

Algbric Topogy

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2026-01-12

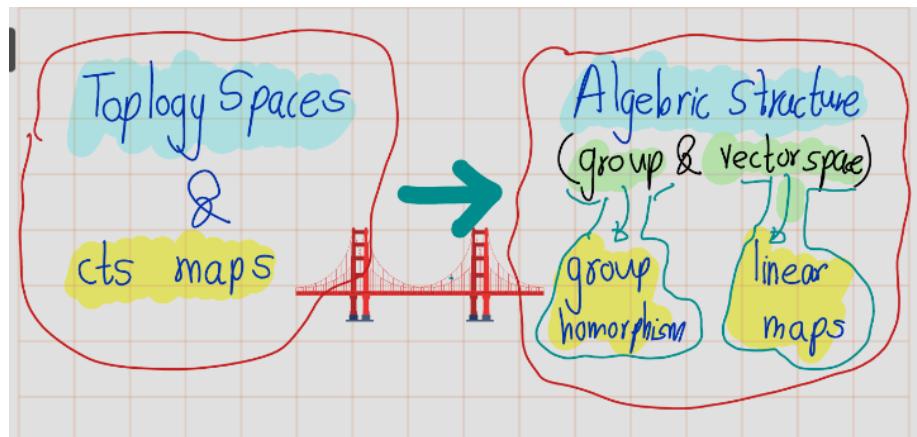
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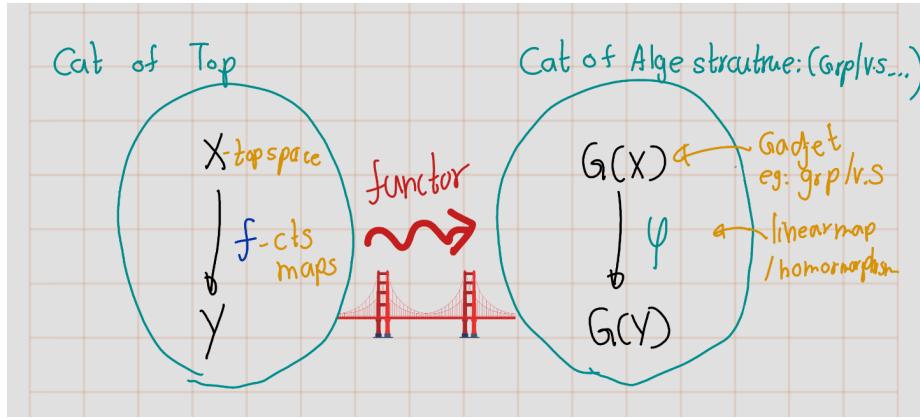
Chapter 1

Introduction

1.0.1 What is Algebraic topology?



Roughly it's about tools/methods connecting **algebra** and **topology**.



The “bridge” is **algebraic topology**.

In order to be useful, the associations

$$X \rightsquigarrow G(X) \quad \text{and} \quad f \rightsquigarrow \varphi_f$$

ought to satisfy some properties.

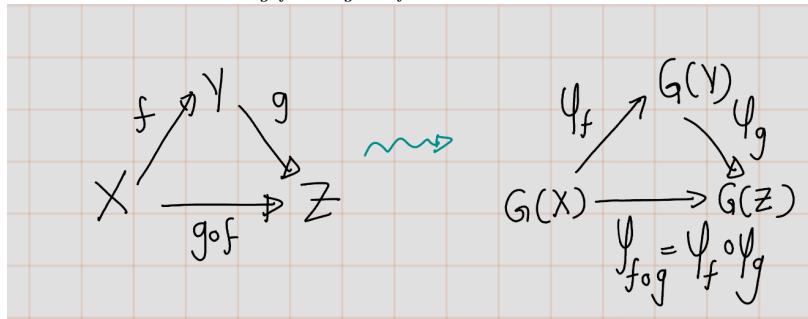
Property Mathematical Form

Identity $\text{id}_X : X \rightarrow X \rightsquigarrow \text{id}_{G(X)} : G(X) \rightarrow G(X)$

Preservation

Composition $X \rightarrow Y$ and $g : Y \rightarrow Z$, then

Compatibility $f : X \rightarrow Z \rightsquigarrow \varphi_{g \circ f} = \varphi_g \circ \varphi_f : G(X) \rightarrow G(Z)$



In somewhat “fancy” language, we are asking that the correspondence above be **functor**. So, what could we do with all this?

