FE Project
IEORE 4706 Foundations for Financial Engineering

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Contents

Perpetual American Options

Exercise Region

For a put option the pay-off are only positive if the value of the underlying security is less than the strike price. Thus if x^* is the optimal price at which we exercise the option then, necessarily for a positive payoff x^* has to be less than K. This means that the range of values for x^* is $(0, K]$, 0 is not included in the range because if the value of the underlying security hit 0 it cannot be traded in the market. We also know that yield of a put option increases with a decrease in the value of the underlying security. Thus if it is optimal to exercise the option at x^* it will always be optimal to exercise the option for all the prices of the security that are below $x*$. Thus the range of the price of the security (S) is $(0, x^*]$. Again 0 is not included because if the value of the security becomes zero it cannot be traded on the market.

Solution of the differential equation

If $x > x^*$ this will mean that the exercise is not optimal, and the payoff of the option is less than the value i.e. $v(x) - h(x) >$. Thus from the Hamilton-Jacobi-Bellman equation v must satisfy the following differential equation.

$$rv(x) - \mu x v'(x) - \frac{1}{2}\sigma^2 x^2 v''(x) = 0$$
$$x^2 v''(x) + \frac{2\mu x}{\sigma^2} v'(x) - \frac{2r}{\sigma^2} v(x) = 0$$

Let the solution to the above general equation be $v(x) = x^\lambda$. Then $v'(x) = \lambda x^{(\lambda-1)}$ and $v''(x) = \lambda(\lambda-1)x^{(\lambda-2)}$. Thus the above equation becomes.

$$\lambda(\lambda-1)x^\lambda + \frac{2\mu = x}{\sigma^2}\lambda x^\lambda - \frac{2r}{\sigma^2}x^\lambda = 0$$
$$\lambda(\lambda-1) + \frac{2\mu x}{\sigma^2}\lambda - \frac{2r}{\sigma^2} = 0$$
$$\sigma^2\lambda^2 + (2\mu - \sigma^2)\lambda - 2r = 0$$
$$\lambda = \frac{-(2\mu - \sigma^2) \pm \sqrt{(2\mu - \sigma^2)^2 + 8\sigma r}}{2\sigma^2}$$

Let m and n be the solution to the equation. Then

$$m = \frac{-(2\mu - \sigma^2) - \sqrt{(2\mu - \sigma^2)^2 + 8r\sigma}}{2\sigma^2}$$
$$n = \frac{-(2\mu - \sigma^2) + \sqrt{(2\mu - \sigma^2)^2 + 8r\sigma}}{2\sigma^2}$$

Where $m \leq 0$ and since $8r\sigma^2 > 0$, $n \geq 0$. Thus $v(x)$ is given by.

$$v(x) = Ax^m + Bx^n$$
$$m = \frac{-(2\mu - \sigma^2) - \sqrt{(2\mu - \sigma^2)^2 + 8r\sigma}}{2\sigma^2}$$
$$n = \frac{-(2\mu - \sigma^2) + \sqrt{(2\mu - \sigma^2)^2 + 8r\sigma}}{2\sigma^2}$$

Boundedness of value of option

Let us assume that $v(x)$ is not bounded then there definitely exists a x such that $v(x) > K$. We sell the put whose value is K and put the money in the bank at time t. Then there are two cases.

Case 1. The put is exercised at time τ . If this happens then we take the money out of the bank to fund the exercise of the put. Thus in this case the profit is given by.

$$-(K - x^*)^+ + K/d(0, \tau) > 0$$

Case 2. The put is not exercised till maturity T. Then in that case the payoff is.

$$-(K - x^*)^+ + K/d(0, \tau) > 0$$

We have an arbitrage opportunity in both the cases. Since there are no arbitrage opportunities in the market thus are assumption that $v(x) > K$ is wrong. Thus we can conclude that $v(x) \leq K$ and bounded.

