Linear Algebra

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Chapter 1

Vector space

1.1 R^n and C^n

$\mathbb{R}^n$ and $\mathbb{C}^n$

We are already be familiar with basic properties of the set of real numbers($\mathbb{R}$). Complex numbers comes when we can take square roots of negative numbers. The idea is to assume we have a square root of -1, denoted by i that obeys the usual rules of arithmetic. Here are the formal definition.

Definition 1.1 (Complex Numbers)**.**

- A complex number is an ordered pair (x, y), where $x, y \in \mathbb{R}$, but we will write this as $x + yi$.
- The set of all complex numbers is denoted by $\mathbb{C}$:

$$\mathbb{C} = \{x + yi \,|\, x, y \in \mathbb{R}\}.$$

- Addition and multiplication on $\mathbb{C}$ are defined by

$$(x + yi) + (u + vi) = (x + u) + (y + v)i,$$

$$(x + yi)(u + vi) = (xu - yv) + (xv + yu)i;$$

  here $x, y, u, v \in \mathbb{R}$

Fun Fact: The symbol i was first used to denote $\sqrt{-1}$ by Leonard Euler in 1777.

- Note that $\mathbb{C} \supseteq \mathbb{R}$ because for all real numbers a $\mathbb{R}$, we can express it as a complex number by writing it as $a + 0i$.

- We usually denote $0 + yi$ simply as yi, and $0 + 1i$ as i.
- The definition of multiplication for complex numbers is based on the assumption that $i^2 = -1$. Using the standard arithmetic rules, we can derive the formula for the product of two complex numbers. This formula can then be used to confirm that i^2 indeed equals -1.

Example 1.1. Let's calculate the product of two complex numbers $(1+2i)$ and $(3+4i)$ using the distributive and commutative properties:

$$\begin{aligned}(1+2i)(3+4i) &= 1\cdot(3+4i) + (2i)(3+4i)\\ &= 1\cdot 3 + 1\cdot 4i + 2i\cdot 3 + (2i)(4i)\\ &= 3 + 4i + 6i - 8\\ &= -5 + 10i\end{aligned}$$

Proposition 1.1 (Properties of Complex Arithmetic)**.**

- ***Commutativity:*** $z_1 + z_2 = z_2 + z_1$ *and* $z_1z_2 = z_2z_1$ *for all* $z_1, z_2 \in \mathbb{C}$.
- ***Associativity:*** $(z_1 + z_2) + z_3 = z_1 + (z_2 + z_3)$ *and* $(z_1z_2)z_3 = z_1(z_2z_3)$ *for all* $z_1, z_2, z_3 \in \mathbb{C}$.
- ***Identities:*** $z + 0 = z$ *and* $z1 = z$ *for all* $z \in \mathbb{C}$.
- ***Additive Inverse:*** *For every* $z \in \mathbb{C}$*, there exists a unique* $-z \in \mathbb{C}$ *such that* $z + (-z) = 0$.
- ***Multiplicative Inverse:*** *For every* $z \in \mathbb{C}$ *with* $z \neq 0$*, there exists a unique* $z^{-1} \in \mathbb{C}$ *such that* $zz^{-1} = 1$.
- ***Distributive Property:*** $z_1(z_2 + z_3) = z_1z_2 + z_1z_3$ *for all* $z_1, z_2, z_3 \in \mathbb{C}$.

The properties above are proved using the familiar properties of real numbers and the definitions of complex addition and multiplication. Here, we are going to prove that commutativity of complex addition and multiplication is proved.

Proof.

- *Addition* : Let $z_1 = a + bi$ and $z_2 = c + di$ be any two complex numbers. Then we have:

  $$z_1 + z_2 = (a + bi) + (c + di) = (a + c) + (b + d)i = z_2 + z_1$$

  This shows that addition is commutative for complex numbers.

- *Multiplication*: Again, let $z_1 = a + bi$ and $z_2 = c + di$ be any two complex numbers. Then we have:

  $$z_1z_2 = (a + bi)(c + di) = ac + adi + bci - bd = (ac - bd) + (ad + bc)i$$

and

$$z_2z_1 = (c + di)(a + bi) = ca + cbi + dai - db = (ca - db) + (da + cb)i$$

As you can see, $(ac-bd)+(ad+bc)i = (ca-db)+(da+cb)i$, which shows that multiplication is also commutative for complex numbers.

So, we have proved that both addition and multiplication are commutative operations in the set of complex numbers. This means that the order in which complex numbers are added or multiplied does not affect the result.

□

Definition 1.2 (Subtraction,Division)**.** Let's suppose $z_1, z_2 \in \mathbb{C}$.

- The negative of a complex number z_1 is denoted as $-z_1$. It is the unique complex number such that
$$z_1 + (-z_1) = 0.$$
- Subtraction in the set of complex numbers is defined as
$$z_1 - z_2 = z_1 + (-z_2).$$
- For $z_1 \neq 0$, let $1/z_1$ denote the multiplicative inverse of z_1. Thus, $1/z_1$ is the unique complex number such that
$$z_1(1/z_1) = 1.$$
- For $z_1 \neq 0$, division by z_1 is defined as
$$z_2/z_1 = z_2(1/z_1).$$

So that we can conveniently make definitions and prove theorems that apply to both real and complex numbers, we adopt the following notation.

Notation :Throughout this note stands $\mathbb{F}$ for either $\mathbb{R}$ or $\mathbb{F}$. The letter $\mathbb{F}$ is used because $\mathbb{R}$ and $\mathbb{C}$ are examples of what are called fields.

Elements of $\mathbb{F}$ are called scalars. The word "scalar" (which is just a fancy word for "number") is often used when we want to emphasize that an object is a number, as opposed to a vector (vectors will be defined soon).

For $\alpha \in \mathbb{F}$ and m a positive integer, we define α^m to denote the product of α with itself m times:
$$\alpha^m = \underbrace{\alpha \cdot \alpha \cdots \alpha}_{m\text{times}}$$
. This definition implies that

$$(\alpha^m)^n = \alpha^{mn} \quad \text{and} \quad (\alpha\beta)^m = \alpha^m\beta^m$$

for all $\alpha\beta \in \mathbb{F}$ and all positive integers m, n.

1.1.1 Lists

Before defining $\mathbb{R}^n$ and $\mathbb{C}^n$, we look at two important examples.

Example 1.2.

- The set $\mathbb{R}^2$, which you can think of as a plane, is the set of all ordered pairs of real numbers:
$$\mathbb{R}^2 := \{(x, y) : x, y \in \mathbb{R}\}$$
- The set $\mathbb{R}^3$, which you can think of as ordinary space, is the set of all ordered triples of real numbers:
$$\mathbb{R}^3 := \{(x, y, z) : x, y, z \in \mathbb{R}\}$$

Let's generalize this examples to higher dimensions.

Definition 1.3 (List,Length)**.**

- Suppose n is a nonnegative integer. A *list* of *length* n is an ordered collection of n elements (which might be numbers, other lists, or more abstract objects).
- Two lists are equal if and only if they have the same length and the same elements in the same order

Note that many mathematicians call a list of length n an n-tuple.

Lists are often written as elements separated by commas and surrounded by parentheses. Thus a list of length two is an ordered pair that might be written as (a, b). A list of length three is an ordered triple that might be written as (x, y, z) A list of length n might look like this: $(z_1, ..., z_n)$

Sometimes we will use the word list without specifying its length. Remember, however, that by definition each list has a finite length that is a non-negative integer. Thus an object that looks like $(x_1, x_2, ...)$, which might be said to have infinite length, is not a list.

A list of length 0 looks like this: $(\ )$.

We consider such an object to be a list so that some of our theorems will not have trivial exceptions.

Lists differ from sets in two ways: in lists, order matters and repetitions have meaning; in sets, order and repetitions are irrelevant.

Proposition 1.2 (Lists versus Sets)**.**

- *The lists* $(3, 5)$ *and* $(5, 3)$ *are not equal, but the sets* $\{3, 5\}$ *and* $\{5, 3\}$ *are equal.*
- *The lists* $(4, 4)$ *and* $(4, 4, 4)$ *are not equal (they do not have the same length), although the sets* $\{4, 4\}$ *and* $\{4, 4, 4\}$ *both equal the set* $\{4\}$*.*

1.1.2 F^n

$\mathbb{F}^n$ Fix a positive integer $\mathbb{F}^n$ for the rest of this chapter.

Definition 1.4. $\mathbb{F}^n$ is the set of all lists of length n of elements of $\mathcal{F}$:

$$\mathbb{F}^n = \{(x_1, \dots, x_n) \mid x_i \in \mathbb{F} \text{ for } i = 1, \dots, n\}.$$

For $(x_1, \dots, x_n) \in \mathbb{F}^n$ and $i \in \{1, \dots, n\}$, we say that x_i is the ith coordinate of $(x_1, \dots, x_n)$.

If $\mathbb{F} = \mathbb{R}$ and n equals 2 or 3, then the definition above of $\mathbb{F}^n$ agrees with our previous notions of $\mathbb{R}^2$ and $\mathbb{R}^3$.

Example 1.3. Let $\mathbb{C}^4$ be the set of all lists of four complex numbers:

$$\mathbb{C}^4 = \{(z_1, z_2, z_3, z_4) \mid z_1, z_2, z_3, z_4 \in \mathbb{C}\}.$$

If $n \geq 4$, we cannot visualize $\mathbb{R}^n$ as a physical object. Similarly, $\mathbb{C}^1$ can be thought of as a plane, but for $n \geq 2$, the human brain cannot provide a full image of $\mathbb{R}^n$. However, even if n is large, we can perform algebraic manipulations in $\mathbb{C}^n$ as easily as in $\mathbb{R}^2$ or $\mathbb{R}^3$. For example, addition in $\mathbb{R}^n$ is defined as follows.

