Topology

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Contents

Chapter 1

Topology

A topology is a geometric structure defined on a set. Basically it is given by declaring which subsets are "open" sets. Thus the axioms are the abstraction of the properties that open sets have.

1.1 Topological Spaces

Definition 1.1. A topology on a set X is a collection $\mathcal{T}$ of subsets of X such that

(T1) ϕ and X are in $\mathcal{T}$;

(T2) Any union of subsets in $\mathcal{T}$ is in $\mathcal{T}$;

(T3) The finite intersection of subsets in $\mathcal{T}$ is in $\mathcal{T}$.

A set X with a topology $\mathcal{T}$ is called a topological space. Denoted by $(X, \mathcal{T})$. An element of $\mathcal{T}$ is called an open set.

Example 1.1. Let X be a three-element set, $X = \{a, b, c\}$ and $\mathcal{T} = \{X, \emptyset, \{a, b\}, \{b\}, \{b, c\}\}$. We can check T1,T2 and T3 conditions.

Example 1.2. Let X be a three-element set, $X = \{a, b, c\}$ as pervoius. There are many possible topologies on X, some of which are indicated schematically in figure 1.1. Furthur, we can see that even a three-element set has many different topologies.

Remark. Not every collection of subsets of X is a topology on X. Observe that Neither of the collections indicated in figure 1.2 is a topology.

First let's consider the left hand coner of figure 1.2. $\{a\}$ and $\{b\}$ in the collection, but $\{a\} \cup \{b\}$ is not in the collection.

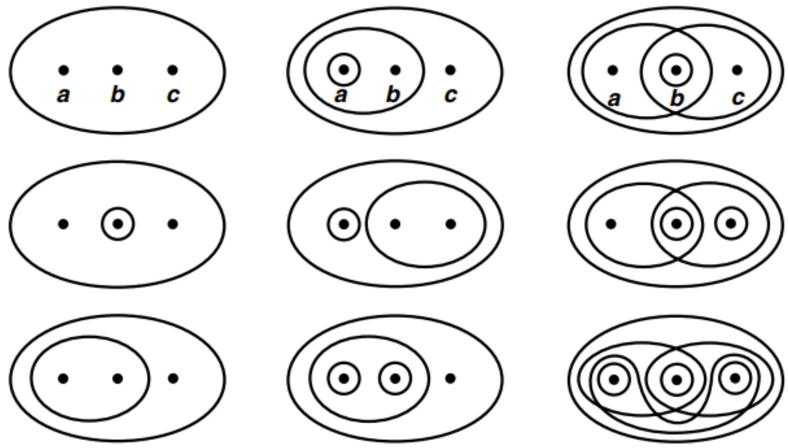


Figure 1.1:

Now consider the right hand coner figure. $\{a, b\}$ and $\{b, c\}$ in collection, but $\{a, b\} \cap \{b, c\} = \{b\}$ is not in the collection.

Figure 1.2:

Example 1.3. If X is any set, the collection of all subsets of X (Power set) is a topology on X. This trivail statified T1 T2 and T3 conditions. Furthur,This is called the *discrete topology*.

Example 1.4. The collection consisting of X and $\emptyset$ only is also a topologyon X. we shall call it the *indiscrete topology*, or the trivial topology.

Example 1.5. Let X be a set and let $\mathcal{T}_f$ be the collection of all subsets U of X such that $X \setminus U$ either is finite or is all of X. In oter words,

$$\mathcal{T}_f := \{U \subseteq X : \text{Either is finite or is all of } X\}$$

Let's check is $\mathcal{T}_f$ a topology. First obseve that both X and $\emptyset$ are in $\mathcal{T}_f$, because $X \setminus X = \emptyset$ is finite and \$X \$ is all of X.So $\mathcal{T}_f$ statified the T1 condition. Now let's check the T2 condition. Let $\{U_\alpha : \alpha \in I, I \text{ is index set}\}$.Now we need to show that $\cup\alpha \in IU_\alpha \in \mathcal{T}_f$. So consider,

$$X \setminus \bigcup_{\alpha\in I} U_\alpha = \bigcap_{\alpha\in I}(X \setminus U_\alpha).$$

Now obsevere that $\cap_{\alpha\in I}(X\setminus U_\alpha)$ is finite, because each set $(X\setminus U_\alpha)$ is finite and arbitary intersection of finite sets is finite. So, $\mathcal{T}_f$ stattified the T2 condition also. Finaly check the last condition, T3 condition. Let $U_1, ..., U_n$ are nonempty elements of $\mathcal{T}_f$, to show that $\bigcup_i U_i \in \mathcal{T}_f$, we compute

$$X \setminus \bigcap_{i=1}^{n} U_i = \bigcup_{i=1}^{n}(X \setminus U_i).$$

Note that the set $\bigcup_{i=1}^n(X \setminus U_i)$ is a finite union of finite sets and, therefore, finite. So it statisfiy the T3 condition also. Thefore $\mathcal{T}_f$ is a topology. Furthur $\mathcal{T}_f$ is called the finite *complement topology.*

Example 1.6. Let X be a set. Define $\mathcal{T}$ to be the collection of all subsets U of X such that $X \setminus U$ either is finite or is all of X. Then $\mathcal{T}$ defines a topology on X, called finite complement topology of X.

1.2 Basis of a Topology

Once we define a structure on a set, often we try to understand what the minimum data you need to specify the structure. In many cases, this minimum data is called a basis and we say that the basis generate the structure. The notion of a basis of the structure will help us to describe examples more systematically.

Definition 1.2. Let X be a set. A basis of a topology on X is a collection $\mathcal{B}$ of subsets in X such that

(B1) For every $x \in X$, there exist an element B in $\mathcal{B}$ such that $x \in B$.

(B2) If $x \in B_1 \cap B_2$ where B_1, B_2 are in $\mathcal{B}$, then there is B_3 in $\mathcal{B}$ such that $x \in B_3 \subseteq B_1 \cap B_2$.

Lemma 1.1 (Generating of a topology)**.** *Let $\mathcal{B}$ be a basis of a topology on X. Define $\mathcal{T}_{\mathcal{B}}$ to be the collection of subsets $U \subset X$ satisfting*

(G1) For every $x \in U$, there is $B \in \mathcal{B}$ such that $x \in B \subset U$.

Then $\mathcal{T}_{\mathcal{B}}$ defines a topology on X. Here we assume that $\emptyset$ trivially satisfies the condition, so that $\emptyset \in \mathcal{T}_{\mathcal{B}}$.

Proof. We need to check the three axioms:

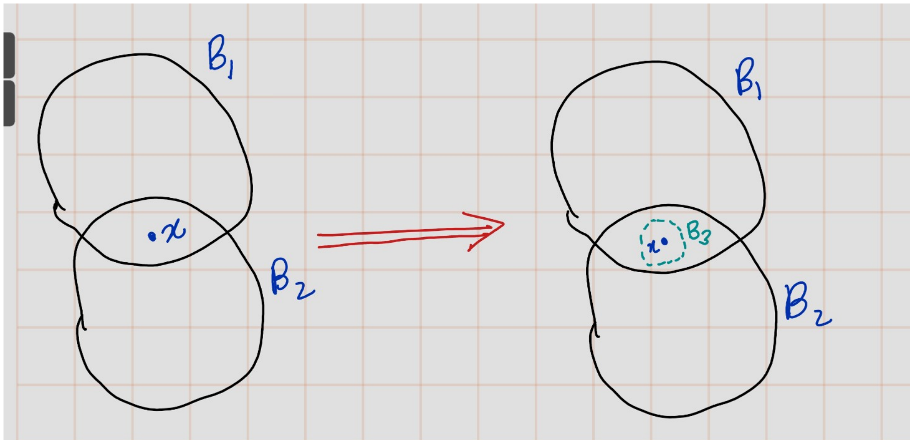


Figure 1.3:

- (T1) $\emptyset \in \mathcal{T}_{\mathcal{B}}$ as we assumed. $X \in \mathcal{T}_{\mathcal{B}}$ by (B1).
- (T2) Consider a collection of subsets $U_\alpha \in \mathcal{T}_{\mathcal{B}}, \alpha \in$ J. We need to show

$$U := \bigcup_{\alpha \in J} U_\alpha \quad \in \mathcal{T}_{\mathcal{B}}$$

By the definition of the union, for each $x \in U$, there is U_α such that $x \in U_\alpha$. Since $U_\alpha \in \mathcal{T}_{\mathcal{B}}$, there is $B \in \mathcal{B}$ such that $x \in B \subset U_\alpha$. Since $U_\alpha \subset U$, we found $B \in \mathcal{B}$ such that $x \in B \subset U$. Thus $U \in \mathcal{T}_{\mathcal{B}}$.

- (T3) Now consider a finite number of subsets $U_1, ..., U_n \in \mathcal{T}_{\mathcal{B}}$. We need to show that

$$U' := \bigcap_{i=1}^{n} U_i \quad \in \mathcal{T}_{\mathcal{B}}$$

- Let's just check for two subsets U_1, U_2 first. For each $x \in U_1 \cap U_2$, there are $B_1, B_2 \in \mathcal{B}$ such that $x \in B_1 \subset U_1$ and $x \in B_2 \subset U_2$. This is because $U_1, U_2 \in \mathcal{T}_{\mathcal{B}}$ and $x \in U_1, x \in U_2$. By (B2), there is $B_3 \in \mathcal{B}$ such that $x \in B_3 \subset B_1 \cap B_2$. Now we found $B_3 \in \mathcal{B}$ such that $x \in B_3 \subset U$.
- We can generalize the above proof to n subsets, but let's use induction to prove it. This is going to be the induction on the number of subsets.
  - When $n = 1$, the claim is trivial.
  - Suppose that the claim is true when we have $n-1$ subsets, i.e. $U_1 \cap \cdots \cap U_{n-1} \in \mathcal{T}_{\mathcal{B}}$. Since

$$U = U_1 \cap \cdots \cap U_n = (U_1 \cap \cdots \cap U_{n-1}) \cap U_n$$

and regarding $U' := U_1 \cap \cdots \cap U_{n-1}$, we have two subsets case $U = U' \cap U_n$. By the first arguments, $U \in \mathcal{T}_{\mathcal{B}}$.

$\square$

Definition 1.3. $\mathcal{T}_{\mathcal{B}}$ is called the **topology generated by a basis** $\mathcal{B}$. On the other hand, if $(X, \mathcal{T})$ is a topological space and $\mathcal{B}$ is a basis of a topology such that $\mathcal{T}_{\mathcal{B}} = \mathcal{T}$, then we say $\mathcal{B}$ is a basis of $\mathcal{T}$. Note that $\mathcal{T}$ itself is a basis of the topology $\mathcal{T}$. So there is always a basis for a given topology.

Example 1.7.

- (Standard Topology of $\mathbb{R}$) Let $\mathbb{R}$ be the set of all real numbers. Let $\mathcal{B}$ be the collection of all open intervals:

  $$(a, b) := \{x \in \mathbb{R} \mid a < x < b\}$$

  Then $\mathcal{B}$ is a basis of a topology and the topology generated by $\mathcal{B}$ is called the standard topology of $\mathbb{R}$.

- Let $\mathbb{R}^2$ be the set of all ordered pairs of real numbers, i.e. $\mathbb{R}^2 := \mathbb{R} \times \mathbb{R}$ (cartesian product). Let $\mathcal{B}$ be the collection of cartesian product of open intervals, $(a, b) \times (c, d)$. Then $\mathcal{B}$ is a basis of a topology and the topology generated by $\mathcal{B}$ is called the standard topology of $\mathbb{R}^2$.

- (Lower limit topology of $\mathbb{R}$) Consider the collection $\mathcal{B}$ of subsets in $\mathbb{R}$:

  $$\mathcal{B} := \{[a, b) := \{x \in \mathbb{R} \mid a \leq x < b\} \mid a, b \in \mathbb{R}\}$$

  This is a basis for a topology on $\mathbb{R}$. This topology is called the lower limit topology.

- (K-topology on $\mathbb{R}$) Finally let K denote the set of all numbers of the form $\frac{1}{n}$, for $n \in \mathbb{Z}^+$. In other words,

  $$K := \left\{\frac{1}{n} : n \in \mathbb{Z}^+\right\}$$

  Let $\mathcal{B}''$ be the collection of all open intervals (a, b), along with all sets of the form $(a, b) \setminus K$.

  $$\mathcal{B}'' := \{(a, b) : a, b \in \mathbb{R}, a < b\} \cup \{(a, b) \setminus K : a, b \in \mathbb{R}, a < b\}$$

  The topology generated by $\mathcal{B}''$ will be called the K-topology on $\mathbb{R}$. When $\mathbb{R}$ is given this topology, we denote it by $\mathbb{R}_K$.

Remark. Let $\mathcal{T}$ be a topology on X. If $\mathcal{B} \subset \mathcal{T}$ and $\mathcal{B}$ satisfies (B1) and (B2), it is easy to see that $\mathcal{T}_{\mathcal{B}} \subset \mathcal{T}$. This is just because of (G1). If $U \in \mathcal{T}_{\mathcal{B}}$, (G1) is satisfied for U so that $\forall x \in U, \exists B_x \in \mathcal{B}$ such that $x \in B_x \subset U$. Therefore $U = \cup_{x \in U} B_x$. By (T2), $U \in \mathcal{T}$.

Lemma 1.2. *Let $(X, \mathcal{T})$ be a topological space. Let $\mathcal{B} \subset \mathcal{T}$. Then $\mathcal{B}$ is a basis and $\mathcal{T}_{\mathcal{B}} = \mathcal{T}$ if and only if $\mathcal{T}$ is the set of all unions of elements in $\mathcal{B}$.*

Proof.

- ($\Rightarrow$) Let $\mathcal{T}'$ be the set of all unions of open sets in $\mathcal{B}$. If $U \in \mathcal{T}$, then U satisfies (G1), i.e. $\forall x \in U, \exists B_x \in \mathcal{B}$ s.t. $x \in B_x \subset U$. Thus $U = \cup_{x \in U} B_x$. Therefore $U \in \mathcal{T}'$. We proved $\mathcal{T} \subset \mathcal{T}'$. It follows from (T2) that $\mathcal{T}' \subset \mathcal{T}$.
- ($\Leftarrow$) Since $X \in \mathcal{T}, X = \cup_\alpha B_\alpha$ some union of sets in $\mathcal{B}$. Thus $\forall x \in X, \exists B_\alpha$ s.t. $x \in B_\alpha$. This proves (B1) for $\mathcal{B}$. If $B_1, B_2 \in \mathcal{B}$, then $B_1 \cap B_2 \in \mathcal{T}$ by (T2). Thus $B_1 \cap B_2 = \cup_\alpha B_\alpha, B_\alpha \in \mathcal{B}$. So $\forall x \in B_1 \cap B_2, \exists B_\alpha \in B$ s.t. $x \in B_\alpha$. This B_α plays the role of B_3 in (B2). Thus $\mathcal{B}$ is a basis. Now it makes sense to consider $\mathcal{T}_{\mathcal{B}}$ and we need to show $\mathcal{T}_{\mathcal{B}} = \mathcal{T}$. By the remark, we already know that $\mathcal{T}_{\mathcal{B}} \subset \mathcal{T}$. On the other hand, if $U \in \mathcal{T}$, then $U = \cup_\alpha B_\alpha, B_\alpha \in \mathcal{B}$. Hence, $\forall x \in U, \exists B_\alpha$ such that $x \in B_\alpha \subset U$. Thus (G1) is satisfied for U. Thus $U \in \mathcal{T}_{\mathcal{B}}$. This proves $\mathcal{T}_{\mathcal{B}} \supset \mathcal{T}$.

□

Lemma 1.3. *Let $(X, \mathcal{T})$ be a topological space. Let $\mathcal{B} \subset \mathcal{T}$. Then $\mathcal{B}$ is a basis and $\mathcal{T}_{\mathcal{B}} = \mathcal{T}$ if and if any $U \in \mathcal{T}$ satisfies (Gl), i.e. $\forall x \in U, \exists B_x \in \mathcal{B}$ s.t. $x \in B_x \subset U$.*

Proof.

- ($\Rightarrow$) Trivial by the definition of $\mathcal{T}_{\mathcal{B}}$.
- ($\Leftarrow X$) satisfies (G1) so $\mathcal{B}$ satisfies (B1). Let $B_1, B_2 \in \mathcal{B} \subset \mathcal{T}$. By (T3), $B_1 \cap B_2 \in \mathcal{T}$. Thus $B_1 \cap B_2$ satisfies (G1). This means (B2) holds for $\mathcal{B}$. Thus $\mathcal{B}$ is a basis. Now the assumption can be rephrased as $\mathcal{T} \subset \mathcal{T}_{\mathcal{B}}$. By the remark above, we already know $\mathcal{T} \supset \mathcal{T}_{\mathcal{B}}$.

□

1.3 Comparing Topologies

Definition 1.4. Let $\mathcal{T}, \mathcal{T}'$ be two topologies for a set X. We say $\mathcal{T}'$ is finer than $\mathcal{T}$ or $\mathcal{T}$ is coarser than $\mathcal{T}'$ if $\mathcal{T} \subset \mathcal{T}'$. The intuition for this notion is " $(X, \mathcal{T}')$ has more open subsets to separate two points in X than $(X, \mathcal{T})$ ".

Lemma 1.4. *Let $\mathcal{B}, \mathcal{B}'$ be bases of topologies $\mathcal{T}, \mathcal{T}'$ on X respectively. Then $\mathcal{T}'$ is finer than $\mathcal{T} \Leftrightarrow \forall B \in \mathcal{B}$ and $\forall x \in B, \exists B' \in \mathcal{B}'$ s.t. $x \in B' \subseteq B$.*

Proof.

- $\Rightarrow$ Since $\mathcal{B} \subset \mathcal{T} \subset \mathcal{T}'$, all subsets in $\mathcal{B}$ satisfies (G1) for $\mathcal{T}'$, which is exactly the statement we wanted to prove.

- $\Leftarrow$ The LHS says $\mathcal{B} \subset \mathcal{T}'$. We need to show that it implies that any $U \in \mathcal{T}$ satisfies (G1) for $\mathcal{T}'$ too.

$$\forall U \in \mathcal{T}, \forall x \in U, \exists B \in \mathcal{B} \text{ s.t. } x \in B \subset U$$

But

$$\forall B \in \mathcal{B}, \forall x \in B, \exists B' \in \mathcal{B}' \text{ s.t. } x \in B' \subset B.$$

Combining those two,

$$\forall U \in \mathcal{T}, \forall x \in U, \exists B' \in \mathcal{B}' \text{ s.t. } x \in B' \subset B \subset U.$$

$\square$

Definition 1.5 (subbasis)**.** Let X be a set.A subbasis $\mathcal{S}$ for a topology on X is a collection of subsets of X whose union equals X.

$$(\text{i.e. } \forall x \in X\ \exists S \in \mathcal{S} \text{ such that } x \in S)$$

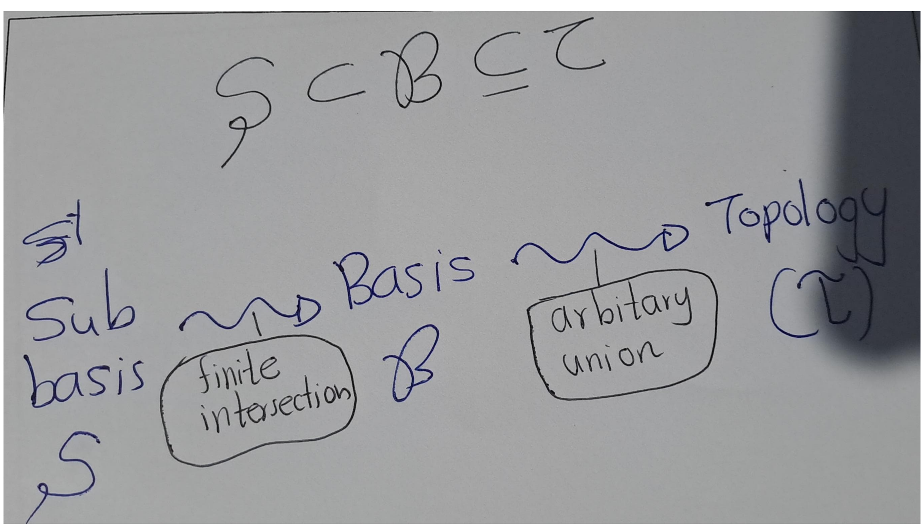


Figure 1.4:

Definition 1.6. The topology generated by the subbasis $\mathcal{S}$ is defined to be the collection $\mathcal{T}$ of all unions of finite intersections of elements of $\mathcal{S}$.

1.4 Order Topology

Definition 1.7 (Linear Order/ Complete Order)**.** Consider order relation "$<$".

1. If $x \neq y$, then either $x < y$ or $y < x$.
2. If $x < y$, then $x \neq y$.
3. If $x < y$ and $y < z$, then $x < z$.

Example 1.8. $\mathbb{R}$ is ordered set with less than relation.

First, let's see intervals in an Ordered Set.

Suppose that X is a set having a simple order relation $<$. Given elements a and b of X such that $a < b$, there are four subsets of X that are called the intervals determined by a and b. They are the following :

- $(a, b) = \{x \in X | a < x < b\}$ (Type: open interval in X),
- $(a, b] = \{x \in X | a < x \leq b\}$(Type: half-open interval in X),
- $[a, b) = \{x \in X | a \leq x < b\}$(Type: half-open interval in X),
- $[a, b] = \{x \in X | a \leq x \leq b\}$ (Type: closed interval in X),

The notation used here is familiar to you already in the case where X is the real line,but these are intervals in an arbitrary ordered set.

The use of the term "open" in this connection suggests that open intervals in X should turn out to be open sets when we put a topology on X. And so they will.

Definition 1.8. Let X be a set with a simple order relation; assume X has more than one element. Let $\mathcal{B}$ be the collection of all sets of the following types:

1. All open intervals (a, b) in X.
2. All intervals of the form $[a_0, b)$, where a_0 is the smallest element (if any) of X.
3. All intervals of the form $(a, b_0]$, where b_0 is the largest element (if any) of X.

i.e.:

$$\begin{aligned}\mathcal{B} :=& \{(a, b) : a < b, a, b \in X\} \\ & \bigcup \{[a, b : a < b, a_0, b \in X \text{ and if } X \text{ has a smallest element and } a_0 \text{ is the smallest element}\} \\ & \bigcup \{(a, b] : a < b, a, b_0 \in X \text{ and if } X \text{ has a largest element and } b_0 \text{ is the largest element}\}\end{aligned}$$

The collection $\mathcal{B}$ is a basis for a topology on X, which is called the order topology.

Notation: Denote an arbitrary element of $\mathbb{R} \times \mathbb{R}$ by $x \times y$, to avoid difficulty with notation.

Definition 1.9 (Dictionary Oreder)**.** Suppose that A and B are two sets with order relations $<_A$ and $<_B$ respectively. Define an order relation $<$ on $A \times B$ by defining $a_1 \times b_1 < a_2 \times b_2$ if $a_1 <_A a_2$, or if $a_1 = a_2$ and $b_1 <_B b_2$. It is called the dictionary order relation on $A \times B$.

Example 1.9. Consider the set $\mathbb{R} \times \mathbb{R}$ in the dictionary order. The set $\mathbb{R} \times \mathbb{R}$ has neither a largest nor a smallest element, so the order topology on $\mathbb{R} \times \mathbb{R}$ has as basis the collection of all open intervals of the form $(a \times b, c \times d)$ for $a < c$, and for $a = c$ and $b < d$.

These two types of intervals are indicated in Figure 14.1. The sub collection consisting of only intervals of the second type is also a basis for the order topology on $\mathbb{R} \times \mathbb{R}$, as you can check.

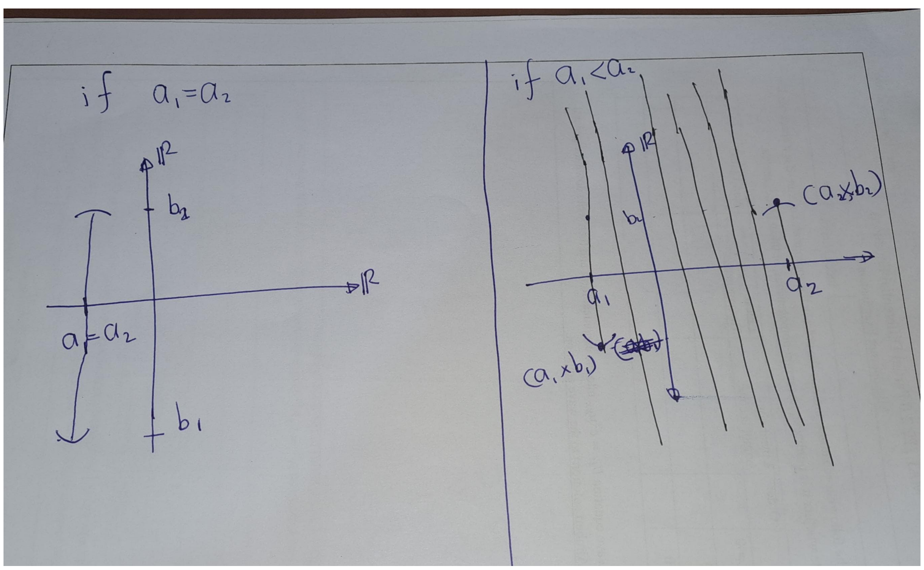


Figure 1.5:

Example 1.10. The standard topology,lower limit topology and upper limit topology on $\mathbb{R}$.

$$\mathcal{B} := \{(x, y) : x < y, x, y \in \mathbb{R}\}$$

$\mathcal{B}$ is a basis which generates the standard topology.

$$\mathcal{B}' := \{[x, y) : x < y, x, y \in \mathbb{R}\}$$

$\mathcal{B}'$ is a basis which generates the lower limit topology on $\mathbb{R}$.

$$\mathcal{B}'' := \{(x, y] : x < y, x, y \in \mathbb{R}\}$$

$\mathcal{B}''$ is a basis which generates the upper limit topology on $\mathbb{R}$.

Lemma 1.5. *The lower limit topology on $\mathbb{R}$ is strictly finer than standard topology on $\mathbb{R}$.*

Proof. Let $(a, b) \in \mathbb{R}$. We are going to use Lemma 1.4. Let (a, b) be a element from basis of standard topology. Let $x \in (a, b)$. Then $a < x < b$. Then observe that $x \in [x, b) \subset (a, b)$. Note that $[x, b)$ is element of basis of lower limit topology. Thus, by lemma 1.4 the lower limit topology on $\mathbb{R}$ is finer than standard topology on $\mathbb{R}$. Now we have to prove that **strictly property**.

Now let $[c, d)$ is element of basis of lower limit topology on $\mathbb{R}$. Now observe that there is no open interval that containing c and contained in $[c, d)$. By lemma 1.4, the lower limit topology on $\mathbb{R}$ is strictly finer than standard topology on $\mathbb{R}$. □

Note that basis element in lower limit topology is **not** open in the standard topology. But other way around is very true. As an example,

$$(1, 2) = \bigcup_{n \in \mathbb{N}} [1 + \frac{1}{n}, 2).$$

Clearly left hand side is open in lower limit topology by T2.

Corollary 1.1. *The lower limit topology on $\mathbb{R}$ is not comparable upper limit topology on $\mathbb{R}$.*

Proof. Exercise

Hint: (try to find counter example) □

Notation:

- $\mathbb{R}_l := \mathbb{R}$ with lower limit topology.
- $\mathbb{R}_u := \mathbb{R}$ with upper limit topology.
- $\mathbb{R} := \mathbb{R}$ with standard topology.

