Option Pricing Theory - Basics to advanced

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2023

Contents

1 Pre-requisite theory 5
1.1 Probability Space and Filtrations 5
1.2 Probability Measures 5
1.3 Equivalent Measures and Radon-Nikdoym Derivative 6
1.4 Girsanov's Theorem for change of measure 8
1.5 Fundamental theorems of Asset Pricing 8

List of Figures

List of Tables

1 Pre-requisite theory

1.1 Probability Space and Filtrations

For any continuous sample space Ω, we can define a $\sigma - algebra$ $\mathcal{F}$ on the space which fulfils the following conditions:

- The algebra contains both the entire set Ω and the null set $\emptyset$, i.e $\Omega, \emptyset \in \mathcal{F}$
- For any set $A_i \in \mathcal{F}$, the complement, A_i^C is also in $\mathcal{F}$
- For any set $A_i \in \mathcal{F}$, the union is also in $\mathcal{F}$
- For any set $A_i \in \mathcal{F}$, the intersection is also in $\mathcal{F}$

Thus for a $\sigma - algebra$ containing the set A_1, A_2, then that set will also contain:

$$\mathcal{F} = \{\Omega, \emptyset, A_1, A_2, A_1^C, A_2^C, A_1 \cup A_2, A_1^C \cup A_1, A_2^C \cup A_1, A_2^C \cup A_2, A_1^C \cup A_2, A_2^C,$$

$\mathrm{A}_1 \cap A_2, A_1^C \cap A_1, A_2^C \cap A_1, A_2^C \cap A_2, A_1^C \cap A_2\}(1)$

The $\sigma - algebra$ is thus closed under complement, union and intersections, meaning any set operation on any set in the algebra will lead to another element in the algebra.

1.2 Probability Measures

Given our space $(\Omega, \mathcal{F})$, we can then define some function $\mathbb{P}$ such that $\mathbb{P} : \omega \rightarrow [0,1] \; \forall \; \omega \in \mathcal{F}$ This probability measure $\mathbb{P}$ will fulfil the following conditions:

- $\mathbb{P}(\Omega) = 1$, $\mathbb{P}(\emptyset) = 0$
- $\mathbb{P}(A_i) \geq 0 \; \forall \; A_i \in \mathcal{F}$

- For any disjoint set $A_i \in \mathcal{F}$, the probability of the unions of sets will equal the sum of the individual probabilities:

$$\mathbb{P}(\bigcup_{i=1}^{n} A_i) = \sum_{i=1}^{n} \mathbb{P}(A_i) \tag{2}$$

1.3 Equivalent Measures and Radon-Nikdoym Derivative

Two Probability Measures $\mathbb{P}$ and $\mathbb{Q}$ are said to be equivalent if they are mutually absolutely continious with respect to each other, i.e:

$$\mathbb{Q} \ll \mathbb{P} \; and \; \mathbb{P} \ll \mathbb{Q} \tag{3}$$

Iff:

$$\mathbb{Q}(A_i) = 0 \; \forall \; A_i \; where \; \mathbb{P}(A_i) = 0 \; and \; \mathbb{P}(A_i) = 0 \; \forall \; A_i \; where \; \mathbb{Q}(A_i) = 0 \tag{4}$$

Put simply, the measure $\mathbb{Q}$ and $\mathbb{P}$ have to agree on what is possible and what is impossible for them to be equivalent.

We can define a new random variable Z which defines the relationship between two equivalent measures such that:

$$\mathbb{Q}(A) = \int_A Z(\omega) d\mathbb{P}(\omega) \tag{5}$$

We can then take the derivative and describe the random variable Z as:

$$Z = \frac{d\mathbb{Q}}{d\mathbb{P}} \tag{6}$$

Given that the random variable Z is the ratio between the changes in the two measures, it has to be non-negative. Furthermore, the random variable is defined such that EZ = 1. We can also define the relationship between expectations under the different measures below:

$$E^{\mathbb{Q}}[X] = E^{\mathbb{P}}[ZX] \ and \ E^{\mathbb{P}}[X] = E^{\mathbb{Q}}[\frac{X}{Z}] \tag{7}$$

Novikovs Theorem

When the Radon-Nikodym derivative is defined as:

$$Z_{\Theta}(t) = e^{\int_0^t \Theta dW - \frac{1}{2}\int_0^t \Theta^2 ds} \tag{8}$$

Then, in order to have EZ = 1 and for Z to be a martingale, Θ, which is some deterministic function, will have to fulfil a certain condition known as Novikovs condition (this is a strong form condition):

$$E[e^{\frac{1}{2}\int_0^t \Theta^2 ds}] < \infty \tag{9}$$

If Θ is a constant, then it is sufficient for the following to be true:

$$\int_0^t \Theta^2 ds < \infty \tag{10}$$

If these are fulfilled, then Girsanovs Theorem will be applicable to the change of measure from one measure to an equivalent measure which is defined by this Radon-Nikodym derivative.

1.4 Girsanov's Theorem for change of measure

1.5 Fundamental theorems of Asset Pricing

In order to start pricing any contract or derivative, we start with the fundamental theorems that are to be fulfilled.

We start with the first theorem:

FTAP 1 - For a market to be arbitrage free, at least one Equivalent Martingale Measure should exist