Topology

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Chapter 1

Topology

A topology is a geometric structure defined on a set. Basically it is given by declaring which subsets are "open" sets. Thus the axioms are the abstraction of the properties that open sets have.

1.1 Topological Spaces

Definition 1.1. A topology on a set X is a collection $\mathcal{T}$ of subsets of X such that

(T1) ϕ and X are in $\mathcal{T}$;

(T2) Any union of subsets in $\mathcal{T}$ is in $\mathcal{T}$;

(T3) The finite intersection of subsets in $\mathcal{T}$ is in $\mathcal{T}$.

A set X with a topology $\mathcal{T}$ is called a topological space. Denoted by $(X, \mathcal{T})$. An element of $\mathcal{T}$ is called an open set.

Example 1.1. Let X be a three-element set, $X = \{a, b, c\}$ and $\mathcal{T} = \{X, \emptyset, \{a, b\}, \{b\}, \{b, c\}\}$. We can check T1,T2 and T3 conditions.

Example 1.2. Let X be a three-element set, $X = \{a, b, c\}$ as pervoius. There are many possible topologies on X, some of which are indicated schematically in figure **??**. Furthur, we can see that even a three-element set has many different topologies.

Remark. Not every collection of subsets of X is a topology on X. Observe that Neither of the collections indicated in figure **??** is a topology.

First let's consider the left hand coner of figure **??**. $\{a\}$ and $\{b\}$ in the collection, but $\{a\} \cup \{b\}$ is not in the collection.

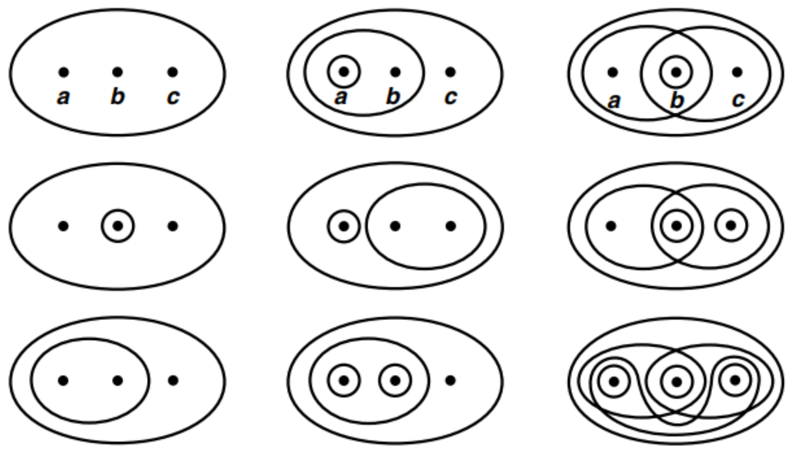

Figure 1.1:

Now consider the right hand coner figure. $\{a, b\}$ and $\{b, c\}$ in collection, but $\{a, b\} \cap \{b, c\} = \{b\}$ is not in the collection.

Figure 1.2:

Example 1.3. If X is any set, the collection of all subsets of X (Power set) is a topology on X. This trivail statified T1 T2 and T3 conditions. Furthur,This is called the *discrete topology*.

Example 1.4. The collection consisting of X and $\emptyset$ only is also a topologyon X. we shall call it the *indiscrete topology*, or the trivial topology.

Example 1.5. Let X be a set and let $\mathcal{T}_f$ be the collection of all subsets U of X such that $X \quad U$ either is finite or is all of X. In oter words,

$$\mathcal{T}_f := \{U \subseteq X : \text{Either is finite or is all of } X\}$$

Let's check is $\mathcal{T}_f$ a topology. First obseve that both X and $\emptyset$ are in $\mathcal{T}_f$, because $X \quad X = \emptyset$ is finite and $X $ is all of X.So $\mathcal{T}_f$ statified the T1 condition. Now let's check the T2 condition. Let $\{U_\alpha : \alpha \in I, I \text{ is index set}\}$.Now we need to show that $\cup\alpha \in IU_\alpha \in \mathcal{T}_f$. So consider,

$$X \quad \bigcup_{\alpha \in I} U_\alpha = \bigcap_{\alpha \in I}(X \quad U_\alpha).$$

