Manifolds

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Contents

Chapter 1

Basic Theroms and Definitions

Definition 1.1 (Topology)**.** A topology on a set X is a collection $\mathcal{T}$ of subsets of X such that

(T1) ϕ and X are in $\mathcal{T}$;

(T2) Any union of subsets in $\mathcal{T}$ is in $\mathcal{T}$;

(T3) The finite intersection of subsets in $\mathcal{T}$ is in $\mathcal{T}$.

A set X with a topology $\mathcal{T}$ is called a topological space. Denoted by $(X, \mathcal{T})$. An element of $\mathcal{T}$ is called an open set.

Definition 1.2. A subset $U \subset M$ is referred to as open in M if $U \in \mathcal{T}$. A subset $A \subset M$ is termed closed if $M \setminus A \in \mathcal{T}$.

Definition 1.3 (Continuity)**.** If both $(M, \mathcal{T}_M)$ and $(N, \mathcal{T}_N)$ are topological spaces, a map $f : M \to N$ is termed continuous if

$$f^{-1}(V) \in \mathcal{T}_M \text{ for all } V \in \mathcal{T}_N$$

. In other words, the preimages of open sets must be open.

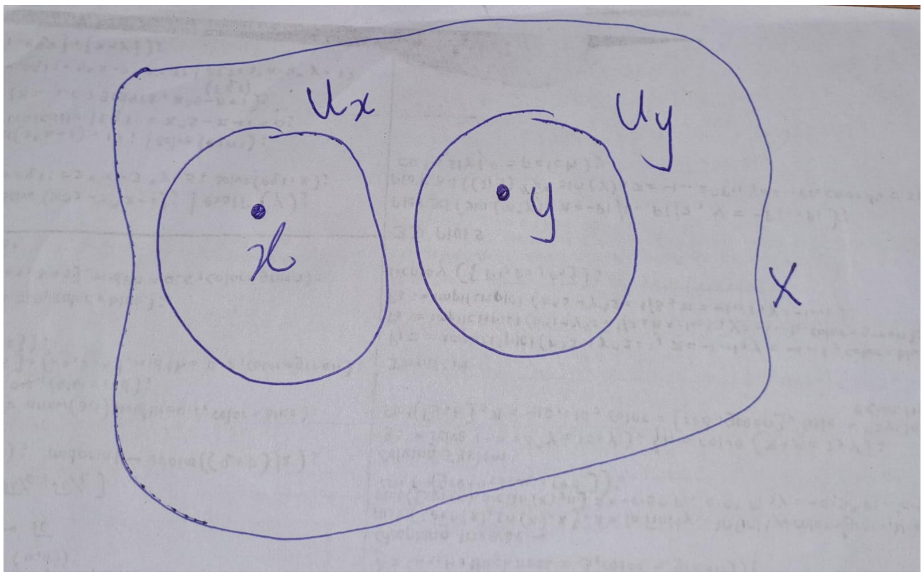
Definition 1.4 (Homemorphism)**.** A map $f : M \to N$ between two topological spaces is called homemorphism if it has following propoties. - f is a bijection, - f is continuous, - the inverse function f^{-1} is continuous.

Two topological spaces M and N are called homeomorphic if there exists a homeomorphism between them.

Definition 1.5 (Hausdorff Space)**.** A topological space $(X, \mathcal{T})$ is called a Hausdorff space if

(H1) $\forall x, y \in X$ such that $x \neq y$, $\exists U_x, U_y \in \mathcal{T}$ such that $x \in U_x$, $y \in U_y$, and $U_x \cap U_y = \emptyset$.

i.e., for every pair of distinct points x, y in X, there are disjoint neighborhoods U_x and U_y of x and y respectively.



::: {.definition #unnamed-chunk-5 name="Countability"} A space X is said to have a **countable basis at the point** x if there is a countable collection $\{U_n\}_{n\in\mathbb{Z}^+}$ of neighborhoods of x such that any neighborhood U of x contains at least one of the sets U_n. A space X that has a countable basis at each of its points is said to satisfy the first countability axiom. :::

