Functinal Analaysis

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Contents

Chapter 1

Banach space

1.1 Lebesgue spaces

1.1.1 a

Definition 1.1. markdown Let X be a set. A **σ-algebra** $\mathcal{J}$ on X is a collection of subsets of X such that:

1. $\emptyset \in \mathcal{J}$,
2. if $E \in \mathcal{J}$, then $X \setminus E \in \mathcal{J}$,
3. if $E_n \in \mathcal{J}$ for every $n \geq 1$, then

$$\bigcup_{n=1}^{\infty} E_n \in \mathcal{J}.$$

- Elements of $\mathcal{J}$ are called $\mathcal{J}$-measurable sets,
- $(X, \mathcal{J})$ is a measurable space.

Definition 1.2. A function $f : X \to \mathbb{C}$ is said to be measurable if

$$f^{-1}(\{\, z \in \mathbb{C} : |z - a| < \delta \,\}) \in \mathcal{T}$$

for every $\delta > 0$ and $a \in \mathbb{C}$.

Definition 1.3. A (positive) measure is a function

$$\mu : \mathcal{T} \to [0, \infty]$$

which is countably additive, in the sense that if $\{E_n\}_{n=1}^{\infty}$ is a countable collection of disjoint measurable sets, then

$$\mu\left(\bigcup_{n=1}^{\infty} E_n\right) = \sum_{n=1}^{\infty} \mu(E_n).$$

- The triple $(X, \mathcal{T}, \mu)$ is called a *measure space*.

Notation :

- Let $0 < p < \infty$,

$$\mathcal{L}^p(X, \mathcal{I}, \mu) := \left\{ f : X \to \mathcal{C} : f \text{ is measurable and } \int_X |f|^p d\mu < \infty \right\}$$

  - Such functions are said to be p**-integrable**.
  - $\mathcal{L}^p$ norm of $f = ||f||_p = \left(\int_X |f|^p \, d\mu\right)^{\frac{1}{p}}$

- $p = \infty$

$$\mathcal{L}^{\infty}(X, \mathcal{T}, \mu) = \{f : X \to \mathbb{C} \ : \ \exists M > 0 \text{ such that } |f| < M \ [\mu]\text{-a.e. on } X\}.$$

- Such functions are said to be **essentially bounded**.
- The essential norm $= \mathcal{L}^{\infty}$ norm of $f = \|f\|_{\infty} = \inf\{M > 0 : |f| < M \ [\mu]\text{-a.e. on } X\}$.

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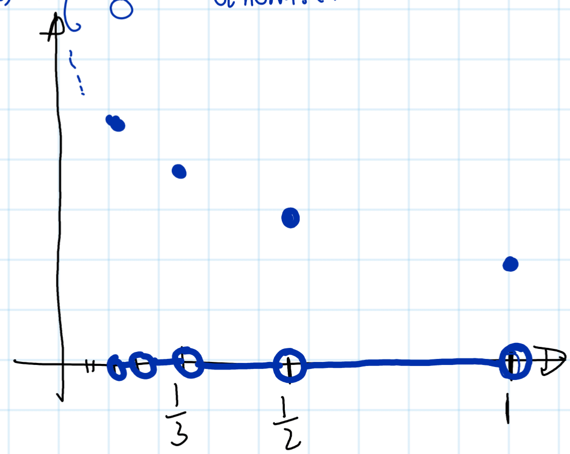
Define

$$f(x) = \begin{cases} \frac{1}{x} & \text{if } x \in \mathbb{N} \\ 0 & \text{otherwise.} \end{cases}$$



- As you can see f is not bdd.
- f is essential bdd.

We can use M as any postive number.

$L^{\infty}(X, \mathcal{T}, \mu) = \{f : X \to \mathbb{C} \ : \ \exists M > 0 \text{ such that } |f| < M \ [\mu]\text{-a.e. on } X\}$.

($M = 1, e, \pi, \frac{1}{2}, 0.0001, 10^{100}, \ldots$)

Further $\|f\|_{\infty} = 0$ (Since we choose inf of such M)

In this section we use the term **"norm."** Strictly speaking, we have not yet verified that the expressions introduced actually satisfy the axioms of a norm. That verification will come later. For now, we use the word "norm" informally, with the understanding that its legitimacy will be established in due course.

