Functinal Analaysis

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Contents

Chapter 1

Banach space

1.1 Lebesgue spaces

1.1.1 a

Definition 1.1. markdown Let X be a set. A σ**-algebra** $\mathcal{J}$ on X is a collection of subsets of X such that:

1. $\emptyset \in \mathcal{J}$,
2. if $E \in \mathcal{J}$, then $X \setminus E \in \mathcal{J}$,
3. if $E_n \in \mathcal{J}$ for every $n \geq 1$, then

$$\bigcup_{n=1}^{\infty} E_n \in \mathcal{J}.$$

- Elements of $\mathcal{J}$ are called $\mathcal{J}$-measurable sets,
- $(X, \mathcal{J})$ is a measurable space.

Definition 1.2. A function $f : X \to \mathbb{C}$ is said to be measurable if

$$f^{-1}(\{\, z \in \mathbb{C} : |z - a| < \delta \,\}) \in \mathcal{T}$$

for every $\delta > 0$ and $a \in \mathbb{C}$.

Definition 1.3. A (positive) measure is a function

$$\mu : \mathcal{T} \to [0, \infty]$$

which is countably additive, in the sense that if $\{E_n\}_{n=1}^{\infty}$ is a countable collection of disjoint measurable sets, then

$$\mu\left(\bigcup_{n=1}^{\infty} E_n\right) = \sum_{n=1}^{\infty} \mu(E_n).$$

- The triple $(X, \mathcal{T}, \mu)$ is called a *measure space*.

Notation :

- Let $0 < p < \infty$,

$$\mathcal{L}^p(X, \mathcal{I}, \mu) := \left\{ f : X \to \mathcal{C} : f \text{ is measurable and } \int_X |f|^p d\mu < \infty \right\}$$

  - Such functions are said to be p**-integrable**.
  - $\mathcal{L}^p$ norm of $f = ||f||_p = \left(\int_X |f|^p \, d\mu\right)^{\frac{1}{p}}$

- $p = \infty$

$$\mathcal{L}^\infty(X, \mathcal{T}, \mu) = \{ f : X \to \mathbb{C} \ : \ \exists M > 0 \text{ such that } |f| < M \ [\mu]\text{-a.e. on } X \}.$$

- Such functions are said to be **essentially bounded**.
- The essential norm $= \mathcal{L}^\infty$ norm of $f = \|f\|_\infty = \inf\{ M > 0 : |f| < M \ [\mu]\text{-a.e. on } X \}$.

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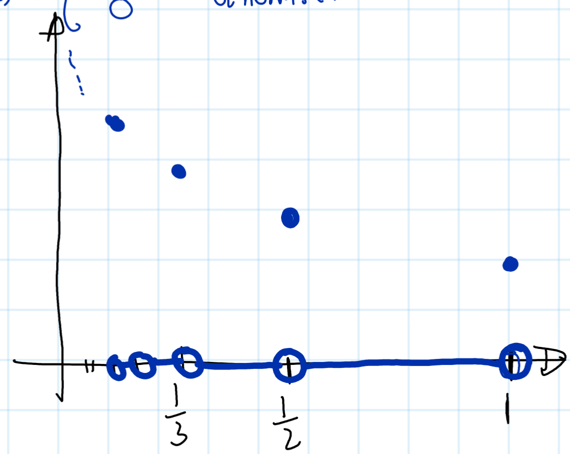
Define

$$f(x) = \begin{cases} \frac{1}{x} & \text{if } x \in \mathbb{N} \\ 0 & \text{otherwise.} \end{cases}$$



- As you can see f is not bdd.
- f is essential bdd.

We can use M as any postive number

$$L^\infty(X, \mathcal{T}, \mu) = \{f : X \to \mathbb{C} \,:\, \exists M > 0 \text{ such that } |f| < M \ [\mu]\text{-a.e. on } X\}.$$

$(M = 1, e, \pi, \frac{1}{2}, 0.0001, 10^{100}, \ldots.)$

Further $\|f\|_\infty = 0$ (Since we choose inf of such M)

In this section we use the term **"norm."** Strictly speaking, we have not yet verified that the expressions introduced actually satisfy the axioms of a norm. That verification will come later. For now, we use the word "norm" informally, with the understanding that its legitimacy will be established in due course.

Lemma 1.1. *Let $(X, \mathcal{T}, \mu)$ be a measure space, let $0 < p < \infty$, and let $f \in \mathcal{L}^p(X, \mathcal{T}, \mu)$. Then*

$$\|f\|_p = 0 \quad \Longleftrightarrow \quad f(x) = 0 \text{ for } [\mu]\text{-a.e. } x \in X.$$

1

($\Leftarrow$) If $f = 0$ $[\mu]$-a.e. then $\int_X |f|^p \, d\mu = 0$

($\Rightarrow$) For $n \geq 1$, let $E_n := \{x \in X : |f(x)| > \frac{1}{n}\}$

Then $E_n \in I$. $\left(\begin{array}{l} \text{Since } f \text{ is measurable and} \\ E_n = f^{-1}\left\{\left(z \in \mathbb{C} : |z| > \frac{1}{n}\right)\right\} \in I \end{array} \right)$

and $\bigcup_{n=1}^{\infty} E_n = \{x \in X \mid f(x) > 0\}$

Now forevery $n \in \mathbb{N}$,

$$0 \underset{\text{hypotheis}}{=} \|f\|_p \underset{\text{def}^{\underline{n}}}{=} \int_X |f|^p \, d\mu \underset{X \supseteq E_n}{\geq} \int_{E_n} |f|^p \, d\mu \underset{\text{In } E_n,\ |f| > \frac{1}{n}}{>} \int_{E_n} \frac{1}{n^p}$$

$$= \int_{E_n} \frac{1}{n^p} \, d\mu = \frac{\mu(E_n)}{n^p} \geq 0$$

Proof.