Definition 1.5 (addition of higer dimesionans)**.** Addition in $\mathbb{F}^n$ is defined by adding corresponding coordinates:

$$(a_1, ..., a_n) + (b_1, ..., b_n) = (a_1 + b_1, ..., a_n + b_n)$$

Often the mathematics of $\mathbb{R}^n$ becomes cleaner if we use a single letter to denote a list of n numbers, without explicitly writing the coordinates. For example, the next result is stated with x and y in $\mathbb{R}^n$ even though the proof requires the more cumbersome notation of $(x_1, \dots, x_n)$ and $(y_1, \dots, y_n)$.

Proposition 1.3. *If $a, b \in \mathbb{F}^n$, then $a + b = b + a$.*

Proof. Suppose $a = (a_1, ..., a_n) \in \mathbb{F}^n$ and $b = (b_1, ..., b_n) \in \mathbb{F}^n$. Then

$$\begin{aligned}
a + b &= (a_1, ..., a_n) + (b_1, ..., b_n) && (1.1)\\
&= (a_1 + b_1, ..., a_n + b_n) && (1.2)\\
&= (b_1 + a_1, ..., b_n + a_n) && (1.3)\\
&= (b_1, ..., b_n) + (a_1, ..., a_n) && (1.4)\\
&= b + a && (1.5)
\end{aligned}$$

where the second and fourth equalities above hold because of the definition of addition in $\mathbb{F}^n$ and the third equality holds because of the usual commutativity of addition in $\mathbb{F}$. □

If a single letter is used to denote an element of F^n, then the same letter with appropriate subscripts is often used when coordinates must be displayed.
For example, if $x \in F^n$, then letting

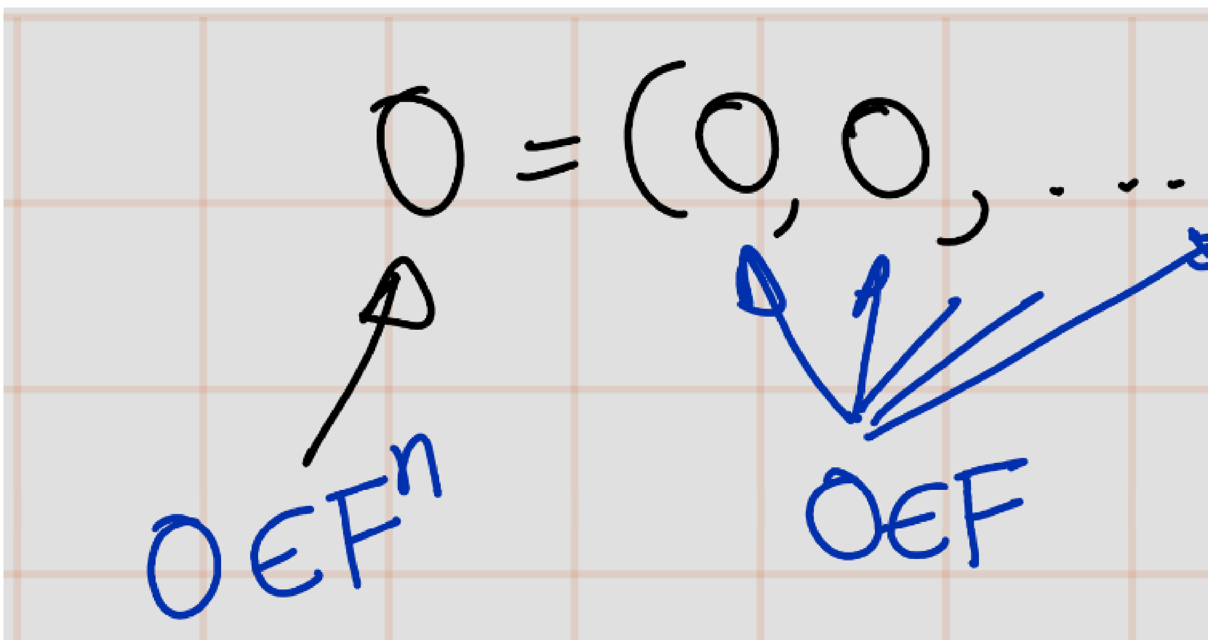
$$x = (x_1, \dots, x_n)$$

is good notation, as shown in the proof above. Even better, work with just x and avoid explicit coordinates when possible.

Notation: 0
Let 0 denote the list of length n whose coordinates are all 0:

$$0 = (0, \dots, 0)$$

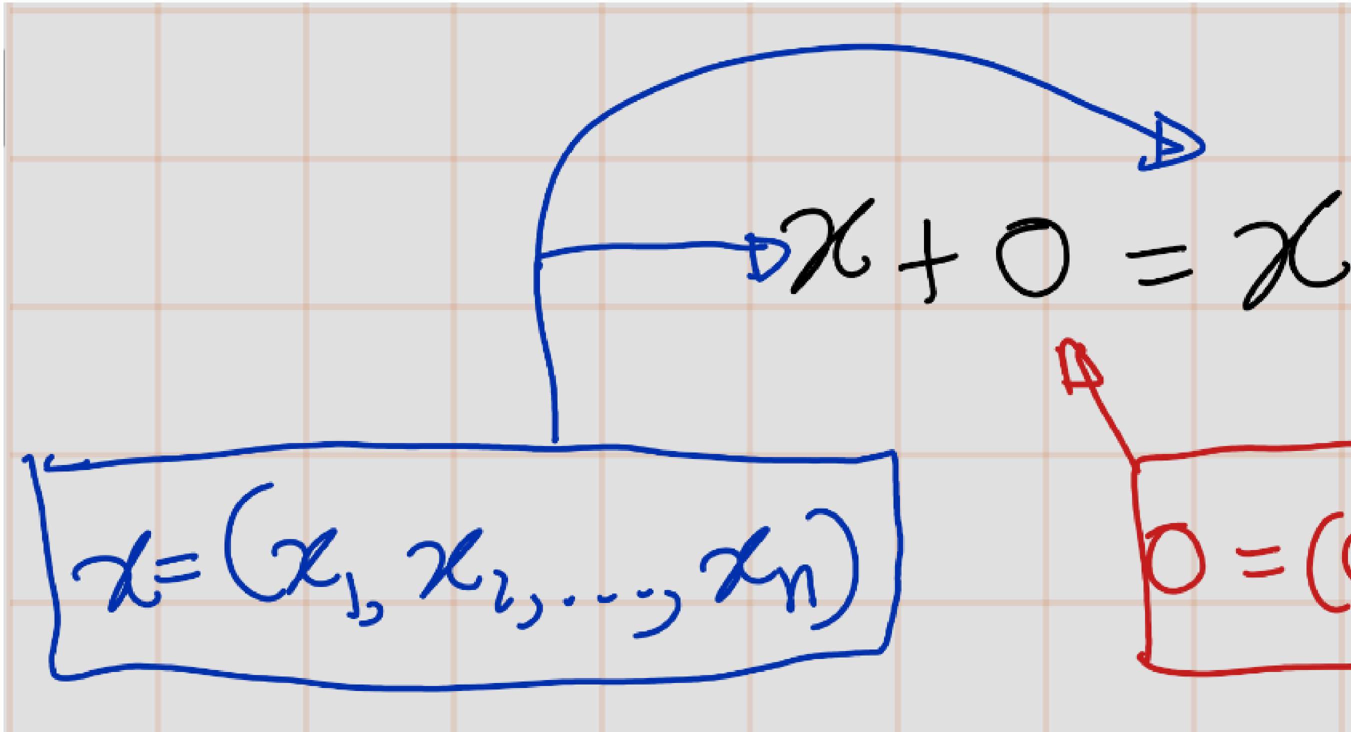
Here we are using the symbol 0 in two different ways—on the left side of the equation above, the symbol 0 denotes a list of length n, which is an element of 0, whereas on the right side, each 0 denotes a number. This potentially confusing practice actually causes no problems because the context should always make clear which 0 is intended.



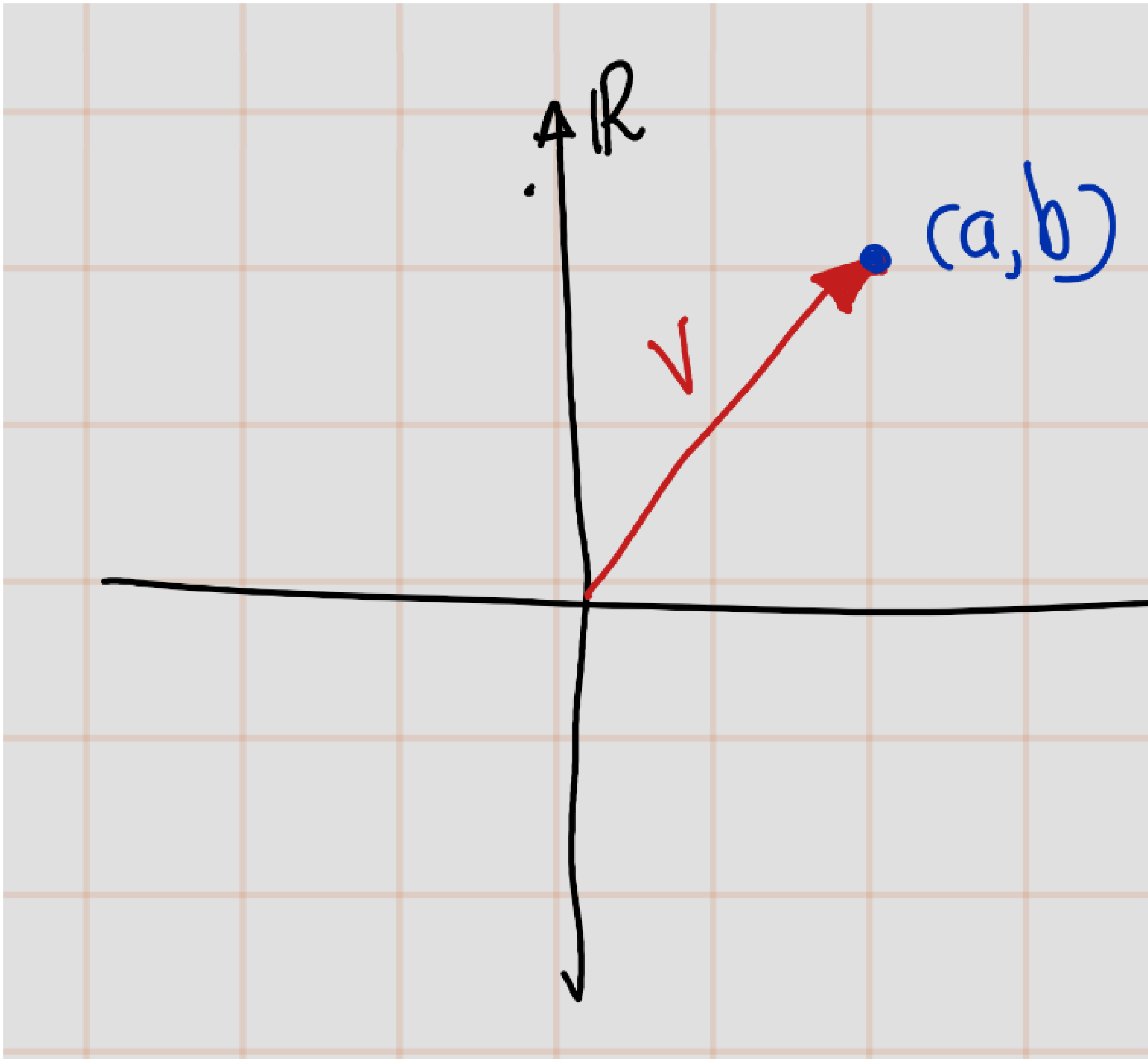
Example 1.4 (Context determines which 0 is intended)**.** Consider the statement that 0 is an additive identity for F^n:

