Exercise 1.1 If r is rational $r \neq 0$ and x is irrational, prove that $r + x$ and rx are irrational.

Suppose that $0 \neq r \in \mathbb{Q}$ and $x \in \mathbb{R} \setminus \mathbb{Q}$

- **Claim 1**: $r + x$ is irrational.
  Assume the contray that $r + x \in \mathbb{Q}$. Since, $\mathbb{Q}$ is field,
  $$x = (r + x) - r \in \mathbb{Q}$$
  . This is a contradiction. Thus, $r + x$ is irrational.

- **Claim 2**: rx is irrational.
  Assume the contray that $rx \in \mathbb{Q}$. Since, $\mathbb{Q}$ is field,
  $$\left(\frac{1}{x}\right) rx = \left(\frac{1}{x}\right) xr = \left(\frac{1}{x}x\right) r = x \in \mathbb{Q}$$
  . This is a contradiction. Thus, rx is irrational.

Exercise 1.2 (1:R2) Prove that there is no rational number whose square is 12.

Proof. First observe that

$$\sqrt{12} = \sqrt{4 \cdot 3} = 2\sqrt{3}$$

Claim 1 : $\forall n \in \mathbb{N} 3|n^2 \implies 3|n$.
Let $n \in \mathbb{N}$. Suppose that $3|n^2$ We know that 3 is prime. Then by number theory resuts we can get $3|n$ or $3|n$. We are done.

Cliam 2: $\sqrt{3}$ is irrartional.
We use inderect proof. Assume contary, $\sqrt{3}$ is rational. In other words,

$$\sqrt{3} = \frac{p}{q} \text{ for some } q \in \mathbb{Z}, q \neq 0 \text{ and } p, q \text{ have no comman factors.}$$

Thus,

$$3q^2 = p^2$$

Then, by above claim we can get that $3|p$. Thus, there exists $k \in \mathbb{Z}$ such that $3k = p$. Thus,

$$\begin{aligned} 3q^2 &= p^2 \\ 3q^2 &= (3k)^2 \\ 3q^2 &= 9k^2 \\ q^2 &= 3k^2 \end{aligned}$$

So, both p and q have comman factor 3.This contract our assumption. Thefore, $\sqrt{(3)}$ is irrational. So we are done proof of claim 2.

Since, $\sqrt{3}$ is not a rational number. Thus $\sqrt{3}$ is irrational number.

Therfore exercise 1.1 we get $\sqrt{(12)}$ is irrational. Hence, there is no rational number whose square is 12. ☐

1.15 Proposition *The axioms for multiplication imply the following statements.*

(*a*) *If* $x \neq 0$ *and* $xy = xz$ *then* $y = z$.
(*b*) *If* $x \neq 0$ *and* $xy = x$ *then* $y = 1$.
(*c*) *If* $x \neq 0$ *and* $xy = 1$ *then* $y = 1/x$.
(*d*) *If* $x \neq 0$ *then* $1/(1/x) = x$.

The proof is so similar to that of Proposition 1.14 that we omit it.

Let $\mathbb{F}$ be an feild and let $x, y, z \in \mathbb{F}$.

a) Suppose that $x \neq 0$ and $xy = xz$

Since $x \neq 0$, there exist $(1/x) \in \mathbb{F}$ suchthat $x(1/x) = 1$ $(*)$

Now,

$$(xy) = (xz)$$

$$\frac{1}{x}(xy) = \frac{1}{x}(xz) \quad \text{(Property of equal sign)}$$

$$\left(\frac{1}{x}x\right)y = \left(\frac{1}{x}x\right)z \quad \text{(By assciativity property in book M3)}$$

$$1y = 1z \quad \text{(By } (*)\text{)}$$

$$y = z \quad \text{(By multiplication identity in book M4.)}$$

~~The~~

b), c), d) can be easily prooved using the result in part a)

b) We apply a) with $z=1$, then b) is trivial

c) We apply ~~b)~~ a) with $z = 1/x$.

$(x \neq 0 \text{ and } xy = xz) \Rightarrow y = z$ [part (a)

If $z = 1/x$

$((x \neq 0) \text{ and } xy = x(1/x) = 1) \Rightarrow y = 1/x$

4. Let E be a nonempty subset of an ordered set; suppose α is a lower bound of E and β is an upper bound of E. Prove that $\alpha \leq \beta$.

Let $x \in E$. (Since $E \neq \phi$, there exist such x.)

Need to show $\alpha \leq \beta$.

