Do Carmo - Differential Geometry of Curves and Surfaces

Ashan J

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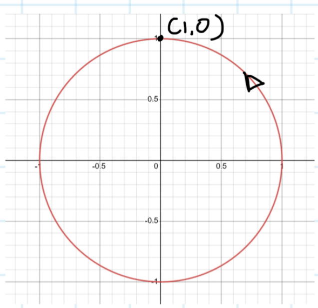
Chapter 1

Curves

1.1 Introduction

1.2 Parametrized Curves

1. Find a parametrized curve $\alpha(t)$ whose trace is the circle $x^2 + y^2 = 1$ such that $\alpha(t)$ runs clockwise around the circle with $\alpha(0) = (0, 1)$.



General version (what we know since All

$\alpha : [0, 2\pi] \longrightarrow \mathbb{R}^2$

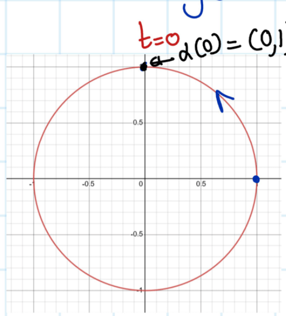
$t \longmapsto (\cos(t), \sin(t))$

$t = 0 \longrightarrow (1, 0)$

$\cancel{t} = \pi/2 \longrightarrow$

$t =$

Now Let's go back to the Problem



$\alpha : [0, 2\pi) \longrightarrow \mathbb{R}^2$

$\alpha(t) = (\cos(-\theta), \sin(-\theta))$

$\alpha(0) = \cos(0), \sin(0)$

$= (1, 0)$

This wrong, we have

2. Let $\alpha(t)$ be a parametrized curve which does not pass through the origin. If $\alpha(t_0)$ is the point of the trace of α closest to the origin and $\alpha'(t_0) \neq 0$, show that the position vector $\alpha(t_0)$ is orthogonal to $\alpha'(t_0)$.

$\boxed{\text{NTS}: \ \alpha(t_0)\cdot\alpha'(t_0) = 0}$

Let $f(t) = |\alpha(t)|$ is the distance from origin and $\alpha(t)$ for all $t \in I$. If $t_0 \in I$ is point such that $\alpha(t_0)$ is the point of trace of $\alpha(t)$ closest to origin. Thus $f(t)$ has minimum at t_0. Thus $f'(t_0) = 0$. So now consider

$$f(t) = |\alpha(t)|^2 = (\alpha(t)\cdot\alpha(t))$$
$$f'(t) = \alpha'(t)\cdot\alpha(t) + \alpha(t)\cdot\alpha'(t)$$
$$= 2\,\alpha(t)\cdot\alpha'(t)$$

Thu $\quad f'(t_0) = 2\,\alpha(t_0)\alpha'(t_0).$
$$0 = \alpha(t_0)\cdot\alpha'(t_0)$$

Note that $\alpha(t_0) \neq 0$ and $\alpha'(t_0) \neq 0$.

given that does not pass through origin

Given that. property of t_0

Thus $\alpha(t_0)$ is orthogonal to $\alpha'(t_0)$.

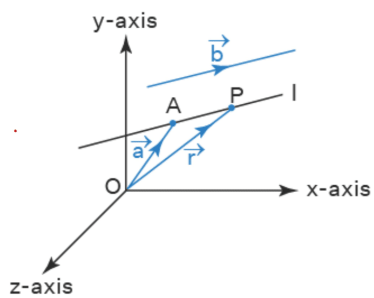
3. A parametrized curve $\alpha(t)$ has the property that its second derivative $\alpha''(t)$ is identically zero. What can be said about α?

$\alpha''(t) = 0 \implies \alpha'(t) = \underline{a}$; $\underline{a}$ is constant vector,

$\implies \alpha(t) = \underline{a}\,t + \underline{b}$, $\underline{b}$ is constant vector

Therefore $\alpha(t) = \underline{a}\,t + \underline{b}$ respent a stright line.