Now obsevere that $\cap_{\alpha \in I}(X \quad U_\alpha)$ is finite, because each set $(X \quad U_\alpha)$ is finite and arbitary intersection of finite sets is finite. So, $\mathcal{T}_f$ stattified the T2 condition also. Finaly check the last condition, T3 condition. Let $U_1, ..., U_n$ are nonempty elements of $\mathcal{T}_f$, to show that $\bigcup_i U_i \in \mathcal{T}_f$, we compute

$$X \quad \bigcap_{i=1}^{n} U_i = \bigcup_{i=1}^{n}(X \quad U_i).$$

Note that the set $\bigcup_{i=1}^{n}(X \ U_i)$ is a finite union of finite sets and, therefore, finite. So it statisfiy the T3 condition also. Thefore $\mathcal{T}_f$ is a topology. Furthur $\mathcal{T}_f$ is called the finite *complement topology.*

Example 1.6. Let X be a set. Define $\mathcal{T}$ to be the collection of all subsets U of X such that $X \quad U$ either is finite or is all of X. Then $\mathcal{T}$ defines a topology on X, called finite complement topology of X.

1.2 Basis of a Topology

Once we define a structure on a set, often we try to understand what the minimum data you need to specify the structure. In many cases, this minimum data is called a basis and we say that the basis generate the structure. The notion of a basis of the structure will help us to describe examples more systematically.

Definition 1.2. Let X be a set. A basis of a topology on X is a collection $\mathcal{B}$ of subsets in X such that

(B1) For every $x \in X$, there exist an element B in $\mathcal{B}$ such that $x \in B$.

(B2) If $x \in B_1 \cap B_2$ where B_1, B_2 are in $\mathcal{B}$, then there is B_3 in $\mathcal{B}$ such that $x \in B_3 \subseteq B_1 \cap B_2$.

Lemma 1.1 (Generating of a topology)**.** *Let $\mathcal{B}$ be a basis of a topology on X. Define $\mathcal{T}_\mathcal{B}$ to be the collection of subsets $U \subset X$ satisfting*

(G1) For every $x \in U$, there is $B \in \mathcal{B}$ such that $x \in B \subset U$.

Then $\mathcal{T}_\mathcal{B}$ defines a topology on X. Here we assume that $\emptyset$ trivially satisfies the condition, so that $\emptyset \in \mathcal{T}_\mathcal{B}$.

Proof. We need to check the three axioms:

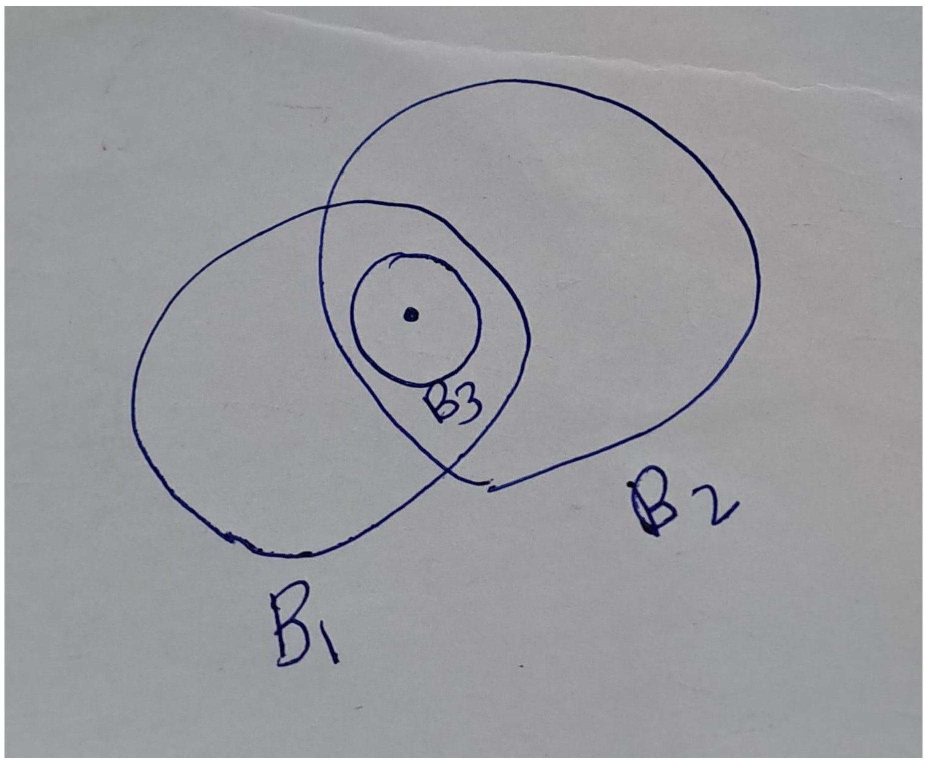

Figure 1.3:

- (T1) $\emptyset \in \mathcal{T}_{\mathcal{B}}$ as we assumed. $X \in \mathcal{T}_{\mathcal{B}}$ by (B1).
- (T2) Consider a collection of subsets $U_\alpha \in \mathcal{T}_{\mathcal{B}}, \alpha \in$ J. We need to show

$$U := \bigcup_{\alpha \in J} U_\alpha \quad \in \mathcal{T}_{\mathcal{B}}$$

By the definition of the union, for each $x \in U$, there is U_α such that $x \in U_\alpha$. Since $U_\alpha \in \mathcal{T}_{\mathcal{B}}$, there is $B \in \mathcal{B}$ such that $x \in B \subset U_\alpha$. Since $U_\alpha \subset U$, we found $B \in \mathcal{B}$ such that $x \in B \subset U$. Thus $U \in \mathcal{T}_{\mathcal{B}}$.

- (T3) Now consider a finite number of subsets $U_1, ..., U_n \in \mathcal{T}_{\mathcal{B}}$. We need to show that
$$U' := \bigcap_{i=1}^{n} U_i \quad \in \mathcal{T}_{\mathcal{B}}$$
- Let's just check for two subsets U_1, U_2 first. For each $x \in U_1 \cap U_2$, there are $B_1, B_2 \in \mathcal{B}$ such that $x \in B_1 \subset U_1$ and $x \in B_2 \subset U_2$. This is because $U_1, U_2 \in \mathcal{T}_{\mathcal{B}}$ and $x \in U_1, x \in U_2$. By (B2), there is $B_3 \in \mathcal{B}$ such that $x \in B_3 \subset B_1 \cap B_2$. Now we found $B_3 \in \mathcal{B}$ such that $x \in B_3 \subset U$.
- We can generalize the above proof to n subsets, but let's use induction to prove it. This is going to be the induction on the number of subsets.
  - When $n = 1$, the claim is trivial.
  - Suppose that the claim is true when we have $n-1$ subsets, i.e. $U_1 \cap \cdots \cap U_{n-1} \in \mathcal{T}_{\mathcal{B}}$. Since
  $$U = U_1 \cap \cdots \cap U_n = (U_1 \cap \cdots \cap U_{n-1}) \cap U_n$$
  and regarding $U' := U_1 \cap \cdots \cap U_{n-1}$, we have two subsets case $U = U' \cap U_n$. By the first arguments, $U \in \mathcal{T}_{\mathcal{B}}$.

□

Definition 1.3. $\mathcal{T}_{\mathcal{B}}$ is called the **topology generated by a basis** $\mathcal{B}$. On the other hand, if $(X, \mathcal{T})$ is a topological space and $\mathcal{B}$ is a basis of a topology such that $\mathcal{T}_{\mathcal{B}} = \mathcal{T}$, then we say $\mathcal{B}$ is a basis of $\mathcal{T}$. Note that $\mathcal{T}$ itself is a basis of the topology $\mathcal{T}$. So there is always a basis for a given topology.

Example 1.7.

- (Standard Topology of $\mathbb{R}$) Let $\mathbb{R}$ be the set of all real numbers. Let $\mathcal{B}$ be the collection of all open intervals:
$$(a, b) := \{x \in \mathbb{R} \mid a < x < b\}$$
Then $\mathcal{B}$ is a basis of a topology and the topology generated by $\mathcal{B}$ is called the standard topology of $\mathbb{R}$.

- Let $\mathbb{R}^2$ be the set of all ordered pairs of real numbers, i.e. $\mathbb{R}^2 := \mathbb{R} \times \mathbb{R}$ (cartesian product). Let $\mathcal{B}$ be the collection of cartesian product of open intervals, $(a,b) \times (c,d)$. Then $\mathcal{B}$ is a basis of a topology and the topology generated by $\mathcal{B}$ is called the standard topology of $\mathbb{R}^2$.

- (Lower limit topology of $\mathbb{R}$) Consider the collection $\mathcal{B}$ of subsets in $\mathbb{R}$:

  $$\mathcal{B} := \{[a,b) := \{x \in \mathbb{R} \mid a \le x < b\} \mid a, b \in \mathbb{R}\}$$

  This is a basis for a topology on $\mathbb{R}$. This topology is called the lower limit topology.

The following two lemma are useful to determine whehter a collection $\mathcal{B}$ of open sets in $\mathcal{T}$ is a basis for $\mathcal{T}$ or not.