We have proved that $v(x)lK$ which implies that $Ax^m + B \leq K \; \forall x$.

$$\begin{aligned} lim_{x\to\infty} v(x) &\leq K \\ \implies lim_{x\to\infty} Ax^m + Bx^n &\leq K \\ \implies lim_{x\to\infty} Ax^m + lim_{x\to\infty} Bx^m &\leq K \qquad \because m < 0 \\ \implies lim_{x\to\infty} Bx^n &\leq K \\ \implies B &= 0 \qquad \because n > 1 \end{aligned}$$

Optimal Exercise

From the second section we can conclude that any $x \leq x^*$ we have $v(x) = h(x) \implies Ax^m = (K - x)$. Since value of a option are continuous we can differentiate wrt x.

$$\begin{aligned}
\frac{dv(x)}{x}|_{x=x^*} &= \frac{dh(x)}{dx}|_{x=x^*} \\
\implies \frac{d(AX^m)}{dx}|_{x=x^*} &= \frac{d(K-x)}{dx}|_{x=x^*} \\
\implies mAx^{*(m-1)} &= -1 \\
\implies mAx^{*m} &= -x^* \\
\implies m(K-x^*) &= -x^* \\
\implies x^* &= \frac{mK}{(m-1)}
\end{aligned}$$

Substituting the value of x^* in the equation $Ax^{*m} = (K - x^*)$.

$$A = \frac{(K-x^*)}{x^{*m}}$$
$$A = \frac{(K - \frac{mK}{(K-1)})}{\left[\frac{mK}{(K-1)}\right]^m}$$

From the above results we can conclude that it is optimal to exercise the option when $x \leq \frac{mK}{(m-1)} \leq K$.

From the previous section we know that m is given by $m = \frac{-(2\mu-\sigma^2)-\sqrt{(2\mu-\sigma^2)^2+8\sigma r}}{2\sigma^2}$. We now need to find the limit when $r \to 0$.

$$\begin{aligned}
&lim_{r\to 0} m \\
&= lim_{r\to 0} m = \frac{-(2\mu-\sigma^2) - \sqrt{(2\mu-\sigma^2)^2 + 8\sigma r}}{2\sigma^2} \\
&= 0
\end{aligned}$$

We now need to find the limit of x^* as r tends to zero.

$$\begin{aligned}
&lim_{r\to 0} x^* \\
&= lim_{r\to 0} \frac{mK}{m-1} \\
&= lim_{m\to 0} \frac{mK}{m-1} \\
&= 0
\end{aligned}$$

Now we need to find A as r tends to zero.

$$\begin{aligned}
& lim_{r\to 0} A \\
& = lim_{m\to 0} A \\
& = lim_{m\to 0} \frac{-K}{m-1}\left[\frac{m-1}{mK}\right]^m
\end{aligned}$$

$$\implies lim_{m\to 0} \log A = lim_{m\to 0} \log \frac{-K}{m-1} + lim_{m\to 0} m \log \frac{m-1}{mk}$$

$$\implies lim_{m\to 0} \log A = \log K + lim_{m\to 0} \frac{\log \frac{m-1}{mk}}{\frac{1}{m}}$$

$$\implies lim_{m\to 0} \log A = \log K$$

$$\implies lim_{m\to 0} A = K$$

$$\implies lim_{r\to 0} A = K$$

Pricing using CRR model

In this question we had use the Cox-Ross-Rubinstein model valuation for valuation of options. The CRR model is a discrete time pricing model in which the evolution of the risky asset is given by.

$$S_{t_{i+1}} = S_{t_i} U^i$$

Where U^i is the up state (u) or the down state (d).

$$u := e^{\sigma \sqrt{\frac{T}{n}}}$$
$$d := e^{-\sigma \sqrt{\frac{T}{n}}}$$

Where σ is the volatility of the underlying security, T is the time at which the security matures and n is the depth of the tree.