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$\Rightarrow \quad \mu(E_n) = 0$

Thus $\quad \mu\left(\bigcup_{n=1}^{\infty} E_n\right) = \sum_{n=1}^{\infty} \mu(E_n)$

$\mu(\{x \in X : f(x) > 0\}) = 0$

$\square$

Fact: $\lambda \in \mathbb{C}, f \in \mathcal{L}^p(X, I, \mu), 0 < p \geq \infty, ||\lambda f||_p = |\lambda|||f||_p$.

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- If $0<p<\infty$;

$$\|\lambda f\|_p = \left(\int_X |\lambda f|^p \, d\mu\right)^{\frac{1}{p}}$$

$$= \left(|\lambda|^p \int_X |f|^p \, d\mu\right)^{\frac{1}{p}} = |\lambda| \, \|f\|_\infty$$

- If $p=\infty$

?

Proof.

□

Lemma 1.2. *Let $(X, \mathcal{T}, \mu)$ be a measure space and let $f \in \mathcal{L}^\infty(X, \mathcal{T}, \mu)$. Then, for*

$$|f(x)| \leq \|f\|_\infty[\mu] - a.e. x \in X.$$

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$\exists$ a sequnce (M_n) of positive numbers s.t.
$M_n \longrightarrow \|f\|_\infty$ and $|f| \leq M_n$ μ-a.e.

(If u need, u can construct the sequnce
$M_n = \|f\|_\infty + \frac{1}{n} = \inf\{M > 0 / |f| < M \ \mu\text{-a.e}\} + \frac{1}{n}$)

By def^n of a.e, $\exists E_n \in I$ s.t. $\mu(E_n) = 0$ s.t.

$$|f(x)| \leq M \quad \text{for} \quad x \in X \setminus E_n$$

Define $E = \bigcup_{n=1}^{\infty} E_n$. Then $\mu(E) = \mu(\bigcup_{n=1}^{\infty} E_n) = \sum_{n=1}^{\infty} \mu(E_n) = 0.$

If $x \in X \setminus E = X \setminus \bigcup_{n=1}^{\infty} E_n = \bigcap_{n=1}^{\infty} (X \setminus E_n)$ then $|f(x)| \leq M_n$, $n \geq 1$

$$\Rightarrow \quad |f| \leq \lim_{n \to \infty} M_n = \|f\|_\infty$$

(our choice of M_n
$M_n \longrightarrow \|f\|_\infty$)

$\blacksquare$

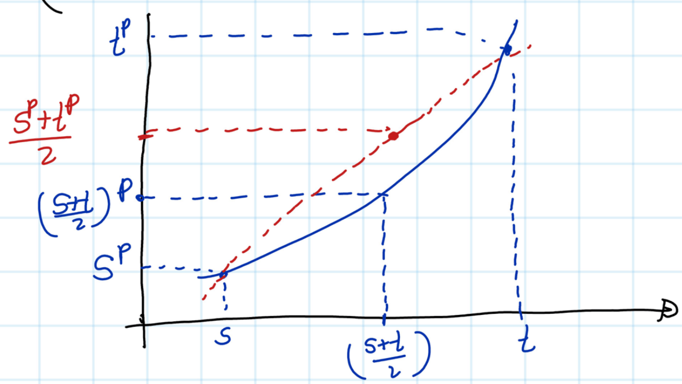
Result : If $f, g \in \mathcal{L}^P(X, I, \mu)$ then $f + g \in \mathcal{L}^P(X, I, \mu)$

$1 \le p < \infty$

$1 \leq p < \infty$

The function $t \longmapsto t^p$, is convex on $(0, \infty)$

(Because 2nd derivate of fun, $p(p-1)t^{p-2} > 0$



$$\left(\frac{s+t}{2}\right)^p \leq \frac{s^p + t^p}{2}$$

$$\left(\frac{|f(x)| + |g(x)|}{2}\right)^p \leq \frac{|f(x)|^p + |g(x)|^p}{2}$$

$$\left(|f(x)| + |g(x)|\right)^p \leq 2^{p-1}\left(|f(x)|^p + |g(x)|^p\right)$$

Then,

$$\int_X |f+g|^p \, d\mu \leq \int_X (|f| + |g|)^p \quad (\Delta \text{ inequlity})$$

$$\leq 2^{p-1}\left[\int (|f|^p + |g|^p) \, d\mu\right] < \infty$$

$p = \infty$

From Lemma

$\mu(E) = \mu($

Then Δ in

$|f(x)+g(x)| \leq |f$

Since $E, F \in I$ th

$f + g \in$

Conclusion: $\|$

Lemma 1.3 (Young's inequality)**.** *Let $a, b \geq 0$ and $1 < p < \infty$. Let q be the conjugate exponent, i.e.*

$\frac{1}{p} + \frac{1}{q} = 1.$

Then

$$ab \leq \frac{a^p}{p} + \frac{b^q}{q}.$$

proof of Young inequality

Case-1) If $a=0$ or $b=0$ this is trivial.