Remark. Let $\mathcal{T}$ be a topology on X. If $\mathcal{B} \subset \mathcal{T}$ and $\mathcal{B}$ satisfies (B1) and (B2), it is easy to see that $\mathcal{T}_{\mathcal{B}} \subset \mathcal{T}$. This is just because of (G1). If $U \in \mathcal{T}_{\mathcal{B}}$, (G1) is satisfied for U so that $\forall x \in U, \exists B_x \in \mathcal{B}$ such that $x \in B_x \subset U$. Therefore $U = \cup_{x \in U} B_x$. By (T2), $U \in \mathcal{T}$.

Lemma 1.2. *Let $(X, \mathcal{T})$ be a topological space. Let $\mathcal{B} \subset \mathcal{T}$. Then $\mathcal{B}$ is a basis and $\mathcal{T}_{\mathcal{B}} = \mathcal{T}$ if and only if $\mathcal{T}$ is the set of all unions of elements in $\mathcal{B}$.*

Proof.

- ($\Rightarrow$) Let $\mathcal{T}'$ be the set of all unions of open sets in $\mathcal{B}$. If $U \in \mathcal{T}$, then U satisfies (G1), i.e. $\forall x \in U, \exists B_x \in \mathcal{B}$ s.t. $x \in B_x \subset U$. Thus $U = \cup_{x \in U} B_x$. Therefore $U \in \mathcal{T}'$. We proved $\mathcal{T} \subset \mathcal{T}'$. It follows from (T2) that $\mathcal{T}' \subset \mathcal{T}$.
- ($\Leftarrow$) Since $X \in \mathcal{T}, X = \cup_\alpha B_\alpha$ some union of sets in $\mathcal{B}$. Thus $\forall x \in X, \exists B_\alpha$ s.t. $x \in B_\alpha$. This proves (B1) for $\mathcal{B}$. If $B_1, B_2 \in \mathcal{B}$, then $B_1 \cap B_2 \in \mathcal{T}$ by (T2). Thus $B_1 \cap B_2 = \cup_\alpha B_\alpha, B_\alpha \in \mathcal{B}$. So $\forall x \in B_1 \cap B_2, \exists B_\alpha \in B$ s.t. $x \in B_\alpha$. This B_α plays the role of B_3 in (B2). Thus $\mathcal{B}$ is a basis. Now it makes sense to consider $\mathcal{T}_{\mathcal{B}}$ and we need to show $\mathcal{T}_{\mathcal{B}} = \mathcal{T}$. By the remark, we already know that $\mathcal{T}_{\mathcal{B}} \subset \mathcal{T}$. On the other hand, if $U \in \mathcal{T}$, then $U = \cup_\alpha B_\alpha, B_\alpha \in \mathcal{B}$. Hence, $\forall x \in U, \exists B_\alpha$ such that $x \in B_\alpha \subset U$. Thus (G1) is satisfied for U. Thus $U \in \mathcal{T}_{\mathcal{B}}$. This proves $\mathcal{T}_{\mathcal{B}} \supset \mathcal{T}$.

□

Lemma 1.3. *Let $(X, \mathcal{T})$ be a topological space. Let $\mathcal{B} \subset \mathcal{T}$. Then $\mathcal{B}$ is a basis and $\mathcal{T}_{\mathcal{B}} = \mathcal{T}$ if and if any $U \in \mathcal{T}$ satisfies (Gl), i.e. $\forall x \in U, \exists B_x \in \mathcal{B}$ s.t. $x \in B_x \subset U$.*

Proof.

- ($\Rightarrow$) Trivial by the definition of $\mathcal{T}_{\mathcal{B}}$.

- ($\Leftarrow X$) satisfies (G1) so $\mathcal{B}$ satisfies (B1). Let $B_1, B_2 \in \mathcal{B} \subset \mathcal{T}$. By (T3), $B_1 \cap B_2 \in \mathcal{T}$. Thus $B_1 \cap B_2$ satisfies (G1). This means (B2) holds for $\mathcal{B}$. Thus $\mathcal{B}$ is a basis. Now the assumption can be rephrased as $\mathcal{T} \subset \mathcal{T}_{\mathcal{B}}$. By the remark above, we already know $\mathcal{T} \supset \mathcal{T}_{\mathcal{B}}$.

□

1.3 Comparing Topologies

Definition 1.4. Let $\mathcal{T}, \mathcal{T}'$ be two topologies for a set X. We say $\mathcal{T}'$ is finer than $\mathcal{T}$ or $\mathcal{T}$ is coarser than $\mathcal{T}'$ if $\mathcal{T} \subset \mathcal{T}'$. The intuition for this notion is " $(X, \mathcal{T}')$ has more open subsets to separate two points in X than $(X, \mathcal{T})$ ".