Since α is an upper bound and β is an lower bound of E.
Then by definition of upper bound and lower bound (Def$^{\underline{n}}$ 1.7 in book)

$$x \leq \alpha \quad \text{and} \quad \beta \leq x$$

Then, $(x = \alpha$ or $x < \alpha)$ and $(\beta = x$ or $\beta < x)$

Now, we have to consider 4-cases

Case-I If $x < \alpha$ and $\beta < x$.
The by part ii) of Def$^{\underline{n}}$ 1.5

$$\beta < \alpha$$

Case-II If $x = \alpha$ and $\beta < x$
Then $\beta < \alpha = x$

Case-III
If $x < \alpha$ and $x = \beta$ then
$x = \beta < \alpha$

Case-IV: If $x = \alpha$ and $\beta = x$
Then $x = \alpha = \beta$

Therefore $\beta < \alpha$ or $\alpha = \beta$. Thus, $\beta \leq \alpha$.

5. Let A be a nonempty set of real numbers which is bounded below. Let $-A$ be the set of all numbers $-x$, where $x \in A$. Prove that

$$\inf A = -\sup(-A).$$

Let $\phi \neq A \subseteq \mathbb{R}$ be a bounded below set. Let

$$-A := \{-x \mid x \in A\}$$

First of all we need to show the existance of $\sup(-A)$. ~~Since $A \neq \phi$, there exist $a \in A$.~~ Since A is bdd below, there exist $l \in \mathbb{R}$ such that,

$$a \geq l \quad \text{for all } a \in A.$$

Then
$$-a \leq -l \quad \text{for all } a \in A$$
$$-a \leq -l \quad \text{for all } -a \in -A.$$

Thus, $-l$ is an upper bound of $(-A)$. Thus $\sup(-A)$ exists. (Since $\mathbb{R}$ have least upper bound property)

Now we are going to proove that $\inf(A) = -\sup(-A)$

<u>claim 1</u>: $\inf(A) \geq -\sup(-A)$.

Since $\sup(-A)$ is an upper bound for $-A$

$$-a \leq \sup(-A) \quad \forall a \in A.$$
$$a \geq -\sup(-A) \quad \forall a \in A.$$

Then $-\sup(-A)$ is an lower bound for (A)

By def$^{\underline{n}}$ of infimum, $\inf(A) \geq -\sup(-A)$ —— $\circledast$

Claim 2: $\inf(A) \leq -\sup(-A)$

Since, $\inf(A)$ is a lower bound of A.

$$\inf(A) \leq a \quad \forall a \in A.$$

$$-a \leq -\inf(A) \quad \forall a \in A.$$

$$-a \leq -\inf(A) \quad \forall -a \in -A.$$

Then $-\inf(A)$ is an upperbound for $-A$.

By def^{n} of supremum $\sup(-A) \leq -\inf(A)$. Thus

$$\inf(A) \leq -\sup(-A) \text{ —— } (**)$$

By claim 1 and 2 we get $\inf(A) = \sup(-A)$

6. Fix $b > 1$.

(*a*) If m, n, p, q are integers, $n > 0$, $q > 0$, and $r = m/n = p/q$, prove that

$$(b^m)^{1/n} = (b^p)^{1/q}.$$

Hence it makes sense to define $b^r = (b^m)^{1/n}$.

(*b*) Prove that $b^{r+s} = b^r b^s$ if r and s are rational.

(*c*) If x is real, define $B(x)$ to be the set of all numbers b^t, where t is rational and $t \leq x$. Prove that

$$b^r = \sup B(r)$$

when r is rational. Hence it makes sense to define

$$b^x = \sup B(x)$$

for every real x.

(*d*) Prove that $b^{x+y} = b^x b^y$ for all real x and y.

First fix $b>1$

a) Suppose that $m, n, p, q \in \mathbb{Z}$ with $n>0, q>0$ and $r = \frac{m}{n} = \frac{p}{q}$

We are going to use thm 1.21 uniqueness property

Recall thm 1.21

$\forall x \in \mathbb{R}^+ \; \forall n \in \mathbb{Z}^+ \; \exists!\, y \in \mathbb{R}$ such that $y^n = x$.

Now consider,

$$\cancel{((b^m)^{1/n})^{nq} = (((b^m)^{1/n})^n)^q = (b^m)^q = b^{mq}}.$$

~~Let~~ First note that $\frac{m}{n} = \frac{p}{q}$ = implies $np = mq$.

$$((b^m)^{1/n})^{nq} = (((b^m)^{1/n})^n)^q = (b^m)^q = b^{mq} = b^{np}$$
$$= b^{pn} = (b^p)^n = (((b^p)^{1/q})^q)^n$$
$$= ((b^p)^{1/q})^{nq}$$

Now by theorm 1.21, we get
$$(b^m)^{1/n} = (b^p)^{1/q}$$
$\blacksquare$

Hence it makes sense to define $b^r = (b^m)^{1/n}$

Suppose that $r, s \in \mathbb{Q}$. Then $r = \frac{m}{n}$ and $s = \frac{p}{q}$ forsome $m, n, p, q \in \mathbb{Z}$ with $n, q \neq 0$.

