Mathematical Logic

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Contents

Chapter 1

Introduction

Mathematical logic is the discipline that formalizes reasoning. It provides a rigorous framework for analyzing statements, constructing arguments, and establishing truth. This textbook is designed to guide students through the foundational principles of logic, beginning with propositional logic and progressing toward predicate logic, proof techniques, and the structure of the real number system.

The study of logic is essential for all areas of mathematics. It enables us to distinguish valid reasoning from fallacy, to express mathematical ideas with precision, and to construct proofs that are both sound and complete. Logic also serves as a bridge between mathematics and computer science, philosophy, and linguistics, where formal reasoning plays a central role.

This book is structured to support both conceptual understanding and technical mastery. Each chapter introduces key definitions, examples, and formal notation, followed by exercises that reinforce the material. The progression is cumulative: later chapters build upon the logical foundations established early on.

1.1 Chapter Overview

- **Chapter 2: Mathematical Logic**
  Introduces propositional and predicate logic, truth tables, logical equivalence, and the algebra of propositions.
- **Chapter 3: Introduction to Proofs**
  Covers terminology, argument structure, validity, and various proof techniques including indirect proofs and proof by cases.

- **Chapter 4: The Real Number System**
  Presents the axioms of real numbers, properties of equality, order, completeness, and the concept of infinity.

- **Chapter 5: Set Theory**
  Introduces the language of sets, operations, and foundational concepts used throughout mathematics.

- **Chapter 6: Exercises**
  Provides practice problems to reinforce the concepts and techniques introduced in earlier chapters.

This textbook reflects a commitment to clarity, rigor, and accessibility. It is intended for undergraduate students beginning their study of mathematical logic, and for anyone seeking a structured and principled approach to formal reasoning.

Chapter 2

Mathematical logic

Mathematical logic is the branch of mathematics that studies the principles and methods of formal reasoning. It is based on symbolic languages that can express statements and arguments in a precise and unambiguous way.

2.1 Propositional logic & Logical operators

Propositional logic is a branch of mathematical logic that studies the logical relationships between propositions, which are statements that can be either true or false.

Definition 2.1 (Proposition (Statement))**.** Proposition (Statement) is a declarative sentence that is either true (T) or false (F), but not both.

Example 2.1.

(i) A square has all its sides equal.
(ii) Every Odd number is not divisible by 2.
(iii) $2 < 3$.
(iv) $\sqrt{2} \notin \mathbb{Q}$.
(v) $\mathbb{Z} \subseteq \mathbb{Q}$.
(vi) The set $\{10, 20, 30\}$ has three elements.

They are all true.

Example 2.2.

(i) Every rectangle is a square.
(ii) $(2+4)^2 = 2^2 4^2$.

(iii) $\sqrt{2} \notin \mathbb{R}$.
(iv) $\mathbb{R} \subseteq \mathbb{Q}$.
(v) $\{10, 11, 12\} \cap \mathbb{N} = \emptyset$

They are all false.

Remark. No sentence can be called a statement if

- It is a question.
- It is an order or request.

Example 2.3.

- "How old are you?" cannot be assigned true or false (In fact, it is a question). So, it is not a statement.
- "Close the door" cannot be assigned u-ue or false (Infect, it is a command). So, it cannot be called a statement.
- "x is a natural number" depends on the value of x. So, it is not considered as a statement. However, often it's referred to as an open statement.

Example 2.4.

NOT a statement	**Statement**
Add 5 to both sides.	Adding 5 to both sides of $x - 5 = 37$ gives $x = 42$.
$\mathbb{Z}$	$42 \in \mathbb{Z}$
42	42 is not a number.
What is the solution of 2x = 84?	The solution of 2x = 84 is 42.

2.2 Statements and Truth Values

Note: The **truth (T)** or **falsity (F)** of a statement is called its **truth value**.

Definition 2.2. A statement is called **simple** (or *atomic*) if it cannot be broken down into two or more statements.

Example 2.5.

- 2 is an even number.
- A square has all its sides equal.
- 7 is an odd number.

Definition 2.3. A **compound statement** is one which is made up of two or more simple statements.

Example 2.6.

- "7 is both an odd and prime number" can be broken into two statements:
  - "7 is an odd number."
  - "7 is a prime number."

- So it is a compound statement.

Note: The simple statements which constitute a compound statement are called **component statements**.

2.3 More About Propositions

- We use letters to denote propositions, such as p, q, r, s.
- The **truth value** of a proposition is denoted as:
  - T for **true**
  - F for **false**

Using these notations, we can form new (compound) propositions from known propositions.
This area of logic is known as **propositional calculus** or **propositional logic**.

> *Note:* Calculus here refers to the manipulation or computation with symbols.

Example 2.7. Let: - p: 7 is an odd number. - q: 7 is a prime number. - r: $5 > 11$

2.4 Logical Operators / Connectives

- A **logical operator** is a rule defined by a **truth table**.

2.4.1 Truth Table

A truth table shows the relationship between the truth values of propositions.
It is useful for: - Visually displaying how a logical operator works. - Determining the truth value of a compound proposition based on its component propositions.

2.5 Logical Operators

Here are some important logical operators:

Operator	**Handle**	**Notation**
Negation	not	$\sim, \neg$
Conjunction	and	$\wedge$
Disjunction	or	$\vee$
Exclusive-or	xor	$\oplus$
Implication	implies	$\rightarrow$
Biconditional	if and only if (iff)	$\leftrightarrow$

2.5.1 The Negation Operator ($\sim, \neg$)

Given any proposition p, we can form a new proposition:
"It is not true that p", which is called the **negation** of p.

Example 2.8. Let p: "The number 2 is even."
This statement is **true**.

Negation:
$\sim p$: "It is not true that the number 2 is even."
This new statement is **false**.

Definition 2.4. Let p be a proposition.
The statement "p is not the case" is another proposition called the **negation** of p.
It is denoted $\sim p$ and read as "not p".

2.5.1.1 Truth Table for Negation

p	$\neg p$
T	F
F	T

2.5.1.2 Alternate Expressions for Negation

Example 2.9. Let P: "The number 2 is even."

Then $\sim P$ can be expressed as:

- "It's not true that the number 2 is even."

- "It is false that the number 2 is even."
- "The number 2 is not even."

Example 2.10. Let: - p: "This book is interesting."

Then the negation $\neg p$ (also written as $\sim p$) can be read as:

1. "This book is not interesting."
2. "This book is uninteresting."
3. "It is not the case that this book is interesting."

