

Conformal Mapping

Ashan J

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Chapter 1

The Convergence Of Sequences of Analytic And Harmonic Functions

1.1 The convergence of sequences of analytic functions

Various branches of complex variable function theory, especially the geometric function theory, utilize the fundamental characteristics of converging sequences of analytic functions in their proofs. These characteristics allow for proofs that are relatively straightforward and refined compared to corresponding proofs in real analysis.

Let us give some definitions.

Definition 1.1 (Point wise convergent). Let $\{f_n(z)\}$, for $n = 1, 2, \dots$, denote a sequence of single-valued functions defined on a set E of points in the z -plane. This sequence is said to **converge at a point** $z_0 \in E$ if the sequence of numbers $f_n(z_0)$ converges.

Definition 1.2. A sequence of such functions $f_n(z)$ is said to **converge on** E if it converges at every point of E .

In such a case, we may speak of the limit function $f(z) = \lim_{n \rightarrow \infty} f_n(z)$ defined on E .

Definition 1.3. The sequence $f_n(z)$ is said to **converge uniformly on** E to a function $f(z)$, which is finite on E , if,

$$\forall \epsilon > 0 \exists N > 0, \forall z \in E \text{ such that } n > N \implies |f_n(z) - f(z)| < \epsilon$$

Definition 1.4. On the other hand, if $f(z) = \infty$ on E , the sequence $f_n(z)$ is said, by definition, to **converge uniformly on E to ∞** if,

$$\forall M > 0 \exists N > 0, \forall z \in E \text{ such that } n > N \implies |f_n(z)| > M$$

Definition 1.5. The sequence $\{f_n(z)\}$ is a **uniform Cauchy sequence in E** , if,

$$\forall \epsilon > 0 \exists N > 0 \text{ such that } \forall z \in D, m, n > N \implies |f_n(z) - f_m(z)| < \epsilon$$

Proposition 1.1. Suppose that $E \subseteq \mathbb{C}$ is a region. A sequence of complex valued functions $\{f_n(z)\}$ converges uniformly in E if and only if it is a uniform Cauchy sequence in E .

Proof. easy

Later I will do it. □

If the functions $f_n(z)$ are defined on a domain B , we shall need, besides the concept of uniform convergence of a sequence in the domain B , the concept of uniform convergence of a sequence in the interior of the domain B .

Definition 1.6. Let functions $f_n(z)$ be defined on a domain B . Then $f_n(z)$ is called **uniform convergent in the interior of the domain B** if there is uniform convergence of $|f_n(z)|$ on every closed set $E \subseteq B$.

Remark. Uniform convergence in the interior of B is a weaker requirement than uniform convergence in B .

Definition 1.7. A single-valued function $f(z)$, defined and finite on a set E not including ∞ , is said to be **continuous on E** if,

$$\forall z_0 \in E \forall \epsilon > 0, \exists \delta > 0 \text{ such that } z \in E \text{ and } |z - z_0| < \delta \implies |f(z) - f(z_0)| < \epsilon$$

Theorem 1.1. Statement: If a sequence $f_n(z)$ of functions that are continuous on a set E converges uniformly to a finite function $f(z)$ defined on E , this function $f(z)$ is also continuous on E .

Proof. To see this, let $z_0 \in E$. Then, since $f_n(z)$ is uniformly continuous, for given $\epsilon > 0$, there exists a number n such that, for all $z \in E$, we have $|f_n(z) - f(z)| < \frac{\epsilon}{3}$.

