

# Manifolds

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2024-03-10



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# Chapter 1

## Basic Theroms and Definitions

**Definition 1.1** (Topology). A topology on a set  $X$  is a collection  $\mathcal{T}$  of subsets of  $X$  such that

- (T1)  $\phi$  and  $X$  are in  $\mathcal{T}$ ;
- (T2) Any union of subsets in  $\mathcal{T}$  is in  $\mathcal{T}$ ;
- (T3) The finite intersection of subsets in  $\mathcal{T}$  is in  $\mathcal{T}$ .

A set  $X$  with a topology  $\mathcal{T}$  is called a topological space. Denoted by  $(X, \mathcal{T})$ . An element of  $\mathcal{T}$  is called an open set.

**Definition 1.2.** A subset  $U \subset M$  is referred to as open in  $M$  if  $U \in \mathcal{T}$ . A subset  $A \subset M$  is termed closed if  $M \setminus A \in \mathcal{T}$ .

**Definition 1.3** (Continuity). If both  $(M, \mathcal{T}_M)$  and  $(N, \mathcal{T}_N)$  are topological spaces, a map  $f : M \rightarrow N$  is termed continuous if

$$f^{-1}(V) \in \mathcal{T}_M \text{ for all } V \in \mathcal{T}_N$$

. In other words, the preimages of open sets must be open.

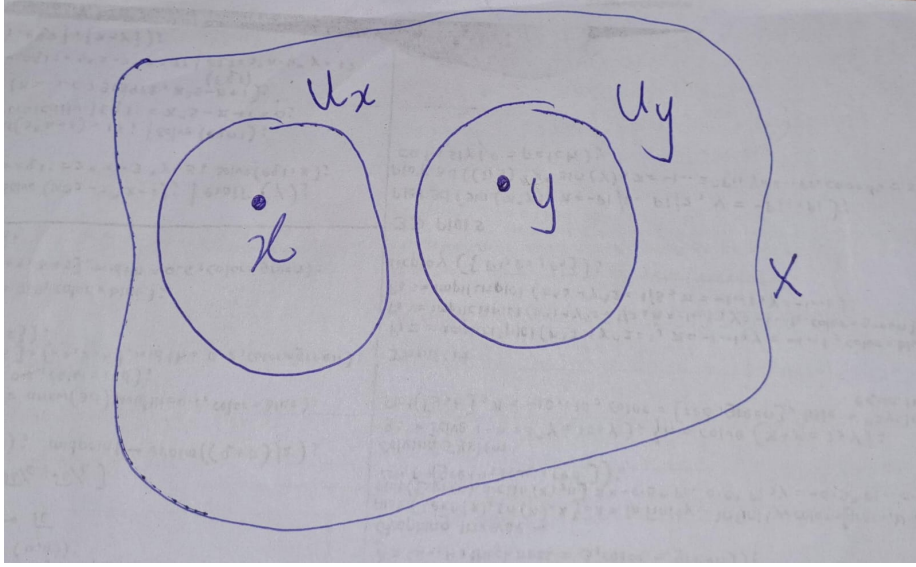
**Definition 1.4** (Homemorphism). A map  $f : M \rightarrow N$  between two topological spaces is called homemorphism if it has following propoties. -  $f$  is a bijection, -  $f$  is continuous, - the inverse function  $f^{-1}$  is continuous.

Two topological spaces  $M$  and  $N$  are called homeomorphic if there exists a homeomorphism between them.

**Definition 1.5** (Hausdorff Space). A topological space  $(X, \mathcal{T})$  is called a Hausdorff space if

**(H1)**  $\forall x, y \in X$  such that  $x \neq y$ ,  $\exists U_x, U_y \in \mathcal{T}$  such that  $x \in U_x$ ,  $y \in U_y$ , and  $U_x \cap U_y = \emptyset$ .

i.e., for every pair of distinct points  $x, y$  in  $X$ , there are disjoint neighborhoods  $U_x$  and  $U_y$  of  $x$  and  $y$  respectively.