Now consider

$$\left(b^{r+s}\right)^{nq} = \left(b^{\frac{m}{n}+\frac{p}{q}}\right)^{nq} = \left(b^{\frac{mq+np}{nq}}\right)^{nq}$$

$$= \left(\left(b^{mq+np}\right)^{1/nq}\right)^{nq}$$

$$= b^{mq+np}$$

$$= b^{mq} \cdot b^{np}$$

$$= \left(b^{m}\right)^{q} \cdot \left(b^{p}\right)^{n}$$

$$= \left(\left(\left(b^{m}\right)^{1/n}\right)^{n}\right)^{q} \cdot \left(\left(\left(b^{p}\right)^{1/q}\right)^{q}\right)^{n}$$

$$= \left(\left(b^{m}\right)^{1/n}\right)^{nq} \cdot \left(\left(b^{p}\right)^{1/q}\right)^{nq}$$

$$= \left(\left(b^{m}\right)^{1/n} \cdot \left(b^{p}\right)^{1/q}\right)^{nq}$$

$$= \left(b^{r} \cdot b^{s}\right)^{nq}$$

By thm 1.21, weget $b^{r+s} = b^{r} \cdot b^{s}$

c/d - Redo It

7. Fix $b > 1$, $y > 0$, and prove that there is a unique real x such that $b^x = y$, by completing the following outline. (This x is called the *logarithm of y to the base b.*)

(a) For any positive integer n, $b^n - 1 \geq n(b - 1)$.

(b) Hence $b - 1 \geq n(b^{1/n} - 1)$.

(c) If $t > 1$ and $n > (b - 1)/(t - 1)$, then $b^{1/n} < t$.

(d) If w is such that $b^w < y$, then $b^{w+(1/n)} < y$ for sufficiently large n; to see this, apply part (c) with $t = y \cdot b^{-w}$.

(e) If $b^w > y$, then $b^{w-(1/n)} > y$ for sufficiently large n.

(f) Let A be the set of all w such that $b^w < y$, and show that $x = \sup A$ satisfies $b^x = y$.

(g) Prove that this x is unique.

Fix $b>1, y>0$

a) claim1: $\forall n \in \mathbb{Z}^+, b^n - 1 \geq n(b-1)$

We are going to use mathematical induction.

~~$n=1$ $b^1 - 1 \geq 1(b-1)$~~

If $n=1$, the inequlity holds

Assume that the given inequlity holds if $n = p \in \mathbb{Z}^+$. i.e:

$$b^p - 1 = p(b-1)$$

Now $n=p+1$

$$\begin{aligned} b^{p+1} - 1 &= b^{p+1} - b + (b-1) \\ &= b(b^p - 1) + (b-1) \\ &\geq b(p(b-1)) + (b-1) \quad \text{(By induction hypothesis)} \\ &= bp(b-1) + (b-1) \\ &= (bp+1)(b-1) \\ &\geq (p+1)(b-1) \quad (\text{Since } b>1) \end{aligned}$$

Therefore, by mathematical induction, for all $n \in \mathbb{Z}^+$ such that $b^n - 1 \geq n(b-1)$,

b) Now replace b with $b^{1/n}$. Then ~~fist giv~~ part a) inequlity becomes.

$$(b-1) \geq n\,(b^{1/n}-1).$$

Note that $b^{1/n} > 1$

c) Now assume that $t > 1$ and $n > \frac{(b-1)}{(t-1)}$

Thus, $n(t-1) > (b-1)$

The part b) becomes,

$$n(t-1) > (b-1) \geq n\,(b^{1/n}-1)$$

$$n(t-1) > n(b^{1/n}-1)$$

$$(t-1) > (b^{1/n}-1) \quad , (\because n \in \mathbb{Z}^{+})$$

$$t > b^{1/n}$$

d) Now suppose that $b^{w} < y$. Then

$$1 < y b^{-w}$$

Apply part(c) with $t = y b^{-w} > 1$ and assume that n is large with

$$n > \frac{(b-1)}{(t-1)},$$

$$b^{1/n} < y b^{-w}$$

$$b^{w+1/n} < y \qquad (\because b > 0)$$

e) Now Suppose that $b^w > y$. Then $b^w\left(\frac{1}{y}\right) > 1$

Apply part c) $t = b^w/y$

~~assu~~ Furthure, assume that $n > \frac{b-1}{t-1}$

By result of part c), $b^{1/n} < b^w\left(\frac{1}{y}\right)$

$$y b^{1/n} < b^w$$

$$y < b^{w - 1/n}$$

f) Let $A := \{w \mid b^w < y\}$

Let $x = \sup(A)$. Since $\mathbb{R}$ is ordered set, one of ~~them~~ is following is true

$$b^x < y \quad ~~or~~, \quad b^x = y, \quad b^x > y$$

Case-1 If $b^x < y$, in part d, we showed that for sufficient large n

$b^{x+\frac{1}{n}} < y$. Then $x + \frac{1}{n} \in A$ ~~is an upper bdd~~

This contradic the fact x is the sup(A)

