0.14892	0.0000E+00	2	16	0.03029	0.0000E+00
5000	**x5**	7	55	0.08252	0.0000E+00	6	44	0.07206	0.0000E+00	22	125	0.19471	0.0000E+00	3	30	0.06496	0.0000E+00
5000	**x6**	4	30	0.04918	0.0000E+00	3	19	0.04112	0.0000E+00	13	95	0.14016	0.0000E+00	2	19	0.03237	0.0000E+00
5000	**x7**	4	30	0.05837	0.0000E+00	5	36	0.06107	0.0000E+00	19	111	0.18132	0.0000E+00	2	20	0.04827	0.0000E+00
5000	**x8**	6	55	0.08952	0.0000E+00	4	24	0.06101	0.0000E+00	68	499	0.65635	0.0000E+00	3	22	0.03871	0.0000E+00
10000	**x1**	10	73	1.18135	0.0000E+00	5	35	0.11482	0.0000E+00	8	76	0.16320	0.0000E+00	3	19	0.05802	0.0000E+00
10000	**x2**	11	75	0.20685	0.0000E+00	212	1703	3.57595	0.0000E+00	8	76	0.20896	0.0000E+00	3	19	0.06237	0.0000E+00
10000	**x3**	14	94	0.24852	0.0000E+00	5	37	0.11556	0.0000E+00	5	51	0.13497	0.0000E+00	4	20	0.06374	0.0000E+00
10000	**x4**	10	74	0.20559	0.0000E+00	3	20	0.05981	0.0000E+00	26	243	0.57624	0.0000E+00	2	16	0.05057	0.0000E+00
10000	**x5**	4	35	0.09897	0.0000E+00	5	31	0.09720	0.0000E+00	31	267	0.65314	0.0000E+00	3	30	0.08397	0.0000E+00
10000	**x6**	3	18	0.05165	0.0000E+00	5	34	0.10333	0.0000E+00	12	108	0.25737	0.0000E+00	2	19	0.05356	0.0000E+00
10000	**x7**	5	42	0.13123	0.0000E+00	4	23	0.07361	0.0000E+00	33	288	0.65692	0.0000E+00	2	20	0.05280	0.0000E+00
10000	**x8**	4	25	0.07246	0.0000E+00	4	23	0.08236	0.0000E+00	7	64	0.17944	0.0000E+00	3	22	0.06539	0.0000E+00
50000	**x1**	10	72	0.86301	0.0000E+00	7	42	0.49996	0.0000E+00	8	76	0.82304	0.0000E+00	3	19	0.24206	0.0000E+00
50000	**x2**	9	71	0.85537	0.0000E+00	7	41	0.49430	0.0000E+00	8	76	0.85630	0.0000E+00	3	19	0.24635	0.0000E+00
50000	**x3**	10	71	0.82141	0.0000E+00	7	39	0.46821	0.0000E+00	4	28	0.35046	0.0000E+00	3	19	0.23377	0.0000E+00
50000	**x4**	8	51	0.64090	0.0000E+00	**	**	**	**	7	60	0.69240	0.0000E+00	2	16	0.16730	0.0000E+00
50000	**x5**	3	24	0.29724	0.0000E+00	7	38	0.40553	0.0000E+00	9	69	0.73666	0.0000E+00	4	31	0.32924	0.0000E+00
50000	**x6**	3	18	0.25253	0.0000E+00	7	40	0.47899	0.0000E+00	10	73	0.88118	0.0000E+00	2	19	0.20408	0.0000E+00
50000	**x7**	3	18	0.22632	0.0000E+00	**	**	**	**	8	64	0.74632	0.0000E+00	2	20	0.23963	0.0000E+00
50000	**x8**	4	24	0.30785	0.0000E+00	6	32	0.33168	0.0000E+00	36	246	2.80129	0.0000E+00	3	22	0.27244	0.0000E+00

TABLE 9. Summary of result from tables 1-8 showing the number of problems/percentage solved with least number of iteration, function values and processing time by each of the four methods

Method	Niter	Percentage	fev	Percentage	Ptime	Percentage	Fails
MDDYM	83	43.22%	91	47.40%	81	40.19%	–
PDYM	19	9.90%	14	7.29%	30	15.63%	4
MDY	36	18.75%	40	20.83%	55	28.65%	7
MFR	23	11.98%	20	10.42%	26	13.53%	27
Undecided	31	16.15%	27	14.06%	–	–	–