Case-11) If $a>0$, $b>0$. then $s = p\log a$ & $t = q\log b$

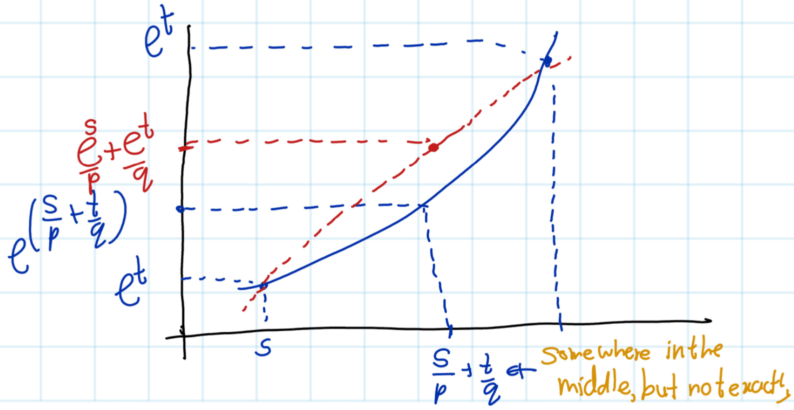
Then,

$$e^{\left(\frac{s}{p}+\frac{t}{q}\right)} = e^{\frac{s}{p}} e^{\frac{t}{q}} = e^{\log a} \cdot e^{\log b} = ab$$

So, $\frac{s}{p} + \frac{t}{q} = \frac{1}{p}s + \frac{1}{q}t = \frac{1}{p}s + \left(1-\frac{1}{p}\right)t = \lambda s + (1-\lambda)t$

it is convex.

e^x is convex



$$e^{\left(\frac{t}{p}+\frac{s}{q}\right)} \leq \frac{1}{p} e^{s} + \frac{1}{q} e^{t} < \frac{1}{p} a^{p} + \frac{1}{q} b^{q}$$

Theorem 1.1 (Holder's inequality)**.** *Fix* $1 \leq p < \infty$ *and let* q *be the conjugate exponent, i.e.*

$\frac{1}{p} + \frac{1}{q} = 1.$

Let $f, g : X \to \mathbb{C}$ *be measurable functions. Then*

$$\int_X |fg| \, d\mu \ \leq \ \left(\int_X |f|^p \, d\mu\right)^{1/p} \left(\int_X |g|^q \, d\mu\right)^{1/q} = ||f||_p ||f||_q.$$

Proof of Hölder's equation.

Case-I $p=1$, in this case $q=\infty$. Then,

$$\int_X |fg|\,d\mu = \int_X |f|\,|g|\,d\mu \leqslant \|g\|_\infty \int_X f\,d\mu = \|g\|_\infty \|f\|$$

(lemma 1.1.2 $|g| \leqslant \|g\|_\infty$)

Case-II, $1<p<\infty$

$$q = 1-\frac{1}{p} = \frac{(p-1)}{p},$$

Then $1<q<\infty$

subcase II.1 If $\|f\|_p=0$ or $\|g\|_q=0$ then,

$fg=0$ $[\mu]$-a.e. Then again inequality is trivial.

subcase II.2 If $\|f\|_p=\infty$ or $\|g\|_q=\infty$ then the resut is trivial.

subcase II.3 If $1<\|f\|_p<\infty$ and $1<\|g\|_q<\infty$

Proof.

Apply young inequlity,

$$a = \frac{|f(x)|}{\|f\|_p} \quad \text{and} \quad b = \frac{|g(x)|}{\|g\|_q}$$

$$ab \leq \frac{a^p}{p} + \frac{b^p}{q}$$

$$\frac{|f(x)||g(x)|}{\|f\|_p \|g\|_q} \leq \frac{1}{p}\left(\frac{|f(x)|}{\|f\|_p}\right)^p + \frac{1}{q}\left(\frac{|g(x)|}{\|g\|_q}\right)^q$$

$$\Rightarrow \int_X \frac{|fg|}{\|f\|_p \|g\|_q} d\mu \leq \frac{1}{p}\int_X \frac{|f(x)|^p}{\|f\|_p^p} d\mu + \frac{1}{q}\int_X \frac{|g(x)|^q}{\|g\|_p^p} d\mu$$

$$= \frac{1}{p\|f\|_p^p}\int_X |f(x)|^p d\mu + \frac{1}{q\|g\|_q}\int |g(x)|^p d\mu$$

$$= \frac{1}{p\|f\|_p^p} \|f\|_p^p + \frac{1}{q\|g\|_q} \|g\|_q^q$$

$$= 1/p + 1/q = 1$$

$$\|f\|_p = \left(\int_X |f| d\mu\right)^{1/p}$$

$\square$

Remark. If $p = 2$ then $q = 2$ then Holder ineqaulty becomes Cauchy -Schawrz inequlity.

Theorem 1.2 (Minkowski's Inequality)**.** *Fix $1 \leq p \leq \infty$. Let $f, g : X \to \mathbb{C}$ be measurable functions. Then*

$$\|f + g\|_p \leq \|f\|_p + \|g\|_p.$$

<u>Minkowski's Inequality proof</u>

First we need to remove some trivial case

- If $\|f+g\|_p = 0$
- If $\|f\|_p = \infty$ and $\|g\|_p = \infty$

Now WOLG we assume that $\|f+g\|_p \neq 0$ and $\|f\|_p < \infty$ and $\|g\|_p < \infty$.