Example 1.11. The positive integers $\mathbb{Z}^+ := \{1, 2, 3, ...\}$ form an ordered set with a smallest element. The order topology on $\mathbb{Z}^+$ is the discrete topology, for every one-point set is open: If $n > 1$, then the one-point set $\{n\} = (n-1, n+1)$ is a basis element; and if $n = 1$, the one-point set $\{1\} = [1, 2)$ is a basis element.

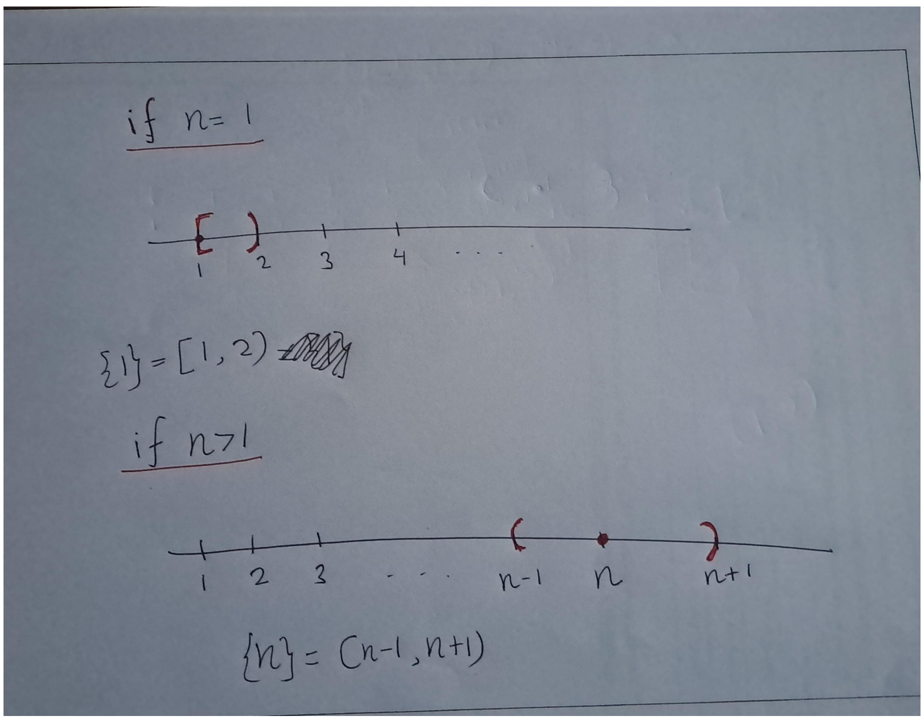


Figure 1.6:

Example 1.12. The set $X = \{1, 2\} \times \mathbb{Z}^+$ in the dictionary order is another example of an ordered set with a smallest element. Denoting $1 \times n$ by a_n and $2 \times n$ by b_n, we can represent X by

$$a_1, a_2, ..., b_1, b_2, ...$$

i.e.:

$$X = \{1 \times 1, 1 \times 2, 1 \times 3, ..., 2 \times 1, 2 \times 2, ...\}$$

Here $a_1 = 1 \times 1$ is the smallest element in X.

The order topology on X is **not** the discrete topology. Most one-point sets are open, but there is an exception-the one-point set $\{b_1\} = 2 \times 1$. Any open set containing b_1 must contain a basis element about b_1 (G1 condition), and any basis element containing b_1 contains points of the a_i sequence. As an example, $b_1 = 2 \times 1 \in (1 \times 7, 2 \times 8)$. Then the sequence $a_8, a_9, a_{10}, ... = 1 \times 8, 1 \times 9, 1 \times 10, ...$ contained in $(a_7, b_8) = (1 \times 7, 2 \times 8)$. So, we cannot find an open interval in X that contains b_1 and is contained in $\{b_1\}$.

Example 1.13.

$$\mathcal{B}''' := \{[x, y] : x \leq y, x, y \in \mathbb{R}\}$$

$\mathcal{B}'''$ is a basis which generates the discrete topology on $\mathbb{R}$. Because, $\{a\} = [a, a]$.

1.5 Product Toplogy on $X \times Y$.

The Cartesian product of two topological spaces has an induced topology called the product topology. There is also an induced basis for it. Here is the example to keep in mind:

Example 1.14. Recall that the standard topology of $\mathbb{R}^2$ is given by the basis

$$\mathcal{B} := \{(a, b) \times (c, d) \subset \mathbb{R}^2 \mid a < b, c < d\}$$

Proof. (Proof of $\mathcal{B}$ is basis.) - (B1) Let $(x, y) \in \mathbb{R}^2$. Then observe that $x \in (x - 1, x + 1) \subseteq \mathbb{R}$, and $y \in (y - 1, y + 1) \subseteq \mathbb{R}$. Thus

$$(x, y) \in (x - 1, x + 1) \times (y - 1, y + 1).$$

See figure 1.7. Therefore, this satisfied the B1 condition.

- Now suppose that $(x, y) \in (a_1, b_1) \times (c_1, d_1) \cap (a_2, b_2) \times (c_2, d_2)$ Now observe $(x, y) \in (a_1, b_1) \times (c_1, d_1) \implies x \in (a_1, b_1)$ and $y \in (c_1, d_1)$

and

$(x, y) \in (a_2, b_2) \times (c_2, d_2) \implies x \in (a_2, b_2)$ and $y \in (c_2, d_2)$ Let $a = \max\{a_1, a_2\}, b = \min\{b_1, b_2\}, c = \max\{c_1, c_2\}$ and $d = \min\{d_1, d_2\}$. Then observe that

$$x \in (a, b) \quad \text{and} y \in (c, d)$$

. Further,

$$(x, y) \in (a, b) \times (c, d) \subseteq (a_2, b_2) \times (c_2, d_2) \implies x \in (a_2, b_2)$$

. See figure 1.8. This satisfies B2 condition. □

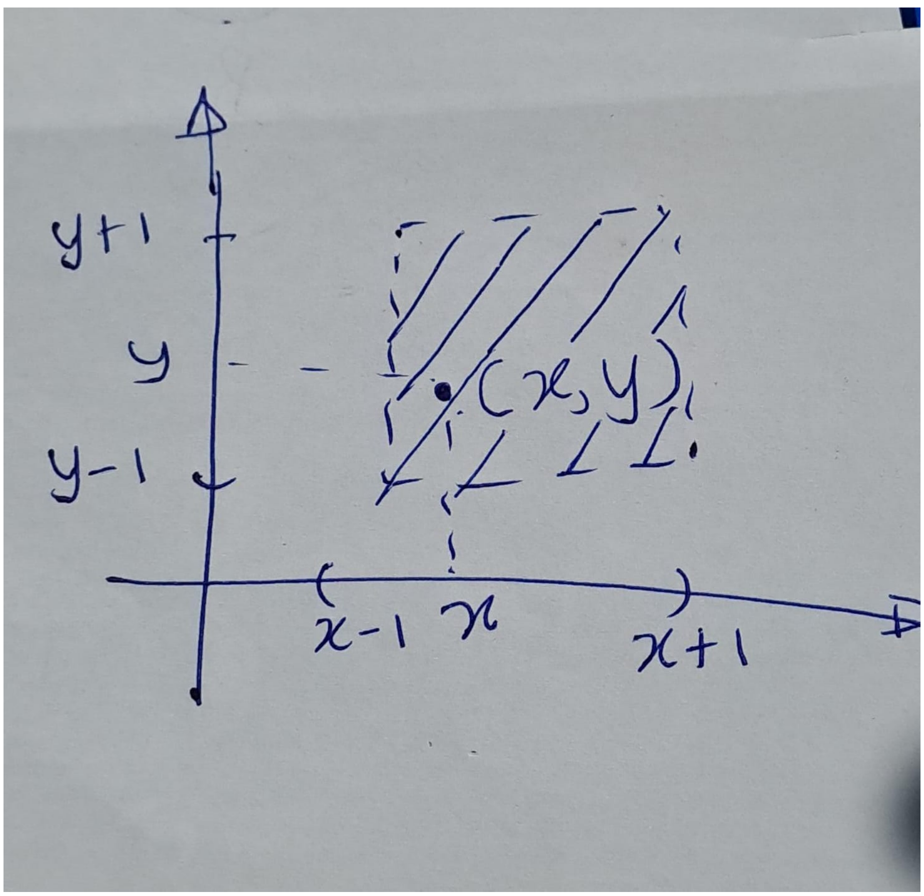


Figure 1.7:

Let's go to a rigid definition.

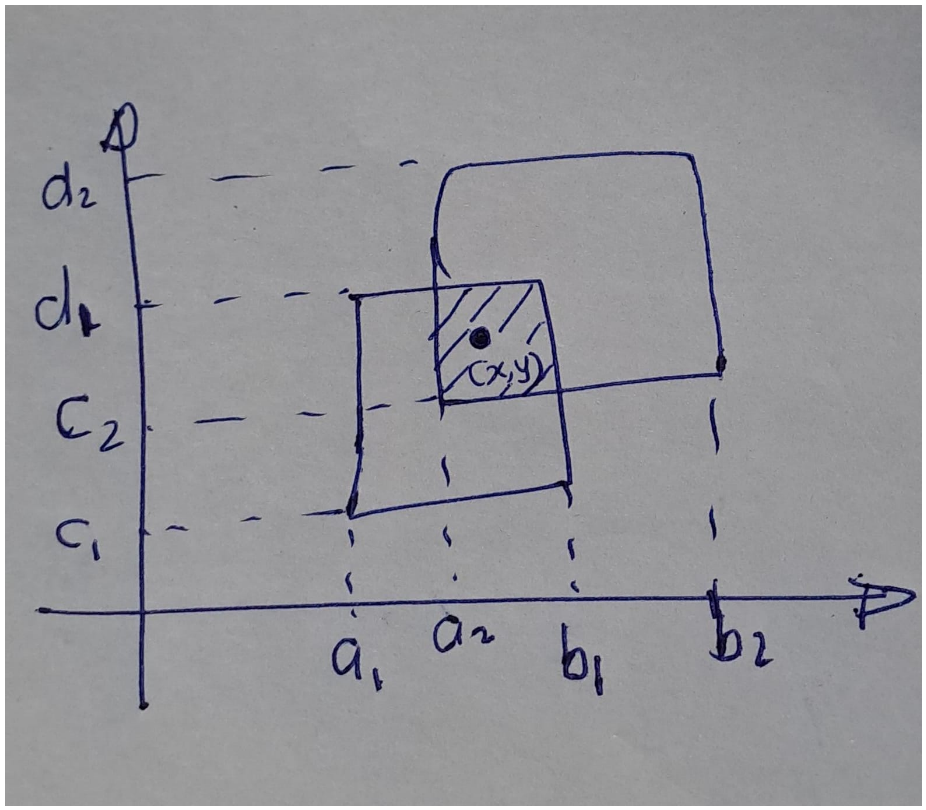


Figure 1.8:

Definition 1.10. If $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ are topological spaces, then the collection $\mathcal{B}$ of subsets of the form $U \times V \subset X \times Y, U \in \mathcal{T}_X, V \in \mathcal{T}_Y$ forms a basis of a topology.

i.e.:

$$\mathcal{B} := \{U \times V \subset X \times Y : U \in \mathcal{T}_X, V \in \mathcal{T}_Y\}$$

The topology generated by $\mathcal{B}$ is called product topology on $X \times Y$.

Proof. (Proof of $\mathcal{B}$ is basis)

- (B1) Let $(x, y) \in X \times Y$ be an arbitrary element. We need to find a subset in $\mathcal{B}$ containing (x, y), but since $X \times Y \in \mathcal{B}$, it is obvious.
- (B2) For any $U_1 \times V_1, U_2 \times V_2 \in \mathcal{B}$, the intersection is $(U_1 \times V_1) \cap (U_2 \times V_2) = (U_1 \cap U_2) \times (V_1 \cap V_2) \in \mathcal{B}$. So it is obvious again.

□

Theorem 1.1. *If $\mathcal{B}_X$ is a basis of $(X, \mathcal{T}_X)$ and $\mathcal{B}_Y$ is a basis of $(Y, \mathcal{T}_Y)$, then $\mathcal{B}_X \times \mathcal{B}_Y$ is a basis of the product topology on $X \times Y$.*

Proof. To check $\mathcal{B}_{X \times Y}$, let's use lemma 1.3 which states that $\mathcal{B}$ is a basis for $\mathcal{T}$ iff for any $U \in T$ and any $x \in U$, there is $B \in \mathcal{B}$ such that $x \in B \subseteq U$.

Let $W \in \mathcal{T}_{prod}$ and $(x, y) \in W$. By the definition of product topology, there are $U \in \mathcal{T}_X$ and $V \in \mathcal{T}_Y$ such that $(x, y) \in U \times V \subseteq W$. Since $\mathcal{B}_X$ and $\mathcal{B}_Y$ are bases, there are $B \in \mathcal{B}_X$ and $C \in \mathcal{B}_Y$ such that $x \in B \subseteq U$ and $y \in C \subseteq V$. Thus we found $B \times C \in \mathcal{B}_{X \times Y}$ such that $(x, y) \in B \times C \subseteq W$. □

Example 1.15. The standard topology on $\mathbb{R}^2 = \mathbb{R} \times \mathbb{R}$ is the product toplogy. (See example 1.14)

Obsereve that basis elements in product topology in $\mathbb{R}^2$ are open recatngles (product of two open intervals.).

Lemma 1.6. *The dictionary topology $\mathbb{R}^2$ is strictly finer than standard toplogy in $\mathbb{R}^2$*

Proof. We are going to use 1.4. Let $(a, b) \times (c, d) \subset \mathbb{R}^2$ be a element of basis of standard toplogy on $\mathbb{R}^2$, and let $x \times y \in (a, b) \times (c, d)$. Now we need to find basis element of the dictionary order topology that contained in $(a, b) \times (c, d)$. So,

$$x \times y \in (x \times c, x \times d) \subset (a, b) \times (c, d).$$

Note that $(x \times c, x \times d)$ is a basis element in the dictonary order topology. Now let's proove the stricly finer condition.

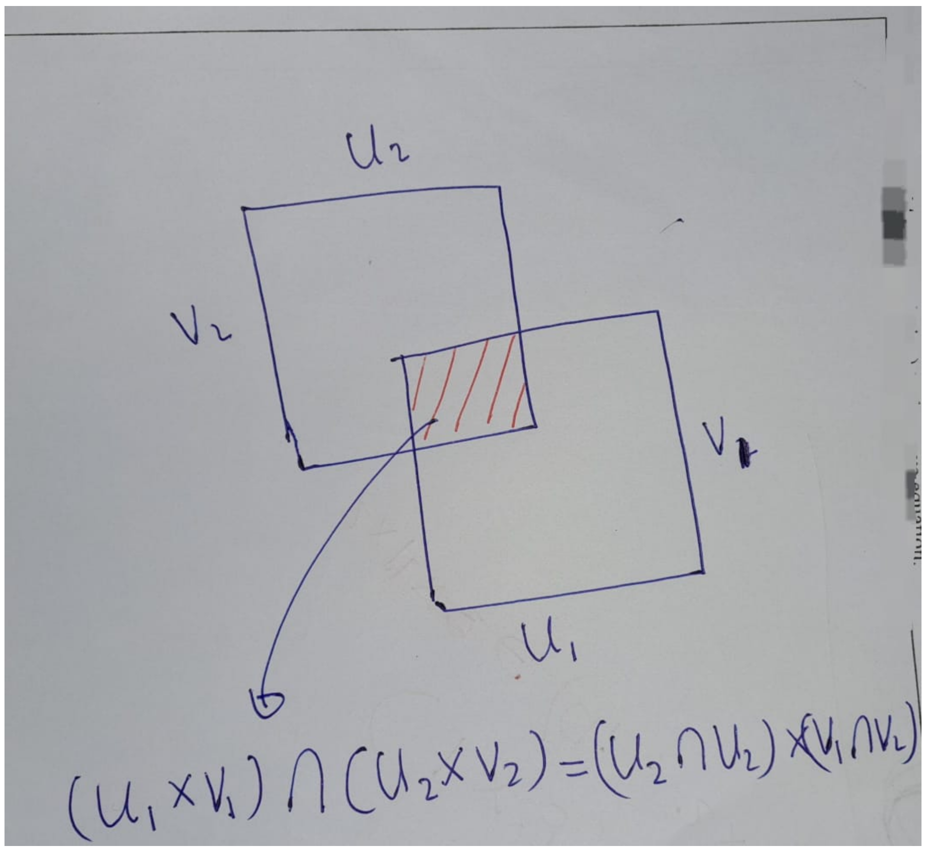


Figure 1.9:

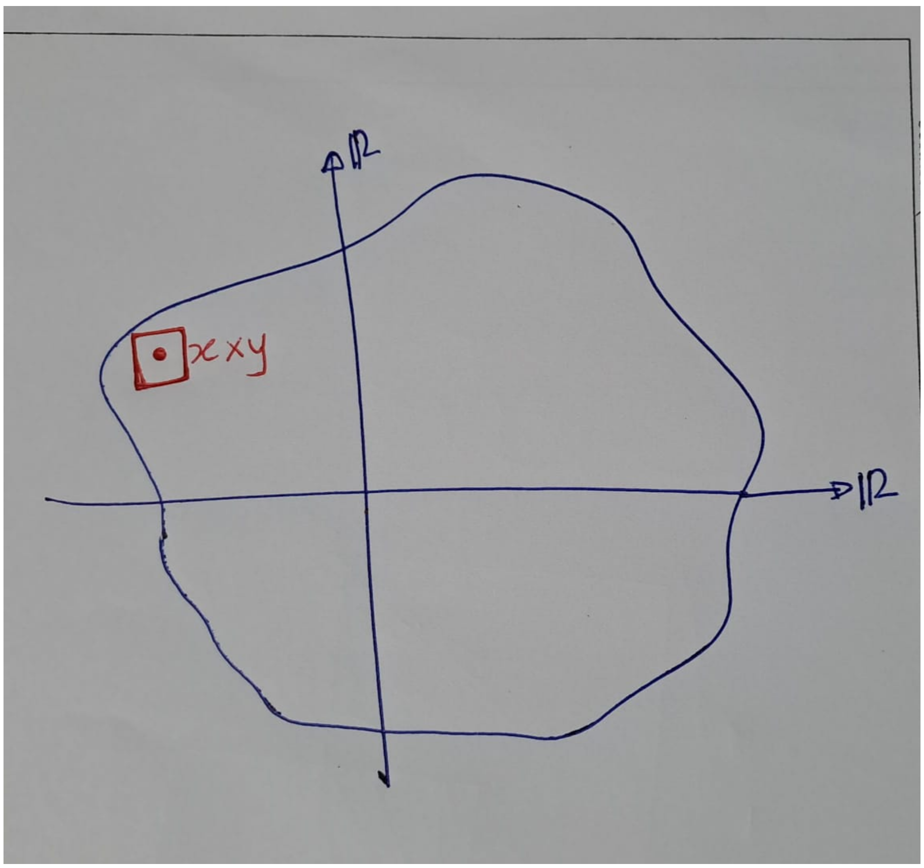


Figure 1.10:

Let $(p \times q, p \times s)$ be basis element of order toplogy. Let $p \times y \in (p \times q, p \times s)$. Now observe that there is no open rectangle that containing $p \times y$ and contained in $(p \times q, p \times s)$. By lemma 1.4, the order topology on $\mathbb{R}^2$ is strictly finer than standard topology on $\mathbb{R}^2$. □

Now I am intersected in a problem. That is can we write dictionary order topology as $\mathbb{R}^2$ as product topology. Actually we can,

$$\mathbb{R}^2_{dictionary} := \mathbb{R}_{discrete} \times \mathbb{R}_{standard}$$

Proof. Let $\{a\} \times (c, d)$ be a basis element in product toplogy $\mathbb{R}_d \times \mathbb{R}$. Let $a \times x \in \{a\} \times (c, d)$ obsereve that

$$a \times x \in \{a\} \times (c, d) = (a \times c, a \times d)$$

and $(a \times c, a \times d)$ is basis element of order topology $\mathbb{R}^2$. Thus by lemma 1.4, order toplogy in $\mathbb{R}^2$ is finer than the product toplogy $\mathbb{R}_d \times \mathbb{R}$.

Now suppose that $(p \times q, r \times s)$ be a basis elemenet in order toplogy on $\mathbb{R}^2$.

- If $p < x$, define $l = y - 1$ and if $p = x$ define $l = r$. In either case we know that $(p \times q) < (x \times l) < (x \times y)$.
- If $x < r$ define $t = y + 1$ and if $x = r$ define $t = s$. In either case we know that $(x \times y) < (x \times t) < (q \times s)$.

See figure 1.12 So

$$(x, y) \in \{x\} \times (l, t) \subseteq (p \times q, r \times s).$$

Thus by lemma 1.4, product toplogy $\mathbb{R}_d \times \mathbb{R}$ is finer than order toplogy in $\mathbb{R}^2$.

Therefore,

$$\mathbb{R}^2_{dictionary} = \mathbb{R}_{discrete} \times \mathbb{R}_{standard}$$

□

Definition 1.11. Let $\pi_1 : X \times Y \to X$ be defined by the equation $\pi_1(x, y) = x$; and let $\pi_2 : X \times Y \to Y$ be defined by the equation $\pi_2(x, y) = y$.

The maps π_1 and π_2 are called the *projections* of $X \times Y$ onto its first and second factors, respectively.

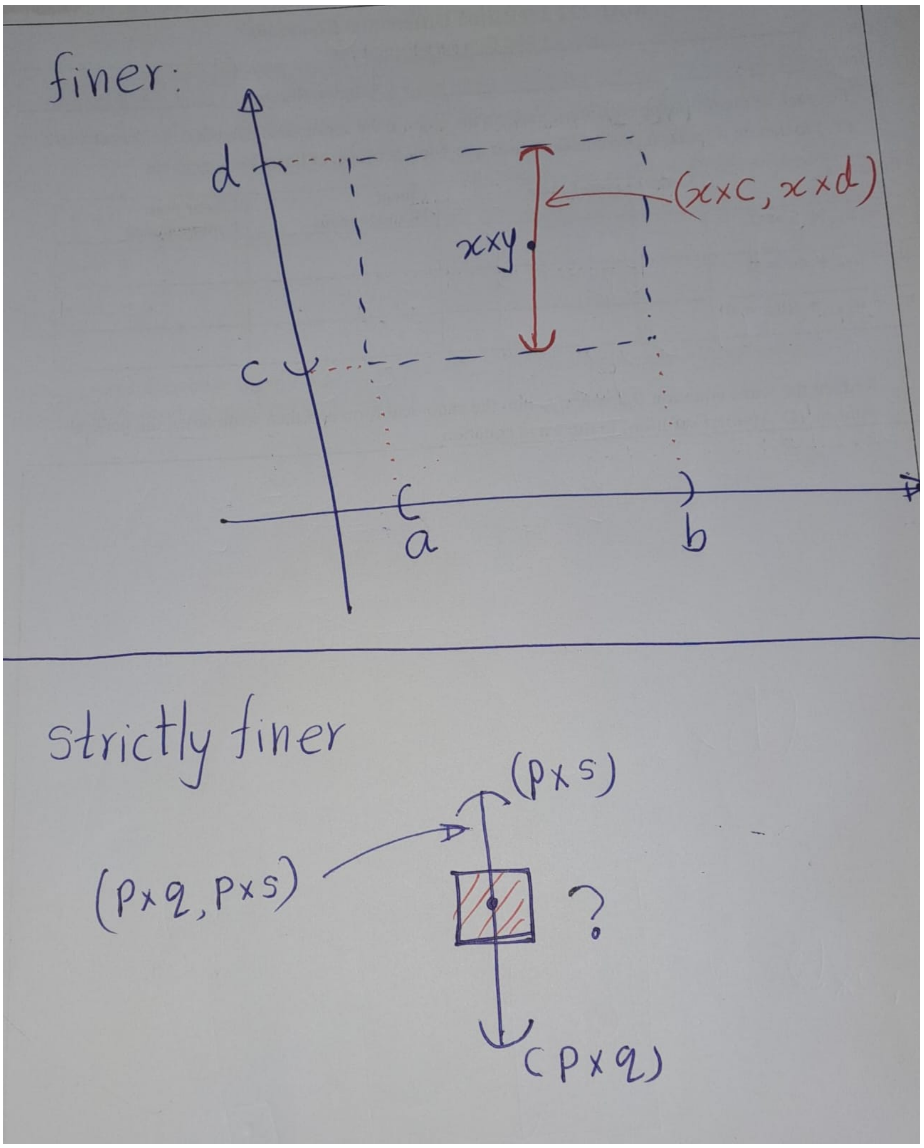


Figure 1.11:

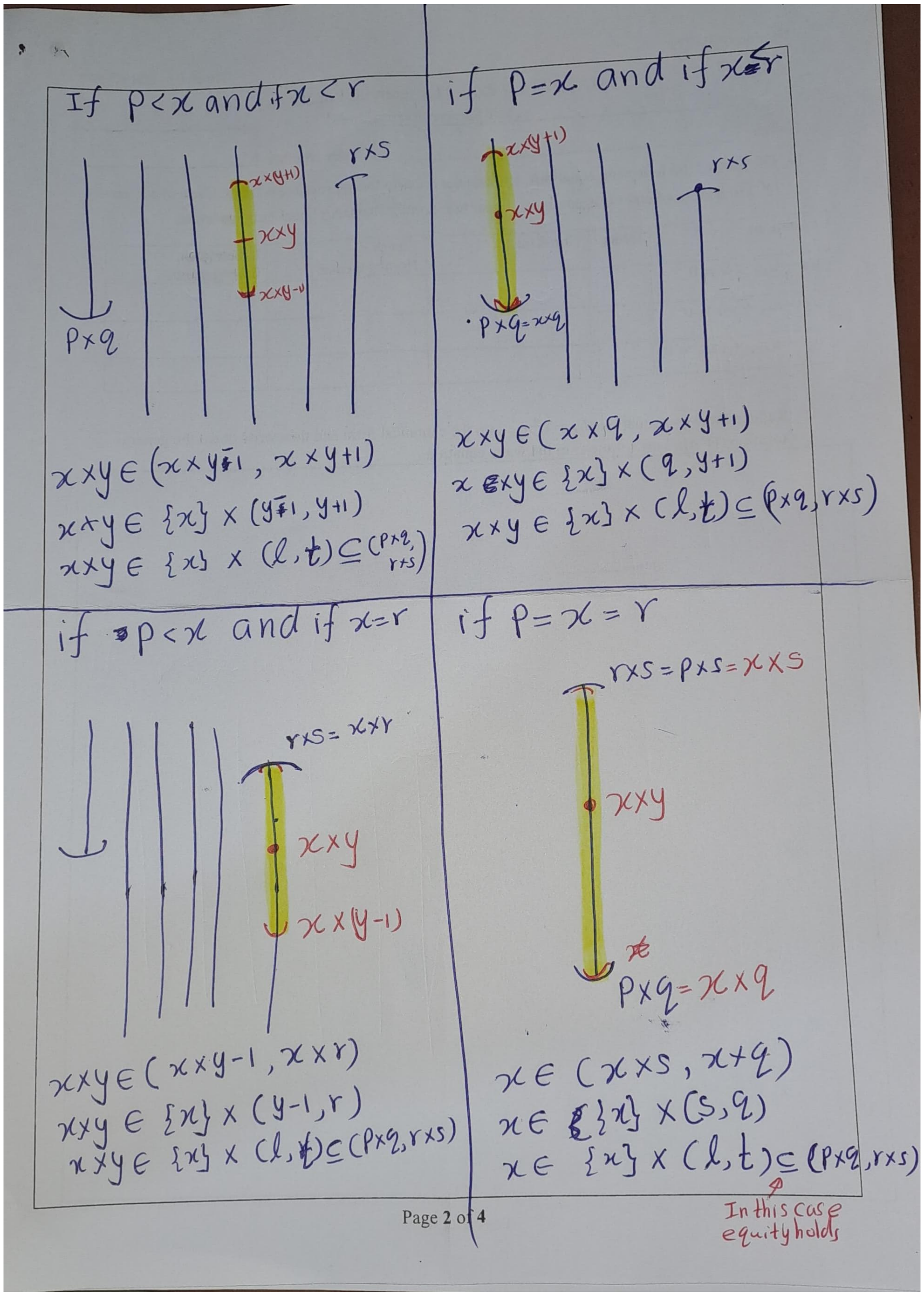

If p<x and if x<r
r×s
x×(y+1)
x×y
x×(y-1)
p×q
x×y ∈ (x×y-1 , x×y+1)
x×y ∈ {x} × (y-1 , y+1)
x×y ∈ {x} × (l , t) ⊆ (p×q, r×s)
if p=x and if x<r
x×(y+1)
x×y
r×s
p×q=x×q
x×y ∈ (x×q , x×y+1)
x×y ∈ {x} × (q , y+1)
x×y ∈ {x} × (l , t) ⊆ (p×q , r×s)
if p<x and if x=r
r×s = x×r
x×y
x×(y-1)
x×y ∈ (x×y-1 , x×r)
x×y ∈ {x} × (y-1 , r)
x×y ∈ {x} × (l , t) ⊆ (p×q , r×s)
if p=x=r
r×s = p×s = x×s
x×y
p×q = x×q
x ∈ (x×s , x×q)
x ∈ {x} × (s , q)
x ∈ {x} × (l , t) ⊆ (p×q , r×s)
In this case equity holds
Page 2 of 4

Figure 1.12:

1.6 The Subspace Topology

A subset of a topological space has a naturally induced topology, called the subspace topology. In geometry, the subspace topology is the source of all funky typologies.