∴ {definition #unnamed-chunk-5 name="Countability"} A space  $X$  is said to have a **countable basis at the point**  $x$  if there is a countable collection  $\{U_n\}_{n \in \mathbb{Z}^+}$  of neighborhoods of  $x$  such that any neighborhood  $U$  of  $x$  contains at least one of the sets  $U_n$ . A space  $X$  that has a countable basis at each of its points is said to satisfy the first countability axiom. ∴

### 1.0.1 Stereographic Projection

- **Stereographic Projection plane  $\mathbb{R}$  and the 1-sphere minus a point**

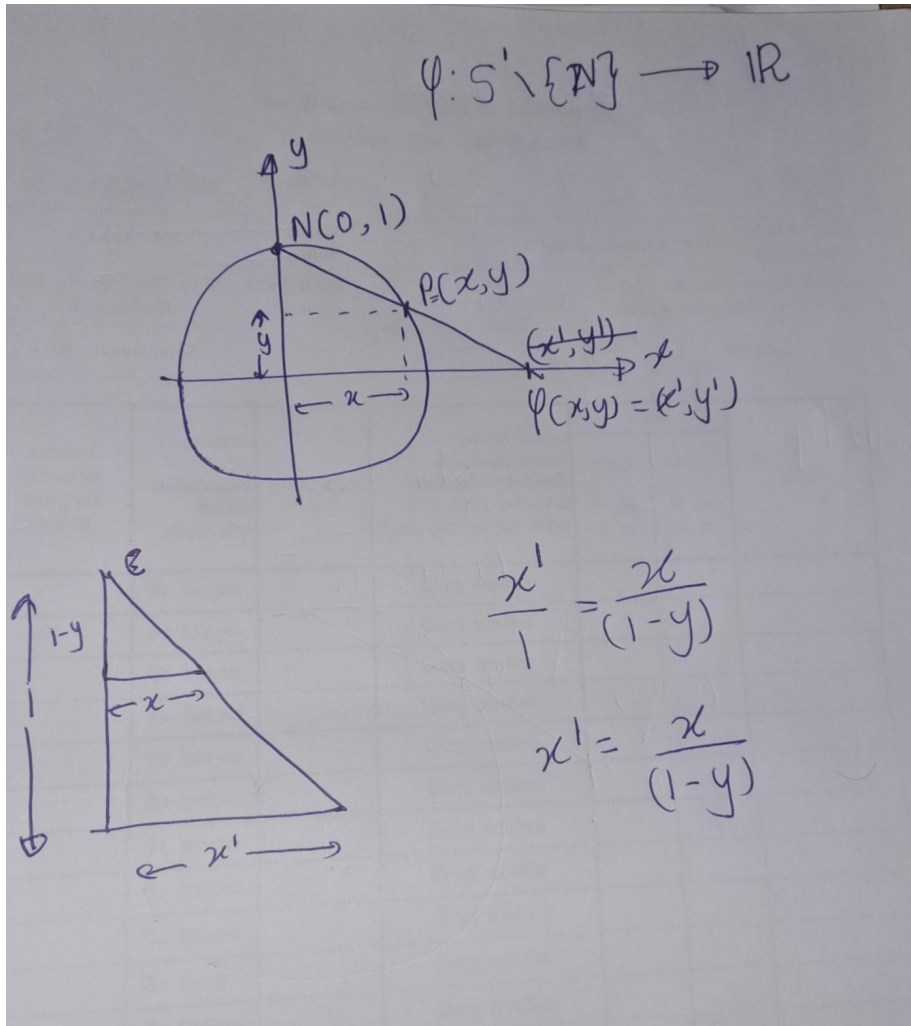
The 1-sphere  $S^1$  is the set of points  $(x, y, z) \in \mathbb{R}^3$  such that  $x^2 + y^2 + z^2 = 1$ .

$$S^1 := \{(x, y) : \|(x, y)\| = 1\}$$

Let  $S^1 \setminus \{N\}$  denote the 1-sphere minus (circle) its north pole, i.e., the point  $(0, 1)$ .

