8. Prove that no order can be defined in the complex field that turns it into an ordered field. *Hint:* -1 is a square.

Assume that $\mathbb{C}$ is one orderd feild.

~~Then (by prop 1.18) Since~~ $i \neq 0$. We consider i

Then by definition 1.5, one of them is true.

$$i<0 \quad , \quad i=0 \quad , \quad i>0.$$

We know that i is non-zero. So we consider two cases $i<0$ and $i>0$

Case-I $i<0$

Then, $-i>0$ (by Prop 1.18)

$$(-i)(-i) = (i)(i) = i^2 = -1 > 0$$

(by def^n 1.17-ii)

This is contradiction.

Case-II $i>0$. Then

$(i)(i) = i^2 = -1 > 0$ (by def^n 1.17-ii)

This is also contradiction.

Therefore, there is no order relation in $\mathbb{C}$

9. Suppose $z = a + bi$, $w = c + di$. Define $z < w$ if $a < c$, and also if $a = c$ but $b < d$. Prove that this turns the set of all complex numbers into an ordered set. (This type of order relation is called a *dictionary order*, or *lexicographic order*, for obvious reasons.) Does this ordered set have the least-upper-bound property?

Let $z = a+bi$ and $w = c+di$

where $a, b, c, d \in \mathbb{R}$. Since $\mathbb{R}$ is anorderd set.

$(a<c$ or $a=c$ or ~~$b<a$~~ $a>c)$

and $(b<d$ or $b=d$ or $b>d)$

Case-I If $a<c$, then $z<w$

Case-II If $a>c$ then $z>w$

Case-III If $a=c$ and $b>d$ then $z>w$

Case-IV If $a=c$ and $b<d$ then $z<w$

Case-V If $a=c$ and $b=d$ then $z=w$

Therefore one of them always true. ie:

either $z<w$ or $z=w$ or $z>w$

Now let's check second condion of orderd set

Let $u = e+fi$.

~~Suppose that $z>w$ and $y>0$~~

Suppose that $z<w$ and $w<u$

$z<w \Rightarrow (a<c)$ or $(a=c$ and $b<d)$

$w<u \Rightarrow (c<e)$ or $(c=e$ and $d<f)$

$w < u \Rightarrow (c < e)$ or $(c = e$ and $d < f)$

Case-I If $a < c$ and $c < e$ then $a < e$

Thus, $z < u$

Case-II If $a < c$ and $c = e$ and $d < f$ then $a < e$. This implies $z < u$.

Case-III If $a = c$ and $b < d$ and $c < e$ then $a < e$ then $z < u$

Case-IV If $a = c$ and $b < d$ and $c = e$ and $d < f$. Then $a = e$ and $b < f$.

This implies $z < u$.

This 4 cases implies ~~$z < w$ and $w < u$~~

If $z < w$ and $w < u$ then $z < w$.

10. Suppose $z = a + bi$, $w = u + iv$, and

$$a = \left(\frac{|w| + u}{2}\right)^{1/2}, \qquad b = \left(\frac{|w| - u}{2}\right)^{1/2}.$$

Prove that $z^2 = w$ if $v \geq 0$ and that $(\bar{z})^2 = w$ if $v \leq 0$. Conclude that every complex number (with one exception!) has two complex square roots.

Suppose that $z = a+bi$ and $w = u+iv$ and ~~where~~ $a = \left(\frac{|w|+u}{2}\right)^{1/2}$, $b = \left(\frac{|w|-u}{2}\right)^{1/2}$

Suppose that $v \geq 0$.

Need to show: $z^2 = w$

$$z^2 = (a+bi)^2 = (a+bi)(a+bi)$$
$$= a^2 + bi + bi + 2i^2 ab$$
$$= (a^2 - b^2) + 2iab$$

$$a^2 - b^2 = \left(\frac{|w|+u}{2}\right) + \left(\frac{|w|-u}{2}\right) = u$$

$$2ab = 2\left(\frac{|w|+u}{2}\right)^{1/2}\left(\frac{|w|-u}{2}\right)^{1/2} = 2\left(\left(\frac{|w|+u}{2}\right)\left(\frac{|w|-u}{2}\right)\right)^{1/2}$$
$$= 2\left(\frac{|w|^2 - u^2}{4}\right)^{1/2} = 2\left(\frac{v^2}{4}\right)^{1/2} = v$$

(Since $v \geq 0$)