Definition 1.12. Let $(X, \mathcal{T})$ be a topological space. Let $Y \subseteq X$. The collection

$$\mathcal{T}_Y = \{Y \cap U | U \in \mathcal{T}\}$$

is a topology on Y, called the subspace topology.

Proof. (Proof of the collection $\mathcal{T}_Y$ is a tology).

- (T1) This is very easy to see. $\mathcal{T}_Y$ contains $\emptyset$ and Y because $\emptyset = Y \cap \emptyset and Y = Y \cap X$, where $\emptyset$ and X are elements of $\mathcal{T}$.
- (T2) Let $\{U_\alpha \cap Y \in \mathcal{T}_y : \alpha \in I,\ \text{I is index set}\}$ be collection of open sets in subspace toplogy of Y, where $U_\alpha \in \mathcal{T}$.
$$\bigcup_{\alpha \in I} (U_\alpha \cap Y) = \bigcup_{\alpha \in I} (U_\alpha) \cap Y.$$
Thus it contains in $\mathcal{T}$. Thus $\mathcal{T}_Y$ is closed under arbitary unions.
- (T3) Let $U_1 \cap Y, U_2 \cap Y, ..., U_n \cap Y$ be finite collection of open sets in subspace toplogy of Y in X.
$$(U_1 \cap Y) \bigcap (U_2 \cap Y) \bigcap \cdots \bigcap (U_n \cap Y) = (U_1 \cap U_2 \cap \cdot U_n) \bigcap Y.$$
Thus it contains in $\mathcal{T}$.Thus $\mathcal{T}_Y$ is closed under finite intersections.

□

Lemma 1.7. *Let $(X, \mathcal{T})$ be a topological space, and $Y \subseteq X$. If $\mathcal{B}$ is a basis for $\mathcal{T}$ then the collection*

$$\mathcal{B}_Y = \{B \cap Y | B \in \mathcal{B}\}$$

is a basis for the subspace topology on Y.

Proof. Let $V \in \mathcal{T}_Y$. Then $V = U \cap Y$ for some $U \in \mathcal{T}$. For every $x \in V$, there is $B \in \mathcal{B}$ such that $x \in B \subset U$ since $\mathcal{B}$ is a basis of $\mathcal{T}$ (Lemma 1.2). Now we found $Y \cap B$ such that $x \in Y \cap B \subset V$. □

Example 1.16. Let $I = [0, 1]$. The dictionary order on $I \times I$ is just the restriction to $I \times I$ of the dictionary order on the plane $\mathbb{R} \times \mathbb{R}$. However, the dictionary order topology on $I \times I$ is **NOT** the same as the subspace topology on $I \times I$ obtained from the dictionary order topology on $\mathbb{R} \times \mathbb{R}$!

For example, the set $\{\frac{1}{2}\} \times (\frac{1}{2}, 1]$ is open in $I \times I$ in the subspace topology, but not in the order topology, as you can check.

See Figure 1.14.

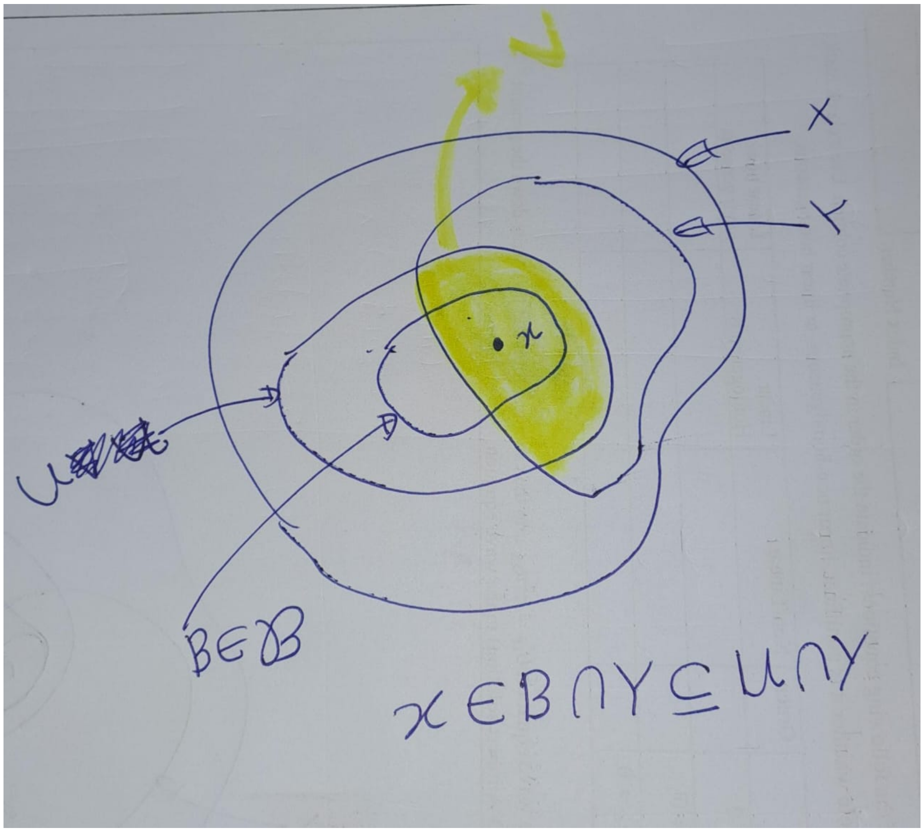


Figure 1.13:

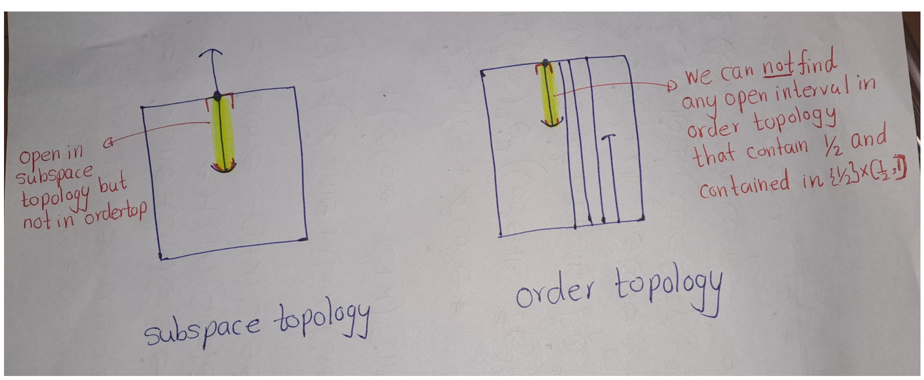


Figure 1.14:

Notation : The set $I \times I$ in the dictionary order topology will be called the ordered square, and denoted by I_o^2.

Let's generalized the idea.

Lemma 1.8. *The subspace topology on $I \times I$ obtained from the dictionary order topology on $\mathbb{R} \times \mathbb{R}$ is strictly finer than the dictionary order toplogy on $I \times I$.*

Proof. So, as pervious we have to proove two things they are finer condtion and strictly condition.

Let $(a_1 \times b_1, a_2 \times b_2)$ be a basis elemenet of order toplogy. and $x \times y \in (a_1 \times b_1, a_2 \times b_2)$

- **Case I** $(a_1 < x < a_2)$:

  $x \times y \in (x \times -1, x \times 2) \cap I^2 = [x \times 0, x \times 1] = \{x\} \times [0, 1] \subset (a_1 \times b_1, a_2 \times b_2)$

  Note that $(x \times -1, x \times 2) \cap I^2$ is a basis element of subspace topology.

- **Case II** $(a_1 = x < a_2)$:

  $x \times y \in (x \times b_1, x \times 2) \cap I^2 = [x \times b_1, x \times 1] = \{x\} \times (b_1, 1] \subset (a_1 \times b_1, a_2 \times b_2)$

  Note that $(x \times b_1, x \times 2) \cap I^2$ is a basis element of subspace topology.

- **Case III** $(a_1 < x = a_2)$:

  $x \times y \in (x \times -1, x \times b_2) \cap I^2 = [x \times 0, x \times b_2] = \{x\} \times [0, b_2) \subset (a_1 \times b_1, a_2 \times b_2)$

  Note that $(x \times -1, x \times b_2) \cap I^2$ is a basis element of subspace topology.

- **Case IV** $(a_1 = x = a_2)$:

  $x \times y \in (x \times b_1, x \times b_2) \cap I^2 = [x \times b_1, x \times b_2] = \{x\} \times [b_1, b_2) \subset (a_1 \times b_1, a_2 \times b_2)$

  Note that $(x \times b_1, x \times b_2) \cap I^2$ is a basis element of subspace topology.

See figure 1.15

In above all four cases, we have found basis element of subspace topology that conatain $x \times y$ and conatined in $(a_1 \times b_1, a_2 \times b_2)$. □

1.7 Closed Sets and Limit Points

Now that we have a few examples at hand, we can introduce some of the basic concepts associated with topological spaces. In this section, we treat the notions of closed set closure of a set, and limit point. These lead naturally to consideration of a certain axiom for topological spaces called the Hausdorff axiom.

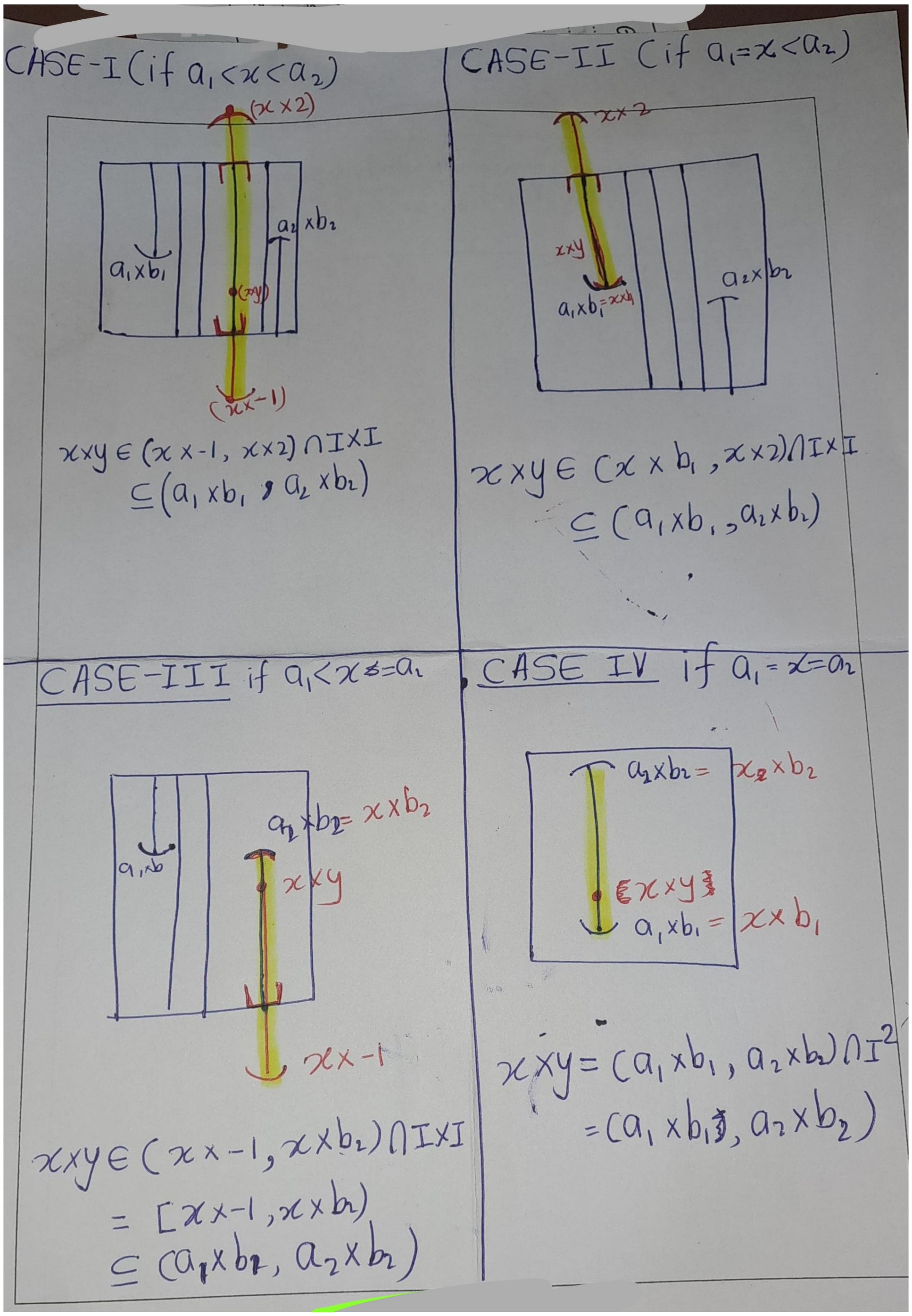

CASE-I (if a₁<x<a₂)
CASE-II (if a₁=x<a₂)
(x×2)
a₂×b₂
a₁×b₁
(x×y)
(x×-1)
x×y ∈ (x×-1, x×2) ∩ I×I
⊆ (a₁×b₁, a₂×b₂)
x×2
x×y
a₁×b₁=x×y
a₂×b₂
x×y ∈ (x×b₁, x×2) ∩ I×I
⊆ (a₁×b₁, a₂×b₂)
CASE-III if a₁<x=a₂
CASE IV if a₁=x=a₂
a₂×b₂= x×b₂
x×y
x×-1
x×y ∈ (x×-1, x×b₂) ∩ I×I
= [x×-1, x×b₂)
⊆ (a₁×b₁, a₂×b₂)
a₂×b₂= x×b₂
{x×y}
a₁×b₁ = x×b₁
x×y = (a₁×b₁, a₂×b₂) ∩ I²
= (a₁×b₁, a₂×b₂)

Figure 1.15:

1.7.1 Closed Set

Closed sets are nothing but complement of open sets. On the other hand, we can also say that open sets are nothing but complement of closed sets. Thus we can actually use closed sets to define topology, although mathematicians usually use open sets to define topology.

Definition 1.13. Let A be a subset of a topological space $(X,)$.

A is a closed set of X if $X \setminus A$ is an open set.

- The closure $\mathrm{Cl}(A) = \overline{A}$ of A in X is the intersection of all closed sets of X, containing A.

$$\mathrm{Cl}(A) = \overline{A} = \bigcap \{C \text{ is closed in } X \ \& \ A \subset C\}$$

  The smallest closed set that containing$A$ is the closure of A.
- The interior $\mathrm{Int}(A) = \mathring{A}$ of A in X is the union of all open sets of X, contained in A.

$$\mathrm{Int}(A) = \mathring{A} = \bigcup \{U \text{ is open in } X \ \& \ U \subset A\}$$

  The largest open set contanied in A is the interior of A.
- $x \in X$ is a limit point (or "cluster point," or "point of accumulation") of A, if every neighborhood of x intersects A in some point other than x itself. In other words, $x \in X$ is a limit point of A, if $x \in \overline{A \setminus \{x\}}$.

Remark.

- $\emptyset, X$ are closed.

I am not going to proove these things. I just give an idea. (This is trivial.) - Fnite unions of closed sets are closed.

$$\bigcup_{i=1}^{n} \underbrace{(X \setminus \underbrace{U_i}_{\text{open}})}_{\text{closed}} = X \setminus \underbrace{\bigcap_{i=1}^{n} U_i}_{\text{open}}$$

- Arbitrary intersection of closed sets are closed.

$$\bigcap_{\alpha \in I} \underbrace{(X \setminus \underbrace{U_\alpha}_{\text{open}})}_{\text{closed}} = X \setminus \underbrace{\bigcup_{\alpha \in I} U_i}_{\text{open}}$$

Example 1.17. The subset $[a, b]$ of $\mathbb{R}$ is closed because its complement

$$\mathbb{R} \setminus [a, b] = (-\infty, a) \cup (b, +\infty)$$

, is open.

Remark. These facts justify our use of the terms "closed interval" and "closed ray." The subset $[a, b)$ of $\mathbb{R}$ is neither open nor closed.

Example 1.18. Similary,the subset $[a, +\infty)$ of $\mathbb{R}$ is closed because its complement

$$\mathbb{R} \setminus [a, +\infty) = (-\infty, a)$$

, is open.

Example 1.19. In the plane $\mathbb{R}^2$, the set

$$\{x \times y | x \geq 0 \text{ and } y \geq 0\}$$

is closed, because its complement is the union of the two sets

$$(-\infty, 0) \times \mathbb{R} \text{ and } \mathbb{R} \times (-\infty, 0),$$

each of which is a product of open sets of $\mathbb{R}$ and is, therefore, open in $\mathbb{R}^2$.

Example 1.20. In the finite complement topology on a set X, the closed sets consist of X itself and all finite subsets of X.

Example 1.21. In the discrete topology on the set X, every set is open; it follows that every set is closed as well.

Example 1.22. Consider the following subset of the real line:

$$Y = [0, 1] \cup (2, 3)$$

, in the subspace topology. In this space,

The set $[0, 1]$ is open, since it is the intersection of the open set $(-\frac{1}{2}, \frac{3}{2})$ of $\mathbb{R}$ with Y . Similarly, $(2, 3)$ is open as a subset of Y ; it is even open as a subset of $\mathbb{R}$. Since $[0, 1]$ and $(2, 3)$ are complements in Y of each other, we conclude that both $[0, 1]$ and $(2, 3)$ are closed as subsets of Y.

Fun Fact: These examples suggest that an answer to the mathematician's riddle: "How is a set different from a door?" should be: "A door must be either open or closed, and cannot be both, while a set can be open, or closed, or both, or neither!

The collection of closed subsets of a space X has properties similar to those satisfied by the collection of open subsets of X;

Theorem 1.2. *Let X be a topological space. Then the following conditions hold:*

(1) $\emptyset$ and X are closed.
(2) Arbitrary intersections of closed sets are closed.
(3) Finite unions of closed sets are closed.

Proof.

1) $\emptyset$ and X are closed because they are the complements of the open sets X and $\emptyset$, respectively.
2) Given a collection of closed sets $\{A_\alpha\}_{\alpha\in J}$, we apply DeMorgan's law,

$$X \setminus \bigcup_{\alpha\in J} A_\alpha = \bigcap_{\alpha\in J} (X \setminus A_\alpha)$$

. Since the sets $X \setminus A_\alpha$ are open by definition, the right side of this equation represents an arbitrary union of open sets, and is thus open. Therefore, $\bigcup A_\alpha$ is closed.
3) Similarly, if A_i is closed for $i = 1, \dots, n$, consider the equation

$$X \setminus \bigcap_{i=1}^{n} A_i = \bigcup_{i=1}^{n} (X \setminus A_i)$$

. The set on the right side of this equation is a finite intersection of open sets. Hence $\bigcup A_i$ is closed.

□

Instead of using open sets, one could just as well specify a topology on a space by giving a collection of sets (to be called "closed sets") satisfying the three properties of this theorem. One could then define open sets as the complements of closed sets and proceed just as before. This procedure has no particular advantage over the one we have adopted, and most mathematicians prefer to use open sets to define topologies.

Now when dealing with subspaces, one needs to be careful in using the term "closed set." If Y is a subspace of X, we say that a set A is closed in Y if A is a subset of Y and if A is closed in the subspace topology of Y (that is, if $Y \setminus A$ is open in Y). We have the following theorem:

Theorem 1.3. *Let X be a toplogical space and Y is a subspace of X, and A is a subset of Y , then A is closed $\iff$ $A = Y \cap C$ forsome C is closed in X.*

Proof.

- ($\Longleftarrow$) Suppose that $A = Y \cap C$ for some closed set C in X. (See Figure 1.16 Thus, $X \setminus C$ is open in X. So, $(X \setminus C) \cap Y$ is open in Y, by definition of the subspace topology. But,

$$(X \setminus C) \cap Y = Y \setminus (C \cap Y) = Y \setminus A.$$

. Hence $Y \setminus A$ is open in Y.Therfore, A is closed in Y.

- ($\Longrightarrow$) Now Suppose that A is closed in Y. (See Figure 1.17 Then, $Y \setminus A$ is open in Y. Thus by definition, $Y \setminus A = U \cap Y$, forsome open set U in X. So, $X \setminus U$ is closed in X. Now consider,

$$Y \cap (X \setminus U) = Y \setminus (Y \cap U) = Y \setminus (Y \setminus U) = A.$$

Thefore, $Y \cap C$ forsome C is closed in X.

□

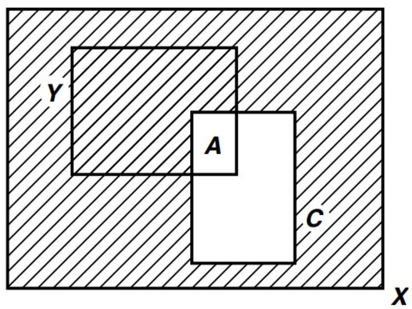


Figure 1.16:

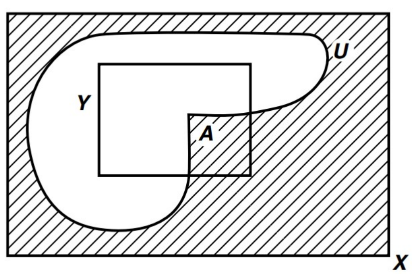


Figure 1.17:

Remark. A set A that is closed in the subspace Y may or may not be closed in the larger space X.

Theorem 1.4. *Let Y be a subspace of X. If A is closed in Y and Y is closed in X, then A is closed in X.*

Proof. Since Y is subspace in X and A is closed in Y, by theorm 1.3, then there exist closed set C in X such that

$$A = Y \cap C$$

Since, Y is closed in X and X is toplogical space, $A = Y \cap C$ is closed in X. □

1.7.2 Closure and Interior of a Set

Definition 1.14. Let A be a subset of a topological space $(X,)$.

- The closure $\text{Cl}(A) = \overline{A}$ of A in X is the intersection of all closed sets of X, containing A.

$$\text{Cl}(A) = \overline{A} = \bigcap \{C \text{ is closed in } X \text{ \& } A \subset C\}$$

  The smallest closed set that containing$A$ is the closure of A.
- The interior $\text{Int}(A) = \mathring{A}$ of A in X is the union of all open sets of X, contained in A.

$$\text{Int}(A) = \mathring{A} = \bigcup \{U \text{ is open in } X \text{ \& } U \subset A\}$$

  The largest open set contanied in A is the interior of A.

Obviously $\text{Int}(A)$ is an open set and $\overline{A}$ is a closed set; furthermore,

$$\text{Int}(A) \subseteq A \subseteq \overline{A}$$

Lemma 1.9. *Let Y be a subspace of X and $A \subset Y$. Let $\bar{A}$ denote the closure of A in X. Then the closure of A in Y equals $\bar{A} \cap Y$.*

Proof. Let B denote the closure of A in Y.

- **Claim**: $B \subseteq \bar{A} \cap Y$

The set $\bar{A}$ is closed in X, so $\bar{A} \cap Y$ is closed in Y (By lemma 1.3). Since $A \subset \bar{A} \cap Y$(Since, $\bar{A}$ is the smallest subset that conatin A, and $A \subseteq Y$), and since by definition B equals the intersection of all closed subsets of Y containing A, we must have $B \subseteq (\bar{A} \cap Y)$.

- **Case-II**: $\bar{A} \cap Y \subseteq B$

we know that B is closed in Y. Hence by lemma 1.3, $B = C \cap Y$ for some set C closed in X. Then C is a closed set of X containing A; because $\bar{A}$ is the intersection of all such closed sets, we conclude that $\bar{A} \subseteq C$. Then $(\bar{A} \cap Y) \subset (C \cap Y) = B$. □

Terminology : We shorten the statement to "U is an open set containing x" to the phrase "U **is a neighbourhood of** x.

Proposition 1.1. *Let A be a subset of the topological space X.*

a) Then $x \in \bar{A}$ if and only if every neighborhood U of x intersects A. (has non-empty intersection with A.)

b) Let $\mathcal{B}$ be a basis for X , then $x \in \bar{A}$ if and only if $B \in \mathcal{B}$ which containing x intersects A.

Proof.

- Statemet a)
  - ($\Longrightarrow$): We are going to use indirect proof. Assume that $x \in \bar{A}$. Then $U = X \backslash \bar{A}$ is neighbohood of x which does not intersect A ($\bar{A} \cap U = \emptyset$). Since, $A \subset \bar{A}$, then $A \cup U = \emptyset$.
  - ($\Longleftarrow$): Let U be a neighborhood of x which does not intersect A. So, $X \setminus U$ is closed and $A \subset X \setminus U$, then $\bar{A} \subset X \setminus U$. (because $\bar{A} is the smallest closed set that containing A.So, $x \in \bar{A}$).
- Statemet b)
  - ($\Longleftarrow$): If every open set containing x intersects A, so does every basis element B containing x, because B is an open set.
  - ($\Longrightarrow$) if every basis element containing x intersects A, so does every open set U containing x, because U contains a basis element that contains x.

$\square$

Example 1.23. Let X be the real line $\mathbb{R}$.

- If $A = (0, 1]$, then $\overline{A} = [0, 1]$, for every neighbourhood of 0 intersects A, while every point outside $[0, 1]$ has a neighbourhood disjoint from A.