Case-III: If $b^x > y$, in part e, we already showed that for sufficiently large n, $b^{x-1/n} > y$

Thus $\left(x - \frac{1}{n}\right)$ ~~is an lower bound~~ is an upper bound for A. This contradict x is $\sup(A)$. (least upper bound of A)

Therefore, $b^x = y$, $x = \sup(A)$

9) Redo It ♀

8. Prove that no order can be defined in the complex field that turns it into an ordered field. *Hint:* -1 is a square.

Assume that $\mathbb{C}$ is one orderd feild.
~~Then (by prop 1.18) Since $i \neq 0$~~ We consider i
Then by definition 1.5, one of them is true.

$$i < 0 \quad , \quad i = 0 \quad , \quad i > 0.$$

We know that i is non-zero. So we consider two cases $i<0$ and $i>0$

Case-I $i<0$
Then, $-i>0$ (by Prop 1.18)

$$(-i)(-i) = (i)(i) = i^2 = -1 > 0$$

(by def^n 1.17-ii)

This is contradiction.

Case-II $i>0$. Then

$(i)(i) = i^2 = -1 > 0$ (by def^n 1.17-ii)

This is also contradiction.
Therefore, there is no order relation in $\mathbb{C}$

9. Suppose $z = a + bi$, $w = c + di$. Define $z < w$ if $a < c$, and also if $a = c$ but $b < d$. Prove that this turns the set of all complex numbers into an ordered set. (This type of order relation is called a *dictionary order*, or *lexicographic order*, for obvious reasons.) Does this ordered set have the least-upper-bound property?

Let $z = a+bi$ and $w = c+di$
where $a, b, c, d \in \mathbb{R}$. Since
$\mathbb{R}$ is anorderd set.

$(a<c$ or $a=c$ or $\cancel{b<a}$ $a>c)$
and $(b<d$ or $b=d$ or $b>d)$

Case-I If $a<c$, then $z<w$
Case-II If $a>c$ then $z>w$
Case-III If $a=c$ and $b>d$ then $z>w$
Case-IV If $a=c$ and $b<d$ then $z<w$
Case-V If $a=c$ and $b=d$ then $z=w$

Therefore one of them always true. ie:
either $z<w$ or $z=w$ or $z>w$

Now let's check second condion of orderd set

Let $u = e+fi$.
~~Suppose that $z>w$ and $y>0$~~
Suppose that $z<w$ and $w<u$
$z<w \Rightarrow (a<c)$ or $(a=c$ and $b<d)$

$w < u \Rightarrow (c < e)$ or $(c = e$ and $d < f)$

<u>Case-I</u> If $a < c$ and $c < e$ then $a < e$

Thus, $z < u$

<u>Case-II</u> If $a < c$ and $c = e$ and $d < f$ then $a < e$. This implies $z < u$.

<u>Case-III</u> If $a = c$ and $b < d$ and $c < e$ then $a < e$ then $z < u$

<u>Case-IV</u> If $a = c$ and $b < d$ and $c = e$ and $d < f$. Then $a = e$ and $b < f$.

This implies $z < u$.

This 4 cases implies ~~$z < w$ and $w < u$~~

If $z < w$ and $w < u$ then $z < w$.

10. Suppose $z = a + bi$, $w = u + iv$, and

$$a = \left(\frac{|w| + u}{2}\right)^{1/2}, \qquad b = \left(\frac{|w| - u}{2}\right)^{1/2}.$$

Prove that $z^2 = w$ if $v \geq 0$ and that $(\bar{z})^2 = w$ if $v \leq 0$. Conclude that every complex number (with one exception!) has two complex square roots.

Suppose that $z = a + bi$ and $w = u + iv$ and

where $a = \left(\frac{|w| + u}{2}\right)^{1/2}$, $b = \left(\frac{|w| - u}{2}\right)^{1/2}$

Suppose that $v \geq 0$.