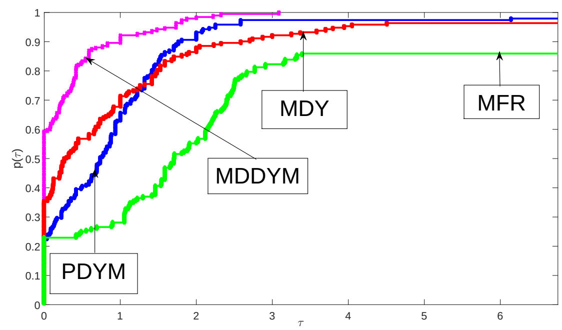

FIGURE 1. Performance profile of Dolan and Moré for number of iterations

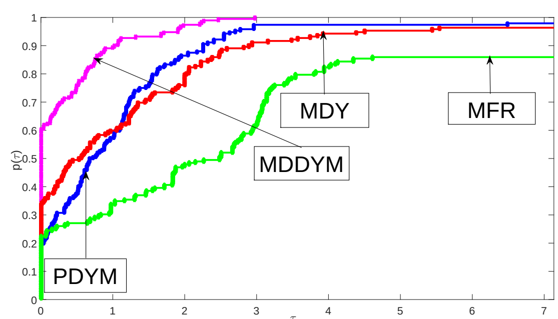

FIGURE 2. Performance profile of Dolan and Moré for function evaluations

In order to explain the results in Tables $1-8$, a summary is drawn in Table 9, which describes performance of each of the four schemes relative to three characteristics, namely number of iteration, function values, and processing time. The summary table clearly indicates that the $MDDYM$ scheme outperforms the $PDYM$, MDY and MFR methods as it successfully solved 43.22% of all the problems in the experiments with less number of iterations compared to the $PDYM$, MDY and MFR solvers, that recorded 9.90%, 18.75%, and 11.98% respectively. Interestingly,

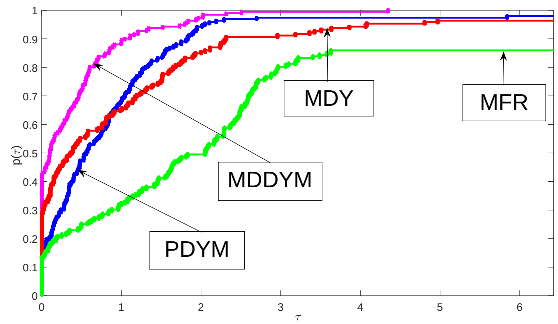

FIGURE 3. Performance profile of Dolan and Moré for processing time

too, the summary table also indicates that some of the methods recorded a tie in 31 problems, which translates to 16.15% and is marked as "UNDECIDED". It is also observed from the table that the $MDDYM$ scheme solves 47.40% of the problems with less function values, as against the $PDYM$, MDY and MFR schemes, that recorded 7.29%, 20.83% and 10.42% respectively. Also, the table indicates that some of the methods solved 27 problems with equal number of functions values, which corresponds to 14.06%. The summarized results also shows that the $MDDYM$ scheme solves 42.19% of all the problems with less processing time as against the $PDYM$, MDY and MFR methods, that recorded 15.63%, 28.65% and 13.53% respectively.

Furthermore, a graphical view and interpretation of the data presented in Tables $1-8$ is drawn in figs $1-3$ by adopting Dolan and Moré [14] performance profile, which is formulated as

$$\Lambda_e(\Theta) = \frac{1}{|\mathcal{C}|}\left|\left\{\tilde{s} \in \mathcal{C} : \frac{t_{\tilde{s},e}}{\min\{t_{\tilde{s},e} : e \in E\}} \leq \Theta\right\}\right|,$$

where $\mathcal{C}$ represent set of experiments conducted, $|\mathcal{C}|$ stands for number of the problems in the set of experiments $\mathcal{C}$, E denotes number of schemes considered while for each $\tilde{s} \in \mathcal{C}$ and $e \in E$, $t_{\tilde{s},e}$ represents either processing time in each iteration, number of iterations or function evaluations obtained. It is observed from all three figures that the $MDDYM$ method is the most successful of all the four methods regarding the three characteristics considered. The top curve in each of the three figures represents the scheme with the best performance in the experiment, which clearly corresponds to the one representing the $MDDYM$ scheme in all three cases. Hence, based on the above discussions it is concluded that the $MDDYM$ scheme outperforms the other three methods as it yields better performance regarding the three characteristics outlined, namely number of iterations, function evaluations and processing time. Therefore, the proposed algorithm is more effective for solving large-scale systems of monotone nonlinear equations with convex constraint than $PDYM$, MDY and MFR methods.