Similar arguments apply to the following subsets of X,

- If $B = \{\frac{1}{n} | n \in \mathbb{Z}^+\}$, then $\overline{B} = \{0\} \cup B$.
- If $C = \{0\} \cup (1, 2)$, then $\overline{C} = \{0\} \cup [1, 2]$.
- If $\mathbb{Q}$ is the set of rational numbers, then $\overline{\mathbb{Q}} = \mathbb{R}$.
- If $\mathbb{Z}^+$ is the set of positive integers, then $\overline{\mathbb{Z}^+} = \mathbb{Z}^+$.
- If $\mathbb{R}^+$ is the set of positive reals, then the closure of $\mathbb{R}^+$ is the set $\mathbb{R}^+ \cup \{0\}$.

Example 1.24. Consider the subspace $Y = (0, 1]$ of the real line $\mathbb{R}$. The set $A = \left(0, \frac{1}{2}\right)$ is a subset of Y; its closure in $\mathbb{R}$ is the set $[0, \frac{1}{2}]$, and its closure in Y is the set $[0, \frac{1}{2}] \cap Y = \left(0, \frac{1}{2}\right]$.

1.7.3 Limit Points

There is yet another way of describing the closure of a set, a way that involves the important concept of limit point, which we consider now.

Definition 1.15 (Limit Point)**.** If A is a subset of the topological space X and if x is a point of X, we say that x is a limit point (or "cluster point," or "point of accumulation") of A if every neighborhood of x intersects A in some point other than x itself.

Said differently, x is a limit point of A if it belongs to the closure of $A \setminus \{x\}$. The point x may lie in A or not; for this definition it does not matter.

Notation: let A' be the set of all limit points of A.

Example 1.25. Consider the real line $\mathbb{R}$.

- If $A = (0, 1]$, then the point 0 is a limit point of A and so is the point $\frac{1}{2}$. In fact, every point of the interval $[0, 1]$ is a limit point of A, but no other point of $\mathbb{R}$ is a limit point of A.
- If $B = \{\frac{1}{n}|n \in \mathbb{Z}^+\}$, then 0 is the only limit point of B. Every other point x of $\mathbb{R}$ has a neighborhood that either does not intersect B at all, or it intersects B only in the point x itself.
- If $C = \{0\} \cup (1, 2)$, then the limit points of C are the points of the interval $[1, 2]$.
- If $\mathbb{Q}$ is the set of rational numbers, every point of $\mathbb{R}$ is a limit point of $\mathbb{Q}$.
- If $\mathbb{Z}^+$ is the set of positive integers, no point of $\mathbb{R}$ is a limit point of $\mathbb{Z}^+$.
- If $\mathbb{R}^+$ is the set of positive reals, then every point of $\{0\} \cup \mathbb{R}^+$ is a limit point of $\mathbb{R}^+$.

Comparison of Examples 1.23 and 1.25 suggests a relationship between the closure of a set and the limit points of a set. That relationship is given in the following theorem:

Theorem 1.5. *Let A be a subset of the topological space X; let A' be the set of all limit points of A. Then*

$$\bar{A} = A \cup A'$$

.

Proof.

- **Claim 1**: $A' \subseteq \bar{A}$
  Supoose that $x \in A'$. Every neighborhood of x intersects A (in a point different from x). Therefore, by propostion 1.1, x belongs to $\overline{A}$. Hence $A' \subseteq \overline{A}$.
- **Claim 2**: $A \cup A' \subseteq \overline{A}$.
  Since by definition $A \subseteq \overline{A}$, it follows that $A \cup A' \subseteq \overline{A}$.
- **Claim 3**:$A \cup A' \supseteq \overline{A}$.
  Let $x \in \overline{A}$.
  - Case I: If $x \in A$ then noting to proove.
  - Case II: If $x \notin A$. Since $x \in \overline{A}$, we know that every neighborhood U of x intersects A; because $x \notin A$, the set U must intersect A in a point different from x. Then $x \in A'$, so that $x \in A \cup A^{\circ}$, as desired.

□

Corollary 1.2. *A subset of a topological space is closed if and only if it contains all its limit points.*

Proof. The set A is closed if and only if $A = \overline{A}$, and the latter holds if and only if $A' \subset A$. □

1.7.4 Boundary Points

Definition 1.16. If $A \subset X$, we define the boundary of A by the equation

$$\operatorname{Bd} A = \overline{A} \cap \overline{(X - A)}.$$

1.7.5 Hausdorff Spaces

Definition 1.17. A topological space $(X, \mathcal{T})$ is called a Hausdorff space if (H1) $\forall x, y \in X$ such that $x \neq y$, $\exists U_x, U_y \in \mathcal{T}$ such that $x \in U_x$, $y \in U_y$, and $U_x \cap U_y = \emptyset$.

i.e., for every pair of distinct points x, y in X, there are disjoint neighborhoods U_x and U_y of x and y respectively.

We called Hausdorff condition as T_2 axiom. So Let's see another definition

Definition 1.18. A space X is a *T1 space* if and only if it satisfies the following condition: For each $x, y \in X$ such that $x \neq y$, there exists an open set $U \subset X$ such that $x \in U$ but $y \notin U$.

These T_1 and T_2 axioms are called the separation axioms.

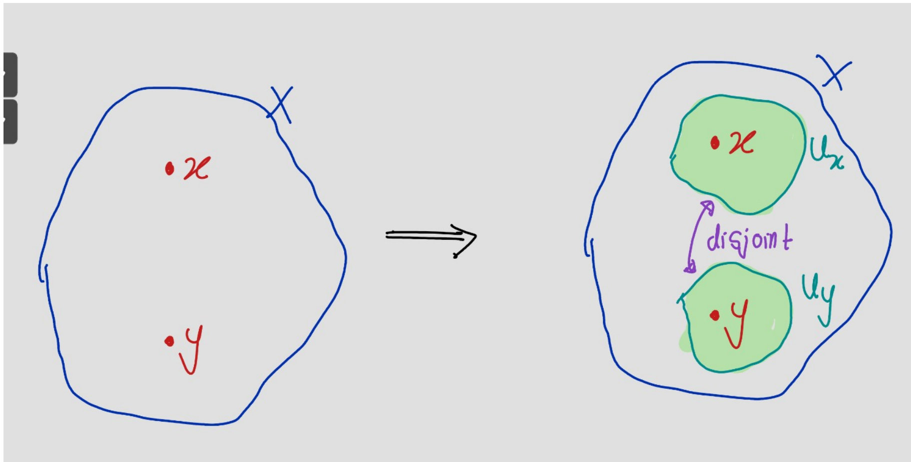


Figure 1.18:

Figure 1.19:

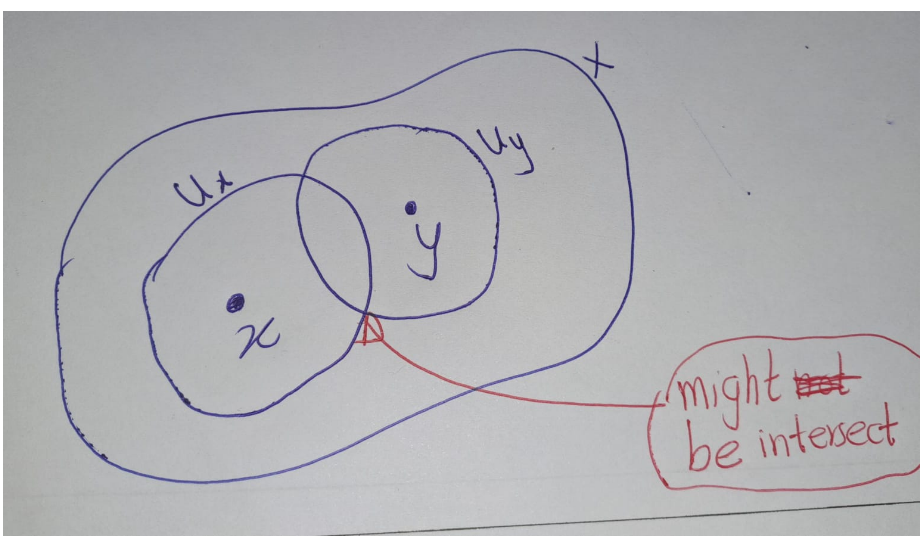


Figure 1.20:

Example 1.26. Let X b a set. Consider X with finite complemenet topology is T_1 Space. (i.e.:$(X, \mathcal{T}_f)$ is T_1 space.) But X is not T_2 if X is infinite.

Claim 1 : $(X, \mathcal{T}_f)$ is T_1 space.

Proof. Let $x, y \in X$. Then

- $X \setminus \{x\}$ is an open set of $\mathcal{T}_f$ containing y but not x.
- $X \setminus \{y\}$ is an open set of $\mathcal{T}_f$ containing x but not y. Hence the result from definition of T_1 space.

□

Claim 2 : X is not T_2 if X is infinite.

Proof. Let $x_1, x_2 \in X$ with $x_1 \neq x_2$ Suppose x_1, x_2 have disjoint neighbhood U_1 and U_2. I.e.: $U_1 \cap U_2 = \emptyset$. Thus

$$X = X \setminus (U_1 \cap U_2) \tag{1.1}$$
$$= \underbrace{(X \setminus U_1)}_{\text{finite}} \cup \underbrace{(X \setminus U_2)}_{\text{finite}} \tag{1.2}$$

Note that both $(X \setminus U_1)$ and $(X \setminus U_2)$ are finite. Thus, X is finite. Hence, if X is T_2 then X is finite. (Contrapostive of given statement). Therefore, if X is infinite then X is not T_2. □

Theorem 1.6. *Every finite point set in a Hausdorff space X is closed.*

Proof. It suffices to show that every one-point set $\{x_0\}$ is closed. If x is a point of X different from x_0, then x and x_0 have disjoint neighborhoods U and V, respectively. Since U does not intersect $\{x_0\}$, the point x cannot belong to the closure of the set $\{x_0\}$. As a result, the closure of the set $\{x_0\}$ is $\{x_0\}$ itself, so that it is closed. □

Definition 1.19. Let X be topological sapce. A sequence $x_1, x_2, x_3, ...$ in X **converges** to $x \in X$ if for every neighborhood U of x there exists $N \in \mathbb{N}$ such that $x_n \in U$ for all $n \geq N$

Notataion: $x_n \to x$

Theorem 1.7. *Let X be a Hausdorff space. If x_n convergence the limit x is unique.*
In other words: The limit of a convergent sequence in a Hausdorff space is unique.

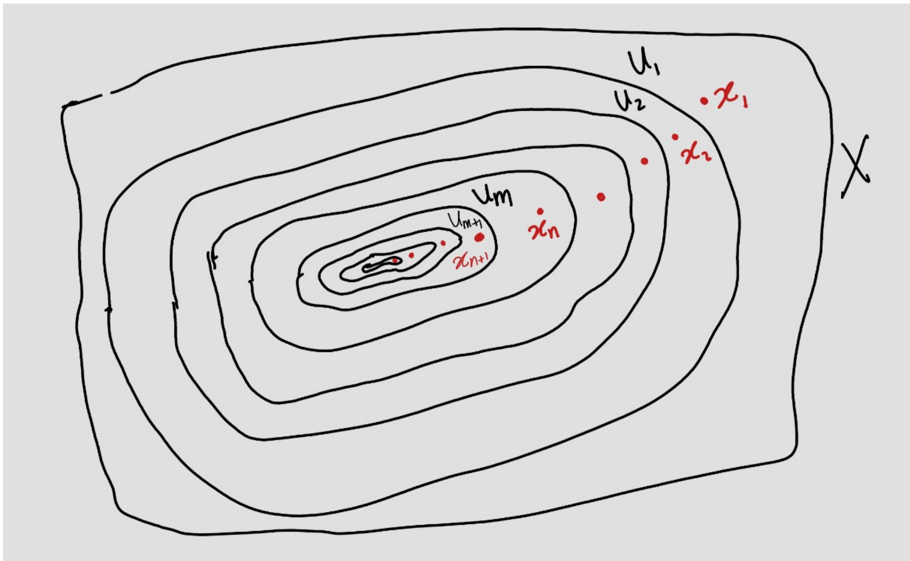


Figure 1.21:

Proof. We are going to indirect proof. Assume that $x_n \to x$ and $x_n \to y$ and $x \neq y$. Since X is a Hausdorff space, there are disjoint open sets U and V of X such that $x \in U$ and \$ y V\$.

- Since $x_n \to x$, U is an open neighbourhood of x, there is a $N_0 \in \mathbb{N}$, such that $x_n \in U$ for all $n \geq N_0$
- Since $x_n \to y$, V is an open neighbourhood of y, there is a $N_1 \in \mathbb{N}$, such that $x_n \in V$ for all $n \geq N_1$

Let $N = max\{N_0, N_1\}$. Since $N > N_0$, then $x_N \in U$. Since $N > N_1$, then $x_N \in V$. Thus, $x_N \in U \cap V$. This contardict disjointness the u and V. Thefore, $x = y$ □

Theorem 1.8. *Let X be a Hausdorff space and $A \subseteq X$. Then,*
$x \in X$ is a limit point $\iff$ every neighbourhood intersect A is infinetly many points.

Proof.

- ($\Longleftarrow$) First suppose that every neighborhood of x intersects A in infinitely many points, it certainly intersects A in some point other than x itself, so that x is a limit point of A.

- ($\Longrightarrow$) Now suppose that x is a limit point of A. We are going to use proof by contradiction. Assume that there is some neighborhood U of x intersects A in only finitely many points. Then U also intersects $A - \{x\}$ in finitely many points. Let $\{x_1, ..., x_m\}$ be the points of $U \cap (A - \{x\})$. i.e.:
$$U \cap (A - \{x\}) = \{x_1, ..., x_m\}$$
The set $X - \{x_1, ..., x_m\}$ is an open set of X, since the finite point set $\{x_1, ..., x_m\}$ is closed. (By pervious theorem) Then
$$U \cap (X - \{x_1, ..., x_m\})$$
is a neighborhood of x that intersects the set $A - \{x\}$ not at all. This contradicts the assumption that x is a limit point of A.

□

Theorem 1.9. *If X is a Hausdorff space, then a sequence of points of X converges to at most one point of X.*

Proof. Suppose that x_n is a sequence of points of X that converges to x. If $y \neq x$, let U and V be disjoint neighbourhoods of x and y, respectively. Since U contains x_n for all but finitely many values of n. (It contain all terms with out some finetely many terms) Since $U \cap V = \emptyset$, the set V cannot caontain those terms. Therefore, x_n cannot converge to y. □

Definition 1.20 (Limit of a sequence)**.** If the sequence x_n of points of the Hausdorff space X converges to the point x of X, we often write $x_n \to x$, and we say that x is the **limit** of the sequence x_n.

Theorem 1.10.

- *Every simply ordered set is a Hausdorff space in the order topology.*
- *The product of two Hausdorff spaces is a Hausdorff space.*
- *A subspace of a Hausdorff space is a Hausdorff space.*

Proof. See exercise 4.27,4.28 and 4.29 □

1.8 Continuous Functions

1.8.1 Continuity of a Function

Definition 1.21. Let X and Y be topological spaces. A function $f : X \to Y$ is said to be continuous if for each open subset V of Y , the set $f^{-1}(V)$ is an open subset of X.

Recall that,

$$f^{-1}(V) = \{x \in X : f(x) \in V\}$$

Remark.

- If it is empty if V does not intersect the image set $f(X)$ of f .
- Continuity of a function depends *not* only upon the function f itself, but also on the topologies specified for its domain and range.
  I like to emphasize this fact, we can say that f is continuous *relative* to specific topologies on X and Y.

Chapter 2

Chapter 2 name

Chapter 3

Chapter 03 name

Up to there is none.

3.1 Post After verify

a. **Interior of a set is open in X:** The interior of a set S, denoted by $\text{int}(S)$, is defined as the union of all open sets contained in S. Since the union of open sets is open, $\text{int}(S)$ is open in X.

b. **If T is open in X then $T \subseteq S$ if and only if $T \subseteq \textbf{int}(S)$:** If T is an open set and $T \subseteq S$, then every point in T is an interior point of S. Therefore, $T \subseteq \text{int}(S)$. Conversely, if $T \subseteq \text{int}(S)$, then T is a subset of S.

c. **Interior of S is an open subset of S when S is given the subspace topology:** The interior of a set S, $\text{int}(S)$, is the union of all open sets contained in S. Therefore, $\text{int}(S)$ is an open subset of S.

d. **S is an open subset of X if and only if $\textbf{int}(S) = S$:** If S is open, then every point of S is an interior point of S, so $\text{int}(S) = S$. Conversely, if $\text{int}(S) = S$, then S is open because the interior of a set is always open.

e. **Interior operator preserves/distributes over binary intersection:** The interior of the intersection of two sets S and T, $\text{int}(S \cap T)$, is equal to the intersection of the interiors of S and T, $\text{int}(S) \cap \text{int}(T)$. This is because the intersection of two open sets is open, and the interior of a set is the union of all open sets contained in it.

f. **The interior operator does not distribute over unions:** The interior of the union of two sets S and T, $\text{int}(S \cup T)$, is a superset of the union of the interiors of S and T, $\text{int}(S) \cup \text{int}(T)$. This is because the union of two open sets is open, and the interior of a set is the union of all open sets contained in it. However, the equality might not hold in general. For example, if $X = \mathbb{R}$, $S = (-\infty, 0]$, and $T = (0, \infty)$, then $\mathrm{int}(S) \cup \mathrm{int}(T) = (-\infty, 0) \cup (0, \infty) = \mathbb{R} \setminus \{0\}$, which is a proper subset of $\mathrm{int}(S \cup T) = \mathrm{int}(\mathbb{R}) = \mathbb{R}$.

Chapter 4

Exercises

4.1 Section 13 in Munkrees book

Exercise 4.1 (Mun 2.13.1)**.** Let X be a topological space; let A be a subset of X. Suppose that for each $x \in A$ there is an open set U containing x such that $U \subset A$. Show that A is open in X.

Proof. Let X be a topological space. Let A be a subset of X. Suppose that for each $x \in A$. Then U_x be the open set that,

$$x \in U_x \subseteq A$$

Now consider,

$$U := \bigcup_{x \in A} U_x.$$

Note that U is open. By defintion of toplogy. Furthur, $A = U$. Hence, A is open set. □

Exercise 4.2 (Mun 2.13.3)**.** Consider the nine topologies on the set X = {a, b, c} indicated in Example 1 of §12. Compare them; that is, for each pair of topologies, determine whether they are comparable, and if so, which is the finer.

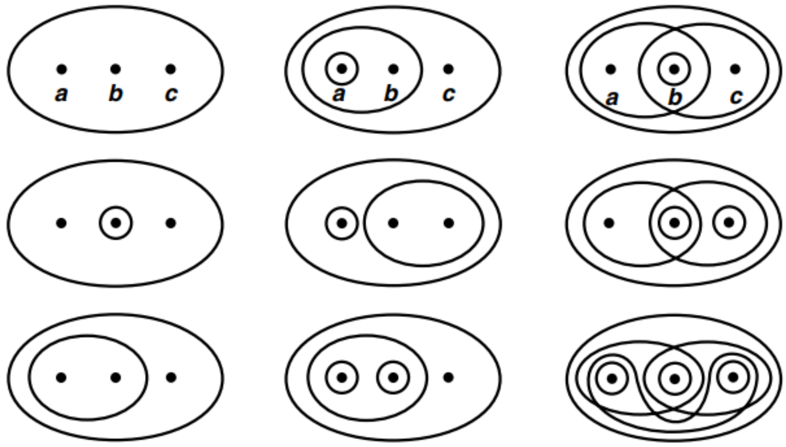


Figure 4.1:

These topologies can be listed as follows,

$$
\begin{aligned}
&\mathcal{T}_1 = \{\emptyset, X\},\\
&\mathcal{T}_2 = \{\emptyset, \{a\}, \{a,b\}, X\},\\
&\mathcal{T}_3 = \{\emptyset, \{b\}, \{a,b\}, \{b,c\}, X\},\\
&\mathcal{T}_4 = \{\emptyset, \{b\}, X\},\\
&\mathcal{T}_5 = \{\emptyset, \{a\}, \{b,c\}, X\},\\
&\mathcal{T}_6 = \{\emptyset, \{b\}, \{c\}, \{a,b\}, \{b,c\}, X\},\\
&\mathcal{T}_7 = \{\emptyset, \{a,b\}, X\},\\
&\mathcal{T}_8 = \{\emptyset, \{a\}, \{b\}, \{a,b\}, X\},\\
&\mathcal{T}_9 = \{\emptyset, \{a\}, \{b\}, \{c\}, \{a,b\}, \{a,c\}, \{b,c\}, X\}.
\end{aligned}
$$

We can get following observations. - $\mathcal{T}_1$ is coarser than any other topology, - $\mathcal{T}_9$ is finer than any other topology. - $\mathcal{T}_7 \subset \mathcal{T}_2 \subset \mathcal{T}_8$ - $\mathcal{T}_4 \subset \mathcal{T}_3 \subset \mathcal{T}_6$ - $\mathcal{T}_7 \subset \mathcal{T}_3 \subset \mathcal{T}_6$ - $\mathcal{T}_4 \subset \mathcal{T}_8$.

Exercise 4.3. Show that the collection $\mathcal{T}_c$ given in Example 4 of §12 (in Munkrees book) is a topology on the set X. Is the collection

$$\mathcal{T}_\infty = \{U | X - U \text{ is infinite or empty or all of } X\}$$

a topology on X?

Proof. (Proof of $\mathcal{T}_c$ is toplogy.) Recall: $\mathcal{T}_c$ is the collection of all subsets U of X such that $X \setminus U$ is either countable or is all of X.

- $\emptyset \in \mathcal{T}_c$. (Beacuse $X \setminus \emptyset = X$)
  $X \in \mathcal{T}_c$. (Beacuse $X \setminus X = \emptyset$ is countable)
  So, $\emptyset$ and X are both in $\mathcal{T}_c$.
- Let $\{U_\alpha\}_{\alpha \in J}$ be a family of sets in $\mathcal{T}_c$. Then

$$X \setminus \bigcup_{\alpha \in J} U_\alpha = \bigcap_{\alpha \in J} (X \setminus U_\alpha) \quad \text{(By De Moragons Law)}$$

  is an intersection of countable sets, hence countable.
- If $U_1, \dots, U_n$ are elements in $\mathcal{T}_c$, then

$$X \setminus \bigcap_{i=1}^{n} U_i = \bigcup_{i=1}^{n} (X \setminus U_i) quad \text{ (By De Moragons Law)}$$

  is countable being a union of countable sets. It follows that $\mathcal{T}_c$ is a topology on X.

$\square$

Note that $\mathcal{T}_\infty$ is in general not a topology on X.

- For example, let $X = \mathbb{R}$,
  $U_1 = (-\infty, 0)$ and $U_2 = (0, \infty)$.
  Then U_1 and U_2 are in $\mathcal{T}_\infty$ but $U_1 \cup U_2 = \mathbb{R} \setminus \{0\}$ is not.

Exercise 4.4 (Mun 2.13.4)**.**

(a) If $\{\mathcal{T}_\alpha\}$ is a family of topologies on X, show that $\bigcap \mathcal{T}_\alpha$ is a topology on X. Is $\bigcup \mathcal{T}_\alpha$ a topology on X?
(b) Let $\{\mathcal{T}_\alpha\}$ be a family of topologies on X. Show that there is a unique smallest topology on X containing all the collections $\mathcal{T}_\alpha$, and a unique largest topology contained in all $\mathcal{T}_\alpha$.
(c) If $X = \{a, b, c\}$, let

$$\mathcal{T}_1 = \{\emptyset, X, \{a\}, \{a, b\}\} \text{ and}$$
$$\mathcal{T}_2 = \{\emptyset, X, \{a\}, \{b, c\}\}.$$

Find the smallest topology containing $\mathcal{T}_1$ and $\mathcal{T}_2$, and the largest topology contained in $\mathcal{T}_1$ and $\mathcal{T}_2$.

Solution:

a)

Suppose that $\{\tau_\alpha\}_{\alpha\in I}$ is a family of Topologies.
Here I is the index set.

* Then $\phi \in \tau_\alpha$ for all $\alpha \in I \Rightarrow \phi \in \bigcap_{\alpha\in I} \tau_\alpha$
$X \in \tau_\alpha$ for all $\alpha \in I \Rightarrow X \in \bigcap_{\alpha \in I} \tau_\alpha$

* Let $\{V_\beta\}_{\beta\in J}$ be a collection of open sets $\bigcap_{\alpha\in I} \tau_\alpha$.
Then $\bigcap_{\beta\in J} V_\beta \in \tau_\alpha$ for any fixed $\alpha \in I$. (Because τ_α is a Topology on X. Thus $\bigcup_{\beta \in J} V_\beta \in \bigcap_{\alpha\in I} \tau_\alpha$.

* Let $V_1, V_2, \ldots, V_n \in \bigcap_{\alpha\in I} \tau_\alpha$.

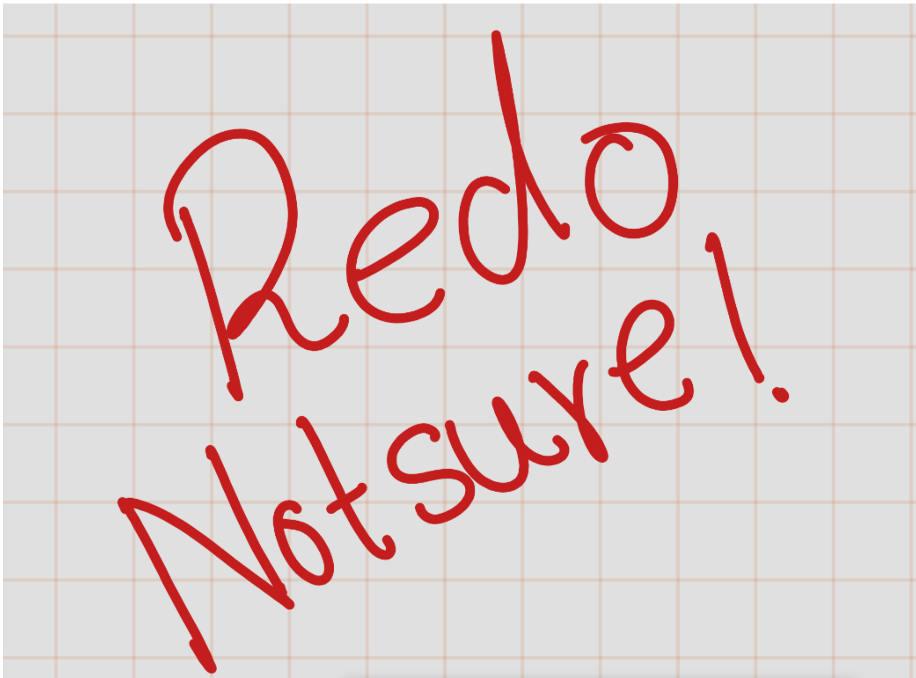
Then for each $\alpha \in I$, $\bigcup_{i=1}^{n} V_i \in \tau_\alpha$.
Thus, $\bigcup_{i=1}^{n} V_i \in \bigcap_{\alpha \in I} \tau_\alpha$.

Thus, $\bigcap_{\alpha\in I} \tau_\alpha$ is Topology.

