Functinal Analaysis

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Chapter 1

Banach space

1.1 Lebesgue spaces

Definition 1.1. markdown Let X be a set. A σ**-algebra** $\mathcal{J}$ on X is a collection of subsets of X such that:

1. $\emptyset \in \mathcal{J}$,
2. if $E \in \mathcal{J}$, then $X \setminus E \in \mathcal{J}$,
3. if $E_n \in \mathcal{J}$ for every $n \geq 1$, then

$$\bigcup_{n=1}^{\infty} E_n \in \mathcal{J}.$$

- Elements of $\mathcal{J}$ are called $\mathcal{J}$-measurable sets,
- $(X, \mathcal{J})$ is a measurable space.

Definition 1.2. A function $f : X \to \mathbb{C}$ is said to be measurable if

$$f^{-1}(\{ z \in \mathbb{C} : |z - a| < \delta \}) \in \mathcal{T}$$

for every $\delta > 0$ and $a \in \mathbb{C}$.

Definition 1.3. A (positive) measure is a function

$$\mu : \mathcal{T} \to [0, \infty]$$

which is countably additive, in the sense that if $\{E_n\}_{n=1}^{\infty}$ is a countable collection of disjoint measurable sets, then

$$\mu\left(\bigcup_{n=1}^{\infty} E_n\right) = \sum_{n=1}^{\infty} \mu(E_n).$$

- The triple $(X, \mathcal{T}, \mu)$ is called a *measure space*.

Notation :

- Let $0 < p < \infty$,

$$L^p(X, \mathcal{I}, \mu) := \left\{ f : X \to \mathcal{C} : f \text{ is measurable and } \int_X |f|^p d\mu < \infty \right\}$$

  - Such functions are said to be p**-integrable**.
  - L^p norm of $f = ||f||_p = \left(\int_X |f|\, d\mu \right)^{\frac{1}{p}}$

- $p = \infty$

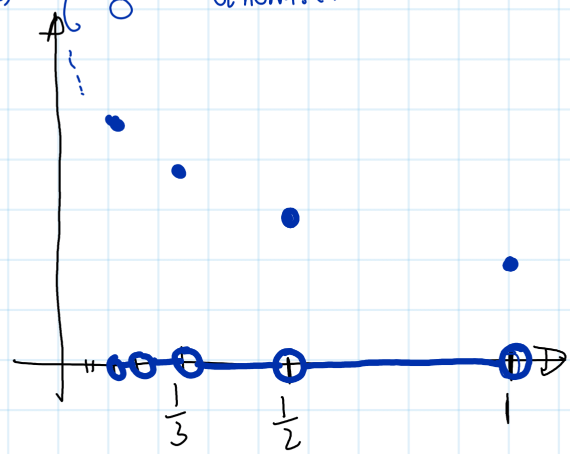
$$L^\infty(X, \mathcal{T}, \mu) = \{ f : X \to \mathbb{C} \ : \ \exists M > 0 \text{ such that } |f| < M \ [\mu]\text{-a.e. on } X \} .$$

- Such functions are said to be **essentially bounded**.
- The essential norm $= L^\infty$ norm of $f = \|f\|_\infty = \inf \{ M > 0 : |f| < M \ [\mu]\text{-a.e. on } X \}$.

3

Define

$$f(x) = \begin{cases} \frac{1}{x} & \text{if } x \in \mathbb{N} \\ 0 & \text{otherwise.} \end{cases}$$



- As you can see f is not bdd.
- f is essential bdd.

We can use M as any postive number.

$$L^\infty(X, \mathcal{T}, \mu) = \{f : X \to \mathbb{C} \,:\, \exists M > 0 \text{ such that } |f| < M \ [\mu]\text{-a.e. on } X\}.$$

$(M = 1, e, \pi, \frac{1}{2}, 0.0001, 10^{100}, \dots.)$

Further $\|f\|_\infty = 0$ (Since we choose inf of such M)

In this section we use the term **"norm."** Strictly speaking, we have not yet verified that the expressions introduced actually satisfy the axioms of a norm. That verification will come later. For now, we use the word "norm" informally, with the understanding that its legitimacy will be established in due course.