Lemma 1.1. *Let $(X, \mathcal{T}, \mu)$ be a measure space, let $0 < p < \infty$, and let $f \in \mathcal{L}^p(X, \mathcal{T}, \mu)$. Then*

$$\|f\|_p = 0 \quad \Longleftrightarrow \quad f(x) = 0 \textit{ for } [\mu]\textit{-a.e. } x \in X.$$

1

$(\Leftarrow)$ If $f = 0$ $[\mu]$-a.e.
then $\int_X |f|^p \, d\mu = 0$

$(\Rightarrow)$ For $n \geq 1$,
let $E_n := \{x \in X : |f(x)| > \frac{1}{n}\}$

Then $E_n \in \mathcal{I}$. $\left(\text{Since } f \text{ is measurable and } E_n = f^{-1}\left\{\left(z \in \mathbb{C} : |z| > \frac{1}{n}\right)\right\} \in \mathcal{I}\right)$

and $\bigcup_{n=1}^{\infty} E_n = \{x \in X \mid f(x) > 0\}$

Now for every $n \in \mathbb{N}$,

$$0 \underset{\text{hypotheis}}{=} \|f\|_p \underset{\text{def}^n}{=} \int_X |f|^p \, d\mu \underset{X \supseteq E_n}{\geq} \int_{E_n} |f|^p \, d\mu \underset{\substack{\text{In } E_n \\ |f| > \frac{1}{n}}}{>} \int_{E_n} \frac{1}{n^p}$$

$$= \int_{E_n} \frac{1}{n^p} \, d\mu = \frac{\mu(E_n)}{n^p} \geq 0$$

Proof.

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$\Rightarrow \quad \mu(E_n) = 0$

Thus $\quad \mu\left(\bigcup_{n=1}^{\infty} E_n\right) = \sum_{n=1}^{\infty} \mu(E_n)$

$\mu(\{x \in X : f(x) > 0\}) = 0$

$\square$

Fact: $\lambda \in \mathbb{C}, f \in \mathcal{L}^p(X, I, \mu), 0 < p \geq \infty, ||\lambda f||_p = |\lambda|||f||_p$.

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- If $0 < p < \infty$,

$$\|\lambda f\|_p = \left(\int_X |\lambda f|^p \, d\mu \right)^{\frac{1}{p}}$$

$$= \left(|\lambda|^p \int_X |f|^p \, d\mu \right)^{\frac{1}{p}} = |\lambda| \, \|f\|_\infty$$

- If $p = \infty$

?

Proof.

□

Lemma 1.2. *Let $(X, \mathcal{T}, \mu)$ be a measure space and let $f \in \mathcal{L}^\infty(X, \mathcal{T}, \mu)$. Then, for*

$$|f(x)| \leq \|f\|_\infty[\mu] - a.e. x \in X.$$

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$\exists$ a sequence (M_n) of positive numbers s.t. $M_n \longrightarrow \|f\|_\infty$ and $|f| \leq M_n$ μ- a.e.

(If u need, u can construct the sequence $M_n = \|f\|_\infty + \frac{1}{n} = \inf\{M > 0 \,|\, |f| < M \ \mu\text{-a.e}\} + \frac{1}{n}$)

By def^n of a.e, $\exists\, E_n \in I$ s.t. $\mu(E_n) = 0$ s.t.

$$|f(x)| \leq M \quad \text{for} \quad x \in X \setminus E_n$$

Define $E = \bigcup_{n=1}^{\infty} E_n$. Then $\mu(E) = \mu\left(\bigcup_{n=1}^{\infty} E_n\right) = \sum_{n=1}^{\infty} \mu(E_n) = 0$.

If $x \in X \setminus E = X \setminus \bigcup_{n=1}^{\infty} E_n = \bigcap_{n=1}^{\infty} (X \setminus E_n)$ then $|f(x)| \leq M_n$, $n \geq 1$

$$\Rightarrow \quad |f| \leq \lim_{n \to \infty} M_n = \|f\|_\infty$$

(our choice of M_n
$M_n \longrightarrow \|f\|_\infty$)

$\blacksquare$

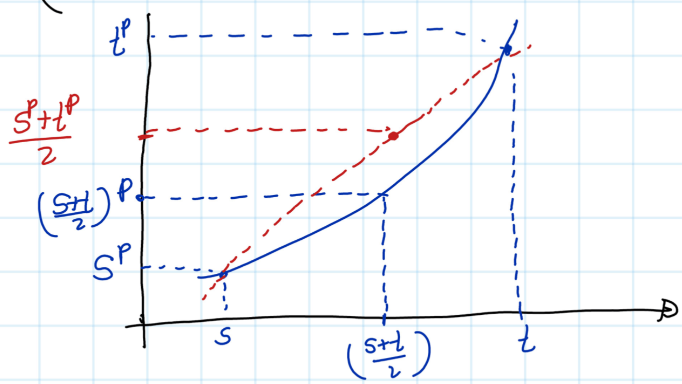
Result : If $f, g \in \mathcal{L}^P(X, I, \mu)$ then $f + g \in \mathcal{L}^P(X, I, \mu)$

$1 \le p < \infty$

$1 \leq p < \infty$

The function $t \longrightarrow t^p$, is convex on $(0,\infty)$

(Because 2nd derivate of fun, $p(p-1)t^{p-2} > 0$



$$\left(\frac{s+t}{2}\right)^p \leq \frac{s^p+t^p}{2}$$

$$\left(\frac{|f(x)|+|g(x)|}{2}\right)^p \leq \frac{|f(x)|^p+|g(x)|^p}{2}$$

$$\left(|f(x)|+|g(x)|\right)^p \leq 2^{p-1}\left(|f(x)|^p+|g(x)|^p\right)$$

Then,

$$\int_X |f+g|^p d\mu \leq \int_X (|f|+|g|)^p \ (\Delta \text{ inequlity})$$
$$\leq 2^{p-1}\left[\int (|f|^p+|g|^p)\,d\mu\right] < \infty$$