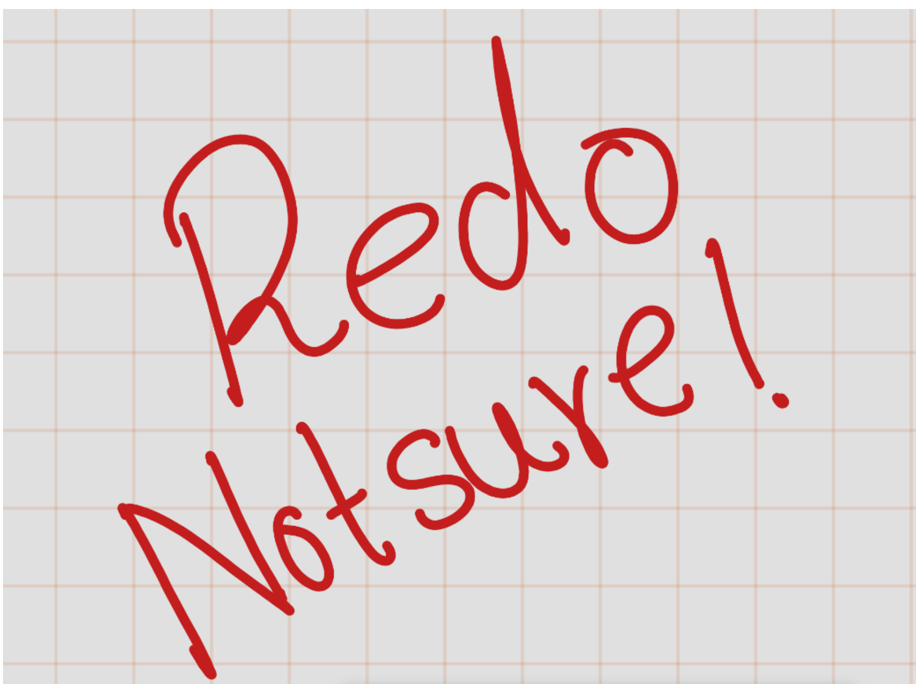
Union of Topologies sometimes might not be a topology
Example: $X := \{a, b, c\}$
$\tau_1 := \{\phi, \{a\}, X\}$
$\tau_2 := \{\phi, \{b\}, X\}$
But $\tau_1 \cup \tau_2 := \{\phi, \{a\}, \{b\}, X\}$ is not a topology.
(Because, $\{a\} \cup \{b\} = \{a, b\} \notin \tau_1 \cup \tau_2$.)

b)

Let Let $\{\mathcal{T}_\alpha\}$ be a family of topologies on X. Let $\mathcal{F} : \{\mathcal{T}_\beta : \mathcal{T}_\beta \supseteq \bigcup \mathcal{T}_\alpha\}$ be family of topology that contain $\bigcup \mathcal{T}_\alpha$.



c)



Exercise 4.5 (Mun 2.13.5)**.** Show that if A is a basis for a topology on X, then

the topology generated by A equals the intersection of all topologies on X that contain A. Prove the same if A is a subbasis.

5. Show that if $\mathcal{A}$ is a basis for a topology on X, then the topology generated by $\mathcal{A}$ equals the intersection of all topologies on X that contain $\mathcal{A}$. Prove the same if $\mathcal{A}$ is a subbasis.

Suppose that A is a basis for Topology τ_0 on X.
Let $B=\{\tau_\alpha\}$ be a family of Topologies on X such that $A \subseteq \tau_\alpha \in \{\tau_\alpha\}$

i.e: $B = \{\tau_\alpha \mid A \subseteq \tau_\alpha\}$

Let τ_A be topology generated by A

N.T.S: $\mathcal{C} = \bigcap \tau_\alpha = \tau_A$

First observe that $\tau_A \in B = \{\tau_\alpha \mid A \subseteq \tau_\alpha\}$

$$\bigcap_\alpha \tau_\alpha \subseteq \tau_A \quad \text{——— (1)} \qquad P \cap Q \subseteq P$$

Now we have proove otherway round.

Let $U \in \tau_A$. Since τ_A is generated by basis A, $\forall x \in U \in \tau_A$

$\exists B_x \in A$ such that $x \in B_x \subseteq U$ ——(i) (Def^n of gen Top)

Futher, $U = \bigcup_{x \in U} B_x$ ——————(ii) Def^n of basis

Since $A \in \tau_\alpha$ for α (Def^n of B)

$U = \bigcup_{x \in U} B_x \in A \in \tau_\alpha$ for a α

$$U \in \bigcap_\alpha \tau_\alpha$$

Thus $\tau_A \subseteq \bigcap_\alpha \tau_\alpha$ ——————— (2)

By (1) an (2) we get $\bigcap_\alpha \tau_\alpha = \tau_A$ $\square$

Part-2. Let $\mathcal{A}$ be a sub basis.
Let $\tau_{\mathcal{A}}$ be a topology generated by subbasis $\mathcal{A}$
Let $D = \{ \tau_\alpha \mid \mathcal{A} \subseteq \tau_\alpha \}$
i.e. D is a collection of topology that contains $\mathcal{A}$
N.T.S: $D = \tau_{\mathcal{A}}$

Observe that $\tau_{\mathcal{A}} \in D = \{\tau_\alpha \mid \mathcal{A} \subset \tau_\alpha\} \Rightarrow \bigcap_\alpha \tau_\alpha \subseteq \tau_{\mathcal{A}}$ ——— (1)

$(A \cap B \subseteq A)$

Conevers: Let $U \in \tau_{\mathcal{A}}$. Then U can be written as union of finite intersections of elements of subbasis $\mathcal{A}$

$$U = \bigcup \left(\bigcap_{i=1}^{n} A_{\alpha_i} \right) \quad \text{——— (i)}$$

where $A_{\alpha_i} \in \mathcal{A}$ for $i = 1, 2, \dots, n$

Futher observe that $A_{\alpha_1}, A_{\alpha_2}, \dots, A_{\alpha_n} \in \bigcap \tau_\alpha$ and $\bigcap \tau_\alpha$ is topolgy. Then finite intersection

$\bigcap_{i=1}^{n} A_{\alpha_i} \in \tau_\alpha$ for α

Futher arbitrary union $\underbrace{\bigcup \left(\bigcap_{i=1}^{n} A_{\alpha_i}\right)}_{U} \in \tau_\alpha$ for all α

$U \in \tau_\alpha$ for all α

Then, $U \in \bigcap_\alpha \tau_\alpha$

Thus, $\tau_A \subseteq \bigcap_\alpha \tau_\alpha$ ——— ②

By ① and ②, $\tau_A = \bigcap_\alpha \tau_\alpha$

$\boxed{\cdot}$

Exercise 4.6 (Mun 2.13.6)**.** Show that the topologies of $\mathbb{R}_l$ and $\mathbb{R}_K$ are not comparable.

Pg: 83 (Similar to $\mathbb{R}_K$)

6. Show that the topologies of $\mathbb{R}_\ell$ and $\mathbb{R}_K$ are not comparable.

Recall: Let $\tau_{\mathbb{R}_\ell}$, $\tau_{\mathbb{R}_K}$ be t

$\tau_{\mathbb{R}_\ell}$ = lower limit topology on $\mathbb{R}_\ell$

It is generated by half-closed, half open interval such as $[a, b)$ with $a<b$

$\tau_{\mathbb{R}_K}$ = It is generated by all collection of all open intervals (a, b), along with the sets of form $(a, b) - K$. where

$$K = \left\{\frac{1}{n} \mid n \in \mathbb{Z}^+\right\}$$

-i.e: Let $\beta = \{(a,b) \mid a,b \in \mathbb{R}\} \cup \{(a,b) \setminus K \mid a,b \in \mathbb{R}\}$

$\tau_{\mathbb{R}_K}$ genered by β

NTS: $\tau_{\mathbb{R}_\ell} \not\subseteq \tau_{\mathbb{R}_K}$ and $\tau_{\mathbb{R}_K} \not\subseteq \tau_{\mathbb{R}_\ell}$

We are going to do show, $\exists U_1 \in \tau_{\mathbb{R}_\ell}$ s.t. $U_1 \notin \tau_{\mathbb{R}_K}$

and $\exists U_2 \in \tau_{\mathbb{R}_K}$ s.t. $U_2 \notin \tau_{\mathbb{R}_\ell}$

Observe that $0 \in [0, 1) \in \tau_{\mathbb{R}_\ell}$ and

But $[0,1) \notin \tau_{\mathbb{R}_K}$.

Because there is No subset of $[0,1)$ in the form (a,b) or $(a,b)-K$ such that contain 0.

Thus $\tau_{\mathbb{R}_\ell} \not\subset \tau_{\mathbb{R}_K}$

Converse: Observe that $\left(\frac{-1}{n}, \frac{1}{n}\right) - K \in \mathbb{R}_K$

Let $a, b \in \mathbb{R}$ and $(a<b$ such that $0 \in [a,b)$

Then we can observe that 0

$\frac{1}{n} \in [a,b)$ for some $n \in \mathbb{N}$

Thus, we can not find any interval $[a,b]$ such that $0 \in [a,b)$ and $[a,b) \subset \left(\frac{-1}{n}, \frac{1}{n}\right)-K$

Exercise 4.7 (Mun 2.13.7)**.** Consider the following topologies on $\mathbb{R}$:

$$\mathcal{T}_1 := \text{the standard topology} \tag{4.1}$$
$$\mathcal{T}_2 := \text{the topology of } \mathbb{R}_K, \tag{4.2}$$
$$\mathcal{T}_3 := \text{the finite complement topology,} \tag{4.3}$$
$$\mathcal{T}_4 := \text{the upper limit topology, having all sets } (a,b] \text{ as basis,} \tag{4.4}$$
$$\mathcal{T}_5 := \text{the topology having all sets } (-\infty, a) = \{x | x < a\} \text{ as basis.} \tag{4.5}$$

Determine, for each of these topologies, which of the others it contains.

Exercise 4.8 (Mun 2.13.8)**.**

(a) Apply Lemma 13.2 to show that the countable collection

$$\mathcal{B} = \{(a,b) | a < b, a \text{ and } b \text{ rational}\}$$

is a basis that generates the standard topology on $\mathbb{R}$.

(b) Show that the collection

$$\mathcal{C} = \{[a,b) | a < b, a \text{ and } b \text{ rational}\}$$

is a basis that generates a topology different from the lower limit topology on $\mathbb{R}$. \end{enumerate}

8

8. (a) Apply Lemma 13.2 to show that the countable collection

$$\mathcal{B} = \{(a,b) \mid a < b, a \text{ and } b \text{ rational}\}$$

is a basis that generates the standard topology on $\mathbb{R}$.

Recall:

Lemma 13.2. *Let X be a topological space. Suppose that $\mathcal{C}$ is a collection of open sets of X such that for each open set U of X and each x in U, there is an element C of $\mathcal{C}$ such that $x \in C \subset U$. Then $\mathcal{C}$ is a basis for the topology of X.*

$(\mathbb{R}, \tau)$ be a top space. Here τ is standerd toplogy. We are going to show that $\forall\ U \in \tau\ \forall x \in U\ \exists G \in \beta$ such that $x \in G \subset U$.

* We use fact rational numbers are dense in $\mathbb{R}$
$\forall a, b \in \mathbb{R}\ \exists c \in \mathbb{Q}$ s.t. $a < c < b$.

Let $U \in \tau$
Choose $\delta > 0$ such that $(x-\delta, x+\delta) \subseteq U$
By *, $\exists a, b \in \mathbb{Q}$ such that
$x - \delta < a < x < b < x + \delta$
Then $x \in (a,b) \subseteq (x-\delta, x+\delta) \subseteq U$
Then $(a,b) \in \beta$.
Thus, β is a basis for standered toplogy

8 b) (b) Show that the collection

$$\mathcal{C} = \{[a, b) \mid a < b,\ a \text{ and } b \text{ rational}\}$$

is a basis that generates a topology different from the lower limit topology on $\mathbb{R}$.

First we are going to show that $\mathcal{C}$ is generat topology

Let $x \in \mathbb{R}$. and let $\delta > 0$. Then $x \in [x-\delta, x+\delta)$

Then by $*$ we can choose $a, b \in \mathbb{Q}$ such that

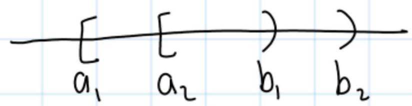
$$x-\delta \leq a \leq x < b < x+\delta$$

$x \in [a, b) \in \mathcal{C}$

Suppose $C_1 = [a_1, b_1)$ and $C_2 = [a_2, b_2)$ and $C_1, C_2 \in \mathcal{C}$. Then

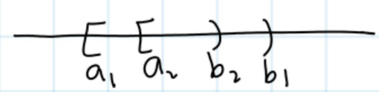
Case-I if $a_1 < b_1 \leq a_2 < b_2$, then $C_1 \cap C_2 = \emptyset$.

Case-II if $a_1 < a_2 < b_1 \leq b_2$



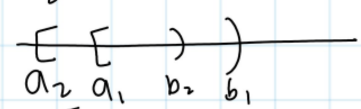
Then $C_1 \cap C_2 = [a_2, b_1) \in \mathcal{C}$

Case-III if $a_1 \leq a_2 < b_2 \leq b_1$



Then, $C_1 \cap C_2 = [a_2, b_2) \in \mathcal{C}$

Case-IV if $a_2 \leq a_1 < b_2 \leq b_1$



$$C_1 \cap C_2 = [a_1, b_2) \in \mathcal{C}$$

Case V if $a_2 < b_2 \leq a_1 < b$ then $C_1 \cap C_2 = \emptyset$.

Thus $\mathcal{C}$ generates a topolgy. τ'

Now we need to show that. τ' is different fra τ_ℓ

Here, τ_ℓ is the lower limit topology.

· Consider

$$[\sqrt{2}, 3) \in \tau_\ell$$

Observe that $(a \in \mathbb{Q} \wedge a \leq \sqrt{2}) \Rightarrow a < \sqrt{2}$

Thus $[\sqrt{2}, 3) \notin \tau'$.

Thus τ' and τ_ℓ is different eachother.

4.2 Section 16 in Munkress Book

Exercise 4.9 (Mun 2.16.2)**.** Show that if Y is a subspace of X, and A is a subset of Y , then the topology A inherits as a subspace of Y is the same as the topology it inherits as a subspace of X.

Solution: Let's denote the topology on X as $\mathcal{T}_X$, the topology on Y as $\mathcal{T}_Y$, and the topology on A as $\mathcal{T}_A$.

We know that Y is a subspace of X, so the topology $\mathcal{T}_Y$ that Y inherits from X is $\mathcal{T}_Y = \{Y \cap U : U \in \mathcal{T}_X\}$.

Similarly, A is a subset of Y, so the topology $\mathcal{T}_A$ that A inherits from Y is $\mathcal{T}_A = \{A \cap V : V \in \mathcal{T}_Y\}$.

Substituting $\mathcal{T}_Y$ into the equation for $\mathcal{T}_A$, we get $\mathcal{T}_A = \{A \cap (Y \cap U) : U \in \mathcal{T}_X\}$.

Since A is a subset of Y, $A \cap Y = A$. So, $\mathcal{T}_A = \{A \cap U : U \in \mathcal{T}_X\}$.

This is exactly the topology that A would inherit as a subspace of X. Therefore, the topology A inherits as a subspace of Y is the same as the topology it inherits as a subspace of X.

Let τ_Y be the topology of A inherit by subspace of Y

Let τ_X be the topology of A inherit by X

claim1: $\tau_Y \subseteq \tau_X$

Let $U_1 \in \tau_Y$, then there exist a set $V_1 \subseteq Y$ which is open in V_1 such that

$$U_1 = V_1 \cap A \quad \text{——— } (1)$$

Since V_1 is open in subspace Y, there exist an open set $W_1 \subseteq X$. Such that

$$V_1 = W_1 \cap Y \quad \text{——— } (2)$$

By (1) and (2)

$$\begin{aligned} U_1 = V_1 \cap A &= (W_1 \cap Y) \cap A \\ &= W_1 \cap (Y \cap A) \\ &= W_1 \cap A \qquad (\because A \subseteq Y) \end{aligned}$$

Thus $U_1 \in \tau_X$. Hence $\tau_Y \subseteq \tau_X$ ——— (*)

Claim: $\mathcal{T}_X \subseteq \mathcal{T}_Y$

Let $U_2 \in \mathcal{T}_X$.
Then there exist open set $V_2 \subseteq X$ such that,

$$U_2 = V_2 \cap A$$
$$= V_2 \cap (Y \cap A) \quad (\because A \subseteq Y)$$
$$= (V_2 \cap Y) \cap A$$

Note that, $V_2 \cap Y$ is open in Y. Thus $U_2 \in \mathcal{T}_Y$
Hence $\mathcal{T}_X \subseteq \mathcal{T}_Y$ — (**)

By (*) and (**) $\mathcal{T}_X = \mathcal{T}_Y$

Exercise 4.10 (Mun 2.16.2)**.** If $\mathcal{T}$ and $\mathcal{T}$ are topologies on X and $\mathcal{T}$ is strictly finer than $\mathcal{T}$, what can you say about the corresponding subspace topologies on the subset Y of X?

Solution:

Exercise 2.16.2

Let τ'_Y and τ_Y be the subspace topology inherit by τ' and τ on Y

~~for~~ claim: $\tau'_Y \supseteq \tau_Y$ (τ'_Y is finer than τ)

~~First we proove finer part.~~

Let $U \in \tau_Y$. Then there exists $V \in \tau$ such that $U = V \cap Y$.

Since $V \in \tau \subseteq \tau'$ ($\because \tau'$ is finer than τ)

Thus $U = V \cap Y \in \tau'_Y$.

Therefore τ'_Y is finer than τ_Y.

But we can not say τ'_Y is nescessarily stricly finer τ_Y. Consider following example

$X = \{a, b, c\}$ and $Y = \{a, b\}$

$\tau = \{\emptyset, X, \{a\}, \{b\}, \{a, b\}\}$ and $\tau' = P(X)$

$\tau_Y = \{\emptyset, \{a, b\}\} = \tau'_Y$

Exercise 4.11 (Mun 2.16.3). Consider the set $Y = [-1, 1]$ as a subspace of $\mathbb{R}$.

Which of the following sets are open in Y ? Which are open in $\mathbb{R}$?

$$A = \{x \mid \frac{1}{2} < |x| < 1\} \tag{4.6}$$

$$B = \{x \mid \frac{1}{2} < |x| \leq 1\} \tag{4.7}$$

$$C = \{x \mid \frac{1}{2} \leq |x| < 1\} \tag{4.8}$$

$$D = \{x \mid \frac{1}{2} \leq |x| \leq 1\} \tag{4.9}$$

$$E = \{x \mid 0 < |x| < 1 \text{ and } \frac{1}{x} \notin \mathbb{Z}^{+}\} \tag{4.10}$$

Exercise 2.6.3

(1) $A := \{x : \frac{1}{2} < |x| < 1\}$

$A = (-1, -\frac{1}{2}) \cup (1, \frac{1}{2})$ is open in $\mathbb{R}$.

In subspace topology on Y.

$A \cap Y = Y$ is open in subspace Y

(2) $B := \{x : \frac{1}{2} < |x| \leq 1\}$

$\cancel{B := \{[-1, 1/2) \cup}$

$B := [-1, -\frac{1}{2}) \cup (\frac{1}{2}, 1]$ is not open in $\mathbb{R}$. But we can consider B as follows

$B = Y \cap (-2, \frac{1}{2}) \cup (\frac{1}{2}, 2)$. Note that $(-2, \frac{1}{2}) \cup (\frac{1}{2}, 2)$ is open in $\mathbb{R}$. Thus B is open in Y.

(3) $C := \{x : \frac{1}{2} \leq |x| < 1\} = (1, -\frac{1}{2}] \cup [\frac{1}{2}, 1)$

There is no basis element B in order topolgy such that $\frac{1}{2} \in B \subseteq C$. Thus, not open in $\mathbb{R}$

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Solution:

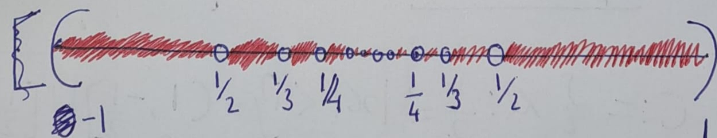
4 $D := \{X : \frac{1}{2} \leq x \leq 1\} = [-1, -\frac{1}{2}] \cup [\frac{1}{2}, 1]$

There exist no ~~open~~ basis element B' such that $x \in B \subseteq D$. So, not open in Y

$\boxed{\text{basis element of order topolgy}}$

So, D is not in $\mathbb{R}$ as well.

5 $E := \{x : 0 < |x| < 1 \text{ and } \frac{1}{x} \notin \mathbb{Z}^+\}$



$E := \bigcup_{n \in \mathbb{Z}^+} (\frac{1}{n+1}, \frac{1}{n})$ is open in $\mathbb{R}$ and Y because it can be written as union of basis element

Exercise 4.12. A map $f : X \to Y$. We say that f is an **open map** if, for every open set U in X, the set $f(U)$ is open in Y. Show that the projections $\pi_1 : X \times Y \to X$ and $\pi_2 : X \times Y \to Y$ are open maps. "

Exercise 4.13. Let X and X' denote a single set in the topologies $\mathcal{T}$ and $\mathcal{T}'$, respectively; let Y and Y' denote a single set in the topologies $\mathcal{U}$ and $\mathcal{U}'$, respectively. Assume these sets are nonempty.

(a) Show that if $\mathcal{T}' \supset \mathcal{T}$ and $\mathcal{U}' \supset \mathcal{U}$, then the product topology on $X' \times Y'$ is finer than the product topology on $X \times Y$.
(b) Does the converse of a. hold? Justify your answer

Exercise 4.14 (Mun 2.16.7)**.** Show that the countable collection

$$\{(a, b) \times (c, d) | a < b \text{ and } c < d, \text{ and } a, b, c, d \text{ are rational}\}$$

is a basis for $\mathbb{R}^2$."

Exercise 4.15. Let X be an ordered set. If Y is a proper subset of X that is convex in X, does it follow that Y is an interval or a ray in X?

Exercise 4.16. If L is a straight line in the plane, describe the topology L inherits as a subspace of $\mathbb{R}_l \times \mathbb{R}$ and as a subspace of $\mathbb{R}_l \times \mathbb{R}_l$. In each case it is a familiar topology.

Exercise 4.17. Show that the dictionary order topology on the set $\mathbb{R} \times \mathbb{R}$ is the same as the product topology $\mathbb{R}_d \times \mathbb{R}$, where $\mathbb{R}_d$ denotes $\mathbb{R}$ in the discrete topology. Compare this topology with the standard topology on $\mathbb{R}^2$.

$$\mathbb{R}^2_{dictionary} := \mathbb{R}_{discrete} \times \mathbb{R}_{standard}$$

Proof. Let $\{a\} \times (c, d)$ be a basis element in product toplogy $\mathbb{R}_d \times \mathbb{R}$. Let $a \times x \in \{a\} \times (c, d)$ obsereve that

$$a \times x \in \{a\} \times (c, d) = (a \times c, a \times d)$$

and $(a \times c, a \times d)$ is basis element of order topology $\mathbb{R}^2$. Thus by lemma 1.4, order toplogy in $\mathbb{R}^2$ is finer than the product toplogy $\mathbb{R}_d \times \mathbb{R}$.

Now suppose that $(p \times q, r \times s)$ be a basis elemenet in order toplogy on $\mathbb{R}^2$.

- If $p < x$, define $l = y - 1$ and if $p = x$ define $l = r$. In either case we know that $(p \times q) < (x \times l) < (x \times y)$.
- If $x < r$ define $t = y + 1$ and if $x = r$ define $t = s$. In either case we know that $(x \times y) < (x \times t) < (q \times s)$.

See figure 1.12 So

$$(x, y) \in \{x\} \times (l, t) \subseteq (p \times q, r \times s).$$

Thus by lemma 1.4, product toplogy $\mathbb{R}_d \times \mathbb{R}$ is finer than order toplogy in $\mathbb{R}^2$.

Therefore,

$$\mathbb{R}^2_{dictionary} = \mathbb{R}_{discrete} \times \mathbb{R}_{standard}$$

□

Let $I = [0, 1]$. Compare the product topology on $I \times I$, the dictionary order topology on $I \times I$, and the topology $I \times I$ inherits as a subspace of $\mathbb{R} \times \mathbb{R}$ in the dictionary order topology.

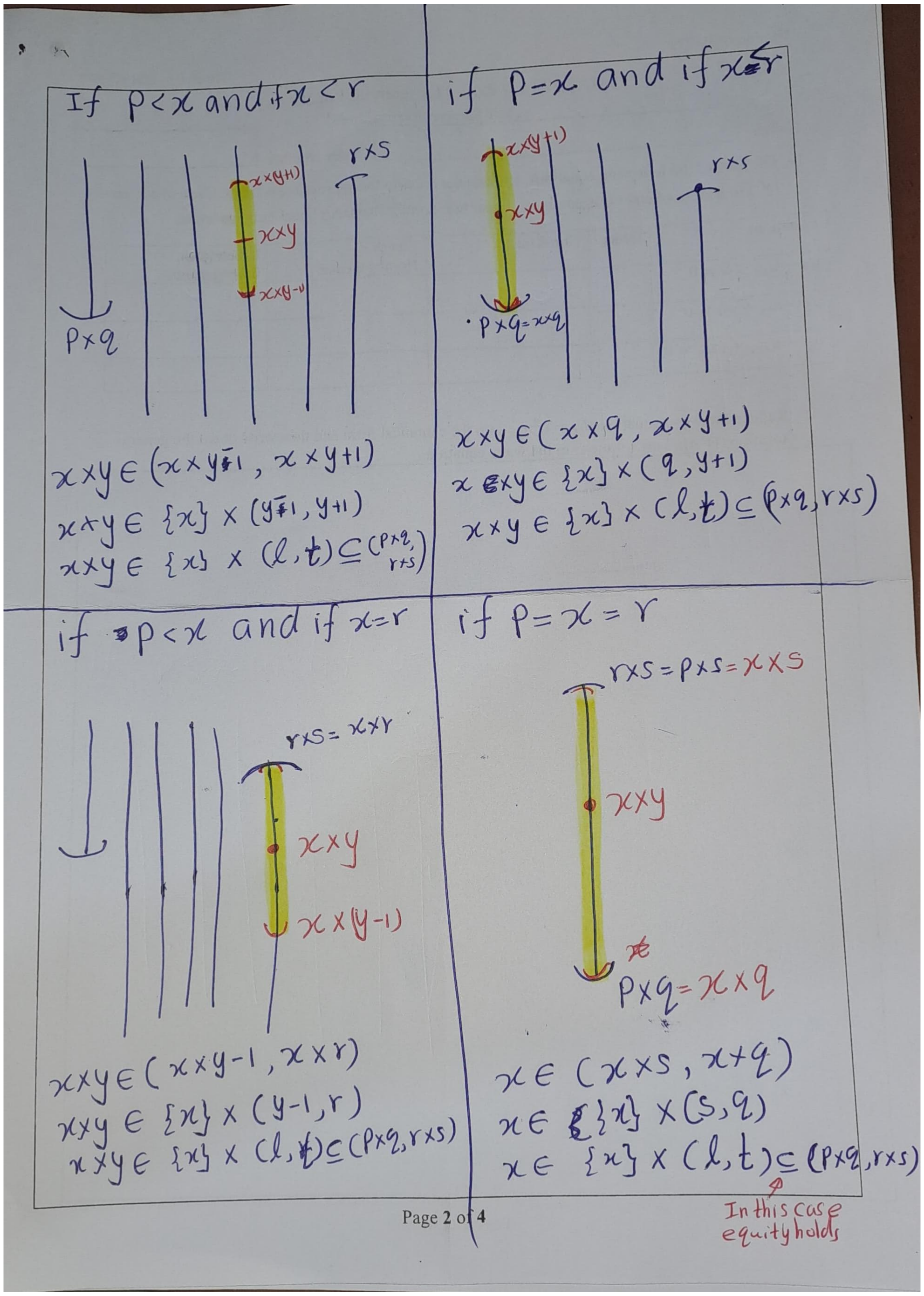

If p<x and if x<r
r×s
x×(y+1)
x×y
x×y-1
p×q
x×y ∈ (x×y-1 , x×y+1)
x×y ∈ {x} × (y-1 , y+1)
x×y ∈ {x} × (l , t) ⊆ (p×q, r×s)
if p=x and if x<r
x×(y+1)
x×y
r×s
p×q=x×q
x×y ∈ (x×q , x×y+1)
x×y ∈ {x} × (q , y+1)
x×y ∈ {x} × (l , t) ⊆ (p×q , r×s)
if p<x and if x=r
r×s= x×r
x×y
x×(y-1)
x×y ∈ (x×y-1 , x×r)
x×y ∈ {x} × (y-1 , r)
x×y ∈ {x} × (l , t) ⊆ (p×q , r×s)
if p=x=r
r×s=p×s=x×s
x×y
p×q=x×q
x ∈ (x×s , x×q)
x ∈ {x} × (s , q)
x ∈ {x} × (l , t) ⊆ (p×q , r×s)
In this case equity holds
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Figure 4.2:

(10) Let consider following topologies on $[0,1]$

$\tau_1 :=$ product topology on $I \times I$

$\tau_2 :=$ dictonary topology on $I \times I$.

