

# Algebraic Topology

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2026-01-12



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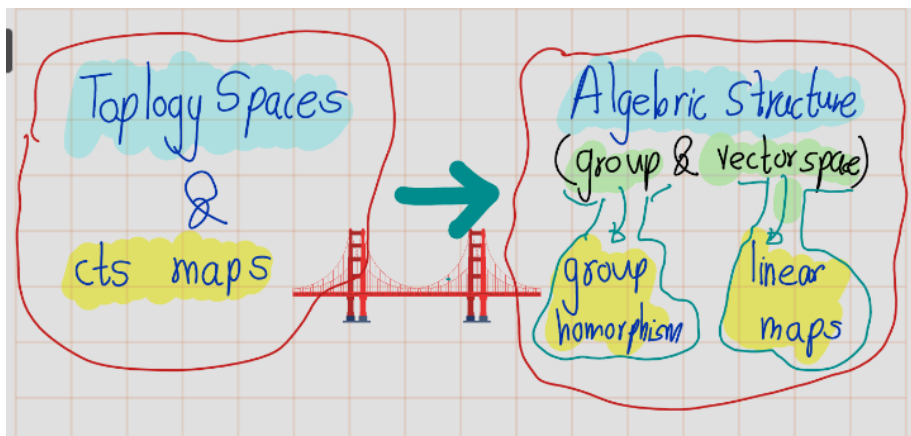
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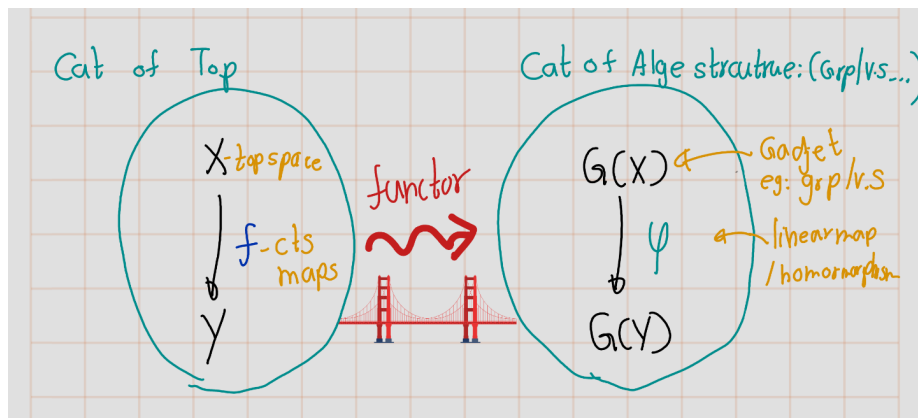
# Chapter 1

## Introduction

### 1.0.1 What is Algebraic topology?



Roughly it's about tools/methods connecting **algebra** and **topology**.



The “bridge” is **algebraic topology**.

In order to be useful, the associations

$$X \rightsquigarrow G(X) \quad \text{and} \quad f \rightsquigarrow \varphi_f$$

ought to satisfy some properties.

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Property Mathematical Form

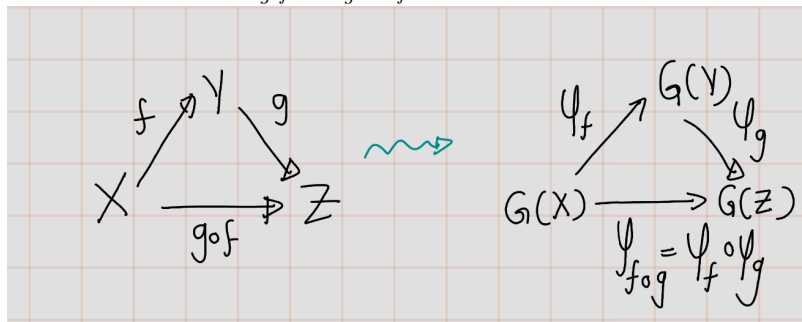
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**Identity**  $\text{id}_X : X \rightarrow X \rightsquigarrow \text{id}_{G(X)} : G(X) \rightarrow G(X)$

**Preservation**

**Composition** If  $f : X \rightarrow Y$  and  $g : Y \rightarrow Z$ , then

**Compatibility**  $f \rightsquigarrow \varphi_{g \circ f} = \varphi_g \circ \varphi_f : G(X) \rightarrow G(Z)$



In somewhat “fancy” language, we are asking that the correspondence above be **functor**. So, what could we do with all this?