1.0.1 Stereographic Projection

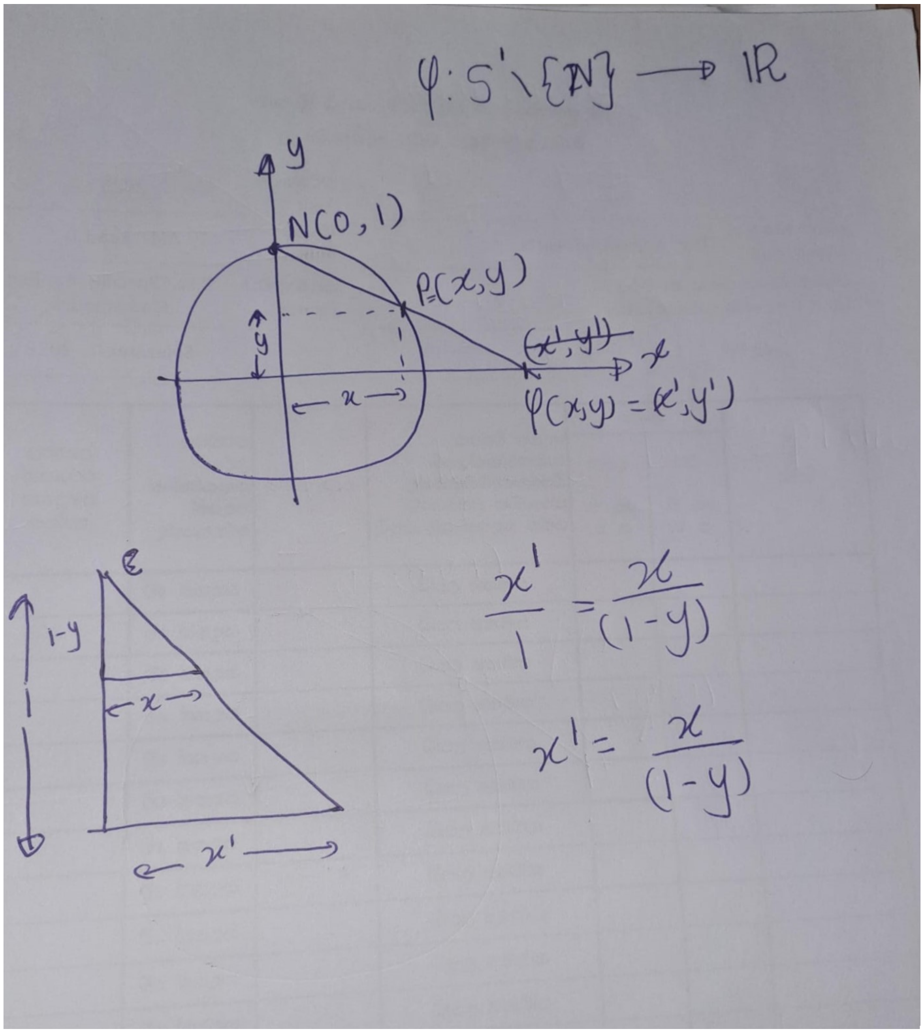
- **Stereographic Projection plane $\mathbb{R}$ and the 1-sphere minus a point** The 1-sphere S^1 is the set of points $(x, y, z) \in \mathbb{R}^3$ such that $x^2+y^2+z^2 = 1$.

$$S^1 := \{(x, y) : ||(x, y)|| = 1\}$$

Let $S^1 \setminus \{N\}$ denote the 1-sphere minus (circle) its north pole, i.e., the point $(0, 1)$.

There exists a homeomorphism $\varphi : S^1 \setminus \{N\} \to \mathbb{R}$, which can be described as follows. In coordinates, this map is precisely

$$\varphi(x, y) = \frac{x}{1-y}$$



- Stereographic Projection plane $\mathbb{R}^2$ and the 2-sphere minus a point

Stereographic projection is an important homeomorphism between the plane $\mathbb{R}^2$ and the 2-sphere minus a point. The 2-sphere S^2 is the set of points $(x, y, z) \in \mathbb{R}^3$ such that $x^2 + y^2 + z^2 = 1$. Let $S^2 \setminus \{N\}$ denote the 2-sphere minus its north pole, i.e., the point $(0, 0, 1)$.

There exists a homeomorphism $\varphi : S^2 \setminus \{N\} \to \mathbb{R}^2$, which can be described as follows.