Building the tree

We had to create the binomial price tree for the risky asset. The tree is generated by using a matrix of size $(n+1) \times (n+1)$, where n is the depth of the tree. The matrix is filled by working forward from the date of valuation to the date of maturity of the underlying security. At each step, it is assumed that the underlying security can move up by a factor of u or down by a factor of d. The code used to generate the price tree of the underlying security is shown below.

```
def buildTree(self):
 for i in range(self.n+1):
   for j in range(i+1):
     self.tree_security[i][j] = self.security_price * math.pow(
   self.u,j) * math.pow(self.d,i-j)

```

Pricing European and American options

We had to come up with a recursive formula for valuation of European and American options at every node of the tree. At the terminals nodes i.e. at the maturity of the option, the option value is simply the exercise value.

$$max[(S_T - K), 0] \quad \text{American and European call option}$$
$$max[(K - S_T), 0] \quad \text{American and European put option}$$

Once the valuation is computed at the terminal nodes, the valuation at every other node can be computed as the discounted expected value of the pay off under the risk neutral measure. In CRR model the risk-neutral measure is given by.

$$q := \mathbb{Q}[U^i = u] = \frac{e^{R\frac{T}{n}} - d}{u - d}$$

And the value of the option at each node for a European options are given by.

$$S_{t_i} = e^{-rT/n}\Big[qS_{t_{i+1}}\{U^i = u\} + (1-q)S_{t_{i+1}}\{U^i = D\}\Big]$$

S in the above recursion represents the price of the option. With America options we have the choice of exercising early and the value at every node is given by.

$$S_{t_i} = max\Big[(SP_{t_i} - K), e^{-rT/n}\big[qS_{t_{i+1}}\{U^i = u\} + (1-q)S_{t_{i+1}}\{U^i = D\}\big]\Big] \text{ American Call}$$
$$S_{t_i} = max\Big[(K - SP_{t_i}), e^{-rT/n}\big[qS_{t_{i+1}}\{U^i = u\} + (1-q)S_{t_{i+1}}\{U^i = D\}\big]\Big] \text{ American Put}$$

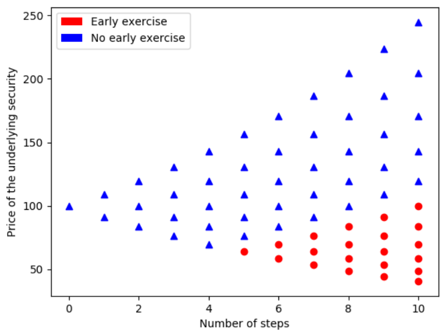
SP in the above recursion is the price of the underlying security which we get from the tree that was computed early. The code that was used to compute the valuation of the American and European options is shown below.

```
  def computePrice(self):
   for i in range(self.n-1,-1,-1):
    for j in range(i+1):
    if self.type == 'European Call':
     self.tree_option[i][j] = math.exp(-1*self.rate*self.deltaT)
                            *((1-self.p) * self.tree_option[i+1][j]
                            + (self.p)*self.tree_option[i+1][j+1])

    elif self.type == 'European Put':
     self.tree_option[i][j] = math.exp(-1*self.rate*self.deltaT)
                            *((1-self.p) * self.tree_option[i+1][j]
                             + (self.p)*self.tree_option[i+1][j+1])

    elif self.type == 'American Call':
     self.tree_option[i][j] = max(self.tree_security[i][j]-
                                 self.strike_price,
                                math.exp(-1*self.rate*self.deltaT)
                               *((1-self.p)*self.tree_option[i+1][j]
                          + (self.p)*self.tree_option[i+1][j+1]))

    elif self.type == 'American Put':
     self.tree_option[i][j] = max(self.strike_price
                             -self.tree_security[i][j],
                             math.exp(-1*self.rate*self.deltaT)
                             *((1-self.p)*self.tree_option[i+1][j]
                             +(self.p)*self.tree_option[i+1][j+1]))
     #Finding the exercise frontier
     if self.tree_option[i][j] == (self.strike_price-self.
 tree_security[i][j]):
```