Thus,

$$z^2 = (a^2 - b^2) + (2ab)i$$
$$= u + vi = w$$

NTS: If $\gamma \leq 0$ then $(\overline{z})^2 = w$

Suppose that $\gamma \leq 0$.

$$\overline{z}^2 = \overline{z}\,\overline{z} = (a-bi)(a-bi)$$
$$= a - bi - bi + i^2 b$$
$$= (a^2 - b^2) - (2ab)i$$

We already show that $(a^2-b^2) = u$ and

$$2ab = 2\left(\frac{\gamma^2}{2}\right)^{1/2}.$$

Since $\gamma \leq 0$, $2ab = 2\left(\frac{\gamma^2}{2}\right)^{1/2} = -\gamma$

Therefore, $(\overline{z})^2 = (a^2-b^2) - (2ab)i$

$$= u - (-\gamma)i$$
$$= u + \gamma i = w$$

11. If z is a complex number, prove that there exists an $r \geq 0$ and a complex number w with $|w| = 1$ such that $z = rw$. Are w and r always uniquely determined by z?

Suppose that $z \in \mathbb{C}$.

- If $z = 0$, choose $r = 0$ and $w = \frac{1}{\sqrt{2}} + \left(\frac{1}{\sqrt{2}}\right)i$

Then $|w| = \sqrt{\left(\frac{1}{\sqrt{2}}\right)^2 + \left(\frac{1}{\sqrt{2}}\right)^2} = \sqrt{\frac{1}{2} + \frac{1}{2}} = 1$

- If $z \neq 0$, choose $r = |z|$ and $w = \frac{z}{|z|}$. Then $r = |z| \geq 0$ and

$$|w| = \left|\frac{z}{|z|}\right| = \frac{|z|}{||z||} = \frac{|z|}{|z|} = 1$$

(by thm 1.33-c)

(by thm 1.33d)

(This idea is get from the unit vectors)

Therefore, $z \in \mathbb{C} \Rightarrow (\exists (r \geq 0$ and $w \in \mathbb{C}$ with $|w| = 1)$ such that $z = rw$

- In firs ~~c~~case, uniqunes does not holds. Because if $z=0$, then we can choose $r=0$ and any $w \in \mathbb{C}$ with $|w|=1$.

- In second case, uniquness holds. If $z \neq 0$ then we can chose

Assume that $z = r_1 w_1 = r_2 w_2$ (1) with $r_1 > 0$, $r_2 > 0$, $|w_1| = 1$, $|w_2| = 1$. Take modulus of both sides

$$|r_1 w_1| = |r_2 w_2|$$

$|r_1||w_1| = |r_2||w_2|$ (thm 1.33 and

$r_1 |w_1| = r_2 |w_2|$ (Since $r_i \in \mathbb{R}$)

$r_1 = r_2$ (2) (Since $|w_1| = |w_2| = 1$

Thus, $\cancel{r_1 w_1 = r_2 w_2}$ ~~By our~~ equation (1), (2) impiles $w_1 = w_2$

Therefore r, w are uniquely determined by z.

12. If $z_1, \ldots, z_n$ are complex, prove that

$$|z_1 + z_2 + \cdots + z_n| \leq |z_1| + |z_2| + \cdots + |z_n|.$$

(12) Suppose that $z_1, z_2, \ldots, z_n \in \mathbb{C}$

We are going to use mathematical inducti

~~$|z_1+z_2+\cdots+z_n| =$~~

If $n=2$, by thm 1.33 trinangle equlity holds. i.e:

$$|z_1+z_2| \leq |z_1|+|z_2|$$

Now assume that following holds for $n=p \in \mathbb{N}$

$$|z_1+z_2+\cdots+z_p| \leq |z_1|+\cdots+|z_p|$$

Now Consider following,

$$|z_1+z_2+\cdots+z_p+z_{p+1}| \leq |z_1+\cdots+z_p|+|z_{p+1}|$$

(by base case)

$$|z_1+\cdots+z_p+z_{p+1}| \leq |z_1|+\cdots+|z_p|+|z_{p+1}|$$

(Induction hypothesis)

Therefore, by mathematical hypothesis, ~~we get~~

$|z_1+z_2+\cdots+z_n| \leq |z_1|+\cdots+|z_n|$ for all $n \in \mathbb{N} \setminus \{0\}$