Note: The symbol $\neg$ is called the **negation operator**.
It operates on a single logical proposition by **complementing its truth value**.
For this reason, it is also called the **logical complement**.

We now introduce logical operators that take **two existing propositions** and form a **new compound proposition**.

These operators are known as **logical connectives**.

2.5.2 The Conjunction Operator — "and"

The word **"and"** can be used to combine two statements to form a new statement.

Example 2.11. Let:

- P: The number 2 is even.
- Q: The number 3 is odd.

Then:

- R_1: "The number 2 is even and the number 3 is odd."
  This is a **true** statement because both P and Q are true.

Example 2.12.

- R_2: "The number 1 is even and the number 3 is odd." $\rightarrow$ **False**
- R_3: "The number 2 is even and the number 4 is odd." $\rightarrow$ **False**
- R_4: "The number 3 is even and the number 2 is odd." $\rightarrow$ **False**

2.5.2.1 Symbolic Notation:

We use the symbol $\wedge$ to represent "and".
So, if P and Q are propositions, then $P \wedge Q$ means "P and Q".

- $P \wedge Q$ is **true** only if both P and Q are true.
- Otherwise, $P \wedge Q$ is **false**.

2.5.2.2 Truth Table for Conjunction

P	Q	$P \wedge Q$
T	T	T
T	F	F
F	T	F
F	F	F

In this table, T stands for "True" and F stands for "False".
These are called **truth values**.

2.5.3 The Disjunction Operator — "or"

Let p and q be propositions.
The proposition "p or q" is called the **disjunction** of p and q, denoted by $p \vee q$.

- $p \vee q$ is **false** only when both p and q are false.
- It is **true** otherwise.

2.5.3.1 Truth Table for Disjunction

p	q	$p \vee q$
T	T	T
T	F	T
F	T	T
F	F	F

This is an **inclusive or**: true if either or both are true.

Example 2.13. Let: - S_1: "The number 2 is even or the number 3 is odd." $\rightarrow$ **True** - S_2: "The number 1 is even or the number 3 is odd." $\rightarrow$ **True** - S_3: "The number 2 is even or the number 4 is odd." $\rightarrow$ **True** - S_4: "The number 3 is even or the number 2 is odd." $\rightarrow$ **False**

2.5.4 The Exclusive OR Operator — "either or"

Let p and q be propositions.
The **exclusive or** of p and q, denoted $p \oplus q$, is:

- **True** when exactly one of p or q is true.
- **False** when both are true or both are false.

2.5.4.1 Truth Table for Exclusive OR

p	q	$p \oplus q$
T	T	F
T	F	T
F	T	T
F	F	F

"Either p or q is true, but not both."

Let p and q be propositions.
The **exclusive or** of p and q, denoted $p \oplus q$, is:

- **True** when exactly one of p or q is true.
- **False** when both are true or both are false.

Example 2.14. Let: - p: This book is interesting. - q: I am staying at home.

Then: - $p \oplus q$: "Either this book is interesting, or I am staying at home, but not both."

2.5.4.2 Truth Table for Exclusive OR

p	q	$p \oplus q$
T	T	F
T	F	T
F	T	T
F	F	F

- p: $3 > 1 \rightarrow$ T

- q: $0 = 1 \rightarrow$ F

- r: $2 = 1 \to \mathrm{F}$

Then:

- $p \oplus q$: T

- $p \oplus r$: T   F = T

2.5.5 Implication Operator — "implies"

Let P and Q be propositions.
The implication "If P, then Q" is written as $P \Rightarrow Q$.

- This is called a **conditional statement**.
- It is **false** only when P is true and Q is false.
- Otherwise, it is **true**.

Example 2.15. Let:

- P: The integer a is a multiple of 6.
- Q: The integer a is divisible by 2.

Then:

- R: "If a is a multiple of 6, then a is divisible by 2."
  This is a **true** statement.

In general, given any two statements P and Q whatsoever, we can form the new statement *"If P, then Q."* This is written symbolically as $P \to Q$, which we read as *"If P, then Q,"* or *"P implies Q."*

Like the symbols $\wedge$ (and) and $\vee$ (or), the symbol $\to$ has a very specific meaning. When we assert that the statement $P \to Q$ is true, we mean that if P is true, then Q must also be true. In other words, the condition of P being true forces Q to be true.

A statement of the form $P \to Q$ is called a *conditional statement* because it means Q will be true under the condition that P is true.

Think of $p \to q$ as a promise: whenever p is true, q will be true also.
There is only one way this promise can be broken—namely, if p is true but q is false.

Definition 2.5. Let p and q be propositions. The implication $p \to q$ is:

- **False** when p is true and q is false
- **True** otherwise

In this implication: - p is called the **hypothesis** (or antecedent or premise) - q is called the **conclusion** (or consequence)

Example 2.16. If $\underbrace{\text{a polygon is a triangle,}}_{\text{hypothesis,p}}$ *then* $\underbrace{\text{the sum of its angle measures is 180°.}}_{\text{conclusion,q}}$

2.5.5.1 Ways to express an implication

- $p \implies q$
- "If p, then q"
- "If p, q"
- "p is sufficient for q"
- "q if p"
- "q when p"
- "p implies q"
- "p only if q"
- "q is necessary for p"
- "q follows from p"

2.5.5.2 Truth Table for $p \implies q$

p	q	$p \implies q$
T	T	T
T	F	**F**
F	T	T
F	F	T

Remark.

- $p \to q$ is **false only when** p is true and q is false.
- $p \to q$ can be **true even if** p is false.
- The truth of $p \to q$ does **not require** that either p or q is true.

Example 2.17. Consider the statement:
"Employee pays taxes only if his income is more than 3 million."

