Linear Algebra

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Contents

Chapter 1

Vector space

1.1 R^n and C^n

$\mathbb{R}^n$ and $\mathbb{C}^n$

We are already be familiar with basic properties of the set of real numbers($\mathbb{R}$).

Some nonconstant polynomials with real coefficients have no real zeros.

Example: the equation

$$x^2 + 1 = 0$$

has no real solutions. Thus we invent a solution, called i, with the property that $i^2 = -i$.

Futhur, complex numbers comes when we can take square roots of negative numbers. The idea is to assume we have a square root of -1, denoted by i that obeys the usual rules of arithmetic. Here are the formal definition.

Definition 1.1 (Complex Numbers).

- A complex number is an ordered pair (x, y), where $x, y \in \mathbb{R}$, but we will write this as $x + yi$.
- The set of all complex numbers is denoted by $\mathbb{C}$:

$$\mathbb{C} = \{x + yi \,|\, x, y \in \mathbb{R}\}.$$

- Addition and multiplication on $\mathbb{C}$ are defined by

$$(x + yi) + (u + vi) = (x + u) + (y + v)i,$$

$$(x + yi)(u + vi) = (xu - yv) + (xv + yu)i;$$

  here $x, y, u, v \in \mathbb{R}$

Fun Fact: The symbol i was first used to denote $\sqrt{-1}$ by Leonard Euler in 1777.

- Note that $\mathbb{C} \supseteq \mathbb{R}$ because for all real numbers a $\mathbb{R}$, we can express it as a complex number by writing it as $a + 0i$.
- We usually denote $0 + yi$ simply as yi, and $0 + 1i$ as i.
- The definition of multiplication for complex numbers is based on the assumption that $i^2 = -1$. Using the standard arithmetic rules, we can derive the formula for the product of two complex numbers. This formula can then be used to confirm that i^2 indeed equals -1.

Example 1.1. Let's calculate the product of two complex numbers $(1+2i)$ and $(3 + 4i)$ using the distributive and commutative properties:

$$\begin{aligned}(1 + 2i)(3 + 4i) &= 1 \cdot (3 + 4i) + (2i)(3 + 4i)\\ &= 1 \cdot 3 + 1 \cdot 4i + 2i \cdot 3 + (2i)(4i)\\ &= 3 + 4i + 6i - 8\\ &= -5 + 10i\end{aligned}$$

Proposition 1.1 (Properties of Complex Arithmetic)**.**

- ***Commutativity****:* $z_1 + z_2 = z_2 + z_1$ *and* $z_1 z_2 = z_2 z_1$ *for all* $z_1, z_2 \in \mathbb{C}$*.*
- ***Associativity****:* $(z_1 + z_2) + z_3 = z_1 + (z_2 + z_3)$ *and* $(z_1 z_2)z_3 = z_1(z_2 z_3)$ *for all* $z_1, z_2, z_3 \in \mathbb{C}$*.*
- ***Identities****:* $z + 0 = z$ *and* $z1 = z$ *for all* $z \in \mathbb{C}$*.*
- ***Additive Inverse****: For every* $z \in \mathbb{C}$*, there exists a unique* $-z \in \mathbb{C}$ *such that* $z + (-z) = 0$*.*
- ***Multiplicative Inverse****: For every* $z \in \mathbb{C}$ *with* $z \neq 0$*, there exists a unique* $z^{-1} \in \mathbb{C}$ *such that* $zz^{-1} = 1$*.*
- ***Distributive Property****:* $z_1(z_2 + z_3) = z_1 z_2 + z_1 z_3$ *for all* $z_1, z_2, z_3 \in \mathbb{C}$*.*

The properties above are proved using the familiar properties of real numbers and the definitions of complex addition and multiplication. Here, we are going to prove that commutativity of complex addition and multiplication is proved.

Proof.

- *Addition* : Let $z_1 = a + bi$ and $z_2 = c + di$ be any two complex numbers. Then we have:

  $$z_1 + z_2 = (a + bi) + (c + di) = (a + c) + (b + d)i = z_2 + z_1$$

  This shows that addition is commutative for complex numbers.

- *Multiplication*: Again, let $z_1 = a + bi$ and $z_2 = c + di$ be any two complex numbers. Then we have:

  $$z_1 z_2 = (a + bi)(c + di) = ac + adi + bci - bd = (ac - bd) + (ad + bc)i$$

  and

  $$z_2 z_1 = (c + di)(a + bi) = ca + cbi + dai - db = (ca - db) + (da + cb)i$$

As you can see, $(ac - bd) + (ad + bc)i = (ca - db) + (da + cb)i$, which shows that multiplication is also commutative for complex numbers.

So, we have proved that both addition and multiplication are commutative operations in the set of complex numbers. This means that the order in which complex numbers are added or multiplied does not affect the result.

$\square$

Definition 1.2 (Subtraction,Division)**.** Let's suppose $z_1, z_2 \in \mathbb{C}$.

- The negative of a complex number z_1 is denoted as $-z_1$. It is the unique complex number such that

  $$z_1 + (-z_1) = 0.$$

- Subtraction in the set of complex numbers is defined as

  $$z_1 - z_2 = z_1 + (-z_2).$$

- For $z_1 \neq 0$, let $1/z_1$ denote the multiplicative inverse of z_1. Thus, $1/z_1$ is the unique complex number such that

  $$z_1(1/z_1) = 1.$$

- For $z_1 \neq 0$, division by z_1 is defined as

  $$z_2/z_1 = z_2(1/z_1).$$

So that we can conveniently make definitions and prove theorems that apply to both real and complex numbers, we adopt the following notation.

Notation :Throughout this note stands $\mathbb{F}$ for either $\mathbb{R}$ or $\mathbb{F}$. The letter $\mathbb{F}$ is used because $\mathbb{R}$ and $\mathbb{C}$ are examples of what are called fields.

Elements of $\mathbb{F}$ are called scalars. The word "scalar" (which is just a fancy word for "number") is often used when we want to emphasize that an object is a number, as opposed to a vector (vectors will be defined soon).

For $\alpha \in \mathbb{F}$ and m a positive integer, we define α^m to denote the product of α with itself m times:

$$\alpha^m = \underbrace{\alpha \cdot \alpha \cdots \alpha}_{m \text{times}}$$

. This definition implies that

$$(\alpha^m)^n = \alpha^{mn} \quad \text{and} \quad (\alpha\beta)^m = \alpha^m \beta^m$$

for all $\alpha\beta \in \mathbb{F}$ and all positive integers m, n.

1.1.1 Lists

Before defining $\mathbb{R}^n$ and $\mathbb{C}^n$, we look at two important examples.

Example 1.2.

- The set $\mathbb{R}^2$, which you can think of as a plane, is the set of all ordered pairs of real numbers:

$$\mathbb{R}^2 := \{(x, y) : x, y \in \mathbb{R}\}$$

- The set $\mathbb{R}^3$, which you can think of as ordinary space, is the set of all ordered triples of real numbers:

$$\mathbb{R}^3 := \{(x, y, z) : x, y, z \in \mathbb{R}\}$$

Let's generalize this examples to higher dimensions.

Definition 1.3 (List,Length)**.**

- Suppose n is a nonnegative integer. A *list* of *length* n is an ordered collection of n elements (which might be numbers, other lists, or more abstract objects).
- Two lists are equal if and only if they have the same length and the same elements in the same order

Note that many mathematicians call a list of length n an n-tuple.

Lists are often written as elements separated by commas and surrounded by parentheses. Thus a list of length two is an ordered pair that might be written as (a, b). A list of length three is an ordered triple that might be written as (x, y, z) A list of length n might look like this: $(z_1, ..., z_n)$

Sometimes we will use the word list without specifying its length. Remember, however, that by definition each list has a finite length that is a non-negative integer. Thus an object that looks like $(x_1, x_2, ...)$, which might be said to have infinite length, is not a list.

A list of length 0 looks like this: $(\ )$.

We consider such an object to be a list so that some of our theorems will not have trivial exceptions.

Lists differ from sets in two ways: in lists, order matters and repetitions have meaning; in sets, order and repetitions are irrelevant.

Proposition 1.2 (Lists versus Sets)**.**

- *The lists* $(3, 5)$ *and* $(5, 3)$ *are not equal, but the sets* $\{3, 5\}$ *and* $\{5, 3\}$ *are equal.*
- *The lists* $(4, 4)$ *and* $(4, 4, 4)$ *are not equal (they do not have the same length), although the sets* $\{4, 4\}$ *and* $\{4, 4, 4\}$ *both equal the set* $\{4\}$*.*

1.1.2 F^n

$\mathbb{F}^n$ Fix a positive integer $\mathbb{F}^n$ for the rest of this chapter.

Definition 1.4. $\mathbb{F}^n$ is the set of all lists of length n of elements of $\mathcal{F}$:

$$\mathbb{F}^n = \{(x_1, \dots, x_n) \mid x_i \in \mathbb{F} \text{ for } i = 1, \dots, n\}.$$

For $(x_1, \dots, x_n) \in \mathbb{F}^n$ and $i \in \{1, \dots, n\}$, we say that x_i is the ith coordinate of $(x_1, \dots, x_n)$.

If $\mathbb{F} = \mathbb{R}$ and n equals 2 or 3, then the definition above of $\mathbb{F}^n$ agrees with our previous notions of $\mathbb{R}^2$ and $\mathbb{R}^3$.

Example 1.3. Let $\mathbb{C}^4$ be the set of all lists of four complex numbers:

$$\mathbb{C}^4 = \{(z_1, z_2, z_3, z_4) \mid z_1, z_2, z_3, z_4 \in \mathbb{C}\}.$$

If $n \geq 4$, we cannot visualize $\mathbb{R}^n$ as a physical object. Similarly, $\mathbb{C}^1$ can be thought of as a plane, but for $n \geq 2$, the human brain cannot provide a full image of $\mathbb{R}^n$. However, even if n is large, we can perform algebraic manipulations in $\mathbb{C}^n$ as easily as in $\mathbb{R}^2$ or $\mathbb{R}^3$. For example, addition in $\mathbb{R}^n$ is defined as follows.

Definition 1.5 (addition of higer dimesionans)**.** Addition in $\mathbb{F}^n$ is defined by adding corresponding coordinates:

$$(a_1, ..., a_n) + (b_1, ..., b_n) = (a_1 + b_1, ..., a_n + b_n)$$

Often the mathematics of $\mathbb{R}^n$ becomes cleaner if we use a single letter to denote a list of n numbers, without explicitly writing the coordinates. For example, the next result is stated with x and y in $\mathbb{R}^n$ even though the proof requires the more cumbersome notation of $(x_1, \dots, x_n)$ and $(y_1, \dots, y_n)$.

Proposition 1.3. *If* $a, b \in \mathbb{F}^n$, *then* $a + b = b + a$.

Proof. Suppose $a = (a_1, ..., a_n) \in \mathbb{F}^n$ and $b = (b_1, ..., b_n) \in \mathbb{F}^n$. Then

$$\begin{aligned} a + b &= (a_1, ..., a_n) + (b_1, ..., b_n) && (1.1) \\ &= (a_1 + b_1, ..., a_n + b_n) && (1.2) \\ &= (b_1 + a_1, ..., b_n + a_n) && (1.3) \\ &= (b_1, ..., b_n) + (a_1, ..., a_n) && (1.4) \\ &= b + a && (1.5) \end{aligned}$$

where the second and fourth equalities above hold because of the definition of addition in $\mathbb{F}^n$ and the third equality holds because of the usual commutativity of addition in $\mathbb{F}$. ☐

If a single letter is used to denote an element of F^n, then the same letter with appropriate subscripts is often used when coordinates must be displayed.
For example, if $x \in F^n$, then letting

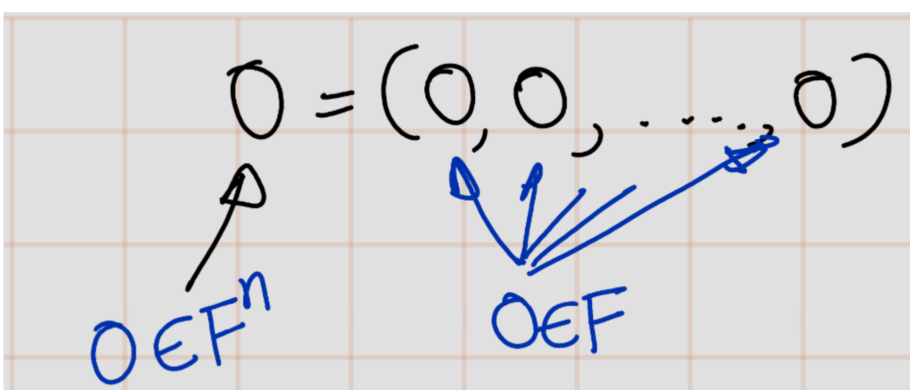
$$x = (x_1, \dots, x_n)$$

is good notation, as shown in the proof above. Even better, work with just x and avoid explicit coordinates when possible.

Notation: 0
Let 0 denote the list of length n whose coordinates are all 0:

$$0 = (0, \dots, 0)$$

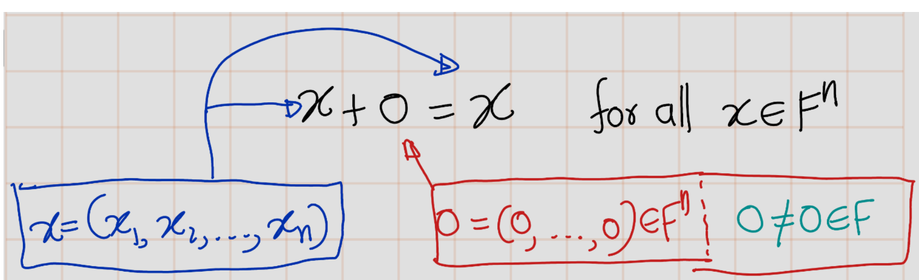
Here we are using the symbol 0 in two different ways—on the left side of the equation above, the symbol 0 denotes a list of length n, which is an element of 0, whereas on the right side, each 0 denotes a number. This potentially confusing practice actually causes no problems because the context should always make clear which 0 is intended.



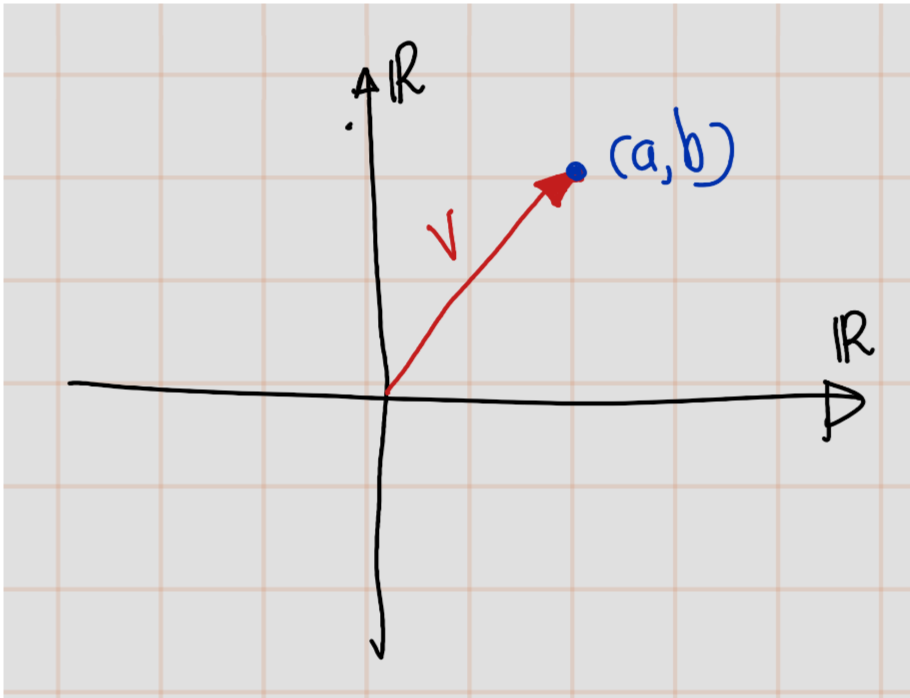
Example 1.4 (Context determines which 0 is intended)**.** Consider the statement that 0 is an additive identity for F^n:

$$x + 0 = x \quad \text{for all } x \in F^n.$$

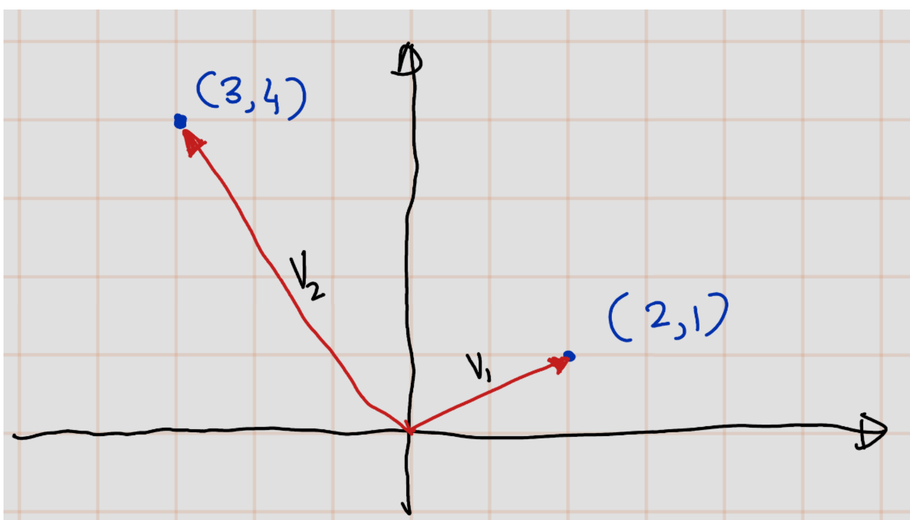
Here, the 0 above refers to the list defined earlier, not the number 0, because we have not defined the sum of an element of F^n (namely, x) and the number 0.



A picture can aid our intuition. We will draw pictures in $\mathbb{R}^2$ because we can sketch this space on two-dimensional surfaces such as paper and computer screens. A typical element of $\mathbb{R}^2$ is a point $\mathbf{v} = (a, b)$. Sometimes we think of $\mathbf{v}$ not as a point but as an arrow starting at the origin and ending at (a, b), as shown here. When we think of an element of $\mathbb{R}^2$ as an arrow, we refer to it as a vector.



Example 1.5. $v_1 := (2,1), v_2 := (3,4) \in \mathbb{R}^2$ can be present as follows,



When we think of vectors in $\mathbb{R}^2$ as arrows, we can move an arrow parallel to itself (without changing its length or direction) and still consider it the same vector. With this viewpoint, you'll often gain a better understanding by dispensing with the coordinate axes and explicit coordinates, simply thinking of the vector itself, as shown in the figure here. The two arrows depicted have the same length and direction, so we regard them as the same vector.

Whenever we use pictures in $\mathbb{R}^2$ or use the somewhat vague language of points and vectors, remember that these are just aids to our understanding, not substitutes for the actual mathematics that we will develop. Although we cannot draw good pictures in high-dimensional spaces, the elements of these spaces are as rigorously defined as elements of $\mathbb{R}^2$.

Example 1.6. $(2, -3, \pi, \ldots, \sqrt{2})$ is an element of $\mathbb{R}^5$, and we may casually refer to it as a point in $\mathbb{R}^5$ or a vector in $\mathbb{R}^5$ without worrying about whether the geometry of $\mathbb{R}^5$ has any physical meaning.