13. If x, y are complex, prove that

$$||x|-|y|| \le |x-y|.$$

Let $x, y \in \mathbb{C}$. By theorm 1.33

$$|z+w| \le |z| + |w| \quad \text{foo all } z, w \in \mathbb{C}$$

We use $z = (x-y)$ and $w = y$. Using triangle inequlity,

$$|(x-y)+y| \le |x-y| + |y|$$

$$|x| \le |x-y| + |y|$$

$$|x| - |y| \le |x-y| \text{ ——— } (*)$$

Now, we use $z = y-x$ and $w = x$. By triangle inequlity,

$$|(y-x)+x| \le |y-x| + |x|$$

$$|y| \le |y-x| + |x|$$

$$|y| - |x| \le |y-x|$$

We know that, $|x-y| = |y-x|$. Thus,

$$|y| - |x| \le |x-y| \text{ ——— } (**)$$

By $(*)$ and $(**)$, $\quad ||x|-|y|| \le |x-y|$

∎

14. If z is a complex number such that $|z| = 1$, that is, such that $z\bar{z} = 1$, compute

$$|1 + z|^2 + |1 - z|^2.$$

Suppose that $z \in \mathbb{C}$ such that $|z| = 1$.

~~P~~ By definit 1.32,

$$\begin{aligned}
|1+z|^2 &= (1+z)\overline{(1+z)} && (\text{def}^n\ 1.32)\\
&= (1+z)(\bar{1}+\bar{z}) && (\text{By thm } 1.31\text{-a})\\
&= (1+z)(1+\bar{z}) && (\because 1 \in \mathbb{R})\\
&= 1+\bar{z}+z+z\bar{z} && (\text{distributi property})\\
&= 1+\bar{z}+z+|z| && (\text{def}^n\ 1.32)\\
&= 1+\bar{z}+z+1 && (\text{by hypothesis})\\
&= 2+\bar{z}+z \quad \text{——— } (1)
\end{aligned}$$

Similarly,

$$\begin{aligned}
|1-z|^2 &= (1-z)(1-\bar{z}) && (\text{def}^n\ 1.32 \text{ and thm } 1.31)\\
&= 1-\bar{z}-z+z\bar{z} && (\text{distributive})\\
&= 1-\bar{z}-\bar{z}+|z| && (\text{def}^n\ 1.32)\\
&= 1-\bar{z}-\bar{z}+1 && (\text{Hypothesis})\\
&= 2-\bar{z}+\bar{z} \quad \text{——— } (2)
\end{aligned}$$

By (1) and (2),

$$|1+z|^2+|1-z|^2 = 4$$

15. **Under what conditions does equality hold in the Schwarz inequality?**

Recall the Schwarz inequality

1.35 Theorem *If $a_1, \ldots, a_n$ and $b_1, \ldots, b_n$ are complex numbers, then*

$$\left|\sum_{j=1}^{n} a_j \bar{b}_j\right|^2 \leq \sum_{j=1}^{n} |a_j|^2 \sum_{j=1}^{n} |b_j|^2.$$

Proof Put $A = \Sigma|a_j|^2$, $B = \Sigma|b_j|^2$, $C = \Sigma a_j \bar{b}_j$ (in all sums in this proof, j runs over the values $1, \ldots, n$). If $B = 0$, then $b_1 = \cdots = b_n = 0$, and the conclusion is trivial. Assume therefore that $B > 0$. By Theorem 1.31 we have

$$\begin{aligned}\sum |Ba_j - Cb_j|^2 &= \sum (Ba_j - Cb_j)(B\bar{a}_j - \overline{Cb_j}) \\ &= B^2 \sum |a_j|^2 - B\bar{C} \sum a_j \bar{b}_j - BC \sum \bar{a}_j b_j + |C|^2 \sum |b_j|^2 \\ &= B^2 A - B|C|^2 \\ &= B(AB - |C|^2).\end{aligned}$$

Since each term in the first sum is nonnegative, we see that

$$B(AB - |C|^2) \geq 0.$$

Since $B > 0$, it follows that $AB - |C|^2 \geq 0$. This is the desired inequality.

If we go through the proof, we can see that, If $A=0$ or $B=0$ implies ~~that~~ the equilitiy. (This case is trivial)

~~Now~~ So, now assume that $A \neq 0$ and $B \neq 0$.

By the prooof thm 1.35

$$\cancel{(AB - |C|^2) = 0}$$

(equlity of Scharatz inequtity) $\Longleftrightarrow \sum |Ba_j - Cb_j|^2 = 0$

$\Longleftrightarrow |Ba_j - Cb_j| = 0$ for all $i = 1, 2, \dots, n$

$\Longleftrightarrow Ba_j = Cb_j$ for all $i = 1, 2 \dots n$

$\Longleftrightarrow a_j = \frac{C}{B} b_j$ for all $i = 1, 2 \dots n$

$\Longleftrightarrow$ the numbers a_j are propostional to b_j

$\Longleftrightarrow$ a_j and b_j are linearly dependent.