$$x + 0 = x \quad \text{for all } x \in F^n.$$

Here, the 0 above refers to the list defined earlier, not the number 0, because we have not defined the sum of an element of F^n (namely, x) and the number 0.



A picture can aid our intuition. We will draw pictures in $\mathbb{R}^2$ because we can sketch this space on two-dimensional surfaces such as paper and computer screens. A typical element of $\mathbb{R}^2$ is a point $\mathbf{v} = (a, b)$. Sometimes we think of $\mathbf{v}$ not as a point but as an arrow starting at the origin and ending at (a, b), as shown here. When we think of an element of $\mathbb{R}^2$ as an arrow, we refer to it as a vector.



Example 1.5. $v_1 := (2,1), v_2 := (3,4) \in \mathbb{R}^2$ can be present as follows,

When we think of vectors in $\mathbb{R}^2$ as arrows, we can move an arrow parallel to itself (without changing its length or direction) and still consider it the same vector. With this viewpoint, you'll often gain a better understanding by dispensing with the coordinate axes and explicit coordinates, simply thinking of the vector itself, as shown in the figure here. The two arrows depicted have the same length and direction, so we regard them as the same vector.

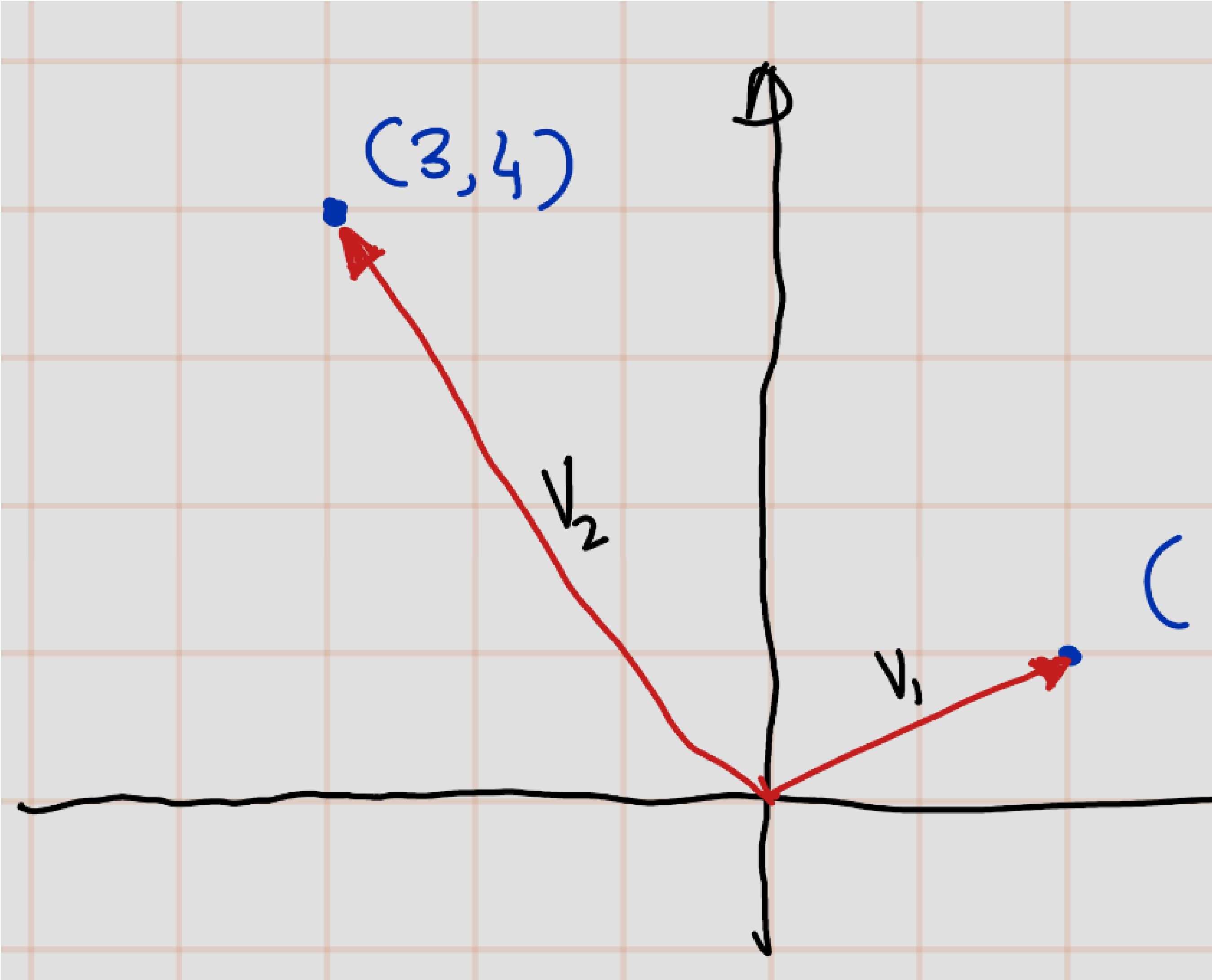


Figure 1.1: Elements of R2 can be thought of as points or as vectors.

1.2 Definition of Vector space

We will define a vector space to be a set V with an addition and a scalar multiplication on V that satisfy the properties in the paragraph above.

Definition 1.6 (addition, scalar multiplication)**.**

- An addition on set V is a function that assigns an element $u+v \in V \quad u, v, \in V$.
- A scalar multiplication on a set V is a function that assigns an element $\lambda v \in V$ to each $\lambda \in F$ and each $v \in V$

Now we are ready to give the formal definition of a vector space

Definition 1.7 (Vector Space)**.** A vector space is a set V along with an addition on V and a scalar multiplication on V such that the following properties hold.

- **Commutativity**: $u + v = v + u$ for all $u, v \in V$.
- **Associativity**: $(u + v) + w = u + (v + w)$ and $(ab)c = a(bc)$ for all $u, v, w \in V$ and for all $a, b \in \mathbb{F}$.
- **Additive Identity**: There exists an element $0 \in V$ such that $v + 0 = v$ for all $v \in V$.
- **Additive Inverse**: For every $v \in V$, there exists $-v \in V$ such that $v + (-v) = 0$.
- **Multiplicative Identity**: $1v = v$ for all $v \in V$.
- **Distributive Properties**: $a(v + w) = av + aw$ and $(a + b)v = av + bv$ for all $v, w \in V$ and all $a, b \in \mathbb{F}$.

The following geometric language sometimes aids our intuition.

Definition 1.8 (Vector)**.** Elements of a vector space are called vectors or points.

Example 1.6. The simplest vector space is $\{0\}$, which contains only one point.

The scalar multiplication in a vector space depends on F. Thus when we need to be precise, we will say that V is a vector space over F instead of saying simply that V is a vector space.

Example 1.7.

- $\mathbb{R}^n$ is a vector space over $\mathbb{R}$
- $\mathbb{C}^n$ is a vector space over $\mathbb{C}$.

Usually the choice of F is either clear from the context or irrelevant. Thus we often assume that F is lurking in the background without specifically mentioning it. With the usual operations of addition and scalar multiplication, F^n is a vector space over F, as you should verify. The example of F^n motivated our definition of vector space.

Example 1.8. F^∞ is defined to be the set of all sequences of elements of F:

$$F^\infty = \{(f_1, f_2, ...) \mid f_i \in F \text{ for } i = 1, 2, ...\}.$$

Addition and scalar multiplication on F^∞ are defined as expected:

$$\begin{aligned}(f_1, f_2, ...) + (g_1, g_2, ...) &= (f_1 + g_1, f_2 + g_2, ...)\\ \lambda(f_1, f_2, ...) &= (cf_1, cf_2, ...).\end{aligned}$$

With these definitions, F^∞ becomes a vector space over F, as you should verify. The additive identity in this vector space is the sequence of all 0's.

Our next example of a vector space involves a set of functions.

Notation:F^S

- If S is a set, then F^S denotes the set of functions from S to F.
- For $f, g \in F^S$, the sum $f + g \in F^S$ is the function defined by $(f + g)(s) = f(s) + g(s)$ for all $s \in S$.
- For $\lambda \in F$ and $f \in F^S$, the product $\lambda f \in F^S$ is the function defined by $(\lambda f)(s) = \lambda f(s)$ for all $s \in S$.