Vector Equations Of A Line



$\overline{OA} = \vec{a}$

$\overline{OP} = \vec{r}$

$\vec{b} \parallel l$

$\boxed{\vec{r} = \vec{a} + \lambda\,\vec{b}}$

4. Let $\alpha : I \longrightarrow R^3$ be a parametrized curve and let $v \in R^3$ be a fixed vector. Assume that $\alpha'(t)$ is orthogonal to v for all $t \in I$ and that $\alpha(0)$ is also orthogonal to v. Prove that $\alpha(t)$ is orthogonal to v for all $t \in I$.

Since, $\alpha'(t)$ is orthogonal to γ for all $t \in I$

$$\alpha'(t) \cdot \gamma = 0 \quad \forall t \in I \text{ — } (1)$$

Since, γ is fixed vector, it does not depend on γ. $\frac{d\gamma}{dt} = 0$ ——— (2)

Now consider;

$$\frac{d(\alpha(t) \cdot \gamma)}{dt} = \alpha'(t) \cdot \gamma + \alpha(t) \cdot \frac{d\gamma}{dt}$$

$$(\alpha(t) \cdot \gamma)' = \alpha'(t) \cdot \gamma + 0 \text{ (by (2))}$$

$$= 0 \text{ (by (1))}$$

Thus, $(\alpha(t) \cdot \gamma)' = 0$ for all $t \in I$.

Therefore $(\alpha(t) \cdot \gamma)$ is a constant for all $t \in I$.

Given that $\alpha(0)$ is orgnal to γ.
Thus $\alpha(0) \cdot \gamma = 0$.
Therefore, $\alpha(t) \cdot \gamma = 0 \quad \forall t \in I$.
Thus, $\alpha(t)$ is perpenducular to γ, for all $t \in I$

5. Let $\alpha : I \longrightarrow R^3$ be a parametrized curve, with $\alpha'(t) \neq 0$ for all $t \in I$. Show that $|\alpha(t)|$ is a nonzero constant if and only if $\alpha(t)$ is orthogonal to $\alpha'(t)$ for all $t \in I$.

"$\Rightarrow$" Suppose that $0 \neq |\alpha(t)| = k$, k is constant

Now consider following

$$(\alpha(t)\cdot\alpha(t))' = \alpha'(t)\cdot\alpha(t) + \alpha(t)\cdot\alpha'(t)$$

$$\frac{d|\alpha(t)|^2}{dt} = (|\alpha(t)|^2)' = 2\alpha(t)\cdot\alpha'(t)$$

$$\frac{dk^2}{dt} = (k^2)' = 2\alpha(t)\cdot\alpha'(t)$$

$$0 = 2\alpha(t)\cdot\alpha'(t)$$

$$0 = \alpha(t)\cdot\alpha'(t)$$

Since $|\alpha(t)| \neq 0$.

Given that $\alpha'(t) \neq 0$

Note that $\alpha(t) \neq 0$ and $\alpha'(t) \neq 0$.

Therefore $\alpha(t)$ is orgnal to $\alpha'(t)$ for all $t \in I$.

$\Leftarrow$ Suppose that $\alpha(t)$ is orthgonal to $\alpha'(t)$ for all $t \in I$

Then $\alpha(t) \cdot \alpha'(t) = 0$ for all $t \in I$.

Similar to above, we can show that $|\alpha(t)|$ is constant.

<u>NTS: $|\alpha(t)| \neq 0$ forall $t \in I$</u>

Assume the contray that $|\alpha(t_0)| = 0$ for some $t_0 \in I$. Then $\alpha(t_0) = 0$ forsome $t_0 \in I$.

Thus, $\alpha'(t_0) = 0$ forsome $t_0 \in I$

It contradict the fact $\alpha'(t) \neq 0$ forall $t \in I$.

Therefore $|\alpha(t)|$ is non-zero constant.