16. Suppose $k \geq 3$, $\mathbf{x}, \mathbf{y} \in R^k$, $|\mathbf{x} - \mathbf{y}| = d > 0$, and $r > 0$. Pr

(*a*) If $2r > d$, there are infinitely many $\mathbf{z} \in R^k$ such that

$$|\mathbf{z} - \mathbf{x}| = |\mathbf{z} - \mathbf{y}| = r.$$

(*b*) If $2r = d$, there is exactly one such $\mathbf{z}$.

(*c*) If $2r < d$, there is no such $\mathbf{z}$.

How must these statements be modified if k is 2 or 1?

17. **Prove that**

$$|\mathbf{x}+\mathbf{y}|^2+|\mathbf{x}-\mathbf{y}|^2=2|\mathbf{x}|^2+2|\mathbf{y}|^2$$

if $\mathbf{x}\in R^k$ and $\mathbf{y}\in R^k$. Interpret this geometrically, as a statement about parallelograms.

Suppose that $x, y \in \mathbb{R}^k$. Here

$x=(x_1,\dots,x_k)$ and $y=(y_1,\dots,y_k)$

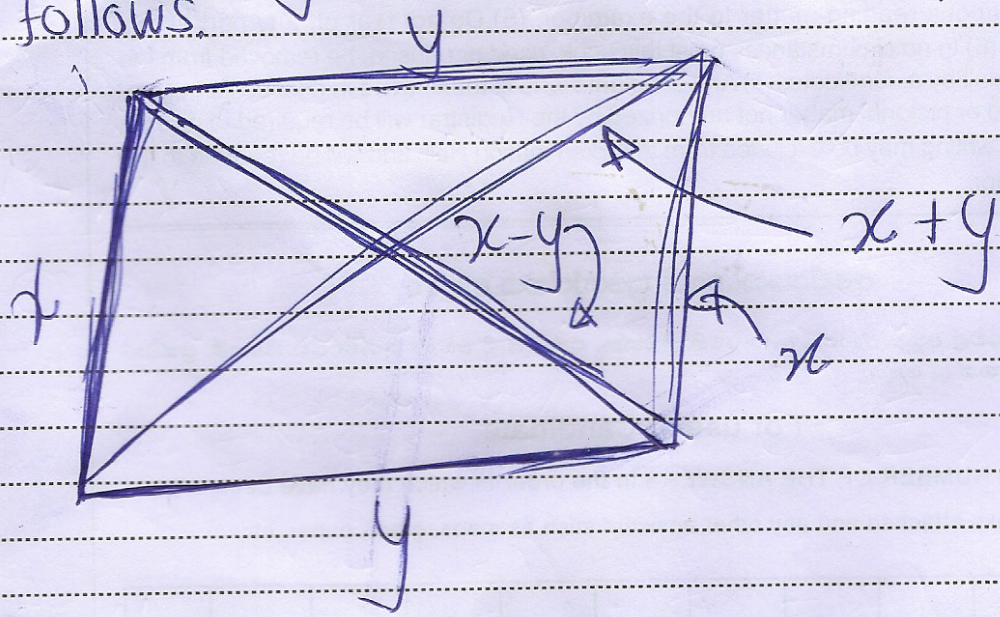
$$|x+y|^2=\big((x+y)\cdot(x+y)\big)$$
$$= x\cdot x+2(x\cdot y)+y\cdot y$$
$$= |x|^2+2(x\cdot y)+|y|^2 \quad \text{--- (1)}$$

$$|x-y|^2=(x-y)\cdot(x-y)$$
$$= x\cdot x-2(x\cdot y)+y\cdot y$$
$$= |x|^2-2(x\cdot y)+|y|^2 \quad \text{--- (2)}$$

By (1) and (2)

$$|x+y|^2+|x-y|^2=|x|^2+2(x\cdot y)+|y|^2$$
$$+|x|^2-2(x\cdot y)+|y|^2$$
$$=2|x|^2+2|y|^2$$

If x and y are sides of parallelgram as follows.



(Similar to A/L vectors)

Then $x-y$ and $x+y$ are diangongols of the parallelgram. Therefore we can conclude that,

The sum of square of diagonals of parallelgram is equal to sum of square of the sides.