Let:

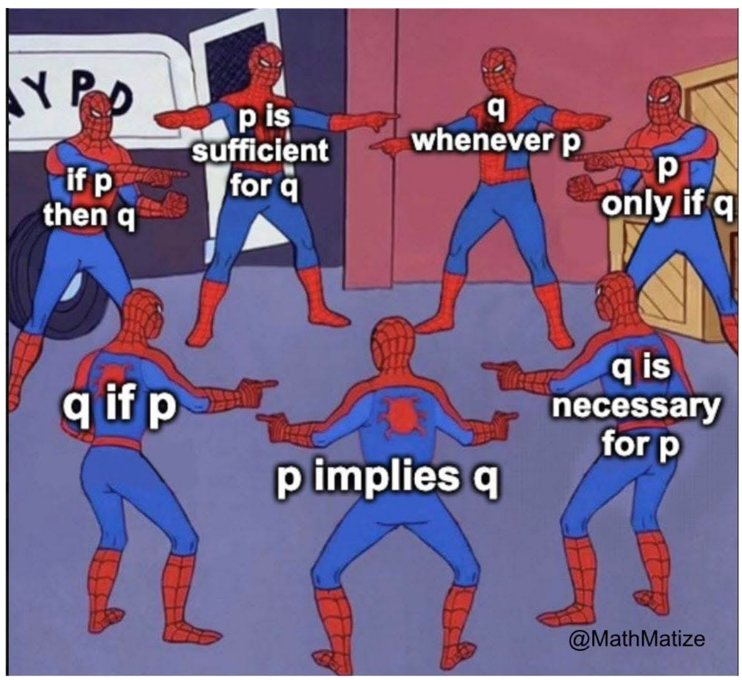


Figure 2.1: Source: Facebook

- p: Employee pays taxes
- q: His income is more than 3 million

Symbolically:

$$p \implies q$$

In other words:

- If employee pays taxes, then his income is more than 3 million.
- Employee's income is more than 3 million, if he pays taxes.

Example 2.18. Consider the statement:
"If n^2 is even, then n is even."

Let:

- p: n^2 is even
- q: n is even

Symbolically:

$$p \implies q$$

2.5.6 Biconditional Logic: $p \iff q$

Definition 2.6. Let p and q be propositions. The biconditional $p \iff q$ is:

- **True** when p and q have the same truth value
- **False** otherwise

Remark. The statement $p \iff q$ is true precisely when both $p \Rightarrow q$ and $q \Rightarrow p$ are true.
This is why we say:

- "p if and only if q"
- $p \iff q \equiv (p \Rightarrow q) \wedge (q \Rightarrow p)$

2.5.6.1 Alternate phrasing

Not surprisingly, there are many ways of saying $P \iff Q$ in English. The following constructions all mean $P \iff Q$:

- P if and only if Q.
- P is necessary and sufficient for Q.
- For P it is necessary and suffcient that Q.
- P is equivalent to Q.
- If P, then Q, and conversely.

The first three of these just combine constructions from the previous section to express that$P \implies Q$ and $Q \implies P$. In the last one, the words "*…and conversely*" mean that in addition to "*If P, then Q*" being true, the converse statement "*If Q, then P" is also true.

2.5.6.2 Truth Table for $p \iff q$

p	q	$p \iff q$
T	T	T
T	F	F
F	T	F
F	F	T

p	q	$p \Rightarrow q$	$q \Rightarrow p$	$(p \Rightarrow q) \wedge (q \Rightarrow p)$
T	T	T	T	T
T	F	F	T	F
F	T	T	F	F
F	F	T	T	T

Example 2.19.

1. "A number is divisible by 2 if and only if it is even."
2. "A number being even is a necessary and sufficient condition for it to be divisible by 2."

2.5.7 Terminology

Let the compound statement be:

"If p, **then** q" — symbolically written as $p \to q$

Components

- p: **Premise**, **Hypothesis**, or **Antecedent**
- q: **Conclusion** or **Consequent**

Transformation	Symbolic Form	Verbal Form
Converse	$q \to p$	If q, then p
Inverse	$\neg p \to \neg q$	If not p, then not q
Contrapositive	$\neg q \to \neg p$	If not q, then not p
Negation	$p \wedge \neg q$	p is true and q is false (i.e., the conditional fails)

2.6 Truth Tables for Compound Propositions

To analyze compound propositions:

- Use separate columns for each sub-expression.
- Evaluate truth values for all combinations of truth values of the atomic propositions.
- The final column shows the truth value of the entire compound proposition.

Example 2.20.

> **If a polygon is a triangle, then the sum of its angle measures is 180°**

Let:

- p: A polygon is a triangle
- q: The sum of the angle measures of a polygon is 180°

Then the compound statement is:

- $p \implies q$

(ii) **Converse**

The converse of a conditional statement $p \implies q$ is $q \implies p$

Statement:
If the sum of the angle measures of a polygon is 180°, **then** the polygon is a triangle.
Symbolically: $q \implies p$

(iii) **Inverse**

The inverse of a conditional statement $p \implies q$ is $\neg p \implies \neg q$

Statement:
If a polygon is **not** a triangle, **then** the sum of its angle measures is **not** 180°.
Symbolically:
$\neg p \to \neg q$

(iv) **Contrapositive**

The contrapositive of a conditional statement $p \implies q$ is $\neg q \implies \neg p$

Statement:
If the sum of the angle measures of a polygon is **not** 180°, **then** the polygon is **not** a triangle.
Symbolically: $\neg q \implies \neg p$

(v) **Negation**

The negation of a conditional statement $p \implies q$ is $p \wedge \neg q$

Statement:
A polygon **is** a triangle **and** the sum of its angle measures is **not** 180°.
Symbolically: $p \wedge \neg q$

2.7 Precedence of Logical Operations

To reduce parentheses in logical expressions, follow this precedence order:

Operation	Symbol	Precedence
Negation	$\neg$	1
Conjunction	$\wedge$	2
Disjunction	$\vee$	3
Implication	$\Rightarrow$	4

Operation	Symbol	Precedence
Biconditional	$\Leftrightarrow$	5

Example 2.21.

- $p \vee q \wedge r$ means:$p \vee (q \wedge r)$
- $(p \vee q) \wedge r$ requires parentheses to override precedence.
- $p \vee q \Rightarrow \neg r$ means: $(p \vee q) \Rightarrow (\neg r)$
- $p \vee (q \Rightarrow \neg r)$ requires parentheses to clarify grouping.

Example 2.22. Parse the statement
$(\neg p) \Rightarrow (p \vee (q \wedge p))$

This uses:

- Negation on p
- Conjunction $q \wedge p$
- Disjunction $p \vee (q \wedge p)$
- Implication from $\neg p$ to the disjunction

2.8 Truth Tables and Logical Analysis

2.8.1 Constructing a Truth Table

To analyze a compound proposition:

1. **Determine the number of atomic propositions**:
   If there are n propositions, the truth table will have 2^n rows.
2. **List all combinations of truth values**:
   Fill the first n columns with all possible combinations of truth values for each proposition.
3. **Evaluate each sub-expression**:
   Add columns for intermediate steps and compute their truth values row by row.

Example 2.23. Construct a truth table for the compound proposition:

$$(p \vee \neg q) \Rightarrow (p \wedge q)$$

p	q	$\neg q$	$p \vee \neg q$	$p \wedge q$	$(p \vee \neg q) \Rightarrow (p \wedge q)$
T	T	F	T	T	T
T	F	T	T	F	F
F	T	F	F	F	T
F	F	T	T	F	F

Remark. To construct the truth table for a given proposition: 1 . Create a table with 2^n rows if a compound proposition involves n. propsitions. 2. Fill in the first n columns with all possible cobinations. 3. Determine and enter the truth value for each combination.