Example 1.1 (Classical Problem). Let $S^1 \subset D^2$ be the unit circle inside the unit disk in \mathbb{R}^2 . Suppose we are given a continuous map $f : S^1 \rightarrow S^1$. Does there exist a continuous extension $g : D^2 \rightarrow S^1$ such that $g(z) = f(z)$ for all $z \in S^1$?

$$\begin{array}{ccc} S^1 & \xrightarrow{f} & S^1 \\ \downarrow i & \nearrow ? \exists g & \\ D^2 & & \end{array}$$

Imagine we had our algebraic topology tools on hand. If there exists a continuous map $g : D^2 \rightarrow S^1$ such that $g \circ j = f$, then the induced homomorphism satisfies

$$\varphi_f = \varphi_{g \circ j} = \varphi_g \circ \varphi_j.$$

That is, the existence of g would imply the existence of a homomorphism φ_g such that the following diagram commutes:

Proof.

$$\begin{array}{ccc} G(S^1) & \xrightarrow{\varphi_f} & G(S^1) \\ \downarrow \varphi_j & \nearrow \varphi_g & \\ G(D^2) & & \end{array}$$

“Commutes” mean $\varphi_f = \varphi_g \circ \varphi_j$.

Suppose we know,

- $G(S^1) \cong \mathbb{Z}$
- $G(D^2) \cong \{e\}$ (the trivial group)
- φ_f is a non-trivial homomorphism, i.e., $\varphi_f(a) \neq 0$ for all $a \in G(S^1)$ is a generator.

This yields a contradiction, since any composition through the trivial group must be trivial. Therefore, **no such map g can exist.** \square

Definition: Invariance (*This is not a formal definition*) Suppose G is a correspondence of the kind we’re looking for:

$$\begin{cases} X & \rightsquigarrow G(X) \\ X \xrightarrow{f} Y & \rightsquigarrow G(X) \xrightarrow{G(f)} G(Y) \end{cases}$$

such that:

$$G(\text{id}_X) = \text{id}_{G(X)}, \quad G(f \circ g) = G(f) \circ G(g).$$

Then observe that it’s **automatic** that

Result :If $f : X \rightarrow Y$ is a homeomorphism, then $G(f) : G(X) \rightarrow G(Y)$ is an isomorphism.

Proof.

- *Homeomorphism:* A function $f : X \rightarrow Y$ is a homeomorphism if
 - f is a bijection (one-to-one and onto),
 - f is continuous,
 - f^{-1} is continuous
- *Isomorphism*
 - Hormphism
 - Bijective

Let $f^{-1} : Y \rightarrow X$ be the inverse of f . Then $f^{-1} \circ f = \text{id}_X$. So,

$$\begin{aligned}\text{id}_{G(X)} &= G(\text{id}_X) = G(f^{-1} \circ f) = G(f^{-1}) \circ G(f) \\ \text{id}_{G(Y)} &= G(\text{id}_Y) = G(f \circ f^{-1}) = G(f) \circ G(f^{-1})\end{aligned}$$

Hence, $G(f^{-1}) = G(f)^{-1}$. Therefore, homeomorphic spaces X, Y yield isomorphic groups, $G(X) \cong G(Y)$

(We say $G(X)$ is an **invariant** of X .)

□

A necessary condition for two spaces X, Y to be homeomorphic is that $G(X) \cong G(Y)$.

Remark. We'll see examples showing this is **NOT** sufficient.

1.1 Some Basic definitions

Definition 1.1 (Homotopy of Maps).

Let X, Y be topological spaces, and let $f, g : X \rightarrow Y$ be continuous maps. We say that f is **homotopic to** g , and write $f \sim g$, if there exists a continuous map

$$H : X \times I \rightarrow Y$$

such that

$$H(x, 0) = f(x), \quad H(x, 1) = g(x) \quad \text{for all } x \in X.$$

Intuitively, the map H defines a continuous deformation of f into g over time $t \in [0, 1]$.

Notations:

- We often write $H_t : X \rightarrow Y$ for the slice $H(\cdot, t)$, so that $H_0 = f, H_1 = g$.
- Sometimes we write $f \sim^H g$ to indicate the specific homotopy H .

Example 1.2.