5. **Application In Image De-Blurring.**

5.1. **Brief introduction of the concept.** Digital image processing often arises in Medical sciences, Biological engineering, file restoration, image and video coding applications (Ref.[5, 7, 42]). In signal and image processing problems, the major interest is to obtain sparse solutions to ill-conditioned linear systems of equations, which involves minimizing the following $\ell_1 - \ell_2$ *norm* problem

$$\min_x \frac{1}{2}\|Ax - q\|_2^2 + \phi\|x\|_1, \tag{68}$$

where ϕ is a nonnegative parameter, $x \in \mathbf{R}^n$, and $q \in \mathbf{R}^m$ is an observed value, and $A \in \mathbf{R}^{m\times n}(m \ll n)$ represents a linear mapping, while $\|x\|_1$ and $\|x\|_2$ denotes the ℓ_1 and ℓ_2 *norms* respectively. Careful observation shows that (68) is an unconstrained optimization problem, popularly referred to as the ℓ_1-regularized least square problem. Various iterative schemes for solving (68) exists (see [15, 36, 34] for instances), however, gradient based methods are the most prominent. In [16], Figueiredo et al. reformulated (68) as a convex quadratic problem, by splitting the vector $x \in \mathbf{R}^n$ in to two parts, namely

$$x = u - v, \quad u \geq 0, \quad v \geq 0, \quad u, v \in \mathbf{R}^n. \tag{69}$$

Let $u_i = (x_i)_+, \quad v_i = (-x_i)_+ \quad \forall i = 1, 2, ..., n$, where $(.)_+$ is a positive operator, which is given as $(.)_+ = \max\{0, x\}$. Now, using the definition of the $\ell_1 - norm$, we obtain $\|x\|_1 = G_n^T u + G_n^T v$, where $G_n = (1, 1, ..., 1)^T \in \mathbf{R}^n$. Applying this representation, (68) can be reformulated as

$$\min_{u,v} \frac{1}{2}\|A(u - v) - q\|_2^2 + \phi G_n u + \phi G_n v, \quad u, v \geq 0. \tag{70}$$

Going by Figueiredo et al. [16], the problem in (70) is reformulated as

$$\min_\chi \frac{1}{2}\chi^T M\chi + D^T M\chi, \quad \chi \geq 0, \tag{71}$$

which represents a quadratic program problem where

$$\chi = \begin{pmatrix} u \\ v \end{pmatrix}, \quad D = \phi G_{2n} + \begin{pmatrix} -h \\ h \end{pmatrix}, \quad h = A^T q, \quad M = \begin{pmatrix} A^T A & -A^T A \\ -A^T A & A^T A \end{pmatrix}, \tag{72}$$

where A denotes a positive semi-definite matrix. So, (71) represents a convex quadratic programming problem, which was translated in [52] as the problem of linear variable inequality (LVI), namely, find the value of $\chi \in \mathbf{R}^n$ such that

$$(\chi' - \chi)^T(M\chi + D) \geq 0, \quad \forall \chi' \geq 0. \tag{73}$$

Also, in [52], the authors considered (71) to be equivalent to the following linear complementarity problem. Find $\chi \in \mathbf{R}^n$,

$$\chi \geq 0, \quad M\chi + D \geq 0, \quad \text{and} \quad \chi^T(M\chi + D) = 0, \tag{74}$$

where $\chi \in \mathbf{R}^n$ denotes the solution of (74) if and only if it satisfies the nonlinear equations

$$F(\chi) = \min\{\chi, M\chi + D\} = 0, \tag{75}$$

where F is a vector-valued mapping. Now, F is also monotone and Lipschitz continuous, hence by [36, 52], the $MDDYM$ scheme can be applied to solve it.