Remark: As an example of the notation above, if S is the interval $[0, 1]$ and $F = \mathbb{R}$, then $\mathbb{R}^{[0,1]}$ is the set of real-valued functions on the interval [0, 1] The elements of the vector space $\mathbb{R}^{[0,1]}$ are real-valued functions on $[0, 1]$, not lists. In general, a vector space is an abstract entity whose elements might be lists, functions, or weird objects.

Now let's verify following example

Example 1.9. F^S **is a vector space**

- If S is a nonempty set, then F^S (with the operations of addition and scalar multiplication as defined above) is a vector space over F.
- The additive identity of F^S is the function $0\colon S \to F$ defined by $0(s) = 0$ for all $s \in S$.
- For $f \in F^S$, the additive inverse of $-f$ is the function $-f\colon S \to F$ defined by $(-f)(s) = -f(s)$ for all $s \in S$.

Proof. Let S be an set non empty set and let $f, g, h \in \mathbb{F}^S$. Let $x \in S$. Let $a, b \in \mathbb{F}$.

- **Commutataive**: $[f + g](x) = f(x) + g(x) = g(x) + f(x) = [g + f](x)$. (Since $f(x), g(x) \in \mathbb{F}$ and $\mathbb{F}$ have comutataive property.)Thus $f+g = g+f$ for all $f, g \in \mathbb{F}$
- **Associativity**:

- $[(f+g)+h](x) = (f(x)+g(x))+h(x) = f(x)+(g(x)+h(x)) = [f+(g+h)](x)$.(Since $f(x), g(x), h(x) \in \mathbb{F}$ and $\mathbb{F}$ have associavity property.)
- $[(a \cdot b) \cdot f](x) = (a \cdot b) \cdot (f(x)) = a \cdot (b \cdot f(x)) = [a \cdot (b \cdot f)](x)$ (Since $f(x) a, b \in \mathbb{F}$ and $\mathbb{F}$ have associavity property.

- **Aadditive Identity**: Let 0 be zero function $0\colon S \to F$ defined by $0(s) = 0$ for all $s \in S$. Then, $[f+0](x) = f(x)+0(x) = f(x)+0 = f(x)$.Thus, $f+0 = f$ for all $f \in F^S$.
- **Addtive Identity**: For $f \in F^S$, the additive inverse of $-f$ is the function $-f\colon S \to F$ defined by $(-f)(s) = -f(s)$ for all $s \in S$. Then $[f+(-f)](x) = f(x)+(-f)(x) = f(x)-f(x) = 0 = 0(x)$. Thus. $f+0 = f$, for all $f \in F^S$.
- **Multiplicative Identity**: $[1 \cdot f](x) = 1 \cdot (f(x)) = f(x)$. Thus, $1f = f$ for all $f \in F^S$.
- **Distributive Property**:

  - $[a(f+g)](x) = a \cdot ((f+g(x))) = a(f(x)+g(x)) = af(x)+ag(x) = [af](x)+[ag](x) = [af+ag](x)$.(Since $f(x), g(x), a \in \mathbb{F}$ and $\mathbb{F}$ have distributive property. Thus, $a(f+g) = af+ag$ for all $f, g \in F^S$ and $a \in F$.
  - $[(a+b)f](x) = (a+b)((f(x))) = a(f(x))+b(f(x)) = [af](x)+[bf](x) = [af+bf](x)$.(Since $f(x) a, b \in \mathbb{F}$ and $\mathbb{F}$ have distributive property. Thus, $(a+b)f = af+bf$ for all $f \in F^S$ and $a, b \in F$

□

The vector space F^n is a special case of the vector space F^S because each $(x_1, \dots, x_n) \in F^n$ can be thought of as a function x from the set $\{1, 2, \dots, n\}$ to F by writing $x(k)$ instead of x_k for the k^{th} coordinate of $(x_1, \dots, x_n)$. In other words, we can think of F^n as $F^{\{1,2,\dots,n\}}$. Similarly, we can think of F^∞ as $F^{\{1,2,\dots\}}$.

Proposition 1.4. *A vector space has a unique additive identity.*

Proof. Suppose 0 and $0'$ are both additive identities for some vector space V. Then,

$$0' = 0' + 0 = 0 + 0' = 0,$$

where the frst equality holds because 0 is an additive identity, the second equality comes from commutativity, and the third equality holds because $0'$ is an additive identity. Thus $0' = 0$, proving that V has only one additive identity. □

Proposition 1.5. *Every element in a vector space has a unique additive inverse.*

Proof. Suppose V is a vector space. Let $v \in V$. Suppose w and w' are additive inverses of v. Then

$$w = w + 0 = w + (v + w') = (w + v) + w' = 0 + w' = w'.$$

Thus $w = w'$, as desired. □

Notations: $-v$, $w - v$
Let $v \in V$. Then - $-v$ denotes the additive inverse of v; - $w - v$ is defined to be $w + (-v)$. "'

For the rest of this note, I will use follwoing noatation. *Notations* V denotes a vector space over F.

Proposition 1.6.

$$0v = 0 \textit{ for all } v \in V$$

0 *denotes a scalar (the number* $0 \in V$*) on the left side of the equation and a vector (the additive identity of* V*) on the right side of the equation.*

Proof. Let $v \in V$. [0v = (0 + 0)v= 0v + 0v] Adding the additive inverse of $0v$ to both sides of the equation above gives $0 = 0v$,as we want □

The result in proposition 1.6 involves the additive identity of V and scalar multiplication. The only part of the defnition of a vector space that connects vector addition and scalar multiplication is the distributive property. Thus the distributive property must be used in the proof of proposition 1.6.

Chapter 2

Exercise1

2.1 Exercise 1A

1. Show that $\alpha + \beta = \beta + \alpha$ for all $\alpha, \beta \in \mathbb{C}$.

Let $\alpha, \beta \in \mathbb{C}$. Then

$\alpha = x_1 + i y_1$ forsome

$\beta = x_2 + i y_2$ forsome

$$
\begin{aligned}
\alpha + \beta &= (x_1 + i y_1) + (x_2 + i y_2) \\
&= (x_1 + x_2) + i(y_1 + y_2) \\
&= (x_2 + x_1) + i(y_2 + y_1) \\
&= (x_2 + i y_2) + (x_1 + i y_1) \\
&= \beta + \alpha
\end{aligned}
$$

2. Show that $(\alpha + \beta) + \lambda = \alpha + (\beta + \lambda)$ for all $\alpha, \beta, \lambda \in \mathbb{C}$.

Let $\alpha, \beta, \lambda \in \mathbb{C}$. Then

$$\alpha = x_1 + iy_1 \text{ forsome } x_1, y_1 \in \mathbb{R}$$
$$\beta = x_2 + iy_2 \text{ forsome } x_2, y_2 \in \mathbb{R}$$
$$\lambda = x_3 + iy_3 \text{ forsome } x_3, y_3 \in \mathbb{R}$$

$$\begin{aligned}(\alpha+\beta)+\lambda &= \big((x_1+iy_1)+(x_2+iy_2)\big)+(x_3+iy_3)\\ &= \big((x_1+x_2)+i(y_1+y_2)\big)+(x_3+iy_3)\\ &= \big((x_1+x_2)+x_3\big)+i\big((y_1+y_2)+y_3\big)\\ &= \big(x_1+(x_2+x_3)\big)+i\big(y_1+(y_2+y_3)\big)\\ &= (x_1+iy_1)+\big((x_2+x_3)+i(y_2+y_3)\big)\\ &= (x_1+iy_1)+\big((x_2+iy_2)+(x_3+iy_3)\big)\\ &= \alpha+(\beta+\lambda)\end{aligned}$$

3. Show that $(\alpha\beta)\lambda = \alpha(\beta\lambda)$ for all $\alpha, \beta, \lambda \in \mathbb{C}$.

4. Show that $\lambda(\alpha+\beta) = \lambda\alpha + \lambda\beta$ for all $\lambda, \alpha, \beta \in \mathbb{C}$.

Let $\alpha, \beta, \lambda \in \mathbb{C}$. Then

$$\alpha = x_1 + i y_1 \text{ for some } x_1, y_1 \in \mathbb{R}$$
$$\beta = x_2 + i y_2 \text{ for some } x_2, y_2 \in \mathbb{R} \qquad \left(\text{def}^n \text{ of Complex numbers}\right)$$
$$\lambda = x_3 + i y_3 \text{ for some } x_3, y_3 \in \mathbb{R}$$

$$\begin{aligned}
\lambda(\alpha+\beta) &= (x_3 + i y_3)\big((x_1 + i y_1) + (x_2 + i y_2)\big) \\
&= (x_3 + i y_3)\big((x_1 + x_2) + i(y_1 + y_2)\big) && \left(\text{Commutative of addition of } \mathbb{R}\right) \\
&= \big(x_3(x_1 + x_2) - y_3(y_1 + y_2)\big) && \left(\text{def}^n \text{ of multiplication of } \mathbb{C}\right) \\
&\qquad + i\big(x_3(y_1 + y_2) + y_3(x_1 + x_2)\big) \\
&= (x_3 x_1 + x_3 x_2 - y_3 y_1 - y_3 y_2) && \left(\text{commutative property}\right) \\
&\qquad + i(x_3 y_1 + x_3 y_2 + y_3 x_1 + y_3 x_2) \\
&= (x_3 x_1 + i x_3 y_1) + (x_3 x_2 + i x_3 y_2) && \left(\text{commutative property}\right) \\
&\qquad - y_3 y_1 + i y_3 x_1 - y_3 y_2 + i y_3 x_2 \\
&= x_3(x_1 + i y_1) + x_3(x_2 + i y_2) && \left(\text{distributive property}\right) \\
&\qquad + y_3(-y_1 + i x_1) + y_3(-y_2 + i x_2) \\
&= x_3 \alpha + x_3 \beta + y_3(i^2 y_1 + i x_1) + y_3(i^2 y_2 + i x_2) && \left(\text{use } i^2 = -1\right) \\
&= x_3 \alpha + x_3 \beta + i y_3 (i y_1 + x_1) + i y_3 (i y_2 + x_2) && \left(\text{distributive}\right) \\
&= x_3 \alpha + x_3 \beta + i y_3 (x_1 + i y_1) + i y_3 (x_2 + i y_2) && \left(\text{distributive}\right) \\
&= x_3 \alpha + x_3 \beta + i y_3 \alpha + i y_3 \beta \\
&= (x_3 + i y_3)\alpha + (x_3 + i y_3)\beta && \left(\text{distributive property}\right) \\
&= \lambda \alpha + \lambda \beta
\end{aligned}$$

5. Show that for every $\alpha \in \mathbb{C}$, there exists a unique $\beta \in \mathbb{C}$ such that $\alpha + \beta = 0$.