Lemma 1.1. *Let $(X, \mathcal{T}, \mu)$ be a measure space, let $0 < p < \infty$, and let $f \in L^p(X, \mathcal{T}, \mu)$. Then*

$$\|f\|_p = 0 \quad \iff \quad f(x) = 0 \text{ for } [\mu]\text{-a.e. } x \in X.$$

1

($\Leftarrow$) If $f=0$ $[\mu]$-a.e.

then $\int_X |f|^p d\mu = 0$

($\Rightarrow$) For $n \geq 1$,

let $E_n := \{x \in X : |f(x)| > \frac{1}{n}\}$

Then $E_n \in \mathcal{I}$. $\left(\begin{array}{l}\text{Since } f \text{ is measurable and} \\ E_n = f^{-1}\{(z \in \mathbb{C} : |z| > \frac{1}{n})\} \in \mathcal{I}\end{array}\right)$

and $\bigcup_{n=1}^{\infty} E_n = \{x \in X \mid f(x) > 0\}$

Now for every $n \in \mathbb{N}$,

$$\underset{\text{hypothesis}}{0} = \|f\|_p \underset{\text{def}^{\text{n}}}{=} \int_X |f|^p d\mu \underset{X \supseteq E_n}{\geq} \int_{E_n} |f|^p d\mu \underset{\text{In } E_n\ |f| > \frac{1}{n}}{>} \int_{E_n} \frac{1}{n^p}$$

$$= \int_{E_n} \frac{1}{n^p} d\mu = \frac{\mu(E_n)}{n^p} \geq 0$$

Proof.

2

$\Rightarrow \quad \mu(E_n) = 0$

Thus $\quad \mu\left(\bigcup_{n=1}^{\infty} E_n\right) = \sum_{n=1}^{\infty} \mu(E_n)$

$\mu(\{x \in X : f(x) > 0\}) = 0$

$\square$

Fact: $\lambda \in \mathbb{C}, f \in L^p(X, I, \mu), 0 < p \geq \infty, ||\lambda f||_p = |\lambda|||f||_p$.

Proof.

4

- If $0<p<\infty$;

$$\|\lambda f\|_p = \left(\int_X |\lambda f|^p \, d\mu\right)^{\frac{1}{p}}$$

$$= \left(|\lambda|^p \int_X |f|^p \, d\mu\right)^{\frac{1}{p}} = |\lambda| \, \|f\|_\infty$$

- If $p=\infty$

?

$\square$

Lemma 1.2. *Let $(X, \mathcal{T}, \mu)$ be a measure space and let $f \in L^\infty(X, \mathcal{T}, \mu)$. Then, for*

$$|f(x)| \leq \|f\|_\infty [\mu] - a.e. x \in X.$$

5

$\exists$ a sequnce (M_n) of positive numbers s.t. $M_n \to \|f\|_\infty$ and $|f| \leq M_n$ μ- a.e.

(If u need, u can construct the sequnce $M_n = \|f\|_\infty + \frac{1}{n} = \inf\{M>0 \,/\, |f| < M \ \mu\text{-a.e}\} + \frac{1}{n}$)

By def^{n} of a.e, $\exists E_n \in I$ s.t. $\mu(E_n) = 0$ s.t.

$$|f(x)| \leq M \quad \text{for} \quad x \in X \setminus E_n$$

Define $E = \bigcup_{n=1}^{\infty} E_n$. Then $\mu(E) = \mu\left(\bigcup_{n=1}^{\infty} E_n\right) = \sum_{n=1}^{\infty} \mu(E_n) = 0.$

If $x \in X \setminus E = X \setminus \bigcup_{n=1}^{\infty} E_n = \bigcap_{n=1}^{\infty} (X \setminus E_n)$ then $|f(x)| \leq M_n$, $n \geq 1$

$$\Rightarrow \quad |f| \leq \lim_{n\to\infty} M_n = \|f\|_\infty$$

(our choice of M_n
$M_n \to \|f\|_\infty$)

$\blacksquare$

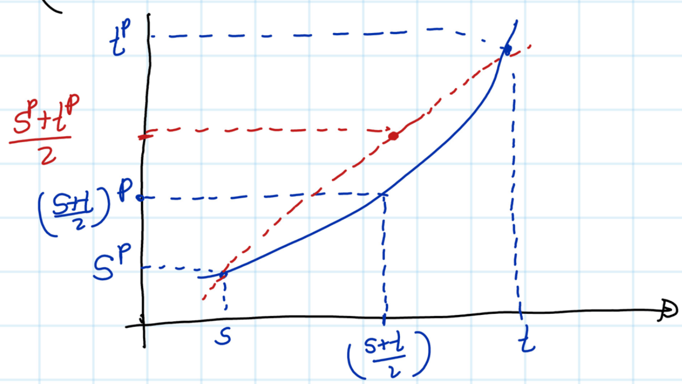
Result : If $f, g \in L^P(X, I, \mu)$ then $f + g \in L^P(X, I, \mu)$