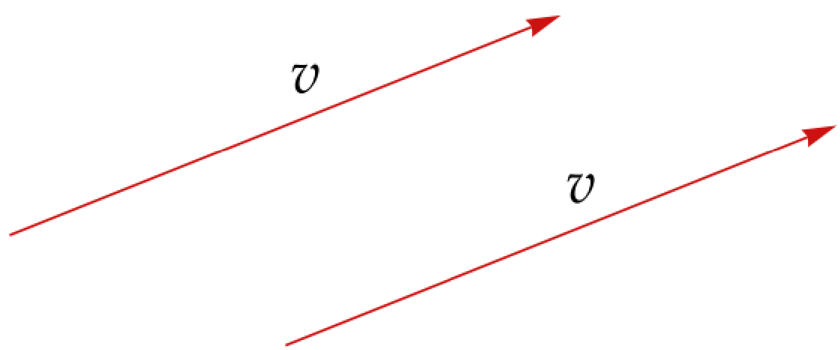


Figure 1.1: A vector

Fun fact: Mathematical models of the economy can have thousands of variables, say (x_1, ..., x_{5000} which means that we must work in $\mathbb{R}^{5000}$. Such a space cannot be dealt with geometrically. However, the algebraic approach works well. Thus our subject is called linear algebra.

Definition 1.6 (Summation of two vectors)**.** We have two vectors, **a** and **b**, in $\mathbb{R}^2$. We want to add them such that the initial point of **b** coincides with the end point of **a**. The sum $\mathbf{a} + \mathbf{b}$ equals the vector whose initial point is the same as that of **a** and whose end point is the same as that of **b**, as shown here (red one is a+b).

Definition 1.7 (The additive inverse in Fn,-v)**.** A vector **v** in $\mathbb{F}^n$, its additive inverse, denoted by $-\mathbf{v}$, is the vector $-\mathbf{v}$ in $\mathbb{F}^n$ such that:

$$\mathbf{v} + (-\mathbf{v}) = \mathbf{0}$$

Therefore, if $\mathbf{v} = (v_1, \ldots, v_n)$, then $-\mathbf{v} = (-v_1, \ldots, -v_n)$.

Given a vector $\mathbf{x}$ in $\mathbb{R}^2$, its additive inverse, denoted by $-\mathbf{x}$, is the vector in $\mathbb{R}^2$ with the same length but pointing in the opposite direction. The figure illustrates this way of thinking about the additive inverse:

$$\mathbf{x} + (-\mathbf{x}) = \mathbf{0}$$

As you can see, the vector labelled $-\mathbf{x}$ has the same length as the vector labelled $\mathbf{x}$ but points in the opposite direction.

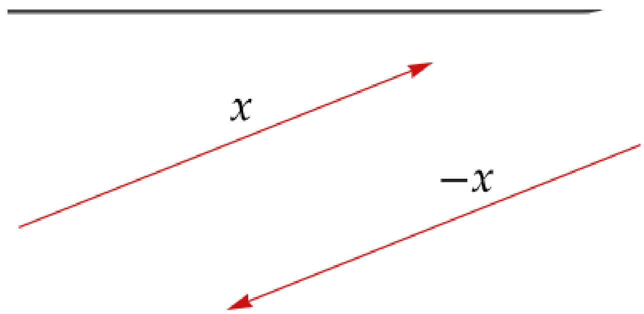


Figure 1.2: A vector and its additive inverse.

We've dealt with addition in $\mathbb{F}^n$, and now we'll explore multiplication. While we could define a coordinate-wise multiplication in $\mathbb{F}^n$, experience shows that it's not useful for our purposes. Instead, we focus on scalar multiplication, which will be central to our subject.

Specifically, we need to define what it means to multiply an element of $\mathbb{F}^n$ by a scalar from $\mathbb{F}$.

1.2 Definition of Vector space

The motivation for vector space definition

The motivation for the definition of a vector space comes from properties of addition and scalar multiplication in F^n:

- Addition is commutative, associative, and has an identity.
- Every element has an additive inverse.
- Scalar multiplication is associative.
- Scalar multiplication by 1 acts as expected.
- Addition and scalar multiplication are connected by distributive properties.

We will define a vector space to be a set V with an addition and a scalar multiplication on V that satisfy the properties in the paragraph above.

Definition 1.8 (addition, scalar multiplication)**.**

- An addition on set V is a function that assigns an element $u+v \in V \quad u, v, \in V$.
- A scalar multiplication on a set $V$is a function that assigns an element $\lambda v \in V$ to each $\lambda \in F$ and each $v \in V$

Example 1.7. Suppose V is the set of real-valued functions on the interval $[0, l]$. For $f, g \in V$ and $\lambda \in R$, define $f + g$ and λf by

$$[f + g](x) = f(x) + g(x)$$

and

$$[\lambda f](x) + = \lambda f(x)$$

. Thus $f + g \in V, \lambda f \in V$.

Now we are ready to give the formal definition of a vector space

Definition 1.9 (Vector Space)**.** A vector space is a set V along with an addition on V and a scalar multiplication on V such that the following properties hold.

- **Commutativity**: $u + v = v + u$ for all $u, v \in V$.
- **Associativity**: $(u + v) + w = u + (v + w)$ and $(ab)c = a(bc)$ for all $u, v, w \in V$ and for all $a, b \in \mathbb{F}$.
- **Additive Identity**: There exists an element $0 \in V$ such that $v + 0 = v$ for all $v \in V$.
- **Additive Inverse**: For every $v \in V$, there exists $-v \in V$ such that $v + (-v) = 0$.
- **Multiplicative Identity**: $1v = v$ for all $v \in V$.
- **Distributive Properties**: $a(v + w) = av + aw$ and $(a + b)v = av + bv$ for all $v, w \in V$ and all $a, b \in \mathbb{F}$.

The following geometric language sometimes aids our intuition.

Definition 1.10 (Vector)**.** Elements of a vector space are called vectors or points.

Example 1.8. The simplest vector space is $\{0\}$, which contains only one point.

The scalar multiplication in a vector space depends on F. Thus when we need to be precise, we will say that V is a vector space over F instead of saying simply that V is a vector space.

Example 1.9.

- $\mathbb{R}^n$ is a vector space over $\mathbb{R}$
- $\mathbb{C}^n$ is a vector space over $\mathbb{C}$.

Usually the choice of F is either clear from the context or irrelevant. Thus we often assume that F is lurking in the background without specifically mentioning it. With the usual operations of addition and scalar multiplication, F^n is a vector space over F, as you should verify. The example of F^n motivated our definition of vector space.

Example 1.10. F^∞ is defined to be the set of all sequences of elements of F:

$$F^\infty = \{(f_1, f_2, ...) \mid f_i \in F \text{ for } i = 1, 2, ...\}.$$

Addition and scalar multiplication on F^∞ are defined as expected:

$$\begin{aligned} (f_1, f_2, ...) + (g_1, g_2, ...) &= (f_1 + g_1, f_2 + g_2, ...) \\ \lambda(f_1, f_2, ...) &= (cf_1, cf_2, ...). \end{aligned}$$

With these definitions, F^∞ becomes a vector space over F, as you should verify. The additive identity in this vector space is the sequence of all 0's.

Our next example of a vector space involves a set of functions.

Notation:F^S

- If S is a set, then F^S denotes the set of functions from S to F.
- For $f, g \in F^S$, the sum $f + g \in F^S$ is the function defined by $(f + g)(s) = f(s) + g(s)$ for all $s \in S$.
- For $\lambda \in F$ and $f \in F^S$, the product $\lambda f \in F^S$ is the function defined by $(\lambda f)(s) = \lambda f(s)$ for all $s \in S$.

Remark: As an example of the notation above, if S is the interval $[0, 1]$ and $F = \mathbb{R}$, then $\mathbb{R}^{[0,1]}$ is the set of real-valued functions on the interval [0, 1] The elements of the vector space $\mathbb{R}^{[0,1]}$ are real-valued functions on $[0, 1]$, not lists. In general, a vector space is an abstract entity whose elements might be lists, functions, or weird objects.

Now let's verify following example

Example 1.11. F^S **is a vector space**

- If S is a nonempty set, then F^S (with the operations of addition and scalar multiplication as defined above) is a vector space over F.
- The additive identity of F^S is the function $0\colon S \to F$ defined by $0(s) = 0$ for all $s \in S$.
- For $f \in F^S$, the additive inverse of $-f$ is the function $-f\colon S \to F$ defined by $(-f)(s) = -f(s)$ for all $s \in S$.

Proof. Let S be an set non empty set and let $f, g, h \in \mathbb{F}^S$. Let $x \in S$. Let $a, b \in \mathbb{F}$.

- **Commutataive**: $[f+g](x) = f(x)+g(x) = g(x)+f(x) = [g+f](x)$. (Since $f(x), g(x) \in \mathbb{F}$ and $\mathbb{F}$ have comutataive property.)Thus $f+g = g+f$ for all $f, g \in \mathbb{F}$
- **Associativity**:
  - $[(f+g)+h](x) = (f(x)+g(x))+h(x) = f(x)+(g(x)+h(x)) = [f+(g+h)](x)$.(Since $f(x), g(x), h(x) \in \mathbb{F}$ and $\mathbb{F}$ have associavity property.)
  - $[(a \cdot b) \cdot f](x) = (a \cdot b) \cdot (f(x)) = a \cdot (b \cdot f(x)) = [a \cdot (b \cdot f)](x)$ (Since $f(x)a, b \in \mathbb{F}$ and $\mathbb{F}$ have associavity property.
- **Aadditive Identity**: Let 0 be zero function $0 \colon S \to F$ defined by $0(s) = 0$ for all $s \in S$. Then, $[f+0](x) = f(x)+0(x) = f(x)+0 = f(x)$.Thus, $f+0 = f$ for all $f \in F^S$.
- **Addtive Identity**: For $f \in F^S$, the additive inverse of $-f$ is the function $-f \colon S \to F$ defined by $(-f)(s) = -f(s)$ for all $s \in S$. Then $[f+(-f)](x) = f(x)+(-f)(x) = f(x)-f(x) = 0 = 0(x)$. Thus. $f+0 = f$, for all $f \in F^S$.
- **Multiplicative Identity**: $[1 \cdot f](x) = 1 \cdot (f(x)) = f(x)$. Thus, $1f = f$ for all $f \in F^S$.
- **Distributive Property**:
  - $[a(f+g)](x) = a \cdot ((f+g(x))) = a(f(x)+g(x)) = af(x)+ag(x) = [af](x)+[ag](x) = [af+ag](x)$.(Since $f(x), g(x), a \in \mathbb{F}$ and $\mathbb{F}$ have distributive property. Thus, $a(f+g) = af+ag$ for all $f, g \in F^S$ and $a \in F$.
  - $[(a+b)f](x) = (a+b)((f(x))) = a(f(x))+b(f(x)) = [af](x)+[bf](x) = [af+bf](x)$.(Since $f(x)a, b \in \mathbb{F}$ and $\mathbb{F}$ have distributive property. Thus, $(a+b)f = af+bf$ for all $f \in F^S$ and $a, b \in F$

□

The vector space F^n is a special case of the vector space F^S because each $(x_1, \dots, x_n) \in F^n$ can be thought of as a function x from the set $\{1, 2, \dots, n\}$ to F by writing $x(k)$ instead of x_k for the k^{th} coordinate of $(x_1, \dots, x_n)$. In other words, we can think of F^n as $F^{\{1,2,\dots,n\}}$. Similarly, we can think of F^∞ as $F^{\{1,2,\dots\}}$.

Proposition 1.4. *A vector space has a unique additive identity.*

Proof. Suppose 0 and $0'$ are both additive identities for some vector space V. Then,

$$0' = 0'+0 = 0+0' = 0,$$

where the frst equality holds because 0 is an additive identity, the second equality comes from commutativity, and the third equality holds because $0'$ is an additive identity. Thus $0' = 0$, proving that V has only one additive identity. □

Proposition 1.5. *Every element in a vector space has a unique additive inverse.*

Proof. Suppose V is a vector space. Let $v \in V$. Suppose w and w' are additive inverses of v. Then

$$w = w + 0 = w + (v + w') = (w + v) + w' = 0 + w' = w'.$$

Thus $w = w'$, as desired. □

Notations: $-v$, $w - v$
Let $v \in V$. Then - $-v$ denotes the additive inverse of v; - $w - v$ is defined to be $w + (-v)$. "'

For the rest of this note, I will use follwoing noatation. *Notations* V denotes a vector space over F.

Proposition 1.6.

$$0v = 0 \textit{ for all } v \in V$$

0 *denotes a scalar (the number* $0 \in V$*) on the left side of the equation and a vector (the additive identity of* V*) on the right side of the equation.*

Proof. Let $v \in V$. [0v = (0 + 0)v= 0v + 0v] Adding the additive inverse of $0v$ to both sides of the equation above gives $0 = 0v$,as we want □

The result in proposition 1.6 involves the additive identity of V and scalar multiplication. The only part of the defnition of a vector space that connects vector addition and scalar multiplication is the distributive property. Thus the distributive property must be used in the proof of proposition 1.6.

1.3 Subspace

By considering subspaces, we can greatly expand our examples of vector spaces.

Definition 1.11. A subset W of a vector space V is called a **subspace** of V if W is also a vector space with the same additive identity, addition, and scalar multiplication as on V.

Fact : Some people use the terminology *linear subspace* , which means the same as subspace.

Example 1.12. $\{(x_1, x_2, 0) : x_1, x_2 \in F\}$ is a subspace of F^3.

The next proposition gives the easiest way to check whether a subset of a vector space is a subspace.

Proposition 1.7. *A subset W of a vector space V is called a **subspace** of V if and only if W satisfies the following three conditions:*

1. ***Additive Identity**: The zero vector $\mathbf{0}$ is in W.*
2. ***Closed Under Addition**: For any vectors $\mathbf{u}$ and $\mathbf{w}$ in W, their sum $\mathbf{u} + \mathbf{w}$ is also in W.*
3. ***Closed Under Scalar Multiplication**: For any scalar a in the field F and any vector $\mathbf{u}$ in W, the product $a\mathbf{u}$ is also in W.*

Proof.

- $\Longrightarrow$
  If U is a subspace of V then U satisfies the three conditions above by the definition of vector space.

- $\Longleftarrow$
  Now suppose U satisfies the three conditions above.

  - The first condition ensures that the additive identity of V is in U.
  - The second condition ensures that addition makes sense on U.
  - The third condition ensures that scalar multiplication makes sense on U. Certainly! Here's the rewritten content in LaTeX:

  If $u \in U$ then $-u$ (which equals $(-1)\mathbf{u}$ by property 1.32) is also in U by the third condition above. Hence every element of U has an additive inverse in U. The other parts of the definition of a vector space, such as associativity and commutativity, are automatically satisfied for U because they hold in the larger space V. Thus, U is a vector space and hence is a subspace of V.

$\square$

Chapter 2

Exercise

2.1 Exercise 1A

1. Show that \(\alpha + \beta = \beta + \alpha\) for all \(\alpha, \beta \in \mathbb{C}\).

Let $\alpha, \beta \in \mathbb{C}$. Then

$$\begin{aligned}
\alpha &= x_1 + i y_1 \text{ for some } x_1, y_1 \in \mathbb{R} && \text{(def}^{n}\text{ of Complex numbers)} \\
\beta &= x_2 + i y_2 \text{ for some } x_2, y_2 \in \mathbb{R} \\
\alpha + \beta &= (x_1 + i y_1) + (x_2 + i y_2) \\
&= (x_1 + x_2) + i(y_1 + y_2) && \text{(def}^{n}\text{ of addition of complex numbers)} \\
&= (x_2 + x_1) + i(y_2 + y_1) && \text{(Commutative law of real number)} \\
&= (x_2 + i y_2) + (x_1 + i y_1) && \text{(def}^{n}\text{ of addition of complex numbers)} \\
&= \beta + \alpha
\end{aligned}$$

2. Show that \((\alpha + \beta) + \lambda = \alpha + (\beta + \lambda)\) for all \(\alpha, \beta, \lambda \in \mathbb{C}\).

Let $\alpha, \beta, \lambda \in \mathbb{C}$. Then

$\alpha = x_1 + i y_1$ for some $x_1, y_1 \in \mathbb{R}$

$\beta = x_2 + i y_2$ for some $x_2, y_2 \in \mathbb{R}$ (def^n of Complex numbers)

$\lambda = x_3 + i y_3$ for some $x_3, y_3 \in \mathbb{R}$

$$
\begin{aligned}
(\alpha + \beta) + \lambda &= \big((x_1 + i y_1) + (x_2 + i y_2)\big) + (x_3 + i y_3) \\
&= \big((x_1 + x_2) + i(y_1 + y_2)\big) + (x_3 + i y_3) \\
&= \big((x_1 + x_2) + x_3\big) + i\big((y_1 + y_2) + y_3\big) \\
&= \big(x_1 + (x_2 + x_3)\big) + i\big(y_1 + (y_2 + y_3)\big) \\
&= (x_1 + i y_1) + \big((x_2 + x_3) + i(y_2 + y_3)\big) \\
&= (x_1 + i y_1) + \big((x_2 + i y_2) + (x_3 + i y_3)\big) \\
&= \alpha + (\beta + \lambda)
\end{aligned}
$$

(def^n of addition of complex numbers)

(asscintive Property of $+$ $\mathbb{R}$)

(def^n of addition of complex numbers)

3. Show that \((\alpha \beta)\lambda = \alpha(\beta \lambda)\) for all \(\alpha, \beta, \lambda \in \mathbb{C}\).

4. Show that \(\lambda(\alpha + \beta) = \lambda\alpha + \lambda\beta\) for all \(\lambda, \alpha, \beta \in \mathbb{C}\).

Let $\alpha, \beta, \lambda \in \mathbb{C}$. Then

$\alpha = x_1 + i y_1$ forsome $x_1, y_1 \in \mathbb{R}$

$\beta = x_2 + i y_2$ forsome $x_2, y_2 \in \mathbb{R}$ (def^n of Complex numbers)

$\lambda = x_3 + i y_3$ forsome $x_3, y_3 \in \mathbb{R}$

$$\lambda(\alpha+\beta) = (x_3 + i y_3)\big((x_1 + i y_1) + (x_2 + i y_2)\big)$$

$$= (x_3 + i y_3)\big((x_1 + x_2) + i(y_1 + y_2)\big) \quad \text{(Commutative of addition of } \mathbb{R})$$

$$= \big(x_3(x_1 + x_2) - y_3(y_1 + y_2)\big) + i\big(x_3(y_1 + y_2) + y_3(x_1 + x_2)\big) \quad \text{(def}^n \text{ of multiplication of } \mathbb{C})$$

$$= (x_3 x_1 + x_3 x_2 - y_3 y_1 - y_3 y_2) + i(x_3 y_1 + x_3 y_2 + y_3 x_1 + y_3 x_2) \quad \text{(commutative property)}$$

$$= (x_3 x_1 + i x_3 y_1) + (x_3 x_2 + i x_3 y_2) - y_3 y_1 + i y_3 x_1 - y_3 y_2 + i y_3 x_2 \quad \text{(Commutative property)}$$

$$= x_3(x_1 + i y_1) + x_3(x_2 + i y_2) + y_3(-y_1 + i x_1) + y_3(-y_2 + i x_2) \quad \text{(distributive property)}$$

$$= x_3\alpha + x_3\beta + y_3(i^2 y_1 + i x_1) + y_3(i^2 y_2 + i x_2) \quad (\text{use } i^2 = -1)$$

$$= x_3\alpha + x_3\beta + i y_3(i y_1 + x_1) + i y_3(i y_2 + x_2) \quad \text{(distributive)}$$

$$= x_3\alpha + x_3\beta + i y_3(x_1 + i y_1) + i y_3(x_2 + i y_2) \quad \text{(distributive)}$$

$$= x_3\alpha + x_3\beta + i y_3\alpha + i y_3\beta$$

$$= (x_3 + i y_3)\alpha + (x_3 + i y_3)\beta \quad \text{(distributive property)}$$

$$= \lambda\alpha + \lambda\beta$$

5. Show that for every $\alpha \in \mathbb{C}$, there exists a unique $\beta \in \mathbb{C}$ such that $\alpha + \beta = 0$.