5) Let $\alpha \in \mathbb{C}$.

Suppose that $\alpha + \beta_1 = 0$ and α

$\beta_1 \neq \beta_2$

$$\alpha + \beta_1 = 0$$
$$-\alpha + (\alpha + \beta_1) = -\alpha + 0$$
$$(-\alpha + \alpha) + \beta_1 = -\alpha$$
$$0 + \beta_1 = -\alpha$$
$$\beta_1 = -\alpha \text{ ——— } (1)$$

Similar we can get $\beta_2 = -\alpha$ ——— $\boxed{2}$

By (1) and (2) $\beta_1 = \beta_2$

Therefore, $\forall \alpha \in \mathbb{C}\ \exists! \beta$ such that $\alpha + \beta = 0$

6. Show that for every $\alpha \in \mathbb{C}$ with $\alpha \neq 0$, there exists a unique $\beta \in \mathbb{C}$ such

that $\alpha\beta = 1$.

6) Let $\alpha \in \mathbb{C} \setminus \{0\}$

Suppose that $\alpha\beta_1 = 1 = \alpha\beta$

$$\alpha\beta_1 = \alpha\beta_2$$

$$\frac{1}{\alpha}(\alpha\beta_1) = \frac{1}{\alpha}(\alpha\beta_2)$$

$$\left(\frac{1}{\alpha}\cdot\alpha\right)\beta_1 = \left(\frac{1}{\alpha}\cdot\alpha\right)\beta_2$$

$$1\cdot\beta_1 = 1\cdot\beta_2$$

$$\beta_1 = \beta_2$$

Therefore, $\forall \alpha \in \mathbb{C}\ \exists!\, \beta$ suchthat α

7. Show that $\frac{-1+\sqrt{3}i}{2}$ is a cube root of 1 (meaning that its cube equals 1).

7) Let $\alpha = \frac{-1}{2} + i\frac{\sqrt{3}}{2}$

Then $\alpha^2 = \left(-\frac{1}{2} + i\frac{\sqrt{3}}{2}\right)^2 = \left(-\frac{1}{2} + i\frac{\sqrt{3}}{2}\right)\left(-\frac{1}{2} + i\frac{\sqrt{3}}{2}\right)$

$= \left(\frac{1}{4} - \frac{3}{4}\right) + i\left(-\frac{\sqrt{3}}{4} - \frac{\sqrt{3}}{4}\right)$

$= -\frac{2}{4} + i\left(-\frac{2\sqrt{3}}{4}\right)$

$= \left(-\frac{1}{2} - \frac{i\sqrt{3}}{2}\right)$

$\alpha^3 = \alpha \cdot \alpha^2 = \left(-\frac{1}{2} + i\frac{\sqrt{3}}{2}\right)\left(-\frac{1}{2} - \frac{i\sqrt{3}}{2}\right)$

$= \left(\frac{1}{4} + \frac{3}{4}\right) + i\left(\frac{\sqrt{3}}{4} - \frac{\sqrt{3}}{4}\right)$

$= 1 + 0i = 1$

Thus α is cube root of 1.

8. Find two distinct square roots of i.

8) We have to find $\alpha \in \mathbb{C}$ such that $\alpha^2 = i$.

Let $\alpha = x + iy$, for some $x, y \in \mathbb{R}$.

$$\alpha^2 = i$$
$$(x+iy)^2 = i$$
$$(x+iy)(x+iy) = i$$
$$(x^2 - y^2) + i(xy + xy) = i$$
$$(x^2 - y^2) + i(2xy) = 0 + i$$

Thus $x^2 - y^2 = 0$. and $2xy = 1$

$((x-y)(x+y) = 0)$ and $2xy = 1$

$(x = y$ or $x = -y)$ and $2xy = 1$

if $x = (-y)$ then $2xy = 2(-y)y = -2y^2 = 1$

Since $y \in \mathbb{R}$ and $y > 0$, this is impossible.

Thus, $x = y$.

Then $2xy = 2x^2 = 1 \Rightarrow x = \pm 1/\sqrt{2}$.

Hence the square root of i is $\left(\frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}}\right)$ and $\left(-\frac{1}{2} - \frac{1}{\sqrt{2}}i\right)$

9. Find $x \in \mathbb{R}^4$ such that $(4, -3, 1, 7) + 2x = (5, 9, -6, 8)$.

9) $(4, -3, 1, 7) +$

$-(4, -3, 1, 7) + (4, -3, 1, 7) +$

$2x = -(4, -3, 1, 7$

$2x = (-4+5, -3$

$x = (1/2, 6,$

10. Explain why there does not exist $\lambda \in \mathbb{C}$ such that $\lambda(2-3i, 5+4i, -6+7i) = (12-5i, 7+22i, -32-9i)$.

(10) Assume the contray there exist $\lambda \in \mathbb{C}$ such that

$$\lambda(2-3i, 5+4i, -6+7i) = (12-5i, 7+22i, -32-9i)$$

$$(\lambda(2-3i), \lambda(5+4i), \lambda(-6+7i)) = (12-5i, 7+22i, -32-9i)$$

Then,

$$\lambda(2-3i) = 12-5i \quad \text{--- } ①$$

$$\lambda(5+4i) = 7+22i$$

$① \times (2+3i)$

$$\lambda(2-3i)(2+3i) = (12-5i)(2+3i)$$

$$\lambda(4+9) = 13\lambda = 39+26i$$

$$\lambda = 3+2i \quad \text{--- } (*)$$

$② \times (5-4i)$

$$\lambda \cdot (5-4i)(5+4i) = (7+22i)(5+4i)$$

$$\lambda(25+16) = 41\lambda = -53+138i$$

$$\lambda = -\frac{53}{41} + \frac{138i}{41} \quad \text{--- } (**)$$

$(*)$ and $(**)$ gives a contradiction.

11. Show that $(x + y) + z = x + (y + z)$ for all $x, y, z \in \mathbb{F}_n$.

12. Show that $(ab)x = a(bx)$ for all $x \in \mathbb{F}_n$ and all $a, b \in \mathbb{F}$.

13. Show that $1x = x$ for all $x \in \mathbb{F}_n$.

14. Show that $\lambda(x+y) = \lambda x + \lambda y$ for all $\lambda \in \mathbb{F}$ and all $x, y \in \mathbb{F}_n$.

15. Show that $(a+b)x = ax + bx$ for all $a, b \in \mathbb{F}$ and all $x \in \mathbb{F}_n$.

2.2 Exercise 1B

1. Prove that $-(-v) = v$ for every $v \in V$. **Solution**: Let $v \in V$. Then there exist a unique additive inverse of v. We denote it by $-v$. Thus,

$$v + (-v) = 0.$$

Then by definition, additive inverse of $(-v)$ is v. We denote it by $-(-v) = v$.

2. Suppose $a \in \mathbb{F}, v \in V$, and $v = 0$. Prove that $a = 0$ or $v = 0$.

(2) Suppose that $a \in F$ ~~and~~ $v \in V$ and

$$av = 0$$

~~Then~~ if $a = 0$, we have nothing to proove.
So assume that $a \neq 0$. Now I need to show that $v = 0$.
Since $a \in F$ and $a \neq 0$. multiplicative inverse of a exist. Let's denote it by a^{-1}. Morever,

$$a^{-1}a = 1 = a a^{-1}$$

$$av = 0$$

$$a^{-1}(av) = (a^{-1}a)v = 1 \cdot v = v = 0 \text{ //}$$

3. (3) Suppose that $v, w \in V$.

First of we need to show that ~~b~~ the existance of such x.

Let $x = \frac{1}{3}w - \frac{1}{3}v$

Note that $\frac{1}{3} \in \mathbb{F}$ and $v, w \in V$, then

$\frac{1}{3}w - \frac{1}{3}v = x \in V$.

Further,

$$v + 3x = v + 3\left(\frac{1}{3}w - \frac{1}{3}v\right)$$

$$= v + 3 \cdot \frac{1}{3} \cdot w - 3 \cdot \frac{1}{3} \cdot v$$ (Since $\frac{1}{3}$ is multiplical inverent)

$$= v + w - v$$

$$= w$$

~~A~~ We are existance of $x \in V$.

Now we have to proove uniquness. Assume that there exist $y \in V$ statis~~t~~fy $v + 3y = w$

Then $w = \cancel{3x}\ v + 3x = v + 3y$

Then $-v + (v + 3x) = -v + (v + 3y)$

$$(-v + v) + 3x = (-v + 3v) + 3y$$

$$3x = 3y$$

$$\frac{1}{3} \cdot (3x) = \frac{1}{3} \cdot (3y)$$

$$\left(\frac{1}{3} \cdot 3\right) \cdot x = 1 \cdot x = x = y = 1 \cdot y = \left(\frac{1}{3} \cdot 3\right) y$$

4.

4 The empty set is not a vector space. The empty set fails to satisfy only one of the requirements listed in the definition of a vector space (1.20). Which one?