$\tau_3 :=$ subspace topology inherit of $\mathbb{R} \times \mathbb{R}$ in the dictonary order topolgy.

Basis for τ_1 is

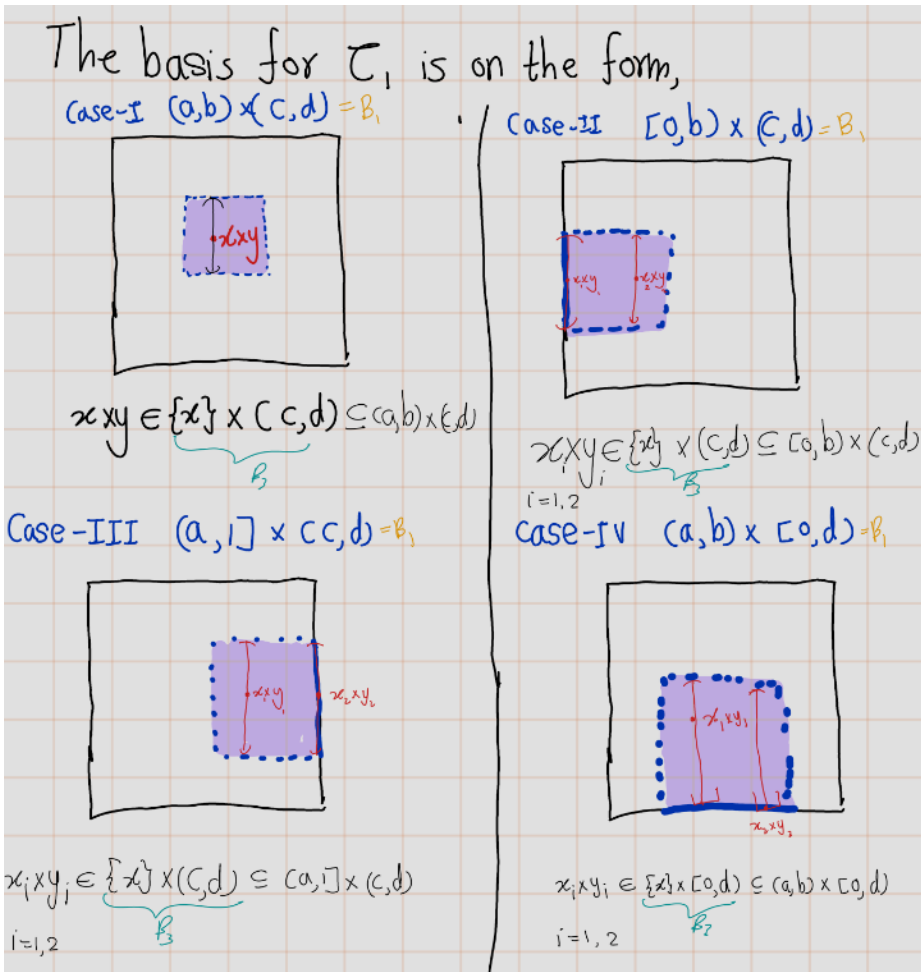
$$\mathcal{B}_1 := \{ (I \cap A) \times (I \cap B) \mid A, B \text{ are open in } \mathbb{R} \}$$

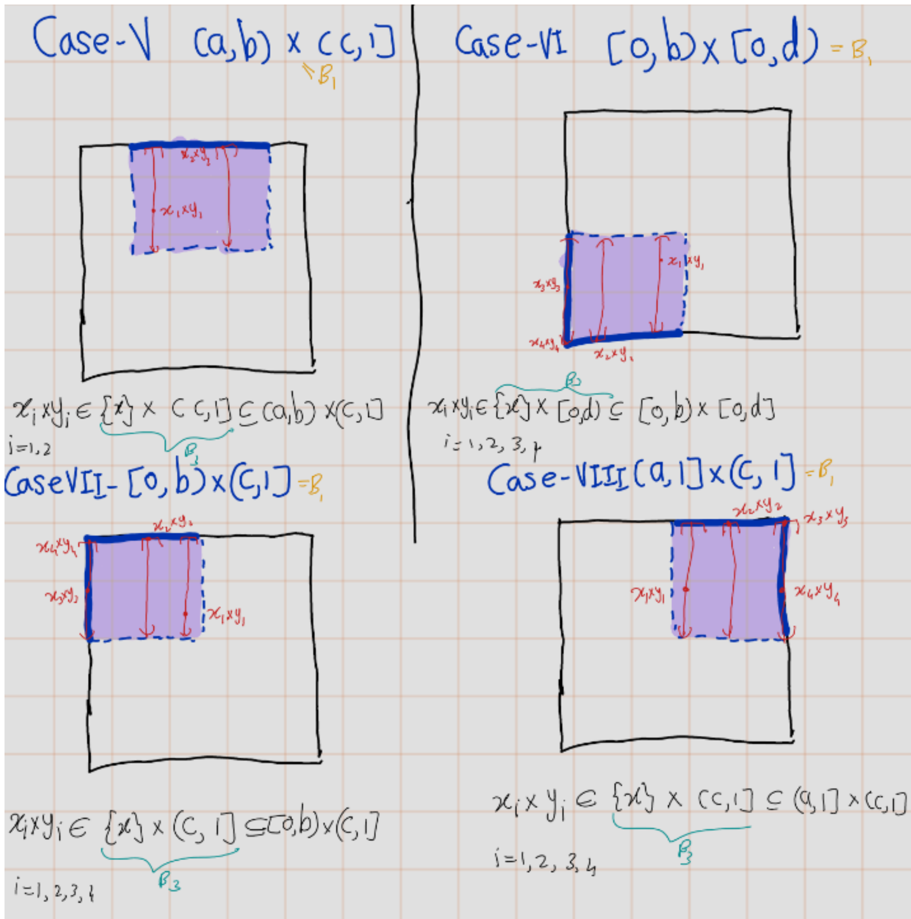
The basis for τ_2

$$\mathcal{B}_2 := \{ (a \times b, c \times d) \mid a, b, c, d \in I \text{ and } a \times b < c \times d \}$$

The basis for τ_3

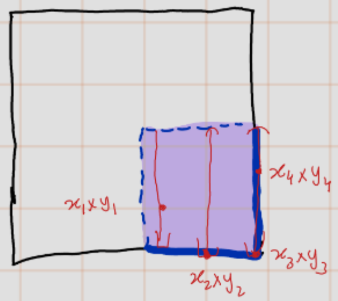
$$\mathcal{B}_3 := \{ \{a\} \times I \cap A \mid a \in I, A \text{ is open in } \mathbb{R} \}$$


The basis for τ₁ is on the form,
Case-I (a,b) x (c,d) = B₁
x x y
x x y ∈ {x} x (c,d) ⊆ (a,b) x (c,d)
B₃
Case-II [0,b) x (c,d) = B₁
x₁xy₁
x₂xy₂
xᵢxyᵢ ∈ {x} x (c,d) ⊆ [0,b) x (c,d)
B₃
i=1,2
Case-III (a,1] x (c,d) = B₁
x₁xy₁
x₂xy₂
xᵢxyᵢ ∈ {x} x (c,d) ⊆ (a,1] x (c,d)
B₃
i=1,2
Case-IV (a,b) x [0,d) = B₁
x₁xy₁
x₂xy₂
xᵢxyᵢ ∈ {x} x [0,d) ⊆ (a,b) x [0,d)
B₂
i=1,2



Case-V (a,b) x (c,1] ≠ B1
x_i x y_i ∈ {x} x (c,1] ⊆ (a,b) x (c,1]
i=1,2
Case-VI [0,b) x [0,d) = B1
x_i x y_i ∈ {x} x [0,d) ⊆ [0,b) x [0,d]
i=1,2,3,4
Case VII- [0,b) x (c,1] = B1
x_i x y_i ∈ {x} x (c,1] ⊆ [0,b) x (c,1]
i=1,2,3,4
Case-VIII (a,1] x (c,1] = B1
x_i x y_i ∈ {x} x (c,1] ⊆ (a,1] x (c,1]
i=1,2,3,4

Case-IX $[0,b)\times(c,1] = B_1$



$x_i\times y_i \in \underbrace{\{x\}\times(c,1]}_{B_3} \subseteq [0,b)\times(c,1]$

$i=1,2,3,4$

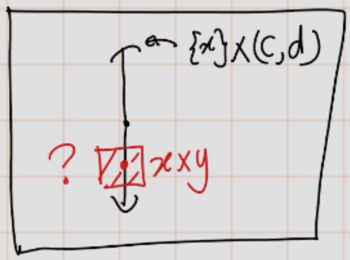
Here, $0\leq a<b\leq 1$ and $0\leq c<d\leq 1$

In above all nine cases we have found $B_3\in\mathcal{B}_3$ such that such that $x\times y\in B_3\subseteq B_1\in\mathcal{B}_1$

Therefore, $\tau_1\subseteq\tau_3$

On the other hand,

$\{a\} \times (I \cap A)$



$0 \leq x \leq 1$ and $0 \leq c < y < d \leq 1$

We can not find any basis element B from τ_1 $(\mathcal{B}_1)$ such that,

$$x \times y \in B \subseteq \{a\} \times (c,d)$$

Thus $\tau_1 \subsetneq \tau_3$.

(τ_3 is strictly finer than τ_1)

Claim: $\mathcal{T}_2 \subsetneq \mathcal{T}3$
So, as previous we have to prove two things they are finer condition and strictly condition.

Let $(a_1 \times b_1, a_2 \times b_2)$ be a basis element of order topology. and $x \times y \in (a_1 \times b_1, a_2 \times b_2)$

- **Case I** $(a_1 < x < a_2)$:

  $x \times y \in (x \times -1, x \times 2) \cap I^2 = [x \times 0, x \times 1] = \{x\} \times [0,1] \subset (a_1 \times b_1, a_2 \times b_2)$

  Note that $(x \times -1, x \times 2) \cap I^2$ is a basis element of subspace topology.

- **Case II** ($a_1 = x < a_2$):

  $$x \times y \in (x \times b_1, x \times 2) \cap I^2 = [x \times b_1, x \times 1] = \{x\} \times (b_1, 1] \subset (a_1 \times b_1, a_2 \times b_2)$$

  Note that $(x \times b_1, x \times 2) \cap I^2$ is a basis element of subspace topology.

- **Case III** ($a_1 < x = a_2$):

  $$x \times y \in (x \times -1, x \times b_2) \cap I^2 = [x \times 0, x \times b_2] = \{x\} \times [0, b_2) \subset (a_1 \times b_1, a_2 \times b_2)$$

  Note that $(x \times -1, x \times b_2) \cap I^2$ is a basis element of subspace topology.

- **Case IV** ($a_1 = x = a_2$):

  $$x \times y \in (x \times b_1, x \times b_2) \cap I^2 = [x \times b_1, x \times b_2] = \{x\} \times [b_1, b_2) \subset (a_1 \times b_1, a_2 \times b_2)$$

  Note that $(x \times b_1, x \times b_2) \cap I^2$ is a basis element of subspace topology.

See figure 4.3

In above all four cases, we have found basis element of subspace topology that contain $x \times y$ and contained in $(a_1 \times b_1, a_2 \times b_2)$. Therefore, $\mathcal{T}_2 \subsetneq \mathcal{T}3$

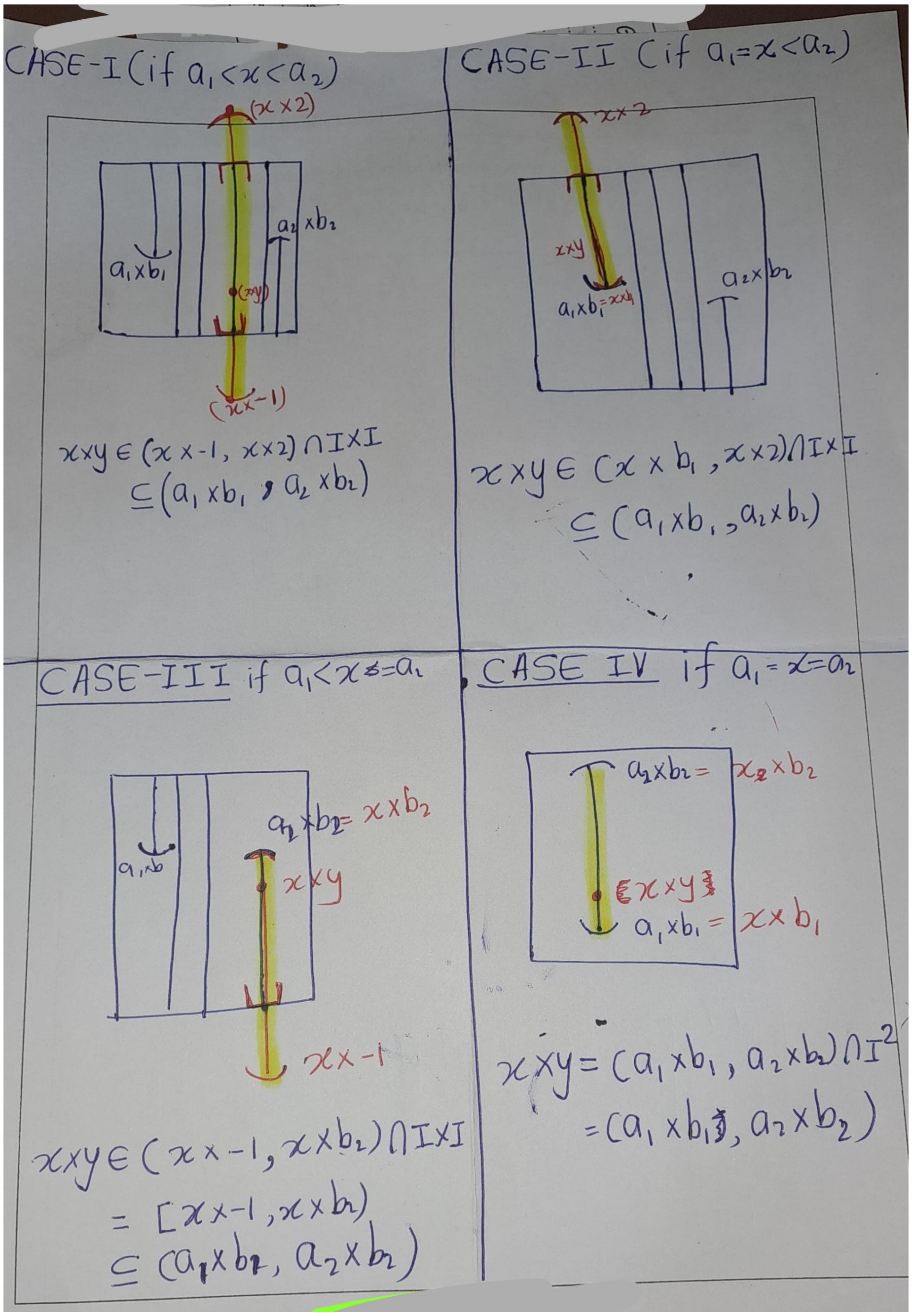
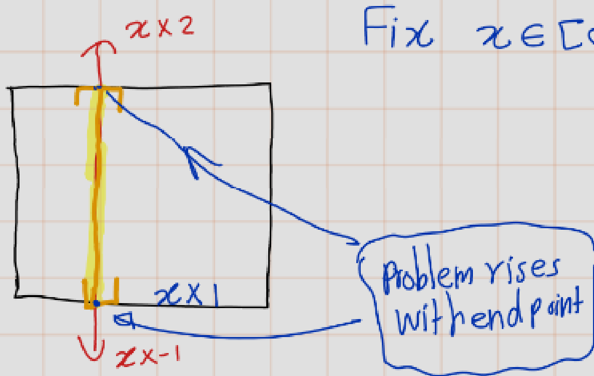

CASE-I (if a1<x<a2)
CASE-II (if a1=x<a2)
CASE-III if a1<x=a2
CASE IV if a1=x=a2
xxy ∈ (x×-1, x×2) ∩ I×I ⊆ (a1×b1, a2×b2)
x×y ∈ (x×b1, x×2) ∩ I×I ⊆ (a1×b1, a2×b2)
xxy ∈ (x×-1, x×b2) ∩ I×I = [x×-1, x×b2) ⊆ (a1×b1, a2×b2)
x×y = (a1×b1, a2×b2) ∩ I^2 = (a1×b1, a2×b2)

Figure 4.3:

Fix $x \in [0,1]$



$x \times 1 \in \{x\} \times \left((x \times -1, x \times 2) \cap I^2\right)$

$x \times 1 \in \{x\} \times [0,1]$

We can not find any basis element B''' from τ_3 ($\mathcal{B}_3$) such that,

$x \times 1 \in B''' \subseteq \{x\} \times [0,1]$

Therefore, $\tau_2 \subsetneq \tau_3$

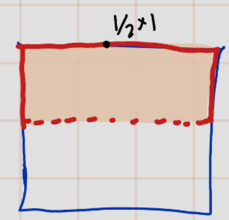
i.e: τ_3 is strictly finer than τ_2

- claim: τ_1 and τ_2 arenot comparble

Let $U = [0,1] \times (\frac{1}{2} \times 1] \in \mathcal{B}_1$ and let $\frac{1}{2} \times 1 \in U$

But there is no basis element in $\mathcal{B}_2$ such that contains $\frac{1}{2} \times 1$ and it contained in U.

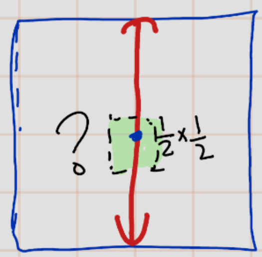
Thus, $\tau_2 \not\subset \tau_1$



Conversly, Let $V=\left\{\frac{1}{2}\right\}\times(0,1)\in\mathcal{B}_2$ and $\frac{1}{2}\times\frac{1}{2}\in V$.

But, there is no basis element in $\mathcal{B}_1$ s containing $1/2\times\frac{1}{2}$ and contained in V. So, $\tau_1\not\in\tau_2$

Therefore, τ_1 and τ_2 arenot comparble



4.3 Section 17 in Munkress Book

Exercise 4.18 (Mun 2.17.1)**.** Let $\mathcal{C}$ be a collection of subsets of the set X. Suppose that $$ and X are in $\mathcal{C}$, and that finite unions and arbitrary intersections of elements of $\mathcal{C}$ are in $\mathcal{C}$. Show that the collection

$$\mathcal{T}=\{X\setminus C|C\in\mathcal{C}\}$$

is a topology on X.

Solution:

Proof.

- (T1)
  - $\emptyset\in\mathcal{C}\implies X\setminus\emptyset=X\in\mathcal{T}$.
  - $X\in\mathcal{C}\implies X\setminus X=\emptyset\in\mathcal{T}$

- (T2) Let $\{U_\alpha\}_{\alpha\in J}$ be family of elements in $\mathcal{C}$. Then $(X \setminus U_\alpha\}_{\alpha\in J} \in \mathcal{T}$. Then,
$$\bigcup_{\alpha\in J}(X \setminus U_\alpha) = X \setminus \bigcup_{\alpha\in J} U_\alpha$$
Since $\mathcal{C}$ is closed arbitrary intersection, $\bigcup_{\alpha\in J} \in \mathcal{C}$. Thus,$\bigcap_{\alpha\in J}(X \setminus U_\alpha) = X \setminus \bigcup_{\alpha\in J} U_\alpha \in \mathcal{T}$.
- (T3) Let $U_1, U_2, ..., U_n \in \mathcal{C}$. Then $(X \setminus U_1), (X \setminus U_2), ..., (X \setminus U_n) \in \mathcal{T}$. Then,
$$\bigcap_{i=1}^{n}(X \setminus U_i) = X \setminus \bigcup_{i=1}^{n} U_i$$
Since $\mathcal{C}$ is closed under finite union, $\bigcup_{i=1}^{n} U_i \in \mathcal{C}$. Thus,$\bigcap_{i=1}^{n}(X \setminus U_i) = X \setminus \bigcup_{i=1}^{n} U_i \in \mathcal{T}$. Therefore $\mathcal{T}$ is a topology on X.

□

Exercise 4.19 (Mun 2.17.2)**.** Show that if A is closed in Y and Y is closed in X, then A is closed in X.

Proof. Since Y is subspace in X and A is closed in Y, by theorm 1.3, then there exist closed set C in X such that
$$A = Y \cap C$$
Since, Y is closed in X and X is toplogical space, $A = Y \cap C$ is closed in X. □

Exercise 4.20 (Mun 2.17.3)**.** Show that if A is closed in X and B is closed in Y , then $A \times B$ is closed in $X \times Y$

Proof. Suppose that A is closed in X and B is closed in Y. Then $X \setminus A$ is open in X and $X \setminus B$ is open in Y.

Claim: $(X \times Y) \setminus (A \times B) = ((X \setminus A) \times Y) \cup ((X \times (Y \setminus B)))$
Proof of Claim:

$$\begin{aligned}
(x \times y) \in (X \times Y) \setminus (A \times B) \quad &\iff \quad (x \times y) \notin (A \times B) && (4.11)\\
&\iff \quad x \notin A \text{ or } y \notin B && (4.12)\\
&\iff \quad x \in X \setminus A \text{ or } y \in Y \setminus B && (4.13)\\
&\iff \quad (x \times y) \in ((X \setminus A) \times Y) \cup ((X \times (Y \setminus B))) && (4.14)
\end{aligned}$$

So, now we are done proof of the claim. Thus, $(X \times Y) \setminus (A \times B)$ is union of open sets and hence, $(X \times Y) \setminus (A \times B)$ is open in $X \times Y$. Thus, $(A \times B)$ is closed in $X \times Y$. □

Exercise 4.21 (Mun 2.17.4)**.** Show that if U is open in X and A is closed in X, then $U \setminus A$ is open in X, and $A \setminus U$ is closed in X.

Proof. Suppose that U is open in X and A is closed in X.

Claim 1: $(U \setminus A) = U \cap (X \setminus A)$.

$$\begin{aligned} x \in (U \setminus A) &\iff x \in U \text{ and } x \notin A && (4.15)\\ &\iff x \in U \text{ and } x \in X \setminus A && (4.16)\\ &\iff x \in U \cap X \setminus A && (4.17)\\ & && (4.18) \end{aligned}$$

Since, A is closed in X, $X \setminus A$ is open in X. Thus, $(U \setminus A) = U \cap (X \setminus A)$ is open in X. (By finite intersection property.)

Claim 2: $(A \setminus U) = A \cap (X \setminus U)$.
Similar to proof of claim 1 we can prove this claim.

Since, U is open in X, $X \setminus U$ is closed in X. Thus, $(A \setminus U) = A \cap (X \setminus U)$ is closed in X. ☐

Exercise 4.22 (Mun 2.17.5)**.** Let X be an ordered set in the order topology. Show that $\overline{(a, b)} \subseteq [a, b]$. Under what conditions does equality hold?

Proof. (Proof of $\overline{(a, b)} \subseteq [a, b]$)

Claim 1: $[a, b]$ is closed in X under order toplology.
Observe that,

$$X \setminus [a, b] = (-\infty, a) \cup (b, \infty).$$

Note that, $(-\infty, a)$ and (b, ∞) are open rays. Then $(-\infty, a) \cup (b, \infty) = X \setminus [a, b]$ is open. Thus, $[a, b]$ is closed in X under toplogy.

Claim 2: $(a, b) \subseteq [a, b]$
Now, let $x \in (a, b)$ then $a < x < b$ so clearly $a \leq x \leq b$ so x is in $[a, b]$. Thus $(a, b) \subseteq [a, b]$.

Therefore, by cliam 1 and 2, $[a, b]$ is a closed set containing (a, b).

Now, to show $\overline{(a, b)} \subseteq [a, b]$, recall that $\overline{(a, b)}$ is the intersection of all closed sets containing (a, b). Let $x \in \overline{(a, b)}$ Since, x is in every closed set containing (a, b) and $[a, b]$ is a closed set containing (a, b),then $x \in [a, b]$ as needed.

Let $a^+ = \inf\{x : x > a\}$ and let $b^- = \sup\{x : x < b\}$. Consider the closed set $[x, y]$. If $[x, y]$ contains (a, b), then $x \leq a^+$ and $y \geq b^-$. Thus there exists a closed set containing (a, b) that does not contain a or b if and only if $a^+ \neq a$ or $b^- \neq b$. This is true exactly when a has an immediate successor or b has an immediate predecessor. ☐

- For example, in the order topology on $\mathbb{Z}$,

$$\overline{(1, 4)} = [2, 3] \subsetneq [1, 4].$$

Exercise 4.23 (Mun 2.17.6)**.** Let A, B, and A_α denote subsets of a space X. Prove the following:

(a) If $A \subset B$, then $\overline{A} \subset \overline{B}$.
(b) $\overline{A \cup B} = \overline{\overline{A}} \cup \overline{\overline{B}}$.
(c) $\overline{\bigcup A_\alpha} \supseteq \bigcup \overline{A_\alpha}$; give an example where equality fails.

Proof. Let A, B, and A_α denote subsets of a space X.

(a) By definition, $\overline{A}$ is the intersection of all closed sets containing A. Since $\overline{B}$ is a closed set that contains B and hence A, it must therefore contain $\overline{A}$.

(b)

- **Claim 1**: $\overline{A \cup B} \supseteq \overline{A} \cup \overline{B}$
  The set $\overline{A \cup B}$ is a closed set that contains $A \cup B$, so it contains both A and B, and therefore it contains both $\overline{A}$ and $\overline{B}$. Therefore, $\overline{A \cup B} \supseteq \overline{A} \cup \overline{B}$.
- **Claim 2**: $\overline{A \cup B} \subseteq \overline{A} \cup \overline{B}$
  $\overline{A} \cup \overline{B}$ is a closed set (since it is the union of two closed sets) that contains $A \cup B$, (Beacuse $A \subseteq \overline{A}$ and $B \subseteq \overline{B} \implies A \cup B \subseteq \overline{A} \cup \overline{B}$.) so it contains $\overline{A \cup B}$.