**Example 1.1** (Classical Problem). Let  $S^1 \subset D^2$  be the unit circle inside the unit disk in  $\mathbb{R}^2$ . Suppose we are given a continuous map  $f : S^1 \rightarrow S^1$ . **Does there exist a continuous extension  $g : D^2 \rightarrow S^1$  such that  $g(z) = f(z)$  for all  $z \in S^1$ ?**

$$\begin{array}{ccc}
S^1 & \xrightarrow{f} & S^1 \\
\downarrow i & \nearrow ? \exists g & \\
D^2 & & 
\end{array}$$

Imagine we had our algebraic topology tools on hand. If there exists a continuous map  $g : D^2 \rightarrow S^1$  such that  $g \circ j = f$ , then the induced homomorphism satisfies

$$\varphi_f = \varphi_{g \circ j} = \varphi_g \circ \varphi_j.$$

That is, the existence of  $g$  would imply the existence of a homomorphism  $\varphi_g$  such that the following diagram commutes:

*Proof.*

$$\begin{array}{ccc}
G(S^1) & \xrightarrow{\varphi_f} & G(S^1) \\
\downarrow \varphi_j & \nearrow \varphi_g & \\
G(D^2) & & 
\end{array}$$

“Commutates” mean  $\varphi_f = \varphi_g \circ \varphi_j$ .

Suppose we know,

- $G(S^1) \cong \mathbb{Z}$
- $G(D^2) \cong \{e\}$  (the trivial group)
- $\varphi_f$  is a non-trivial homomorphism, i.e.,  $\varphi_f(a) \neq 0$  for all  $a \in G(S^1)$  is a generator.

This yields a contradiction, since any composition through the trivial group must be trivial. Therefore, **no such map  $g$  can exist.**  $\square$

**Definition:** Invariance (*This is not a formal definition*) Suppose  $G$  is a correspondence of the kind we’re looking for:

$$\begin{cases} X & \rightsquigarrow G(X) \\ X \xrightarrow{f} Y & \rightsquigarrow G(X) \xrightarrow{G(f)} G(Y) \end{cases}$$

such that:

$$G(\text{id}_X) = \text{id}_{G(X)}, \quad G(f \circ g) = G(f) \circ G(g).$$

Then observe that it’s **automatic** that

**Result** :If  $f : X \rightarrow Y$  is a homeomorphism, then  $G(f) : G(X) \rightarrow G(Y)$  is an isomorphism.

*Proof.*

- *Homeomorphism:* A function  $f : X \rightarrow Y$  is a homeomorphism if
  - $f$  is a bijection (one-to-one and onto),
  - $f$  is continuous,
  - $f^{-1}$  is continuous
- Isomorphism
  - Homomorphism
  - Bijective

Let  $f^{-1} : Y \rightarrow X$  be the inverse of  $f$ . Then  $f^{-1} \circ f = \text{id}_X$ . So,

$$\begin{aligned}\text{id}_{G(X)} &= G(\text{id}_X) = G(f^{-1} \circ f) = G(f^{-1}) \circ G(f) \\ \text{id}_{G(Y)} &= G(\text{id}_Y) = G(f \circ f^{-1}) = G(f) \circ G(f^{-1})\end{aligned}$$

Hence,  $G(f^{-1}) = G(f)^{-1}$ . Therefore, homeomorphic spaces  $X, Y$  yield isomorphic groups,  $G(X) \cong G(Y)$

(We say  $G(X)$  is an **invariant** of  $X$ .)

□

A necessary condition for two spaces  $X, Y$  to be homeomorphic is that  $G(X) \cong G(Y)$ .

*Remark.* We'll see examples showing this is **NOT** sufficient.

## 1.1 Some Basic definitions

**Definition 1.1** (Homotopy of Maps).

Let  $X, Y$  be topological spaces, and let  $f, g : X \rightarrow Y$  be continuous maps.

We say that  $f$  is **homotopic to**  $g$ , and write  $f \sim g$ , if there exists a continuous map

$$H : X \times I \rightarrow Y$$

such that

$$H(x, 0) = f(x), \quad H(x, 1) = g(x) \quad \text{for all } x \in X.$$

Intuitively, the map  $H$  defines a continuous deformation of  $f$  into  $g$  over time  $t \in [0, 1]$ .

**Notations:**



- We often write  $H_t : X \rightarrow Y$  for the slice  $H(\cdot, t)$ , so that  $H_0 = f$ ,  $H_1 = g$ .
- Sometimes we write  $f \stackrel{H}{\sim} g$  to indicate the specific homotopy  $H$ .

**Example 1.2.**

Let  $f, g : X \rightarrow \mathbb{R}^n$  be any continuous functions.