Recall the defn of vectorspace

1.20 definition: *vector space*

A *vector space* is a set V along with an addition on V and a scalar multiplication on V such that the following properties hold.

commutativity

$u + v = v + u$ for all $u, v \in V$.

associativity

$(u + v) + w = u + (v + w)$ and $(ab)v = a(bv)$ for all $u, v, w \in V$ and for all $a, b \in \mathbf{F}$.

additive identity

There exists an element $0 \in V$ such that $v + 0 = v$ for all $v \in V$.

additive inverse

For every $v \in V$, there exists $w \in V$ such that $v + w = 0$.

multiplicative identity

$1v = v$ for all $v \in V$.

distributive properties

$a(u + v) = au + av$ and $(a + b)v = av + bv$ for all $a, b \in \mathbf{F}$ and all $u, v \in V$.

A vector space must contain $0 \in V$ (additive identity)
So, the emptyset fails the addtive identity property

5.

5 Show that in the definition of a vector space (1.20), the additive inverse condition can be replaced with the condition that

$$0v = 0 \text{ for all } v \in V.$$

Here the 0 on the left side is the number 0, and the 0 on the right side is the additive identity of V.

The phrase a "condition can be replaced" in a definition means that the collection of objects satisfying the definition is unchanged if the original condition is replaced with the new condition.

additive inverse

For every $v \in V$, there exists $w \in V$ such that $v + w = 0$.

Need to show:

$$(\forall v \in V\ \exists w \in V \text{ s.t. } v+w=0_F) \Longleftrightarrow \forall v \in V,\ 0_F v = 0_V$$

proof: $\Leftarrow$

Let $v \in V$. Suppose that $0_F w = 0_V$ for all $w \in V$.

Since $1 \in F$, addtive inverse of 1 is exists.

Further, $1 + (-1) = 0_F$

$$(1 + (-1))v = 0_F v = 0_V$$

$$v + (-1)v = 1 \cdot v + (-1)v = 0_F v = 0_V$$

Thus, adative identity of V is "$(-1)v$". So, we are done backward direction. Let's proove the forward direction.

"$\Leftarrow$"

Now suppose that $\forall v \in V\ \exists w \in V$ s.t. $v+w = 0_F$

$$0_F v = (0_F + 0_F) v \quad \text{(addtive identiy of feild)}$$
$$= 0_F v + 0_F v \quad \text{(distributive property)} \quad ①$$

Since $0_F v \in V$, there exist additive identity of $0_F v$. Let's all that as w. Furthur

$$0_F v + w = 0 \quad ②$$

Now consider, (By ①)

$$0_F v = 0_F v + 0_F v$$
$$\underbrace{(0_F v + w)}_{0_V} = 0_F v + \underbrace{(0_F v + w)} \quad \text{(property of equality)}$$
$$0_V = 0_F v + 0_V \quad \text{(By ②)}$$
$$0_V = 0_F v \quad \text{(additive identity)}$$

6.

6 Let ∞ and $-\infty$ denote two distinct objects, neither of which is in $\mathbf{R}$. Define an addition and scalar multiplication on $\mathbf{R} \cup \{\infty, -\infty\}$ as you could guess from the notation. Specifically, the sum and product of two real numbers is as usual, and for $t \in \mathbf{R}$ define

$$t\infty = \begin{cases} -\infty & \text{if } t < 0, \\ 0 & \text{if } t = 0, \\ \infty & \text{if } t > 0, \end{cases} \qquad t(-\infty) = \begin{cases} \infty & \text{if } t < 0, \\ 0 & \text{if } t = 0, \\ -\infty & \text{if } t > 0, \end{cases}$$

and

$$\begin{aligned} t + \infty = \infty + t = \infty + \infty &= \infty, \\ t + (-\infty) = (-\infty) + t = (-\infty) + (-\infty) &= -\infty, \\ \infty + (-\infty) = (-\infty) + \infty &= 0. \end{aligned}$$

With these operations of addition and scalar multiplication, is $\mathbf{R} \cup \{\infty, -\infty\}$ a vector space over $\mathbf{R}$? Explain.

$\mathbb{R} \vee (-\infty, \infty)$ is NOT a vector space.

- Asscitivity property fails. Let $0 \neq v \in V$

Consider,

$$v + (\infty + (-\infty)) = v + 0 = v \neq 0 = \infty + (-\infty) = (v + \infty) + (-\infty)$$

- Unique additive identity fails (Extra part)

Consider.

$$\infty = (2 + (-1)) \cdot \infty = 2\infty + (-1)\infty = \infty - \infty = 0$$

Ther for any $t \in \mathbb{R}$.

$$t = 0 + t = \infty + t = \infty = 0.$$

So, this fails, uniqueness property of additive identity

7. Suppose S is a non-empty set. Let V^S denote the set of functions from

S to V. Define a natural addition and scalar multiplication on V^S, and show that V^S is a vector space with these definitions.

7 Suppose S is a nonempty set. Let V^S denote the set of functions from S to V. Define a natural addition and scalar multiplication on V^S, and show that V^S is a vector space with these definitions.

Let S be a set with $S \neq \emptyset$. Let

$$V^S := \{ f : S \longrightarrow V \mid f \text{ is functions to } V \}$$

Let $f, g \in V^S$. Let $\lambda \in S$ Define

- addition : $V^S \times V^S \longrightarrow V^S$

$$f+g(x) = f(x) + g(x) \quad \forall x \in S.$$

- Scaler multiplication : $V \times V^S \longrightarrow V^S$

$$[\lambda f](x) = \lambda \cdot (f(x)) \quad \forall x \in S.$$

NTS: V^S is a vector space

Commutative

claim 1: $f+g = g+f \quad \forall f, g \in V^S$

Let $f, g \in V^S$

$$[f+g](x) = f(x)+g(x) \quad \forall x \in V$$
$$= g(x)+f(x) \quad \forall x \in V \quad (\because f(x), g(x) \in V)$$
$$= [g+f](x) \quad \forall x \in V$$

Thus, $f, g \in V$

Assciativity

claim 2.1: $(f+g)+h = f+(g+h) \quad \forall f, g, h \in V^S$

Let $f, g, h \in V^S$. Note that

$$[(f+g)+h](x) = [f+g](x) + h(x)$$
$$= (f(x)+g(x)) + h(x)$$
$$= f(x) + (g(x)+h(x))$$
$$= [f+(g+h)](x)$$

claim 2.2: $(ab)f = a(bf) \quad \forall f \in V^S, \forall a, b \in V$

Let $a, b \in V$ and $f \in V^S$

$$[(ab)f](x) = (ab)f(x)$$
$$= a(b f(x))$$
$$= [a(bf)](x)$$

Addtive identity.

claim: $\left.\begin{aligned} 0_{map} : S &\longrightarrow V \\ x &\longmapsto 0 \end{aligned}\right\}$ is the addtive identity

such that $f + 0_{map} = f \quad \forall f \in V^S$ (Note that $0_{map} \in V^S$)

$$[f + 0_{map}](x) = f(x) + 0_{map}(x) = f(x) + 0 = f(x)$$

Addtive inverse

Let $f \in V^S$. Let define,

$$\begin{aligned} g : S &\longrightarrow V \\ x &\longmapsto -f(x) \end{aligned}$$

Note that $g \in V^S$,

$$\begin{aligned} [f+g](x) &= f(x) + g(x) \\ &= f(x) + (-f(x)) \\ &= 0. \end{aligned}$$

multiplicative inverse

claims. $1f = f \quad \forall f \in V^S$

Let $f \in V^S$.

$$[1 \cdot f](x) = 1 \cdot f(x) = f(x).$$

Distributiv-

claim.6.1: $a(f+g) = af + ag \quad \forall a \in V, \forall f, g \in V^S$.

Let $f, g \in V^S$, and let $a \in V$.

$$\begin{aligned}[a(f+g)](x) &= a([f+g](x)) \\ &= a(f(x) + g(x)) \\ &= af(x) + ag(x) \\ &= [af](x) + [ag](x) \\ &= [af + ag](x)\end{aligned}$$

Claim 6.2: $(a+b)f = af + bf, \ \forall f \in V^S \ \forall a, b \in V$

$$\begin{aligned}[(a+b)f](x) = (a+b)f(x) &= af(x) + bf(x) \\ &= [af](x) + [bf](x) = \\ &= [af + bf](x)\end{aligned}$$

Therefore, V^S is a Vector space

8.

8 Suppose V is a real vector space.

- The *complexification* of V, denoted by $V_{\mathbf{C}}$, equals $V \times V$. An element of $V_{\mathbf{C}}$ is an ordered pair (u, v), where $u, v \in V$, but we write this as $u + iv$.
- Addition on $V_{\mathbf{C}}$ is defined by

$$(u_1 + iv_1) + (u_2 + iv_2) = (u_1 + u_2) + i(v_1 + v_2)$$

for all $u_1, v_1, u_2, v_2 \in V$.

- Complex scalar multiplication on $V_{\mathbf{C}}$ is defined by

$$(a + bi)(u + iv) = (au - bv) + i(av + bu)$$

for all $a, b \in \mathbf{R}$ and all $u, v \in V$.

Prove that with the definitions of addition and scalar multiplication as above, $V_{\mathbf{C}}$ is a complex vector space.

Think of V as a subset of $V_{\mathbf{C}}$ by identifying $u \in V$ with $u + i0$. The construction of $V_{\mathbf{C}}$ from V can then be thought of as generalizing the construction of $\mathbf{C}^n$ from $\mathbf{R}^n$.