Need to show: $z^2 = w$

$$z^2 = (a+bi)^2 = (a+bi)(a+bi)$$
$$= a^2 + bi + bi + 2i^2 ab$$
$$= (a^2 - b^2) + 2iab$$

$$a^2 - b^2 = \left(\frac{|w| + u}{2}\right) + \left(\frac{|w| - u}{2}\right) = u$$

$$2ab = 2\left(\frac{|w| + u}{2}\right)^{1/2}\left(\frac{|w| - u}{2}\right)^{1/2} = 2\left(\left(\frac{|w|+u}{2}\right)\left(\frac{|w|-u}{2}\right)\right)^{1/2}$$
$$= 2\left(\frac{|w|^2 - u^2}{4}\right)^{1/2} = 2\left(\frac{v^2}{4}\right)^{1/2} = v$$

(Since $v \geq 0$)

Thus,

$$z^2 = (a^2 - b^2) + (2ab)i$$
$$= u + vi = w$$

NTS: If $\gamma \le 0$ then $(\bar{z})^2 = w$

Suppose that $\gamma \le 0$.

$\bar{z}^2 = \bar{z}\,\bar{z} = (a-bi)(a-bi)$

$= a - bi - bi + i^2 b$

$= (a^2 - b^2) - (2ab)i$

We already show that $(a^2 - b^2) = u$ and

$2ab = 2\left(\frac{\gamma^2}{2}\right)^{1/2}$.

Since $\gamma \le 0$, $2ab = 2\left(\frac{\gamma^2}{2}\right)^{1/2} = -\gamma$

Therefore, $(\bar{z})^2 = (a^2 - b^2) - (2ab)i$

$= u - (-\gamma)i$

$= u + \gamma i = w$

11. If z is a complex number, prove that there exists an $r \geq 0$ and a complex number w with $|w| = 1$ such that $z = rw$. Are w and r always uniquely determined by z?

Suppose that $z \in \mathbb{C}$.

- If $z = 0$, choose $r = 0$ and $w = \frac{1}{\sqrt{2}} + \left(\frac{1}{\sqrt{2}}\right)i$

Then $|w| = \sqrt{\left(\frac{1}{\sqrt{2}}\right)^2 + \left(\frac{1}{\sqrt{2}}\right)^2} = \sqrt{\frac{1}{2} + \frac{1}{2}} = 1$

- If $z \neq 0$, choose $r = |z|$ and $w = \frac{z}{|z|}$. Then $r = |z| \geq 0$ and

$$|w| = \left|\frac{z}{|z|}\right| = \frac{|z|}{||z||} = \frac{|z|}{|z|} = 1$$

(by thm 1.33-c)

(by thm 1.33d)

(This idea is get from the unit vectors)

Therefore, $z \in \mathbb{C} \Rightarrow (\exists (r \geq 0$ and $w \in \mathbb{C}$ with $|w| = 1)$ such that $z = rw$

- In first ~~cb~~case, uniquness does not holds. Because if $z=0$, then we can choose $r=0$ and any $w \in \mathbb{C}$ with $|w|=1$.

- In second case, uniquness holds. If $z \neq 0$ then we can chose

Assume that $z = r_1 w_1 = r_2 w_2$ — (1) with $r_1 > 0$, $r_2 > 0$, $|w_1| = 1$, $|w_2| = 1$. Take modulus of both sides

$$|r_1 w_1| = |r_2 w_2|$$

$|r_1||w_1| = |r_2||w_2|$ (thm 1.33 and

$r_1 |w_1| = r_2 |w_2|$ (Since $r_i \in \mathbb{R}$)

$r_1 = r_2$ — (2) (Since $|w_1| = |w_2| = 1$

Thus, ~~$r_1 w_1 = r_2 w_2$~~ By ~~our~~ equation (1), (2) impiles $w_1 = w_2$

Therefore r, w are uniquely determined by z.

12. If $z_1, \ldots, z_n$ are complex, prove that

$$|z_1 + z_2 + \cdots + z_n| \le |z_1| + |z_2| + \cdots + |z_n|.$$

(12) Suppose that $z_1, z_2, \ldots, z_n \in \mathbb{C}$

We are going to use mathematical inducti.