For a point $p \in S^2 \setminus \{N\}$, let $\varphi(p)$ denote the unique point in P such that the intersection of the segment $\overline{Nf(p)}$ and S^2 is p. In coordinates, this map is precisely

$$\varphi(x, y, z) = \left(\frac{x}{1-z}, \frac{y}{1-z}\right).$$

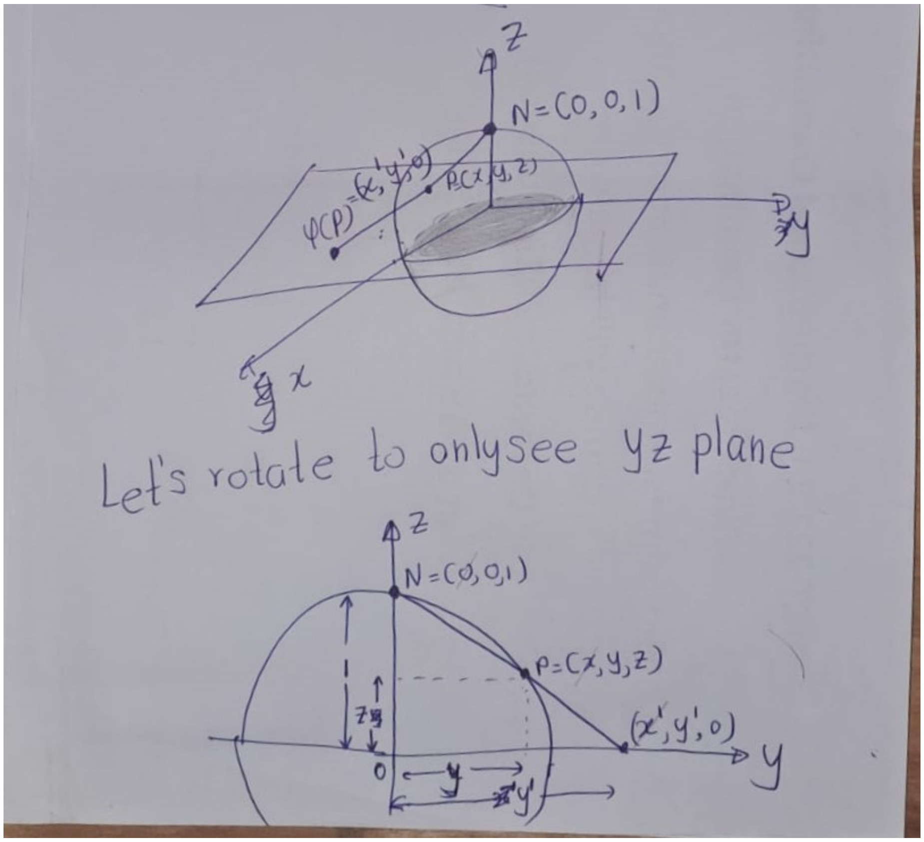


Figure 1.1:

Definition 1.6. If X is a space, a point x of X is said to be an **isolated point** of X if the one-point set $\{x\}$ is open in X.

