IEOR 4735 – Fall 2020 Project

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Our objective is to price an option with the following payoff:

$$max\left[0,\left(\frac{S(T)}{S(0)}-\ k\right)\cdot\left(\frac{L(T\ -\ \Delta,T\ -\ \Delta,\ T)}{L(0,T\ -\ \Delta,T)}-\ k'\right)\right]$$

in USD at the predetermined expiry date T where S_T is the price of the STOXX50E quantoed from EUR to USD, and strike prices k and k' in USD. $L(t,T\ -\ \Delta,T)$ is the 3-month LIBOR forward rate quoted at time t, starting at time $T-\Delta$. Our risk-free rate, the rate on one-year treasury bills, as well as our parameter estimation for the short rate model used for the LIBOR rate are based on data obtained from FRED.

We first represent the STOXX50E process and the EUR/USD exchange rate process as two geometric Brownian motions.

As such, we define the following:

$$S_t\ =\ STOXX50E\ ,\ X_t = EUR/USD,\ Q_t = X_t S_t,$$

$$dS_t = \mu_S S_t dt +\ \sigma_S S_t dW_t^S,$$

$$dX_t = \mu_X X_t dt +\ \sigma_X X_t dW_t^X$$

$$dW_t^X dW_t^S = \rho dt,$$

where ρ is the correlation between the STOXX50E and the USD/EUR exchange rate.

We can now present the price process for our variable of interest, Q_t, the STOXX50E quantoed to USD:

$$dQ_t = Q_t(\mu_X + \mu_S + \rho\sigma_X\sigma_S)dt + \sigma_S Q_t dW_t^S + \sigma_X Q_t dW_t^X.$$

After an application of Ito's lemma, we have:

$$Q_t =\ Q_0 e^{\left(\mu_X + \mu_S - \frac{\sigma_S^2+\sigma_X^2}{2}\right)t + \sigma_S W_t^S + \sigma_S W_t^X}.$$

Through defining the process this way, we save memory and time by avoiding direct simulations of S_t and X_t separately.

To model the LIBOR short rate, we use the Vasicek model. We simulate the short rate since $L(T\ -\ \Delta T,T\ -\ \Delta T,T)$ is a short rate (a forward rate with forward start date equal to the date of the quote), rather than a forward rate, and $L(0,T\ -\ \Delta,T)$ is a parameter which can be observed from the current LIBOR forward curve (derived from Eurodollar futures and swap rates) at the time of pricing of our option. We then simulate the rate according to the Euler – Maruyama discretization scheme, as the Vasicek model does not have a closed form solution.

We assume LIBOR is uncorrelated with the STOXX50E as they are rooted in different countries and thus are affected by different macro factors. Additionally, we assume no correlation with the EUR/USD exchange rate as the exchange rate is generally only affected by the sovereign risk-free rate, whereas LIBOR is simply an interbank loan rate not set by the government.

Given that the processes for STOXX50E and the exchange rate are both expressed as GBMs with correlated Brownian motions, it follows that S_t and X_t are jointly normal:

$$p(X,S) = \frac{1}{2\pi\sigma_S\sigma_X\sqrt{1-\rho^2}} exp\left[\frac{-\left(X-\left(\mu_X-\frac{\sigma_X^2}{2}\right)\right)^2}{2\sigma_X^2(1-\rho^2)} - \frac{\left(S-\left(\mu_S-\frac{\sigma_S^2}{2}\right)\right)^2}{2\sigma_S^2(1-\rho^2)} + \frac{\left(X-\left(\mu_X-\frac{\sigma_X^2}{2}\right)\right)\left(S-\left(\mu_S-\frac{\sigma_S^2}{2}\right)\right)}{\sigma_X\sigma_S(1-\rho^2)}\right]$$

We can estimate the $\mu's$ and $\sigma's$ using maximum likelihood estimation on the log-return data of S_t and X_t and estimate ρ by sample correlation of log-returns of the two processes. Note, however, that we first run a Pearson correlation test to see if estimation on ρ is necessary. In the most recent estimation, we found the correlation to be ~.036.

Finally, we discretize the Vasicek SDE, $dr_t = (b - ar_t)dt + \sigma dW_t$, where r_t is the short rate at time t as follows:

$$r_{t+1} = b\Delta t + (1 - a\Delta t)r_t + \sigma\sqrt{\Delta t}\epsilon, \epsilon \sim \mathrm{N}(0,1)$$

Note that this is a variation of the AR(1) model, and therefore can be parametrized as such. We consequently use OLS estimators of $r_{t+1} \sim r_t$ which will prove to be easier to estimate than Maximum Likelihood estimators (as there are 3 parameters to estimate according to a normal distribution which has only two) and we find:

$$\hat{a} = 1 - \frac{\sum(r_t - \bar{r}_t)(r_{t-1} - \bar{r}_t)}{\sum(r_t - \bar{r}_t)^2},$$

$$\hat{b} = \overline{r_t} - (1 - a)\overline{r_{t-1}},$$

$$\hat{\sigma} = \sqrt{\sum \frac{(r_t - b - (1+a)r_{t-1})^2}{n}},$$

where we have used the fact that $\Delta t = 1$, since our data is daily.

Finally, we run Monte Carlo simulations on different sets of strike rates to find their corresponding option prices. The results are summarized in the table below along with a graph of selected paths from the simulations:

K	K’	Price of Quanto
1	1	0.162865033
1	0.75	0.146837313
0.75	0.75	0.220911508
0.5	0.75	0.358196166
0.5	0.5	0.379893412
0.25	0.5	0.538399194
0.25	0.25	0.570957849
0.1	0.1	0.696082494
0.1	0.01	0.710096546
0.01	0.01	0.774825979