First, observe that (triangle ineqlity)

$|f(x)+g(x)| \leq |f(x)| + |g(x)|$ for all $x \in X$

$p=1$ $\|f+g\|_1 = \int_X |f+g| d\mu \leq \int_X |f| + |g| d\mu$

$= \int_X |f| d\mu + \int_X |g| d\mu$

$= \|f\|_1 + \|g\|_1$

$p=\infty$ $|f| < \|f\|_\infty$ $[\mu]$-a.e on X (lemm 1.1.2)

Thus, $|f| < \|f\|_\infty < \infty$ on $E \in \Sigma$ s.t. $\mu(X \setminus E) = 0$

$|g| < \|g\|_\infty < \infty$ on $F \in \Sigma$ s.t. $\mu(X \setminus F) = 0$

Proof.

Then $|f+g| \leqslant |f|+|g| < \|f\|_\infty + \|g\|_\infty$ on $E \cap F \in \Sigma$

(finite intersection of measurable set is measurable)

Furthur,

$\mu(X \setminus (E \cap F)) = \mu((X \setminus E) \cup (X \setminus F))$ (De morgan)

$\leqslant \mu(X \setminus E) + \mu(X \setminus F) = 0+0=0$

$\mu(X \setminus E \cap F) = 0$

Thus, $|f+g| < \|f\|_\infty + \|g\|_\infty$ on $[\mu]$ a.e on X.

$\|f\|_\infty = \inf\{M > 0 : |f| < M \; [\mu]\text{-a.e. on } X\}.$

$\Rightarrow \|f+g\|_\infty \leqslant \|f\|_\infty + \|g_\infty\|$

If $1 < p < \infty$. Now we come to the intersting part of proof. (Other cases are some what trivial).

We know $|f+g| \leqslant |f|+|g|$ (Δ inequlity)

$|f+g|^p \leqslant (|f|+|g|)^p$ (Since $t \longmapsto t^p$ is increasing $p \geqslant 1, t \in (0,\infty)$)

Then, $\|f+g\|_p = \left(\int_X |f+g|^p\right)^{1/p} \leqslant \left(\int_X (|f|+|g|)^p\right)^{1/p}$

$$= \| |f|+|g| \|_p$$

Thus it sufficient to show that,

$$\| |f|+|g| \|_p \leqslant \| |f| \|_p + \| |g| \|_p$$

Thus WLOG, we may assume that $f \geqslant 0, g \geqslant 0$ on X.

We have, $(f+g)^p = (f+g)(f+g)^{p-1}$

$$= f(f+g)^{p-1} + g(f+g)^{p-1}$$

Let's focus in first half of above and compute

$$\int_X f(f+g)^{p-1} d\mu \leqslant \|f\|_p \, \|(f+g)^{p-1}\|_q$$

$$= \|f\|_p \left(\int_X (f+g)^{(p-1)q} d\mu\right)^{1/q}$$

$$= \|f\|_p \left(\int_X (f+g)^{p} d\mu\right)^{1/q}$$

$\frac{1}{p} + \frac{1}{q} = 1$

$q = \frac{1}{1-\frac{1}{p}} = \frac{p}{p-1}$

Similarly $\int_X g(f+g)^{p-1} d\mu \leq \|g\|_p \left(\int_X (f+g)^p d\mu\right)^{\frac{1}{q}}$

Note that $0 < \int_X (f+g)^p d\mu < \infty$

$\|f+g\|_p \neq 0 \Rightarrow \left(\int_X (f+g)^p\right)^{1/p} \neq 0$

$\Rightarrow \int_X (f+g)^p d\mu \neq 0$

$$\frac{\left(\int_X (f+g)^p d\mu\right)}{\left(\int_X (f+g)^p d\mu\right)^{\frac{1}{q}}} \leq \|f\|_p + \|g\|_p$$

$$\|f+g\|_p = \left(\int_X (f+g)^p d\mu\right)^{\frac{1}{p}} = \left(\int_X (f+g)^p d\mu\right)^{1-\frac{1}{q}} \leq \|f\|_p + \|g\|_p$$

□

Next, we consider the following question:

Question : For which measurable functions $f : X \to \mathbb{C}$ do we have $\|f\|_p = 0$?

Answer: By lemma **??** $||f||_p = 0 \iff f = 0\ [\mu]-$ a.e. Precisely those functions such that $f(x) = 0$ for μ-almost every $x \in X$.

In particular, there are some functions f which are not identically zero but have zero $\mathscr{L}^p$-norm. This is unfortunate, so we typically consider the following quotient space:

We define

$$L^p(X, \mathcal{T}, \mu) = \frac{\mathscr{L}^p(X, \mathcal{T}, \mu)}{N_p},$$

where

$$N_p = \{\, f \in \mathscr{L}^p(X, \mathcal{T}, \mu) : \|f\|_p = 0 \,\}.$$

We have seen that for any $\lambda \in \mathbb{C}$ and any $f, g \in \mathscr{L}^p(X, \mathcal{T}, \mu)$, we always have

$$\begin{aligned} \|\lambda f\|_p &= |\lambda| \, \|f\|_p, \\ \|f+g\|_p &\underset{\substack{\uparrow \\ \text{By Mink}}}{\le} \|f\|_p + \|g\|_p. \end{aligned}$$

Claim: $\mathscr{L}^p$ is vector space over $\mathbb{C}$.

Proof.

1. Zero function

Let $0(x) := 0$ for all x. Then $|0|^p = 0$ and $\int_X |0|^p \, d\mu = 0 < \infty$, so $0 \in \mathscr{L}^p$.