Commutative

Let $\alpha_1 + i\beta, \alpha_2 + i\beta_2 \in V_c$.

$$(\alpha_1 + i\beta_1) + (\alpha_2 + i\beta_2) = (\alpha_1 + \alpha_2) + i(\beta_1 + \beta_2)$$
$$= (\alpha_2 + \alpha_1) + i(\beta_2 + \beta_1) \quad \left(\begin{array}{l}\alpha_1, \alpha_2, \beta_1 \beta_2 \in V \\ \text{Commutative pr.}\end{array}\right)$$
$$= (\alpha_2 + i\beta_2) + (\alpha_1 + i\beta_1)$$

Associativity

Let $\alpha_1+i\beta_1, \alpha_2+i\beta_2, \alpha_3+i\beta_3 \in V_{\mathbb{C}}$

$(\alpha_1+i\beta_1)+\left[(\alpha_2+i\beta_2)+(\alpha_3+i\beta_3)\right]$

$$= (\alpha_1+i\beta_1)\left[(\alpha_2+\alpha_3)+i(\beta_2+\beta_3)\right]$$
$$= \alpha_1+(\alpha_2+\alpha_3)+i(\beta_1+(\beta_2+\beta_3))$$
$$= (\alpha_1+\alpha_2)+\alpha_3+i((\beta_1+\beta_2)+\beta_3)$$
$$= \left[(\alpha_1+\alpha_2)+i(\beta_1+\beta_2)\right]+(\alpha_3+i\beta_3)$$
$$= \left[(\alpha_1+i\beta_1)+(\alpha_2+i\beta_2)\right]+(\alpha_3+i\beta_3)$$

Let $x_1+iy_1, x_2+iy_2 \in \mathbb{C}$ and $(\alpha+i\beta) \in V_{\mathbb{C}}$

$$= \left[(x_1+iy_1)\cdot(x_2+iy_2)\right](\alpha+i\beta)$$
$$= \left[(x_1x_2-y_1y_2)+i(x_1y_2+x_2y_1)\right](\alpha+i\beta)$$
$$= (x_1x_2-y_1y_2)\alpha-(x_1y_2+x_2y_1)\beta + i(\beta(x_1x_2-y_1y_2)+\alpha(x_1y_2+x_2y_1))$$
$$= x_1x_2\alpha-y_1y_2\alpha-x_1y_2\beta-x_2y_1\beta + i(x_1x_2\beta-y_1y_2\beta+x_1y_2\alpha+\boxed{x_2y_1\alpha}$$

$$= x_1x_2\alpha + x_1y_2\beta - y_1x_2\beta - y_1y_2\alpha$$
$$+ i(x_1x_2\beta + x_1y_2\alpha + \boxed{y_1x_2\alpha} - y_1y_2\beta$$
$$= x_1(x_2\alpha + y_2\beta) - y_1(x_2\beta + y_2\alpha)$$
$$+ i\left[x_1(x_2\beta + y_2\alpha) + y_1(x_2\alpha - y_2\beta)\right]$$
$$= (x_1 + iy_1)\left[(x_2\alpha - y_2\beta) + i(x_2\beta + y_2\alpha)\right]$$
$$= (x_1 + iy_1)\left[(x_2 + iy_2)\cdot(\alpha + i\beta)\right]$$

Additive Identity

Let $\underline{0}$ be the zero vector. ($\underline{0} \in V$) Let $\alpha + i\beta \in V_{\mathbb{C}}$

$$(\alpha + i\beta) + (\underline{0} + i\underline{0}) = (\alpha + \underline{0}) + i(\beta + \underline{0})$$
$$= \alpha + i\beta$$

Additive inverse,

Let $\alpha + i\beta \in V_{\mathbb{C}}$. Now consider,

$$\alpha + i\beta + (-\alpha + i(-\beta)) = (\alpha + (-\alpha)) + i(\beta + (-\beta)) = 0$$

multiplicative identity

$$(1 + i\cdot 0)\cdot(\alpha + i\beta) = (1\cdot\alpha - 0\,\beta) + i(1\cdot\beta + 0\cdot\alpha) = \alpha + i\beta$$

Distributive

Let $(x+iy) \in \mathbb{C}$ and, $(\alpha_1+i\beta_1), (\alpha_2+i\beta_2) \in V_{\mathbb{C}}$

$$
\begin{aligned}
(x+iy)\ \big((\alpha_1+i\beta_1)+(\alpha_2+i\beta_2)\big)
&= (x+iy)\left[(\alpha_1+\alpha_2)+i(\beta_1+\beta_2)\right] \\
&= x(\alpha_1+\alpha_2) - y(\beta_1+\beta_2) \\
&\quad + i\left[x(\beta_1+\beta_2)+y(\alpha_1+\alpha_2)\right] \\
&= x\alpha_1 + x\alpha_2 - y\beta_1 - y\beta_2 \\
&\quad + i\left[x\beta_1 + x\beta_2 + y\alpha_1 + y\alpha_2\right] \\
&= (x\alpha_1 - y\beta_1) + i(x\beta_1 + y\alpha_1) \\
&\quad + (x\alpha_2 - y\beta_2) + i(x\beta_2 - y\alpha_2) \\
&= (x+iy)(\alpha_1+i\beta_1) + (x+iy)(\alpha_2+i\beta_2)
\end{aligned}
$$

Let $(x_1+iy_1), (x_2+iy_2) \in \mathbb{C}$ and $(\alpha+i\beta) \in V_{\mathbb{C}}$

$$
\begin{aligned}
\left[(x_1+iy_1)+(x_2+iy_2)\right](\alpha+i\beta)
&= \left[(x_1+x_2)+i(y_1+y_2)\right](\alpha+i\beta) \\
&= \left[(x_1+x_2)\alpha - (y_1+y_2)\beta\right] + i\left[(x_1+x_2)\beta + (y_1+y_2)\alpha\right]
\end{aligned}
$$

$$= [x_1\alpha + x_2\alpha - y_1\beta - y_2\beta] + i[x_1\beta + x_2\beta + y_1\alpha + y_2\alpha]$$

$$= [(x_1\alpha - y_2\beta) + i(x_1\beta + y_2\alpha)] + [(x_2\alpha - y_2\beta) + i(x_2\beta + y_2\alpha)]$$

$$= (x_1 + iy_2)(\alpha + i\beta) + (x_2 + iy_2)(\alpha + i\beta)$$

2.3 Exercise 1C

1.

1 For each of the following subsets of $\mathbf{F}^3$, determine whether it is a subspace of $\mathbf{F}^3$.

(a) $\{(x_1, x_2, x_3) \in \mathbf{F}^3 : x_1 + 2x_2 + 3x_3 = 0\} = A$

(b) $\{(x_1, x_2, x_3) \in \mathbf{F}^3 : x_1 + 2x_2 + 3x_3 = 4\} = B$

(c) $\{(x_1, x_2, x_3) \in \mathbf{F}^3 : x_1 x_2 x_3 = 0\} = C$

(d) $\{(x_1, x_2, x_3) \in \mathbf{F}^3 : x_1 = 5x_3\} = D$

a) A is subspace of $\mathbb{F}^3$?

- Claim1: $(0,0,0) \in \mathbb{F}^3$

  Note that $0+2\cdot 0+3\cdot 0=0$. Thus, $(0,0,0) \in A$

- Claim2: $\forall (x_1, x_2, x_3), (y_1, y_2, y_3) \in A$; $(x_1, x_2, x_3)+(y_1, y_2, y_3) \in A$

Let $(x_1, x_2, x_3), (y_1, y_2, y_3) \in A$.

Then, $x_1+2x_2+3x_3=0$ — ①

$y_1+2y_2+3y_3=0$ — ②

①+② $(x_1+x_2+3x_3)+(y_1+2y_2+3y_3)=0$

$(x_1+y_1)+2(x_2+y_2)+3(x_3+y_3)=0$

Thus, $(x_1+y_1, x_2+y_2, x_3+y_3)=0$

- Claim3: Closed under sclaer multiplication

  Let $a \in \mathbb{F}$ and $(x_1, x_2, x_3) \in A$. Then,

$x_1+2x_2+3x_3=0$

$a\cdot(x_1+2x_2+3x_3) = ax_1+a(2x_2)+a(3x_3) = ax_1+2(ax_2)+3(ax_3)=0$

Then $a(x_1, x_2, x_3) = 0 \in A$.

b) B is not a subspace

Note that, $(0,0,0) \notin B$.

Because, $0+2\cdot 0+3\cdot 0=0 \neq 4$.

c) C is a NOT subspace.

Note that $(1,1,0), (0,0,1) \in C$. $\left(\text{Because } \begin{array}{l} 1\cdot 1\cdot 0=0 \text{ and} \\ 0\cdot 0\cdot 1=0 \end{array}\right)$

But $(1,1,0)+(0,0,1)=(1,1,1) \notin F$

$(\text{Because } 1\cdot 1\cdot 1=1\neq 0)$

d) D is a subspace.

- claim 1: $(0,0,0) \in D$

(Because, $0=5\cdot 0$)

- claim 2: $\forall (x_1,x_2,x_3), (y_1,y_2,y_3) \in D$,

  $(x_1,x_2,x_3)+(y_1,y_2,y_3) \in D$

Let $(x_1,x_2,x_3), (y_1,y_2,y_3) \in D$.

Then, $x_1=5x_3$ and $y_1=5y_3$

So, $x_1+x_2=5x_3+5y_3=5(x_3+y_3)$

Thus, $(x_1,x_2,x_3)+(y_1,y_2,y_3)=(x_1+y_1, x_2+y_2, x_3+y_3) \in D$

• Sclar multiplication

Let $a \in F$ and $(x_1, x_2, x_3, x_4) \in$

Then $x_2 = 5x_4$

So, $ax_2 = a(5x_4) = 5(ax_4$

Then,

$a(x_1, x_2, x_3, x_4) = (ax_1, ax_2, ax_3, a$

Hence, U is subspace

Therefore, U is subspace iff

2.

Statement 1: If $b \in \mathbb{F}$, then

$$\{(x_1, x_2, x_3, x_4) \in \mathbb{F}^4 : x_3 = 5x_4 + b\}$$

is a subspace of $\mathbb{F}^4$ if and only if $b = 0$.

Proof of Statement 1

1.35 example: *subspaces*

(a) If $b \in \mathbf{F}$, then

$$\{(x_1, x_2, x_3, x_4) \in \mathbf{F}^4 : x_3 = 5x_4 + b\}$$

is a subspace of $\mathbf{F}^4$ if and only if $b = 0$.