1.1. THE CONVERGENCE OF SEQUENCES OF ANALYTIC FUNCTIONS 7

Furthermore, since $f_n(z)$ is continuous, there exists a number $\delta > 0$ such that, for all $z \in E$ satisfying the inequality $|z - z_0| < \delta$, we have $|f_n(z) - f_n(z_0)| < \frac{\epsilon}{3}$ (because of the continuity of $f_n(z)$ on E). Therefore, for $z \in E$ and $|z - z_0| < \delta$, we have

$$|f(z) - f(z_0)| \leq |f(z) - f_n(z)| + |f_n(z) - f_n(z_0)| + |f_n(z_0) - f(z_0)| < \epsilon,$$

which means that $f(z)$ is continuous at the point $z_0 \in E$. It then follows that, if the functions $f_n(z)$ are continuous in the domain B and if a sequence of them converges in the interior of B to a finite function $f(z)$, then $f(z)$ is continuous in B . \square

Chapter 2

UNIVALENT MAPPING OF MULTIPLY CONNECTED DOMAINS

2.1 Univalent conformal mapping of a doubly connected domain onto an annulus

2.1.1 page 206

Let us now begin with the simplest case of the problem posed, namely, the case of doubly connected domains. Let us show that every doubly connected domain can be mapped univalently onto some circular annulus, whose boundary circles may degenerate to points.

Let B denote a doubly connected domain in the z -plane.

If it has an isolated boundary point z_0 , then, by adjoining it to the domain B , we obtain a simply connected domain that we can then map univalently either onto the disk $I(I < I)$ or onto the $(-)$ -plane with the point $(= \infty)$ excluded. We can do this in such a way that the point $z = z_0$ is mapped into $(= 0)$. The domain B is thus univalently mapped either onto the annulus $0 < |z| < I$ or onto the annulus $0 < |z| < \infty$.

Suppose now that the boundary of the domain B consists of two continua K_1 and K_2 . One of these, let us say K_1 is necessarily bounded. The complement of K_1 in the z -plane is an open set consisting of two disjoint domains. One of these domains, let us say B_1 contains the domain B . The domain B_1 is simply connected. Therefore, it can be mapped conformally onto the disk $|z'| < I$.

Under this mapping, the continuum K_2 is mapped into a continuum K contained in $|z'| < I$, and the domain B is mapped into a domain B' .

Let us now map whichever of the simply connected domains complementary to K contains B' onto the domain $|z''| > I$ in such a way that $z' = \infty$ is mapped into $z'' = \infty$. Under this mapping, the circle $|z'| = I$ is mapped into an analytic Jordan curve contained in the domain $|z''| > I$ and the domain B' is mapped into a doubly connected domain B'' that does not include ∞ and that is bounded by this curve and by the circle $|z''| = I$.

The composite of these two mappings constitutes a univalent mapping of the domain B onto the domain B'' . Furthermore, on the basis of §3 of Chapter II, we conclude that this mapping sets up a one-to-one correspondence between the prime ends of the domain B and the boundary points of the domain B'' . Here, just as in §3 of Chapter II, by the prime end of the domain B that corresponds to the point z on the boundary of the domain B'' , we mean the set of all cluster points of all sequences of points of the domain B that approach the boundary point z .

Chapter 3

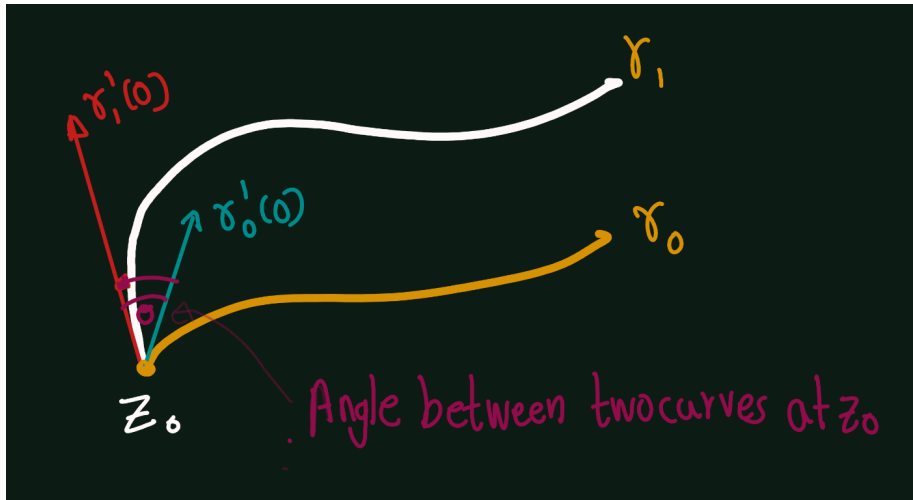
Conformal Mappings

Definition 3.1 (Tangent Vector). Let $\gamma(t) = x(t) + iy(t), 0 \leq t \leq 1$, be a smooth parameterized curve terminating at $z_0 = \gamma(0)$. We refer to

$$\gamma'(0) = \lim_{t \rightarrow 0} \frac{\gamma(t) - \gamma(0)}{t} = x'(0) + iy'(0)$$