There exists a homeomorphism  $\varphi : S^1 \setminus \{N\} \rightarrow \mathbb{R}$ , which can be described as follows. In coordinates, this map is precisely

$$\varphi(x, y) = \frac{x}{1 - y}$$



### - Stereographic Projection plane $\mathbb{R}^2$ and the 2-sphere minus a point

Stereographic projection is an important homeomorphism between the plane  $\mathbb{R}^2$  and the 2-sphere minus a point. The 2-sphere  $S^2$  is the set of points  $(x, y, z) \in \mathbb{R}^3$  such that  $x^2 + y^2 + z^2 = 1$ . Let  $S^2 \setminus \{N\}$  denote the 2-sphere minus its north pole, i.e., the point  $(0, 0, 1)$ .

There exists a homeomorphism  $\varphi: S^2 \setminus \{N\} \rightarrow \mathbb{R}^2$ , which can be described as follows.

For a point  $p \in S^2 \setminus \{N\}$ , let  $\varphi(p)$  denote the unique point in  $\mathbb{R}^2$  such that the intersection of the segment  $N\varphi(p)$  and  $S^2$  is  $p$ . In coordinates, this map is precisely

$$\varphi(x, y, z) = \left( \frac{x}{1-z}, \frac{y}{1-z} \right).$$

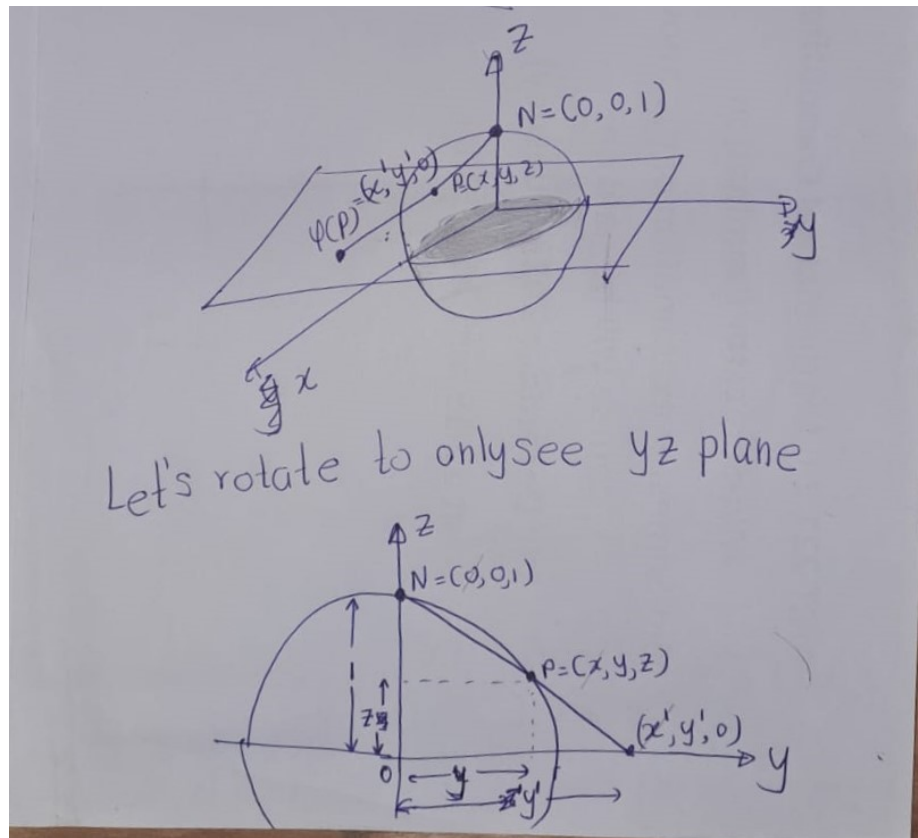


Figure 1.1:

**Definition 1.6.** If  $X$  is a space, a point  $x$  of  $X$  is said to be an **isolated point** of  $X$  if the one-point set  $\{x\}$  is open in  $X$ .