$p=\infty$

From Lemm

$\mu(E)=\mu($

Then Δ in

$|f(x)+g(x)| \leq |$

Since $E,F\in I$ th

$f+g\in$

Conclusion: $\|$

Lemma 1.3 (Young's inequality)**.** *Let $a,b \geq 0$ and $1 < p < \infty$. Let q be the conjugate exponent, i.e.*

$\frac{1}{p} + \frac{1}{q} = 1.$

Then

$$ab \le \frac{a^p}{p} + \frac{b^q}{q}.$$

proof of Young inequality

Case-I) If $a=0$ or $b=0$ this is trivial.

Case-II) If $a>0$, $b>0$. then $s=p\log a$ & $t=q\log b$

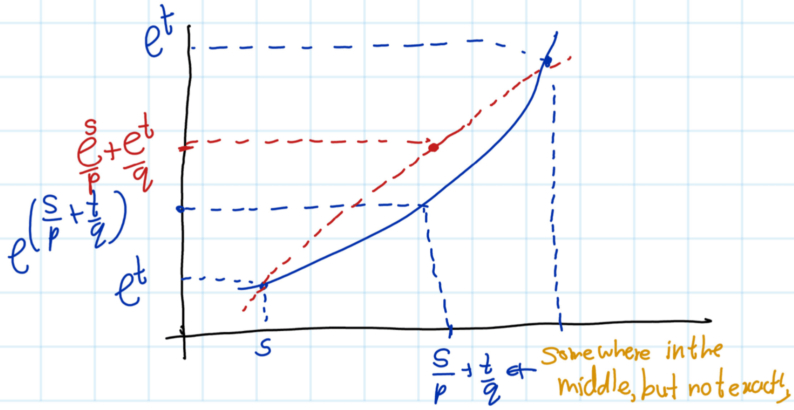
Then,

$$e^{(\frac{s}{p}+\frac{t}{q})} = e^{\frac{s}{p}}\, e^{\frac{t}{q}} = e^{\log a}\cdot e^{\log b} = ab$$

So, $\frac{s}{p}+\frac{t}{q} = \frac{1}{p}s+\frac{1}{q}t = \frac{1}{p}s+\left(1-\frac{1}{p}\right)t = \lambda s+(1-\lambda)t$

it is convex.

e^x is convex



$$e^{\left(\frac{t}{p}+\frac{s}{q}\right)} \leq \frac{1}{p} e^{s} + \frac{1}{q} e^{t} < \frac{1}{p} a^{p} + \frac{1}{q} b^{q}$$

Theorem 1.1 (Holder's inequality)**.** *Fix $1 \leq p < \infty$ and let q be the conjugate exponent, i.e.*

$\frac{1}{p} + \frac{1}{q} = 1.$

Let $f, g : X \to \mathbb{C}$ be measurable functions. Then

$$\int_X |fg| \, d\mu \ \leq \ \left(\int_X |f|^p \, d\mu\right)^{1/p} \left(\int_X |g|^q \, d\mu\right)^{1/q} = ||f||_p ||f||_q.$$

Proof of Hölder's equation.

Case-I $p=1$, in this case $q=\infty$. Then,

$$\int_X |fg|d\mu = \int_X |f||g|\, d\mu \leq \|g\|_\infty \int_X f\, d\mu = \|g\|_\infty \|f\|$$

(lemma 1.1.2
$|g| \leq \|g\|_\infty$)

Case-II, $1<p<\infty$

$$q = 1-\frac{1}{p} = \frac{(p-1)}{p},$$

Then $1<q<\infty$

subcase II.1 If $\|f\|_p=0$ or $\|g\|_q=0$ then,

$fg=0$ $[\mu]$-a.e. Then again inequlity is trivial.

subcase II.2 If $\|f\|_p=\infty$ or $\|g\|_q=\infty$ then the resut is trivial.

subcase II.3 If $1<\|f\|_p<\infty$ and $1<\|g\|_q<\infty$

Proof.