2. Closed under scalar multiplication

Let $f \in \mathscr{L}^p$ and $\lambda \in \mathbb{C}$. Then

$$|\lambda f|^p = |\lambda|^p |f|^p,$$

so

$$\int_X |\lambda f|^p \, d\mu = |\lambda|^p \int_X |f|^p \, d\mu < \infty.$$

Thus $\lambda f \in \mathscr{L}^p$.

3. Closed under addition

Let $f, g \in \mathcal{L}^p$. Use the standard inequality for $p \geq 1$:

$$|f+g|^p \leq 2^{p-1}(|f|^p + |g|^p).$$

Integrate:

$$\int_X |f+g|^p \, d\mu \leq 2^{p-1} \left(\int_X |f|^p \, d\mu + \int_X |g|^p \, d\mu \right) < \infty,$$

since both integrals on the right are finite. Hence $f + g \in \mathcal{L}^p$.

4. Vector space axioms

The pointwise operations

$$(f+g)(x) := f(x) + g(x), \quad (\lambda f)(x) := \lambda f(x)$$

inherit associativity, commutativity, distributivity, etc., from $\mathbb{C}$. Together with steps 1–3, this shows $\mathcal{L}^p(X, \mathcal{T}, \mu)$ is a vector space over $\mathbb{C}$.

Then

$$L^p(X, \mathcal{T}, \mu) = \frac{\mathcal{L}^p(X, \mathcal{T}, \mu)}{N_p}$$

is the quotient of this vector space by the subspace N_p, so it is also a vector space.

$\square$

Claim: N^p is subspace of $\mathcal{L}^p$

Proof. Let $f, g \in N^p$ and $\lambda \in \mathbb{C}$,

- $0_{map} \in N^p \implies N^p \neq \emptyset$
- $\|\lambda f\|_p = |\lambda| \, \|f\|_p = 0$
- $\|f+g\|_p \leq \|f\|_p + \|g\|_p = 0 \implies \|f+g\|_p = 0.$

Thus, N^p is a subspace of $\mathcal{L}^p$. $\square$

Thus, L^p is subspace. Hence, N_p is a subspace of $L^p(X, \mathcal{T}, \mu)$; therefore $L^p(X, \mathcal{T}, \mu)$ is a vector space over $\mathbb{C}$.

If for $f \in L^p(X, \mathcal{T}, \mu)$ we denote by $[f]$ its image in the quotient space $L^p(X, \mathcal{T}, \mu)$, then

$$\lambda[f] + [g] = [\,\lambda f + g\,].$$

Define

$$\|[f]\|_p = \|f\|_p$$

More ever, $\|[\cdot]\|_p$ well defined.

Proof. Let $f, g \in \mathcal{L}^p$ By Minkowski's inequlirty we can get,

$$\big|\, \|f\|_p - \|g\|_p \,\big| \;\leq\; \|\, f - g \,\|_p, \qquad f, g \in L^p(X, \mathcal{T}, \mu). \tag{1.1}$$

Suppos that $[f] = [g]$. Then, $f - g \in N^p \implies f - g \in N^p$. Then $\|\, f - g \,\|_p = 0$. Thus,

$$\begin{aligned} \big|\, \|f\|_p - \|g\|_p \,\big| \;\leq\; \|\, f - g \,\|_p = 0 &\implies \|[f - g]\|_p = 0 \\ &\implies \big|\, \|f\|_p - \|g\|_p \,\big| = 0 \\ &\implies \|f\|_p = \|g\|_p \end{aligned}$$

□

Note that $\|[f]\|_p = 0$ if and only $[f] = 0_{L^p}$ in $L^p(X, I, \mu)$.

Proof.

- $\implies$:

$$\begin{aligned} \|[f]\|_p = 0 &\implies \|f\|_p = 0. \\ &\implies f \in N^p \\ &\implies [f] = [0] = 0_{L^p} \end{aligned}$$

- $\impliedby$:

$$\begin{aligned} [f] = 0_{L^p} &\implies [f] = [0] \\ &\implies f - 0 \in N^p \\ &\implies f \in N^p \\ &\implies \|f\|_p = 0 \\ &\implies \|[f]\|_p = \|f\|_p = 0. \end{aligned}$$

□

Now we can avoid the problem that have earlier. Now we can defnie the norm.

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1.2 A point of notation

For convenience, mathematicians agree to write f instead of $[f]$. This causes very little confusion; the only thing to keep in mind is that one can capture the behaviour of an element in $L^p(X, \mathcal{T}, \mu)$ only up to sets of zero μ-measure. For the rest of this course, we will use this convention and write elements of the quotient space L^p simply as functions.

1.2.1 Summary

For $1 \le p \le \infty$:

1. **Vector space:** $L^p(X, \mathcal{T}, \mu)$ is a vector space over $\mathbb{C}$.
2. **Definition of the p-norm:** To every $f \in L^p(X, \mathcal{T}, \mu)$ we associate a non-negative number defined by

   $$\|f\|_p = \left(\int_X |f|^p \, d\mu \right)^{1/p}, \qquad 1 \le p < \infty,$$

   and for $p = \infty$,

   $$\|f\|_\infty = \inf\{M \ge 0 : |f(x)| \le M \text{ for almost every } x\}.$$

3. **Homogeneity:** For every $\lambda \in \mathbb{C}$ and $f \in L^p(X, \mathcal{T}, \mu)$,

   $$\|\lambda f\|_p = |\lambda| \, \|f\|_p.$$

4. **Triangle inequality:** For every $f, g \in L^p(X, \mathcal{T}, \mu)$,

   $$\|f + g\|_p \le \|f\|_p + \|g\|_p.$$

5. **Definiteness:** For every $f \in L^p(X, \mathcal{T}, \mu)$,

   $$\|f\|_p \ge 0,$$

   with equality if and only if $f = 0$ almost everywhere.