Example 2.24. For $n = 3$,

This table lists all possible combinations of truth values for three atomic propositions: p, q, and r.

p	q	r
T	T	T
F	T	T
T	F	T
F	F	T
T	T	F
F	T	F
T	F	F
F	F	F

Exercise 2.1. Construct truth tables for the following compound propositions

a)
$$\neg(P \vee Q) \vee (\neg P)$$

b)
$$\neg(P \Rightarrow Q)$$

c)
$$P \vee (Q \Rightarrow R)$$

Exercise 2.2. Suppose the statement $((P \wedge Q) \vee R) \Rightarrow (R \vee S)$ is **false**.
Find the truth values of P, Q, R, and S without constructing a full truth table.

2.9 Tautologies and Contradictions

Definition 2.7.

- A **tautology** is a compound proposition that is always true, regardless of the truth values of its components.
- A **contradiction** is a compound proposition that is always false.
- A **contingency** is a compound proposition that is neither a tautology nor a contradiction.

Example 2.25. Let p: “This course is easy.”

(i) Contradiction:
“This course is easy **and** this course is not easy”
Expression: $p \wedge \neg p$

p	$\neg p$	$p \wedge \neg p$
T	F	F
F	T	F

(ii) Tautology:
“This course is easy **or** this course is not easy”
Expression: $p \vee \neg p$

p	$p \vee \neg p$
T	T
F	T

2.10 Logical Equivalence

Definition 2.8. Two compound propositions p and q are **logically equivalent** if they yield the same truth values for all combinations of truth values of their components. Denoted as:

$$p \equiv q$$

Remark.

- $p \equiv q$ means that $p \iff q$ is a tautology.
- The symbol $\equiv$ is not a logical connective; it asserts that the biconditional $p \iff q$ is always true.

2.10.1 Useful Logical Equivalences

These equivalences are foundational in propositional logic and are often used to simplify or transform logical expressions.

1. **Double Negation**

$$\neg(\neg p) \equiv p$$

2. **De Morgan's Law (Conjunction)**

$$\neg(p \land q) \equiv \neg p \lor \neg q$$

3. **De Morgan's Law (Disjunction)**

$$\neg(p \lor q) \equiv \neg p \land \neg q$$

4. **Implication**

$$p \Rightarrow q \equiv \neg p \lor q$$

5. **Negation of Implication**

$$\neg(p \Rightarrow q) \equiv p \land \neg q$$

2.11 The Algebra of Propositions

2.12 The Algebra of Propositions

2.12.1 Logical Laws and Their Equivalences

Law Name	Disjunction-related Expression(s)	Conjunction-related Expression(s)
Idempotent Laws	$p \lor p \equiv p$	$p \land p \equiv p$
Associative Laws	$(p \lor q) \lor r \equiv p \lor (q \lor r)$	$(p \land q) \land r \equiv p \land (q \land r)$
Commutative Laws	$p \lor q \equiv q \lor p$	$p \land q \equiv q \land p$
Distributive Laws	$p \lor (q \land r) \equiv (p \lor q) \land (p \lor r)$	$p \land (q \lor r) \equiv (p \land q) \lor (p \land r)$

Law Name	Disjunction-related Expression(s)	Conjunction-related Expression(s)
Identity Laws	$p \vee F \equiv p, \quad p \vee T \equiv T$	$p \wedge F \equiv F, \quad p \wedge T \equiv p$
Involution Law	—	$\neg\neg p \equiv p$
Complement Laws	$\neg p \vee p \equiv T$	$\neg p \wedge p \equiv F$
De Morgan's Laws	$\neg(p \wedge q) \equiv \neg p \vee \neg q$	$\neg(p \vee q) \equiv \neg p \wedge \neg q$
Conditional Identities	$p \Rightarrow q \equiv \neg p \vee q$	$p \Leftrightarrow q \equiv (p \Rightarrow q) \wedge (q \Rightarrow p)$

Example 2.26.

1. Show that:

$$p \wedge \neg(q \Rightarrow \neg r) \equiv p \wedge (q \wedge r)$$

2. Show that:

$$p \Leftrightarrow q \equiv \neg q \Leftrightarrow \neg p$$

3. Using logically equivalent statements (without truth tables), show:

$$\neg(\neg p \wedge q) \wedge (p \vee q) \equiv p$$

2.13 Predicate Logic and Quantifiers

Consider the following statements:

$$x > 3, \quad x = y + 3, \quad x + y = z$$

The truth value of these statements has no meaning without specifying the values of x, y, z.

However, we can make propositions out of such statements.

A **predicate** is a property that is true or false about the subject (in logic, we say "variable") of a statement.

For example:

$$\text{``} \underbrace{x}_{\text{subject}} \quad \underbrace{\text{is greater than 3''}}_{\text{predicate}}$$

2.13.1 Predicates

A **predicate** is a declarative sentence whose truth value depends on one or more variables.

The statement:

$$x \text{ is greater than } 3$$

has two parts:

- The **predicate**: "is greater than 3"
- The **variable**: x

We denote this statement by $P(x)$, where:

- P is the predicate "is greater than 3"
- x is the variable

By assigning a value to x, $P(x)$ becomes a **proposition** with a definite truth value:

- $P(5)$: "5 is greater than 3" $\rightarrow$ **True**
- $P(2)$: "2 is greater than 3" $\rightarrow$ **False**

Note:

- A **predicate** is neither true nor false on its own.
- A predicate becomes a **proposition** when its variables are substituted with specific values.
- The **domain** (also called the universe or universe of discourse) of a predicate variable is the set of all values that may be substituted for the variable.

Definition 2.9. Let A be a nonempty set. An expression $P(x)$ defined on A is called a **predicate** if:

$$P(a) \text{ is either true or false for each } a \in A$$

That is, $P(a)$ becomes a **statement** (i.e., a proposition with a definite truth value) whenever any element $a \in A$ is substituted for the variable x.

Example 2.27. **(i)** Let $P(x) : x^2 > 6$, where $x \in \mathbb{N}$

- $P(1)$: $1^2 = 1 \ngtr 6 \rightarrow$ False

- $P(2)$: $2^2 = 4 \not> 6 \to$ False
- For $x \in \mathbb{N}$ and $x \neq 1, x \neq 2$, $P(x)$ is true.