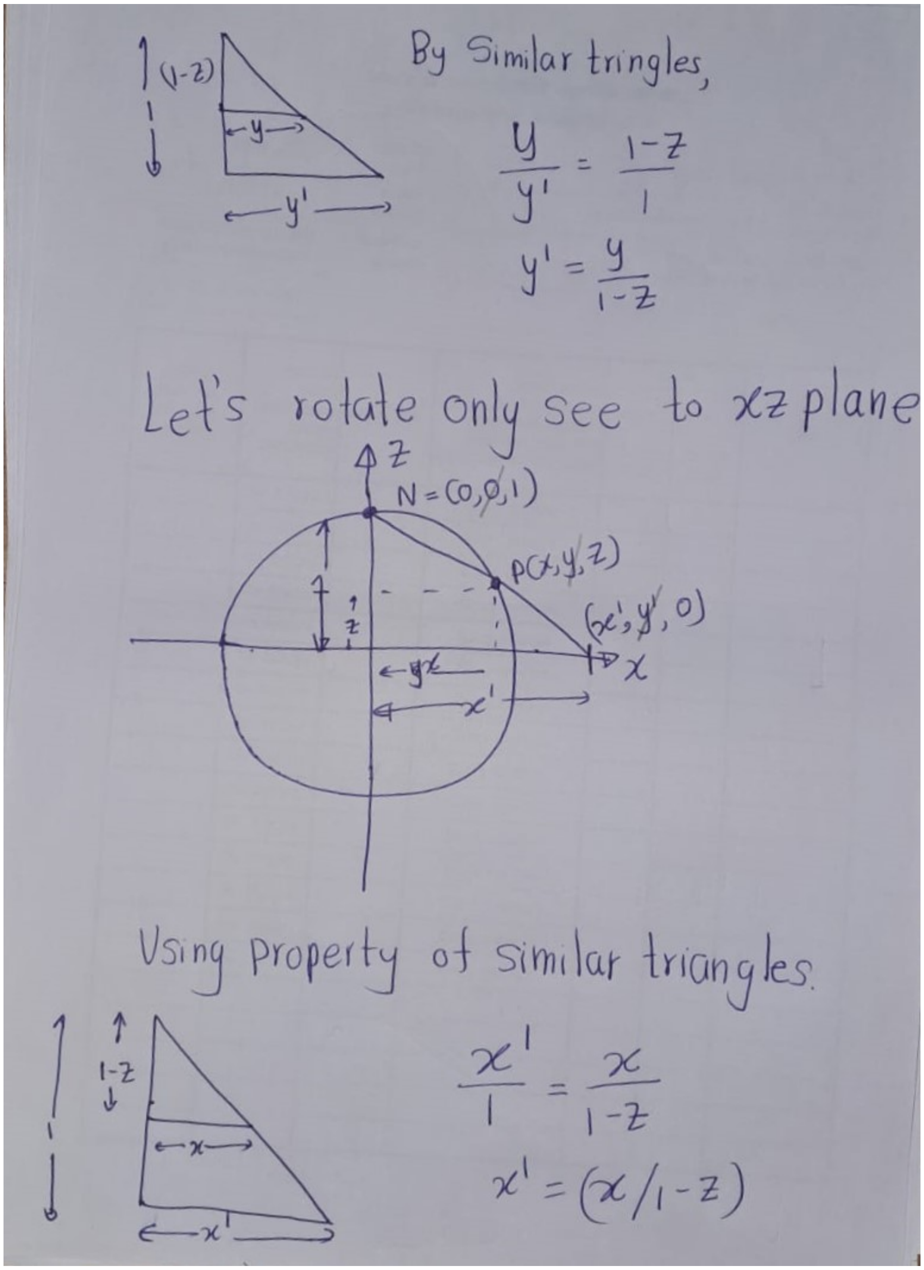

(1-z)
y
y'
By Similar tringles,
$\frac{y}{y'} = \frac{1-z}{1}$
$y' = \frac{y}{1-z}$
Let's rotate only see to xz plane
z
N = (0,0,1)
P(x,y,z)
$(x', y', 0)$
z
x
x'
Using property of similar triangles.
1-z
x
x'
$\frac{x'}{1} = \frac{x}{1-z}$
$x' = (x/1-z)$

Figure 1.2:

Chapter 2

Manifolds

2.1 Topological Manifolds

Definition 2.1. Let $(M, \mathcal{T})$ be a topological space with topology $\mathcal{T}$. Then M is called an n-dimensional topological manifold, if the following holds:

- **(TM1)**: M is Hausdorff.
- **(TM2)**: The topology of M has a countable basis.
- **(TM3)**: M is locally homeomorphic to $\mathbb{R}^n$, that is, for all $p \in M$ exists an open subset $U \subset M$ with $p \in U$, an open subset $V \subset \mathbb{R}^n$ and a homeomorphism $\varphi : U \to V$.

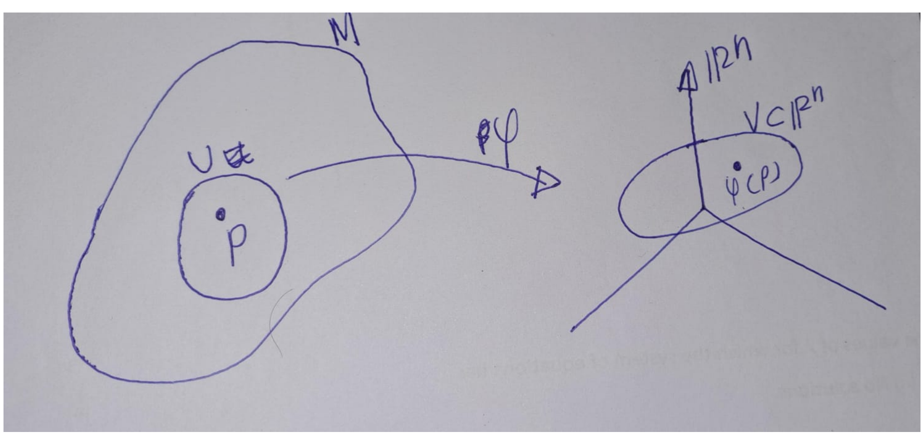


Figure 2.1:

Remark. The first two conditions in the definition 2.1 are more of a technical nature and are sometimes neglected. The important fact is that a topological manifold is locally homeomorphic to $\mathbb{R}^n$. Loosely speaking, manifolds look locally like Euclidean space. If the topology on M is induced by a metric, then the first condition is satisfied automatically. If M is given as a subset of $\mathbb{R}^N$ with the subset topology, then both conditions M1 and M2 are satisfied automatically.

Let's see some examples.

Example 2.1. Euclidean space $M = \mathbb{R}^n$ itself is an n-dimensional topological manifold:

- (TM1): We know that $\mathbb{R}^n$ is metrc space. Let's say the metric as d. Let $x, y \in \mathbb{R}^n$ with $x \neq y$. Let $r = d(x, y)$. Since $x \neq y$,$r > 0$. Let $U_x = B(x, r/2)$ and $U_y = B(y, r/2)$. So, $x \in U_x$ and $y \in U_y$ We need to show that $U_x \cap U_y \neq \emptyset$. We are going to proof by contrdiction. So, asssume the contray, there exist $z \in U_x \cap U_y$. Thus, $d(x, z) < r/2$ and $d(y, z) < r/2$. Then,

$$r = d(x, y) \leq d(x, z) + d(z, y) = d(x, z) + d(y, z) < \frac{r}{2} + \frac{r}{2} = r$$

  This is contradiction. Hence $U_x \cap U_y \neq \emptyset$. Therefore $M = \mathbb{R}^n$ is Hausdorff.
- (TM2): Later I will update this part

Problem **:(**.

- (TM3): Let $U = \mathbb{R}^n = M$ and $V = \mathbb{R}^n$ and $\varphi = id$. We can easily tell that idenntity map is bijective. Furthur, we can observe that inverse of identity map is itself and it is well defined. So, Let $U' \subset U = \mathbb{R}^n$ be an open set

$$\forall x \in U' \quad id^{-1}(x) = id(x) = x$$

  . Thus,

$$id(U') = id^{-1}(U') = U'.$$

  Hence, by definition of continuous mapping, id and id^{-} are continuous.

Example 2.2. Let $M \subset \mathbb{R}^n$ be an open subset. Then M is an n-dimensional topological manifold.
(TM1), (TM2) Obvious.
(TM3) Holds true with $U = M$, $V = M$ and $x = \text{id}$.

Here Ia m not going to proove this. It is very similar to first example.

Example 2.3. The standard sphere $M = S^n = \{\underline{y} = (y^0, ..., y^n) \in \mathbb{R}^{n+1} : ||\underline{y}|| = 1\}$ is an n-dimensional topological manifold.

- (TM1) and (TM2) , since S^n is a subset of $\mathbb{R}^{n+1}$.

- (TM3) We construct two homeomorphisms with the help of the stereographic projection. Let N be north pole of the n-sphere, that is $(\underbrace{0,...,0}_{n\ times},1) \in \mathbb{R}^{n+1}$. Let $U_1 := S^n \setminus \{N\}$ and $v_1 = \mathbb{R}^{n+1}$. We define

$$\begin{aligned} \varphi : U_1 &\to V_1 && (2.1) \\ \underline{y} = (y^0, y^1, ..., y^n) &\mapsto \frac{(y^0, y^1, ..., y^n)}{1 - y^{n+1}} && (2.2) \end{aligned}$$

- Cliam 1: *varphi* is injective.
Let $(x^0, ..., x^n), (y^0, y^1, ..., y^n) \in \mathbb{R}^n$. Suppose that $\varphi(x^0, ..., x^n) = \varphi(y^0, y^1, ..., y^n)$.