5) Let $\alpha \in \mathbb{C}$.

Suppose that $\alpha+\beta_1=0$ and $\alpha+\beta_2=0$ with $\beta_1 \neq \beta_2$

$$\alpha+\beta_1=0$$
$$-\alpha+(\alpha+\beta_1)=-\alpha+0$$
$$(-\alpha+\alpha)+\beta_1=-\alpha$$
$$0+\beta_1=-\alpha$$
$$\beta_1=-\alpha \quad \text{—— } (1)$$

Similar we can get $\beta_2=-\alpha$ —— $\boxed{2}$

By (1) and (2) $\beta_1=\beta_2$

Therefore, $\forall \alpha \in \mathbb{C}\ \exists! \beta$ such that $\alpha+\beta=0$

6. Show that for every \(\alpha \in \mathbb{C}\) with \(\alpha \neq 0\), there exists a unique \(\beta \in \mathbb{C}\) such that \(\alpha\beta = 1\).

6) Let $\alpha \in \mathbb{C} \setminus \{0\}$

Suppose that $\alpha\beta_1 = 1 = \alpha\beta_2$ with $\beta_1 \neq \beta_2$.

$$\alpha\beta_1 = \alpha\beta_2$$

$$\frac{1}{\alpha}(\alpha\beta_1) = \frac{1}{\alpha}(\alpha\beta_2)$$

$$\left(\frac{1}{\alpha}\cdot\alpha\right)\beta_1 = \left(\frac{1}{\alpha}\cdot\alpha\right)\beta_2$$

$$1\cdot\beta_1 = 1\cdot\beta_2$$

$$\beta_1 = \beta_2$$

Therefore, $\forall \alpha \in \mathbb{C}\ \exists!\, \beta$ such that $\alpha\beta = 1$

7. Show that \(\frac{-1 + \sqrt{3}i}{2}\) is a cube root of 1 (meaning that its cube equals 1).

7) Let $\alpha = -\frac{1}{2} + i\frac{\sqrt{3}}{2}$

Then $\alpha^2 = \left(-\frac{1}{2} + i\frac{\sqrt{3}}{2}\right)^2 = \left(-\frac{1}{2} + i\frac{\sqrt{3}}{2}\right)\left(-\frac{1}{2} + i\frac{\sqrt{3}}{2}\right)$

$$= \left(\frac{1}{4} - \frac{3}{4}\right) + i\left(-\frac{\sqrt{3}}{4} - \frac{\sqrt{3}}{4}\right)$$

$$= -\frac{2}{4} + i\left(-\frac{2\sqrt{3}}{4}\right)$$

$$= \left(-\frac{1}{2} - \frac{i\sqrt{3}}{2}\right)$$

$$\alpha^3 = \alpha \cdot \alpha^2 = \left(-\frac{1}{2} + i\frac{\sqrt{3}}{2}\right)\left(-\frac{1}{2} - \frac{i\sqrt{3}}{2}\right)$$

$$= \left(\frac{1}{4} + \frac{3}{4}\right) + i\left(\frac{\sqrt{3}}{4} - \frac{\sqrt{3}}{4}\right)$$

$$= 1 + 0i = 1$$

Thus α is cube root of 1.

8. Find two distinct square roots of \(i\).

8) We have to find $\alpha \in \mathbb{C}$ such that

$$\alpha^2 = i.$$

Let $\alpha = x + iy$, for some $x, y \in \mathbb{R}$.

$$\alpha^2 = i$$
$$(x+iy)^2 = i$$
$$(x+iy)(x+iy) = i$$
$$(x^2 - y^2) + i(xy + xy) = i$$
$$(x^2 - y^2) + i(2xy) = 0 + i$$

Thus $x^2 - y^2 = 0$. and $2xy = 1$

$((x-y)(x+y) = 0)$ and $2xy = 1$

$(x = y$ or $x = -y)$ and $2xy = 1$

if $x = (-y)$ then $2xy = 2(-y)y = -2y^2 = 1$

Since $y \in \mathbb{R}$ and $y > 0$, this is impossible.

Thus, $x = y$.

Then $2xy = 2x^2 = 1 \Rightarrow x = \pm 1/\sqrt{2}$.

Hence the square root of i is $\left(\frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}}\right)$ and $\left(-\frac{1}{2} - \frac{1}{\sqrt{2}}i\right)$

9. Find \(x \in \mathbb{R}^4\) such that \((4, -3, 1, 7) + 2x = (5, 9, -6, 8)\).

9) $(4,-3,1,7) + 2x = (5,9,-6,8)$

$$-(4,-3,1,7) + (4,-3,1,7) + 2x = -(4,-3,1,7) + (5,9,-6,8)$$

$$2x = -(4,-3,1,7) + (5,9,-6,8)$$

$$2x = (-4+5, -3+9, -1-6, -7+8)$$

$$x = (1/2, 6, -7/2, 1/2) \in \mathbb{R}^4$$

10. Explain why there does not exist $\lambda \in \mathbb{C}$ such that $\lambda(2 - 3i, 5 + 4i, -6 + 7i) = (12 - 5i, 7 + 22i, -32 - 9i)$.

10) Assume the contray there exist $\lambda \in \mathbb{C}$ such that

$$\lambda(2-3i, 5+4i, -6+7i) = (12-5i, 7+22i, -32-9i)$$

$$(\lambda(2-3i), \lambda(5+4i), \lambda(-6+7i)) = (12-5i, 7+22i, -32-9i)$$

Then,

$$\lambda(2-3i) = 12-5i \quad \text{—— } ①$$

$$\lambda(5+4i) = 7+22i$$

$① \times (2+3i)$

$$\lambda(2-3i)(2+3i) = (12-5i)(2+3i)$$

$$\lambda(4+9) = 13\lambda = 39+26i$$

$$\lambda = 3+2i \quad \text{—— } (*)$$

$② \times (5-4i)$

$$\lambda \cdot (5-4i)(5+4i) = (7+22i)(5+4i)$$

$$\lambda(25+16) = 41\lambda = -53+138i$$

$$\lambda = -\frac{53}{41} + \frac{138}{41}i \quad \text{—— } (**)$$

$(*)$ and $(**)$ gives a contradiction.

11. Show that \((x + y) + z = x + (y + z)\) for all \(x, y, z \in \mathbb{F}_n\).

12. Show that \((ab)x = a(bx)\) for all \(x \in \mathbb{F}_n\) and all \(a, b \in \mathbb{F}\).

13. Show that \(1x = x\) for all \(x \in \mathbb{F}_n\).

14. Show that \(\lambda(x + y) = \lambda x + \lambda y\) for all \(\lambda \in \mathbb{F}\) and all \(x, y \in \mathbb{F}_n\).

15. Show that \((a + b)x = ax + bx\) for all \(a, b \in \mathbb{F}\) and all \(x \in \mathbb{F}_n\).

2.2 Exercise 1B

1. Prove that \(−(−v) = v\) for every \(v\in V\). **Solution**: Let \(v\in V\). Then there exist a unique additive inverse of \(v\). We denote it by \(-v\). Thus, \

$$v + (-v) = 0.$$

Then by definition, additive inverse of \((-v)\) is v. We denote it by \(-(-v)=v\).

2. Suppose \(a \in \mathbb{F} ,v\in V\), and \(v= 0\). Prove that \(a=0\) or \(v=0\).

(2) Suppose that $a \in \mathbb{F}$ ~~and~~ $v \in V$ and

$$a v = \underline{0}$$

~~Then~~ if $a = 0$, we have nothing to proove.
So assume that $a \neq 0$. Now I need to show that $v = 0$.
Since $a \in F$ and $a \neq 0$. multiplicative inverse of a exist. Let's denote it by a^{-1}. Morever,

$$a^{-1} a = 1 = a a^{-1}$$

$$a v = 0$$

$$a^{-1}(a v) = (a^{-1}(a))v = 1 \cdot v = v = 0 \;//$$

3\. ③ Suppose that $v, w \in V$.

First of we need to show that ~~b~~ the existance of such x.

Let $x = \frac{1}{3}w - \frac{1}{3}v$

Note that $\frac{1}{3} \in \mathbb{F}$ and $v, w \in V$, then

$\frac{1}{3}w - \frac{1}{3}v = x \in V$

Further,

$$v + 3x = v + 3\left(\frac{1}{3}w - \frac{1}{3}v\right)$$

$$= v + 3\cdot\frac{1}{3}\cdot w - 3\cdot\frac{1}{3}\cdot v$$

(Since $\frac{1}{3}$ is multiplical inverse)

$$= v + w - v$$

$$= w$$

~~AN~~ We are existance of $x \in V$.

Now we have to proove uniquness. Assume that there exist $y \in V$ statistfy $v + 3y = w$

Then $w = \cancel{3x}\ v + 3x = v + 3y$

Then $-v + (v + 3x) = -v + (v + 3y)$

$$(-v + v) + 3x = (-v + \cancel{3}v) + 3y$$

$$3x = 3y$$

$$\frac{1}{3}\cdot(3x) = \frac{1}{3}\cdot(3y)$$

$$\left(\frac{1}{3}\cdot 3\right)\cdot x = 1\cdot x = x = y = 1\cdot y = \left(\frac{1}{3}\cdot 3\right)y$$

4 The empty set is not a vector space. The empty set fails to satisfy only one of the requirements listed in the definition of a vector space (1.20). Which one?

Recall the defn of vectorspace

1.20 definition: *vector space*

A *vector space* is a set V along with an addition on V and a scalar multiplication on V such that the following properties hold.

commutativity
$u + v = v + u$ for all $u, v \in V$.

associativity
$(u + v) + w = u + (v + w)$ and $(ab)v = a(bv)$ for all $u, v, w \in V$ and for all $a, b \in \mathbf{F}$.

additive identity
There exists an element $0 \in V$ such that $v + 0 = v$ for all $v \in V$.

additive inverse
For every $v \in V$, there exists $w \in V$ such that $v + w = 0$.

multiplicative identity
$1v = v$ for all $v \in V$.

distributive properties
$a(u + v) = au + av$ and $(a + b)v = av + bv$ for all $a, b \in \mathbf{F}$ and all $u, v \in V$.

A vector space must contain $0 \in V$ (additive identity)
So, the emptyset fails the addtive identity property

5 Show that in the definition of a vector space (1.20), the additive inverse condition can be replaced with the condition that

$$0v = 0 \text{ for all } v \in V.$$

Here the 0 on the left side is the number 0, and the 0 on the right side is the additive identity of V.

The phrase a "condition can be replaced" in a definition means that the collection of objects satisfying the definition is unchanged if the original condition is replaced with the new condition.

additive inverse

For every $v \in V$, there exists $w \in V$ such that $v + w = 0$.

Need to show:

$$(\forall v \in V \, \exists w \in V \text{ s.t. } v+w=0_F) \Longleftrightarrow \forall v \in V, \; 0_F v = 0_V$$

proof: $\Leftarrow$

Let $v \in V$. Suppose that $0_F w = 0_V$ for all $w \in V$.

Since $1 \in F$, addtive inverse of 1 is exists.

Further, $1 + (-1) = 0_F$

$$(1+(-1))\,v = 0_F\,v = 0_V$$

$$v + (-1)v = 1 \cdot v + (-1)v = 0_F v = 0_V$$

Thus, adative identity of V is "$(-1)\,v$". So, we are done backward direction. Let's proove the forward direction.

5.

"$\Leftarrow$"

Now suppose that $\forall v \in V \; \exists w \in V$ s.t. $v + w = 0_F$

$$0_F V = (0_F + 0_F) V \quad \text{(addtive identiy of feild)}$$
$$= 0_F V + 0_F V \quad \text{(distributive property)} \quad — (1)$$

Since $0_F V \in V$, there exist additive identity of $0_F V$. Let's call that as w. Further

$$0_F V + W = 0 \quad — (2)$$

Now consider, (By (1))

$$0_F V = 0_F V + 0_F V$$
$$\underbrace{(0_F V + W)}_{0_V} = 0_F V + \underbrace{(0_F V + W)}_{0_V} \quad \text{(property of equality)}$$
$$= 0_F V + 0_V \quad \text{(By (2))}$$
$$0_V = 0_F V \quad \text{(additive identity)}$$

6 Let ∞ and $-\infty$ denote two distinct objects, neither of which is in $\mathbf{R}$. Define an addition and scalar multiplication on $\mathbf{R} \cup \{\infty, -\infty\}$ as you could guess from the notation. Specifically, the sum and product of two real numbers is as usual, and for $t \in \mathbf{R}$ define

$$t\infty = \begin{cases} -\infty & \text{if } t < 0, \\ 0 & \text{if } t = 0, \\ \infty & \text{if } t > 0, \end{cases} \qquad t(-\infty) = \begin{cases} \infty & \text{if } t < 0, \\ 0 & \text{if } t = 0, \\ -\infty & \text{if } t > 0, \end{cases}$$

and

$$\begin{aligned} t + \infty = \infty + t = \infty + \infty &= \infty, \\ t + (-\infty) = (-\infty) + t = (-\infty) + (-\infty) &= -\infty, \\ \infty + (-\infty) = (-\infty) + \infty &= 0. \end{aligned}$$

With these operations of addition and scalar multiplication, is $\mathbf{R} \cup \{\infty, -\infty\}$ a vector space over $\mathbf{R}$? Explain.

$\mathbb{R} \cup (-\infty, \infty)$ is NOT a vector space.

- Asscitivity property fails. Let $0 \neq v \in V$

Consider,

$v + (\infty + (-\infty)) = v + 0 = v \neq 0 = \infty + (-\infty) = (v + \infty) + (-\infty)$

- Unique additive identity fails (Extra part)

Consider.

$\infty = (2 + (-1)) \cdot \infty = 2\infty + (-1)\infty = \infty - \infty = 0$

Then for any $t \in \mathbb{R}$,

$t = 0 + t = \infty + t = \infty = 0.$

6. So, this fails, uniqueness property of additive identity

7. Suppose S is a non-empty set. Let V^S denote the set of functions from S to V. Define a natural addition and scalar multiplication on V^S, and show that V^S is a vector space with these definitions.

7 Suppose S is a nonempty set. Let V^S denote the set of functions from S to V. Define a natural addition and scalar multiplication on V^S, and show that V^S is a vector space with these definitions.

Let S be a set with $S \neq \emptyset$. Let

$$V^S := \{ f : S \to V \mid f \text{ is functions to } V \}$$

Let $f, g \in V^S$. Let $\lambda \in S$ Define

- addition : $V^S \times V^S \longrightarrow V^S$

$$f+g(x) = f(x) + g(x) \quad \forall x \in S.$$

- scaler multiplication : $V \times V^S \longrightarrow V^S$

$$[\lambda f](x) = \lambda \cdot (f(x)) \quad \forall x \in S.$$

NTS: V^S is a vector space

Commutative

claim1 : $f+g = g+f \quad \forall f, g \in V^S$

Let $f, g \in V^S$

$$\begin{aligned}[f+g](x) &= f(x)+g(x) \quad \forall x\in V \\ &= g(x)+f(x) \quad \forall x\in V \quad (\because f(x), g(x)\in V) \\ &= [g+f](x) \quad \forall x\in V\end{aligned}$$

Thus, $f, g\in V$

Assciativity

claim 2.1: $(f+g)+h = f+(g+h) \ \forall f, g, h\in V^S$

Let $f, g, h\in V^S$. Note that

$$\begin{aligned}[(f+g)+h](x) &= [f+g](x)+h(x) \\ &= (f(x)+g(x))+h(x) \\ &= f(x)+(g(x)+h(x)) \\ &= [f+(g+h)](x)\end{aligned}$$

claim 2.2: $(ab)f = a(bf) \quad \forall f\in V^S, \forall a, b\in V$

Let $a, b\in V$ and $f\in V^S$

$$\begin{aligned}[(ab)f](x) &= (ab)f(x) \\ &= a(b f(x)) \\ &= [a(bf)](x)\end{aligned}$$

Addtive identity.

claim: $\left.\begin{aligned} 0_{map}: S &\longrightarrow V \\ x &\longmapsto 0 \end{aligned}\right\}$ is the addtive identity

such that $f + 0_{map} = f \quad \forall f \in V^S$ (Note that $0_{map} \in V^S$)

$$[f + 0_{map}](x) = f(x) + 0_{map}(x) = f(x) + 0 = f(x)$$

Addtive inverse

Let $f \in V^S$. Let define,

$$\begin{aligned} g: S &\longrightarrow V \\ x &\longmapsto -f(x) \end{aligned}$$

Note that $g \in V^S$,

$$\begin{aligned} [f+g](x) &= f(x) + g(x) \\ &= f(x) + (-f(x)) \\ &= 0. \end{aligned}$$

multiplicative inverse

claims. $1f = f \quad \forall f \in V^S$

Let $f \in V^S$.

$$[1 \cdot f](x) = 1 \cdot f(x) = f(x).$$

Distributiv-

claim 6.1: $a(f+g) = af + ag \quad \forall a \in V, \forall f, g \in V^S$.

Let $f, g \in V^S$, and let $a \in V$.

$$\begin{aligned}[a(f+g)](x) &= a([f+g](x)] \\ &= a(f(x) + g(x)) \\ &= af(x) + ag(x) \\ &= [af](x) + [ag](x) \\ &= [af+ag](x)\end{aligned}$$

Claim 6.2: $(a+b)f = af + bf, \ \forall f \in V^S \ \forall a, b \in V$

$$\begin{aligned}[(a+b)f](x) = (a+b)f(x) &= af(x) + bf(x) \\ &= [af](x) + [bf](x) = \\ &= [af+bf](x)\end{aligned}$$

Therefore, V^S is a Vector space

8 Suppose V is a real vector space.

- The *complexification* of V, denoted by $V_{\mathbf{C}}$, equals $V \times V$. An element of $V_{\mathbf{C}}$ is an ordered pair (u, v), where $u, v \in V$, but we write this as $u + iv$.
- Addition on $V_{\mathbf{C}}$ is defined by

$$(u_1 + iv_1) + (u_2 + iv_2) = (u_1 + u_2) + i(v_1 + v_2)$$

  for all $u_1, v_1, u_2, v_2 \in V$.
- Complex scalar multiplication on $V_{\mathbf{C}}$ is defined by

$$(a + bi)(u + iv) = (au - bv) + i(av + bu)$$

  for all $a, b \in \mathbf{R}$ and all $u, v \in V$.

Prove that with the definitions of addition and scalar multiplication as above, $V_{\mathbf{C}}$ is a complex vector space.

Think of V as a subset of $V_{\mathbf{C}}$ by identifying $u \in V$ with $u + i0$. The construction of $V_{\mathbf{C}}$ from V can then be thought of as generalizing the construction of $\mathbf{C}^n$ from $\mathbf{R}^n$.