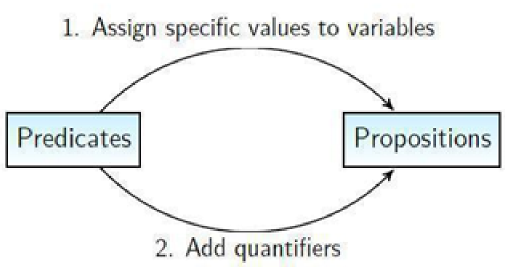
(ii) Let $P(x, y) : x^2 + y^2 = 4$, where $x, y \in \mathbb{R}$

- $P(0, 2)$: $0^2 + 2^2 = 4 \to$ True
- $P(1, 1)$: $1^2 + 1^2 = 2 \to$ False

2.13.2 Converting Predicates to Propositions

There are two standard methods:

1. **Assign specific values to variables**
   Example: $P(3)$, $P(0, 2)$
2. **Add quantifiers**
   - Universal: $\forall x \in \mathbb{N},\ P(x)$
   - Existential: $\exists x \in \mathbb{R},\ P(x)$



2.14 Quantifiers

A predicate becomes a proposition when we assign it fixed values. However, another way to convert a predicate into a proposition is by **quantifying** it.

Quantification expresses whether a predicate is true for:

- **All** values in the universe of discourse
- **Some** values in the universe of discourse

Definition 2.10. Quantifiers are words that refer to quantities such as "all" or "some." They indicate how many elements in the domain satisfy a given predicate.

Two Types of Quantifiers

1. **Universal Quantifier**
   Symbol: $\forall$
   Meaning: "For all" or "For every," or "For each," Example:

   For every $n \in \mathbb{Z}$, $2n$ is even.

This can be expressed symbolically as:

$$\forall n \in \mathbb{Z},\ 2n \text{ is even}$$

2. **Existential Quantifier**
   Symbol: $\exists$
   Meaning: "There exists a" or "There is a."
   Example:

   There exists an integer $n \in \mathbb{Z}$ such that $n^2 = 2$.

This can be expressed symbolically as:

$$\exists n \in \mathbb{Z},\ n^2 = 2$$

2.14.1 Universal Quantifier $\forall$

Let $p(x)$ be a predicate, where $x \in D$.

Suppose the statement
> "For any x in a non-empty subset $S \subseteq D$, $p(x)$ is true"
is valid.

Then the expression

$$\forall x \in S,\ p(x)$$

is a **proposition**.

This means:
> "For every x in the subset S, the predicate $p(x)$ holds."

Example 2.28. Let $p(x)$: "x is even"
Let $D = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$
Let $S = \{2, 4, 6, 8, 10\} \subseteq D$

Then:

$$\forall x \in S,\ p(x)$$

is true, since all elements of S are even.

2.14.1.1 Notation

- $\forall x\, P(x)$: Read as "for all x, $P(x)$" or "for every x, $P(x)$"
- $\forall$: Universal quantifier

Definition 2.11. An element x for which $P(x)$ is false is called a counterexample of $\forall x P(x)$.

2.14.1.2 Other ways to say $\forall$

- All of…
- For each…
- Given any…
- For any…
- For arbitrary…

Example 2.29. Let $P(x) : x^2 > x$, where x is a real number.

Then "$\forall x\, P(x)$" in English means "for all x, $x^2 > x$", or we can say "for any x, $x^2 > x$", or in other words "given any x, $x^2 > x$".

An element $x = \frac{1}{2}$, since $\left(\frac{1}{2}\right)^2 = \frac{1}{4} < \frac{1}{2}$, is false. That is, $P\left(\frac{1}{2}\right)$ is false.

So, $x = \frac{1}{2}$ is a counterexample to the statement "for all x, $x^2 > x$", where x is a real number.

Example 2.30. Consider the predicate $|x| \geq 0$, where $x \in \mathbb{R}$.

Let $P(x) : |x| \geq 0$.

Then, $\forall x \in \mathbb{R}$, $P(x)$ is true.

i.e., $\forall x \in \mathbb{R}$, $|x| \geq 0$.

Example 2.31. Consider the predicate $x > \frac{1}{2}$, where $x \in \mathbb{R}$.

Let $P(x) : x > \frac{1}{2}$.

Then $\forall x \in \mathbb{R}$, $P(x)$ is false, since for instance $0 \in \mathbb{R}$, but $P(0)$ is false.

i.e., $\neg(\forall x \in \mathbb{R},\ P(x))$.

However, $\forall x \in \mathbb{N}$, $P(x)$ is true.

i.e., $\forall x \in \mathbb{N}$, $P(x)$.

2.14.2 The Existential Quantifier "∃"

Let $p(x)$ be a predicate, where $x \in D$.

Suppose:
"There is $x_0 \in S \subseteq D$, such that $p(x_0)$" is true.

Then:
"There is $x_0 \in S \subseteq D$, such that $p(x_0)$" is a proposition.

We write this proposition as:

$$\exists x_0 \in S,\ p(x_0)$$

Which means:
"There exists $x_0 \in S$ such that $p(x_0)$" is true.

i.e.,

$$\exists x_0 \in S,\ p(x_0)$$

2.14.2.1 Notation

$\exists x\, P(x)$, where $\exists$ is called the existential quantifier.

2.14.2.2 Other ways to read ∃:

- "There is an x such that $P(x)$."
- "There is at least one x such that $P(x)$."
- "For some x, $P(x)$."

Example 2.32. Consider the predicate $2x > 7$, where $x \in \mathbb{N}$.

Let $P(x) : 2x > 7$.

When $x \in \mathbb{N}$ and $2x > 7$ is true (since $10 > 7$),

i.e., when $x = 5$, $x \in \mathbb{N}$ and $P(5)$ is true.

Therefore,

$$\exists x \in \mathbb{N},\ P(x) \text{ is true}$$

i.e.,

$$\exists x \in \mathbb{N},\ 2x > 7$$

Example 2.33. Consider the predicate $x^2 > 1$, where $x \in \mathbb{R}$.

Let $A = \{0, 1, -1\}$ and define $P(x) : x^2 > 1$.

Then: - $P(0)$ is false, - $P(1)$ is false, - $P(-1)$ is false.

Therefore,

$$\exists x \in A,\ P(x)$$

is false,

i.e.,

$$\neg(\exists x \in A,\ P(x))$$

.

However,

$$\exists x \in \mathbb{N},\ P(x)$$

is true,
since when $x = 2$, $x \in \mathbb{N}$ and $P(2)$ is true.

i.e.,

$$\exists x \in \mathbb{N},\ x^2 > 1.$$

Let $p(x)$ be a predicate and D be the domain of x.