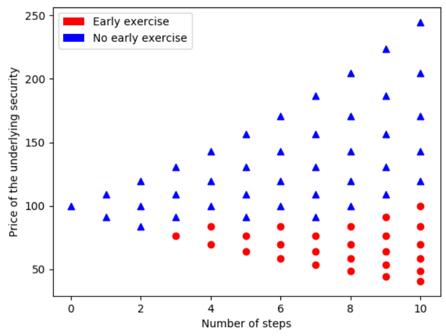
```
            self.EPA[i][j] = 1
```

Exercise Frontier of an American Put

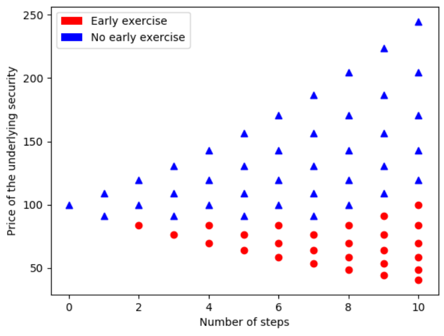
Once the valuation of the American put option had been done we had to find the optimal exercise frontier of the American Put and study its relation to the interest rate and the depth of the tree in CRR model. We plotted the exercise frontier for different values of depth and interest rate.



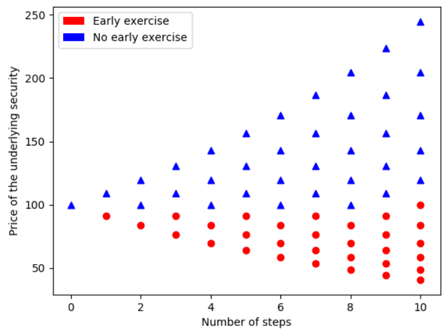
(a) 0.1% interest



(b) 5% interest



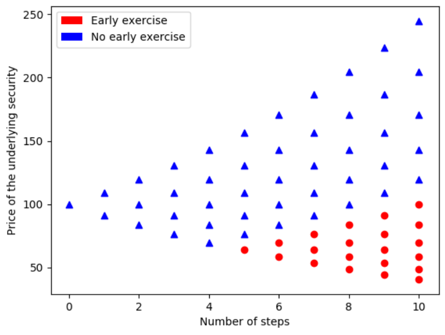
(c) 10% interest



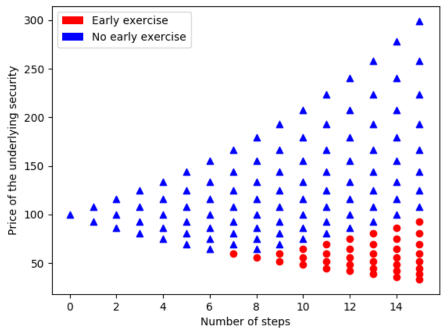
(d) 15% interest

Figure 1: Interest rate variation in a tree of depth 10

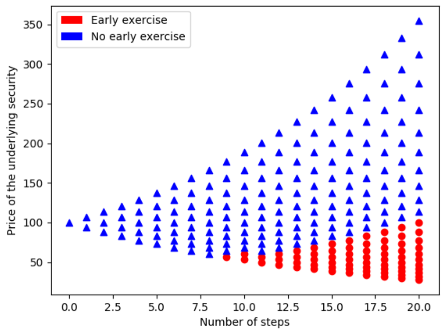
It can be observed from the figures that for the same depth the exercise frontier moves up in the tree i.e. when we increase the interest rate we can optimally exercise an American option earlier.



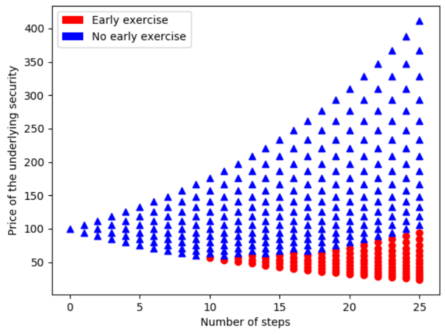
(a) Tree with depth 10



(b) Tree with depth 15



(c) Tree with depth 20



(d) Tree with depth 25

Figure 2: Depth variation with interest rate 0.1%

From the figures we can observe that with the variation of the depth of the tree for the same interest rate, the exercise frontier gets delayed. This means that in the depth of the tree is increased the points of early exercise move further down in the tree.