Apply yang inequlity,

$$a = \frac{|f(x)|}{\|f\|_p} \quad \text{and} \quad b = \frac{|g(x)|}{\|g\|_q}$$

$$ab \leq \frac{a^p}{p} + \frac{b^p}{q}$$

$$\frac{|f(x)||g(x)|}{\|f\|_p \|g\|_q} \leq \frac{1}{p}\left(\frac{|f(x)|}{\|f\|_p}\right)^p + \frac{1}{q}\left(\frac{|g(x)|}{\|g\|_q}\right)^q$$

$$\Rightarrow \int_X \frac{|fg|}{\|f\|_p \|g\|_q} d\mu \leq \frac{1}{p}\int_X \frac{|f(x)|^p}{\|f\|_p^p} d\mu + \frac{1}{q}\int_X \frac{|g(x)|^q}{\|g\|_p^p} d\mu$$

$$= \frac{1}{p\|f\|_p^p}\int_X |f(x)|^p d\mu + \frac{1}{q\|g\|_q}\int |g(x)|^p d\mu$$

$$= \frac{1}{p\|f\|_p^p} \|f\|_p^p + \frac{1}{q\|g\|_q} \|g\|_q^q$$

$$= 1/p + 1/q = 1$$

$\|f\|_p = \left(\int_X |f| d\mu\right)^{1/p}$

□

Remark. If $p = 2$ then $q = 2$ then Holder ineqaulty becomes Cauchy -Schawrz inequlity.

Theorem 1.2 (Minkowski's Inequality)**.** *Fix* $1 \leq p \leq \infty$*. Let* $f, g : X \to \mathbb{C}$ *be measurable functions. Then*

$$\|f + g\|_p \leq \|f\|_p + \|g\|_p.$$

Mikowski's Inequlity proof

First we need to remove some trivial case

- If $\|f+g\|_p = 0$
- If $\|f\|_p = \infty$ and $\|g\|_p = \infty$

Now WOLG we assume that $\|f+g\|_p \neq 0$ and $\|f\|_p < \infty$ and $\|g\|_p < \infty$.

First, observe that (triangle ineqlity)

$|f(x)+g(x)| \leq |f(x)| + |g(x)|$ for all $x \in X$

$p=1$ $\quad \|f+g\|_1 = \int_X |f+g| \, d\mu \leq \int_X |f| + |g| \, d\mu$

$= \int_X |f| \, d\mu + \int_X |g| \, d\mu$

$= \|f\|_1 + \|g\|_1$

$p=\infty$ $\quad |f| < \|f\|_\infty$ $\quad [\mu]$-a.e on X (lemma 1.1.2)

Thus, $|f| < \|f\|_\infty < \infty$ on $E \in \Sigma$ s.t. $\mu(X \setminus E) = 0$

$|g| < \|g\|_\infty < \infty$ on $F \in \Sigma$ s.t. $\mu(X \setminus F) = 0$

Proof.

Then $|f+g| \leqslant |f|+|g| < \|f\|_\infty + \|g\|_\infty$ on $E \cap F \in \Sigma$

(finite intersection of measurable set is measurable)

Furthur,

$$\mu(X \setminus (E \cap F)) = \mu((X \setminus E) \cup (X \setminus F)) \text{ (De morgan)}$$
$$\leqslant \mu(X \setminus E) + \mu(X \setminus F) = 0+0=0$$

$$\mu(X \setminus E \cap F) = 0$$

Thus, $|f+g| < \|f\|_\infty + \|g\|_\infty$ on $[\mu]$ a.e on X.

$\|f\|_\infty = \inf\{M > 0 : |f| < M \; [\mu]\text{-a.e. on } X\}$.

$$\Rightarrow \|f+g\|_\infty \leqslant \|f\|_\infty + \|g_\infty\|$$

If $1 < p < \infty$. Now we come to the interesting part of proof. (Other cases are some what trivial).

We know $|f+g| \leq |f|+|g|$ (Δ inequlity)

$$|f+g|^p \leqslant (|f|+|g|)^p$$ (Since $t \longmapsto t^p$ is increasing $p \geqslant 1, t \in (0,\infty)$)

Then, $\|f+g\|_p = \left(\int_X |f+g|^p\right)^{1/p} \leqslant \left(\int_X (|f|+|g|)^p\right)^{1/p}$

$= \| \, |f|+|g| \, \|_p$

Thus it sufficient to show that,

$$\| \, |f|+|g| \, \|_p \leqslant \| \, |f| \, \|_p + \| \, |g| \, \|_p$$

Thus WLOG, we may assume that $f \geqslant 0, g \geqslant 0$ on X.

We have, $(f+g)^p = (f+g)(f+g)^{p-1}$

$= f(f+g)^{p-1} + g(f+g)^{p-1}$

Let's focus in first half of above and compute

$$\begin{aligned}\int_X f(f+g)^{p-1} d\mu &\leqslant \|f\|_p \, \|(f+g)^{p-1}\|_q \\ &= \|f\|_p \left(\int_X (f+g)^{(p-1)q} d\mu\right)^{1/q} \\ &= \|f\|_p \left(\int_X (f+g)^{p} d\mu\right)^{1/q}\end{aligned}$$

$\frac{1}{p} + \frac{1}{q} = 1$

$q = \frac{1}{1-\frac{1}{p}} = \frac{p}{p-1}$

Similarly $\int_X g(f+g)^{p-1} d\mu \leq \|g\|_p \left(\int_X (f+g)^p d\mu\right)^{\frac{1}{q}}$