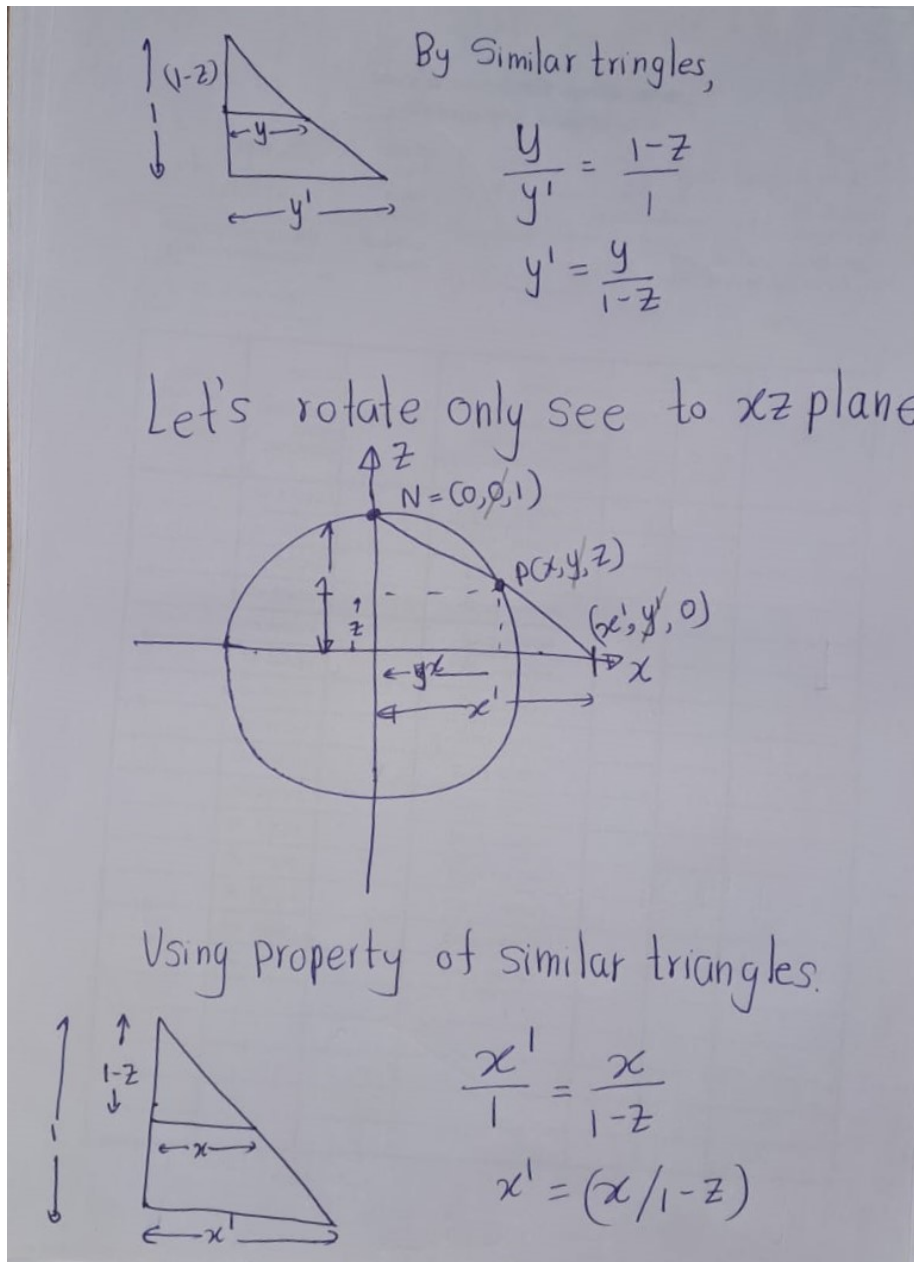


Figure 1.2:



## Chapter 2

# Manifolds

### 2.1 Topological Manifolds

**Definition 2.1.** Let  $(M, \mathcal{T})$  be a topological space with topology  $\mathcal{T}$ . Then  $M$  is called an  $n$ -dimensional topological manifold, if the following holds:

- (TM1):  $M$  is Hausdorff.
- (TM2): The topology of  $M$  has a countable basis.
- (TM3):  $M$  is locally homeomorphic to  $\mathbb{R}^n$ , that is, for all  $p \in M$  exists an open subset  $U \subset M$  with  $p \in U$ , an open subset  $V \subset \mathbb{R}^n$  and a homeomorphism  $\varphi : U \rightarrow V$ .