~~$|z_1+z_2+\cdots+z_n| =$~~

If $n=2$, by thm 1.33 trinangle equlity holds. i.e:

$$|z_1+z_2| \leq |z_1| + |z_2|$$

Now assume that following holds for $n = p \in \mathbb{N}$

$$|z_1+z_2+\cdots+z_p| \leq |z_1| + \cdots + |z_p|$$

Now Consider following,

$$|z_1+z_2+\cdots+z_p+z_{p+1}| \leq |z_1+\cdots+z_p| + |z_{p+1}|$$

(by base case)

$$|z_1+\cdots+z_p+z_{p+1}| \leq |z_1| + \cdots + |z_p| + |z_{p+1}|$$

(Induction hypothesis)

Therefore, by mathematical hypothesis, ~~we get~~

$|z_1+z_2+\cdots+z_n| \leq |z_1| + \cdots + |z_n|$ for all $n \in \mathbb{N} \setminus \{0\}$

13. If x, y are complex, prove that

$$||x| - |y|| \leq |x - y|.$$

Let $x, y \in \mathbb{C}$. By theorm 1.33

$$|z+w| \leq |z| + |w| \text{ for all } z, w \in \mathbb{C}$$

We use $z = (x-y)$ and $w = y$. Using triangle inequlity,

$$|(x-y)+y| \leq |x-y| + |y|$$

$$|x| \leq |x-y| + |y|$$

$$|x| - |y| \leq |x-y| \text{ ——— } (*)$$

Now, we use $z = y-x$ and $w = x$. By triangle inequlity,

$$|(y-x)+x| \leq |y-x| + |x|$$

$$|y| \leq |y-x| + |x|$$

$$|y| - |x| \leq |y-x|$$

We know that, $|x-y| = |y-x|$. Thus,

$$|y| - |x| \leq |x-y| \text{ ——— } (**)$$

By $(*)$ and $(**)$, $||x| - |y|| \leq |x-y|$

■

14. If z is a complex number such that $|z| = 1$, that is, such that $z\bar{z} = 1$, compute

$$|1+z|^2 + |1-z|^2.$$

Suppose that $z \in \mathbb{C}$ such that $|z| = 1$.

P By definit 1.32,

$$\begin{aligned}
|1+z|^2 &= (1+z)(\overline{1+z}) && (\text{def}^n\ 1.32) \\
&= (1+z)(\bar{1}+\bar{z}) && (\text{By thm } 1.31\text{-a}) \\
&= (1+z)(1+\bar{z}) && (\because 1 \in \mathbb{R}) \\
&= 1+\bar{z}+z+z\bar{z} && (\text{distributi property}) \\
&= 1+\bar{z}+z+|z| && (\text{def}^n\ 1.32) \\
&= 1+\bar{z}+z+1 && (\text{by hypothesis}) \\
&= 2+\bar{z}+z && \text{——— } 1
\end{aligned}$$

Similarly,

$$\begin{aligned}
|1-z|^2 &= (1-z)(1-\bar{z}) && (\text{def}^n\ 1.32 \text{ and thm } 1.31) \\
&= 1-\bar{z}-z+z\bar{z} && (\text{distributive}) \\
&= 1-\bar{z}-\bar{z}+|z| && (\text{def}^n\ 1.32) \\
&= 1-\bar{z}-\bar{z}+1 && (\text{Hypothesis}) \\
&= 2-\bar{z}+\bar{z} && \text{——— } 2
\end{aligned}$$

By 1 and 2,

$$|1+z|^2 + |1-z|^2 = 4$$

15. **Under what conditions does equality hold in the Schwarz inequality?**

Recall the Schwarz inequality

1.35 Theorem *If $a_1, \ldots, a_n$ and $b_1, \ldots, b_n$ are complex numbers, then*

$$\left|\sum_{j=1}^{n} a_j \bar{b}_j\right|^2 \leq \sum_{j=1}^{n} |a_j|^2 \sum_{j=1}^{n} |b_j|^2.$$

Proof Put $A = \Sigma |a_j|^2$, $B = \Sigma |b_j|^2$, $C = \Sigma a_j \bar{b}_j$ (in all sums in this proof, j runs over the values $1, \ldots, n$). If $B = 0$, then $b_1 = \cdots = b_n = 0$, and the conclusion is trivial. Assume therefore that $B > 0$. By Theorem 1.31 we have

$$\begin{aligned}\sum |Ba_j - Cb_j|^2 &= \sum (Ba_j - Cb_j)(B\bar{a}_j - \overline{Cb_j}) \\ &= B^2 \sum |a_j|^2 - B\bar{C} \sum a_j \bar{b}_j - BC \sum \bar{a}_j b_j + |C|^2 \sum |b_j|^2 \\ &= B^2 A - B|C|^2 \\ &= B(AB - |C|^2).\end{aligned}$$

Since each term in the first sum is nonnegative, we see that

$$B(AB - |C|^2) \geq 0.$$

Since $B > 0$, it follows that $AB - |C|^2 \geq 0$. This is the desired inequality.

If we go through the proof, we can See that, If $A=0$ or $B=0$ implies ~~that~~ the equilitiy. (This case is trivial)

~~Now~~ So, now assume that $A \neq 0$ and $B \neq 0$.