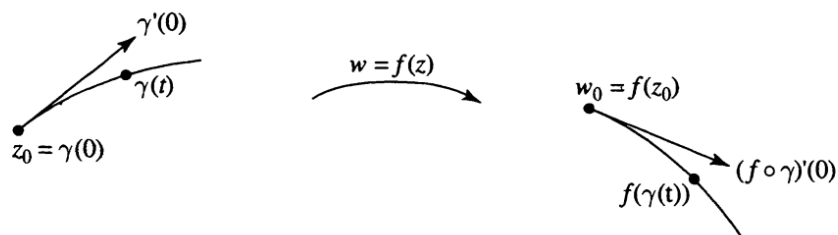
as the tangent vector to the curve γ at z_0

Definition 3.2 (Angle between two curves). We define the angle between two curves at z_0 to be the angle between their tangent vectors at z_0 .



Theorem 3.1. If $\gamma(t), 0 \leq t \leq 1$, is a smooth parameterized curve terminating at $z_0 = \gamma(0)$, and $f(z)$ is analytic at z_0 , then the tangent to the curve $f(\gamma(t))$ terminating at $f(z_0)$ is given by:

$$(f \circ \gamma)'(0) = f'(z_0)\gamma'(0) \quad (3.1)$$



Proof.

- If $\gamma'(0) \neq 0$, then $\gamma(t) \neq \gamma(0)$ for t near 0, $t \neq 0$, so we may write

$$\frac{f(\gamma(t)) - f(\gamma(0))}{t} = \frac{f(\gamma(t)) - f(\gamma(0))}{\gamma(t) - \gamma(0)} \cdot \frac{\gamma(t) - \gamma(0)}{t} \quad (3.2)$$

and pass to the limit, to obtain the formula (6.1).

- If $\gamma'(0) = 0$, then proceeding as in Section 2, we obtain $(f \circ \gamma)'(0) = 0$, and again the formula holds.

□

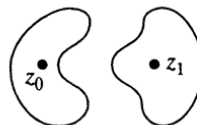
Chapter 4

Uniformization by square domains

Definition 4.1 (Domain). A subset D of the complex plane is a domain if D is open and if any two points of D can be connected by a broken line segment in D .



domain



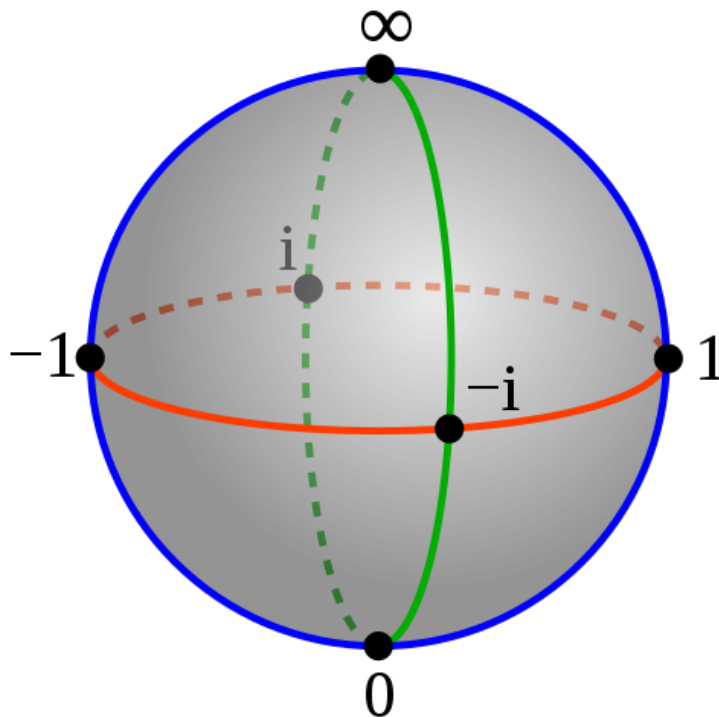
not a domain

Example 4.1.