Finite difference method for American options

Generating the matrix

We know that the matrix is A is defined by.

$$(AP)_j = \frac{\sigma^2 x_j^2}{2}\left(\frac{-P_{j-1}^n + 2P_j^n - P_{j+1}^n}{h^2}\right) - rx_j\left(\frac{P_{j+1}^n - P_j^n}{h}\right) + rP_j^n$$

$(AP)_j$ is a linear function of the components of the components of P. The code to generate the matrix is shown below.

```
x = np.array([h*j for j in range(M+1)])
t = np.array([DELTA_T*j for j in range(N)])
A = [[0 for i in range(M+1)] for j in range(M+1)]
for i in range(M):
 if(i>0):
  A[i][i-1] = - (((SIGMA**2)*(x[i]**2))/(2*(h**2)))
  A[i][i] =  (((SIGMA**2)*(x[i]**2))/((h**2))) + (r*x[i]/h) + r
  A[i][i+1] = - (((SIGMA**2)*(x[i]**2))/(2*(h**2))) - (r*x[i]/h
 )
A = np.array(A)
```

Discretization Scheme

The explicit Euler scheme given to us is as follows.

$$\min\left\{\frac{P_j^{n+1} - P_j^n}{\Delta t} + \frac{\sigma^2 x_j^2}{2}\left(\frac{-P_{j-1}^n + 2P_j^n - P_{j+1}^n}{h^2}\right) - rx_j\left(\frac{P_{j+1}^n - P_j^n}{h}\right) + rP_j^n,\ P_j^{n+1} - g(x_j)\right\} = 0$$

$$P_{M+1}^{n+1} = 0$$

Not considering the stopping condition we can see that the explicit scheme is given by.

$$
\begin{aligned}
&\min\left\{\frac{P_j^{n+1}-P_j^n}{\Delta t}+\frac{\sigma^2 x_j^2}{2}\left(\frac{-P_{j-1}^n+2P_j^n-P_{j+1}^n}{h^2}\right)-rx_j\left(\frac{P_{j+1}^n-P_j^n}{h}\right)+rP_j^n,\ P_j^{n+1}-g(x_j)\right\}=0\\
&\implies \min\left\{\frac{P_j^{n+1}-P_j^n}{\Delta t}+AP^n, P_j^{n+1}-g(x_j)\right\}=0\\
&\implies P_j^{n+1}+\min\left\{-P_j^n+\Delta t AP^n, -g(x_j)\right\}=0\\
&\implies P_j^{n+1}=-\min\left\{-P_j^n+\Delta t AP^n, -g(x_j)\right\}\\
&\implies P_j^{n+1}=\max\left\{P_j^n-\Delta t AP^n, g(x_j)\right\}\\
&\implies P^{n+1}=\max\left\{P^n+\Delta t AP^n, \phi\right\}
\end{aligned}
$$

Stability of explicit scheme

The solution to the explicit scheme is unstable and blows up as the value of M is increased.