By the prooof thm 1.35

$\cancel{(AB - |C|^2) = 0}$

(equlity of Schwartz inequtity) $\Longleftrightarrow \sum |Ba_j - Cb_j|^2 = 0$

$\Longleftrightarrow |Ba_j - Cb_j| = 0$ for all $i = 1, 2, \ldots, n$

$\Longleftrightarrow Ba_j = Cb_j$ for all $i = 1, 2 \ldots n$

$\Longleftrightarrow a_j = \frac{C}{B} b_j$ for all $i = 1, 2 \ldots n$

$\Longleftrightarrow$ the numbers a_j are propostional to b_j

$\Longleftrightarrow$ a_j ~~are~~ and b_j are linearly dependent.

16. Suppose $k \geq 3$, $\mathbf{x}, \mathbf{y} \in R^k$, $|\mathbf{x}-\mathbf{y}| = d > 0$, and $r > 0$. Pr

(*a*) If $2r > d$, there are infinitely many $\mathbf{z} \in R^k$ such that

$$|\mathbf{z}-\mathbf{x}| = |\mathbf{z}-\mathbf{y}| = r.$$

(*b*) If $2r = d$, there is exactly one such $\mathbf{z}$.

(*c*) If $2r < d$, there is no such $\mathbf{z}$.

How must these statements be modified if k is 2 or 1?

17. Prove that

$$|\mathbf{x}+\mathbf{y}|^2+|\mathbf{x}-\mathbf{y}|^2=2|\mathbf{x}|^2+2|\mathbf{y}|^2$$

if $\mathbf{x}\in R^k$ and $\mathbf{y}\in R^k$. Interpret this geometrically, as a statement about parallelograms.

Suppose that $x, y \in \mathbb{R}^k$. Here $x=(x_1,\dots,x_k)$ and $y=(y_1,\dots,y_k)$

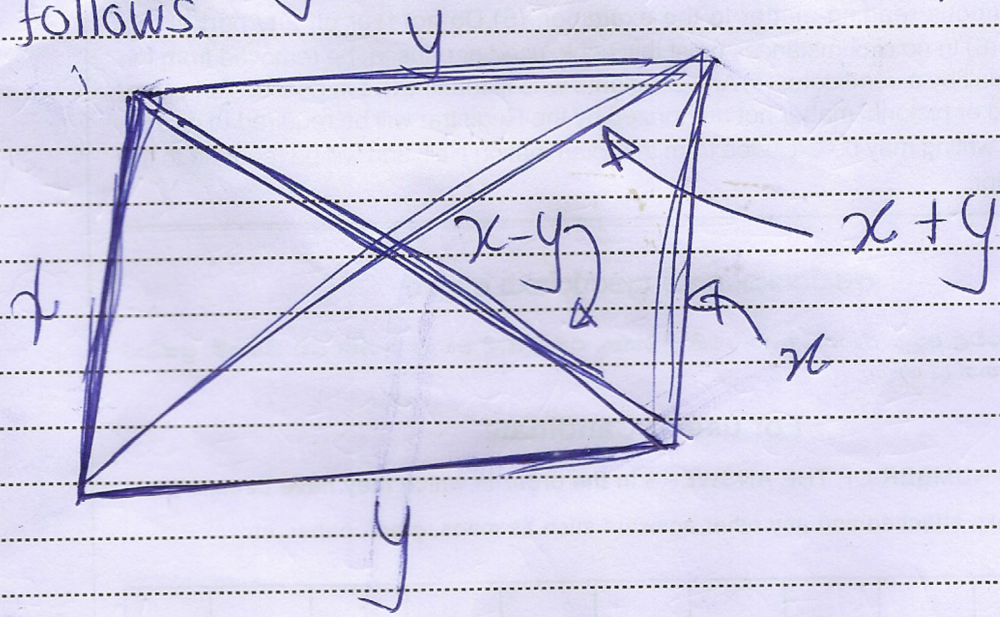
$$|x+y|^2 = \big((x+y)\cdot(x+y)\big)$$
$$= x\cdot x + 2(x\cdot y) + y\cdot y$$
$$= |x|^2 + 2(x\cdot y) + |y|^2 \quad \text{——} \; (1)$$

$$|x-y|^2 = (x-y)\cdot(x-y)$$
$$= x\cdot x - 2(x\cdot y) + y\cdot y$$
$$= |x|^2 - 2(x\cdot y) + |y|^2 \quad \text{——} \; (2)$$

By (1) and (2)

$$|x+y|^2+|x-y|^2 = |x|^2 + 2(x\cdot y) + |y|^2 + |x|^2 - 2(x\cdot y) + |y|^2$$
$$= 2|x|^2 + 2|y|^2$$

If x and y are sides of parallelgram as follows.