- **Examples:**
 - Open half planes
 - Open disks
 - Open sectors
 - Open annuli,
 - Open punctured disks.
- **Non Examples:**
 - Union of the open upper and lower half-planes = $U = \mathbb{C} \setminus \mathbb{R}$.
(It is impossible to connect a point in the upper half-plane to a point in the lower half-plane by a broken line segment that does not cross the real line.)

Definition 4.2 (Riemann Sphere). The Riemann sphere, also called the extended complex plane consist of the complex numbers \mathbb{C} together with ∞ . The set of extended complex numbers may be written as $\mathbb{C} \cup \{\infty\}$.

Notation: $\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$



Definition 4.3. A domain in the plane is “simply connected” if it has no “holes.”

Example 4.2.

- **Example**
 - Disks
 - Rectangles
- ***Non Example**
 - Annuli
 - Punctured disks
 - Punctured plane
 - (Becasue they have “holes”)

Later we discuss this more precise

Definition 4.4 (Meromorphic). A function $f(z)$ is meromorphic on a domain D if $f(z)$ is analytic on D except possibly at isolated singularities, each of which is a pole.

Proof. (Proof of Theorem 1.1 in the paper)

Let Ω be a finitely connected domain in $\hat{\mathbb{C}}$ with $\infty \in \Omega$. It is a known fact that there exists a conformal map g of Ω onto a square domain $\tilde{\Omega}$ with the normalization

$$g(z) = z + \frac{b_1}{z} + \dots$$

near ∞ .

The map $f \in \mathcal{F} \mapsto \tilde{f} := f \circ g^{-1}$ is a bijection between \mathcal{F} and $\tilde{\mathcal{F}}$. Moreover, if

$$f(z) = z + \frac{a_1}{z} + \dots \quad \text{and} \quad \tilde{f}(z) = z + \frac{\tilde{a}_1}{z} + \dots$$

near ∞ , then $\tilde{a}_1 = a_1 - b_1$. □

Let f be as in the statement. We consider the rectangle $R = [-l, l] \times [-r, r] \subset \mathbb{C}$ for large $r > 0$. Here we chose $l = r^{2/3}$ so that

$$\frac{l}{r} \rightarrow 0 \quad \text{and} \quad \frac{r}{l^2} \rightarrow 0$$

as $r \rightarrow \infty$.

In the following, we assume that r is so large that $\tilde{C} \setminus \Omega$ is contained in the interior of R . Then $\partial R \subset \tilde{C} \setminus \Omega$ and $j = f(R)$ is a Jordan curve in \mathbb{C} . We want to

$$A = \frac{1}{2i} \int_J \bar{w} dw \tag{4.1}$$

$$= \frac{1}{2i} \int_{\partial R} f(\bar{z}) f'(z) dz \tag{4.2}$$

$$= \frac{1}{2i} \int_{\partial R} \overline{\left(z + \frac{a_1}{z} + \dots\right)} \left(1 - \frac{a_1}{z^2} + \dots\right) dz \tag{4.3}$$

$$= \frac{1}{2i} \int_{\partial R} \left(\bar{z} + \frac{\bar{a}_1}{\bar{z}} - \frac{\bar{a}_1 \bar{z}}{z^2} + O\left(\frac{1}{|z|^2}\right)\right) dz \tag{4.4}$$

$$= 4rl + \int_{\partial R} \operatorname{Im}\left(\frac{\bar{a}_1 z}{\bar{z}}\right) \frac{dz}{z} + o(1). \tag{4.5}$$

Sure, I can provide a proof for the proposition. Here it goes:

Proposition: If a set A is a subset of the interior of another set B (denoted as $A \subseteq \operatorname{Int}(B)$), then the complement of A (denoted as A^c) is a subset of the boundary of B (denoted as $\operatorname{bd}(B)$).