Commutative

Let $\alpha_1 + i\beta, \alpha_2 + i\beta_2 \in V_{\mathbb{C}}$.

$$\begin{aligned}(\alpha_1 + i\beta_1) + (\alpha_2 + i\beta_2) &= (\alpha_1 + \alpha_2) + i(\beta_1 + \beta_2) \\ &= (\alpha_2 + \alpha_1) + i(\beta_2 + \beta_1) \quad \left(\begin{matrix}\alpha_1, \alpha_2, \beta_1\beta_2 \in V \\ \text{commutative pro}\end{matrix}\right) \\ &= (\alpha_2 + i\beta_2) + (\alpha_1 + i\beta_1)\end{aligned}$$

Associativity

Let $\alpha_1+i\beta_1,\ \alpha_2+i\beta_2,\ \alpha_3+i\beta_3 \in V_{\mathbb{C}}$

$$(\alpha_1+i\beta_1)+\left[(\alpha_2+i\beta_2)+(\alpha_3+i\beta_3)\right]$$

$$= (\alpha_1+i\beta_1)\left[(\alpha_2+\alpha_3)+i(\beta_2+\beta_3)\right]$$
$$= \alpha_1+(\alpha_2+\alpha_3)+i(\beta_1+(\beta_2+\beta_3))$$
$$= (\alpha_1+\alpha_2)+\alpha_3+i((\beta_1+\beta_2)+\beta_3)$$
$$= \left[(\alpha_1+\alpha_2)+i(\beta_1+\beta_2)\right]+(\alpha_3+i\beta_3)$$
$$= \left[(\alpha_1+i\beta_1)+(\alpha_2+i\beta_2)\right]+(\alpha_3+i\beta_3)$$

Let $x_1+iy_1,\ x_2+iy_2 \in \mathbb{C}$ and $(\alpha+i\beta) \in V_{\mathbb{C}}$

$$= \left[(x_1+iy_1)\cdot(x_2+iy_2)\right](\alpha+i\beta)$$
$$= \left[(x_1x_2-y_1y_2)+i(x_1y_2+x_2y_1)\right](\alpha+i\beta)$$
$$= (x_1x_2-y_1y_2)\alpha-(x_1y_2+x_2y_1)\beta$$
$$+i(\beta(x_1x_2-y_1y_2)+\alpha(x_1y_2+x_2y_1))$$
$$= x_1x_2\alpha-y_1y_2\alpha-x_1y_2\beta-x_2y_1\beta$$
$$+i(x_1x_2\beta-y_1y_2\beta+x_1y_2\alpha+\boxed{x_2y_1\alpha}$$

$$= x_1x_2\alpha + x_1y_2\beta - y_1x_2\beta - y_1y_2\alpha$$
$$+ i(x_1x_2\beta + x_1y_2\alpha + \boxed{y_1x_2\alpha} - y_1y_2\beta$$
$$= x_1(x_2\alpha + y_2\beta) - y_1(x_2\beta + y_2\alpha)$$
$$+ i\left[x_1(x_2\beta + y_2\alpha) + y_1(x_2\alpha - y_2\beta)\right]$$
$$= (x_1 + iy_1)\left[(x_2\alpha - y_2\beta) + i(x_2\beta + y_2\alpha)\right]$$
$$= (x_1 + iy_1)\left[(x_2 + iy_2)\cdot(\alpha + i\beta)\right]$$

Additive Identity

Let $\underline{0}$ be the zero vector. ($\underline{0} \in V$) Let $\alpha + i\beta \in V_{\mathbb{C}}$

$$(\alpha + i\beta) + (\underline{0} + i\underline{0}) = (\alpha + \underline{0}) + i(\beta + \underline{0})$$
$$= \alpha + i\beta$$

Additive inverse,

Let $\alpha + i\beta \in V_{\mathbb{C}}$. Now consider,

$$\alpha + i\beta + (-\alpha + i(-\beta)) = (\alpha + (-\alpha)) + i(\beta + (-\beta)) = 0$$

multiplicative identity

$$(1 + i\cdot 0)\cdot(\alpha + i\beta) = (1\cdot\alpha - 0\,\beta) + i(1\cdot\beta + 0\cdot\alpha) = \alpha + i\beta$$

Distributive

Let $(x+iy) \in \mathbb{C}$ and, $(\alpha_1+i\beta_1), (\alpha_2+i\beta_2) \in V_{\mathbb{C}}$

$$\begin{aligned}
(x+iy)\ \big((\alpha_1+i\beta_1)+(\alpha_2+i\beta_2)\big) &= (x+iy)\left[(\alpha_1+\alpha_2)+i(\beta_1+\beta_2)\right] \\
&= x(\alpha_1+\alpha_2)-y(\beta_1+\beta_2) \\
&\quad +i\left[x(\beta_1+\beta_2)+y(\alpha_1+\alpha_2)\right] \\
&= x\alpha_1+x\alpha_2-y\beta_1-y\beta_2 \\
&\quad +i\left[x\beta_1+x\beta_2+y\alpha_1+y\alpha_2\right] \\
&= (x\alpha_1-y\beta_1)+i(x\beta_1+y\alpha_1) \\
&\quad +(x\alpha_2-y\beta_2)+i(x\beta_2-y\alpha_2) \\
&= (x+iy)(\alpha_1+i\beta_1)+(x+iy)(\alpha_2+i\beta_2)
\end{aligned}$$

Let $(x_1+iy_1), (x_2+iy_2) \in \mathbb{C}$ and $(\alpha+i\beta) \in V_{\mathbb{C}}$

$$\begin{aligned}
\left[(x_1+iy_1)+(x_2+iy_2)\right](\alpha+i\beta) &= \left[(x_1+x_2)+i(y_1+y_2)\right](\alpha+i\beta) \\
&= \left[(x_1+x_2)\alpha-(y_1+y_2)\beta\right]+i\left[(x_1+x_2)\beta+(y_1+y_2)\alpha\right] \\
&= \left[x_1\alpha+x_2\alpha-y_1\beta-y_2\beta\right]+i\left[x_1\beta+x_2\beta+y_1\alpha+y_2\alpha\right] \\
&= \left[(x_1\alpha-y_2\beta)+i(x_1\beta+y_2\alpha)\right]+\left[(x_2\alpha-y_2\beta)+i(x_2\beta+y_2\alpha)\right] \\
&= (x_1+iy_2)(\alpha+i\beta)+(x_2+iy_2)(\alpha+i\beta)
\end{aligned}$$

2.3 Exercise 1C

1.

1 For each of the following subsets of $\mathbf{F}^3$, determine whether it is a subspace of $\mathbf{F}^3$.

(a) $\{(x_1, x_2, x_3) \in \mathbf{F}^3 : x_1 + 2x_2 + 3x_3 = 0\} = A$

(b) $\{(x_1, x_2, x_3) \in \mathbf{F}^3 : x_1 + 2x_2 + 3x_3 = 4\} = B$

(c) $\{(x_1, x_2, x_3) \in \mathbf{F}^3 : x_1 x_2 x_3 = 0\} = C$

(d) $\{(x_1, x_2, x_3) \in \mathbf{F}^3 : x_1 = 5x_3\} = D$

a) A is subspace of $\mathbb{F}^3$?

- Claim1: $(0,0,0)\in \mathbb{F}^3$

Note that $0+2\cdot 0+3\cdot 0=0$. Thus, $(0,0,0)\in A$

- Claim2: $\forall (x_1,x_2,x_3), (y_1,y_2,y_3)\in A$; $(x_1,x_2,x_3)+(y_1,y_2,y_3)\in A$

Let $(x_1,x_2,x_3), (y_1,y_2,y_3)\in A$.

Then, $x_1+2x_2+3x_3=0$ —①

$y_1+2y_2+3y_3=0$ —②

①+② $(x_1+x_2+3x_3)+(y_1+2y_2+3y_3)=0$

$(x_1+y_1)+2(x_2+y_2)+3(x_3+y_3)=0$

Thus, $(x_1+y_1, x_2+y_2, x_3+y_3)=0$

- Claim3: Closed under sclaer multiplication

Let $a\in F$ and $(x_1,x_2,x_3)\in A$. Then,

$x_1+2x_2+3x_3=0$

$a\cdot(x_1+2x_2+3x_3)=ax_1+a(2x_2)+a(3x_3)=ax_1+2(ax_2)+3(ax_3)=0$

Then $a(x_1,x_2,x_3)=0\in A$.

b) B is not a subspace

Note that, $(0,0,0) \notin B$.

Because, $0+2\cdot0+3\cdot0=0 \neq 4$.

c) C is a NOT subspace.

Note that $(1,1,0), (0,0,1) \in C$. $\left(\text{Because } 1\cdot1\cdot0=0 \text{ and } 0\cdot0\cdot1=0\right)$

But $(1,1,0)+(0,0,1)=(1,1,1) \notin F$

$\left(\text{Because } 1\cdot1\cdot1=1\neq0\right)$

d) D is a subspace.

- claim 1: $(0,0,0) \in D$

$\left(\text{Because, } 0=5\cdot0\right)$

- claim 2: $\forall (x_1,x_2,x_3), (y, y_2, y_3) \in D$, $(x_1,x_2,x_3)+(y_1,y_2,y_3) \in D$

Let $(x_1,x_2,x_3), (y_1,y_2,y_3) \in D$.

Then, $x_1=5x_3$ and $y_1=5y_3$

So, $x_1+x_2 = 5x_3+5y_3 = 5(x_3+y_3)$

Thus, $(x_1,x_2,x_3)+(y_1,y_2,y_3)=(x_1+y_1, x_2+y_2, x_3+y_3) \in D$

• Sclar multiplication

Let $a \in F$ and $(x_1, x_2, x_3, x_4) \in U$.

Then $x_2 = 5x_4$

So, $ax_2 = a(5x_4) = 5(ax_4)$

Then,

$a(x_1, x_2, x_3, x_4) = (ax_1, ax_2, ax_3, ax_4) \in U.$

Hence, U is subspace

Therefore, U is subspace iff $b = 0$.

Statement 1: If \(b\in \mathbb{F}\), then \

$$left(x\_1, x\_2, x\_3, x\_4) in mathbb F^4 : x\_3 = 5x\_4 + b right$$

is a subspace of \(\mathbb{F}^4\) if and only if \(b = 0\).

Proof of Statement 1

1.35 example: *subspaces*

(a) If $b \in \mathbf{F}$, then
$$\{(x_1, x_2, x_3, x_4) \in \mathbf{F}^4 : x_3 = 5x_4 + b\}$$
is a subspace of $\mathbf{F}^4$ if and only if $b = 0$.

Let $b \in F$. and let
$$V := \{(x_1, x_2, x_3, x_4) \in F^4 : x_3 = 5x_4 + b\}$$

claim: V is subspace of $F^4 \Leftrightarrow b = 0$

"$\Rightarrow$"

Suppose that V is subspace. Then $(0,0,0,0) \in V$

Then, $0 = 5 \cdot 0 + b \Rightarrow b = 0$.

"$\Leftarrow$"

Now suppose that $b = 0$. Then, $x_3 = 5x_4$ — 1

• Additive id

Note that $(0,0,0,0) \in D$. Because $0 = 5 \cdot 0 = 0$.

• Closed under addition

Let $(x_1, x_2, x_3, x_4), (y_1, y_2, y_3, y_4) \in D$.

Then $x_1 = 5x_4$ and $y_1 = 5y_4$

• Sclar multiplication

Let $a \in F$ and $(x_1, x_2, x_3, x_4) \in U$.

Then $x_2 = 5x_4$

So, $ax_2 = a(5x_4) = 5(ax_4)$

Then,

$a(x_1, x_2, x_3, x_4) = (ax_1, ax_2, ax_3, ax_4) \in U$.

Hence, U is subspace

Therefore, U is subspace iff $b = 0$.

Statement 2 : The set of continuous real-valued functions on the interval \\(

$$0, 1$$

\\) is a subspace of \\(\mathbb{R}^

$$0, 1$$

\\)

Proof of statement 2:

(b) The set of continuous real-valued functions on the interval $[0, 1]$ is a subspace of $\mathbf{R}^{[0,1]}$.

- Let $0_{map} : [0,1] \longrightarrow \mathbb{R}$ defined by $x \longmapsto 0$

0_{map} is continous map. (Since it constant function.) Hence, $0_{map} \in \mathbb{R}^{[0,1]}$ (See claim1)

- Let $f, g \in \mathbb{R}^{[0,1]}$. Then $[f+g]$ is also continous function on $[0,1]$. Thus $f+g \in \mathbb{R}^{[0,1]}$ (Claim2)
- Let $c \in \mathbb{R}$ and $f \in \mathbb{R}^{[0,1]}$. Then cf is also continuous function on $[0,1]$. Thus $cf \in \mathbb{R}^{[0,1]}$ (Claim3)

Claim: 0_{map} is continous on $\mathbb{R}^{[0,1]}$

Let $x_0 \in [0,1]$

Let $\varepsilon > 0$. choose $\delta = 1 > 0$. Let $x \in [0,1]$.

Suppose that $|x - x_0| < 1$.

Now consider,

$$|0_{map}(x) - 0_{map}(x_0)| = |0 - 0| = 0 < \varepsilon$$

Therefore, 0_{map} is continous map.

Claim 2: If $f, g \in R^{[0,1]}$ then $f+g \in R^{[0,1]}$

Suppose that $f, g \in R^{[0,1]}$. Let $x_0 \in [0,1]$

Then If f is continous at x_0,

$\forall \varepsilon > 0 \, \exists \delta_1 > 0$ s.t. $\forall x \in [0,1]$,

$$|x - x_0| \leq \delta \Rightarrow |f(x) - f(x_0)| < \varepsilon/2 \quad (1)$$

If g is continous at x_0, where $x_0 \in [0,1]$

$\forall \varepsilon > 0 \, \exists \delta_2 > 0$ s.t. $\forall x \in [0,1]$

$$|x - x_0| < \delta_2 \Rightarrow |g(x) - g(x_0)| < \varepsilon/2 \quad (2)$$

Let $\varepsilon > 0$

Choose $\delta_3 = \max\{\delta_1, \delta_2\}$. Let $x \in [0,1]$.

Suppose that $|x - x_0| < \delta$. Now consider,

$$
\begin{aligned}
|[f+g](x) - [f+g](x_0)| &= |f(x) + g(x) - (f(x_0) + g(x_0)] \\
&= |f(x) + g(x) - f(x_0) \; g(x_0)| \\
&= |f(x) - f(x_0) + g(x) - g(x_0)| \\
&\leq |f(x) - f(x_0)| + |g(x) - g(x_0)| \\
&= \varepsilon/2 + \varepsilon/2 = \varepsilon
\end{aligned}
$$

Thus, $f+g$ is continous at every point in $[0,1]$

claim 3: If $c \in \mathbb{R}$ and $f \in \mathbb{R}^{[0,1]}$ then $[cf] \in \mathbb{R}^{[0,1]}$

Suppose that $f, g \in \mathbb{R}^{[0,1]}$ and $\in \mathbb{R}$ Let $x_0 \in [0,1]$

Then If f is continous at x_0,

$\forall \varepsilon > 0 \, \exists \delta_0 > 0$ s.t. $\forall x \in [0,1]$,

$$|x - x_0| \le \delta_0 \Rightarrow |f(x) - f(x_0)| < \varepsilon/|c| \quad (1)$$

Let $\varepsilon > 0$. Choose $\delta = \delta_0 > 0$

Suppose that $|x - x_0| < \delta$.

$$\begin{aligned} |[cf](x) - [cf](x_0)| &= |c(f(x)) - c \cdot (f(x_0))| \\ &= |c(f(x) - f(x_0))| \\ &\le |c|\,|f(x) - f(x_0)| \\ &\le |c| \cdot \frac{\varepsilon}{|c|} = \varepsilon \end{aligned}$$

Thus, $[cf] \in \mathbb{R}^{[0,1]}$.

Statement 3 : The set of differentiable real-valued functions on \(\mathbb{R}\) is a subspace of \(\mathbb{R}^{\mathbb{R}}\).

Proof of Statement 3

(c) The set of differentiable real-valued functions on $\mathbf{R}$ is a subspace of $\mathbf{R}^{\mathbf{R}}$.

- 0_{map} is differntible. So, $0_{map} \in \mathbb{R}^{\mathbb{R}}$
- Since, sum of differntiable functions are differntiable real-value function.
  If $f, g \in \mathbb{R}^{\mathbb{R}}$ then $f+g \in \mathbb{R}^{\mathbb{R}}$.
- If $c \in \mathbb{R}$ and $f \in \mathbb{R}^{\mathbb{R}}$, then $cf \in \mathbb{R}^{\mathbb{R}}$. Because (cf) is differntiable.

Statement 4 : The set of differentiable real-valued functions \(\mathbb{R}\) on the interval (0, 3) such that \(f'(2)=b\) is a subspace of \(\mathbb{R}^{(0,3)}\) if and only if \(b = 0\).

Proof of Statement 4

(d) The set of differentiable real-valued functions f on the interval $(0,3)$ such that $f'(2) = b$ is a subspace of $\mathbf{R}^{(0,3)}$ if and only if $b = 0$.

Let $U := \{f : (0,3) \to \mathbb{R} \mid f'(2) = b\}$

NTS: U is subspace of $\mathbb{R}^{(0,3)} \Leftrightarrow b = 0$

"$\Rightarrow$"

Suppose that U is subspace of $\mathbb{R}^{(0,3)}$

Then, $0_{map} \in U$. Then $[(0_{map})'](2) = 0_{map}(2) = 0$

"$\Leftarrow$"

Now suppose that, $b = 0$.

- Observe that 0_{map} is differentiable and $[(0_{map})'](2) = [0_{map}](2) = 0$
- Let $f, g \in U$. Then f, g are differentiable and $f'(2) = g'(2)$

  Then $f+g$ are differentiable.

  Further, $[f+g](2) = f(2) + g(2) = 0 + 0 = 0$.

  Thus, $f+g \in U$.
- Let $f \in U$ and $a \in \mathbb{R}$. So $f'(2) = 0$.

  Then af are differentiable. Moreover, $[(af)'](2) = a(f'(2)) = a \cdot 0 = 0$. Thus $af \in U$.

Therefore, U is a subspace

Statement 5 : The set of all sequences of complex numbers with limit \(0\) is a subspace of \(\mathbb{C}^{\infty}\)

Proof of Statement 5:

(e) The set of all sequences of complex numbers with limit 0 is a subspace of $\mathbf{C}^\infty$.

Let $U := \{ (x_n)_{n\in\mathbb{N}} \mid x_n \in \mathbb{C}$ and $x_n \to 0$ as $n \to \infty \}$

- Consider, a sequnce with all components are 0. Observe that sequnce $(0,0,0,\dots) \in U$.
- Let $(x_n), (y_n) \in U$. So,
  $\lim_{n\to\infty} x_n = 0$ and $\lim_{n\to\infty} y_n = 0$.
  Then, $\lim_{n\to\infty} (x_n + y_n) = \lim_{n\to\infty} x_n + \lim_{n\to\infty} y_n = 0$
  Therefore, $(x_n + y_n)_{n\in\mathbb{N}} \in U$.
- Let $a \in \mathbb{C}$ and $(x_n)_{n\in\mathbb{N}}$. So, $\lim_{n\to\infty} x_n = 0$.
  Now observe
  $\lim_{n\to\infty} c x_n = c \lim_{n\to\infty} x_n = c \cdot 0 = 0.$
  Thus, $cf \in U$.