Then  $f \sim g$ , because we can define a homotopy:

$$H(x, t) = t g(x) + (1 - t) f(x)$$

This map  $H : X \times I \rightarrow \mathbb{R}^n$  is continuous  
(since vector space operations in  $\mathbb{R}^n$  are continuous),  
and satisfies:

$$H_0 = f, \quad H_1 = g$$

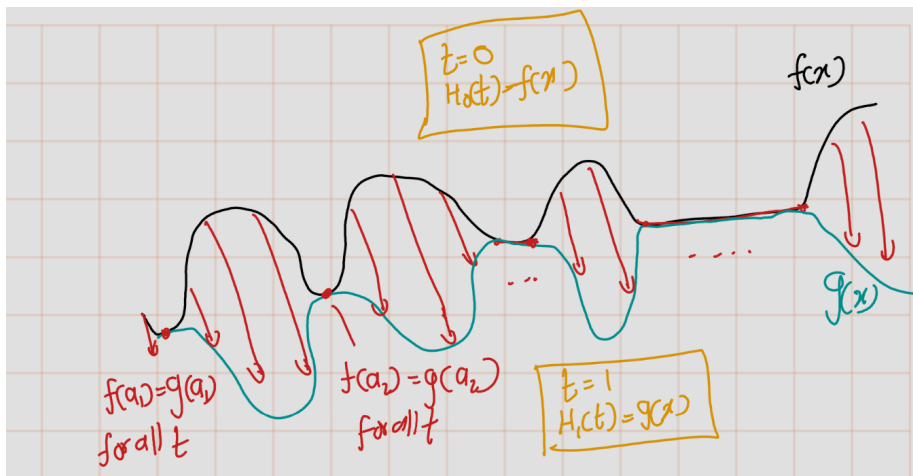
**Definition 1.2.** Let  $A \subset X$ , and suppose  $f|_A = g|_A$ , i.e.,  $f(a) = g(a)$  for all  $a \in A$ .

We say that  $f$  is **homotopic to  $g$  relative to  $A$**  if there exists a homotopy

$$H : X \times I \rightarrow Y$$

such that

$$H(a, t) = f(a) = g(a) \quad \text{for all } a \in A, \text{ and all } t \in [0, 1].$$



**Example 1.3.**

Let  $f, g : I \rightarrow \mathbb{R}^2$  be continuous maps (paths), as illustrated.

Then the interpolation homotopy is a homotopy from  $f$  to  $g$ , and it satisfies

$$H(0, t) = f(0) = g(0), \quad H(1, t) = f(1) = g(1)$$

for all  $t \in [0, 1]$ .

Hence,  $H$  is a **homotopy relative to the endpoints**  $\{0, 1\} \subset I$ .

**Notation:**

When  $f$  is homotopic to  $g$  **relative to a subset**  $A \subset X$ , we write:  $f \sim g \text{ rel } A$

This means there exists a homotopy  $H : X \times I \rightarrow Y$  such that

$$H(a, t) = f(a) = g(a) \quad \text{for all } a \in A, t \in [0, 1].$$

**Example 1.4.** Let  $f, g, h : X \rightarrow Y$  be continuous maps between topological spaces.

We show that the relation  $f \sim g$  (homotopy) satisfies reflexivity, symmetry, and transitivity.

## 1. Reflexivity

$$f \sim f \quad \text{via } H(x, t) = f(x) \text{ for all } t \in [0, 1].$$

This is a constant homotopy.

2. Symmetry Suppose  $f \sim g$  via homotopy  $H(x, t)$ .

Then define:

$$G(x, t) = H(x, 1 - t)$$

This reverses the deformation and gives a homotopy from  $g$  to  $f$ , so  $g \sim f$ .

## 3. Transitivity

Suppose  $f \sim g$  via  $F(x, t)$ , and  $g \sim h$  via  $G(x, t)$ .

Define a new homotopy  $H : X \times I \rightarrow Y$  by:

$$H(x, t) = \begin{cases} F(x, 2t) & \text{if } 0 \leq t \leq \frac{1}{2}, \\ G(x, 2t - 1) & \text{if } \frac{1}{2} \leq t \leq 1. \end{cases}$$

Then:

$$H(x, 0) = F(x, 0) = f(x), \quad H(x, 1) = G(x, 1) = h(x).$$

By the pasting lemma (Lemma 1.1),  
 $F(x, 1) = g(x) = G(x, 0)$ , so  $H$  is continuous.  
Thus,  $f \sim h$ .

Let me recall pasting lemma, if you can not remember,

**Lemma 1.1** (Pasting Lemma). *Let  $X, Y$  be both closed (or both open) subsets of a topological space  $A$  such that*

$$A = X \cup Y,$$

*and let  $B$  be a topological space. If  $f : A \rightarrow B$  is continuous when restricted to both  $X$  and  $Y$ , then  $f$  is continuous.*



## Chapter 2

hj