Note that $0 < \int_X (f+g)^p d\mu < \infty$

$\|f+g\|_p \neq 0 \Rightarrow \left(\int_X (f+g)^p\right)^{1/p} \neq 0$

$\Rightarrow \int_X (f+g)^p d\mu \neq 0$

$$\frac{\left(\int_X (f+g)^p d\mu\right)}{\left(\int_X (f+g)^p d\mu\right)^{\frac{1}{q}}} \leq \|f\|_p + \|g\|_p$$

$$\|f+g\|_p = \left(\int_X (f+g)^p d\mu\right)^{\frac{1}{p}} = \left(\int_X (f+g)^p d\mu\right)^{1-\frac{1}{q}} \leq \|f\|_p + \|g\|_p$$

□

Next, we consider the following question:

Question : For which measurable functions $f : X \to \mathbb{C}$ do we have $\|f\|_p = 0$?

Answer: By lemma @ref(lemm:lemma1.1.1) $||f||_p = 0 \iff f = 0\ [\mu]-$ a.e. Precisely those functions such that $f(x) = 0$ for μ-almost every $x \in X$.

In particular, there are some functions f which are not identically zero but have zero $\mathcal{L}^p$-norm. This is unfortunate, so we typically consider the following quotient space:

We define

$$L^p(X, \mathcal{T}, \mu) = \frac{\mathcal{L}^p(X, \mathcal{T}, \mu)}{N_p},$$

where

$$N_p = \{\, f \in \mathcal{L}^p(X, \mathcal{T}, \mu) : \|f\|_p = 0 \,\}.$$

We have seen that for any $\lambda \in \mathbb{C}$ and any $f, g \in \mathcal{L}^p(X, \mathcal{T}, \mu)$, we always have

$$\begin{aligned} \|\lambda f\|_p &= |\lambda|\, \|f\|_p, \\ \|f+g\|_p &\underset{\substack{\uparrow \\ \text{By Mink}}}{\leq} \|f\|_p + \|g\|_p. \end{aligned}$$

Claim: $\mathcal{L}^p$ is vector space over $\mathbb{C}$.

Proof.

1. Zero function

Let $0(x) := 0$ for all x. Then $|0|^p = 0$ and $\int_X |0|^p \, d\mu = 0 < \infty$, so $0 \in \mathcal{L}^p$.

2. Closed under scalar multiplication

Let $f \in \mathcal{L}^p$ and $\lambda \in \mathbb{C}$. Then

$$|\lambda f|^p = |\lambda|^p |f|^p,$$

so

$$\int_X |\lambda f|^p \, d\mu = |\lambda|^p \int_X |f|^p \, d\mu < \infty.$$

Thus $\lambda f \in \mathcal{L}^p$.

3. Closed under addition

Let $f, g \in \mathcal{L}^p$. Use the standard inequality for $p \geq 1$:

$$|f+g|^p \leq 2^{p-1}(|f|^p + |g|^p).$$

Integrate:

$$\int_X |f+g|^p \, d\mu \leq 2^{p-1} \left(\int_X |f|^p \, d\mu + \int_X |g|^p \, d\mu \right) < \infty,$$

since both integrals on the right are finite. Hence $f + g \in \mathcal{L}^p$.

4. Vector space axioms

The pointwise operations

$$(f+g)(x) := f(x) + g(x), \quad (\lambda f)(x) := \lambda f(x)$$

inherit associativity, commutativity, distributivity, etc., from $\mathbb{C}$. Together with steps 1–3, this shows $\mathcal{L}^p(X, \mathcal{T}, \mu)$ is a vector space over $\mathbb{C}$.

Then

$$L^p(X, \mathcal{T}, \mu) = \frac{\mathcal{L}^p(X, \mathcal{T}, \mu)}{N_p}$$

is the quotient of this vector space by the subspace N_p, so it is also a vector space.

$\square$

Claim: N^p is subspace of $\mathcal{L}^p$

Proof. Let $f, g \in N^p$ and $\lambda \in \mathbb{C}$,

- $0_{map} \in N^p \implies N^p \neq \emptyset$
- $\|\lambda f\|_p = |\lambda| \, \|f\|_p = 0$
- $\|f+g\|_p \leq \|f\|_p + \|g\|_p = 0 \implies \|f+g\|_p = 0.$

Thus, N^p is a subspace of $\mathcal{L}^p$. $\square$

Thus, L^p is subspace. Hence, N_p is a subspace of $L^p(X, \mathcal{T}, \mu)$; therefore $L^p(X, \mathcal{T}, \mu)$ is a vector space over $\mathbb{C}$.

If for $f \in L^p(X, \mathcal{T}, \mu)$ we denote by $[f]$ its image in the quotient space $L^p(X, \mathcal{T}, \mu)$, then

$$\lambda[f] + [g] = [\,\lambda f + g\,].$$

Define

$$\|[f]\|_p = \|f\|_p$$

More ever, $\|[\cdot]\|_p$ well defined.