Therefore, U is a subspace.

3.

3 Show that the set of differentiable real-valued functions f on the interval $(-4,4)$ such that $f'(-1) = 3f(2)$ is a subspace of $\mathbf{R}^{(-4,4)}$.

Let $U := \{ f : (-4,4) \to \mathbb{R} : f'(-1) = 3f(2) \}$

• ~~Let~~ Consider $0_{map} : (-4,4) \to \mathbb{R}$ defined like
$x \mapsto 0$

$$[(0\,map)'](-1) = 0 = 3 \cdot 0 = 3\,(0_{map}(2))$$

• Let $f, g \in U$. So, $f'(-1) = 3f(2)$ and
$g'(-1) = 3g(2)$

addition of
Hence, $[(f+g)'](-1) = [f' + g'](-1) = f'(-1) + g'(-1) = 3f(2) + 3g(2)$
$= 3(f(2) + g(2))$

Thus, $(f+g) \in U$.

• Let $a \in \mathbb{R}$ and $f \in U$. Then $f'(-1) = 3f(2)$
Then, $[(af)'](-1) = a(f'(-1)) = a(3f(2)) = 3(af(2))$
$= 3([af](2))$

So, $af \in U$.

Therefore by def^n of subspace of U.

4 Suppose $b \in \mathbf{R}$. Show that the set of continuous real-valued functions f on the interval $[0,1]$ such that $\int_0^1 f = b$ is a subspace of $\mathbf{R}^{[0,1]}$ if and only if $b = 0$.

Suppose $b \in \mathbb{R}$.

Let $A := \left\{ f : [0,1] \to \mathbb{R} \;\middle|\; \begin{array}{l} f \text{ is continous on } [0,1] \text{ and} \\ \int_0^1 f = b \end{array} \right\}$

NTS: A is subspace of $\mathbb{R}^{[0,1]} \iff b = 0$

- "$\Rightarrow$" First suppose that A is subspace of $\mathbb{R}^{[0,1]}$

  Since A is subspace addtive identity exists

  Let's say that "0_{map}".

$$b = \int_0^1 0_{map}(x)\,dx = \int_0^1 0\,dx = 0$$

- "$\Leftarrow$". Now suppose that $b=0$.

  – Addtive identity

  Recall "0_{map}" is the addtive identity of $\mathbb{R}^{[0,1]}$

  Then observe, $\int_0^1 0_{map}(x)\,dx = \int_0^1 0\,dx = 0 = b$

  Thus, $0_{map} \in A$.

4.

- Closed under addtion.

Let $f, g \in A$. This gives

$$\int_0^1 f(x)\,dx = 0 \quad \text{---} (1)$$

and $$\int_0^1 g(x)\,dx = 0 \quad \text{---} (2)$$

So we have,

$$\int_0^1 [f+g](x)\,dx = \int_0^1 f(x) + g(x)\,dx$$
$$= \underbrace{\int_0^1 f(x)\,dx}_{0} + \underbrace{\int_0^1 g(x)\,dx}_{0}$$
$$= 0 + 0$$
$$= 0$$

Thus, $f+g \in A$.

- Closed under Scaler multiplication.

Let $\lambda \in \mathbb{R}$ and $f \in U$. Then it statisfy $\int_0^1 f(x)dx = 0$

So observe,

$$\int_0^1 [\lambda f](x)\,dx = \int_0^1 \lambda f(x)\,dx = \lambda \int_0^1 f(x)\,dx = \lambda \cdot 0 = 0$$

Thus $\lambda f \in A$.

Thus A is subspace of $\mathbb{R}^{[0,1]}$.

5 Is $\mathbf{R}^2$ a subspace of the complex vector space $\mathbf{C}^2$?

$\mathbb{R}^2$ is NOT a subspace of $\mathbb{C}^2$.
Becaus it fails closed under scaler multiplication.

Note that $(1,1) \in \mathbb{R}^2$ and
$i(1,1) = (i, i) \notin \mathbb{R}^2$

5.

6 (a) Is $\{(a, b, c) \in \mathbf{R}^3 : a^3 = b^3\}$ a subspace of $\mathbf{R}^3$?
(b) Is $\{(a, b, c) \in \mathbf{C}^3 : a^3 = b^3\}$ a subspace of $\mathbf{C}^3$?

a)
Let $U := \{(a,b,c) \in \mathbb{R}^3 \mid a^3 = b^3\}$.
claim a) : U is a subspace of $\mathbb{R}^3$

- Addtive identity: Note that $(0,0,0) \in U$.
(Because $0^3 = 0^3 = 0$)
- Closed Under addtion
Let $u = (u_1, u_2, u_3)$, $v = (v_1, v_2, v_3) \in U$
Then $u_1^3 = u_3^3$ and $v_1^3 = v_3^3$
So, $u_1 = u_3$ and $v_1 = v_2$ (See the following claim)

Thus $u+v = (u_1+v_1, u_2+v_2, u_3+v_3)$
Note that $(u+v_1)^3 = (u_3+v_3)^3$
Therefore, $u+v \in \mathbb{R}^3$
- <u>closed under scaler multiplication</u>
Let $\lambda \in \mathbb{R}$ and $u = (u_1, u_2, u_3) \in U$. Note that.
$u_1^3 = u_3^3 \Rightarrow \lambda u_1^3 = \lambda u_3^3$.
$\lambda u = \lambda(u_1, u_2, u_3) = (\lambda u_1, \lambda u_2, \lambda u_3) \in U$

Claim: $\forall a, b \in \mathbb{R}$, $a^3 = b^3 \iff a = b$

<u>proof of claim:</u> Let $a, b \in \mathbb{R}$.

" $\Longrightarrow$ "

Suppose that $a^3 = b^3$

<u>Case-I</u>: If $a = 0$ or $b = 0$, then it is trivial that $a = b = 0$.

<u>Case-II</u>: If $a \neq 0$ and $b \neq 0$, then

$$0 = (a^3 - b^3) = (a-b)(a^2 + ab + b^2)$$

This implies $a - b = 0 \Rightarrow a = b$. (subclaim)

<u>subclaim: $(a^2 + ab + b^2) > 0$</u>

$$a^2 + ab + b^2 = \left(a + \frac{1}{2}b\right)^2 - \frac{1}{4}b^2 + b^2$$

$$= \left(a + \frac{1}{2}b\right)^2 + \frac{3}{4}b^2 > 0$$

"$\Leftarrow$" Suppose that $a=b$. Now consider follw

$$0=(a-b)=(a-b)^3=(a^3-3a^2b+3ab^2-b^3)$$
$$=(a^3-3a^3+3b^3-b^3)=-2a^3+2b^3=2(-a^3+b^3)$$

So, $-a^3+b^3=0 \Rightarrow b^3=a^3$

6b) Claim: $U_2=\{(a,b,c)\in\mathbb{C}^3 : a^3=b^3\}$ is NOT subspace of $\mathbb{C}^3$

counterexample: Easits way is deal with 3rd rooot unit. $\left(1, \frac{1}{2}-\frac{\sqrt{3}}{2}i, \frac{1}{2}+\frac{\sqrt{3}}{2}i\right)$

Notethat

$$\left(1, \frac{1}{2}+\frac{\sqrt{3}}{2}i, 0\right) \left(1, \frac{1}{2}-\frac{\sqrt{3}}{2}i, 0\right) \in U_2$$

Because, $\left(\frac{1}{2}+\frac{\sqrt{3}}{2}i\right)^3=1$ and $\left(\frac{1}{2}+\frac{\sqrt{3}}{2}i\right)^3=1$

$$\left(1, \frac{1}{2}+\frac{\sqrt{3}}{2}i, 0\right) \left(1, \frac{1}{2}-\frac{\sqrt{3}}{2}i, 0\right) = (2, 1, 0)$$

But $8=2^3\neq 1^3=1$.

Thefore U_2 is subspace of $\mathbb{C}^3$.

7 Prove or give a counterexample: If U is a nonempty subset of $\mathbf{R}^2$ such that U is closed under addition and under taking additive inverses (meaning $-u \in U$ whenever $u \in U$), then U is a subspace of $\mathbf{R}^2$.

Counter example

Let $U := \{(a,b) \in \mathbb{R}^2 \mid a, b \in \mathbb{Z}\}$

First Note that $(1,1) \in U$. Thus, $U \neq \emptyset$.

Claim 1: U is closed under addtion

Let $(x,y), (x',y') \in U$ then $x, y, x', y' \in \mathbb{Z}$

Note that $x+y, x'+y' \in \mathbb{Z}$.

Thus, $(x,y)+(x',y') = (x+x', y+y') \in U$

claim 2: U is closed under additive inverse.

Let $(a,b) \in U$. Then $a,b \in \mathbb{Z}$. Then $-a, -b \in \mathbb{Z}$.
So, $(-a,-b) \in U$. Further,

$$(a,b)+(-a,-b) = (a-a, b-b) = (0,0) \in U$$

But U is NOT subspace. because
$(1,1) \in U \subseteq \mathbb{R}^2$ and $1/\sqrt{3} \in \mathbb{R}$, but $\frac{1}{\sqrt{3}}(1,1) = (\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}) \notin U$

8 Give an example of a nonempty subset U of $\mathbf{R}^2$ such that U is closed under scalar multiplication, but U is not a subspace of $\mathbf{R}^2$.

$U := \{ (a,b) \in \mathbb{R}^2 \mid ab = 0 \}$

claim: <u>U is closed under scaler multiplication</u>

Let $\lambda \in \mathbb{R}$ and $(x,y) \in U$

$\lambda(x,y) = (\lambda x, \lambda y) \in U$ (since $ab = 0 \Rightarrow \lambda ab = 0$)

But U is <u>not</u> a closed under scaler multiplicatio

because $(1,0), (0,1) \in U$. but

$(1,0) + (0,1) = (1,1) \notin U$ (becaue $1 \cdot 1 = 1 \neq 0$)

9 A function $f: \mathbf{R} \to \mathbf{R}$ is called *periodic* if there exists a positive number p such that $f(x) = f(x+p)$ for all $x \in \mathbf{R}$. Is the set of periodic functions from $\mathbf{R}$ to $\mathbf{R}$ a subspace of $\mathbf{R}^{\mathbf{R}}$? Explain.

Let $V := \{ f : \mathbb{R} \to \mathbb{R} \mid f \text{ is periodic} \}$

But V is NOT a subspace of $\mathbb{R}^{\mathbb{R}}$

before construct counter example let's go through following rough work.

Let f, g are periodic with period P_1, P_2 respectively. i.e: $f(x+P_1) = f(x)$ and

$g(x+P_2) = g(x)$

$$\frac{P_1}{P_2} \in \mathbb{Q} \iff \frac{P_1}{P_2} = \frac{n_1}{n_2} \text{ for some } n_1, n_2 \in \mathbb{Z}$$

$$\iff P_1 n_2 = P_2 n_1$$

$$\iff [f+g](x+P_1n_2) = f(x+P_1n_2) + g(x+P_1 n_2)$$

$$= f(x+P_1n_2) + g(x+P_2n_1)$$

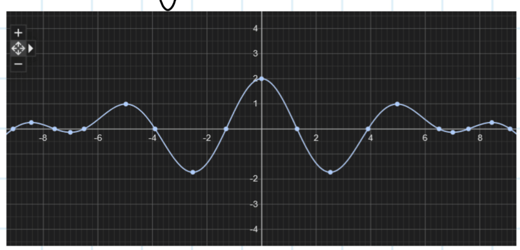
$$= f(x) + g(x)$$

$$= [f+g](x)$$

So, what we need to get ratio of periods should not be rational. Let's consider

$f(x) = \cos(\sqrt{2}x)$ have period $\sqrt{2}\pi$
$g(x) = \cos(x)$ have period 2π

$h(x) = [f+g](x) = f(x) + g(x) = \cos(\sqrt{2}x) + \cos(x)$



claim: h is not periodic function
Assume that contray, h has period p. $(p > 0)$
i.e: $h(x+p) = h(x)$

$$h(0) = h(0+p)$$
$$h(0) = h(p)$$
$$\cos(0) + \cos(0) = \cos(\sqrt{2}p) + \cos(p)$$
$$2 = \cos(\sqrt{2}p) + \cos p$$

Note that 1 is upper bound of Cos .
that implies,

$$p = 2\pi k \quad \text{for some } k \in \mathbb{Z}$$
$$\sqrt{2}p = 2\pi l \quad \text{for some } l \in \mathbb{Z}$$

Thus,

$$p = \cancel{2\pi} k = \frac{\cancel{2\pi} l}{\sqrt{2}}$$
$$\frac{l}{k} = \sqrt{2} \#$$

Since $l, k \in \mathbb{Z}$, $l/k \neq \sqrt{2}$. Therefore this is a contradiction. Thus h is not periodic function.

10 Suppose V_1 and V_2 are subspaces of V. Prove that the intersection $V_1 \cap V_2$ is a subspace of V.

- Non-empty.: Note that $0 \in V_1$ and $0 \in V_2$. Then $0 \in V_1 \cap V_2$. Thus $V_1 \cap V_2 \neq \emptyset$
- Addative identity.. perivoalsy, we proved that $0 \in V_1 \cap V_2$.
- Closed under addition: Let $u \in V_1 \cap V_2$ and $w \in V_1 \cap V_2$.

Thus, $u \in V_1$ and $u \in V_2$
$w \in V_1$ and $w \in V_2$

Hence $u + w \in V_1$ and $u + w \in V_2$

Therfore, $u + w \in V_1 \cap V_2$

Closed under scaler multiplication

Let $u \in V_1 \cap V_2$ and $\lambda \in \mathbb{F}$. Then

$$\left.\begin{array}{l} u \in V_1 \Rightarrow \lambda u \in V_1 \\ u \in V_2 \Rightarrow \lambda u \in V_2 \end{array}\right\} \Rightarrow \lambda u \in V_1 \cap V_2$$

Thus, $V_1 \cap V_2$ is a subspace of V

11 Prove that the intersection of every collection of subspaces of V is a subspace of V.

Let $\{U_\alpha\}_\alpha$ be an arbitary collection of subspaces of V. here $\alpha \in I$, I- index set

- non-emptyness: $0 \in U_\alpha$ $\forall \alpha \in I$. Thus

  $0 \in \bigcap_{\alpha \in I} U_\alpha$. Thus $\bigcap_{\alpha \in I} U_\alpha \neq \emptyset$

- additive identity: Perivosly prooved that $0 \in \bigcap_{\alpha \in I} U_\alpha$.

- Closed under addition: Let $v, w \in \bigcap_{\alpha \in I} U_\alpha$. Thu

  $$\left.\begin{array}{l} v \in U_\alpha \text{ for all } \alpha \in I \\ w \in U_\alpha \text{ for all } \alpha \in I \end{array}\right\} v + u \in U_\alpha \text{ for all } \alpha \in I$$

  Therefore $v + u \in \bigcap_{\alpha \in I} U_\alpha$.

- Closed under scaler multiplication

Let $\lambda \in \mathbb{F}$ and $u \in \bigcap_{\alpha \in I} U_\alpha$. Thus,

$u \in U_\alpha$ for $\alpha \in I$. $\Rightarrow \lambda u \in U_\alpha$ for all $\alpha \in I$

$\Rightarrow \lambda u \in \bigcap_{\alpha \in I} U_\alpha$.

12 Prove that the union of two subspaces of V is a subspace of V if and only if one of the subspaces is contained in the other.

Let U_1, U_2 subspace of V.

N.T.S: $U_1 \cup U_2$ are subspace of $V \Leftrightarrow U_1 \subseteq U_2$ or $U_2 \subseteq U_1$

Backward direction:

Without lost of generality assume that $U_1 \subseteq U_2$
Then $U_1 \cup U_2 = U_2$. We already know that U_2 is a subspace of V

Forward direction:

Suppose that $U_1 \cup U_2$ is a subspace. Assume the contrary $U_1 \nsubseteq U_2$ and $U_2 \nsubseteq U_1$. Thus $U_1 \setminus U_2 \neq \emptyset$ and $U_2 \setminus U_1 \neq \emptyset$. Then we can choose $v \in U_1 \setminus U_2 \subseteq U_1 \cup U_2$

$w \in U_2 \setminus U_1 \subseteq U_1 \cup U_2$

By our hypothesis, $v + w \in U_1 \cup U_2$ — 1

Then $v+w \in U_1$ and $v+w \in U_2$

Since U_1, U_2 are subspaces, $-v \in U_1$ and $-w \in U_2$

$\rightarrow$ If $(v+w) \in U_1$ then $(v+w)-v = w \in U_1$ but $w \in U_2 \setminus U_1$

$\rightarrow$ If $(v+w) \in U_2$ then $(v+w)-w = v \in U_1$ but $v \in U_1 \setminus U_2$

This gives that contradiction. Therefore

$U_1 \subseteq U_2$ or $U_2 \subseteq U_1$.

13 Prove that the union of three subspaces of V is a subspace of V if and only if one of the subspaces contains the other two.

This exercise is surprisingly harder than Exercise 12, possibly because this exercise is not true if we replace **F** *with a field containing only two elements.*

Let U_1, U_2, U_3 be subspace of V.
Let $U = U_1 \cup U_2 \cup U_3$.
NTS: U is subspace $\iff$ one of subspaces contain the other two

backward direction.
Without loss of generality suppose that $U_1 \cup U_2 \subseteq U_3$. Then $U_1 \cup U_2 \cup U_3 = U_3$.
We know that U_3 is a subspace.

Forward direction
Suppose that $U = U_1 \cup U_2 \cup U_3$ is subspace.
Assume the contray there is no subspace is subset either of other two. — $\circledast$

Let $w \in U_1 \setminus (U_2 \cup U_3)$. (We can choose such that elements by hypothesis $\circledast$).
Let $x \in U_2 \setminus U_1$ (by $\circledast$)
We know that $w, x \in U$.
Then $(w+x) \in U$ and $(w-x) \in U$
(Since U is a subspace.

~~If $(w+x) \in U_1$~~

Then $(w+x) \in U_1$ or $(w+x) \in U_2$ or $(w+x) \in U_3$.

If $(w+x) \in U_1$, then $(w+x) - w = x \in U_1$

This is ~~a contradiction~~. impossible

If $(w+x) \in U_2$ then $(w+x) - x = w \in U_2$

But this ~~contration~~ not possibl

Therefore, ~~only thing~~ $w+x \in U_3$ —— (**)

Then, $(w-x) \in U_1$ or $(w-x) \in U_2$ or $(w-x) \in U_3$

If $(w-x) \in U_1$, then $(w-x) - w = -x \in U_1$

This is ~~contradiction~~ impossibl

If $(w-x) \in U_2$ then $(w-x) + x = w \in U_2$

This is impossible.

Therefore $(w-x) \in U_3$ —— (***)

By (**) and (***). $(w-x)+(w-x) = 2w \in U_3$

This is contradiction. Thefore, at least one of subspace must be asubset of another.

14 Suppose

$$U = \{(x, -x, 2x) \in \mathbf{F}^3 : x \in \mathbf{F}\} \quad \text{and} \quad W = \{(x, x, 2x) \in \mathbf{F}^3 : x \in \mathbf{F}\}.$$

Describe $U + W$ using symbols, and also give a description of $U + W$ that uses no symbols.

$$U+W := \{(2x, 0, 4x) \in \mathbb{F}^3 \mid x \in \mathbb{F}\}$$

$$= \{2x(1, 0, 2) \in \mathbb{F}^3 \mid x \in \mathbb{F}\}$$

U+W is set of elements of $\mathbb{F}^3$
first element ~~and~~ is half of 3rd element
2nd element is zero.