The image de-blurring experiments are conducted using four images, namely Barbara, Girl, Lena, and Cameraman. Codes for the experiments are generated on MATLAB $R2015a$ with parameter values as applied in the last experiment with $\mu = 1.5, \rho = 0.2, \beta = 0.8$. Also, to further investigate the performance of the $MDDYM$ method, it is compared with the $CG\_DESCENT - type$ algorithm [53], simply denoted as $MFRM$, which performs well in image de-blurring problems.

The parameters for the $MFRM$ method are the same as used by the authors. The iteration for the two schemes is set to terminate whenever the following condition is reached:

$$\frac{|f(x_k) - f(x_{k-1})|}{f(x_{k-1})} < 10^{-5}, \tag{76}$$

where $f(x_k)$ represents a merit function $f(x) = \frac{1}{2}\|Ax - q\|_2^2 + \phi\|x\|_1$. The performance of both schemes is observed in terms of functions evaluations (Obj), processing time (PT(s)), mean square error (MSE), signal to noise ratio (SNR), which is given by

$$SNR = 20 \times \log_{10}\left(\frac{\|\tilde{x}\|}{\|x - \tilde{x}\|}\right),$$

where $\tilde{x}$ denotes the recovered image, x the original image, and the structural similarity index (SSIM), which computes the similarity between original image and the restored one in each of the experiments is conducted. For the experiment conducted, the codes were implemented with $x_0 = A^T q$. A represents a partial Discrete Wavelet Transform (DWT) matrix, for which the m rows are selected randomly from the $n \times n$ DWT matrix. The encoding matrix A is able to be tested on large images without storing any matrix, since it doesn't require storage and also enables fast matrix-vector multiplications involving A and A^T. Results of the experiments carried are given in Table 10, while Figure 4 displays the original, blurred, and reconstructed images obtained by the $MDDYM$ and $MFRM$ schemes respectively. Careful observation of the results displayed in Table 10 indicates that the $MDDYM$ outperformed the $MFRM$ scheme in all the four metrics considered, namely objective function (ObjFun), mean square error (MSE), signal to noise ratio (SNR) and structural similarity index (SSIM). Figure 4 also shows that images restored from the blurred images by $MDDYM$ scheme appears slightly closer to the original one than the restored images by $MFRM$ method. Therefore, taking everything into consideration, it can be concluded that the $MDDYM$ method is suitable for reconstruction of the images considered. The MATLAB implementation of the SSIM index can be obtained at http://www .cns .nyu .edu / lcv /ssim/.

Table 10. Performance results for MDDYM and MFRM methods based on objective function (ObjFun) value, mean square error (MSE), SNR and SSIM index

Image & size	ObjFun		MSE		SNR		SSIM	
	MDDYM	MFRM	MDDYM	MFRM	MDDYM	MFRM	MDDYM	MFRM
Barbara 256×256	1.530×10^6	1.585×10^6	1.5967×10^2	2.0627×10^2	20.66	19.55	0.80	0.75
Girl 256×256	6.224×10^6	6.321×10^6	1.4978×10^2	1.6071×10^2	21.73	21.42	0.77	0.75
Lena 256×256	1.459×10^6	1.513×10^6	6.5691×10^1	9.0026×10^1	24.29	22.93	0.90	0.87
Cameraman 256×256	1.415×10^6	1.473×10^6	1.2579×10^2	1.7757×10^2	21.55	20.05	0.87	0.83

6. **Conclusion.** We have presented a modified Dai-Yuan scheme for solving constrained system of monotone nonlinear equations in this article. It was achieved by developing a new CG update parameter, which was inspired by recent works in [54] and [57] for unconstrained optimization. The scheme requires less memory to implement and avoids computing derivatives, which makes it appropriate for large dimension and non-smooth problems. The scheme's global convergence was established by employing basic assumptions. Also, numerical experiments conducted with some benchmark problems indicates that the proposed scheme is promising

FIGURE 4. Original and blurred images (First and second columns from the left). Restored images by the two methods (Third and Fourth columns)

as it is competitive and more efficient compared to the recent $PDYM$, MDY, and MFR schemes. Moreover, as part of its novelty, the scheme is applied to solve image de-blurring problems in compressed sensing. The experiments conducted show that the scheme produces better results than the $MFRM$ scheme. As a further research, we intend to explore application of the scheme in other real life problems like signal processing and robotic control.

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