(Similar to A/L vectors)

Then $x-y$ and $x+y$ are diangongols of the parallelgram. Therefore we can conclude that,

The sum of square of diagonals of parallelgram is equal to sum of square of the sides.

18. If $k \geq 2$ and $\mathbf{x} \in R^k$, prove that there exists $\mathbf{y} \in R^k$ such that $\mathbf{y} \neq \mathbf{0}$ but $\mathbf{x} \cdot \mathbf{y} = 0$. Is this also true if $k = 1$?

Suppose that $k \geq 2$ and $x \in \mathbb{R}^k$

Here $x = (x_1, \ldots, x_k) \in \mathbb{R}^k$

- If $x = 0$, then we can ~~get~~ choose ~~what~~ any nonzero vector.

- If ~~some~~ $x_i = 0$ for some $i = 1, 2, \ldots, k$

but not all component zero $(x \neq 0)$

Then we can choose $y = (y_1, \ldots, y_k)$ as

$y_i = 1$ and other every component equal to 0. Then y_i is nonzero. Further

$x \cdot y = 0$

eg: $(x_1, x_2, \ldots, x_n, 0, x_{n+2}, \ldots x_k) \cdot (0, \ldots, 1, \ldots 0)$

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$= x_1 \cdot 0 + x_2 \cdot 0, \ldots, 0 \cdot 1, x_{n+2} \cdot 0 + \cdots + x_k 0$

$= 0$

- If all component of x are nonzero

$$x_i = 0 \quad \text{for all} \quad i = 1, 2, \ldots, k$$

We can choose $y = (-x_2, x_1, 0, \ldots, 0)$

Then $y \neq 0$.

$x \cdot y$

$= (x_1, x_2, \ldots, x_n) \cdot (-x_2, x_1, 0, \ldots, 0)$

$= (-x_1 x_2 + x_2 x_1 + 0 + \cdots + 0)$

$= 0$

By proposition 1.6-b), The given statement doesnot hold when $k = 1$

19. Suppose $\mathbf{a} \in R^k$, $\mathbf{b} \in R^k$. Find $\mathbf{c} \in R^k$ and $r > 0$ such that

$$|\mathbf{x} - \mathbf{a}| = 2|\mathbf{x} - \mathbf{b}|$$

if and only if $|\mathbf{x} - \mathbf{c}| = r$.
(*Solution:* $3\mathbf{c} = 4\mathbf{b} - \mathbf{a}$, $3r = 2|\mathbf{b} - \mathbf{a}|$.)

Suppose that $a, b \in \mathbb{R}^k$. We are going to use ~~that~~ the given answers.

First observe that

~~Forwa~~

$$x \cdot y = \sum_{i=1}^{k} x_i y_i = \sum_{i=1}^{k} y_i x_i = y \cdot x$$

(Since $x_i y_i$ is real number)

$$|x-a| = 2|x-b| \iff |x-a|^2 = 4|x-b|^2$$

$$\iff (x-a)\cdot(x-a) = 4(x-b)\cdot(x-b)$$

$$\iff x\cdot x - 2a\cdot x + a\cdot a = 4(x\cdot x - 2b\cdot x + b\cdot b)$$

$$\iff |x|^2 - 2(a\cdot x) + |a|^2 = 4|x|^2 - 8(b\cdot x) + 4|b|^2$$

$$\iff 3|x|^2 - 8(b\cdot x) + 2(a\cdot x) - |a|^2 + 4|b|^2 = 0$$

$$\iff |x|^2 - \frac{8}{3}(b\cdot x) + \frac{2}{3}(a\cdot x) - \frac{1}{3}|a|^2 + \frac{4}{3}|b|^2 = 0$$

$$\iff |x|^2 - \frac{8}{3}(b\cdot x) + \frac{2}{3}(a\cdot x) + \frac{16}{9}|b|^2 + \frac{1}{9}|a|^2 - \frac{8}{9}(b\cdot a) + \left(\frac{8}{9}(b\cdot a) - \frac{4}{9}|b|^2 - \frac{4}{9}|a|^2\right) = 0$$

$$\iff \left|x - \frac{4}{3}b + \frac{1}{3}a\right|^2 - \frac{4}{9}|b-a|^2 = 0$$

$$\iff \left|x - \frac{4}{3}b + \frac{1}{3}a\right|^2 = \frac{2}{3}|b-a|^2$$

$$\Leftrightarrow \quad |x-c|^2 = 4|x-b|^2$$

$$\Leftrightarrow \quad |x-c| = 2|x-b|$$

20) Redo It