Linear Algebra

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Chapter 1

Vector space

1.1 $\mathbb{R}^n$ and $\mathbb{C}^n$

We are already be familiar with basic properties of the set of real numbers($\mathbb{R}$). Complex numbers comes when we can take square roots of negative numbers. The idea is to assume we have a square root of -1, denoted by i that obeys the usual rules of arithmetic. Here are the formal defnition.

Definition 1.1 (Complex Numbers)**.**

- A complex number is an ordered pair (x, y), where $x, y \in \mathbb{R}$, but we will write this as $x + yi$.
- The set of all complex numbers is denoted by $\mathbb{C}$:

$$\mathbb{C} = \{x + yi \,|\, x, y \in \mathbb{R}\}.$$

- Addition and multiplication on $\mathbb{C}$ are defined by

$$(x + yi) + (u + vi) = (x + u) + (y + v)i,$$

$$(x + yi)(u + vi) = (xu - yv) + (xv + yu)i;$$

  here $x, y, u, v \in \mathbb{R}$

Fun Fact: The symbol i was first used to denote $\sqrt{-1}$ by Leonard Euler in 1777.

- Note that $\mathbb{C} \supseteq \mathbb{R}$ because for all real numbers a $\mathbb{R}$, we can express it as a complex number by writing it as $a + 0i$.
- We usually denote $0 + yi$ simply as yi, and $0 + 1i$ as i.

- The definition of multiplication for complex numbers is based on the assumption that $i^2 = -1$. Using the standard arithmetic rules, we can derive the formula for the product of two complex numbers. This formula can then be used to confirm that i^2 indeed equals -1.

Example 1.1. Let's calculate the product of two complex numbers $(1+2i)$ and $(3+4i)$ using the distributive and commutative properties:

$$\begin{aligned}(1+2i)(3+4i) &= 1\cdot(3+4i) + (2i)(3+4i)\\ &= 1\cdot 3 + 1\cdot 4i + 2i\cdot 3 + (2i)(4i)\\ &= 3 + 4i + 6i - 8\\ &= -5 + 10i\end{aligned}$$

Proposition 1.1 (Properties of Complex Arithmetic)**.**

- ***Commutativity:*** *$z_1 + z_2 = z_2 + z_1$ and $z_1 z_2 = z_2 z_1$ for all $z_1, z_2 \in \mathbb{C}$.*
- ***Associativity:*** *$(z_1 + z_2) + z_3 = z_1 + (z_2 + z_3)$ and $(z_1 z_2) z_3 = z_1(z_2 z_3)$ for all $z_1, z_2, z_3 \in \mathbb{C}$.*
- ***Identities:*** *$z + 0 = z$ and $z1 = z$ for all $z \in \mathbb{C}$.*
- ***Additive Inverse:*** *For every $z \in \mathbb{C}$, there exists a unique $-z \in \mathbb{C}$ such that $z + (-z) = 0$.*
- ***Multiplicative Inverse:*** *For every $z \in \mathbb{C}$ with $z \neq 0$, there exists a unique $z^{-1} \in \mathbb{C}$ such that $zz^{-1} = 1$.*
- ***Distributive Property:*** *$z_1(z_2 + z_3) = z_1 z_2 + z_1 z_3$ for all $z_1, z_2, z_3 \in \mathbb{C}$.*

The properties above are proved using the familiar properties of real numbers and the definitions of complex addition and multiplication. Here, we are going to prove that commutativity of complex addition and multiplication is proved.

Proof.

- *Addition* : Let $z_1 = a + bi$ and $z_2 = c + di$ be any two complex numbers. Then we have:

  $$z_1 + z_2 = (a + bi) + (c + di) = (a + c) + (b + d)i = z_2 + z_1$$

  This shows that addition is commutative for complex numbers.

- *Multiplication*: Again, let $z_1 = a + bi$ and $z_2 = c + di$ be any two complex numbers. Then we have:

  $$z_1 z_2 = (a + bi)(c + di) = ac + adi + bci - bd = (ac - bd) + (ad + bc)i$$