18. If $k \geq 2$ and $\mathbf{x} \in R^k$, prove that there exists $\mathbf{y} \in R^k$ such that $\mathbf{y} \neq \mathbf{0}$ but $\mathbf{x} \cdot \mathbf{y} = 0$. Is this also true if $k = 1$?

Suppose that $k \geq 2$ and $x \in \mathbb{R}^k$

Here $x = (x_1, \dots, x_k) \in \mathbb{R}^k$

- If $x = 0$, then we can ~~get~~ choose ~~what~~ any nonzero vector.

- If ~~some~~ $x_i = 0$ forsome $i = 1, 2, \dots, k$

but not all component zero ($x \neq 0$)

Then we can choose $y = (y_1, \dots, y_k)$ as $y_i = 1$ and other every component equal to 0. Then y_i is nonzero. Further $x \cdot y = 0$

eg: $(x_1, x_2, \dots, \underset{n}{x_{n}}, 0, \underset{n+2}{x}, \dots x_k) \cdot (0, \dots, 1, \dots 0)$

nth Componet

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$= x_1 \cdot 0 + x_2 \cdot 0, \dots, 0 \cdot 1, x_{n+2} \cdot 0 + \dots + x_k$

$= 0$

- If all component of x are nonzero

$$x_i = 0 \quad \text{for all} \quad i = 1, 2, \ldots, k$$

We can choose $y = (-x_2, x_1, 0, \ldots, 0)$
Then $y \neq 0$.

$$x \cdot y$$
$$= (x_1, x_2, \ldots, x_n) \cdot (-x_2, x_1, 0, \ldots, 0)$$
$$= (-x_1 x_2 + x_2 x_1 + 0 + \cdots + 0)$$
$$= 0$$

By proposition 1.6-b.), The given statement doesnot hold when $k=1$

19. Suppose $\mathbf{a} \in R^k$, $\mathbf{b} \in R^k$. Find $\mathbf{c} \in R^k$ and $r > 0$ such that

$$|\mathbf{x} - \mathbf{a}| = 2|\mathbf{x} - \mathbf{b}|$$

if and only if $|\mathbf{x} - \mathbf{c}| = r$.
(*Solution:* $3\mathbf{c} = 4\mathbf{b} - \mathbf{a}$, $3r = 2|\mathbf{b} - \mathbf{a}|$.)

Suppose that $a, b \in \mathbb{R}^k$. We are going to use ~~that~~ the given answers.

First observe that

~~Forwa~~

$$x \cdot y = \sum_{i=1}^{k} x_i y_i = \sum_{i=1}^{k} y_i x_i = y \cdot x$$

(Since x_i, y_i is real number)

$$|x-a| = 2|x-b| \iff |x-a|^2 = 4|x-b|^2$$

$$\iff (x-a)\cdot(x-a) = 4(x-b)\cdot(x-b)$$

$$\iff x\cdot x - 2a\cdot x + a\cdot a = 4(x\cdot x - 2b\cdot x + b\cdot b)$$

$$\iff |x|^2 - 2(a\cdot x) + |a|^2 = 4|x|^2 - 8(b\cdot x) + 4|b|^2$$

$$\iff 3|x|^2 - 8(b\cdot x) + 2(a\cdot x) - |a|^2 + 4|b|^2 = 0$$

$$\iff |x|^2 - \frac{8}{3}(b\cdot x) + \frac{2}{3}(a\cdot x) - \frac{1}{3}|a|^2 + \frac{4}{3}|b|^2 = 0$$

$$\iff |x|^2 - \frac{8}{3}(b\cdot x) + \frac{2}{3}(a\cdot x) + \frac{16}{9}|b|^2 + \frac{1}{9}|a|^2 - \frac{8}{9}(b\cdot a) \left(+ \frac{8}{9}(b\cdot a) - \frac{4}{9}|b|^2 - \frac{4}{9}|a|^2 \right) = 0$$

$$\iff \left|x - \frac{4}{3}b + \frac{1}{3}a\right|^2 - \frac{4}{9}|b-a|^2 = 0$$

$$\iff \left|x - \frac{4}{3}b + \frac{1}{3}a\right|^2 = \frac{2}{3}|b-a|^2$$

$$\Leftrightarrow \quad |x-c|^2 = 4|x-b|^2$$

$$\Leftrightarrow \quad |x-c| = 2|x-b|$$

20) Redo It