Topology

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Chapter 1

Topology

A topology is a geometric structure defined on a set. Basically it is given by declaring which subsets are "open" sets. Thus the axioms are the abstraction of the properties that open sets have.

1.1 Topological Spaces

Definition 1.1. A topology on a set X is a collection $\mathcal{T}$ of subsets of X such that

(T1) ϕ and X are in $\mathcal{T}$;

(T2) Any union of subsets in $\mathcal{T}$ is in $\mathcal{T}$;

(T3) The finite intersection of subsets in $\mathcal{T}$ is in $\mathcal{T}$.

A set X with a topology $\mathcal{T}$ is called a topological space. Denoted by $(X, \mathcal{T})$. An element of $\mathcal{T}$ is called an open set.

Example 1.1. Let X be a three-element set, $X = \{a, b, c\}$ and $\tau = \{X, \emptyset, \{a, b\}, \{b\}, \{b, c\}\}$. We can check T1,T2 and T3 conditions.

Example 1.2. Let X be a three-element set, $X = \{a, b, c\}$ as pervoius. There are many possible topologies on X, some of which are indicated schematically in figure 1.1. Furthur, we can see that even a three-element set has many different topologies.

Remark. Not every collection of subsets of X is a topology on X. Observe that Neither of the collections indicated in figure 1.2 is a topology.

First let's consider the left hand coner of figure 1.2. $\{a\}$ and $\{b\}$ in the collection, but $\{a\} \cup \{b\}$ is not in the collection.

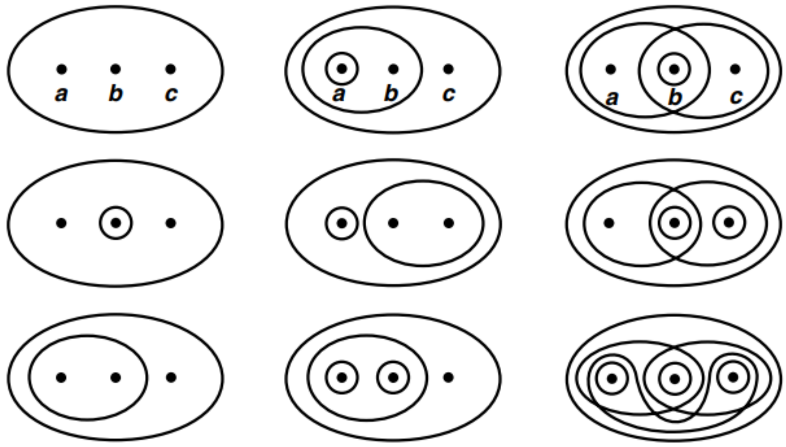

Figure 1.1:

Now consider the right hand coner figure. $\{a, b\}$ and $\{b, c\}$ in collection, but $\{a, b\} \cap \{b, c\} = \{b\}$ is not in the collection.

Figure 1.2:

Example 1.3. If X is any set, the collection of all subsets of X (Power set) is a topology on X. This trivail statified T1 T2 and T3 conditions. Furthur,This is called the *discrete topology*.

Example 1.4. The collection consisting of X and $\emptyset$ only is also a topologyon X. we shall call it the *indiscrete topology*, or the trivial topology.

Example 1.5. Let X be a set and let τ_f be the collection of all subsets U of X such that X U either is finite or is all of X. In oter words,

$$\tau_f := \{U \subseteq X : \text{Either is finite or is all of } X\}$$

Let's check is τ_f a topology. First obseve that both X and $\emptyset$ are in τ_f , because X X is finite and \$X \$ is all of X.So τ_f statified the T1 condition. Now let's check the T2 condition. Let $\{U_\alpha : \alpha \in I, I \text{is index set}\}$.Now we need to show that $\cup\alpha \in IU_\alpha \in \tau_f$. So consider,

$$X \ \cup_{\alpha\in I} U_\alpha = \cap_{\alpha\in I}(X \ \ U_\alpha).$$

Now obsevere that *{ I} (X U)* is finite, because each set $(X \ \ U_\alpha)$ is finite and arbitary intersection of finite sets is finite. So, τ_f stattified the T2 condition also. Finaly check the last condition, T3 condition. Let $U_1, ..., U_n$ are nonempty elements of τ_f , to show that $\Cap_i U_i \in \tau_f$, we compute

$$X \ \cap_{i=1}^n U_i = \cup_{i=1}^n (X \ \ U_i).$$

Note that the set $\cup_{i=1}^n (X \ U_i)$ is a finite union of finite sets and, therefore, finite. So it statisfiy the T3 condition also. Thefore τ_f is a topology. Furthur τ_f is called the finite *complement topology.*

Example 1.6. Let X be a set. Define $\mathcal{T}$ to be the collection of all subsets U of X such that $X \ \ U$ either is finite or is all of X. Then $\mathcal{T}$ defines a topology on X, called finite complement topology of X.

1.2 Basis of a Topology

Once we define a structure on a set, often we try to understand what the minimum data you need to specify the structure. In many cases, this minimum data is called a basis and we say that the basis generate the structure. The notion of a basis of the structure will help us to describe examples more systematically.

Definition 1.2. Let X be a set. A basis of a topology on X is a collection $\mathcal{B}$ of subsets in X such that

(B1) For every $x \in X$, there exist an element B in $\mathcal{B}$ such that $x \in B$.

(B2) If $x \in B_1 \cap B_2$ where B_1, B_2 are in $\mathcal{B}$, then there is B_3 in $\mathcal{B}$ such that $x \in B_3 \subseteq B_1 \cap B_2$.

Lemma 1.1 (Generating of a topology)**.** *Let $\mathcal{B}$ be a basis of a topology on X. Define $\mathcal{T}_{\mathcal{B}}$ to be the collection of subsets $U \subset X$ satisfting*

(G1) For every $x \in U$, there is $B \in \mathcal{B}$ such that $x \in B \subset U$.

Then $\tau_{\mathcal{B}}$ defines a topology on X. Here we assume that $\emptyset$ trivially satisfies the condition, so that $\emptyset \in \tau_{\mathcal{B}}$.

Proof. We need to check the three axioms: □

Chapter 2

Chapter 2 name

Chapter 3

Chapter 03 name

Up to there is none.