$$\begin{aligned} \varphi(x^0, ..., x^n) &= \varphi(y^0, y^1, ..., y^n) && (2.3) \\ \frac{(x^0, x^1, ..., x^n)}{1 - x^{n+1}} &= \frac{(y^0, y^1, ..., y^n)}{1 - y^{n+1}} && (2.4) \\ (y^0, y^1, ..., y^n)(1 - x^{n+1}) &= (x^0, x^1, ..., x^n)(1 - y^{n+1}) && (2.5) \\ (y^0 - y^0x^{n+1}, y^1 - y^1x^{n+1}, ..., y^n - y^nx^{n+1}) &= (x^0 - x^0y^{n+1}, x^1 - x^1y^{n+1}, ..., x^n - x^ny^{n+1}) && (2.6) \\ y^0(1 - x^{n+1}), y^1(1 - x^{n+1}), ..., y^n(1 - y^nx^{n+1}) &= x^0(1 - y^{n+1}), x^1(1 - y^{n+1}), ..., x^n(1 - y^{n+1}) && (2.7) \\ & && (2.8) \end{aligned}$$

Thus, $y^i(1-x^{n+1}) = x^i(1-y^{n+1})$ for all $i = 0, 1, ..., n$. Since $1-y^{n+1}, 1-x^{n+1} > 0$, ?

?

?

?

Problem HOW INJECTIVITY COMES**:(**.

CHECK**:(**.
- Claim 2: φ is surejctive. Surjectivity means that for every $\underline{v} \in V_1 = \mathbb{R}^n$, there exists some $\underline{y} \in U_1$ such that $\varphi(\underline{y}) = \underline{v}$.

So, let $\underline{v} = (v^0, v^1, ..., v^n) \in V_1$. We need to find $\underline{y} = (y^0, y^1, ..., y^n) \in U_1$ such that

$$\frac{(y^0, y^1, ..., y^n)}{1 - y^{n+1}} = \underline{v}.$$

We can solve this equation for $\underline{y}$ as follows:

$$\underline{y} = (1 - y^{n+1})\underline{v} = \underline{v} - y^{n+1}\underline{v}.$$

We know that $\underline{y} \in U_1 = S^n \setminus \{N\}$, so $y^{n+1} = 1 - \|\underline{y}\|^2$. Substituting this into the equation gives us

$$\underline{y} = \underline{v} - (1 - \|\underline{y}\|^2)\underline{v} = \|\underline{y}\|^2 \underline{v}.$$

Solving this equation for $\|\underline{y}\|^2$ gives us

$$\|\underline{y}\|^2 = \frac{\|\underline{v}\|^2}{1 + \|\underline{v}\|^2}.$$

Substituting this back into the equation for $\underline{y}$ gives us

$$\underline{y} = \frac{\underline{v}}{1 + \|\underline{v}\|^2}.$$

This is a well-defined point in U_1 for every $\underline{v} \in V_1$, so φ is surjective.

- Claim: φ is continuous.
  Note that the inverse map ϕ is given by,

$$\begin{aligned} \phi : V_1 &\to U_1 & (2.9) \\ \underline{x} = (x^0, x^1, ..., x^n) &\mapsto \frac{(x^0, x^1, ..., x^{n-1})}{1 + x^n} & (2.10) \end{aligned}$$

  I will update this proof. I want some to to write rigirs proof**:(**.

Analogously, we define the homeomorphism, which omits the south pole: Let now $U_2 := S^n \setminus \{S\}$ with $S := (0, ..., 0, -1) \in \mathbb{R}^{n+1}$ and $V_2 := \mathbb{R}^n$. Then

$$\begin{aligned} \varphi : U_2 &\to V_2, & (2.11) \\ \underline{y} = (y^0, y^1, ..., y^n) &\mapsto \frac{(y^0, y^1, ..., y^n)}{1 + y^n} & (2.12) \end{aligned}$$

Therefore, n-sphere S^n is an n-dimensional topological manifold.