Properties (iii), (iv), and (v) show that $\| \cdot \|_p$ defines a norm, so $L^p(X, \mathcal{I}, \mu)$ is a **normed linear space**.

Definition 1.4. Banach spaces are normed linear spaces with an additional property: they are *complete*, meaning every Cauchy sequence converges.

Our next task is to show that $L^p(X, \mathcal{I}, \mu)$ is a Bannch space. (We need to show that complete space)

Before doing so, we recall some important results from measure theory.

Lemma 1.4 (Chebyshev's Inequality)**.** *Let $(X, \mathcal{T}, \mu)$ be a measure space and let f be a non-negative measurable function on X.*
Then, for every $\lambda > 0$,

$$\mu\{x \in X : f(x) \geq \lambda\} \;\leq\; \frac{1}{\lambda}\int_X f\, d\mu.$$

Proof. Let $E_\lambda = \{x \in X : f(x) \geq \lambda\}$, Then,

$$\int_X f\, d\mu \geq \int_{E_\lambda} f\, d\mu \geq \int_{E_\lambda} \lambda\, d\mu = \lambda \int_{E_\lambda} d\mu = \lambda\mu(E_\lambda)$$

□

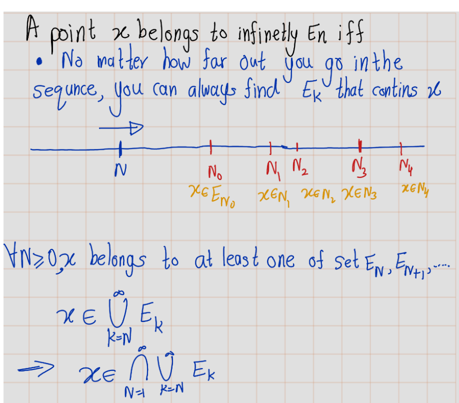
Lemma 1.5 (Borel–Cantelli Lemma)**.** *Let $(X, \mathcal{T}, \mu)$ be a measure space and let $\{E_n\}_{n=1}^{\infty}$ be a collection of measurable sets such that $\sum_{n=1}^{\infty} \mu(E_n) < \infty$. Then μ-almost every $x \in X$ belongs to at most finitely many of the sets E_n.*

Proof.

$$\begin{aligned} S &:= \{x \in X : x \text{ belongs to infinitly many } E_n\} \\ &= \bigcap_{N=1}^{\infty} \bigcup_{k=N}^{\infty} E_k \text{ (See latter picture.)} \end{aligned}$$

$$\begin{aligned} \mu(S) &\leq \mu\left(\cup_{k=N}^{\infty}\right) \\ &\leq \sum_{k=N}^{\infty} \mu(E_k) \text{ for all } N \end{aligned}$$

Then left hand side is goes to zero as $N \to \infty$.



□

Lemma 1.6 (Fatou's Lemma)**.** *Let $(X, \mathcal{T}, \mu)$ be a measure space and let $\{f_n\}_{n=1}^{\infty}$ be a sequence of non-negative measurable functions on X. Then,*

$$\int_X \liminf_{n\to\infty} f_n \, d\mu \;\le\; \liminf_{n\to\infty} \int_X f_n \, d\mu.$$

Lemma 1.7. *Let $(X, \mathcal{T}, \mu)$ be a measure space and let $1 \le p \le \infty$. Let $\{f_n\}_{n=1}^{\infty} \subset L^p(X, \mathcal{T}, \mu)$ be a sequence such that there exists a sequence of positive numbers $\{\varepsilon_n\}_{n=1}^{\infty}$ with*

$$\sum_{n=1}^{\infty} \varepsilon_n < \infty, \text{ and } \|f_n - f_{n+1}\|_p \le \varepsilon_n^2, \qquad n \ge 1.$$

Then there exists $f \in L^p(X, \mathcal{T}, \mu)$ such that

- *pointwise a.e. convergence:*

$$\lim_{n\to\infty} f_n(x) = f(x) \quad \text{for } \mu\text{-almost every } x \in X,$$

- *convergence in L^p:*

$$\lim_{n\to\infty} \|f - f_n\|_p = 0.$$

Proof.

- **Claim 1**: (f_n) is cauchy.
  For $n, m \ge 1$, consider follwing

$$\begin{aligned}
\|f_n - f_{n+m}\| &= \|f_n - f_{n+1} + f_{n+1} + \cdots + f_{n+m-1} - f_{n+m}\| \\
&= \|f_n - f_{n+1}\| + \|f_{n+1} - f_{n+2}\| + \cdots + \|f_{n+m-1} - f_{n+m}\| \\
&\ge \sum_{k=n}^{n+m-1} \|f_k - f_{k+1}\| \\
&\ge \sum_{k=n}^{n+m-1} \epsilon_k^2
\end{aligned}$$

  Then left hand side is goes to zero as $n \to 0$. Thus, (f_n) is cauchy.