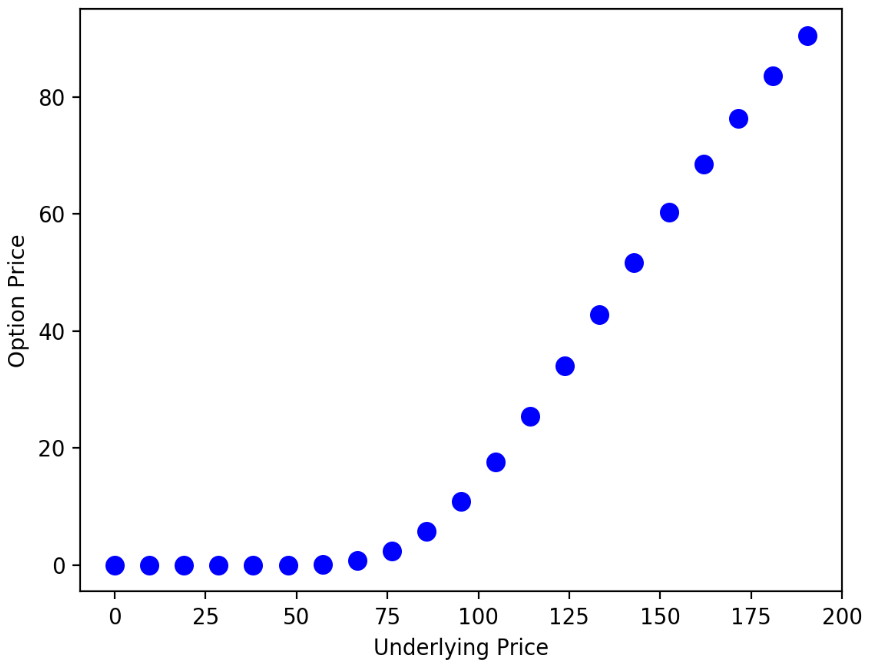


Figure 3: Explicit Scheme Solution for Different Values of S at $t = 0$

The figure shows the solution to the explicit solution scheme for $M = 20$. The code used to generate the solution is shown below.

```
def phi_call(x):
 payoff = max(x-K,0)
 return payoff

def phi_put(x):
 payoff = max(K-x,0)
 return payoff

P = np.array([[0.0 for i in range(M+1)] for j in range(N+1)])

for i in range(M+1):
 P[0][i] = phi_call(x[i])


for i in range(1,N+1):
 P[i] = np.maximum(P[i-1]-DELTA_T*np.matmul(A,P[i-1]),np.array([
   phi_call(x[j]) for j in range(M+1)]))

print("Explicit Euler Pricing Solution:")
print(P[N])
explicit_sol = P[N]
```

Implicit Euler Scheme

We know that.

$$\left[\frac{\partial p}{\partial q}\right]_{ij} = \frac{\partial p_i}{\partial q_j} \qquad \text{for any vector } p \text{ and } q$$

$$(Bx - b)_i = \sum_{a=1}^{M+1} (B_{ia} x_a) - b_i \qquad \text{for } i = 0, 1 \ldots M + 1$$

From the above results we can conclude that. If $(Bx - b)_i \leq (x - \phi)$.

$$\begin{aligned}
\left[\frac{\partial F(x)}{\partial x}\right]_{ij} &= \left[\frac{\partial(Bx-b)}{\partial x}\right]_{ij} \\
&= \frac{\partial(Bx-b)_i}{\partial x_j} \\
&= \frac{\partial(\sum_{a=1}^{M+1}(B_{ia}x_a)-b_i)}{\partial x_j} \\
&= B_{ij}
\end{aligned}$$

Now considering the case when $(Bx-b)_i > (x-\phi)$.

$$\begin{aligned}
&\left[\frac{\partial F(x)}{\partial x}\right]_{ij} \\
&= \left[\frac{\partial(x-\phi)}{\partial x}\right]_{ij} \\
&= \frac{\partial(x-\phi)_i}{\partial x_j} \\
&= \frac{\partial(x_i-\phi_i)}{\partial x_j} \\
&= \delta_{ij}
\end{aligned}$$

The above is the required result.

Newton's method for implicit scheme

The implementation of Newton-Raphson algorithm for the implicit scheme is shown below.

```
B = np.identity(M+1) + DELTA_T*A
P = [np.array([np.random.uniform() for i in range(M+1)]) for i in
    range(N+1)]
for i in range(M+1):
 P[0][i] = phi_call(x[i])

def F(a,b):
 return np.minimum(np.matmul(B,a)-b,a-P[0])

def deriv(a,b,i,j):
 p = np.matmul(B,a)-b
 q = a - P[0]
 if(p[i]<=q[i]):
   return B[i][j]
  else:
   if(i==j):
    return 1
   else:
    return 0

def F_dash(a,b):
 K = [[0 for i in range(M+1)] for i in range(M+1)]
 for i in range(M+1):
  for j in range(M+1):
   K[i][j] = deriv(a,b,i,j)
 return np.array(K)

for j in range(1,N+1):
 for i in range(num_iterations):
  P[j] = P[j] - np.matmul(np.linalg.pinv(F_dash(P[j],P[j-1])),F(P
   [j],P[j-1]))
```

We run the code for 1000 iterations and find that solution remains stable. Shown below are the results for the option price for M=20 and M=50.