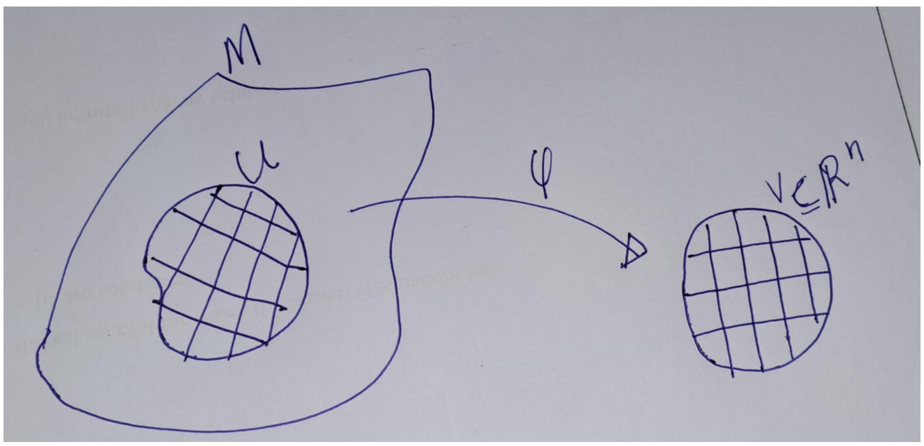
Example 2.4 (Non-Example)**.** We consider $M := \{(y^1, y^2, y^3) \in \mathbb{R}^3 | (y^1)^2 = (y^2)^2 + (y^3)^2\}$, the double cone.

Since $M \subset \mathbb{R}^3$, both (i) and (ii) are satisfied.

But M is **not** a 2-dimensional manifold. Assume it were, then there would exist an open subset $U \subset M$ with $0 \in U$, an open subset $V \subset \mathbb{R}^2$ and a homeomorphism $\varphi : U \to V$ with $\varphi(0) = 0$. How do we Gruntee that such hormouphsim exsist that maps 0 to 0**:(** Without losss of generality assume $V = B_r(x(0))$ with $r > 0$. Choose $(p^1, p^2, p^3), (q^1, q^2, q^3) \in U$ with $p^1 > 0$ and $q^1 < 0$. Furthermore, choose a continuous path $c : [0, 1] \to V$ with $c(0) = x(q_1)$, $c(1) = x(q_2)$ and $c(t) \neq x(0)$ for all $t \in [0, 1]$.

Define the continuous path $\tilde{c} := x^{-1} \circ c : [0, 1] \to U$. Then $\tilde{c}(0) = q_1$, $\tilde{c}(1) = q_2$, that is, we have $\tilde{c}_1(0) > 0$ while $\tilde{c}_1(1) < 0$. Applying the mean value theorem we find, that there exists a $t \in (0, 1)$ with $\tilde{c}_1(t) = 0$. Then $\tilde{c}(t) = (0, 0, 0)$ and consequently $c(t) = x(\tilde{c}(t)) = x(0)$, which contradicts the choice of c. Hence, M is not a 2-dimensional topological manifold.

Definition 2.2 (charts)**.** If M is an n-dimensional topological manifold, the homeomorphisms $\varphi : U \to V$ are called charts (or local coordinate systems) of M.



After choosing a local coordinate system $\varphi : U \to V$ every point $p \in U$ is uniquely characterized by its coordinates $(\varphi^1(p), \ldots, \varphi^n(p))$.

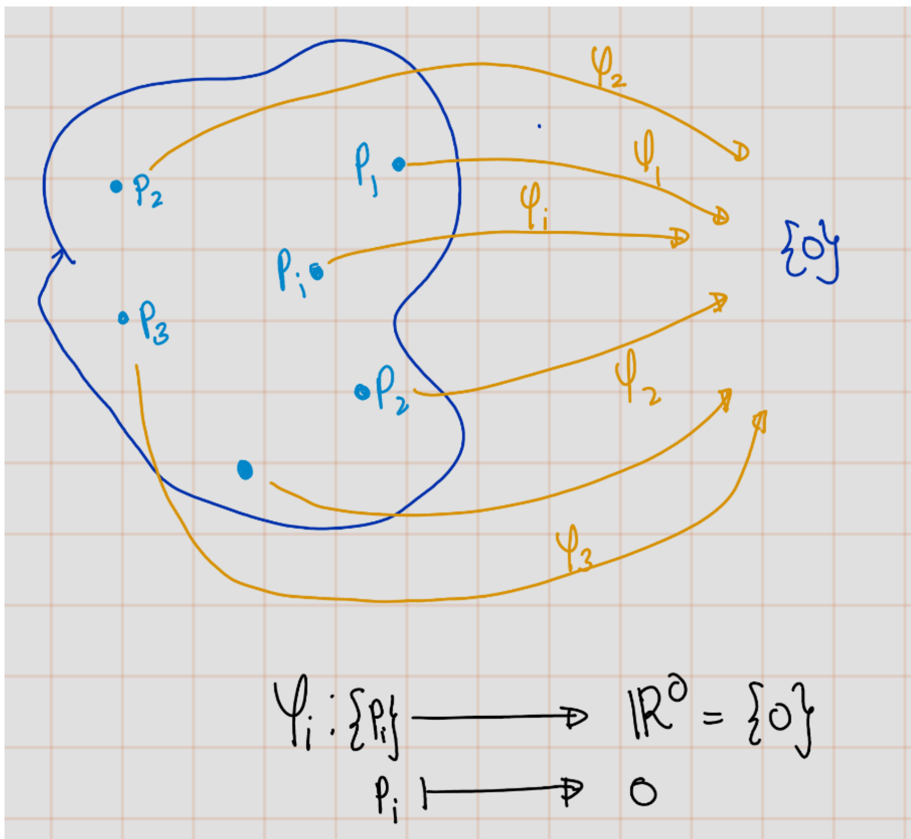
Example 2.5 (0-dimesional manifold)**.** In a 0-dimensional manifold M every point $p \in M$ has an open neighborhood U, which is homeomorphic to $R^0 = \{0\}$. Consequently $\{p\} = U$ is an open subset of M for all $p \in M$, that is, M carries the discrete topology. Since there exists a countable basis for the topology on M and the topology is discrete in addition, M has to be countable itself.