- **Claim 2**:

  - $p = \infty$: For $n, m \ge 1$ then we have ,

$$\|f_n(x) - f_m(x)\| \le \|f_n - f_m\|_\infty \text{ for [] -almost every} x \in X.$$

    Hence, there is a measurable set $E_{n,m} \subset X$ such that

    $\mu(X \backslash E_{n,m}) = 0$ and $|f_n(x) - f_m(x)| \le \|f_n - f_m\|_\infty$ for every $x \in E_{n,m}$ (By lemmareflem:lem

Then the set $E = \bigcap_{n,m} E_{n,m}$ is measurable and satisfies

$$\mu(X \backslash E) = \mu\left(X \setminus \bigcap_{n,m=1}^{\infty} E_{n,m} \right) = \mu\left(\bigcup_{n,m=1}^{\infty} (X \setminus E_{n,m}) \right) \leq \sum_{n,m=1}^{\infty} \mu(X \backslash E_{n,m}) = 0.$$

and moreover,

$$|f_n(x) - f_m(x)| \leq \|f_n - f_m\|_\infty \quad \text{for every } x \in E \text{ and every } n, m \in \mathbb{N}.$$

Thus, for every $x \in E$, the sequence $\{f_n(x)\}_n \subset \mathbb{C}$ is Cauchy.
Since $\mathbb{C}$ is complete, we may define a measurable function $f : X \to \mathbb{C}$ by

$$f(x) = \lim_{n\to\infty} f_n(x), \qquad x \in E.$$

- $p < \infty$

$\square$

Theorem 1.3 (Riesz–Fischer)**.** *Let $(X, \mathcal{T}, \mu)$ be a measure space and let $1 \leq p \leq \infty$.*
Then the space $L^p(X, \mathcal{T}, \mu)$ is complete.

Proof. Let $\{f_n\}_{n\in\mathbb{N}} \subset L^p(X, \mathcal{T}, \mu)$ be a Cauchy sequence.

Claim 1: There exist a subsequnce g_n of f_n such that $\|g_n - g_{n+1}\|_p \leq \frac{1}{4^n}$ for all n. Defnie

- Let $\varepsilon = \frac{1}{4}$. Then there exists $N_1 \in \mathbb{N}$ such that $n, m \geq N_1 \implies \|f_n - f_m\|_p \leq \frac{1}{4}$.
  Define $\mathbf{g_1} = \mathbf{f_{N_1}}$.
- Let $\varepsilon = \frac{1}{4^2}$. Then there exists $N_2 \in \mathbb{N}$ and $N_2 \geq N_1$ such that $n, m \geq N_2 \implies \|f_n - f_m\|_p \leq \frac{1}{4^2}$.
  Define $\mathbf{g_2} = \mathbf{f_{N_2}}$.

$\vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots$

- Let $\varepsilon = \frac{1}{4^k}$. Then there exists $N_k \in \mathbb{N}$ and $N_k \geq N_{k-1}$ such that $n, m \geq N_k \implies \|f_n - f_m\|_p \leq \frac{1}{4^k}$.
  Define $\mathbf{g_k} = \mathbf{f_{N_k}}$.
  $\vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots$
  Then $\|g_k - g_{k+1}\| = \|f_{N_k} - f_{N_{k+1}}\| \leq \frac{1}{4^k}$ for all k, because $N_k, N_{k+1} \geq N_k$.

  Then by lemma 1.7, there exist $f \in L^p$ such that $\|g - g_n\|_p \to 0$ and $g_n(x) \to f[\mu]-$ a.e on X. (i.e.: $\{g_n\}_n$ converges to f in the p-norm.) But Cauchy sequences can have at most one cluster point, so that the original sequence $\{f_n\}_n$ must converge to f.

Therefore, we see that $L^p(X, \mathcal{T}, \mu)$ is a Banach space (complete normed space)for $1 \leq p \leq \infty$. You may notice a useful corollary of Lemma 1.7 and the proof above: Every Cauchy sequence in $L^p(X, \mathcal{T}, \mu)$ has a subsequence converging pointwise almost everywhere to some function in the space. $\square$

1.3 Banach spaces and linear operators

1.3.1 Terminology

Let X be a vector space over $\mathbb{C}$.

Definition 1.5 (norm)**.** A **norm** is a function $\|\cdot\| : X \to [0, \infty)$ with the following properties:

- If $x \in X$, then $\|x\| = 0$ if and only if $x = 0$.
- If $x, y \in X$, then $\|x + y\| \leq \|x\| + \|y\|$.
- If $x \in X$ and $\lambda \in \mathbb{C}$, then $\|\lambda x\| = |\lambda| \, \|x\|$.

The pair $(X, \|\cdot\|)$ is a **normed space**.

It is easily verified that each normed space is a metric space, if we set $d(x, y) = \|x - y\|$.

In particular, normed spaces are topological spaces, where the topology is induced by this metric. For instance, a sequence $\{x_n\}_n \subset X$ converges to $x \in X$ if and only if

$$\lim_{n\to\infty} \|x - x_n\| = 0.$$

It is useful to note that the function

$$X \to [0, \infty), \qquad x \mapsto \|x\|$$

is continuous, in view of the inequality

$$\big|\, \|x\| - \|y\| \,\big| \leq \|x - y\|, \qquad x, y \in X.$$

Norms are not necessarily unique.