$1 \leq p < \infty$	

$1 \leqslant p < \infty$

The function $t \longrightarrow t^p$, is convex on $(0,\infty)$

(Because 2nd derivate of fun, $p(p-1)t^{p-2} > 0$



$$\left(\frac{s+t}{2}\right)^p \leqslant \frac{s^p+t^p}{2}$$

$$\left(\frac{|f(x)|+|g(x)|}{2}\right)^p \leqslant \frac{|f(x)|^p+|g(x)|^p}{2}$$

$$\left(|f(x)|+|g(x)|\right)^p \leqslant 2^{p-1}\left(|f(x)|^p+|g(x)|^p\right)$$

Then,

$$\int_X |f+g|^p d\mu \leqslant \int_X (|f|+|g|)^p \quad (\Delta \text{ inequlity})$$
$$\leqslant 2^{p-1}\left[\int (|f|^p+|g|^p)\, d\mu\right] < \infty$$

$p = \infty$

From Lemm

$\mu(E) = \mu($

Then Δ in

$|f(x)+g(x)| \leqslant |$

Since $E, F \in I$ th

$f+g \in$

Conclusion: $\|$

Lemma 1.3 (Young's inequality)**.** *Let $a, b \geq 0$ and $1 < p < \infty$. Let q be the conjugate exponent, i.e.*

$\frac{1}{p} + \frac{1}{q} = 1.$

Then

$$ab \leq \frac{a^p}{p} + \frac{b^q}{q}.$$

proof of Young inequality

Case-I) If $a=0$ or $b=0$ this is trivial.

Case-II) If $a>0$, $b>0$. then $s = p\log a$ & $t = q\log b$

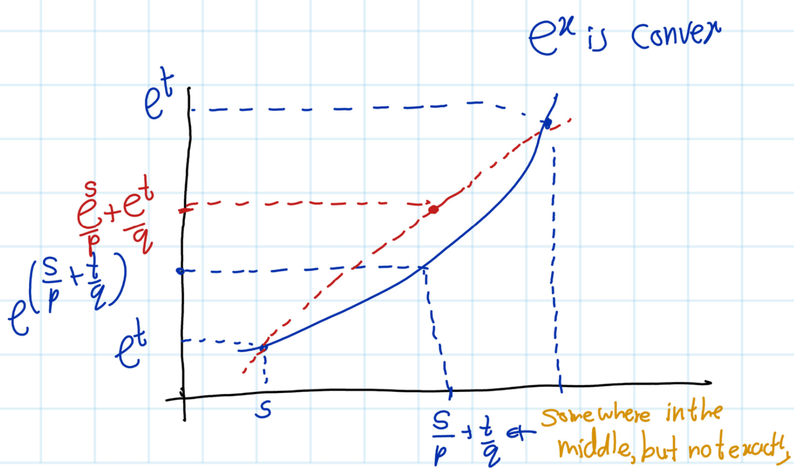
Then,

$$e^{\left(\frac{s}{p}+\frac{t}{q}\right)} = e^{\frac{s}{p}}\, e^{\frac{t}{q}} = e^{\log a} \cdot e^{\log b} = ab$$

So,

$$\frac{s}{p} + \frac{t}{q} = \frac{1}{p}s + \frac{1}{q}t = \frac{1}{p}s + \left(1-\frac{1}{p}\right)t = \lambda s + (1-\lambda)t$$

it is convex.



$$e^{\left(\frac{t}{p}+\frac{s}{q}\right)} \leq \frac{1}{p} e^{s} + \frac{1}{q} e^{t} < \frac{1}{p} a^{p} + \frac{1}{q} b^{q}$$

Theorem 1.1 (Holder's inequality)**.** *Fix $1 \leq p < \infty$ and let q be the conjugate exponent, i.e.*

$\frac{1}{p} + \frac{1}{q} = 1.$

Let $f, g : X \to \mathbb{C}$ be measurable functions. Then

$$\int_X |fg|\, d\mu \;\leq\; \left(\int_X |f|^p\, d\mu\right)^{1/p} \left(\int_X |g|^q\, d\mu\right)^{1/q} = ||f||_p ||f||_q.$$

Proof of Hölder's equation.