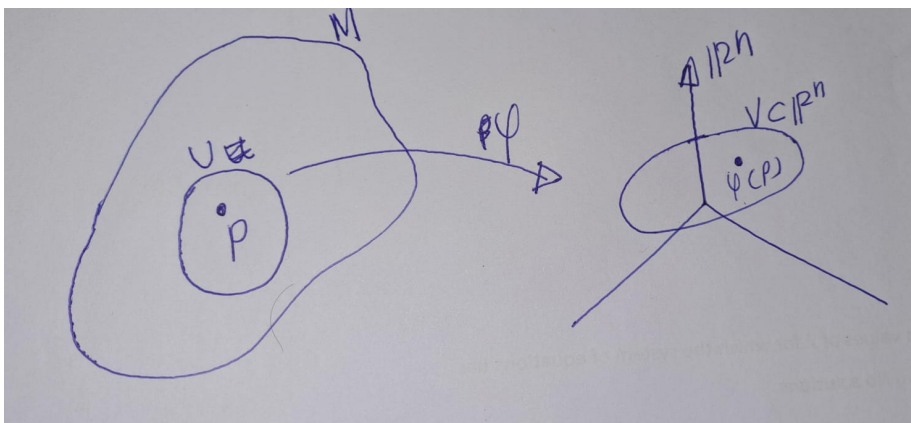


Figure 2.1:

*Remark.* The first two conditions in the definition ?? are more of a technical nature and are sometimes neglected. The important fact is that a topological manifold is locally homeomorphic to  $\mathbb{R}^n$ . Loosely speaking, manifolds look locally like Euclidean space. If the topology on  $M$  is induced by a metric, then the first condition is satisfied automatically. If  $M$  is given as a subset of  $\mathbb{R}^N$  with the subset topology, then both conditions M1 and M2 are satisfied automatically.

Let's see some examples.

**Example 2.1.** Euclidean space  $M = \mathbb{R}^n$  itself is an  $n$ -dimensional topological manifold:

- (TM1): We know that  $\mathbb{R}^n$  is metric space. Let's say the metric as  $d$ . Let  $x, y \in \mathbb{R}^n$  with  $x \neq y$ . Let  $r = d(x, y)$ . Since  $x \neq y, r > 0$ . Let  $U_x = B(x, r/2)$  and  $U_y = B(y, r/2)$ . So,  $x \in U_x$  and  $y \in U_y$ . We need to show that  $U_x \cap U_y \neq \emptyset$ . We are going to proof by contradiction. So, assume the contrary, there exist  $z \in U_x \cap U_y$ . Thus,  $d(x, z) < r/2$  and  $d(y, z) < r/2$ . Then,

$$r = d(x, y) \leq d(x, z) + d(z, y) = d(x, z) + d(y, z) < \frac{r}{2} + \frac{r}{2} = r$$

This is contradiction. Hence  $U_x \cap U_y \neq \emptyset$ . Therefore  $M = \mathbb{R}^n$  is Hausdorff.

- (TM2): Later I will update this part

Problem :(.

- (TM3): Let  $U = \mathbb{R}^n = M$  and  $V = \mathbb{R}^n$  and  $\varphi = id$ . We can easily tell that identity map is bijective. Further, we can observe that inverse of identity map is itself and it is well defined. So, Let  $U' \subset U = \mathbb{R}^n$  be an open set

$$\forall x \in U' \quad id^{-1}(x) = id(x) = x$$

. Thus,

$$id(U') = id^{-1}(U') = U'.$$

Hence, by definition of continuous mapping,  $id$  and  $id^{-1}$  are continuous.

**Example 2.2.** Let  $M \subset \mathbb{R}^n$  be an open subset. Then  $M$  is an  $n$ -dimensional topological manifold.

(TM1), (TM2) Obvious.

(TM3) Holds true with  $U = M$ ,  $V = M$  and  $x = id$ .

Here I am not going to prove this. It is very similar to first example.

**Example 2.3.** The standard sphere  $M = S^n = \{\underline{y} = (y^0, \dots, y^n) \in \mathbb{R}^{n+1} : \|\underline{y}\| = 1\}$  is an  $n$ -dimensional topological manifold.

- (TM1) and (TM2), since  $S^n$  is a subset of  $\mathbb{R}^{n+1}$ .