Proof. Let $f, g \in \mathcal{L}^p$ By Minkowski's inequlirty we can get,

$$\big|\, \|f\|_p - \|g\|_p \,\big| \;\leq\; \|\, f - g \,\|_p, \qquad f, g \in L^p(X, \mathcal{T}, \mu). \tag{1.1}$$

Suppos that $[f] = [g]$. Then, $f - g \in N^p \implies f - g \in N^p$. Then $\|\, f - g \,\|_p = 0$. Thus,

$$\begin{aligned}
\big|\, \|f\|_p - \|g\|_p \,\big| \;\leq\; \|\, f - g \,\|_p = 0 &\implies \|[f-g]\|_p = 0 \\
&\implies \big|\, \|f\|_p - \|g\|_p \,\big| = 0 \\
&\implies \|f\|_p = \|g\|_p
\end{aligned}$$

□

Note that $\|[f]\|_p = 0$ if and only $[f] = 0_{L^p}$ in $L^p(X, I, \mu)$.

Proof.

- $\implies$:

$$\begin{aligned}
\|[f]\|_p = 0 &\implies \|f\|_p = 0. \\
&\implies f \in N^p \\
&\implies [f] = [0] = 0_{L^p}
\end{aligned}$$

- $\impliedby$:

$$\begin{aligned}
[f] = 0_{L^p} &\implies [f] = [0] \\
&\implies f - 0 \in N^p \\
&\implies f \in N^p \\
&\implies \|f\|_p = 0 \\
&\implies \|[f]\|_p = \|f\|_p = 0.
\end{aligned}$$

□

Now we can avoid the problem that have earlier. Now we can defnie the norm.

Here's a clean, well-structured Markdown version of your text, with mathematical expressions formatted clearly and consistently.

1.2 A point of notation

For convenience, mathematicians agree to write f instead of $[f]$. This causes very little confusion; the only thing to keep in mind is that one can capture the behaviour of an element in $L^p(X,\mathcal{T},\mu)$ only up to sets of zero μ-measure. For the rest of this course, we will use this convention and write elements of the quotient space L^p simply as functions.

1.2.1 Summary

For $1 \le p \le \infty$:

1. **Vector space:** $L^p(X,\mathcal{T},\mu)$ is a vector space over $\mathbb{C}$.
2. **Definition of the p-norm:** To every $f \in L^p(X,\mathcal{T},\mu)$ we associate a non-negative number defined by

   $$\|f\|_p = \left(\int_X |f|^p \, d\mu\right)^{1/p}, \qquad 1 \le p < \infty,$$

   and for $p = \infty$,

   $$\|f\|_\infty = \inf\{M \ge 0 : |f(x)| \le M \text{ for almost every } x\}.$$

3. **Homogeneity:** For every $\lambda \in \mathbb{C}$ and $f \in L^p(X,\mathcal{T},\mu)$,

   $$\|\lambda f\|_p = |\lambda| \, \|f\|_p.$$

4. **Triangle inequality:** For every $f, g \in L^p(X,\mathcal{T},\mu)$,

   $$\|f+g\|_p \le \|f\|_p + \|g\|_p.$$

5. **Definiteness:** For every $f \in L^p(X,\mathcal{T},\mu)$,

   $$\|f\|_p \ge 0,$$

   with equality if and only if $f = 0$ almost everywhere.

Properties (iii), (iv), and (v) show that $\|\cdot\|_p$ defines a norm, so $L^p(X,\mathcal{J},\mu)$ is a **normed linear space**.

Definition 1.4. Banach spaces are normed linear spaces with an additional property: they are *complete*, meaning every Cauchy sequence converges.

Our next task is to show that $L^p(X,\mathcal{J},\mu)$ is a Bannch space. (We need to show that complete space)

Before doing so, we recall some important results from measure theory.

Lemma 1.4 (Chebyshev's Inequality)**.** *Let $(X, \mathcal{T}, \mu)$ be a measure space and let f be a non-negative measurable function on X.*
Then, for every $\lambda > 0$,

$$\mu\{x \in X : f(x) \geq \lambda\} \;\leq\; \frac{1}{\lambda}\int_X f\, d\mu.$$

Proof. Let $E_\lambda = \{x \in X : f(x) \geq \lambda\}$, Then,

$$\int_X f\, d\mu \geq \int_{E_\lambda} f\, d\mu \geq \int_{E_\lambda} \lambda\, d\mu = \lambda \int_{E_\lambda} d\mu = \lambda\mu(E_\lambda)$$

□

Lemma 1.5 (Borel–Cantelli Lemma)**.** *Let $(X, \mathcal{T}, \mu)$ be a measure space and let $\{E_n\}_{n=1}^\infty$ be a collection of measurable sets such that $\sum_{n=1}^\infty \mu(E_n) < \infty$. Then μ-almost every $x \in X$ belongs to at most finitely many of the sets E_n.*

Proof.

$$\begin{aligned} S &:= \{x \in X : x \text{ belongs to infinitly many } E_n\} \\ &= \bigcap_{N=1}^{\infty} \bigcup_{k=N}^{\infty} E_k (I will explain this later) \end{aligned}$$

$$\begin{aligned} \mu(S) &\leq \mu\left(\cup_{k=N}^{\infty}\right) \\ &\leq \sum_{k=N}^{\infty} \mu(E_k) \text{ for all } N \end{aligned}$$

Then left hand side is goes to zero as $N \to \infty$.