Definition 1.6 (Equivalent of norms)**.** Given two norms $\|\cdot\|_1$ and $\|\cdot\|_2$ on X, we say that they are equivalent if there are positive constants C_1, C_2 such that

$$C_1 \|x\|_1 \leq \|x\|_2 \leq C_2 \|x\|_1, \qquad x \in X.$$

1.3.2 Linear operators

Definition 1.7. Let X and Y be two vector spaces. A linear operator is a mapping $T : X \to Y$ which preserves the linear structure, in other words

$$T(\alpha x + y) = \alpha Tx + Ty, \qquad x, y \in X, \ \alpha \in \mathbb{C}.$$

Definition 1.8 (bounded Linear Opeartor)**.** Suppose in addition that X and Y are normed spaces, and let $T : X \to Y$ be a linear operator. We say that T is bounded provided that there is a constant $M \geq 0$ satisfying

$$\|Tx\| \leq M\|x\|, \qquad \forall x \in X.$$

Definition 1.9. The **operator norm of** T is the infimum of such constants M, and is denoted by $\|T\|$. i.e.:

$$\text{Operator norm of } T = ||T|| := \inf\{M > 0 : \|Tx\| \leq M\|x\|, \qquad \forall x \in X\}$$

Remark. We note that to check that a given operator $T : X \to Y$ is bounded, it suffices to check the condition

$$\|Tx\| \leq M\|x\|$$

for every $x \neq 0$, since $T0 = 0$ by linearity.

Notation: $B(X, Y)$ = the collection of all bounded linear operators from X to Y.

Lemma 1.8. *Let X, Y be normed spaces and $T \in B(X, Y)$. Then,*

$$\|Tx - Ty\| \leq \|T\| \, \|x - y\|, \qquad x, y \in X.$$

Proof. There is a sequence $\{M_n\}_n$ of positive constants such that

$$\|Tx\| \leq M_n\|x\|, \qquad x \in X, \ n \in \mathbb{N},$$

and $\lim_{n\to\infty} M_n = \|T\|$. Thus, we conclude that

$$\|Tx\| \leq \|T\| \, \|x\|, \qquad x \in X.$$

The claim now follows by linearity:

$$\|Tx - Ty\| = \|T(x - y)\| \leq \|T\| \, \|x - y\|.$$

□

Theorem 1.4. *Let X, Y be normed spaces and $T : X \to Y$. TFAE*

1. *T is continuous.*
2. *T is continuous at 0.*
3. *$T \in B(X, Y)$ (T is bounded)*

Chapter 2

Hello bookdown

All chapters start with a first-level heading followed by your chapter title, like the line above. There should be only one first-level heading (`#`) per .Rmd file.

2.1 A section

All chapter sections start with a second-level (`##`) or higher heading followed by your section title, like the sections above and below here. You can have as many as you want within a chapter.

An unnumbered section

Chapters and sections are numbered by default. To un-number a heading, add a `{.unnumbered}` or the shorter `{-}` at the end of the heading, like in this section.

Chapter 3

Cross-references

Cross-references make it easier for your readers to find and link to elements in your book.

3.1 Chapters and sub-chapters

There are two steps to cross-reference any heading:

1. Label the heading: `# Hello world {#nice-label}`.
    - Leave the label off if you like the automated heading generated based on your heading title: for example, `# Hello world` = `# Hello world {#hello-world}`.
    - To label an un-numbered heading, use: `# Hello world {-#nice-label}` or `{# Hello world .unnumbered}`.
2. Next, reference the labeled heading anywhere in the text using `\@ref(nice-label)`; for example, please see Chapter 3.
    - If you prefer text as the link instead of a numbered reference use: any text you want can go here.

3.2 Captioned figures and tables

Figures and tables *with captions* can also be cross-referenced from elsewhere in your book using `\@ref(fig:chunk-label)` and `\@ref(tab:chunk-label)`, respectively.

See Figure 3.1.

```
par(mar = c(4, 4, .1, .1))
plot(pressure, type = 'b', pch = 19)
```

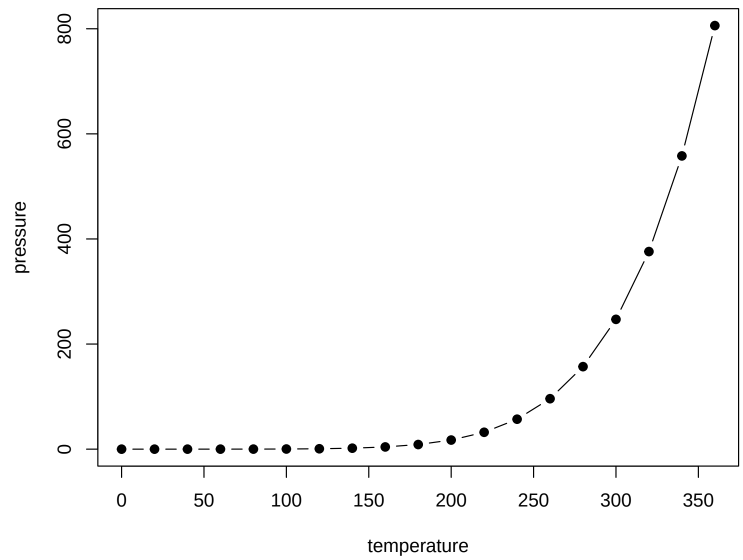


Figure 3.1: Here is a nice figure!

Don't miss Table 3.1.

```
knitr::kable(
  head(pressure, 10), caption = 'Here is a nice table!',
  booktabs = TRUE
)
```

Table 3.1: Here is a nice table!

temperature	pressure
0	0.0002
20	0.0012
40	0.0060
60	0.0300
80	0.0900
100	0.2700
120	0.7500
140	1.8500
160	4.2000
180	8.8000

Chapter 4

Parts

Chapter 5

Footnotes and citations

Chapter 6

Blocks

Chapter 7

Sharing your book