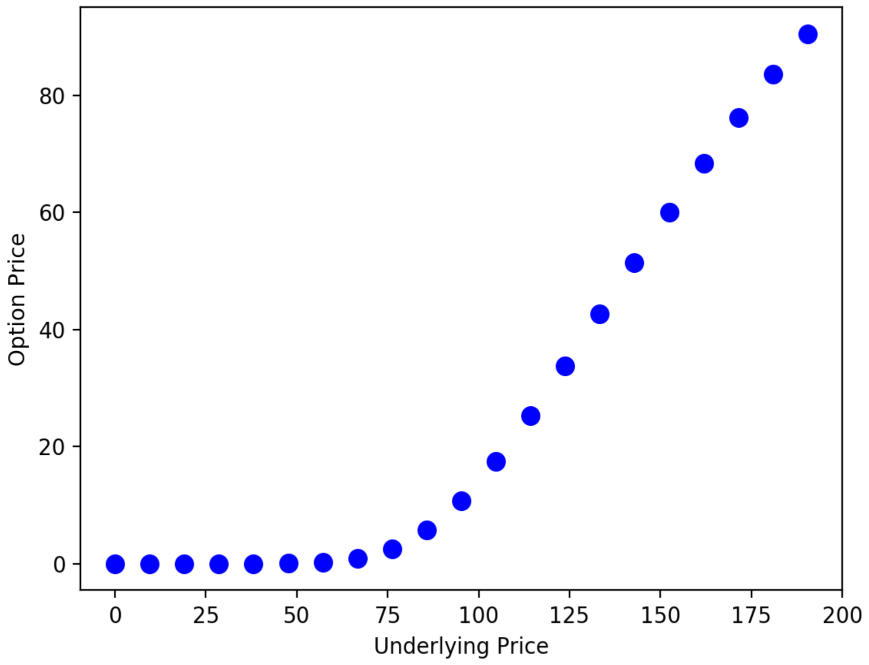


Figure 4: Implicit Scheme Solution for Different Values of S at $t = 0$ and $M = 20$

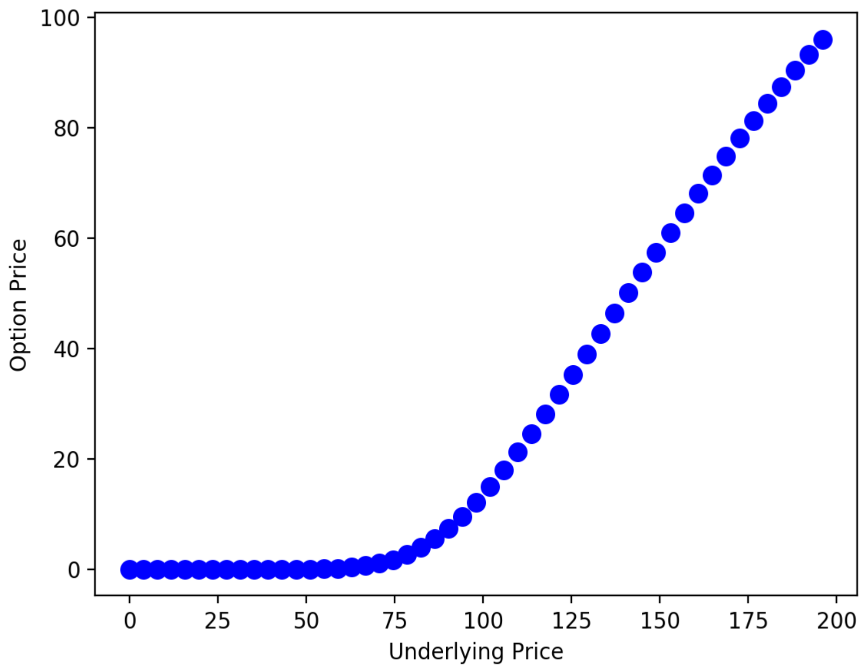


Figure 5: Implicit Scheme Solution for Different Values of S at $t = 0$ and $M = 50$