Apply young inequality,

$$a = \frac{|f(x)|}{\|f\|_p} \quad \text{and} \quad b = \frac{|g(x)|}{\|g\|_q}$$

$$ab \leq \frac{a^p}{p} + \frac{b^p}{q}$$

$$\frac{|f(x)||g(x)|}{\|f\|_p \|g\|_q} \leqslant \frac{1}{p}\left(\frac{|f(x)|}{\|f\|_p}\right)^p + \frac{1}{q}\left(\frac{|g(x)|}{\|g\|_q}\right)^q$$

$$\Rightarrow \int_X \frac{|fg|}{\|f\|_p \|g\|_q} d\mu \leqslant \frac{1}{p}\int_X \frac{|f(x)|^p}{\|f\|_p^p} d\mu + \frac{1}{q}\int_X \frac{|g(x)|^q}{\|g\|_p^p} d\mu$$

$$= \frac{1}{p\|f\|_p^p}\int_X |f(x)|^p d\mu + \frac{1}{q\|g\|_q}\int |g(x)|^p d\mu$$

$$= \frac{1}{p\|f\|_p^p} \|f\|_p^p + \frac{1}{q\|g\|_q} \|g\|_q^q$$

$$\|f\|_p = \left(\int_X |f| d\mu\right)^{1/p}$$

$$= 1/p + 1/q = 1$$

□

Lemma 1.6 (Fatou's Lemma)**.** *Let $(X, \mathcal{T}, \mu)$ be a measure space and let $\{f_n\}_{n=1}^{\infty}$ be a sequence of non-negative measurable functions on X. Then,*

$$\int_X \liminf_{n\to\infty} f_n \, d\mu \;\leq\; \liminf_{n\to\infty} \int_X f_n \, d\mu.$$

Lemma 1.7 (A Technical Convergence Lemma)**.** *Let $(X, \mathcal{T}, \mu)$ be a measure space and let $1 \leq p \leq \infty$.*

Let $\{f_n\}_{n=1}^{\infty} \subset L^p(X, \mathcal{T}, \mu)$ be a sequence such that there exists a sequence of positive numbers $\{\varepsilon_n\}_{n=1}^{\infty}$ with

$$\sum_{n=1}^{\infty} \varepsilon_n < \infty,$$

and

$$\|f_n - f_{n+1}\|_p \leq \varepsilon_n^2, \qquad n \geq 1.$$

Then there exists $f \in L^p(X, \mathcal{T}, \mu)$ such that

- *pointwise a.e. convergence:*

$$\lim_{n \to \infty} f_n(x) = f(x) \quad \text{for } \mu\text{-almost every } x \in X,$$

- *convergence in L^p:*

$$\lim_{n \to \infty} \|f - f_n\|_p = 0.$$

Chapter 2

Hello bookdown

All chapters start with a first-level heading followed by your chapter title, like the line above. There should be only one first-level heading (`#`) per .Rmd file.

2.1 A section

All chapter sections start with a second-level (`##`) or higher heading followed by your section title, like the sections above and below here. You can have as many as you want within a chapter.

An unnumbered section

Chapters and sections are numbered by default. To un-number a heading, add a `{.unnumbered}` or the shorter `{-}` at the end of the heading, like in this section.

Chapter 3

Cross-references

Cross-references make it easier for your readers to find and link to elements in your book.

3.1 Chapters and sub-chapters

There are two steps to cross-reference any heading:

1. Label the heading: `# Hello world {#nice-label}`.
   - Leave the label off if you like the automated heading generated based on your heading title: for example, `# Hello world` = `# Hello world {#hello-world}`.
   - To label an un-numbered heading, use: `# Hello world {-#nice-label}` or `{# Hello world .unnumbered}`.
2. Next, reference the labeled heading anywhere in the text using `\@ref(nice-label)`; for example, please see Chapter 3.
   - If you prefer text as the link instead of a numbered reference use: any text you want can go here.

3.2 Captioned figures and tables

Figures and tables *with captions* can also be cross-referenced from elsewhere in your book using `\@ref(fig:chunk-label)` and `\@ref(tab:chunk-label)`, respectively.