- (TM3) We construct two homeomorphisms with the help of the stereographic projection. Let  $N$  be north pole of the  $n$ -sphere, that is  $(0, \dots, 0, 1) \in \mathbb{R}^{n+1}$ . Let  $U_1 := S^n \setminus \{N\}$  and  $v_1 = \mathbb{R}^{n+1}$ . We define  
 $n \text{ times}$

$$\varphi : U_1 \rightarrow V_1 \quad (2.1)$$

$$\underline{y} = (y^0, y^1, \dots, y^n) \mapsto \frac{(y^0, y^1, \dots, y^n)}{1 - y^{n+1}} \quad (2.2)$$

- Claim 1:  $\varphi$  is injective.

Let  $(x^0, \dots, x^n), (y^0, y^1, \dots, y^n) \in \mathbb{R}^n$ . Suppose that  $\varphi(x^0, \dots, x^n) = \varphi(y^0, y^1, \dots, y^n)$ .

$$\varphi(x^0, \dots, x^n) = \varphi(y^0, y^1, \dots, y^n) \quad (2.3)$$

$$\frac{(x^0, x^1, \dots, x^n)}{1 - x^{n+1}} = \frac{(y^0, y^1, \dots, y^n)}{1 - y^{n+1}} \quad (2.4)$$

$$(y^0, y^1, \dots, y^n)(1 - x^{n+1}) = (x^0, x^1, \dots, x^n)(1 - y^{n+1}) \quad (2.5)$$

$$(y^0 - y^0 x^{n+1}, y^1 - y^1 x^{n+1}, \dots, y^n - y^n x^{n+1}) = (x^0 - x^0 y^{n+1}, x^1 - x^1 y^{n+1}, \dots, x^n - x^n y^{n+1}) \quad (2.6)$$

$$y^0(1 - x^{n+1}), y^1(1 - x^{n+1}), \dots, y^n(1 - y^n x^{n+1}) = x^0(1 - y^{n+1}), x^1(1 - y^{n+1}), \dots, x^n(1 - y^{n+1}) \quad (2.7)$$

Thus,  $y^i(1 - x^{n+1}) = x^i(1 - y^{n+1})$  for all  $i = 0, 1, \dots, n$ . Since  $1 - y^{n+1}, 1 - x^{n+1} > 0$ , ?

?

?

?

Problem HOW INJECTIVITY COMES: (.

CHECK: (.

- Claim 2:  $\varphi$  is surjective. Surjectivity means that for every  $\underline{v} \in V_1 = \mathbb{R}^n$ , there exists some  $\underline{y} \in U_1$  such that  $\varphi(\underline{y}) = \underline{v}$ .

So, let  $\underline{v} = (v^0, v^1, \dots, v^n) \in V_1$ . We need to find  $\underline{y} = (y^0, y^1, \dots, y^n) \in U_1$  such that

$$\frac{(y^0, y^1, \dots, y^n)}{1 - y^{n+1}} = \underline{v}.$$

We can solve this equation for  $\underline{y}$  as follows:

$$\underline{y} = (1 - y^{n+1})\underline{v} = \underline{v} - y^{n+1}\underline{v}.$$

We know that  $\underline{y} \in U_1 = S^n \setminus \{N\}$ , so  $y^{n+1} = 1 - \|\underline{y}\|^2$ . Substituting this into the equation gives us

$$\underline{y} = \underline{v} - (1 - \|\underline{y}\|^2)\underline{v} = \|\underline{y}\|^2\underline{v}.$$

Solving this equation for  $\|\underline{y}\|^2$  gives us

$$\|\underline{y}\|^2 = \frac{\|\underline{v}\|^2}{1 + \|\underline{v}\|^2}.$$

Substituting this back into the equation for  $\underline{y}$  gives us

$$\underline{y} = \frac{\underline{v}}{1 + \|\underline{v}\|^2}.$$

This is a well-defined point in  $U_1$  for every  $\underline{v} \in V_1$ , so  $\varphi$  is surjective.

- Claim:  $\varphi$  is continuous.