Case-I $p=1$, in this case $q=\infty$. Then,

$$\int_X |fg|d\mu = \int_X |f||g|\, d\mu \leq \|g\|_\infty \int_X f\, d\mu = \|g\|_\infty \|f\|$$

(lemma 1.1.2
$|g| \leq \|g\|_\infty$)

Case-II, $1<p<\infty$

$$q = 1-\frac{1}{p} = \frac{(p-1)}{p},$$

Then $1<q<\infty$

Subcase II.1 If $\|f\|_p=0$ or $\|g\|_q=0$ then,

$fg=0$ $[\mu]$-a.e. Then again inequlity is trivial.

Subcase II.2 If $\|f\|_p=\infty$ or $\|g\|_q=\infty$ then the resut is trivial.

Subcase II.3 If $1<\|f\|_p<\infty$ and $1<\|g\|_q<\infty$

Proof.

Apply yaung inequlity,

$$a = \frac{|f(x)|}{\|f\|_p} \quad \text{and} \quad b = \frac{|g(x)|}{\|g\|_q}$$

$$ab \leq \frac{a^p}{p} + \frac{b^p}{q}$$

$$\frac{|f(x)||g(x)|}{\|f\|_p\|g\|_q} \leqslant \frac{1}{p}\left(\frac{|f(x)|}{\|f\|_p}\right)^p + \frac{1}{q}\left(\frac{|g(x)|}{\|g\|_q}\right)^q$$

$$\Rightarrow \int_X \frac{|fg|}{\|f\|_p\|g\|_q}\,d\mu \leqslant \frac{1}{p}\int_X \frac{|f(x)|^p}{\|f\|_p^p}\,d\mu + \frac{1}{q}\int_X \frac{|g(x)|^q}{\|g\|_p^p}\,d\mu$$

$$= \frac{1}{p\|f\|_p^p}\int_X |f(x)|^p\,d\mu + \frac{1}{q\|g\|_q}\int |g(x)|^p\,d\mu$$

$$= \frac{1}{p\|f\|_p^p}\|f\|_p^p + \frac{1}{q\|g\|_q}\|g\|_q^q$$

$$\|f\|_p = \left(\int_X |f|\,d\mu\right)^{1/p}$$

$$= 1/p + 1/q = 1$$

□

Remark. If $p = 2$ then $q = 2$ then Holder ineqaulty becomes Cauchy -Schawrz inequlity.

Chapter 2

Hello bookdown

All chapters start with a first-level heading followed by your chapter title, like the line above. There should be only one first-level heading (`#`) per .Rmd file.

2.1 A section

All chapter sections start with a second-level (`##`) or higher heading followed by your section title, like the sections above and below here. You can have as many as you want within a chapter.

An unnumbered section

Chapters and sections are numbered by default. To un-number a heading, add a `{.unnumbered}` or the shorter `{-}` at the end of the heading, like in this section.

Chapter 3

Cross-references

Cross-references make it easier for your readers to find and link to elements in your book.

3.1 Chapters and sub-chapters

There are two steps to cross-reference any heading:

1. Label the heading: `# Hello world {#nice-label}`.
    - Leave the label off if you like the automated heading generated based on your heading title: for example, `# Hello world` = `# Hello world {#hello-world}`.
    - To label an un-numbered heading, use: `# Hello world {-#nice-label}` or `{# Hello world .unnumbered}`.
2. Next, reference the labeled heading anywhere in the text using `\@ref(nice-label)`; for example, please see Chapter 3.
    - If you prefer text as the link instead of a numbered reference use: any text you want can go here.

3.2 Captioned figures and tables

Figures and tables *with captions* can also be cross-referenced from elsewhere in your book using `\@ref(fig:chunk-label)` and `\@ref(tab:chunk-label)`, respectively.

See Figure 3.1.

```r
par(mar = c(4, 4, .1, .1))
plot(pressure, type = 'b', pch = 19)
```