Proof:

Let's denote the interior of B as $Int(B)$ and the boundary of B as $bd(B)$. By definition, we have:

1. $Int(B) = B - bd(B)$
2. $A^c = U - A$ where U is the universal set.

Given that $A \subseteq Int(B)$, we can say that A does not contain any points from $bd(B)$. Therefore, all points in $bd(B)$ must be in A^c .

Hence, $A^c \subseteq bd(B)$.

This completes the proof. Please note that this is a general proof and the specifics might vary depending on the exact definitions and properties of the sets and the topological space they are in. If you have a specific example or further questions, feel free to ask!

Chapter 5

Riemann Mapping Theorem

Theorem 5.1 (Riemann Mapping Theorem). *If D is a simply connected domain in the complex plane, and D is not the entire complex plane, then there is a conformal map of D onto the open unit disk \mathbb{D} .*

5.1 Hyperbolic Geomeery

Suppose $w = f(z)$ is a conformal self-map of the open unit disk \mathbb{D} . From Pick's lemma we then have equality,

$$\left| \frac{dw}{dz} \right| = \frac{1 - |\omega|^2}{1 - |z|^2}$$

In differential form this becomes

$$\frac{|dw|}{1 - |w|^2} = \frac{|dz|}{1 - |z|^2},$$

which means that if γ is any smooth curve in \mathbb{D} , and $\omega = f(z)$ is a conformal self-map of \mathbb{D} , then

$$\int_{f \circ \gamma} \frac{|dw|}{1 - |w|^2} = \int_{\gamma} \frac{|dz|}{1 - |z|^2}.$$

Thus to obtain a length function that is invariant under conformal self-maps of \mathbb{D} , we are led to make the following definition.

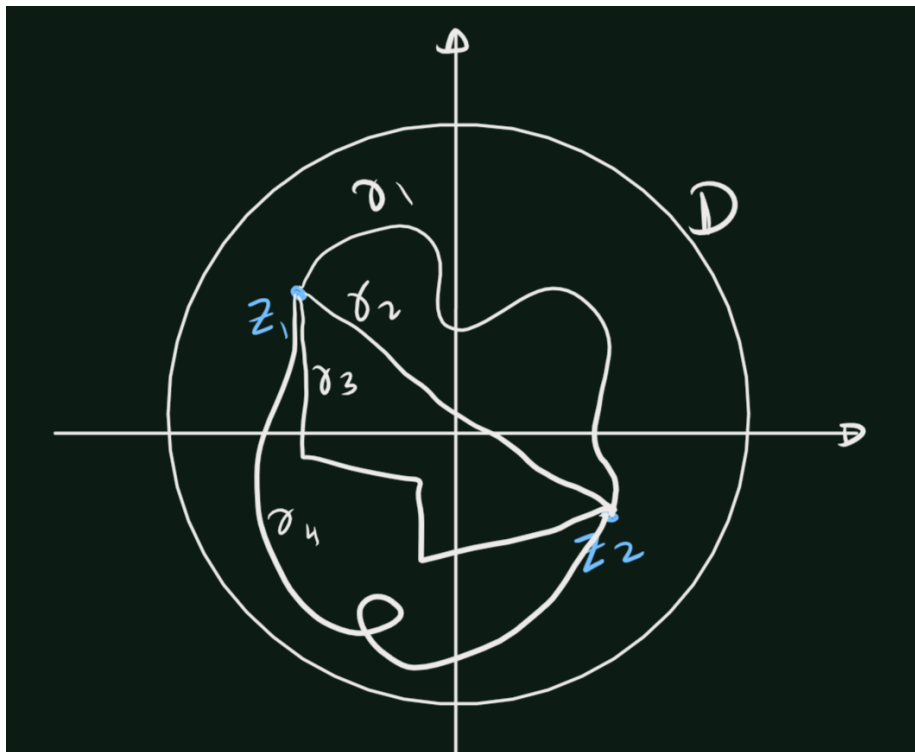
Definition 5.1. We define the length of γ in the hyperbolic metric by

$$\text{hyperbolic length of } \gamma = 2 \int_{\gamma} \frac{|dz|}{1 - |z|^2}$$

The factor 2 is a harmless factor, which is often omitted. (It adjusts the metric so that its curvature is -1.)