See Figure 3.1.

```r
par(mar = c(4, 4, .1, .1))
plot(pressure, type = 'b', pch = 19)
```

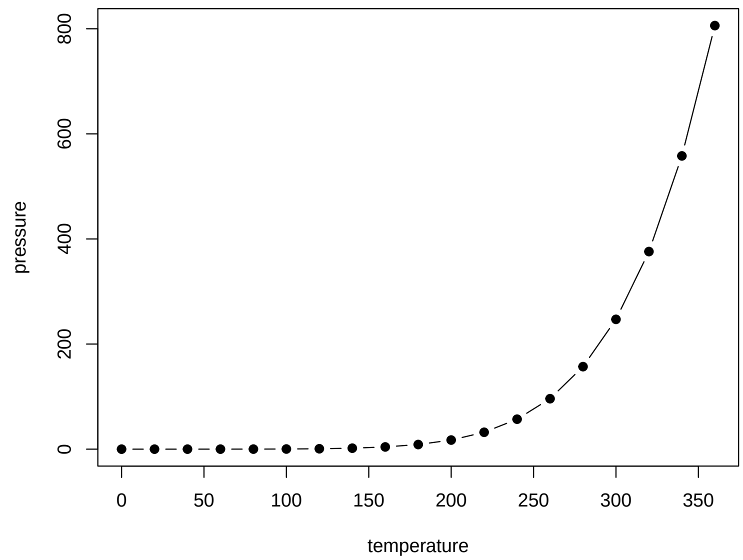


Figure 3.1: Here is a nice figure!

Don't miss Table 3.1.

```r
knitr::kable(
  head(pressure, 10), caption = 'Here is a nice table!',
  booktabs = TRUE
)
```

Table 3.1: Here is a nice table!

temperature	pressure
0	0.0002
20	0.0012
40	0.0060
60	0.0300
80	0.0900
100	0.2700
120	0.7500
140	1.8500
160	4.2000
180	8.8000

Chapter 4

Parts

You can add parts to organize one or more book chapters together. Parts can be inserted at the top of an .Rmd file, before the first-level chapter heading in that same file.

Add a numbered part: `# (PART) Act one {-}` (followed by `# A chapter`)

Add an unnumbered part: `# (PART\*) Act one {-}` (followed by `# A chapter`)

Add an appendix as a special kind of un-numbered part: `# (APPENDIX) Other stuff {-}` (followed by `# A chapter`). Chapters in an appendix are prepended with letters instead of numbers.

Chapter 5

Footnotes and citations

5.1 Footnotes

Footnotes are put inside the square brackets after a caret `^[]`. Like this one [1].

5.2 Citations

Reference items in your bibliography file(s) using `@key`.

For example, we are using the **bookdown** package [**?**] (check out the last code chunk in index.Rmd to see how this citation key was added) in this sample book, which was built on top of R Markdown and **knitr** [**?**] (this citation was added manually in an external file book.bib). Note that the `.bib` files need to be listed in the index.Rmd with the YAML `bibliography` key.

The RStudio Visual Markdown Editor can also make it easier to insert citations: https://rstudio.github.io/visual-markdown-editing/#/citations

[1] This is a footnote.

Chapter 6

Blocks

6.1 Equations

Here is an equation.

$$f(k) = \binom{n}{k} p^k (1-p)^{n-k} \tag{6.1}$$

You may refer to using `\@ref(eq:binom)`, like see Equation (6.1).

6.2 Theorems and proofs

Labeled theorems can be referenced in text using `\@ref(thm:tri)`, for example, check out this smart theorem 6.1.

Theorem 6.1. *For a right triangle, if c denotes the* length *of the hypotenuse and a and b denote the lengths of the* ***other*** *two sides, we have*

$$a^2 + b^2 = c^2$$

Read more here https://bookdown.org/yihui/bookdown/markdown-extensions-by-bookdown.html.

6.3 Callout blocks

The R Markdown Cookbook provides more help on how to use custom blocks to design your own callouts: https://bookdown.org/yihui/rmarkdown-cookbook/custom-blocks.html

Chapter 7

Sharing your book

7.1 Publishing

HTML books can be published online, see: https://bookdown.org/yihui/bookdown/publishing.html

7.2 404 pages

By default, users will be directed to a 404 page if they try to access a webpage that cannot be found. If you'd like to customize your 404 page instead of using the default, you may add either a `_404.Rmd` or `_404.md` file to your project root and use code and/or Markdown syntax.

7.3 Metadata for sharing

Bookdown HTML books will provide HTML metadata for social sharing on platforms like Twitter, Facebook, and LinkedIn, using information you provide in the `index.Rmd` YAML. To setup, set the `url` for your book and the path to your `cover-image` file. Your book's `title` and `description` are also used.

This `gitbook` uses the same social sharing data across all chapters in your book-all links shared will look the same.

Specify your book's source repository on GitHub using the `edit` key under the configuration options in the `_output.yml` file, which allows users to suggest an edit by linking to a chapter's source file.

Read more about the features of this output format here:

https://pkgs.rstudio.com/bookdown/reference/gitbook.html

Or use:

```
?bookdown::gitbook
```