Proposition 2.1. *A topological space M is a 0-dimensional topological manifold, if and only if M is countable and carries the discrete topology.*

Proof.

- ($\Longrightarrow$) By definition, a 0-dimensional topological manifold is a topological space where every point has a neighborhood homeomorphic to the 0-dimensional Euclidean space, which is a single point $\{0\}$. This implies that for every point $p \in M$, there exists an open neighborhood U such that $\{p\} = U$. This is exactly the definition of a discrete topology.

  Since, there exists a countable basis for the topology on M, and every point in M is an open set (i.e., the topology is discrete), then M must be countable. This is because every point in M corresponds to an open set in the basis, and since the basis is countable, M must also be countable.



φ_2
p_1
φ_1
p_2
φ_i
p_i
$\{0\}$
p_3
p_2
φ_2
φ_3
$\varphi_i : \{p_i\} \longrightarrow \mathbb{R}^0 = \{0\}$
$p_i \longmapsto 0$

Figure 2.2:

- ($\Longleftarrow$) If M carries the discrete topology, then every subset of M is open. In particular, for every point $p \in M$, the set $\{p\}$ is an open set. This means that every point in M has a neighborhood homeomorphic to the 0-dimensional Euclidean space, which is a single point $\{0\}$. This is exactly the definition of a 0-dimensional topological manifold.

  If M is countable, then there exists a countable basis for the topology on M. Since every point in M is an open set (i.e., the topology is discrete), this basis can be taken to be the set of all singletons $\{p\}$, where $p \in M$.

Therefore, a topological space M is a 0-dimensional topological manifold if and only if M is countable and carries the discrete topology.

□

Definition 2.3. A topological manifold M is said to be **connected**, if for every two points $p, q \in M$ there exists a continuous map $c : [0, 1] \to M$ with $c(0) = p$ and $c(1) = q$.

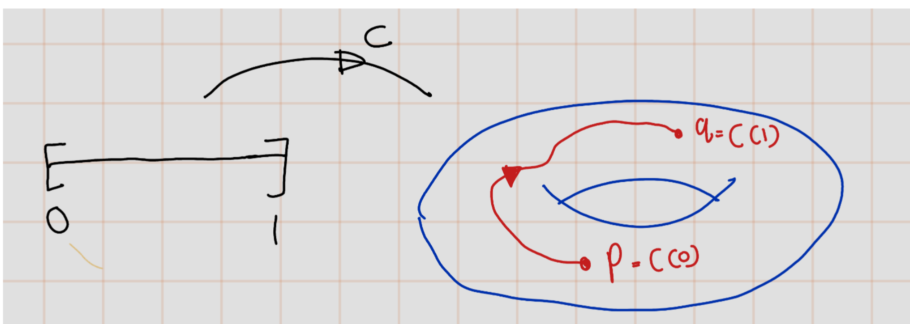


Figure 2.3:

Given two points, there has to be a continuous curve in M which connects both. Usually, in Topology one calls this path-connected, which is in the case of manifolds equivalent to being connected. We do not want to go deeper into this subject at this point.