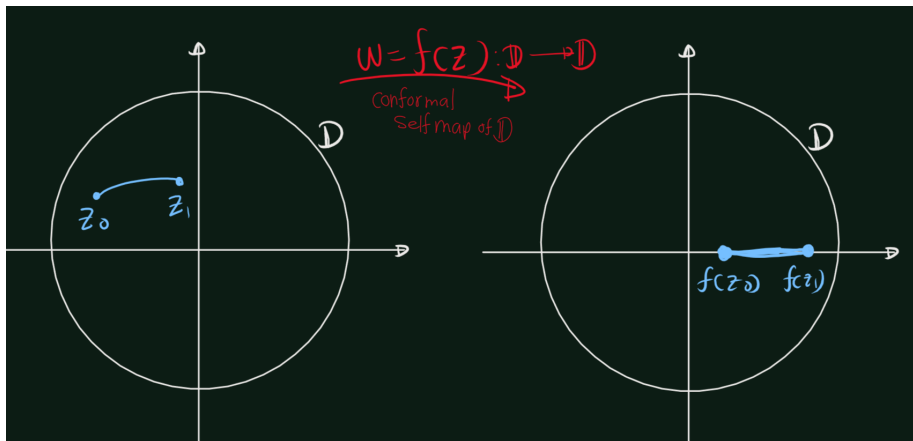
Definition 5.2. The hyperbolic distance $\rho(z_0, z_1)$ from z_0 to z_1 is defined as the infimum (greatest lower bound) of the hyperbolic lengths of all piecewise smooth curves in \mathbb{D} from z_0 to z_1 .

$$\rho(z_0, z_1) = \inf_{\gamma} \{\text{hyperbolic length of } \gamma\} = \inf_{\gamma} \left\{ 2 \int_{\gamma} \frac{|dz|}{1 - |z|^2} \right\}$$



Since conformal self-maps of \mathbb{D} preserve the hyperbolic lengths of curves, they also preserve hyperbolic distances. That is, for any conformal self-map $w = J(z)$ of \mathbb{D} ,

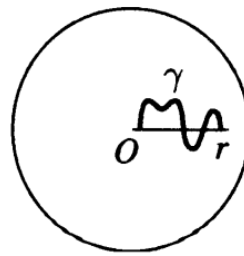
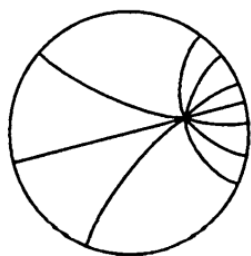
$$\rho(f(z_0), f(z_1)) = \rho(z_0, z_1), \quad \text{where } z_0, z_1 \in \mathbb{D}.$$



Theorem 5.2. For any two distinct points z_0, z_1 in the open unit disk \mathbb{D} , there exists a unique shortest curve in \mathbb{D} from z_0 to z_1 in the hyperbolic metric. Specifically, this curve corresponds to the arc of the circle passing through z_0 and z_1 that is orthogonal to the unit circle.

Definition 5.3. The paths of shortest hyperbolic length between points are called **hyperbolic geodesics**.

