(1.1) Since the option is a put, the option yields a
(1) profit that increases as the price of the underlying decreases. So if we deem it optimal to exercise at a price x^*, then surely it is also optimal to exercise at any $S_t \leq x^*$ So our exercise region will be of the form $[0, x^*]$ for some ~~xxx~~ x^* (we can't exercise at $S_T = 0$, since stock will be delisted. Now, the payoff will only be positive if $S_t < K$ and so a rational individual will only ever exercise at an "optimal" price x^* if x^* is $\leq K$ (at K, person will be indifferent). So, we expect $S = (0, x^*]$ where $x \in (0, K]$.

(2) If ~~xxx~~ $x > x^*$ then exercise isn't optimal

$\iff v(x) - h(x) > 0 \implies rv(x) - \mu x v'(x) - \frac{1}{2}\sigma^2 x^2 v''(x) = 0$ by 1.3

We note that the equation is a version of the general Cauchy-Euler eqn which has solution $v(x) = Ax^m + Bx^n$ where m and n are the roots of char. eqn.: $\lambda^2 + (2\sigma^2\mu - 1)\lambda - 2\sigma^2 r = 0$

Using given ODE we have:

$$x^2 v''(x) + 2\frac{\mu x v'(x)}{\sigma^2} - \cancel{2\mu x}\frac{2r}{\sigma^2} v(x) = 0$$

let $v(x) = x^\lambda$ $\quad v'(x) = \lambda x^{\lambda - 1}$ $\quad v''(x) = \lambda(\lambda - 1)v^{\lambda - 2}$

$\implies$

$$x^\lambda\left(\lambda(\lambda-1) + \frac{2\mu}{\sigma^2}\lambda - \frac{\cancel{2r}\,2r}{\sigma^2}\right) = 0$$

$$\lambda^2 - \lambda + \frac{2\mu}{\sigma^2}\lambda - \frac{2r}{\sigma^2} = 0$$

$$\lambda^2 + \frac{(2\mu - \sigma^2)}{\sigma^2}\lambda - \frac{2r}{\sigma^2} = 0$$

$$\sigma^2\lambda^2 + (2\mu - \sigma^2)\lambda - 2r = 0$$

$$\lambda = \frac{-(2\mu-\sigma^2) \pm \sqrt{(2\mu-\sigma^2)^2 + 8r\sigma^2}}{2\sigma^2}$$

$$= \frac{-(2\mu-\sigma^2) \pm \sqrt{\sigma^4 - 4\mu\sigma^2 + 4\mu^2 + 8r\sigma^2}}{2\sigma^2}$$

$$m = \frac{-(2\mu-\sigma^2) - \sqrt{\sigma^4 - 4\mu\sigma^2 + 4\mu^2 + 8r\sigma^2}}{2\sigma^2}$$

$$n = \frac{-(2\mu-\sigma^2) + \sqrt{\sigma^4 - 4\mu\sigma^2 + 4\mu^2 + 8r\sigma^2}}{2\sigma^2}$$

since $8\sigma^2 r \geq 0$, $n \geq 2/2 = 1$, $m \leq 0$

(3) Suppose v unbounded then $\forall M \in \mathbb{R}^+$, $\exists x$ s.t. $v(x) > M$. In particular x s.t. $v(x) > K$.

If $v(x) > K$ then sell put, lend \$K for profit > 0

Two cases.

C1 - put exercised then $\pi_T = (K - x^*) + \frac{K}{d(t,T)} \geq 0$

C2 - put never exercised then made money initially w/ no payment later

$\Rightarrow$ arb so $v(x)$ bounded.

Since $v(x) \leq K \Leftrightarrow Ax^m + Bx^n \leq K \;\; \forall x$

$$\Rightarrow \lim_{x \to \infty} v(x) \leq K$$

Since $m \leq 1$, $\lim_{x \to \infty} Ax^m = 0$

so ~~xxxx~~ $\lim_{x \to \infty} Bx^n \leq K$ but $\lim_{x \to \infty} x^n = \infty$ since $n \geq 1$

Hence, $B = 0$

(4) $v(x^*) = h(x^*) \implies A(x^*)^m = \cancel{K-x^*}\ K - x^*$ (i) since soln for $v(x)$ holds for $x > x^*$ and put value is continuous.

$$\frac{dv}{dx}\Big|_{x=x^*} = \frac{dh}{dx}\Big|_{x=x^*} \iff Am(x^*)^{m-1} = -1 \iff Am(x^*)^m = -$$

$$(i) \implies Am(x^*)^m \frac{(K-x^*)}{(x^*)^m} = -x^*$$

$$\iff mK = (m-1)x^* \implies x^* = \frac{mK}{m-1}, \quad A = \frac{\left(K - \frac{mK}{m-1}\right)}{\left(\frac{mK}{m-1}\right)^m}$$

Optimal to exercise if $x \leq \frac{mK}{m-1} < K$

$$\lim_{r \to 0^+} m = \lim_{r \to 0^+} \frac{\sigma^2 - 2\mu - \sqrt{\sigma^4 - 4\mu\sigma^2 + 4\mu^2 + 8r\sigma^2}}{2\sigma^2}$$

Note since $\mu \leq r$, $r \to 0 \implies \mu \to 0$

$$\lim_{r \to 0^+} m = 0$$

$$\lim_{r \to 0} x^* = \lim_{r \to 0} \frac{mK}{m-1} = K \lim_{m \to 0^-} \frac{m}{m-1} = 0$$

and $$\lim_{r \to 0} A = \lim_{m \to 0^-} \frac{\left(K - \frac{mK}{m-1}\right)}{\left(\frac{mK}{m-1}\right)^m} = K \lim_{m \to 0^-} \frac{1}{\left(\frac{mK}{m-1}\right)^m} - \lim_{m \to 0^-} \frac{1}{\left(\frac{mK}{m-1}\right)^{m-1}}$$

$$\lim_{m \to 0^-} \left(\frac{mK}{m-1}\right)^{-m} = \lim_{m \to 0^-} \left(\frac{K}{1 - 1/m}\right)^{-m} = \lim_{m \to 0^-} e^{-m(\log K - \log(1 - \frac{1}{m}))}$$

$$= \lim_{m \to 0^-} e^{m \log(1 - \frac{1}{m})} = \lim_{m \to 0^-} e^{\log(1-\frac{1}{m})^m} = \cancel{K} e^{\log(1)} = \cancel{K} e^0 = 1$$

and $$\lim_{m \to 0^-} \frac{1}{\left(\frac{mK}{m-1}\right)^{m-1}} = \left(\frac{0}{-1}\right)^1 = 0$$

Hence, $A \to 1$ as $r \to 0^-$