An existential statement is a statement of the form:

$$\exists x \in D,\ p(x)$$

- Forms:
  - "There exists an x such that $p(x)$"
  - "For some x, $p(x)$"
  - "We can find an x such that $p(x)$"
  - "There is some x such that $p(x)$"
  - "There is at least one x such that $p(x)$"
- The statement is **true** if $p(x)$ is true for **at least one** $x \in D$.
- The statement is **false** if $p(x)$ is false for **all** $x \in D$.
- A **Counterproof** to disprove an existential statement, one must show that $p(x)$ is false for **every** $x \in D$.

2.14.3 Quantifiers and Negation

Suppose A is a non-empty set, $P(x)$ is a predicate where $x \in A$, and $B \subseteq A$ is a non-empty subset.

We consider the negations of quantified statements:

1. Negation of a Universal Statement

$$\neg(\forall x \in B,\ P(x))$$

This means:
It is not the case that for every $x \in B$, $P(x)$ holds.
So, there exists some $x \in B$ such that $P(x)$ does not hold:

$$\exists x \in B,\ \neg P(x)$$

2. Negation of an Existential Statement

$$\neg(\exists x \in B,\ P(x))$$

This means:
It is not the case that there exists some $x \in B$ such that $P(x)$ holds.
So, for every $x \in B$, $P(x)$ does not hold:

$$\forall x \in B,\ \neg P(x)$$

The following equivalences hold:

$$\neg(\forall x \in B,\ P(x)) \equiv \exists x \in B,\ \neg P(x)$$
$$\neg(\exists x \in B,\ P(x)) \equiv \forall x \in B,\ \neg P(x)$$

- $\neg(\forall x \in D,\ p(x)) \equiv \exists x \in D,\ \neg p(x)$
- $\neg(\exists x \in D,\ p(x)) \equiv \forall x \in D,\ \neg p(x)$
- The negation of a universal statement ("all are") is logically equivalent to an existential statement ("there is at least one that is not").
- The negation of an existential statement ("some are") is logically equivalent to a universal statement ("all are not").

Example 2.34.

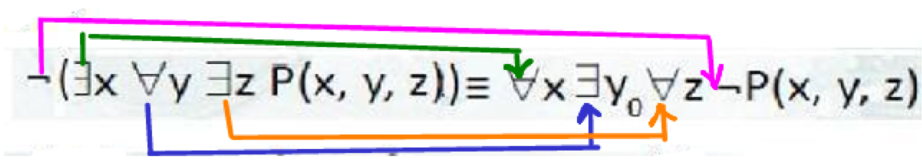
- $\forall$ primes p, p is odd
  Negation: $\exists$ prime p, p is even
- $\exists$ triangle T, the sum of angles of T equals 200°
  Negation: $\forall$ triangles T, the sum of angles of T does not equal 200°

2.14.3.1 Truth Value of Quantified Statements

Statement	When True	When False
$\forall x \in D,\ P(x)$	$P(x)$ is true for every $x \in D$	There exists at least one $x \in D$ such that $P(x)$ is false
$\exists x \in D,\ P(x)$	There exists at least one $x \in D$ such that $P(x)$ is true	$P(x)$ is false for every $x \in D$

Example 2.35. We are given the logical equivalence:

$$\neg(\exists x\ \forall y\ \exists z\ P(x,y,z)) \equiv \forall x\ \exists y\ \forall z\ \neg P(x,y,z)$$



Note: Logical operations from propositional logic — such as $\neg$, $\wedge$, $\vee$, $\Rightarrow$, $\Leftrightarrow$ — can also be applied to quantified statements.

Here is a clean transcription of the image content, showing only the questions and definitions without any answers or reformulations:

2.14.4 Negation of Quantification

Given a quantified statement of the form:

$$\neg(\forall x \in D,\, P(x) \wedge Q(x))$$

Apply logical transformations to express its equivalent forms.

2.14.4.1 Conditional Quantification

Consider the conditional quantification:

$$\forall x \in D,\ P(x) \Rightarrow Q(x)$$

Define the following related forms:

- **Converse**: $\forall x \in D$, $Q(x) \Rightarrow P(x)$
- **Contrapositive**: $\forall x \in D$, $\neg Q(x) \Rightarrow \neg P(x)$
- **Inverse**:$\forall x \in D$, $\neg P(x) \Rightarrow \neg Q(x)$

Note: : A conditional proposition is logically equivalnce to its contrapositive.

$$\forall x \in D, P(x) \implies Q(x) \equiv \forall x \in D, \neg Q(x) \implies \neg P(x)$$

2.14.5 Negation of Conditional Quantification

$$\neg(\forall x \in D, P(x) \implies Q(x)) \equiv \exists x_0 \in D \neg(P(x_0) \implies Q(x_0)) \tag{2.1}$$
$$\equiv \exists x_0 \in D P(x_0) \wedge \neg Q(x_0) \tag{2.2}$$

Example 2.36. Let $x \in \mathbb{R}$.

- $x = 1$ is a sufficient condition for $x^2 = 1$
  i.e., $\forall x \in \mathbb{R},\ x = 1 \Rightarrow x^2 = 1$
- $x^2 = 1$ is a necessary condition for $x = 1$
  i.e., $\forall x \in \mathbb{R},\ x^2 \neq 1 \Rightarrow x \neq 1$
- $x = 1$ only if $x^2 = 1$
  i.e., $\forall x \in \mathbb{R},\ x^2 \neq 1 \Rightarrow x \neq 1$

Example 2.37. Let $x \in \mathbb{R}$.

Statement: $\forall x,\ x > 10 \Rightarrow x^2 > 100$

Negation: $\exists x \in \mathbb{R}$ such that $x > 10$ and $x^2 \leq 100$

Example 2.38. Let $x \in \mathbb{R}$.

Statement:
If $x > 1$, then $x^2 > 1$

Define:
$P(x) : x > 1$
$Q(x) : x^2 > 1$

Symbolic form:

$$\forall x \in \mathbb{R},\ P(x) \Rightarrow Q(x)$$

Negation:

$$\exists x_0 \in \mathbb{R},\ P(x_0) \wedge \neg Q(x_0)$$

i.e.,

$$x_0 > 1 \quad \text{and} \quad x_0^2 \leq 1$$

Chapter 3

Introduction to Proofs

3.0.1 Terminology

- **Theorem**: A statement that can be shown to be true (i.e., a fact or result).
- **Proposition**: A smaller or less important theorem.
- **Axiom / Postulate**: A statement assumed to be true.
- **Lemma**: A less important theorem used in the proof of another theorem.
- **Corollary**: A less important theorem that follows from a larger theorem.
- **Conjecture**: A statement proposed as true but not yet proven.