Lemma 1.4. *Let $\mathcal{B}, \mathcal{B}'$ be bases of topologies $\mathcal{T}, \mathcal{T}'$ on X respectively. Then $\mathcal{T}'$ is finer than $\mathcal{T}$ $\Leftrightarrow$ $\forall B \in \mathcal{B}$ and $\forall x \in B, \exists B' \in \mathcal{B}'$ s.t. $x \in B' \subseteq B$.*

Proof.

- $\Rightarrow$ Since $\mathcal{B} \subset \mathcal{T} \subset \mathcal{T}'$, all subsets in $\mathcal{B}$ satisfies (G1) for $\mathcal{T}'$, which is exactly the statement we wanted to prove.

- $\Leftarrow$ The LHS says $\mathcal{B} \subset \mathcal{T}'$. We need to show that it implies that any $U \in \mathcal{T}$ satisfies (G1) for $\mathcal{T}'$ too.

$$\forall U \in \mathcal{T}, \forall x \in U, \exists B \in \mathcal{B} \text{ s.t. } x \in B \subset U$$

But

$$\forall B \in \mathcal{B}, \forall x \in B, \exists B' \in \mathcal{B}' \text{ s.t. } x \in B' \subset B.$$

Combining those two,

$$\forall U \in \mathcal{T}, \forall x \in U, \exists B' \in \mathcal{B}' \text{ s.t. } x \in B' \subset B \subset U.$$

□

Definition 1.5 (subbasis)**.** Let X be a set.A subbasis $\mathcal{S}$ for a topology on X is a collection of subsets of X whose union equals X.

$$\text{(i.e. } \forall x \in X\ \exists S \in \mathcal{S} \text{ such that } x \in S)$$

Definition 1.6. The topology generated by the subbasis $\mathcal{S}$ is defined to be the collection $\mathcal{T}$ of all unions of finite intersections of elements of $\mathcal{S}$.

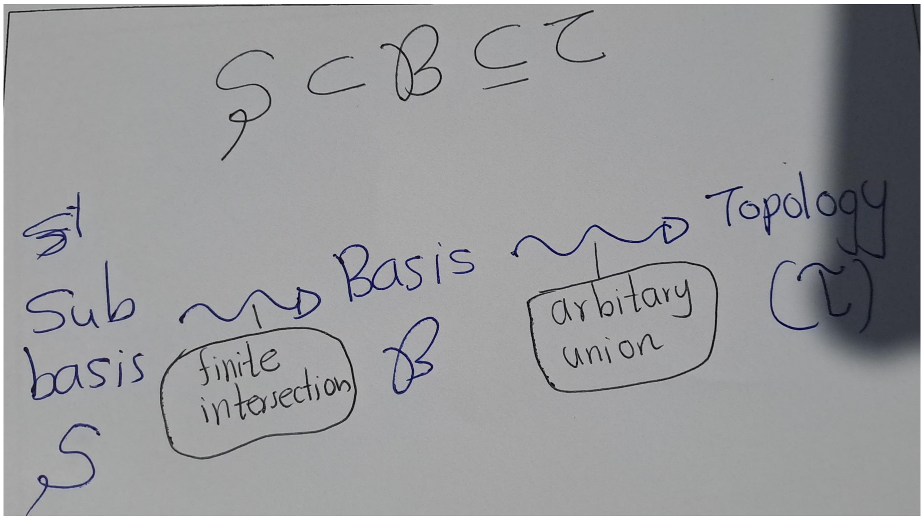

Figure 1.4:

1.4 Order Topology

Definition 1.7 (Linear Order/ Complete Order)**.** Consider order relation "<".

1. If $x \neq y$, then either $x < y$ or $y < x$.
2. If $x < y$, then $x \neq y$.
3. If $x < y$ and $y < z$, then $x < z$.

Example 1.8. $\mathbb{R}$ is ordered set with less than relation.

First, let's see intervals in an Ordered Set.

Suppose that X is a set having a simple order relation $<$. Given elements a and b of X such that $a < b$, there are four subsets of X that are called the intervals determined by a and b. They are the following :

- $(a, b) = \{x \in X | a < x < b\}$ (Type: open interval in X),
- $(a, b] = \{x \in X | a < x \leq b\}$(Type: half-open interval in X),
- $[a, b) = \{x \in X | a \leq x < b\}$(Type: half-open interval in X),
- $[a, b] = \{x \in X | a \leq x \leq b\}$ (Type: closed interval in X),

The notation used here is familiar to you already in the case where X is the real line,but these are intervals in an arbitrary ordered set.

The use of the term "open" in this connection suggests that open intervals in X should turn out to be open sets when we put a topology on X. And so they will.