15 Suppose U is a subspace of V. What is $U + U$?

Let U be as subspace of V.

$U+U = \{(u+v) \mid u, v \in U\}$

Claim: $U+U=U$

Subclaim1: $U+U \subseteq U$

Let $w \in U+U$. Then $w = u_1+u_2$ for some $u_1, u_2 \in U$. ~~The~~ Since U is subspace, $w = u_1+u_2 \in U$.

Subclaim2: $U \subseteq U+U$

Let $u \in U$. Since U is subspace of V, $0 \in U$ ~~and~~ then $u = u+0 \in U$

$U+U=U$. Thus U is a subspace of V $\square$

16 Is the operation of addition on the subspaces of V commutative? In other words, if U and W are subspaces of V, is $U + W = W + U$?

Let U_1, U_2 are subspaces of V.

claim: $U_1 + U_2 \subseteq U_2 + U_1$

Suppose $y \in U_1 + U_2$. Then $y = u_1 + u_2$ for some $u_1 \in U_1$, $u_2 \in U_2$

Since $u_1, u_2 \in V$. and V is vectorspace and it has commutative property. Thus,

$$u_1 + u_2 = u_2 + u_1$$

Thus $y \in U_1 + U_2 = u_2 + u_1 \in U_2 + U_1$

Thereofee $U_1 + U_2 \subseteq U_2 + U_1$.

Similary, we can show that $U_1 + U_2 \supseteq U_2 + U_1$.

Thefore, $U_1 + U_2 = U_2 + U_1$

17 Is the operation of addition on the subspaces of V associative? In other words, if V_1, V_2, V_3 are subspaces of V, is

$$(V_1 + V_2) + V_3 = V_1 + (V_2 + V_3)?$$

Suppose that U_1, U_2, U_3 are subspaces of V

NTS: $U_1 + (U_2 + U_3) = (U_1 + U_2) + U_3$

Claim) $V_1 + (V_2 + V_3) \subseteq (V_1 + V_2) + V_3$

Let $x \in U_1 + (U_2 + U_3)$. ~~Then $x = u_1 + u_2 + u_3 \in$~~

Then $x = u_1 + (u_2) + u_3)$ forsome $u_1 \in U_1$
$u_2 \in U_2$
$u_3 \in U_3$

Since $u_1, u_2, u_3 \in V$ and V is vector space, it has assciavity property.

~~Vectors of vectors associative~~. Then

$x = (u_1 + u_2) + u_3$

Then $x \in (V_1 + V_2) + V_3$.

Similary ~~sh~~ we can show that,

$V_1 + (V_2 + V_3) \supseteq (V_1 + V_2) + V_3$

Thefore, $V_1 + (V_2 + V_3) = (V_1 + V_2) + V_3$.

18 Does the operation of addition on the subspaces of V have an additive identity? Which subspaces have additive inverses?

claim1: Addtive identity is the subspace $\{0\}$ of V. Note that,

$$U+\{0\} = \{u+0 : u \in U\} = \{u : u \in U\} = U$$

Similarly we can show that $\{0\}+U = U$

Thefore, Addtive identity is $\{0\}$.

claim2: The subspace $\{0\}$ has addtive inverse

Assume that U is subspace of V and ~~W is~~ ~~it~~ U has addtive inverse W.

$$U+W = \{0\}.$$

Note that $0 \in U$. Then $0+w=0$ $\forall w \in W$

Then $W = \{0\}$.

Hence $\cancel{\{0\}+U}$ $U+\{0\} = \{0\}$

Thefore, $U = \{0\}$

19 Prove or give a counterexample: If V_1, V_2, U are subspaces of V such that

$$V_1 + U = V_2 + U,$$

then $V_1 = V_2$.

The counter example:

$$U_1 := \{(x,x) \in \mathbb{R}^2\}$$
$$U_2 := \{(x,2x) \in \mathbb{R}^2\}.$$
$$V := \mathbb{R}^2$$

claim 1: $U_1 + V = \mathbb{R}^2$

subclaim 1.1: $U_1 + V \supseteq \mathbb{R}^2$

Let $(x,y) \in \mathbb{R}^2$.

Then $(x,y) = (0,0) + (x,y) \in U_1 + V$

Thus, $U_1 + V \supseteq \mathbb{R}^2$.

otherway round is trival.

i.e: $U_1 + V \subseteq \mathbb{R}^2$.

Therefore, $U_1 + V = \mathbb{R}^2$.

Similary, we can show that

$$U_2 + V = \mathbb{R}^2.$$

Let's summrize.

$$U_1 + V = \mathbb{R}^2 = U_2 + V$$

but it is clear that, $U_1 \neq U_2$.

Therefore given statment is false.

Claim 0: V_1, V_2, V are subspace

- additive identity
  $(0,0) \in U_1, U_2, V$.
- closed under addition.
  $(x,x), (y,y) \in U_1 \Rightarrow (x+x, y+y) \in U_1$
  $(u,2u), (v,2v) \in U_2 \Rightarrow (u+v, 2(u+v)) \in U_2$
  $(w_1, w_2), (w_3, w_4) \in \mathbb{R}^2 \Rightarrow (w_1+w_2, w_3+w_4) \in \mathbb{R}^2$

• Closed under scaler multiplication

Let $\lambda \in F$.

$(u,u) \in U_1 \Rightarrow \lambda(u,u) = (\lambda u, \lambda u) \in U_1$

$(v,2v) \in U_2 \Rightarrow \lambda(v,2v) = (\lambda v, \lambda(2v)) = (\lambda v, 2(\lambda v)) \in U_2$

$(x,y) \in \mathbb{R}^2 \Rightarrow \lambda(x,y) = (\lambda x, \lambda y) \in \mathbb{R}^2$.

Hence U_1, U_2, V are subspaes

20 Suppose

$$U = \{(x, x, y, y) \in \mathbf{F}^4 : x, y \in \mathbf{F}\}.$$

Find a subspace W of $\mathbf{F}^4$ such that $\mathbf{F}^4 = U \oplus W$.

Let $W = \{(0, w_2, w_3, 0) \mid w_2, w_3 \in \mathbb{F}\}$

Now let's verify that $U \cap W = \{0\}$

~~Let~~ $\cancel{(u_1, u_1, v, v) \in U}$

~~Thus, observe that~~ $\cancel{(0,0,0,0) \in}$

Let $(a,b,c,d) \in U \cap W$

The $(a,b,c,d) \in U \Rightarrow a = b$ and $c = d$

$(a,b,c,d) \in W \Rightarrow a = 0$ and $c = 0$

Thefore, $(a,b,c,d) = (0,0,0,0) \in \{0\}$.

Thus $U \cap W \subseteq \{0\}$

Therefore, $U \cap W = \{0\}$

Now let's verify $U \oplus W = \mathbb{F}^4$

Let $(a,b,c,d) \in \mathbb{F}^4$

~~$(a,b,c,d) = (a, b-a, c-d, d)$~~

~~$= (a,a,d,d) + (0, b-a\ c$~~

$(a,b,c,d) = (a, b-a+a, c-d+d, d)$

$= (a,a,d,d) + (0, b-a, c-\cancel{d},)$

Note that $(a,a,d,d) \in U$ and

$(0, b-a, c-d, 0) \in W$

Thus, $\mathbb{F}^4 \subseteq U \oplus W$

It is trivial that $\mathbb{F}^4 \supseteq U \oplus W$.

Hence, $U \oplus W = \mathbb{F}^4$

21 Suppose

$$U = \{(x, y, x + y, x - y, 2x) \in \mathbf{F}^5 : x, y \in \mathbf{F}\}.$$

Find a subspace W of $\mathbf{F}^5$ such that $\mathbf{F}^5 = U \oplus W$.

Let $W = \{(0, 0, a, b, c) \in \mathbb{F}^5 \mid a, b, c \in \mathbb{F}\}$

Claim 1: $U \cap W = \{0\}$

Let $(x_1, x_2, x_3, x_4, x_5) \in U \cap W$

Since $(x_1, x_2, x_3, x_4, x_5) \in W$ then

$x_1 = 0$, $x_2 = 0$

Since $(x_1, x_2, x_3, x_4, x_5) \in U$ then

$x_3 = x_1 + x_2 = 0 + 0 = 0$

$x_4 = x_1 - x_2 = 0 - 0 = 0$

$x_5 = 2x_1 = 2 \cdot 0 = 0$

Therefore $(x_1, x_2, x_3, x_4, x_5) = (0, 0, 0, 0, 0) \in \{0\}$

Thus, $\cancel{(x_1, x_2, x_3}$ $U \cap W \subseteq \{0\}$.

It is trivial ~~date~~ that $U \cap W \supseteq \{0\}$.

Therefore, $U \cap W = \{0\}$

Claim 2: $U \oplus W = \mathbb{F}^5$

~~Let $(x_1, x_2, x_3, x_4, x_5) \in$~~

Let $(y_1, y_2, y_3, y_4, y_5) \in \mathbb{F}^5$

$(y_1, y_2, y_3, y_4, y_5) = (y_1, y_2, y_3 - y_1 - y_2, y_4 - y_1 + y_2, y_5 - 2y_1)$
$+ (0, 0, y_1 + y_2, y_1 - y_2, 2y_1)$

~~The~~ Here,

$$(x_1, x_2, x_3 - x_1 - x_2, x_4 - x_1 + x_2, x_5 - 2x_1) \in U$$

and

$$(0, 0, x_1 + x_2, x_1 - x_2, 2x_1) \in W$$

Thus, $\mathbb{F}^5 \subseteq U + W$. It is trivial that

$$\mathbb{F}^5 = U + W.$$

22 Suppose

$$U = \{(x, y, x + y, x - y, 2x) \in \mathbf{F}^5 : x, y \in \mathbf{F}\}.$$

Find three subspaces W_1, W_2, W_3 of $\mathbf{F}^5$, none of which equals $\{0\}$, such that $\mathbf{F}^5 = U \oplus W_1 \oplus W_2 \oplus W_3$.

Let $W_1 := \{(x,0,0,0,0) \in \mathbb{F}^5 \mid x \in \mathbb{F}\}$

$W_2 := \{(0,x,0,0,0) \in \mathbb{F}^5 \mid x \in \mathbb{F}\}$

$W_3 := \{(0,0,x,0,0) \in \mathbb{F}^5 \mid x \in \mathbb{F}\}$

Suppose that

$$(0,0,0,0,0) = (a_1, a_2, a_1+a_2, a_1-a_2, 2a_1) + (a_3,0,0,0,0) + (0,a_4,0,0,0) + (0,0,a_5,0,0)$$

$$(0,0,0,0,0) = (a_1+a_3, a_2+a_4, a_1+a_2+a_5, a_1-a_2, 2a_1)$$

We can see that $a_1 = 0$.

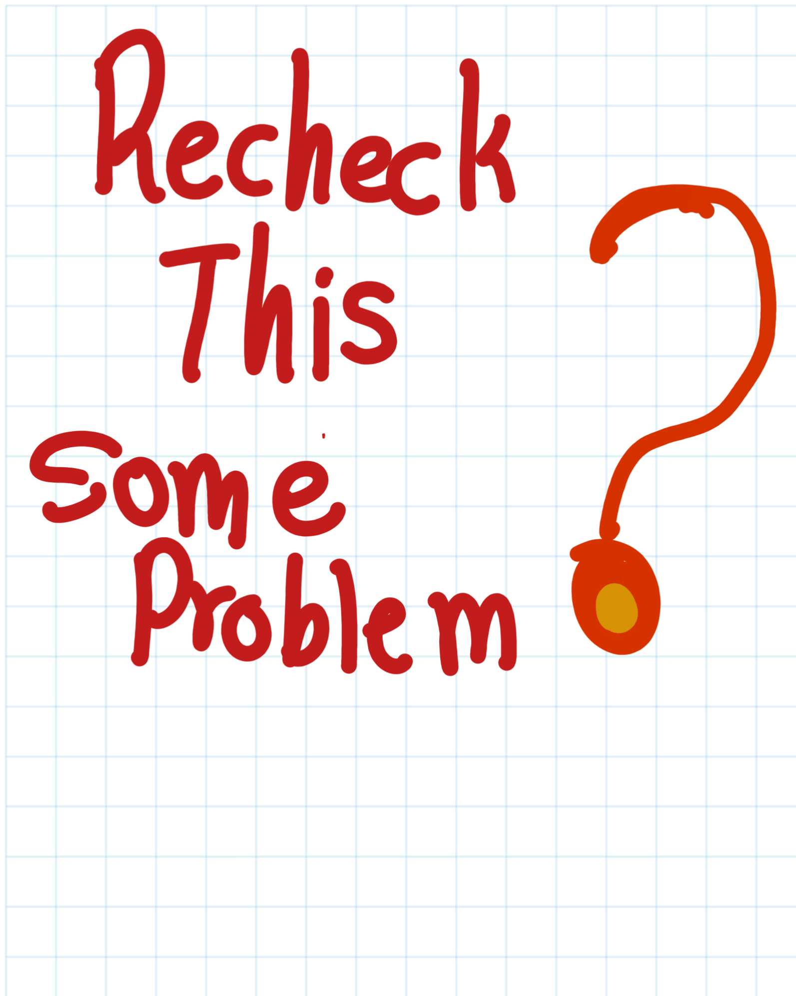
$a_1 - a_2 = 0 \Rightarrow a_2 = 0$.

$a_1 + a_2 + a_5 = 0 \Rightarrow a_5 = 0$

$a_2 + a_4 = 0 \Rightarrow a_4 = 0$

$a_1 + a_3 = 0 \Rightarrow a_3 = 0$

Theefore, the only way to write 0 as a sum $U + W_1 + W_2 + W_3$, where each $u \in U$, $w_k \in W_k$ $k = 1,2,3$. is by taking each $u = w_1 = w_2 = w_3 = 0$. Theerefore the sum $U + W_1 + W_2 + W_3$ is a direct sum.


Recheck
This
Some
Problem

23 Prove or give a counterexample: If V_1, V_2, U are subspaces of V such that

$$V = V_1 \oplus U \quad \text{and} \quad V = V_2 \oplus U,$$

then $V_1 = V_2$.

Hint: When trying to discover whether a conjecture in linear algebra is true or false, it is often useful to start by experimenting in $\mathbf{F}^2$.

Let $V := \mathbb{R}^2$,
$V_1 := \{(0, x) \in \mathbb{R}^2 \mid x \in \mathbb{R}\}$
$V_2 := \{(x, x) \in \mathbb{R}^2 \mid x \in \mathbb{R}\}$
$U := \{(x, 0) \in \mathbb{R}^2 \mid x \in \mathbb{R}\}$

claim 1: $V_1 + U = \mathbb{R}^2 = V$
Let $(x, y) \in \mathbb{R}^2$,
$(x, y) = (0, x) + (x, 0) \in V_1 + U$
Since ~~Note that~~ $(0, x) \in V_1$ and $(x, 0) \in U$. Thus $\mathbb{R}^2 \subseteq V_1 + U$
Thus, $V_1 + U = V$ ——— (*).

claim2: $V_2 + U = \mathbb{R}^2 = V$
Let $(\bar{x}, \bar{y}) \in \mathbb{R}^2$
$(\bar{x}, \bar{y}) = (\bar{x}, \bar{x}) + (0, \bar{y} - \bar{x}) \in V_2 + U$
Since, $(\bar{x}, \bar{x}) \in V_2$ and $(0, \bar{y} - \bar{x}) \in U$
~~It is trivial that~~ Thus, $V_2 + U \supseteq \mathbb{R}^2$
Thefore. $V_2 + U = \mathbb{R}^2 = V$ ——— (**).

claim3: $V_1 \cap U = \{0\}$.
Let $(a, b) \in V_1 \cap U$. Then since $(a, b) \in V_1$, $a = 0$
Since $(a, b) \in U$ then $b = 0$. Therefore, ~~$(0,0) \in$~~
$(a, b) = (0, 0) \in \{0, 0\}$. Thus. $V_1 \cap U \subseteq \{0\}$
Therefore, $V_1 \cap U = \{0\}$. ——— (***)

Claim 4: $V_2 \cap U = \{0\}$.

Let $(c,d) \in V_2 \cap U$. ~~Then~~ Since $(c,d) \in V_2$, then $c=d$

Since $(c,d) \in U$, then $d=0$. Thus,

$(c,d)=(0,0) \in \{0\}$. Thus, $V_2 \cap U \subseteq \{0\}$.

Thefore, $V_2 \cap U = \{0\}$. ————— ****

Therefore $V_1 \oplus U = \mathbb{R}^2 = V = V_2 \oplus U$.

But $V_1 \neq V_2$ (Since $(2,2) \in V_2$, $(0,3) \in V_1$
$(2,2) \notin V_2$, $(0,3) \notin V_1$)

24 A function $f\colon \mathbf{R} \to \mathbf{R}$ is called *even* if

$$f(-x) = f(x)$$

for all $x \in \mathbf{R}$. A function $f\colon \mathbf{R} \to \mathbf{R}$ is called *odd* if

$$f(-x) = -f(x)$$

for all $x \in \mathbf{R}$. Let V_e denote the set of real-valued even functions on $\mathbf{R}$ and let V_o denote the set of real-valued odd functions on $\mathbf{R}$. Show that $\mathbf{R}^{\mathbf{R}} = V_e \oplus V_o$.

Claim 1: V_e, V_o is a subspace

- Addtive identity: Zero function is the addtive identity. ~~map~~. ~~Not~~ i.e:

$$0_{map} : \mathbb{R} \to \mathbb{R}$$
$$x \mapsto 0$$

$$0_{map}(x) = 0 = 0_{map}(-x)$$

$$-0_{map}(x) = 0 = 0_{map}(-x)$$

• Closed under addition

Let $g_1 g_2 \in V_e$ and $h_1, h_2 \in V_o$

$g_1(x) = g_1(-x) \ \forall x \in \mathbb{R}$

$g_2(x) = g_2(-x) \ \forall x \in \mathbb{R}$

Then $\forall x \in \mathbb{R}$

$[g_1 + g_2](x) = g_1(x) + g_2(x)$

$= g_1(-x) + g_2(-x)$

$= [g_1 + g_2](x)$

Thus, $g_1 + g_2 \in V_e$

$h_1(-x) = -h_1(x) \ \forall x \in \mathbb{R}$

$h_2(-x) = -h_2(x) \ \forall x \in \mathbb{R}$

Then $\forall x \in \mathbb{R}$.

$[h_1 + h_2](-x) = h_1(-x) + h_2(-x)$

$= (-h_1(x) - h_2(x))$

$= -[h_1 + h_2](x)$

Thus $h_1 + h_2 \in V_o$

Therefore, V_e and V_o are closed under addition.

• Cosled under scaler multiplication

Let $\lambda \in \mathbb{F}$, $g \in V_e$, $h \in V_o$, Then $\forall x \in \mathbb{R}$.

$g(x) = g(-x)$

$[\lambda g](-x) = \lambda g(-x)$

$= \lambda g(x)$

$= [\lambda g](x)$

$h_1(-x) = -h_1(x)$

$[\lambda h_1](-x) = \lambda \cancel{h_1}(-x)$

$= \lambda(-h_1(x))$

$= -\lambda h_1(x)$

$= -[\lambda h_1](x)$

Thus, V_e and V_o are closed under scaler mulitiplication.

~~Le~~ Therefore V_e and V_o are subspaces.