Note: A **proof** is a valid argument that establishes the truth of a theorem (or any statement that can be true or false).
Axioms and postulates do not require proof—they serve as foundational assumptions, akin to basic words in a dictionary that help define others.

3.1 Arguments

- An **argument** is an assertion that a given set of propositions $p_1, p_2, \dots, p_n$, called **premises**, yields another proposition q, called the **conclusion**.

- This is denoted symbolically as:

$$p_1, p_2, \dots, p_n \vdash q$$

- The argument is said to be **valid** if q is true whenever all the premises $p_1, p_2, \dots, p_n$ are true.

- An argument that is not valid is called a **fallacy**.

- The argument $p_1, p_2, \dots, p_n \vdash q$ is **valid** if and only if the compound proposition:

$$(p_1 \wedge p_2 \wedge \dots \wedge p_n) \Rightarrow q$$

is a **tautology**.

Example 3.1. Let $n \in \mathbb{N}$. Consider the following argument:

1. $(-1)^n$ is positive or $(-1)^n$ is negative.
2. If $(-1)^n$ is positive, then $(-1)^{2n} > 0$.
3. If $(-1)^n$ is negative, then $(-1)^{2n} > 0$.

Therefore, $(-1)^{2n} > 0$.

Let: - p: $(-1)^n$ is positive
- q: $(-1)^n$ is negative
- r: $(-1)^{2n} > 0$

We examine the validity of the argument:

$$\underbrace{(p \vee q)}_{Premise}, \underbrace{(p \Rightarrow r)}_{Premise}, \underbrace{(q \Rightarrow r)}_{Premise} \vdash \underbrace{r}_{Conclusion}$$

This corresponds to the compound proposition:

$$((p \vee q) \wedge (p \Rightarrow r) \wedge (q \Rightarrow r)) \Rightarrow r$$

p	q	r	$p \vee q$	$p \Rightarrow r$	$q \Rightarrow r$	$(p \vee q) \wedge (p \Rightarrow r) \wedge (q \Rightarrow r)$	$((p \vee q) \wedge (p \Rightarrow r) \wedge (q \Rightarrow r)) \Rightarrow r$
T	T	T	T	T	T	T	T
F	T	T	T	T	T	T	T
T	F	T	T	T	T	T	T
F	F	T	F	T	T	F	T
T	T	F	T	F	F	F	F
T	F	F	T	F	T	F	F
F	T	F	T	T	F	F	F
F	F	F	F	T	T	F	T

This table confirms that the compound proposition is **not** a tautology, hence the argument is **not valid** in all cases. However, in the specific example from Section 8.3, the premises are true and the conclusion follows, so the argument is valid **in that instance**, though not universally.

3.2 Valid Argument

By definition, a argument is **valid** if: > If the premises are true, then the conclusion is true.

This corresponds to the tautology:

$$((p \Rightarrow q) \wedge p) \Rightarrow q$$

Example 3.2. Premises: 1. $p \Rightarrow q$ 2. p

Conclusion: q

Example 3.3. Is the following argument valid?

Premises: 1. If the door is open, then I must close it. 2. The door is open.

Conclusion: I must close it.

Let: - p: The door is open
- q: I must close it

This matches the form:

$$(p \Rightarrow q),\ p \vdash q$$

Since the compound proposition $((p \Rightarrow q) \wedge p) \Rightarrow q$ is a tautology, the argument is valid.

We consider the argument:

$$[(p \Rightarrow q) \wedge p] \Rightarrow q$$

This structure represents a **valid argument**, often referred to as a **rule of inference**.

3.2.0.1 Method 1: Truth Table

p	q	$[(p \Rightarrow q) \wedge p] \Rightarrow q$
T	T	T
T	F	T
F	T	T
F	F	T

This confirms that the compound proposition is **not always true**, but when both premises are true, the conclusion follows — hence the argument is valid in that case.

3.2.0.2 Method 2: Direct Reasoning

If:

- p is true, and
- $p \Rightarrow q$ is true,

Then:

- q must be true.

Remark. A **logical rule of inference** is defined to be any valid argument — that is, one where the conclusion necessarily follows from the premises.

(i) Modus Ponens This rule affirms the consequent based on a conditional and its antecedent:

$$\begin{array}{ll} p \Rightarrow q & \text{(premise 1)} \\ p & \text{(premise 2)} \\ \hline q & \text{(conclusion)} \end{array}$$

If "p implies q" and "p" is true, then "q" must also be true.

Example 3.4. Claim:$\sqrt{2} < \sqrt{3}$

$$\begin{aligned} 2 < 3 &\implies 2 - 3 < 0 && (3.1) \\ &\implies (\sqrt{2} - \sqrt{3})(\sqrt{2} + \sqrt{3}) < 0 && (3.2) \\ &\implies (\sqrt{2} - \sqrt{3}) < 0 && (3.3) \\ &\implies \sqrt{2} < \sqrt{3} && (3.4) \end{aligned}$$

(ii) Modus Tollens This rule denies the antecedent based on a conditional and the negation of its consequent:

$$\begin{array}{ll} p \Rightarrow q & \text{(premise 1)} \\ \neg q & \text{(premise 2)} \\ \hline \neg p & \text{(conclusion)} \end{array}$$

If "p implies q" and "q" is false, then "p" must also be false.

Example 3.5. Claim: $\sqrt{2} + \sqrt{3} < \sqrt{11}$

$$
\begin{aligned}
\sqrt{2}+\sqrt{3} \geq \sqrt{11} &\implies (\sqrt{2}+\sqrt{3})^2 < (\sqrt{11})\text{Since } \sqrt{2}+\sqrt{3} \geq 11 > 0 && (3.5)\\
&\implies 2+2\sqrt{6}+3 \geq 11 && (3.6)\\
&\implies 5+2\sqrt{6} \geq 11 && (3.7)\\
&\implies 2\sqrt{6} \geq 6 && (3.8)\\
&\implies 4\times 6 \geq 36 \text{ Since } 4\times 6 \geq 36 > 0 && (3.9)\\
\underbrace{\sqrt{2}+\sqrt{3} \geq \sqrt{11}}_{p} &\implies \underbrace{24 \geq 36}_{q} \text{ is true} && (3.10)\\
& && (3.11)\\
\text{But } 24 < 36 \text{ is true} & && (3.12)\\
\therefore 24 \geq 36 \text{ is false} & && (3.13)\\
\therefore \underbrace{\neg(24 \geq 36)}_{\neg q} \text{ is true} & && (3.14)\\
\therefore \neg p & && (3.15)\\
\text{i.e. } \sqrt{2}+\sqrt{3} < \sqrt{11} & && (3.16)
\end{aligned}
$$

3.3 Mathematical proofs

There are several standard techniques used to establish mathematical theorems. Below are five commonly used methods:

1. Direct proofs
2. Proof by Contradiction
3. Indirect proofs
4. proof by cases
5. Proof by Principle of Mathematical Induction

3.3.1 Direct proofs

Direct proof is probably the easiest approach to establish the theorems, as it does not require knowledge of any special techniques. The argument is constructed using a series of simple statements, where each one should follow directly from the previous one. It is important not to miss out any steps as this may lead to a gap in reasoning. To prove the hypothesis, one may use axioms, as well as the previously established statements of different theorems.