Let $b \in F$. and let

$$V := \{(x_1, x_2, x_3, x_4) \in F^4 : x_3 = 5x_4 + b\}$$

claim: V is subspace of $F^4 \Leftrightarrow b = 0$

"$\Rightarrow$"

Suppose that V is subspae. Then $(0,0,0,0) \in V$

Then, $0 = 5 \cdot 0 + b \Rightarrow b = 0$.

"$\Leftarrow$"

Now suppose that $b = 0$. Then, $x_3 = 5x_4$ — (1)

• Addtive id

Note that $(0,0,0,0) \in D$. Because $0 = 5 \cdot 0 = 0$.

• Closed under addition

Let $(x_1, x_2, x_3, x_4), (y_1, y_2, y_3, y_4) \in D$.

Then $x_1 = 5x_4$ and $y_1 = 5y_4$

- Sclar multiplication

Let $a \in F$ and $(x_1, x_2, x_3, x_4) \in U$.

Then $x_2 = 5x_4$

So, $ax_2 = a(5x_4) = 5(ax_4)$

Then,

$a(x_1, x_2, x_3, x_4) = (ax_1, ax_2, ax_3, ax_4) \in U$.

Hence, U is subspace

Therefore, U is subspace iff $b = 0$.

Statement 2 : The set of continuous real-valued functions on the interval $[0, 1]$ is a subspace of $\mathbb{R}^[0, 1]$

Proof of statement 2:

(b) The set of continuous real-valued functions on the interval [
of $\mathbf{R}^{[0,1]}$.

- Let $O_{map} : [0,1] \longrightarrow \mathbb{R}$ defined
$x \longmapsto 0$

O_{map} is continous map. (Since it cons
Hence, $O_{map} \in \mathbb{R}^{[0,1]}$ (See claim1)

- Let $f, g \in \mathbb{R}^{[0,1]}$. Then $[f+g]$
Continous function on $[0,1]$. Thus $f+$
- Let $c \in \mathbb{R}$ and $f \in \mathbb{R}^{[0,1]}$. Then cf i
Continuous function on $[0,1]$. Thus

Claim: O_{map} is continous on $\mathbb{R}^{[0,1]}$

Let $x_0 \in [0,1]$

Let $\varepsilon > 0$. Choose $\delta = 1 > 0$. Let $x \in [$

Suppose that $|x - x_0| < 1$.

Now consider,

$$|O_{map}(x) - O_{map}(x_0)| = |0 - 0| = 0 < \varepsilon$$

Therefore, O_{map} is continous map.

Claim 2: If $f, g \in R^{[0,1]}$ then $f+g \in R^{[0,1]}$

Suppose that $f, g \in R^{[0,1]}$. Let $x_0 \in [0,1]$

Then Iff is continous at x_0,

$\forall \varepsilon > 0 \exists \delta_1 > 0$ s.t. $\forall x \in [0,1]$,

$$|x - x_0| \leq \delta \Rightarrow |f(x) - f(x_0)| < \varepsilon/2 \quad (1)$$

If g is continous at x_0, where $x_0 \in [0,1]$

$\forall \varepsilon > 0 \exists \delta_2 > 0$ s.t. $\forall x \in [0,1]$

$$|x - x_0| < \delta_2 \Rightarrow |g(x) - g(x_0)| < \varepsilon/2 \quad (2)$$

Let $\varepsilon > 0$

Choose $\delta_3 = \max\{\delta_1, \delta_2\}$. Let $x \in [0,1]$.

Suppose that $|x - x_0| < \delta$. Now consider,

$$\begin{aligned}
|[f+g](x) - [f+g](x_0)| &= |f(x) + g(x) - (f(x_0) + g(x_0))| \\
&= |f(x) + g(x) - f(x_0) \; g(x_0)| \\
&= |f(x) - f(x_0) + g(x) - g(x_0)| \\
&\leq |f(x) - f(x_0)| + |g(x) - g(x_0)| \\
&= \varepsilon/2 + \varepsilon/2 = \varepsilon
\end{aligned}$$

Thus, $f+g$ is continous at every point in $[0,1]$

claim3: If $c \in \mathbb{R}$ and $f \in \mathbb{R}^{[0,1]}$ then $[cf] \in \mathbb{R}^{[0,1]}$

Suppose that $f, g \in \mathbb{R}^{[0,1]}$ and $c \in \mathbb{R}$ Let $x_0 \in [0,1]$

Then If f is continous at x_0,

$\forall \varepsilon > 0 \, \exists \delta_0 > 0$ s.t. $\forall x \in [0,1]$,

$$|x - x_0| \leq \delta_0 \Rightarrow |f(x) - f(x_0)| < \varepsilon/|c| \quad (1)$$

Let $\varepsilon > 0$. Choose $\delta = \delta_0 > 0$

Suppose that $|x - x_0| < \delta$.

$$\begin{aligned} |[cf](x) - [cf](x_0)| &= |c(f(x)) - c \cdot (f(x_0))| \\ &= |c(f(x) - f(x_0))| \\ &\leq |c|\,|f(x) - f(x_0)| \\ &\leq |c| \cdot \frac{\varepsilon}{|c|} = \varepsilon \end{aligned}$$

Thus, $[cf] \in \mathbb{R}^{[0,1]}$.

Statement 3 : The set of differentiable real-valued functions on $\mathbb{R}$ is a subspace of $\mathbb{R}^{\mathbb{R}}$.

Proof of Statement 3

(c) The set of differentiable real-valued functi

- 0 map is differntible. So,
- Since, sum of differnti
  differntiable real-value func
  If $f, g \in \mathbb{R}^{\mathbb{R}}$ then $f+g \in \mathbb{R}$
- If $c \in \mathbb{R}$ and $f \in \mathbb{R}^{\mathbb{R}}$, then
  (cf) is differntiable.

Statement 4 : The set of differentiable real-valued functions $\mathbb{R}$ on the interval $(0, 3)$ such that $f'(2) = b$ is a subspace of $\mathbb{R}^{(0,3)}$ if and only if $b = 0$.

Proof of Statement 4

(d) The set of differentiable real-valued functions f on the interval $(0,3)$ such that $f'(2) = b$ is a subspace of $\mathbf{R}^{(0,3)}$ if and only if $b = 0$.

Let $U := \{ f : (0,3) \to \mathbb{R} \mid f'(2) = b \}$

NTS: U is subspace of $\mathbb{R}^{(0,3)} \iff b = 0$

"$\Rightarrow$"

Suppose that U is subspace of $\mathbb{R}^{(0,3)}$

Then, $0_{map} \in U$. Then $[(0_{map})'](2) = 0_{map}(2) = 0$

"$\Leftarrow$"

Now suppose that, $b = 0$.

- Observe that 0_{map} is differentiable and $[(0_{map})'](2) = [0_{map}](2) = 0$
- Let $f, g \in U$. Then f, g are differntiable and
  $f'(2) = g'(2)$
  Then $f+g$ are differntiable.
  Furthur, $[f+g](2) = f(2) + g(2) = 0 + 0 = 0$.
  Thus, $f+g \in U$.
- Let $f \in U$ and $a \in \mathbb{R}$. So $f'(2) = 0$.
  Then af are differntiable. Moreover,
  $[(af)'](2) = a(f'(2)) = a \cdot 0 = 0$. Thus $af \in U$.

Therefore, U is a subspace

Statement 5 : The set of all sequences of complex numbers with limit 0 is a subspace of $\mathbb{C}^\infty$

Proof of Statement 5:

(e) The set of all sequences of complex numbers with limit 0 is a subspace of $\mathbb{C}$

Let $U := \{(x_n)_{n\in\mathbb{N}} \mid x_n \in \mathbb{C}$ and $x_n \to 0$ as $n \to \infty\}$

- Consider, a sequnce with all components are 0
  Observe that sequnce $(0, 0, 0, \ldots) \in U$.
- Let $(x_n), (y_n) \in U$. So,
  $\lim_{n\to\infty} x_n = 0$ and $\lim_{n\to\infty} y_n = 0$.
  Then, $\lim_{n\to\infty}(x_n + y_n) = \lim_{n\to\infty} x_n + \lim_{n\to\infty} y_n = 0$
  Therefore, $(x_n + y_n)_{n\in\mathbb{N}} \in U$.
- Let $a \in \mathbb{C}$ and $(x_n)_{n\in\mathbb{N}}$. So, $\lim_{n\to\infty} x_n = 0$.
  Now observe
  $\lim_{n\to\infty} c x_n = c \lim_{n\to\infty} x_n = c \cdot 0 = 0$.
  Thus, $cf \in U$.

Therefore, U is a subspace.

3.

3 Show that the set of differentiable real-valued functions f on the interval $(-4, 4)$ such that $f'(-1) = 3f(2)$ is a subspace of $\mathbf{R}^{(-4,4)}$.

Let $U := \{ f : (-4,4) \to \mathbb{R} : f'(-1) = 3f(2) \}$

• ~~Let~~ Consider $0_{map} : (-4,4) \to \mathbb{R}$ defined like

$$x \mapsto 0$$

$$[(0\,map)'](-1) = 0 = 3 \cdot 0 = 3\,(0\,map(2))$$

• Let $f, g \in U$. So, $f'(-1) = 3f(2)$ and $g'(-1) = 3g(2)$

addition of

Hence, $[(f+g)'](-1) = [f' + g'](-1) = f'(-1) + g'(-1) = 3f(2) + 3g(2)$
$= 3(f(2) + g(2))$

Thus, $(f+g) \in U$.

• Let $a \in \mathbb{R}$ and $f \in U$. Then $f'(-1) = 3f(2)$

Then, $[(af)'](-1) = a(f'(-1)) = a(3f(2)) = 3(af(2))$
$= 3([af](2))$

So, $af \in U$.

Therefore by def^{n} of subspace of U.