Note that the inverse map  $\phi$  is given by,

$$\phi : V_1 \rightarrow U_1 \tag{2.9}$$

$$\underline{x} = (x^0, x^1, \dots, x^n) \mapsto \frac{(x^0, x^1, \dots, x^{n-1})}{1 + x^n} \tag{2.10}$$

I will update this proof. I want some to to write rigirs proof:(

Analogously, we define the homeomorphism, which omits the south pole: Let now  $U_2 := S^n \setminus \{S\}$  with  $S := (0, \dots, 0, -1) \in \mathbb{R}^{n+1}$  and  $V_2 := \mathbb{R}^n$ . Then

$$\varphi : U_2 \rightarrow V_2, \tag{2.11}$$

$$\underline{y} = (y^0, y^1, \dots, y^n) \mapsto \frac{(y^0, y^1, \dots, y^n)}{1 + y^n} \tag{2.12}$$

Therefore,  $n$ -sphere  $S^n$  is an  $n$ -dimensional topological manifold.

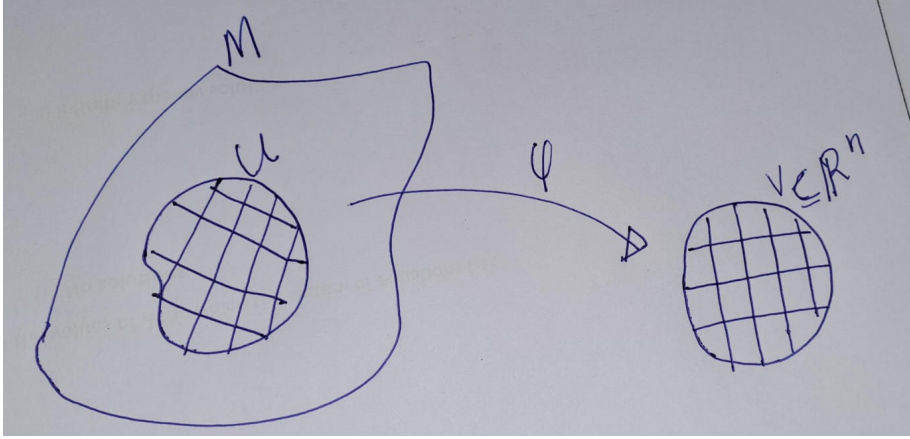
**Example 2.4** (Non-Example). We consider  $M := \{(y^1, y^2, y^3) \in \mathbb{R}^3 | (y^1)^2 = (y^2)^2 + (y^3)^2\}$ , the double cone.

Since  $M \subset \mathbb{R}^3$ , both (i) and (ii) are satisfied.

But  $M$  is **not** a 2-dimensional manifold. Assume it were, then there would exist an open subset  $U \subset M$  with  $0 \in U$ , an open subset  $V \subset \mathbb{R}^2$  and a homeomorphism  $\varphi : U \rightarrow V$  with  $\varphi(0) = 0$ . How do we Gruntee that such hormouphsim exist that maps 0 to 0:( Without loss of generality assume  $V = B_r(x(0))$  with  $r > 0$ . Choose  $(p^1, p^2, p^3), (q^1, q^2, q^3) \in U$  with  $p^1 > 0$  and  $q^1 < 0$ . Furthermore, choose a continuous path  $c : [0, 1] \rightarrow V$  with  $c(0) = x(q_1)$ ,  $c(1) = x(q_2)$  and  $c(t) \neq x(0)$  for all  $t \in [0, 1]$ .

Define the continuous path  $\tilde{c} := x^{-1} \circ c : [0, 1] \rightarrow U$ . Then  $\tilde{c}(0) = q_1$ ,  $\tilde{c}(1) = q_2$ , that is, we have  $\tilde{c}_1(0) > 0$  while  $\tilde{c}_1(1) < 0$ . Applying the mean value theorem we find, that there exists a  $t \in (0, 1)$  with  $\tilde{c}_1(t) = 0$ . Then  $\tilde{c}(t) = (0, 0, 0)$  and consequently  $c(t) = x(\tilde{c}(t)) = x(0)$ , which contradicts the choice of  $c$ . Hence,  $M$  is not a 2-dimensional topological manifold.

**Definition 2.2** (charts). If  $M$  is an  $n$ -dimensional topological manifold, the homeomorphisms  $\varphi : U \rightarrow V$  are called charts (or local coordinate systems) of  $M$ .