3.3.1.1 Proving Conditional Statements: $p \implies q$

- If p is **false**, then the **implication** is always Thus, show that if p is true, then q is true.
- *Direct Proof*: Assume that p is true. Use rules of inference, axioms, and logical equivalences to show that q must also be true.

Proof of "$p \Rightarrow q$":

Proof of "$p \Rightarrow q$":

Suppose that p. ---------(i)

$$\left.\begin{array}{l} - \\ - \\ - \\ - \\ - \\ - \end{array}\right\} \begin{array}{l} Definitions, \\ Axioms, \\ Logical\ equivalences, \\ Rules\ of\ inference \end{array}$$

Therefore, q. -----------(ii)

By (i), (ii) we get $p \Rightarrow q$ ■

Before we begin exploring example proofs, it's essential to clearly define the foundational terms we'll be using. Based on your notes and the image provided, here are the formal definitions:

Definition 3.1 (Even and Odd Integers)**.**

- An integer n is **even** if there exists an integer $k \in \mathbb{Z}$ such that:

$$n = 2k$$

- An integer n is **odd** if there exists an integer $k \in \mathbb{Z}$ such that:

$$n = 2k + 1$$

- Every integer is either even or odd, and no integer is both.

Example 3.6. 5 is odd integer and 10 is even integer.

- Since $5 = 2 \times 2 + 1$
- Since $10 = 2 \times 5$

Definition 3.2 (Divisibility)**.** A integer \$b 0 \$ b **divides another integer** a if: $\exists k \in \mathbb{Z}$ such that $a = k \cdot b$ ** Notation**: $b \neq 0$ divides a , written $b \mid a$)

In this case, b is called a **factor** or **divisor** of a.

Example 3.7.

- $3 \mid 12$ because $12 = 4 \cdot 3$
- $5 \nmid 12$ because there is no integer k such that $12 = 5k$

Example 3.8. Give a direct proof of the theorem "If n is an odd integer, then n^2 is odd."

Proof. Let $n \in \mathbb{Z}$ be an odd integer.
By definition of oddness, there exists an integer $k \in \mathbb{Z}$ such that:

$$n = 2k + 1$$

Now compute n^2:

$$n^2 = (2k+1)^2 = 4k^2 + 4k + 1 = 2(2k^2 + 2k) + 1$$

Let $m = 2k^2 + 2k \in \mathbb{Z}$. Then:

$$n^2 = 2m + 1$$

Thus, n^2 is of the form $2m + 1$, which is odd. $\square$

Theorem 3.1. *Let n and m be integers. Then*

i. if n and m are both even, then $n + m$ is even,
ii. if n and m are both odd, then $n + m$ is even,
iii. if one of n and m is even and the other is odd, then $n + m$ is odd.

Claim 1: If n and m are both even, then $n + m$ is even.

Proof. If n and m are even, then there exist integers $a, b \in \mathbb{Z}$ such that:

$$n = 2a, \quad m = 2b$$

Then:

$$n + m = 2a + 2b = 2(a + b)$$

Since $a + b \in \mathbb{Z}$, $n + m$ is divisible by 2, hence even. $\square$

Claim 2: If n and m are both odd, then $n + m$ is even.

Proof. If n and m are odd, then there exist integers $a, b \in \mathbb{Z}$ such that:

$$n = 2a + 1, \quad m = 2b + 1$$

Then:

$$n + m = (2a + 1) + (2b + 1) = 2a + 2b + 2 = 2(a + b + 1)$$

Since $a + b + 1 \in \mathbb{Z}$, $n + m$ is even. $\square$

Claim 3: If one of n and m is even and the other is odd, then $n+m$ is odd.

Proof. Assume without loss of generality that n is even and m is odd.
Then there exist integers $a, b \in \mathbb{Z}$ such that:

$$n = 2a, \quad m = 2b + 1$$

Then:

$$n + m = 2a + (2b + 1) = 2(a + b) + 1$$

Since $a + b \in \mathbb{Z}$, $n + m$ is of the form $2k + 1$, hence odd. $\square$

Consider the following definitions

- A integer$a \in \mathbb{Z}$. said to be Type 0: if there exist integr n such that $a = 3n$.
- A integer$a \in \mathbb{Z}$. said to be Type 1: if there exist integr n such that $a = 3n + 1$.
- A integer$a \in \mathbb{Z}$. said to be Type 2: if there exist integr n such that $a = 3n + 2$.

(i) If a and b are both type 1 integers, then $a + b$ is a type 2 integer.

Proof. Let $a = 3m + 1$ and $b = 3n + 1$ for some $m, n \in \mathbb{Z}$.
Then:

$$a + b = (3m + 1) + (3n + 1) = 3(m + n) + 2$$

So $a + b$ is of the form $3k + 2$, hence type 2. $\square$

(ii) If a and b are both type 2 integers, then $a + b$ is a type 1 integer.

Proof. Let $a = 3m + 2$ and $b = 3n + 2$ for some $m, n \in \mathbb{Z}$.
Then:

$$a + b = (3m + 2) + (3n + 2) = 3(m + n) + 4 = 3(m + n + 1) + 1$$

So $a + b$ is of the form $3k + 1$, hence type 1. $\square$

(iii) If a is type 1 and b is type 2, then ab is type 2.

Proof. Let $a = 3m + 1$, $b = 3n + 2$ for some $m, n \in \mathbb{Z}$.
Then:

$$ab = (3m + 1)(3n + 2) = 9mn + 6m + 3n + 2 = 3(3mn + 2m + n) + 2$$

So ab is of the form $3k + 2$, hence type 2. $\square$