Let $f, g : X \rightarrow \mathbb{R}^n$ be any continuous functions.
Then $f \sim g$, because we can define a homotopy:

$$H(x, t) = t g(x) + (1 - t) f(x)$$

This map $H : X \times I \rightarrow \mathbb{R}^n$ is continuous
(since vector space operations in \mathbb{R}^n are continuous),
and satisfies:

$$H_0 = f, \quad H_1 = g$$

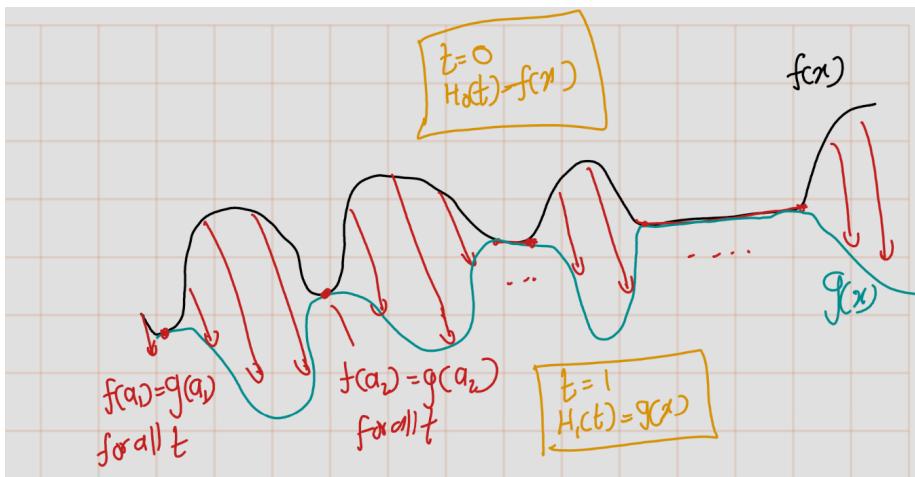
Definition 1.2. Let $A \subset X$, and suppose $f|_A = g|_A$, i.e., $f(a) = g(a)$ for all $a \in A$.

We say that f is **homotopic to g relative to A** if there exists a homotopy

$$H : X \times I \rightarrow Y$$

such that

$$H(a, t) = f(a) = g(a) \quad \text{for all } a \in A, \text{ and all } t \in [0, 1].$$



Example 1.3.

Let $f, g : I \rightarrow \mathbb{R}^2$ be continuous maps (paths), as illustrated.

Then the interpolation homotopy is a homotopy from f to g , and it satisfies

$$H(0, t) = f(0) = g(0), \quad H(1, t) = f(1) = g(1)$$

for all $t \in [0, 1]$.

Hence, H is a **homotopy relative to the endpoints**
 $\{0, 1\} \subset I$.

Notation:

When f is homotopic to g **relative to a subset** $A \subset X$,
we write: $f \sim g$ rel A

This means there exists a homotopy $H : X \times I \rightarrow Y$ such that

$$H(a, t) = f(a) = g(a) \quad \text{for all } a \in A, t \in [0, 1].$$

Example 1.4. Let $f, g, h : X \rightarrow Y$ be continuous maps between topological spaces.

We show that the relation $f \sim g$ (homotopy) satisfies reflexivity, symmetry, and transitivity.

1. Reflexivity

$$f \sim f \quad \text{via } H(x, t) = f(x) \text{ for all } t \in [0, 1].$$

This is a constant homotopy.

2. Symmetry Suppose $f \sim g$ via homotopy $H(x, t)$.

Then define:

$$G(x, t) = H(x, 1 - t)$$

This reverses the deformation and gives a homotopy from g to f , so $g \sim f$.

3. Transitivity

Suppose $f \sim g$ via $F(x, t)$, and $g \sim h$ via $G(x, t)$.

Define a new homotopy $H : X \times I \rightarrow Y$ by:

$$H(x, t) = \begin{cases} F(x, 2t) & \text{if } 0 \leq t \leq \frac{1}{2}, \\ G(x, 2t - 1) & \text{if } \frac{1}{2} \leq t \leq 1. \end{cases}$$

Then:

$$H(x, 0) = F(x, 0) = f(x), \quad H(x, 1) = G(x, 1) = h(x).$$

By the pasting lemma (Lemma 1.1),
 $F(x, 1) = g(x) = G(x, 0)$, so H is continuous.
 Thus, $f \sim h$.

Let me recall pasting lemma, if you can not remember,

Lemma 1.1 (Pasting Lemma). *Let X, Y be both closed (or both open) subsets of a topological space A such that*

$$A = X \cup Y,$$

and let B be a topological space. If $f : A \rightarrow B$ is continuous when restricted to both X and Y , then f is continuous.

Chapter 2

hj