Now N.T.S: ~~$V_o + V_e$~~ $V_o + V_e = \mathbb{R}^{\mathbb{R}}$

Claim: $\mathbb{R}^{\mathbb{R}} \subseteq V_o + V_e$

Let $f \in \mathbb{R}^{\mathbb{R}}$. Define $g, h \in \mathbb{R}^{\mathbb{R}}$

$$g(x) = \frac{f(x) - f(-x)}{2} \quad \forall x \in \mathbb{R}$$

$$h(x) = \frac{f(x) - f(-x)}{2} \quad \forall x \in \mathbb{R}$$

Note that

$$g(x) + h(x) = \frac{f(x) - f(-x)}{2} + \frac{f(x) - f(-x)}{2}$$

$$= f(x) \quad \forall x \in \mathbb{R}.$$

Furthur note that g is even function.

$$g(-x) = \frac{f(-x) - f(-(-x))}{2} = \frac{f(-x) + f(x)}{2} = g(x).$$

Furthur note that h is oddfuncti

$$h(-x) = \frac{f(-x) - f(-(-x))}{2} = \frac{-f(x) + f(-x)}{2}$$

$$= -\left(\frac{f(x) - f(-x)}{2}\right) = -h(x)$$

Thus h is odd function. ($h \in V_o$)

Therefore, $V_o + V_e \supseteq \mathbb{R}^{\mathbb{R}}$. Otherway around is trival. (i.e: $V_e + V_o \subseteq \mathbb{R}^{\mathbb{R}}$)

Thus, $V_e + V_o = \mathbb{R}^{\mathbb{R}}$

Finally, we need to show that $\mathbb{R}^{\mathbb{R}} = V_e \oplus V_o$

Claim: $V_e \cap V_o = \{0\}$

Let $f \in V_e \cap V_o$.

$f \in V_e \Rightarrow f(-x) = f(x) \quad \forall x \in \mathbb{R}$ — (1)

$f \in V_o \Rightarrow f(-x) = -f(x) \quad \forall x \in \mathbb{R}$ — (2)

By (1) and (2) $\quad f(x) = 0 \quad \forall x \in \mathbb{R}$

Thus $f = 0_{map} \in \{0\}$. Thus $V_e \cap V_o \subseteq \{0\}$

It is trivial that $V_e \cap V_o \supseteq \{0\}$

Therefore $V_e \cap V_o = \{0\}$

Therefore $\mathbb{R}^{\mathbb{R}} = V_e \oplus V_o$

Chapter 3

Exercise 02 (Finite-Dimensional Vector Spaces)

3.1 Exercise 2A

1 Find a list of four distinct vectors in $\mathbf{F}^3$ whose span equals

$$\{(x, y, z) \in \mathbf{F}^3 : x + y + z = 0\}.$$

Let $\gamma_1 = (1, 0, 0)$

$\gamma_2 = (0, 1, 0)$

$\gamma_3 = (0, 2, 1)$

$\gamma_4 = (0, 1, 1)$.

~~Then~~ Claim: $\gamma_1, \gamma_2, \gamma_3, \gamma_4$ are linearly independent

Suppose that $a, b, c, d \in \mathbb{F}$ such that

$$a\gamma_1 + b\gamma_2 + c\gamma_3 + d\gamma_4 = 0$$

$$a(1,0,0) + b(0,1,0) + c(0,2,1) + d(0,1,1) = (0,0,0)$$

$$(a, b+2c+d, c+d) = (0,0,0)$$

Thus $a = 0$, $b + 2c + d = 0$ — (1), $c + d = 0$ — (2)

Furthar $a + b + 2c + d + c + d = b + 3c + 2d = 0$ — (3)

(3) − (1) $d = 0$. Then $c = 0$. Thus $\gamma_1, \gamma_2, \gamma_3, \gamma_4$ are linearly independent.

3 Suppose $v_1, \ldots, v_m$ is a list of vectors in V. For $k \in \{1, \ldots, m\}$, let

$$w_k = v_1 + \cdots + v_k.$$

Show that $\operatorname{span}(v_1, \ldots, v_m) = \operatorname{span}(w_1, \ldots, w_m)$.

Claim 1: $\operatorname{span}(w_1, w_2, \ldots, w_m) \subseteq \operatorname{span}(v_1, \ldots, v_m)$

Let $a \in \operatorname{span}(w_1, \ldots, w_m)$. Then there exist $a_1, a_2, \ldots, a_m \in \mathbb{F}$ such that

$$\begin{aligned} a &= a_1 w_1 + a_2 w_2 + \cdots + a_m w_m \\ &= a_1 v_1 + a_2 (v_1 + v_2) + \cdots + a_m (v_1 + \cdots + v_m) \\ &= (a_1 + \cdots + a_m) v_1 + (a_2 + \cdots + a_m) v_2 \\ &\quad + \cdots + a_m v_m \end{aligned}$$

Thus $a \in \operatorname{span}(v_1, v_2, \ldots, v_m)$

Hence $\operatorname{span}(w_1, \ldots, w_m) \subseteq \operatorname{span}(v_1, \ldots, v_m)$ — (*)

Claim 2: $\operatorname{span}(v_1, \ldots, v_m) \subseteq \operatorname{span}(w_1, \ldots, w_m)$

Let $b \in \operatorname{span}(v_1, \ldots, v_m)$. Then there exist $b_1, b_2, \ldots, b_m \in \mathbb{F}$ such that

$b = b_1 v_1 + b_2 v_2 + \cdots + b_m v_m$

We re writere $v_1, \ldots, v_m$ as follows.

$v_1 = w_1$, $v_2 = (w_2 - w_1)$, $v_3 = (w_3 - w_2)$, $\ldots$,

$v_m = w_m - w_{m-1}$

Then

$$b = b_1 w_1 + b_2 (w_2 - w_1) + \cdots + b_m (w_m - w_{m-1})$$

$$= (b_1 - b_2) w_1 + \cdots + (b_m) w_m$$

Thus, $b \in \operatorname{span}(w_1, w_2, \ldots, w_m)$

Therefore, $\operatorname{span}(v_1, \ldots, v_m) \subseteq \operatorname{span}(w_1, \ldots, w_m)$ — (**)

~~By (*) and~~

By claim (1) and (2) such that

$$\operatorname{span}(v_1, v_2, \ldots, v_m) = \operatorname{span}(w_1, \ldots, w_m)$$

4 (a) Show that a list of length one in a vector space is linearly independent if and only if the vector in the list is not 0.

a) Let V be a vector space.
Let $\gamma \in V$ with $\gamma \neq 0$.

Forward direction: This is trivial.
Suppose that there exist $a \in \mathbb{F}$ such that
$a\gamma = 0$.
By Exercise 1B ② and $\gamma \neq 0$, we get $a = 0$.
Thus ~~it is~~ it is linearly independent.

Backward direction:
Suppose that $\gamma \in V$ is linearly independent
~~Then $\gamma = 0$. Because otherwis~~
Assume contray $\gamma = 0$. Then $5 \cdot \gamma = 0$
This contradicti linear independent. Thus $\gamma \neq 0$.

(b) Show that a list of length two in a vector space is linearly independent if and only if neither of the two vectors in the list is a scalar multiple of the other.

b) Let V be a vector space and ~~let~~ $v_1, v_2 \in V$. ~~Suppose that v_1, v_2 are linearly independent.~~

<u>Forwrd direction</u>: Suppose that v_1, v_2 are linearly independent. Assume the contray that vectors are scaler multipliers of others. Without loss of generality we assume that $\exists k \in \mathbb{F}, v_1 = k v_2$ ~~for some $k \in \mathbb{F}$~~ $k \neq 0$.

Then.

$$v_1 = k v_2$$

$$1 \cdot v_1 + (-k) v_2 = 0$$

Thus $v_1, v_2 \in V$ are linearly dependent. This is contradiction.

<u>Backward direction</u>:

We are going to proove ~~that~~ this by proof by ~~contradicti~~ contrapositive.

~~N.T.S: If $v_1, v_2 \in V$ are linear dependent then there exist $a, b \in \mathbb{F}$ such that $a v_1 + b v_2 = 0$~~

Suppose that $v_1, v_2 \in V$ are linearly dependent. Then there exist $a_1, a_2 \in V$ ~~with $\{a_1 \neq 0, a_2 \neq 0\}$~~ (but not ~~zero~~ both zero),

$$a_1 v_1 + a_2 v_2 = 0$$

Without loss generality $a_1 \neq 0$. Then

$$a_1 v_1 + a_2 v_2 = 0$$

$$v_1 = -\frac{a_2}{a_1} v_2 \quad \blacksquare$$

5 Find a number t such that

$$(3, 1, 4), (2, -3, 5), (5, 9, t)$$

is not linearly independent in $\mathbf{R}^3$.

Since the vectors $(3,1,4), (2,-3,5)$ and $(5,9,t)$ are linearly ~~in~~denpent. Thus there exist $a, b \in \mathbb{R}$ such that

$$(5,9,t) = a(3,1,4) + b(2,-3,5)$$

$$5 = 3a + 2b \quad \text{—} (1)$$

$$9 = a - 3b \quad \text{—} (2)$$

$$t = 4a + 5b$$

By (1) and (2) $a = 3$ and $b = (-2)$

Thus $t = 4(3) + 5(-2) = 2$ //

6 Show that the list $(2,3,1), (1,-1,2), (7,3,c)$ is linearly dependent in $\mathbf{F}^3$ if and only if $c = 8$.

Backward direction

If $c=8$ then observe that

$$(0,0,0) = 2(2,3,1) + 3(1,-1,2) - 1(7,3,8)$$

Therefore, the given list of vectors are linearly independent.

Forward direction: Now suppose that the given list is linearly dependent. ~~Then there~~

Then there exist $p,q,r \in \mathbb{F}$ with not all 0 such that

$$p(2,3,1)+q(1,-1,2)+r(7,3,c)=(0,0,0)$$

$$2p+q+7r=0 \quad \text{——} \ (1)$$
$$3p-q+3r=0 \quad \text{——} \ (2)$$
$$p+2q+rc=0 \quad \text{——} \ (3)$$

$(1)+(2)$: $5p+10r=0 \Rightarrow p=-2r$

$3\times(1) - 2\times(2)$: $3q+21r+2q-6r=0 \Rightarrow 5q=-15r$
$\Rightarrow q=-3r$

Since $p,q,r \neq 0$.

By (3) $(-2r)+2(-3r)+r(c)=0$
$r(c-8)=0$

Since $r\neq 0$, $c=8$ //

7 (a) Show that if we think of **C** as a vector space over **R**, then the list $1 + i, 1 - i$ is linearly independent.

(b) Show that if we think of **C** as a vector space over **C**, then the list $1 + i, 1 - i$ is linearly dependent.

a) Suppose that there exist $a, b \in \mathbb{R}$ suchthat

$$a(1+i)+b(1-i)=0+0i$$
$$(a+b)+(a-b)i = 0+0i$$

Thus $\left.\begin{matrix} a+b=0 \\ a-b=0 \end{matrix}\right\} \Rightarrow a=0, b=0$

Therefore, $(1+i), (1-i)$ are linearly independent over $\mathbb{R}$.

b) Observe that,

$$i(1+i)+1(1-i) = i+(-1)+1-i=0$$

Thus $(1+i), (1-i)$ are linearly dependent over $\mathbb{C}$.

8 Suppose v_1, v_2, v_3, v_4 is linearly independent in V. Prove that the list

$$v_1 - v_2, v_2 - v_3, v_3 - v_4, v_4$$

is also linearly independent.

(8) Suppose that v_1, v_2, v_3, v_4 is linearly independent.

If $a_1 v_1 + \dots + a_4 v_4 = 0$ for some $a_1, \dots, a_4 \in \mathbb{F}$

then $a_1 = a_2 = a_3 = a_4 = 0$. Let

$u_1 = (v_1 - v_2)$, $u_2 = (v_2 - v_3)$, $u_3 = (v_3 - v_4)$, $u_4 = v_4$

Suppoe that

$$b_1 u_1 + b_2 u_2 + b_3 u_3 + b_4 u_4 = 0 \quad \text{for some } b_1, \dots, b_4 \in \mathbb{F}$$

$$b_1(v_1 - v_2) + b_2(v_2 - v_3) + b_3(v_3 - v_4) + b_4 v_4 = 0$$

$$b_1 v_1 + (b_2 - b_1) v_2 + (b_3 - b_2) v_3 + (b_4 - b_3) v_4 = 0$$

Since $v_1, \dots, v_4$ are linearly idendendt.

$b_1 = 0$

$b_2 - b_1 = 0 \Rightarrow b_2 = 0$

$b_3 - b_2 = 0 \Rightarrow b_3 = 0$

$b_4 - b_3 = 0 \Rightarrow b_4 = 0$

Therefore u_1, u_2, u_3, u_4 are linearly idepend

9 Prove or give a counterexample: If $v_1, v_2, \dots, v_m$ is a linearly independent list of vectors in V, then

$$5v_1 - 4v_2, v_2, v_3, \dots, v_m$$

is linearly independent.

⑧ Suppose that v_1, v_2, v_3, v_4 is linearly ~~id~~independent.

If $a_1 v_1 + \dots + a_4 v_4 = 0$ for some $a_1, \dots, a_4 \in \mathbb{F}$ then $a_1 = a_2 = a_3 = a_4 = 0$. Let

$u_1 = (v_1 - v_2)$, $u_2 = (v_2 - v_3)$, $u_3 = (v_3 - v_4)$, $u_4 = v_4$

Suppose that

$$b_1 u_1 + b_2 u_2 + b_3 u_3 + b_4 u_4 = 0 \quad \text{for some } b_1, \dots, b_4 \in \mathbb{F}$$

$$b_1(v_1 - v_2) + b_2(v_2 - v_3) + b_3(v_3 - v_4) + b_4 v_4 = 0$$

$$b_1 v_1 + \cancel{b_4}(b_2 - b_1) v_2 + (b_3 - b_2) v_3 + (b_4 - b_3) v_4 = 0$$

Since $v_1, \dots, v_4$ are linearly idendendt.

$b_1 = 0$

$b_2 - b_1 = 0 \Rightarrow b_2 = 0$

$b_3 - b_2 = 0 \Rightarrow b_3 = 0$

$b_4 - b_3 = 0 \Rightarrow b_4 = 0$

Therefore u_1, u_2, u_3, u_4 are linearly idepend

10 Prove or give a counterexample: If $v_1, v_2, \dots, v_m$ is a linearly independent list of vectors in V and $\lambda \in \mathbf{F}$ with $\lambda \neq 0$, then $\lambda v_1, \lambda v_2, \dots, \lambda v_m$ is linearly independent.

(10). Suppose that $\gamma_1, \dots, \gamma_m \in V$ is a linearly independt and $\lambda \in \mathbb{F}$ with $\lambda \neq 0$.

Suppose that ~~$a_1, a_2, \dots, a_m \in \mathbb{F}$~~ there exist $a_1, a_2, \dots, a_m \in \mathbb{F}$ such that,

$$a_1(\lambda \gamma_1) + a_2(\lambda \gamma_2) + \dots + a_m(\lambda \gamma_m) = 0$$

Then $(a_1\lambda)\gamma_1 + (a_2\lambda)\gamma_2 + \dots + (a_m\lambda)\gamma_m = 0$

Since $\gamma_1, \dots, \gamma_m$ are linearly independent.

$$a_1\lambda = a_2\lambda = \dots = a_m\lambda = 0$$

Since $\lambda \neq 0$, $a_1 = a_2 = \dots = a_m = 0$

Therefore $\lambda\gamma_1, \lambda\gamma_2, \dots \lambda\gamma_m$ is linearly indepent.

11 Prove or give a counterexample: If $v_1, \ldots, v_m$ and $w_1, \ldots, w_m$ are linearly independent lists of vectors in V, then the list $v_1 + w_1, \ldots, v_m + w_m$ is linearly independent.

The given statement is false.

Couter example:

$v_1 = (-1,0) \qquad v_2 = (0,-1)$

$w_1 = (1,0) \qquad w_2 = (0,1)$

Suppose that $a_1 v_1 + a_2 v_2 = 0$

$$a_1(-1,0) + a_2(0,-1) = (0,0)$$
$$(-a_1, -a_2) = (0,0)$$

Thus, $a_1 = 0$, $a_2 = 0$

Thus $v_1, v_2 \in V$ are linearly independent.

~~Similary we can show that $w_1, w_2 \in V$ ar~~

Suppose that $b_1 w_1 + b_2 w_2 = 0$

$$b_1(1,0) + b_2(0,1) = (0,0)$$
$$(b_1, b_2) = (0,0)$$

Thus $b_1 = b_2 = 0$. Thus, w_1, w_2 are linearly indepent.

Now note that.

$$1(v_1 + w_1) + 1(v_2 + w_2) = ((-1,0) + (1,0)) + 1((0,-1) + (0,1))$$
$$= (0,0)$$

Therefore, $(v_1 + w_1)$ and $(v_2 + w_2)$ are linearly dependent.

12 Suppose $v_1, \ldots, v_m$ is linearly independent in V and $w \in V$. Prove that if $v_1 + w, \ldots, v_m + w$ is linearly dependent, then $w \in \operatorname{span}(v_1, \ldots, v_m)$.

Suppose that $v_1, v_2, \ldots, v_m \in V$ are linearly independent, $w \in V$. First suppose that $v_1+w, \ldots, v_m+w$ is linear dependent.

There exist scalers $a_1, a_2, \ldots, a_n \in \mathbb{F}$ ~~st~~ with $a_i \neq 0$ for all $i=1,2,\ldots,m$. Such that

$$a_1(v_1+w)+\cdots+a_m(v_m+w)=0$$

$$(a_1v_1+\cdots+a_m v_m)+(a_1+a_2+\cdots+a_m)w=0$$

If $(a_1+a_2+\cdots+a_m)=0$ then

$$a_1v_1+a_2v_2+\cdots+a_mv_m=0$$

Since $v_1,\ldots,v_m$ are linearly independent.

$$a_1=a_2=\cdots=a_m=0$$

But this cannot happen because of our hypothesis. of linear dependent of $v_1+w, \ldots, v_2+w$

Therefore $(a_1+a_2+\cdots+a_m)\neq 0$. Hence.

$$w=\frac{1}{(a_1+a_2+\cdots+a_m)}(a_1v_1+a_2v_2+\cdots+a_nv_n)$$

Thus $w \in \operatorname{span}\{v_1, \ldots, v_m\}$

13 Suppose $v_1, ..., v_m$ is linearly independent in V and $w \in V$. Show that

$$v_1, ..., v_m, w \text{ is linearly independent} \iff w \notin \operatorname{span}(v_1, ..., v_m).$$

Suppose that $v_1, ..., v_m \in V$ are linear independent and $w \in W$.

Backward direction

Suppose that $w \in span\{v_1, ..., v_m\}$.
Assume the contray that $v_1, v_2, ..., v_m, w$ is linear dependent. Then there exist $a_1, ..., a_m, b \in \mathbb{F}$, ($a_i \neq 0$ for all $i = 1, 2, ..., m$, $b \neq 0$). such that

$$a_1 v_1 + \cdots + a_m v_m + bw = 0 \quad (1)$$

• If $b = 0$, then $a_1 v_1 + \cdots + a_m v_m = 0$
Since $v_1, ..., v_m$ are linearly independent
$a_1, = a_2 = \cdots = a_m = 0$
Therefore, $b = a_1 = a_2 = \cdots = a_m = 0$. Thus $v_1, ..., v_m, w$ is linearly independt.
This contradict ~~fact~~ hypothesis, $v_1, ..., v_m, w$ is linearly dependent.