(c) The set $\overline{\bigcup A_\alpha}$ is a closed set that contains each A_α, so it contains each $\overline{A_\alpha}$, and thus it contains $\bigcup \overline{A_\alpha}$. Thus, $\overline{\bigcup A_\alpha} \supseteq \bigcup \overline{A_\alpha}$
(But we can't do the converse because an arbitrary union of closures isn't necessarily closed!)

- **Example 1** that equality fails,
  Let A_n be the closed set $[\frac{1}{n}, 1] \subset \mathbb{R}$. Then

$$\bigcup_{n \in \mathbb{N}} \overline{A_n} = \bigcup_{n \in \mathbb{N}} A_n = \bigcup_{n \in \mathbb{N}} [\frac{1}{n}, 1] = (0, 1], \text{ But } \bigcup_{n \in \mathbb{N}} A_n = [0, 1]$$

.

- **Example 2** that equality fails,
  Consider $\mathbb{R}$ with the standard topology and all one-point sets $\{q\}$ with q rational. Then $\overline{\{q\}} = \{q\}$ for every such set, so that $\bigcup_{q \in \mathbb{Q}} \overline{\{q\}} = \mathbb{Q}$, but $\overline{\bigcup_{q \in \mathbb{Q}} \{q\}} = \overline{\mathbb{Q}} = \mathbb{R}$.

□

Exercise 4.24 (Mun 2.17.7)**.** Criticize the following "proof" that $\overline{\cup A_\alpha} \subset \cup \overline{A_\alpha}$: if $\{A_\alpha\}$ is a collection of sets in X and if $x \in \overline{\cup A_\alpha}$, then every neighborhood U of x intersects $\cup A_\alpha$.Thus U must intersect some A_α, so that x must belong to the closure of some A_α. Therefore, $x \in \cup \overline{A_\alpha}$.

Solution: If $\{A_\alpha\}$ is a collection of sets in X and if $x \in \overline{\cup A_\alpha}$, then every neighborhood U of x intersects $\cup A_\alpha$.Thus U must intersect some A_α. But this A_α may be distinct for different neighbourhoods, so not necessarily every neighbourhood U of A_α intersects the same A_α. Hence we cannot conclude that x must belong to the closure of some fixed A_α.

Exercise 4.25 (Mun2.17.8)**.** Let A, B, and A_α denote subsets of a space X. Determine whether the following equations hold; if an equality fails, determine whether one of the inclusions $\supset$ or $\subset$ holds. (a) $\overline{A \cap B} = \overline{A} \cap \overline{B}$. (b) $\overline{\bigcup A_\alpha} = \bigcup \overline{A_\alpha}$. (c) $\overline{A - B} = \overline{A} - \overline{B}$.

Solution

(a)

a) claim 1: $\overline{A \cap B} \subseteq \bar{A} \cap \bar{B}$

Note that $\bar{A}, \bar{B}$ are closed sets, then $\bar{A} \cap \bar{B}$ is also closed set

Since, $\bar{A} \supseteq A$ and $\bar{B} \supseteq B$, then $\bar{A} \cap \bar{B} \supseteq A \cap B$

Hence, $\bar{A} \cap \bar{B}$ is closed set that contain $A \cap B$. Therefore,

$$\bar{A} \cap \bar{B} \supseteq \overline{A \cup B}$$

But other inclusion does not holds in general

Example that fails equlity

Let $A = \mathbb{Q}$ and $B = \mathbb{R} \setminus \mathbb{Q}$

Then $\bar{A} = \mathbb{R}$ and $\bar{B} = \mathbb{R}$. But $\overline{A \cup B} = \bar{\emptyset} = \emptyset$

So,

$$\bar{A} \cap \bar{B} \neq \overline{A \cap B}$$

(b)

b) claim: $\overline{\bigcap_\alpha A_\alpha} \subseteq \bigcap_\alpha (\overline{A_\alpha})$

Since, $\overline{A_\alpha}$ is closed and arbitary intersection of closed sets is closed $\bigcap_\alpha \overline{A_\alpha}$ is closed.

Since $\overline{A_\alpha} \supseteq A_\alpha$, then $\bigcap_\alpha \overline{A_\alpha} \supseteq \bigcap_\alpha A_\alpha$

Thus, $\bigcap_\alpha \overline{A_\alpha}$ is closed set that contain $\bigcap_\alpha A_\alpha$

Therefore, $\bigcap_\alpha \overline{A_\alpha} \supseteq \overline{\bigcap_\alpha A_\alpha}$

The reverse inclusion does not hold ingeneral

Example that not holding equlity

Let $A_n = \left(0, \frac{1}{n}\right)$, where $n \in \mathbb{N} \setminus \{0\}$

Then $\bigcap_{n \in \mathbb{N}} A_n = \bigcap_{n \in \mathbb{N}} \left(0, \frac{1}{n}\right) = \emptyset$. Thus. $\overline{\bigcap_{n \in \mathbb{N}} A_n} = \emptyset$

But, $\overline{A_n} = [0, 1/n]$. Then $\bigcap_{n \in \mathbb{N}} \overline{A_n} = \{0\}$

So, $\bigcap_{n \in \mathbb{N}} \overline{A_n} \neq \overline{\left(\bigcap_{n \in \mathbb{N}} A_n\right)}$

(c)

c) Claim: $\bar{A} \setminus \bar{B} \subseteq \overline{A \setminus B}$

Let $x \in \bar{A} \setminus \bar{B}$.

Assume the contray $x \notin \overline{A \setminus B}$.

That implies there exist open neighbourhood of U_x of x, such that,

$$\phi = U_x \cap (A \setminus B) = (U_x \cap A) \setminus (U_x \cap B)$$

Thus, $U_x \cap A \subseteq U_x \cap B$

Since $x \in \bar{A} \setminus \bar{B}$, then $x \notin \bar{B}$.

Hence there exist an neighbauhood V_x of x such that

$$\phi = V_x \cap B$$

Let $W = V_x \cap U_x$. Note that, W is an open neighbauhood of x. (Finite intersection of open set is open)

But

$$W \cap A = (U_x \cap V_x) \cap A \subseteq (U_x \cap V_x) \cap B = U_x \cap \underbrace{(V_x \cap B)}$$

$$= U_x \cap \phi = \phi$$

This implies $x \notin \overline{A}$. (Because we found an open neighbourhood of x (W) such that $W \cap A = \emptyset$.)

This contradict with $x \in \overline{A} \setminus \overline{B} \subseteq \overline{A}$

Therefore $x \in \overline{A \setminus B}$. Hence $\overline{A} \setminus \overline{B} \subseteq \overline{A \setminus B}$. ■

The reverse inclusion is not true in general.

Example that not hold the reverse inclusion.

Let $A := \mathbb{Q}$ and $B := (\mathbb{R} \setminus \mathbb{Q})$. Then $\overline{A} = \overline{B} = \mathbb{R}$

Then, $A \setminus B = \mathbb{Q} \setminus (\mathbb{R} \setminus \mathbb{Q}) = \mathbb{Q}$. Thus, $\overline{A \setminus B} = \overline{\mathbb{Q}} = \mathbb{R}$

But, $\overline{A} \setminus \overline{B} = \mathbb{R} \setminus \mathbb{R} = \emptyset$. Therefore,

$$\overline{A} \setminus \overline{B} \neq \overline{A \setminus B}$$

Exercise 4.26 (Mun 2.17.9)**.** Let $A \subset X$ and $B \subset Y$. Show that in the space $X \times Y$,

$$\overline{A \times B} = \overline{A} \times \overline{B}$$

.

Claim 1: $\bar{A} \times \bar{B} \supseteq \overline{A \times B}$

By exerise 2.17.3 $\bar{A} \times \bar{B}$ is closed in $X \times Y$.

Since, $\bar{A} \supseteq A$ and $\bar{B} \supseteq A$, then $\bar{A} \times \bar{B} \supseteq A \times B$.

Thus $\bar{A} \times \bar{B}$ is a closed set that contain $A \times B$. So,

$$\bar{A} \times \bar{B} \supseteq \overline{A \times B}$$

We can easily verify this result

$A \subseteq X$ and $B \subseteq Y \Rightarrow A \times B \subseteq X \times Y$

Let $a \times b \in A \times B$. Then,

$$\left.\begin{matrix} a \in A \subseteq X \\ b \in B \subseteq Y \end{matrix}\right\} \Rightarrow a \times b \in X \times Y$$

Claim 2: $\bar{A} \times \bar{B} \subseteq \overline{A \times B}$.

Let $a \times b \in \bar{A} \times \bar{B}$. Then $a \in \bar{A}$ and $b \in \bar{B}$

Let U be neighbourhood of a in X.

Let V be neighbourhood of b in Y.

Since $a \in \bar{A}$, $U \cap A \neq \emptyset$

Since $b \in \bar{B}$, $V \cap A \neq \emptyset$.

$$(U \times V) \cap (A \times B) = \underbrace{(U \cap A)}_{\neq \emptyset} \times \underbrace{(V \cap B)}_{\neq \emptyset} \neq \emptyset$$

Thus, every basis element that contain $a \times b$ intersect $A \times B$. $((U \times V) \cap (A \times B) \neq \emptyset)$. Thus $a \times b \in \overline{A \times B}$. Therefore

$$\bar{A} \times \bar{B} \subseteq \overline{A \times B}$$

By claim 1 & 2, $\bar{A} \times \bar{B} = \overline{A \times B}$

Exercise 4.27 (Mun 2.17.10)**.** Show that every order topology is Hausdorff.

Let X be a top space with order topolgy.
Let $x, y \in X$ be two distinct points.
Without loss of generality, $x < y$.

Let $A := \{x \mid a < x < b\}$
i.e: A is the set of elements of between a & b.

- If $A = \emptyset$, then
  $a \in (-\infty, b)$ and $b \in (a, \infty)$.
  So, $(-\infty, b) \cap (a, \infty) = \emptyset$.
  Hence X is Hausdorff

- If $A \neq \emptyset$. then,
  $a \in (-\infty, x)$ and $b \in (x, +\infty)$ for all $x \in A$
  So, $(-\infty, x) \cap (x, \infty) = \emptyset$. for all $x \in A$
  Hence X is Hausdorff

Exercise 4.28 (Mun 2.17.11)**.** Show that the product of two Hausdorff spaces is Hausdorff.

Let X and Y be Haurdorff spaces.
Let $a_1 \times b_1, a_2 \times b_2 \in X \times Y$ be distinct points in $X \times Y$
Since, $a_1 \times b_1 \neq a_2 \times b_2$, Then $a_1 \neq a_2$ or $b_1 \neq b_2$

- Case-I If $a_1 \neq a_2$

  Since X is Hausdorff, There exist open sets $U_1, U_2 \subseteq X$ such that $a_1 \in U_1$ and $a_2 \in U_2$ and $U_1 \cap U_2 = \emptyset$.
  Then $U_1 \times Y$ and $U_2 \times Y$ are open sets in $X \times Y$
  Note that $a_1 \times b_1 \in U_1 \times Y$ and $a_2 \times b_2 \in U_2 \times Y$
  Furthur,

  $$(U_1 \times Y) \cap (U_2 \times Y) = \underbrace{(U_1 \cap U_2)} \times (Y_1 \cap Y_2)$$
  $$= \emptyset \times (Y_1 \cap Y_2) = \emptyset$$

  Hence $X \times Y$ is Haursdorff

- Case-II If $a_1 = a_2$ then $b_1 \neq b_2$.

  This case is very similar to case-I.

Hence $X \times Y$ is Haursdorff.

Exercise 4.29 (Mun 2.17.12). Show that a subspace of a Hausdorff space is Hausdorff.

Let X be a top space. Let Y be a subspace of X. Let $a, b \in Y$. Since $a, b \in X$, there exist two open sets U_a and U_b such that $a \in U_a$ and $b \in U_b$ and $U_a \cap U_b = \phi$.

Let $V_a = U_a \cap Y$ and $V_b = U_b \cap Y$. Note that V_a and V_b is open. Futhur

$V_a \cap V_b = (U_a \cap Y) \cap (U_b \cap Y) = (U_a \cap U_b) \cap Y = \phi \cap Y = \phi$

Thus, Y is haursdorff.

Exercise 4.30 (Mun 2.17.13)**.** Show that X is Hausdorff if and only if the diagonal $\Delta = \{x \times x | x \in X\}$ is closed in $X \times X$.

"$\Longrightarrow$" Suppose that X is Haurdoff.
Let $x\times y \in (X\times X)\setminus\Delta$. Then $x\neq y$.
Since X is Haurdorff Space. There exist opensets U_x and U_y such that $x\in U_x$ and $y\in U_y$ and $U_x\cap U_y=\emptyset$. Thus,

$$x\times y\in (U_x\times U_y)\subseteq (X\times X)\setminus\Delta$$

Thus, $(X\times X)\setminus\Delta$ is open. So, Δ is closed in $X\times X$

"$\Longleftarrow$" Now suppose that Δ is closed in $X\times X$.
Let $a,b\in X$ with $a\neq b$. Then $a\times b\in (X\times X)\setminus\Delta$.
Since Δ is closed, $X\times X\setminus\Delta$ is open. Thus, there exist open set V_a and V_b of X Such that $a\times b\in V_a\times V_b\subseteq (X\times X)\setminus\Delta$. Now observe that
$a\in V_a$ and $b\in V_b$
(V_a is neighbourhood of a & V_b is neighbourhood of b)
Since $V_a\times V_b\subseteq (X\times X)\setminus\Delta$ then $V_a\cap V_b=\emptyset$.
Thus X is Haursdorff.

We can easily verify this result assume the contrary $V_a\cap V_b\neq\emptyset$. Then we can find $z\in V_a\cap V_b$. Then $z\times z\in\Delta$
But this contradict $z\times z\subseteq V_a\times V_b\subseteq X\times X\setminus\Delta$.
Thus, $V_a\cap V_b=\emptyset$

Exercise 4.31 (Mun 2.17.14)**.** In the finite complement topology on $\mathbb{R}$, to what point or points does the sequence $x_n=\frac{1}{n}$ converge?

Claim: $x_n = 1/n$ cgt to every real number $\mathbb{R}$.

Let $r \in \mathbb{R}$ and let U be an open neighbourhood of r. Since U is open in co-finite topology, $\mathbb{R} \setminus U$ is finite.

Let $S := \{n \in \mathbb{N} \mid 1/n \in \mathbb{R} \setminus U\}$. So, S is finite

- Case-I ($S = \phi$)

  Let $N = 1$. Then we have forall $n \geqslant N$, $1/n \in U$.

  Thus $x_n \longrightarrow r$.

- Case-II (If $S \neq \phi$)

  Since S is finite, we can find the $\max(S)$.

  Let $N = \max(S) + 1$. Then,

  If $n \geqslant N$, then $\frac{1}{n} \notin S$. So, $1/n \in U$. Thus

  $x_n \longrightarrow r$.

Since $r \in \mathbb{R}$ is arbitary. Thus, $x_n = 1/n$ is cgt to forall $r \in \mathbb{R}$

Exercise 4.32 (Mun 2.17.15)**.** Show the T_1 axiom is equivalent to the condition that for each pair of points of X, each has a neighborhood not containing the other.

First, we are going to proove following result.

Lemma: A top space is a T_1 space iff each singleton set is closed set

proof of Lemma:

Claim 1: If X is T_1 space then all singletons are closed

Suppose that X is T_1 space. Let $x \in X$. Let $y \in X \setminus \{x\}$

Since X is T_1 space, then there exist an open neighbourhood U_y of y such that $x \notin U_y$. So, $\{x\} \cap U_y = \emptyset$.

Let $U := \bigcup_{y \in X \setminus \{x\}} U_y$. Since arbitary union of open set is open, then U is open. Further, $X \setminus \{x\} = U$ is open.

Therefore, $\{x\}$ is closed.

claim 2: If in X every singleton set is closed then X is T_1 space.

Suppose that every singleton in X is closed.

Let $x, y \in X$ with $x \neq y$. Observe that $X \setminus \{x\}$ is open.

and $y \in X \setminus \{x\}$ and $x \notin X \setminus \{x\}$.

Therefore, X is T_1 space.

Therefore, $\begin{pmatrix} X \text{ is} \\ T_1 \text{ space} \end{pmatrix} \iff \begin{pmatrix} \text{every singleton set of } X \\ \text{is closed} \end{pmatrix}$ ∎

Now let's go back to problem.

"$\Rightarrow$" Suppose X is T_1 space. Let $x, y \in X$ with $x \neq y$. By claim 1, $\{x\}, \{y\}$ is closed.

So, $X \setminus \{x\}$ and $Y \setminus \{y\}$ is open and

$x \in X \setminus \{y\}$ and $y \in X \setminus \{x\}$

$(y \notin X \setminus \{y\}$ and $y \notin X \setminus \{x\})$

Therefore, each pair of points of X, each has neighbourhood not containing the other.

"$\Leftarrow$" Now suppose that for each distinct $a, b \in X$ has neighoods not containing the other point. It is enough to show that every singleton set is closed.

Let $x \in X$ and let $y \in X \setminus \{x\}$. By our hypothesis, there is a neighbourhood U of y that does not contain x ($x \notin U$ and $y \in U$). Thus,

$y \in U \subseteq X \setminus \{x\}$

Therefore, $X \setminus \{x\}$ is open.

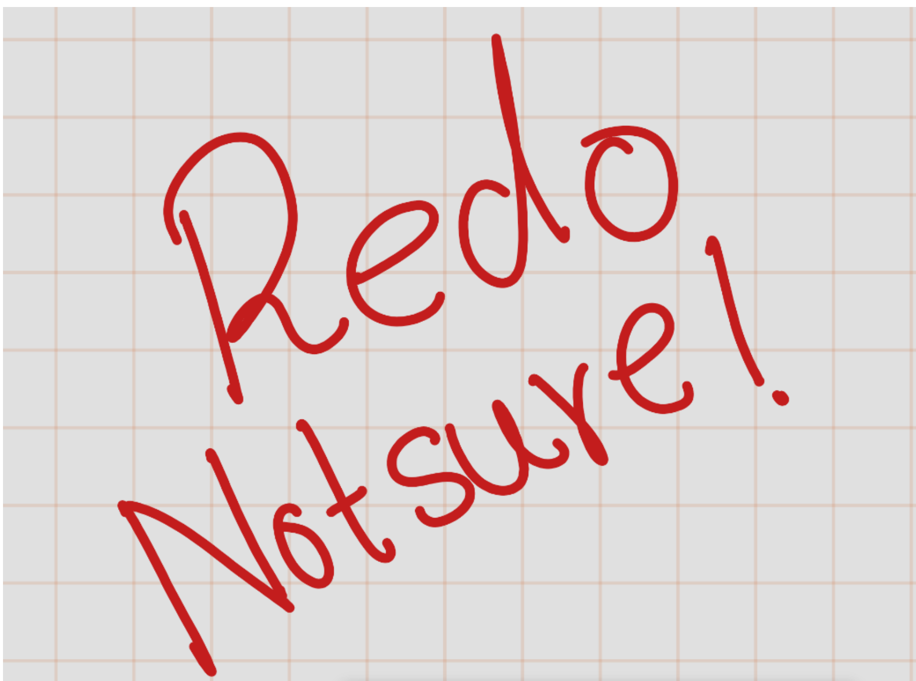
$\{x\}$ is closed.

By above lemma, X is a T_1 space. $\blacksquare$

Exercise 4.33. Consider the following five topologies on $\mathbb{R}$ given

$$\begin{aligned}
\mathcal{T}_1 &:= \text{the standard topology,} && (4.19)\\
\mathcal{T}_2 &:= \text{the topology of } \mathbb{R}_K, && (4.20)\\
\mathcal{T}_3 &:= \text{the finite complement topology,} && (4.21)\\
\mathcal{T}_4 &:= \text{the upper limit topology, having all sets } (a,b] \text{ as basis,} && (4.22)\\
\mathcal{T}_5 &:= \text{the topology having all sets } (-\infty, a) = \{x | x < a\} \text{ as basis.} && (4.23)\\
& && (4.24)
\end{aligned}$$

(a) Determine the closure of the set $K = \{1/n | n \in \mathbb{Z}^+\}$ under each of these topologies.
(b) Which of these topologies satisfy the Hausdorff axiom? the T_1 axiom?

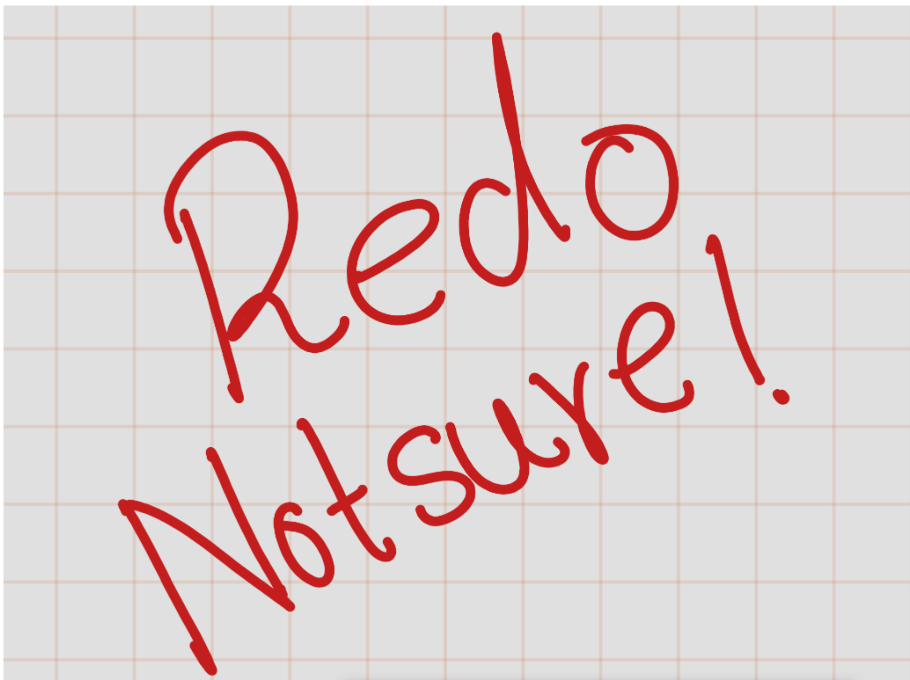


As a summary

Topologies	$\overline{K}$	Is T_2 space ?	Is T_1 Space
$\mathcal{T}_1$	$K \cup \{0\}$	✓	×
$\mathcal{T}_2$	K	✓	✓
$\mathcal{T}_2$	0	✓	✓
$\mathcal{T}_2$	K	✓	✓
$\mathcal{T}_2$	$[0, \infty)$	✓	✓

Topologies	$\overline{K}$	Is T_2 space ?	Is T_1 Space
$\mathcal{T}_2$	0	✓	✓

Exercise 4.34 (Mun 2.17.17)**.** Consider the lower limit topology on $\mathbb{R}$ and the topology given by the basis C of Exercise 8 of §13. Determine the closures of the intervals $A = (0, \sqrt{2})$ and $B = (\sqrt{2}, 3)$ in these two topologies.



Exercise 4.35 (Mun 2.17.18)**.** Determine the closures of the following subsets of the ordered square:

$$A := \left\{ \frac{1}{n} \times 0 \mid n \in \mathbb{Z}^+ \right\}, \tag{4.25}$$

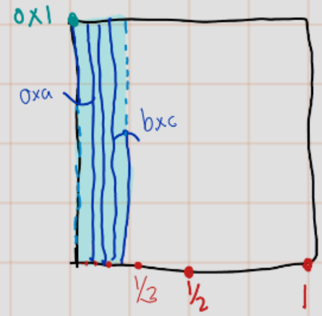
$$B := \left\{ \left(1 - \frac{1}{n}\right) \times \frac{1}{2} \mid n \in \mathbb{Z}^+ \right\}, \tag{4.26}$$

$$C := \{ x \times 0 \mid 0 < x < 1 \}, \tag{4.27}$$

$$D := \left\{ x \times \frac{1}{2} \mid 0 < x < 1 \right\}, \tag{4.28}$$

$$E := \left\{ \frac{1}{2} \times y \mid 0 < y < 1 \right\}. \tag{4.29}$$

$A := \{ \frac{1}{n} \times 0 \mid n \in \mathbb{Z}^+ \}$



claim: $A' = \{0 \times 1\}$

- First candidate: point 0×1

The neighbourhood of 0×1 is in the form $(0 \times a, b \times c)$ for some $a < 1$ and $0 < b \leq 1$ and $0 \leq c \leq 1$.

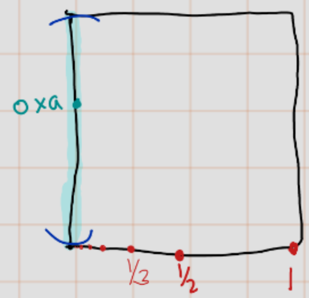
By archimeadian property we can find $n_0 \in \mathbb{N}$ such that $\frac{1}{n_0} < b$. Then if $n \geq n_0$ then $1/n < b$

So $(0 \times a, b \times c)$ contains all points $(1/n) \times 0$ for sufficiently large n. Thus 0×1 is a limit point.

- Next possible candidate: Points in the form $0 \times a$, $0 < a < 1$

No points $0 \times a$ for $a < 1$ is a limit point.

$(0 \times a) \in (0 \times 0, 0 \times 1)$
but $\frac{1}{n} \times 0 \notin (0 \times 0, 0 \times 1)$ for all $n \in \mathbb{N}$
Thus,

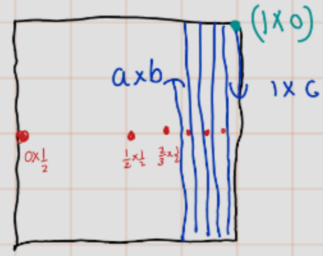


- Next candidate: 0×0 point.

Then $0 \times 0 \in [0 \times 0, 0 \times 1)$
but $\frac{1}{n} \times 0 \notin [0 \times 0, 0 \times 1)$ for all $n \in \mathbb{N}$

Thus, the possible limit point is 0×1.
Hence $A' = \{0 \times 1\}$.
Therefore $\overline{A} = A \cup A' = A \cup \{0 \times 1\}$

$B := \{ 1 - 1/n \mid n \in \mathbb{Z}^+ \}$



- First candidate: 1×0

The neighbourhood of (1×0) is in the form $(a \times b, 1 \times c)$ for some $c < 1$ and $0 \leq a < 1$ and $0 \leq b \leq 1$.

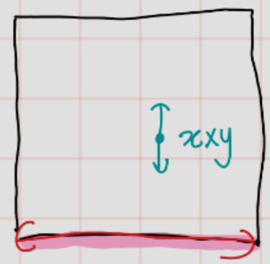
By archimeadian property we can find $n_0 \in \mathbb{N}$ such that $1 - \frac{1}{n_0} > a$. Then if $n \geq n_0$ then $1 - \frac{1}{n_0} > a$

So $(a \times b, 1 \times c)$ contains all points $1 - \frac{1}{n} \times \frac{1}{2}$ for sufficiently large n.
Thus 1×0 is a limit point

Recall Archmiedian Property
If $x, y \in \mathbb{R}$ with $x > 0$, then $\exists n \in \mathbb{N}$ s.t. $nx > y$
$x = (1-a) > 0$, $y = 1$.
$\exists n \in \mathbb{N}$ $n(1-a) > 1$
$1 - a > \frac{1}{n}$
$1 - \frac{1}{n} > a$

- Similar to above case we can show that the $1 \times c \notin B'$, where $0 \leq c < 1$. Thus only possible limit point is 1×0. Thus, $B' = \{1 \times 0\}$. Thus $\overline{B} = B \cup B' = B \cup \{1 \times 0\}$

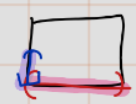
$C := \{x<0 \mid 0<x<1\}$



- Candiate no 1

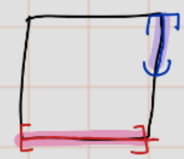
Any point $x\times y$ with $0<y<1$ and $0\leq x\leq 1$.