After choosing a local coordinate system  $\varphi : U \rightarrow V$  every point  $p \in U$  is uniquely characterized by its coordinates  $(\varphi^1(p), \dots, \varphi^n(p))$ .

**Example 2.5** (0-dimensional manifold). In a 0-dimensional manifold  $M$  every point  $p \in M$  has an open neighborhood  $U$ , which is homeomorphic to  $\mathbb{R}^0 = \{0\}$ . Consequently  $\{p\} = U$  is an open subset of  $M$  for all  $p \in M$ , that is,  $M$  carries the discrete topology. Since there exists a countable basis for the topology on  $M$  and the topology is discrete in addition,  $M$  has to be countable itself.

**Proposition 2.1.** A topological space  $M$  is a 0-dimensional topological manifold, if and only if  $M$  is countable and carries the discrete topology.

*Proof.*

- ( $\Rightarrow$ ) By definition, a 0-dimensional topological manifold is a topological space where every point has a neighborhood homeomorphic to the 0-dimensional Euclidean space, which is a single point  $\{0\}$ . This implies that for every point  $p \in M$ , there exists an open neighborhood  $U$  such that  $\{p\} = U$ . This is exactly the definition of a discrete topology.

Since, there exists a countable basis for the topology on  $M$ , and every point in  $M$  is an open set (i.e., the topology is discrete), then  $M$  must be countable. This is because every point in  $M$  corresponds to an open set in the basis, and since the basis is countable,  $M$  must also be countable.

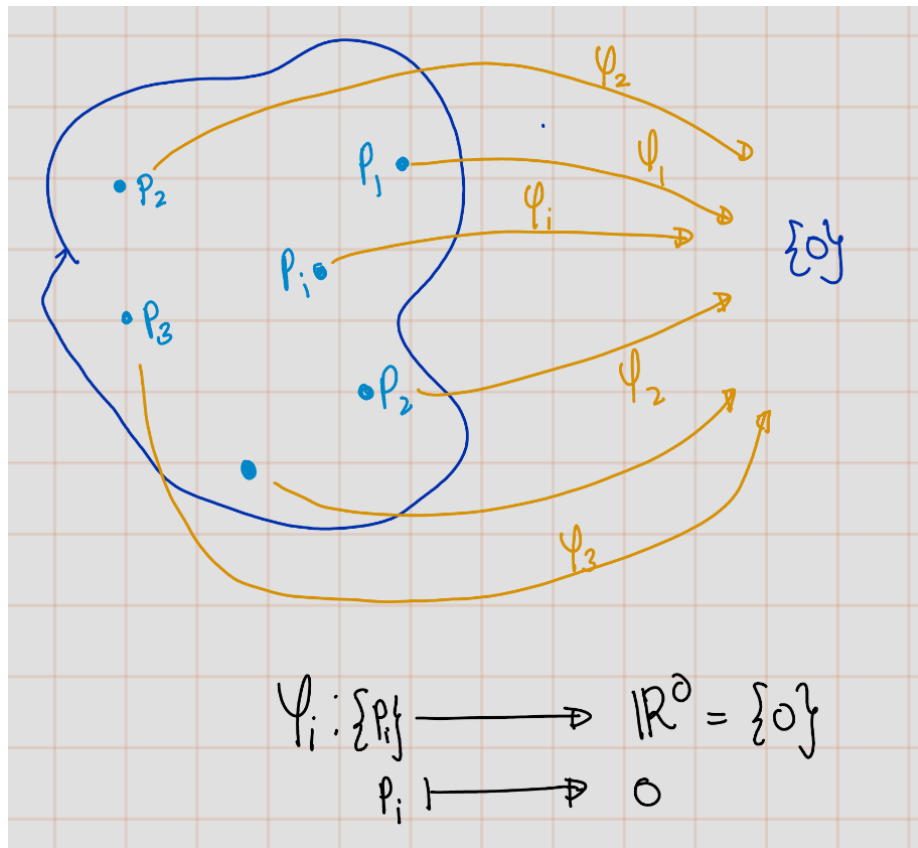


Figure 2.2: