Algbric Topogy

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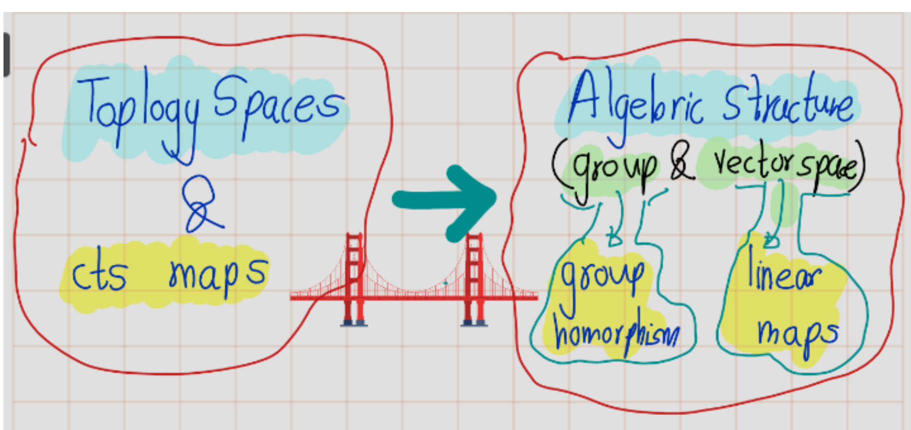
Contents

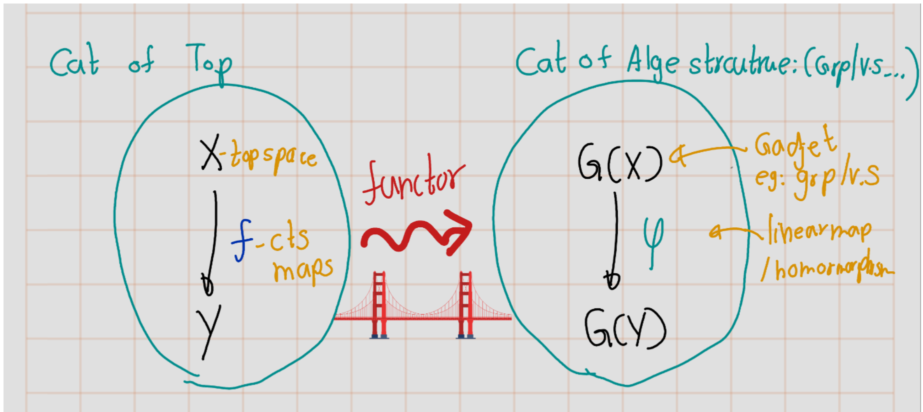
Chapter 1

Introduction

1.0.1 What is Algberic topolgy?



Roughly it's about tools/methods connecting **algebra** and **topology**.



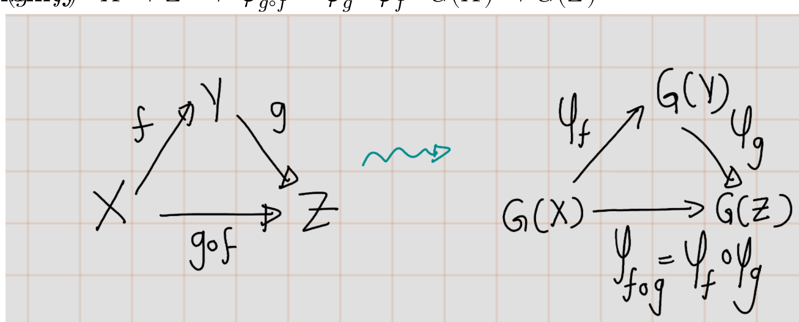
The "*bridge*" is **algebraic topology**.

In order to be useful, the associations

$$X \rightsquigarrow G(X) \quad \text{and} \quad f \rightsquigarrow \varphi_f$$

ought to satisfy some properties.

Property	Mathematical Form
Identity Preservation	$\mathrm{id}_X : X \to X \ \rightsquigarrow\ \mathrm{id}_{G(X)} : G(X) \to G(X)$
Composition Compatibility	If $f : X \to Y$ and $g : Y \to Z$, then $(g \circ f) : X \to Z \ \rightsquigarrow\ \varphi_{g\circ f} = \varphi_g \circ \varphi_f : G(X) \to G(Z)$



In somewhat "*fancy*" language, we are asking that the correspondence above be **functor** So, what could we do with all this?

Example 1.1 (Classical Problem)**.** Let $S^1 \subset D^2$ be the unit circle inside the unit disk in $\mathbb{R}^2$.Suppose we are given a continuous map $f : S^1 \to S^1$. **Does there exist a continuous extension** $g : D^2 \to S^1$ **such that** $g(z) = f(z)$ **for all** $z \in S^1$**?**

$$\begin{array}{ll} S^1 \xrightarrow{f} S^1 & \\ \downarrow i \quad \nearrow ?\ \exists\ g & \\ D^2 & \end{array}$$

Imagine we had our algebraic topology tools on hand. If there exists a continuous map $g : D^2 \to S^1$ such that $g \circ j = f$, then the induced homomorphism satisfies

$$\varphi_f = \varphi_{g\circ j} = \varphi_g \circ \varphi_j.$$

That is, the existence of g would imply the existence of a homomorphism φ_g such that the following diagram commutes:

Proof.

$$\begin{array}{l} G(S^1) \xrightarrow{\varphi_f} G(S^1) \\ \downarrow \varphi_j \quad \nearrow \varphi_g \\ G(D^2) \end{array}$$

"Commutes" mean $\varphi_f = \varphi_g \circ \varphi_j$.

Suppose we know,

- $G(S^1) \cong \mathbb{Z}$
- $G(D^2) \cong \{e\}$ (the trivial group)
- φ_f is a non-trivial homomorphism, i.e., $\varphi_f(a) \neq 0$ for all $a \in G(S^1)$ is a generator.

This yields a contradiction, since any composition through the trivial group must be trivial. Therefore, **no such map** g **can exist**. □

Definition: Invaraince (*This is not a formal definition*) Suppose G is a correspondence of the kind we're looking for:

$$\begin{cases} X & \rightsquigarrow G(X) \\ X \xrightarrow{f} Y & \rightsquigarrow G(X) \xrightarrow{G(f)} G(Y) \end{cases}$$

such that:

$$G(\mathrm{id}_X) = \mathrm{id}_{G(X)}, \quad G(f \circ g) = G(f) \circ G(g).$$

Then observe that it's **automatic** that

Result :If $f : X \to Y$ is a homeomorphism, then
$G(f) : G(X) \to G(Y)$ is an isomorphism.

Proof.

- *Homeomorphism*: A function $f : X \to Y$ is a homeomorphism if
  - f is a bijection (one-to-one and onto),
  - f is continuous,
  - f^{-1} is continuous
- Isomorphism
  - Hormphism
  - Bijective

Let $f^{-1} : Y \to X$ be the inverse of f. Then $f^{-1} \circ f = \mathrm{id}_X$. So,

$$\mathrm{id}_{G(X)} = G(\mathrm{id}_X) = G(f^{-1} \circ f) = G(f^{-1}) \circ G(f)$$
$$\mathrm{id}_{G(Y)} = G(\mathrm{id}_Y) = G(f \circ f^{-1}) = G(f) \circ G(f^{-1})$$

Hence,$G(f^{-1}) = G(f)^{-1}$. Therefore, homeomorphic spaces X, Y yield isomorphic groups,$G(X) \cong G(Y)$

(We say $G(X)$ is an ***invariant*** *of X.)*

□

A necessary condition for two spaces X, Y to be homeomorphic is that $G(X) \cong G(Y)$.

Remark. We'll see examples showing this is **NOT** sufficient.

1.1 Some Basic defnitions

Definition 1.1 (Homotopy of Maps)**.**

Let X, Y be topological spaces, and let $f, g : X \to Y$ be continuous maps.
We say that f **is homotopic to** g, and write $f \sim g$, if there exists a continuous map

$$H : X \times I \to Y$$

such that

$$H(x, 0) = f(x), \qquad H(x, 1) = g(x) \quad \text{for all } x \in X.$$

Intuitively, the map H defines a continuous deformation of f into g over time $t \in [0, 1]$.

Notations:

- We often write $H_t : X \to Y$ for the slice $H(\cdot, t)$, so that $H_0 = f, \quad H_1 = g$.
- Sometimes we write $f \overset{H}{\sim} g$ to indicate the specific homotopy H.

Example 1.2.

Let $f, g : X \to \mathbb{R}^n$ be any continuous functions.
Then $f \sim g$, because we can define a homotopy:

$$H(x,t) = t\,g(x) + (1-t)\,f(x)$$

This map $H : X \times I \to \mathbb{R}^n$ is continuous
(since vector space operations in $\mathbb{R}^n$ are continuous),
and satisfies:

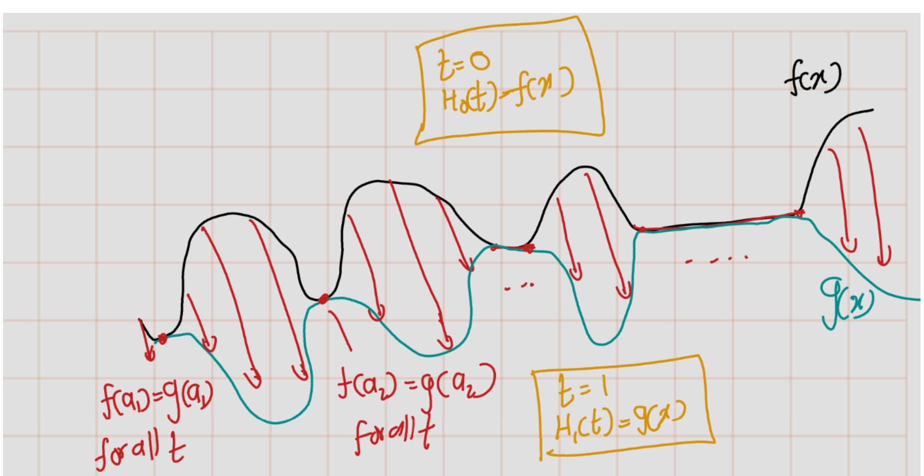
$$H_0 = f, \qquad H_1 = g$$

Definition 1.2. Let $A \subset X$, and suppose $f|_A = g|_A$, i.e., $f(a) = g(a)$ for all $a \in A$.
We say that f **is homotopic to** g **relative to** A if there exists a homotopy

$$H : X \times I \to Y$$

such that

$$H(a,t) = f(a) = g(a) \quad \text{for all } a \in A, \text{ and all } t \in [0,1].$$



Example 1.3.

Let $f, g : I \to \mathbb{R}^2$ be continuous maps (paths), as illustrated.

Then the interpolation homotopy is a homotopy from f to g, and it satisfies

$$H(0,t) = f(0) = g(0), \qquad H(1,t) = f(1) = g(1)$$

for all $t \in [0,1]$.

Hence, H is a **homotopy relative to the endpoints**
$\{0,1\} \subset I$.

Notation:

When f is homotopic to g **relative to a subset** $A \subset X$,
we write:$f \sim g$ rel A

This means there exists a homotopy $H : X \times I \to Y$ such that

$$H(a,t) = f(a) = g(a) \quad \text{for all } a \in A,\ t \in [0,1].$$

Example 1.4. Let $f, g, h : X \to Y$ be continuous maps between topological spaces.
We show that the relation $f \sim g$ (homotopy) satisfies reflexivity, symmetry, and transitivity.

1. Reflexivity
$$f \sim f \quad \text{via } H(x,t) = f(x) \text{ for all } t \in [0,1].$$
This is a constant homotopy.

2. Symmetry Suppose $f \sim g$ via homotopy $H(x,t)$.
Then define:

$$G(x,t) = H(x, 1-t)$$

This reverses the deformation and gives a homotopy from g to f, so $g \sim f$.

3. Transitivity

Suppose $f \sim g$ via $F(x,t)$, and $g \sim h$ via $G(x,t)$.
Define a new homotopy $H : X \times I \to Y$ by:

$$H(x,t) = \begin{cases} F(x,2t) & \text{if } 0 \le t \le \frac{1}{2}, \\ G(x,2t-1) & \text{if } \frac{1}{2} \le t \le 1. \end{cases}$$

Then:

$$H(x,0) = F(x,0) = f(x), \qquad H(x,1) = G(x,1) = h(x).$$

By the pasting lemma (Lemma 1.1),
$F(x,1) = g(x) = G(x,0)$, so H is continuous.
Thus, $f \sim h$.

Let me recall pasting lemma, if you can not remember,

Lemma 1.1 (Pasting Lemma)**.** *Let X, Y be both closed (or both open) subsets of a topological space A such that*

$$A = X \cup Y,$$

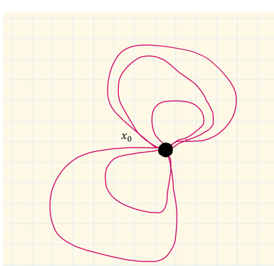
and let B be a topological space. If $f : A \to B$ is continuous when restricted to both X and Y, then f is continuous.

Definition 1.3. Let X b topological space and $x_0 \in X$. Let $\Pi_1(X, x_0)$ denote the set of homotpy equivalnce classes of loops based at x_0. (i.e.: $\sigma I \to X, \sigma(0) = sigma(1) = x_0$, where homotopy relative to $\{0,1\}$)

Definition 1.4. Let X be a topological space, and let $x_0 \in X$. Define $\pi_1(X, x_0)$ to be the set of homotopy equivalence classes of loops based at x_0 (i.e., maps $\alpha : I \to X \quad$ with $\alpha(0) = \alpha(1) = x_0$, where homotopy is taken **rel** $\{0,1\}$).

i.e.:

$$\Pi(X, x_0) = \{[\sigma] : \sigma \text{ is a loop at } x_0\}\,, \text{ where.}$$
$$[\sigma] = \{\alpha : \sigma \text{ is a loop at } x_0 \text{ and } \sigma \sim \alpha\}$$



Remark. The previous lemma shows that concatenation gives a well-defined binary operation on $\pi_1(X, x_0)$

$$[\sigma][\tau] \mapsto [\sigma * \tau],$$

where $*$ denotes concatenation and the constant path acts as identity.

Theorem 1.1. $\pi_1(X, x_0)$ *is a **group** under this operation, with:*

- ***Neutral element:*** *$[c]$, where c is the constant loop at x_0.*
- ***Inverse:*** *For $[\sigma]$, the inverse is $[\sigma^{-1}]$, where*

$$\sigma^{-1}(t) = \sigma(1-t).$$

Proof.

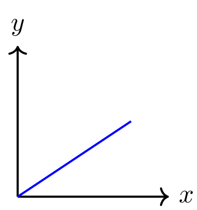
- Closedness
- **Associvity**: NTS: $([ ][ ]) [ ] = $.
  First note that We need to show $( )   ( ) $

$$(\alpha\beta)(s) := \begin{cases} \alpha(2s), & 0 \le s \le \frac{1}{2}, \\ \beta(2s-1), & \frac{1}{2} \le s \le 1. \end{cases}$$

$$(\alpha\beta)\gamma(s) := \begin{cases} \alpha\beta(2s), & 0 \le s \le \frac{1}{2}, \\ \gamma(2s-1), & \frac{1}{2} \le s \le 1. \end{cases}$$

$$:= \begin{cases} \alpha(4s), & 0 \le s \le \frac{1}{4}, \\ \beta(4s-1), & \frac{1}{4} \le s \le \frac{1}{2}. \\ \gamma(2s-1), & \frac{1}{2} \le s \le 1. \end{cases}$$

We picture these paths by the diagrams,



- **Idenitity**: NTS: $[c_{x_0}][\sigma] = [\sigma]$ for all $\sigma \in \Pi_1(X, x_0)$
  We need to show $c_{x_0}\sigma \sim \sigma$ for all $\sigma \in \Pi_1(X, x_0)$

$\square$

Chapter 2

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