Then $x\times y$ has a neighbourhood that is not intersect with C. Thus it is not a limit point.



- candidate. 2 : 0×0 point

$0\times 0 \in [0\times 0, \frac{1}{2}\times 0]$ but $[0\times 0, \frac{1}{2}\times 0] \cap C = \emptyset$. Thus, 0×0 is NOT a limit point



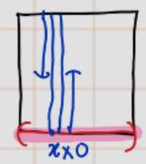
- Candidate 3: 1×1

$1\times 1 \in (1\times\frac{1}{2}, 1\times 1]$ and $(1\times\frac{1}{2}, 1\times 1] \cap C$

Thus 1×1 is NOT a limit point

- Candidate 4: $x \times 0$ with $0 < x \leq 1$

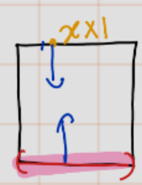
Let $(p \times q, r \times s)$ be an open neighbohood of $x \times 0$.
Then $p < r$. Then we can find $t \in I$ s.t.; $p < t < r$
Then $t \times 0 \in (p \times q, r \times s)$ and $t \times 0 \in C$.



Any neighbourhood of $x \times 0$ intersect the C.

Candidates: $x \times 1$, with $0 \leq x < 1$

Any neighbourhood of $x \times 1$ intersect the
(Similar to above case)



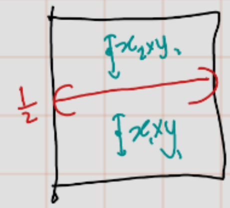
Thus $C' = \{x \times 0 \mid 0 < x \leq 1\} \cup \{x \times 1 \mid 0 \leq x < 1\}$

So, $C' = ((0,1] \times \{0\}) \cup ([0,1) \times \{1\})$

Therefore $\overline{C} = C \cup C'$

$= C \cup (0,1] \times \{0\} \cup [0,1) \times \{1\}$

$D = \left\{ x \times \frac{1}{2} \mid 0 < x < 1 \right\}$



Candidate1: Any point $x \times y$ with $0 < y < 1/2$

choose $\varepsilon = \min\left\{ \left|\frac{1}{2} - y\right|, |y| \right\}$. Then $x \times y \in (x \times y - \varepsilon, x \times y + \varepsilon)$
and $(x \times y - \varepsilon, x \times y + \varepsilon) \cap D$. Thus these points are not limit point.

Candidate no: 2 Any point $x \times y$ with $1/2 < y < 1$

Similary we can show that this is not a limit point.

Candidate no3: Any point $0 \times y$ and $0 \le y < 1$

$0 \times y \in [0 \times 0, 0 \times 1)$ but
$[0 \times 0, 0 \times 1] \cap D$. So, these
points are not limit points.

candidate no 4: Any point $1 \times y$ with $0 < y \leq 1$

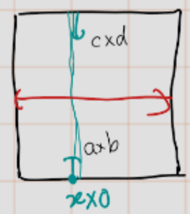
$1 \times y \in (1 \times 0, 1 \times 1]$ but $(1 \times 0, 1 \times 1] \cap D$. Thus this are not limit points

Candidate no 5: $x \times 0$ with $0 < x \leq 1$

Let $(a \times b, c \times d)$ be an open set such that $x \times 0 \in (a \times b, c \times d)$

Thus, $a < c$. Then we can find $p \in I \subseteq \mathbb{R}$ suchthat $a < p < c$. Then

$$p \times \tfrac{1}{2} \in (a \times b, c \times d)$$



So any neighbourhood of $x \times 0$ must include a entire vertical segment that intersect D.

Thus $x \times 0$ with $0 < x \leq 1$ is a limit point.

candidate no 6: $x \times 1$ with $0 \leq x < 1$

Similar to case 5, we can proove that these type of points are limit points.

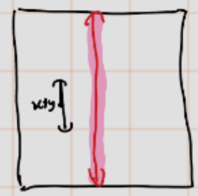
Hence, $D' = (0, 1] \times \{0\} \cup [0, 1) \times \{1\}$

$$= \{x \times 0 \mid 0 < x \leq 1\} \cup \{x \times 0 \mid 0 \leq x < 1\}$$

$$\overline{D} = D \cup D'$$

$$= D \cup (0, 1] \times \{0\} \times [0, 1) \times \{1\}$$

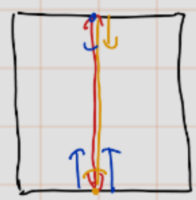
e) $E = \{\frac{1}{2} \times y \mid 0 < y < 1\}$



Candidate no1: $x \times y$ with
$0 \leq x < 1/2$. or $1/2 < x \leq 1$.

choose $\varepsilon = \min\{|y|, |1-y|\}$

Then $x \times y \in (x \times y - \varepsilon, x \times y + \varepsilon)$ and $(x \times y - \varepsilon, x \times y + \varepsilon) \cap E$.
Thus, there exist a neighbood disjont form E.



Candidate no 2: $\frac{1}{2} \times 0$ and $\frac{1}{2} \times 1$

Then every neighbood of above points must intersect the set E. Thus these points are limit points.

Candidate no 3: $\frac{1}{2} \times x$ with $0 < x < 1$

Then every neighbourhood that intesect with C other than point

Thus $E' = \{\frac{1}{2} \times 0, \frac{1}{2} \times 1\} \cup E$
Therefore $\bar{E} = E \cup E'$
$= E \cup \{\frac{1}{2} \times 0, \frac{1}{2} \times 1\}$

Exercise 4.36 (Mun 2.17.19)**.** If $A \subset X$, we define the boundary of A by the equation

$$\operatorname{Bd} A = \overline{A} \cap \overline{(X \setminus A)}.$$

(a) Show that $\operatorname{Int} A$ and $\operatorname{Bd} A$ are disjoint, and $\overline{A} = \operatorname{Int} A \cup \operatorname{Bd} A$.
(b) Show that $\operatorname{Bd} A = \emptyset \Leftrightarrow A$ is both open and closed.
(c) Show that U is open $\Leftrightarrow \operatorname{Bd} U = \overline{U} \setminus U$.
(d) If U is open, is it true that $U = \operatorname{Int}(\overline{U})$? Justify your answer.

Solution

(a) Show that Int A and Bd A are disjoint, and $\overline{A} = \text{Int}\, A \cup \text{Bd}\, A$.

claim: $\text{Int}(A) \cap Bd(A) = \emptyset$

Assume the contray. $\text{Int}(A) \cap Bd(A) \neq \emptyset$. Then, we can choose $x \in \text{Int}(A) \cap Bd$

Since $x \in \text{Int}(A)$, then there exist an neighbourhood U of x such that

$$x \in U \subseteq A. \text{ --- } (*)$$

Since $x \in Bd(A) = \bar{A} \cap \overline{(X \setminus A)}$. Thus, $x \in \overline{(X \setminus A)}$.

By Mun theorm 2.17.5 every neighbaurhood U of x intersect wit $(X \setminus A)$

$$U \cap (X \setminus A) = \emptyset$$

But this contrdict $(*)$ $(U \subseteq A)$.

Thus, $\text{Int}(A) \cap Bd(A) = \emptyset$.

claim 2: $\bar{A} = \text{Int}(A) \cup Bd(A)$

subclaim 2.1 $\bar{A} \subseteq \text{Int}(A) \cup Bd(A)$

Let $x \in \bar{A}$.

- If there exist an open neighbour hood U of x such that $U \subseteq A$, then $x \in U \subseteq \text{Int}(A)$

  Then $x \in \text{Int}(A)$ --- $(*)$

- Otherwise every open neighbourhood of x intersect $X \setminus A$. Thus $x \in \overline{X \setminus A}$. Thus $x \in \overline{(X \setminus A)} \cap \overline{X} = Bd(A)$ (**)

By (*) and (**), $x \in Int(A) \cup Bd(A)$.
Thus $\overline{A} \subseteq Int(A) \cup Bd(A)$ — (1)

Subclaim 2.2: $\overline{A} \supseteq Int(A) \cup Bd(A)$.

Let $y \in Int(A) \cup Bd(A)$.

- If $y \in Int(A)$. Thus, $y \in A$. So, $y \in \overline{A}$
- If $y \in Bd(A)$. Then by definition of boundary points
  $y \in Bd(A) = \overline{A} \cap \overline{(X \setminus A)}$

Thus $y \in \overline{A}$. Therefore, $\overline{A} \supseteq Int(A) \cup Bd(A)$. — (2)

By (1) and (2), $\overline{A} = Int(A) \cup Bd(A)$

(b) Show that Bd $A = \emptyset \Leftrightarrow A$ is both open and closed.

Claim: $bd(A) = \phi$ iff (A is both open & closed)

"$\Rightarrow$": Suppose that $Bd(A) = \phi$. Then,

$\bar{A} = Int(A) \cup Bd(A) = \bar{A} = Int(A) \cup \phi$

$\bar{A} = Int(A)$.

We know that, $Int(A) \subseteq A \subseteq \bar{A}$. Thus, $A = \bar{A} = Int(A)$

Since $A = Int(A)$, A is open
Since $A = \bar{A}$, A is closed.
$\Rightarrow$ A is both open and closed.

"$\Leftarrow$" Suppose that A is closed & open.

A is open $\Rightarrow int(A) = A$

A is closed $\Rightarrow A = \bar{A}$

Thus, $\bar{A} = Int(A) \cup Bd(A)$

$A = A \cup Bd(A)$

By part a) we know that $Int(A) \cap Bd(A) = \phi$

Thus $Bd(A) = \phi$

Therefore, $Bd(A) = \phi \iff$ A is both open, closed claim: U is open $\Leftrightarrow Bd(U) = \overline{U} \setminus U$

$\Rightarrow$ Suppose that U is open. Then $(X \setminus U)$ is closed. Thus, $\overline{(X\setminus U)} = (X \setminus U)$. Then,

$$\begin{aligned} bd(U) &= \overline{U} \cap \overline{(X\setminus U)} \\ &= \overline{U} \cap (X \setminus U) \\ &= (\overline{U} \setminus U). \end{aligned}$$

"$\Leftarrow$" Now Suppose that $bd(U) = \overline{U} \setminus U$ — ①

By def^{n} of $Bd(A)$
$$\begin{aligned} Bd(A) &= \overline{A} \cap \overline{(X\setminus A)} \\ &= \overline{(X\setminus A)} \cap \overline{(X\setminus(X\setminus A))} \\ &= Bd(X\setminus A) \end{aligned}$$

$Bd(X\setminus U) = \overline{U} \setminus U = \overline{U} \cap (X\setminus U) \subseteq X \setminus U$ — (i)

By part a) we know that
$\operatorname{Int}((X\setminus U) \cup Bd(X\setminus U) = \overline{X\setminus U}$

Note that all that $\operatorname{Int}(X\setminus U) \subseteq X \setminus U$
$Bd(X\setminus U) \subseteq X\setminus U$ By (i)

(c) Show that U is open $\Leftrightarrow \operatorname{Bd} U = \overline{U} \setminus U$.

Then $(X\setminus U) \supseteq \underline{\mathrm{Int}(X\setminus U) \cup Bd(X\setminus U)} = \overline{X\setminus U}$

i.e: $(X\setminus U) \supseteq \overline{X\setminus U}$

So, $X\setminus U = \overline{X\setminus U}$

Thus, $X\setminus U$ is closed. Thefore U is open.

Therefore U is open iff $Bd(U) = \overline{U}\setminus U$.

d)(d) If U is open, is it true that $U = \mathrm{Int}(\overline{U})$? Justify your answer.

No!

Counter example1

Let $U = (1,2) \cup (2,3) \subseteq \mathbb{R}$ (with usual topology).

$\overline{U} = [1,3]$

$\mathrm{Int}(\overline{U}) = (1,3) \neq (1,2) \cup (2,3) = U$

Counter example 2:

Let $U = \mathbb{R}\setminus\{0\}$. Then $\overline{U} = \mathbb{R}$. But $\mathrm{Int}(\overline{U}) = \mathbb{R} \neq U$

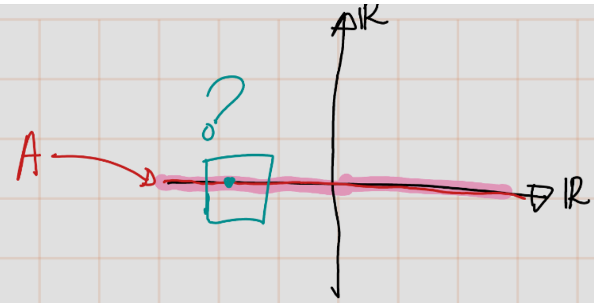
Exercise 4.37 (Mun 2.17.20)**.** Find the boundary and the interior of each of the following subsets of $\mathbb{R}^2$:

(a) $A = \{(x,y)|y = 0\}$
(b) $B = \{(x,y)|x > 0 \text{ and } y \neq 0\}$
(c) $C = A \cup B$
(d) $D = \{(x,y)|x \text{ is rational}\}$
(e) $E = \{(x,y)|0 < x^2 - y^2 \leq 1\}$
(f) $F = \{(x,y)|x \neq 0 \text{ and } y \leq \frac{1}{x}\}$

Solution

(a) $A = \{(x,y)|y = 0\}$

a)



Observe that any neighbourhood of a point A not contains in A. Thus $\text{int}(A) = \emptyset$.

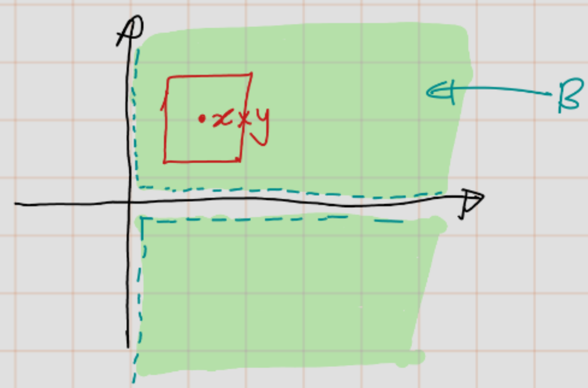
Furthur, A is closed. So, $\bar{A} = A = \text{Int}(A) \cup \text{Bd}(A)$

$$A = \emptyset \cup \text{Bd}(A)$$

$$A = \text{Bd}(A)$$

(b) $B = \{(x, y) | x > 0 \text{ and } y \neq 0\}$

b)



claim: B is open

Let $x \times y \in B$.

Choose $\varepsilon = x/2$ and $\delta = |y|/2$

Then $x \times y \in (x-\varepsilon, x+\varepsilon) \times (y-\delta, y+\delta) = U$

This U is open in product topology on $\mathbb{R}^2$ which is standard topology.

Thus B is open.

Thus $\mathrm{Int}(B) = B$.

(c) $C = A \cup B$

(d) $D = \{(x, y) | x \text{ is rational}\}$

(e) $E = \{(x, y) | 0 < x^2 - y^2 \leq 1\}$

(f) $F = \{(x, y) | x \neq 0 \text{ and } y \leq \frac{1}{x}\}$

4.4 Section 18 in Munkress Book

Exercise 4.38 (Mun 2.18.1)**.** Proove that the function $f : \mathbb{R} \to \mathbb{R}$, the $\epsilon - \delta$ defintion of continuty implies the open set definition.

$(\mathbb{R}, \tau_{stand})$ consider

① Let $f : \mathbb{R} \longrightarrow \mathbb{R}$, the $\varepsilon-\delta$ ddefintio implies opensetdefintio

Suppose that f is continousat $x \in \mathbb{R}$

Suppose f $\varepsilon-\delta$ definition for continuity.

Let (a, b) be a basis element of standerd topolgy of $\mathbb{R}$

Let $x \in f^{-1}((a,b)) \Rightarrow f(x) \in (a,b)$

Let $\varepsilon = \min\{|f(x)-a|, |f(x)-b|\}$

Then f is continuous under ε-δ definition,

$\exists\, \delta > 0$ suchthat

$|x-y| < \delta \Rightarrow |f(x) - f(y)| < \varepsilon$ — ⊛

⊛ statment equivallent followinstatmen

$f(x-\delta, x+\delta) \subseteq (f(x)-\varepsilon, f(x)+\varepsilon) \subset (a,b)$

$\Rightarrow (x-\delta, x+\delta) \subseteq f^{-1}((a,b))$

$x \in (x-\delta, x+\delta)$ isopen subsetof $\mathbb{R}$. So

Thus. $f^{-1}(a,b)$ is open.

Therefore f is continous underthe opensetdefintion.

Exercise 4.39 (Mun 2.18.2)**.** Suppose that $f : X \to Y$ is continous. If x is a limit point of sub set of A of X, is it necessary true that $f(x)$ is a limit point of $f(A)$?

(2) Suppose $f: X \longrightarrow Y$ is continuous. If x is a limit point of subset A of X.
Is it nessary true that $f(x)$ is limit point of $f(A)$

No. See following example.

$f: \mathbb{R} \longrightarrow \mathbb{R}$,

$f(x) = 2$ for all $x \in \mathbb{R}$ and

c is a constant.

$\cancel{f(\tfrac{3}{2}) = 2}$

$3 \in \mathbb{R}$ is a limit point of $\mathbb{R}$

$f(3) = 2$

$f(\mathbb{R}) = \{2\}$ ~~since $f(\mathbb{R})$~~

Since $f(\mathbb{R})$ is singlton it has no limit points

Thus $f(3) = 2$ is not a limit point.

Exercise 4.40 (Mun 2.18.3)**.**

Let X and X' denote a single set in the two topologies $\mathcal{T}$ and $\mathcal{T}'$, respectively. Let $i : X' \to X$ be the identity function.

1. Show that i is continuous $\Leftrightarrow$ $\mathcal{T}'$ is finer than $\mathcal{T}$.

2. Show that i is a homeomorphism $\Leftrightarrow$ $\mathcal{T}' = \mathcal{T}$.

③a) Let X and X' be a single set in two topologgies τ and τ' respectivel,

Let define $i: X \longrightarrow X$ be identity ma

a) N.TS: i is continus $\Longleftrightarrow$ τ' is finer than τ

" $\Longrightarrow$ "

Suppose that i is continous

~~$\Rightarrow$ Then $i^{-1}(U)$ is open in X' for all U open set in X~~

~~Since i is identity~~

Let U is opensentin X.

Then $i^{-1}(U) = U$ also open in X'

Thus $\tau \subset \tau'$

Thus τ' is fnear than τ

" $\Longleftarrow$ " Suppose that $\tau' \supset \tau$

~~let V be an openset in X'~~

~~$i^{-1}(V) = V$ i.e: $V \in \tau'$~~

Let V be an open set in X.

ie: $V \in \tau \subset \tau' \Rightarrow$

$\Rightarrow$ V is open set in X'

$\Rightarrow$ $i^{-1}(V) = V$ is also in X'

$\Rightarrow$ Thu, i is continuuoüs

) N.T.S: ~~i is hoemo~~

i is ~~tmo~~ homeomorphism $\Leftrightarrow \tau = \tau'$

$\Rightarrow$

Suppose that i is homemorphism.

Then i is continuous map.

Then part (a) gives that $\tau \subset \tau'$ — 1

~~Since~~ Futher i^{-1} is also continaous map

Smilary using part (a) we can get that

$\tau \subset \tau'$ — 2

By 1 and 2 $\tau = \tau'$

$\Leftarrow$ Now suppose that $\tau = \tau'$

Let U' be an open set int X'. Then

$i(U') = U'$ is ~~also~~ open in X also.

i^{-1} is continous. — 3

Let V' be an open in X. Then

$i^{-1}(V') = V'$ is open in X' also. Then

i is ~~also~~ also continuos map — 4

Clearly i is abijection function — 5

By 3, 4, and 5 we get, i is homemorphism

Exercise 4.41 (Mun 2.18.4)**.** Given $x_0 \in X$ and $y_0 \in Y$, show that the maps $f : X \to X \times Y$ and $g : Y \to X \times Y$ defined by $f(x) = x \times y_0$ and $g(y) = x_0 \times y$

are imbeddings.

NTs: the restriction of f and g to their images to leads to homeomorphism.

$Img(f) := X \times \{y_0\} \subseteq X \times Y$
$Img(g) := \{x_0\} \times X \subseteq X \times Y$

Let $f' : X \longrightarrow X \times \{y_0\}$
$x \longmapsto f(x)$

Let $g' : Y \longrightarrow \{x_0\} \times Y$
$x \longmapsto g(x)$

surjective

- clearly, f' and g' are surjective to its image.

injective

- Suppose $f'(x_1) = f'(x_2)$
$f(x_1) = f(x_2)$
$(x_1 \times y_0) = (x_2 \times y_0)$
Thus $x_1 = x_2$
$\Rightarrow$ Therefore, f' is injective

Similary we can show that g' is injective
Thefore f', g' are bijective.

claim: f' is continous.

Let U be an open set in $X \times \{y_0\}$

By subspace topology,

$$U := (X \times \{y_0\}) \cap (V \times W)$$

forsome $V \times W$ is open in $X \times Y$, where V is open in X and W is open in Y.

$$\begin{aligned} \text{So, } f'^{-1}(U) &:= f'^{-1}\big((X \times \{y_0\}) \cap (V \times W)\big) \\ &= f'^{-1}\big((X \cap V) \times (\{y_0\} \cap W)\big) \\ &= f'^{-1}(X \cap V) \\ &= (X \cap V) \\ &\Rightarrow V \text{ is open in } X \end{aligned}$$

Hence, f' is continous.

Similarly we can show that g' is also continous

Claim: $(f')^{-1}$ is continous

Let $\tilde{U}$ be an open set in X

$$((f')^{-1})^{-1}(\tilde{U}) = f'(U) = U \times \{y_0\}$$
$$= (X \cap U) \times (Y \cap \{y_0\})$$
$$= (X \times \{y_0\}) \cap (U \times Y)$$

Note that this set is open in $X \times \{y_0\}$
Hence $(f')^{-1}$ is continuous

Similary we can show that $(g')^{-1}$ is continuous

Thus f and g imbeddings.

Exercise 4.42 (Mun 2.18.5)**.** Show that the subspace (a, b) of $\mathbb{R}$ is homeomorphic with $(0, 1)$ and the subspace $[a, b]$ of $\mathbb{R}$ is homeomorphic with $[0, 1]$.

$\boxed{18.5}$

5. Show that the subspace (a, b) of $\mathbb{R}$ is homeomorphic with $(0, 1)$ and the subspace $[a, b]$ of $\mathbb{R}$ is homeomorphic with $[0, 1]$.

Consider $(\mathbb{R}, \tau_{stand})$ $(a \neq b)$

Define $f : (a,b) \longrightarrow (0,1)$

$x \longmapsto \dfrac{x-a}{b-a} \quad \forall x \in (a,b)$

claim 1: f is continuous

f^{-1} can be defined as follows

$f^{-1} : (0,1) \longrightarrow (a,b)$

$y \longmapsto a + (b-a)y$

Let $(a_0, b_0) \subseteq (0,1)$ open set in $(0,1)$

$f^{-1}((a_0, b_0)) = (a + (b-a)a_0 \,,\, a + (b-a)b_0)$

$$0 < a_0 < b_0 < 1$$

$$0 < (b-a)a_0 < (b-a)b_0 < b-a$$

$$a < a+(b-a)a_0 < a+(b-a)b_0 < a+(b-a)$$

$$a < a+(b-a)a_0 < a+(b-a)b_0 < b$$

Thus $(a+(b-a)a_0, a+(b-a)b_0)$ is an open set in subspace of (a,b)

thus f is continous map

Claim: f is one to one

Let $x, y \in (a,b)$

Suppose $f(x) = f(y)$

$$\frac{x-a}{b-a} = \frac{y-a}{b-a}$$

$$x = y$$

Thus, f is one to one.

Claim: f is onto (surjective)

Let $z \in (0,1)$ and
let $z_0 = a + (b-a)z$

$$f(z_0) = \frac{(\cancel{a} + \cancel{(b-a)}z) - \cancel{a}}{\cancel{(b-a)}}$$

$$f(z_0) = z$$

if $z \in (0,1)$ then $\exists z_0 \in (a,b)$ suchthat $f(z_0) = z$

Claim: f^{-1} is also continuous

Let $(\tilde{a}, \tilde{b}) \subseteq (a,b)$ be a opensetin
in subspace (a,b)

$$f((\tilde{a}, \tilde{b})) = \left(\frac{\tilde{a}-a}{(b-a)}, \frac{\tilde{b}-a}{(b-a)}\right)$$

Then $\left(\frac{\tilde{a}-a}{b-a}, \frac{\tilde{b}-a}{b-a}\right)$ is a openset
in $(0,1)$ Thus f^{-1} isalso continuous
Thus f is a homeomorphism. Thus, $(0,1)$ is
and (a,b) is homeopohic

Using the same map $f : [a,b] \to [0,1]$
$f(x) = \frac{x-a}{b-a}$ we can similary showthat

$[a,b]$ is homeoomorphic to $[0,1]$

Exercise 4.43 (Mun 2.18.6)**.** Find a function $f : \mathbb{R} \to \mathbb{R}$ that is continuous at precisely one point.

Let $f : \mathbb{R} \longrightarrow \mathbb{R}$

$$f(x) := \begin{cases} x & \text{if } x \in \mathbb{Q} \\ 0 & \text{if } x \notin \mathbb{Q} \end{cases}$$

Claim 1: f is continuous at 0.

Let U be neighbourhood of $f(0) = 0$.

Cose $U = V$. $\qquad f^{-1}(U)$

Sub claim $f(U) \subseteq V$

$x \in f(U) \rightarrow$ if $x \notin \mathbb{Q} \Rightarrow x = 0 \in U = V$

if $x \in \mathbb{Q} \Rightarrow x = f(x) \in f(U)$

$\Rightarrow x \in U = V$

Hence, $f(U) \subseteq V$.

By Mun thm 18.1, we can get f is continuous at 0 claim2: f is not not continuous at any point other than 0.

subclaim 2.1: Discontinuntiy at non-zero rational point

Let $x_0 \in \mathbb{Q}$. with $x_0 \neq 0$. Let U be an open neighbourhood of $f(x_0)$ defined by,

$$f(x_0) = x_0 \in U = \left(x_0 - \frac{1}{2}|x_0|,\ x_0 + \frac{1}{2}|x_0|\right)$$

Then note that $0 \notin U$. — (*)

But forevery open neighbourhood V_{x_0} of x_0, contain $p \in \mathbb{R}\setminus\mathbb{Q}$ suchthat $p \in V_{x_0}$.

But $f(p) = 0 \in f(V_{x_0}) \not\subseteq U$ (By (*))

Thus, f is not continuous at x_0

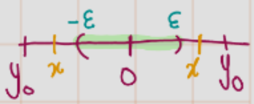
Therefore, f is not continous at any rational number other than 0.

subclaim 2.2: Discontinuity at non-zero irrational number

Let $y_0 \in \mathbb{R} \setminus \mathbb{Q}$. Let V be an open neighbourhood of $f(y_0) = 0$, where $0 < \varepsilon < |y_0|$

For every open neighbourhood U of y_0, we can find $x \in \mathbb{Q}$ such that $x \in U$ and $x \notin V$. So,

$$f(x) = x \notin V. \text{ So, } f(x) \notin V.$$



Thus f is not continous at y_0

Therefore f is not continuous at every irrational numbers