• If $b \neq 0$, equation (1) becomes,

$$w = \left(-\frac{a_1}{b}\right) v_1 + \cdots + \left(-\frac{a_m}{b}\right) v_m$$

Since $\left(-\frac{a}{b}\right) \in \mathbb{F}$, $w \in span\{v_1, ..., v_m\}$. But this contract hypothesis $w \notin span\{x_1, ..., v_m\}$.

Therefore, both cases of $b = 0$ and $b \neq \phi$ give a contradiction. So, we conclude that $v_1, v_2, ..., v_m, w$ is linearly dependent.

forward direction.

Suppose that $\gamma_1, \dots, \gamma_m, w$ are linearly independent. Assume the contray that we have $w \in \operatorname{span}\{\gamma_1, \dots, \gamma_m\}$. Then

$$w = a_1\gamma_1 + \dots + a_m\gamma_m \text{ for some } a_i \in \mathbb{F}, i=1,2,\dots,m$$

Then $a_1\gamma_1 + \dots + a_m\gamma_m - 1\cdot w = 0$

~~This means that $\gamma_1, \dots, \gamma_m, w$ are linearly independent.~~

~~Since $\gamma_1, \dots, \gamma_m, w$ are linearly independent~~

This implies that $\gamma_1, \gamma_2, \dots, \gamma_m, w$ are linearly dependent. This is contradiction.

Thus, $w \notin \operatorname{span}\{\gamma_1, \dots, \gamma_m\}$

~~This implies that~~

Thefore, $\gamma_1, \dots, \gamma_m, w$ is linearly indepent

$\Longleftrightarrow w \notin \operatorname{span}\{\gamma_1, \dots, \gamma_m\}$.

14 Suppose $v_1, \ldots, v_m$ is a list of vectors in V. For $k \in \{1, \ldots, m\}$, let

$$w_k = v_1 + \cdots + v_k.$$

Show that the list $v_1, \ldots, v_m$ is linearly independent if and only if the list $w_1, \ldots, w_m$ is linearly independent.

Forward direction

Suppose that the list $v_1, \ldots, v_m \in V$ is linearly independent.

Suppose that ~~$a_1, a_2, \ldots, a_m \in \mathbb{F}$~~ there exist $a_1, a_2, \ldots a_m \in \mathbb{F}$ such that.

$$a_1 w_1 + a_2 w_2 + \cdots + a_m w_m = 0$$

$$a_1 v_1 + a_2(v_1 + v_2) + \cdots + a_m(v_1 + \cdots + v_m) = 0$$

$$(a_1 + \cdots + a_m) v_1 + \cdots + a_m v_m = 0$$

Since $v_1, \ldots, v_m$ are linearly indepent

$$a_m = 0,\ a_{m-1} + a_m = 0,\ \ldots,\ a_1 + a_2 + \cdots + a_n = 0$$

Thus. $a_1 = a_2 = \cdots = a_m = 0$.

Thus the list $w_1, w_2, \ldots, w_m$ are linearly independent.

Backward direction

Now suppose that $w_1, \ldots, w_m$ are linearly independent.

Suppose that there exist $b_1, \ldots, b_m \in \mathbb{F}$ such that,

$$b_1 v_1 + b_2 v_2 + \cdots + b_n v_n = 0$$

We express $v_1, v_2, \ldots, v_n$ are in terms of $w_1, \ldots, w_m$

~~$b_1 w_1 + b_2 \in$~~

$v_1 = w_1$, $v_2 = w_2 - w_1$, $\ldots$, $v_m = w_m - w_{m-1}$. Then,

$$b_1 w_1 + b_2(w_2 - w_1) + \cdots + b_m(w_m - w_{m-1}) = 0$$

$$\cancel{b_1 w_1 +} (b_1 - b_2) w_1 + (b_2 - b_3) w_2 + \cdots + b_n w_n = 0$$

Since $w_1, w_2, \ldots, w_m$ are linearly independent.

$$b_m = 0,\ b_{m-1} - b_m = 0,\ \ldots,\ b_1 - b_2 = 0$$

Thus, $b_1 = b_2 = \cdots = b_m = 0$

Thus $v_1, v_2, \ldots, v_n$ are linearly independent.

15 Explain why there does not exist a list of six polynomials that is linearly independent in $\mathcal{P}_4(\mathbf{F})$.

First note that,

$$\operatorname{span}(1, x, x^2, x^3, x^4) = P_4(x)$$

length of spaning list $= 5$

2.22 *length of linearly independent list $\leq$ length of spanning list*

In a finite-dimensional vector space, the length of every linearly independent list of vectors is less than or equal to the length of every spanning list of vectors.

Therefore length of linear independent is at most 5. So list of polynomials 6 that linear independent cannot be existed.

16 Explain why no list of four polynomials spans $\mathcal{P}_4(\mathbf{F})$.

Note that $1, x, x^2, x^3, x^4 \in \mathcal{P}_4(\mathbb{F})$ are linearly independent.

$p(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 = 0$

$p(0) = a_0 = 0$

$p(1) = a_0 + a_1 + a_2 + a_3 + a_4 = a_1 + a_2 + a_3 + a_4 = 0$ — ①

$p(-1) = a_0 - a_1 + a_2 - a_3 + a_4 = -a_1 + a_2 - a_3 + a_4 = 0$ — ②

① + ②

$2a_2 + 2a_4 = 0$

$a_2 = -a_4$ — ⑤

① − ②

$2a_1 + 2a_3 = 0$

$a_1 = -a_3$ — ⑥

$p(2) = a_0 + 2a_1 + 4a_2 + 8a_3 + 16a_4 = 0$ — ③

$p(-2) = a_0 - 2a_1 + 4a_2 - 8a_3 + 16a_4 = 0$ — ④

③ − ④, $4a_1 + 16a_3 = 0$ (by 6)

$+4a_1 - 16a_1 = 0$

$-12a_1 = 0$

$a_1 = 0$

$\Rightarrow a_3 = 0$

③ + ④ $\quad 8a_2 + 32a_4 = 0$

$8a_2 + 32(-a_2) = 0 \quad$ (by ⑤)

$a_2 = 0$

$\Rightarrow a_1 = 0$

Therefore $1, x, x^2, x^3, x^4$ are linear independent.

$\text{length}(1, x, x^2, x^3, x^4) = 5$

2.22 *length of linearly independent list $\leq$ length of spanning list*

In a finite-dimensional vector space, the length of every linearly independent list of vectors is less than or equal to the length of every spanning list of vectors.

Therefore, $5 \leq$ length of spaning list

Hence, no list of length four can be spanning

17 Prove that V is infinite-dimensional if and only if there is a sequence $v_1, v_2, \dots$ of vectors in V such that $v_1, \dots, v_m$ is linearly independent for every positive integer m.

Forward Direction.

Suppose that V is ~~finite~~ infinte dimensional

Then V cannot be spanned by finite many vectors.

Let ~~v_1~~ $0 \neq v_1 \in V$.

Since V is infinite dimensional there must exist $v_2 \in V$ such that $v_2 \in \text{span}(v_1)$ and in general $v_k \notin \text{span}(v_1, \dots, v_{k-1})$

~~Because~~ (We can alway find v_k since no finite list spans V)

Hence we can obtain that a sequnce $v_1, v_2, \dots \in V$.

~~Reca~~ By Lineat depend lemma (Book 2.19)

Let $v_1, \dots, v_m \in V$

If $V_k \in \text{span}(v_1, \dots, v_{k-1})$ $\forall k = 1, 2, \dots, m$

then $v_1, \dots, v_m$ is a linearly independent.

Therefore $v_1, v_2, \dots, v_m$ is linearly independent.

Forward Direction.

Suppose that V is ~~finite~~ infinte dimensional

Then V cannot be spanned by finite many vectors.

Let $0 \neq \gamma_1 \in V$.

Since V is infinite dimensional there must exist $\gamma_2 \in V$ suchthat $\gamma_2 \in \operatorname{span}(\gamma_1)$ and ingeneral $\gamma_k \notin \operatorname{span}(\gamma_1, \dots, \gamma_{k-1})$

(We can alaway find γ_k since no finite list spans V)

Hence we can obtain that a sequnce $\gamma_1, \gamma_2, \dots \in V$.

By Linear depend lemma (Book 2.19)

Let $\gamma_1, \dots, \gamma_m \in V$

If $V_k \in \operatorname{span}(\gamma_1, \dots, \gamma_{k-1})$ $\forall k = 1, 2, \dots, m$

then $\gamma_1, \dots, \gamma_m$ is a linearly independent.

Therefore $\gamma_1, \gamma_2, \dots, \gamma_m$ is linearly independent.

18 Prove that $\mathbf{F}^\infty$ is infinite-dimensional.

Let $e_i = (0, \ldots, 0, 1, 0 0 \ldots)$ be the vector 1 in the ith components and 0 elsewhere.
Observe that $e_1, e_2, \ldots, e_m$ are linear indepent for any $m \in \mathbb{N}$.

Suppose that there exist $a_1, \ldots, a_m \in \mathbb{F}$ such that $a_1 e_1 + a_2 e_2 + \cdots + a_m e_m = 0$

$a_1(1, 0, 0, \ldots) + a_2(0, 1, 0 \ldots) + \cdots + a_m(0, \ldots) = (0, 0 \ldots,)$

$(a_1, a_2, \ldots, a_m, 0, \ldots) = (0, 0, \ldots, 0)$

Thus $a_1 = a_2 = \cdots = a_m = 0$

By previous exerise,

17 Prove that V is infinite-dimensional if and only if there is a sequence $v_1, v_2, \ldots$ of vectors in V such that $v_1, \ldots, v_m$ is linearly independent for every positive integer m.

We can conclude that F^∞ has infinite-dimensional.

19 Prove that the real vector space of all continuous real-valued functions on the interval $[0, 1]$ is infinite-dimensional.

Let V be a vector space of all contininus real-valued functions on interval $[0,1]$

Define $g_k(x): \mathbb{R} \longrightarrow [0,1]$ $\forall k \in \mathbb{N} \cup \{0\}$

$x \longmapsto x^k$

Note that $g_k(x) \in V$.

Claim: $g_0(x), g_1(x), g_2(x), \ldots, g_m(x)$ are linearly independent.

Suppose that there exist $a_0, a_1, \ldots, a_m \in \mathbb{F}$ such that

$$f(x) = a_0 1 + a_1 g_1(x) + \cdots + a_m g_m(x) = 0$$

$$f(x) = a_0 + a_1 x + \cdots + a_m x^m = 0$$

$f(0) = a_0 = 0$. Then

$$f'(x) = a_1 + 2a_2 x + \cdots + a_m m x^{m-1} = 0$$

$$f'(0) = a_1 = 0$$

We can go on this proceed same process, we can obtain $a_0 = a_1 = a_2 = \cdots = a_m = 0$

Thus $g_1(x), g_2(x), \ldots, g_m(x)$ are linear indepent

Recall previous exercise 2A(17)

17 Prove that V is infinite-dimensional if and only if there is a sequence $v_1, v_2, \ldots$ of vectors in V such that $v_1, \ldots, v_m$ is linearly independent for every positive integer m.

Thus V is infinite-dimenisinal vector space.

20 Suppose $p_0, p_1, ..., p_m$ are polynomials in $\mathcal{P}_m(\mathbf{F})$ such that $p_k(2) = 0$ for each $k \in \{0, ..., m\}$. Prove that $p_0, p_1, ..., p_m$ is not linearly independent in $\mathcal{P}_m(\mathbf{F})$.

Assume the contray, that $p_0, p_1, \dots, p_m$ are linearly independent.

Note that $1, x, x^2, \dots, x^m$ are spaning list of $\mathcal{P}_m(\mathbb{F})$

i.e: $\operatorname{span}(1, x, x^2, \dots, x^m) = \mathcal{P}(\mathbb{F})$

$\begin{pmatrix}\text{length of linear} \\ \text{indepent list}\end{pmatrix} \leq \begin{pmatrix}\text{length of spanning} \\ \text{list in } \mathcal{P}(\mathbb{F})\end{pmatrix} = m+1$

Thus $p_0, p_1, \dots, p_m, x$ is linearly dependent ← (It contains m+2 element)

~~Note~~ Then note that $p_0, p_1, \dots, p_m$ is linear independent and $x \in \mathcal{P}_m(\mathbb{F})$

Question number 13,

~~$\begin{pmatrix}p_0, p_1, \dots, p_m, x \text{ is} \\ \text{linearly independent}\end{pmatrix}$ iff $\Longleftrightarrow$ $\begin{pmatrix}x \in \operatorname{span} \\ \operatorname{span}(p_0, \dots, p_m)\end{pmatrix}$~~

$\begin{pmatrix}p_0, p_1, \dots, p_m, x \text{ is} \\ \text{linearly dependent}\end{pmatrix} \Longleftrightarrow \left(x \in \operatorname{span}(p_0, \dots, p_m)\right)$

Thus there exist $a_0, a_1, \dots, a_m \in \mathbb{F}$ such tha[t]

$x = a_0 p_0(x) + a_1 p_1(x) + \dots + a_m p_m(x)$

$2 = a_0 p_0(2) + a_1 p_1(2) + \dots + a_m p_m(2)$

$2 = 0$

This is contradiction.

3.2 Exercise 2B (Bases)

1. Find all vector spaces that have exactly one basis.
2. Verify all assertions in Example
3. a. Let W be the subspace of $\mathbb{R}^5$ defined by

$$W = \{(x_1, x_2, x_3, x_4, x_5) \in \mathbb{R}^5 \mid x_1 = 3x_2 \text{ and } x_3 = 7x_4\}.$$

Find a basis of W.

b. Extend the basis in (a) to a basis of $\mathbb{R}^5$.

c. Find a subspace$V$of$\mathbb{R}^5$such that $\mathbb{R}^5 = W \oplus V$.

Answer: a. Let $(x_1, x_2, x_3, x_4, x_5) \in U$. Then $x_1 = 3x_2$ and $x_3 = 7x_4$. Thus,

$$\begin{aligned}(x_1, x_2, x_3, x_4, x_5) &= (3x_2, x_2, 7x_4, x_4, x_5) && (3.1)\\ &= x_2(3,1,0,0,0) + x_4(0,0,7,1,0) + (0,0,0,0,1) && (3.2)\end{aligned}$$

Let $v_1 := (3,1,0,0,0), v_2 := (0,0,7,1,0)$ and $v_3 := (0,0,0,0,1)$. Since $x_2, x_4, x_5 \in \mathbb{R}$,then

$$\text{span}(v_1, v_2, v_3) = U$$

Now, we need to show that v_1, v_2, v_3 are linearly independent.Suppose that there exist $a_1, a_2, a_3 \in \mathbb{R}$ such that,

$$\begin{aligned}a_1(3,1,0,0,0) + a_2(0,0,7,1,0) + a_3(0,0,0,0,1) &= (0,0,0,0,0) && (3.3)\\ (3a_1, a_1, 7a_2, a_2, a_3) &= (0,0,0,0,0). && (3.4)\end{aligned}$$

Thus, $a_1 = a_2 = a_3 = 0$. Hence, v_1, v_2, v_3 are linearly independent.

Therefore, the list v_1, v_2, v_3 is a basis of U.

b. Let $v_4 := (0,1,0,0,0)$ and $v_5 := (0,0,1,0,0)$.

Claim 1: $\text{span}(v_1, v_2, v_3, v_4, v_5) = \mathbb{R}^5$.

Let $(y_1, y_2, y_3, y_4, y_5) \in \mathbb{R}^5$. Then,

$$\begin{aligned}(y_1, y_2, y_3, y_4, y_5) &= \left(y_1, y_2 + \frac{y_1}{3} - \frac{y_1}{3}, y_3 + 7y_4 - 7y_4, y_4, y_5\right) && (3.5)\\ &= \frac{y_1}{3}(3,1,0,0,0) + y_4(0,0,7,1,0) + y_5(0,0,0,0,1) + \left(y_2 - \frac{y_1}{3}\right)(0,1,0,0,0) + (y_3 - 7y_4) && (3.6)\end{aligned}$$

Since $y_1, ..., y_5 \in \mathbb{R}$, we have established that the list $(3,1,0,0,0), (0,0,7,1,0), (0,0,0,0,1), (0,1,0,0,0), (0$
spans $\mathbb{R}^5$

Claim 2: the list of vectors v_1, v_2, v_3, v_4, v_5 are linearly independent.

Suppose that there exist $b_1, ..., b_5 \in \mathbb{R}$ such that

$$b_1v_1 + b_2v_2 + b_3v_3 + b_4v_4 + b_5v_5 = 0 \tag{3.7}$$

$$b_1(3,1,0,0,0) + b_2(0,0,7,1,0) + b_3(0,0,0,0,1) + b_4(0,1,0,0,0) + b_5(0,0,1,0,0) = (0,0,0,0,0) \tag{3.8}$$

$$(3b1, b1 + b4, 7b2 + b5, b2, b3) = (0,0,0,0,0). \tag{3.9}$$

This implies $b_1 = b_2 = b_3 = b_4 = b_5 = 0$.

c.

4. (a) Let W be the subspace of $\mathbb{R}^5$ defined by

   $$W = \{(x_1, x_2, x_3, x_4, x_5) \in \mathbb{R}^5 \mid 6x_1 = x_2 \text{ and } x_3 + 2x_4 + 3x_5 = 0\}.$$

   Find a basis of W.

   (b) Extend the basis in (a) to a basis of $\mathbb{R}^5$.

   (c) Find a subspace V of $\mathbb{R}^5$ such that $\mathbb{R}^5 = W \oplus V$.

5. Suppose V is finite-dimensional and U, W are subspaces of V such that $V = U + W$. Prove that there exists a basis of V consisting of vectors in $U \cup W$.

6. Prove or give a counterexample: If p_0, p_1, p_2, p_3 is a list in $P_3(\mathbb{R})$ such that none of the polynomials p_0, p_1, p_2, p_3 has degree 2, then p_0, p_1, p_2, p_3 is not a basis of $P_3(\mathbb{R})$.

7. Suppose v_1, v_2, v_3, v_4 is a basis of V. Prove that $v_1 + v_2, v_2 + v_3, v_3 + v_4, v_4$ is also a basis of V.

8. Prove or give a counterexample: If v_1, v_2, v_3, v_4 is a basis of V and W is a subspace of V such that $v_1, v_2 \in W$ and $v_3 \notin W$ and $v_4 \notin W$, then v_1, v_2 is a basis of W.

9. Suppose $v_1, \dots, v_n$ is a list of vectors in V. For $i \in \{1, \dots, n\}$, let

   $$w_i = v_1 + \cdots + v_i.$$

   Show that $v_1, \dots, v_n$ is a basis of V if and only if $w_1, \dots, w_n$ is a basis of V.

10. Suppose U and W are subspaces of V such that $V = U \oplus W$. Suppose also that $u_1, \dots, u_m$ is a basis of U and $w_1, \dots, w_n$ is a basis of W. Prove that $u_1, \dots, u_m, w_1, \dots, w_n$ is a basis of V.

11. Suppose V is a real vector space. Show that if $v_1, \ldots, v_n$ is a basis of V (as a real vector space), then $v_1, \ldots, v_n$ is also a basis of the complexification $V_{\mathbb{C}}$ (as a complex vector space). See Exercise 8 in Section 1B for the definition of the complexification $V_{\mathbb{C}}$.