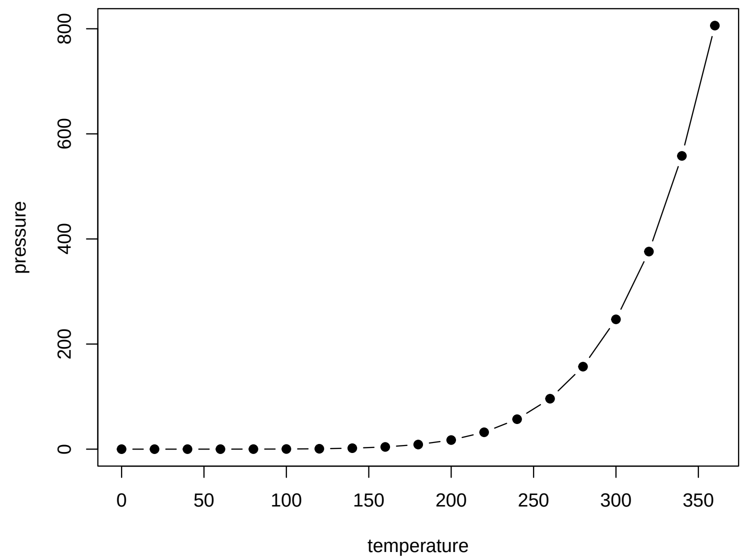


Figure 3.1: Here is a nice figure!

Don't miss Table 3.1.

```r
knitr::kable(
  head(pressure, 10), caption = 'Here is a nice table!',
  booktabs = TRUE
)
```

Table 3.1: Here is a nice table!

temperature	pressure
0	0.0002
20	0.0012
40	0.0060
60	0.0300
80	0.0900
100	0.2700
120	0.7500
140	1.8500
160	4.2000
180	8.8000

Chapter 4

Parts

You can add parts to organize one or more book chapters together. Parts can be inserted at the top of an .Rmd file, before the first-level chapter heading in that same file.

Add a numbered part: `# (PART) Act one {-}` (followed by `# A chapter`)

Add an unnumbered part: `# (PART\*) Act one {-}` (followed by `# A chapter`)

Add an appendix as a special kind of un-numbered part: `# (APPENDIX) Other stuff {-}` (followed by `# A chapter`). Chapters in an appendix are prepended with letters instead of numbers.

Chapter 5

Footnotes and citations

5.1 Footnotes

Footnotes are put inside the square brackets after a caret `^[]`. Like this one [1].

5.2 Citations

Reference items in your bibliography file(s) using `@key`.

For example, we are using the **bookdown** package [**?**] (check out the last code chunk in index.Rmd to see how this citation key was added) in this sample book, which was built on top of R Markdown and **knitr** [**?**] (this citation was added manually in an external file book.bib). Note that the `.bib` files need to be listed in the index.Rmd with the YAML `bibliography` key.

The RStudio Visual Markdown Editor can also make it easier to insert citations: https://rstudio.github.io/visual-markdown-editing/#/citations

[1] This is a footnote.

Chapter 6

Blocks

6.1 Equations

Here is an equation.

$$f\left(k\right) = \binom{n}{k} p^k \left(1-p\right)^{n-k} \tag{6.1}$$

You may refer to using `\@ref(eq:binom)`, like see Equation (6.1).

6.2 Theorems and proofs

Labeled theorems can be referenced in text using `\@ref(thm:tri)`, for example, check out this smart theorem 6.1.

Theorem 6.1. *For a right triangle, if c denotes the* length *of the hypotenuse and a and b denote the lengths of the **other** two sides, we have*

$$a^2 + b^2 = c^2$$

Read more here https://bookdown.org/yihui/bookdown/markdown-extensions-by-bookdown.html.

6.3 Callout blocks

The R Markdown Cookbook provides more help on how to use custom blocks to design your own callouts: https://bookdown.org/yihui/rmarkdown-cookbook/custom-blocks.html

Chapter 7

Sharing your book

7.1 Publishing

HTML books can be published online, see: https://bookdown.org/yihui/bookdown/publishing.html

7.2 404 pages

By default, users will be directed to a 404 page if they try to access a webpage that cannot be found. If you'd like to customize your 404 page instead of using the default, you may add either a `_404.Rmd` or `_404.md` file to your project root and use code and/or Markdown syntax.

7.3 Metadata for sharing

Bookdown HTML books will provide HTML metadata for social sharing on platforms like Twitter, Facebook, and LinkedIn, using information you provide in the `index.Rmd` YAML. To setup, set the `url` for your book and the path to your `cover-image` file. Your book's `title` and `description` are also used.

This `gitbook` uses the same social sharing data across all chapters in your book-all links shared will look the same.

Specify your book's source repository on GitHub using the `edit` key under the configuration options in the `_output.yml` file, which allows users to suggest an edit by linking to a chapter's source file.

Read more about the features of this output format here:

https://pkgs.rstudio.com/bookdown/reference/gitbook.html

Or use:

```
?bookdown::gitbook
```