- These hyperbolic geodesics play a role similar to that of straight lines in Euclidean geometry. They satisfy all the axioms of Euclidean geometry except for the parallel axiom (which states that through each point not on a given line, there passes a unique straight line through the point and parallel to the given line).



hyperbolic geodesics

Proof. (proof of theorem), Let $w = f(z)$ be a conformal self-map of \mathcal{D} such that $f(z_0) = 0$. By multiplying by a unimodular constant, we can arrange that $f(z_1) = r > 0$. Since $f(z)$ preserves hyperbolic lengths, and since $f(z)$ maps circles orthogonal to the unit circle onto circles orthogonal to the unit circle, it

suffices to show that the straight line segment from 0 to r is a unique path of shortest hyperbolic length from 0 to r . For this, let $\gamma(t) = x(t) + iy(t)$, $0 \leq t \leq 1$, be a piecewise smooth path in \mathbb{C} from 0 to r . Then $\alpha(t) = \Re(\gamma(t)) = x(t)$ defines a path in \mathbb{D} from 0 to r along the real axis, and

$$\int_{\alpha} \frac{|dz|}{1-|z|^2} = \int_0^1 \frac{|dx(t)|}{1-(x(t))^2} \leq \int_0^1 \frac{|dx(t)|}{1-(\gamma(t))^2} \leq \int_{\gamma} \frac{|dz|}{1-|z|^2}$$

If $y(t) \neq 0$ for some t , then $|\gamma(t)| > |x(t)|$, and the first inequality above is strict. In this case, the path $\alpha(t)$ on the real axis is strictly shorter than the path $\gamma(t)$. Further, if $\alpha(t)$ is decreasing on some interval, we could reduce the integral by deleting a parameter interval over which $\alpha(t)$ starts and ends at the same value. We conclude that the integral is a minimum exactly when $\gamma(t)$ is real and nondecreasing, in which case the path is the straight line segment from 0 to r . \square

5.2 Proof of Riemann Mapping Theorem

Recall the following theorem.

Theorem 5.3. *The following properties are equivalent, for a domain D in the complex plane:*

- i. D is simply connected,
- ii. every closed differential on D is exact,
- iii. for each $z_0 \in \mathbb{C} \setminus D$, there is an analytic branch of $\log(z - z_0)$ defined on D ,
- iv. each closed curve γ in D has winding number $W(\gamma, z_0) = 0$ about all points $z_0 \in \mathbb{C} \setminus D$,

(v) the complement of D in the extended complex plane $\hat{\mathbb{C}}$ is connected.

Suppose that D is simply connected and that $D \neq \mathbb{C}$. Choose $a \in \mathbb{C} \setminus D$. By the characterization of simple connectivity, By theorem 5.3, there is an analytic branch $g(z)$ of $\log(z - a)$ in D . Then,

$$h(z) = e^{g(z)/2} = e^{\left(\frac{\log(z-a)}{2}\right)} = (z-a)^{\frac{1}{2}} = \sqrt{z-a}$$

So, $h(z) = e^{g(z)/2}$ is an analytic branch of $\sqrt{z-a}$ in D , and $(h(z))^2 = z-a \neq 0$ (Since $a \notin D$) in D . If $h(z_1) = h(z_2)$, then

$$z_1 = (h(z_1))^2 + a = (h(z_2))^2 + a = z_2$$

. Thus $h(z)$ is univalent, and $h(z)$ maps D conformally onto $h(D)$. Finally, note that if $w_0 \in h(D)$, then $-w_0 \notin h(D)$. Indeed, if $w_0 = h(z_0)$ and $-w_0 = h(z_1)$ for $z_0, z_1 \in D$, then $z_0 = h(z_0)^2 + a = w_0^2 + a = h(z_1)^2 + a = z_1$, which is impossible. We summarize.

Lemma 5.1. *Let D be a simply connected domain. Suppose $a \notin D$, and let $h(z)$ be an analytic branch of $\sqrt{z-a}$ in D . Then $h(z)$ is univalent on D , and further, $h(D)$ is disjoint from $-h(D)$.*

Proof. Done it earlier. □

5.2.1 Adlof Proof

Theorem 5.4 (Riemann Mapping Theorem). *Given any simply connected region Ω which is not the whole plane, and a point z_0 in Ω , there exists a unique analytic function $f(z)$ in Ω , normalized by the conditions $f(z_0) = 0$, $f'(z_0) > 0$, such that $f(z)$ defines a one-to-one mapping of Ω onto the disk $|w| < 1$.*

Proof.

- **Uniqueness:**

□