Observatorio **A**stronomico **N**acional **B**lack hole **I**maging **T**ool **OAN-BIT**

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Chapter 1

Initial Conditions of the Photons

In order to calculate the image of a black hole we will trace back the path of photons from the observer's camera which can be an image plane or a point camera, onto the emission structure around the black hole.

1.1 Initial Conditions for Position and Momentum

1.1.1 Coordinate Systems

We will take Cartesian coordinates (X, Y, Z) centered at observer's position and Cartesian coordinates (x, y, z) centered at the black hole. These two systems will be related by a rotation and a translation. As seen in Figure 1.1 we may define a rotation between the system (X, Y, Z) and an intermediate system (X', Y', Z') as

$$\begin{pmatrix} X' \\ Y' \\ Z' \end{pmatrix} = \begin{pmatrix} \cos\alpha & 0 & -\sin\alpha \\ 0 & 1 & 0 \\ \sin\alpha & 0 & \cos\alpha \end{pmatrix} \begin{pmatrix} X \\ Y \\ Z \end{pmatrix}. \tag{1.1}$$

If the observer is located at a distance D from the black hole as in Figure 1.1, we introduce the spatial translation from the intermediate system (X', Y', Z') to the system (x, y, z),

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} X' \\ Y' \\ Z' \end{pmatrix} + \begin{pmatrix} D\cos\alpha \\ 0 \\ D\sin\alpha \end{pmatrix} \tag{1.2}$$

The composition of these two transformations give

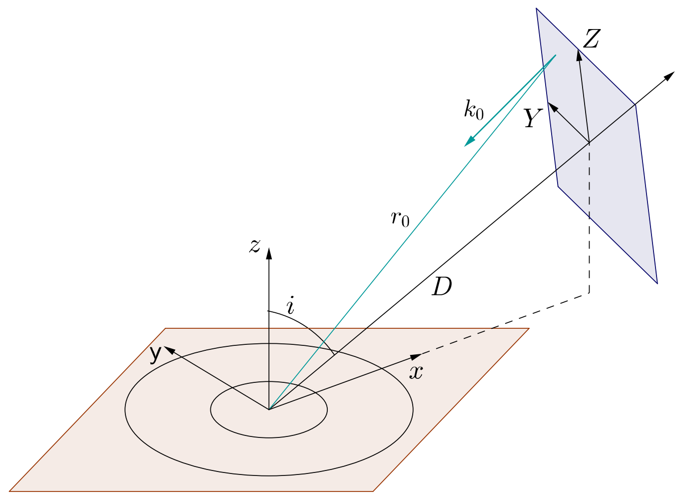

Figure 1.1: Coordinate Transformation relating the observer's system with the black hole's system.

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} \cos\alpha & 0 & -\sin\alpha \\ 0 & 1 & 0 \\ \sin\alpha & 0 & \cos\alpha \end{pmatrix} \begin{pmatrix} X \\ Y \\ Z \end{pmatrix} + \begin{pmatrix} D\cos\alpha \\ 0 \\ D\sin\alpha \end{pmatrix} \tag{1.3}$$

or more explicitly,

$$\begin{cases} x &= X\cos\alpha - Z\sin\alpha + D\cos\alpha \\ y &= Y \\ z &= X\sin\alpha + Z\cos\alpha + D\sin\alpha. \end{cases} \tag{1.4}$$

Since the angle α is related with the inclination angle by $\alpha = \frac{\pi}{2} - i$ we replace the trigonometric functions in the coordinate transformations by

$$\begin{cases} x &= X\sin i - Z\cos i + D\sin i \\ y &= Y \\ z &= X\cos i + Z\sin i + D\cos i \end{cases} \tag{1.5}$$

or better

$$\begin{cases} x &= (X + D)\sin i - Z\cos i \\ y &= Y \\ z &= (X + D)\cos i + Z\sin i. \end{cases} \tag{1.6}$$

Since the black hole's metric will be given in spherical coordinates, we introduce the relation between the Cartesian system (x, y, z) and the coordinates (r, θ, ϕ) by

$$\begin{cases} r &= \sqrt{x^2+y^2+z^2} \\ \theta &= \arccos\left(\frac{z}{r}\right) \\ \phi &= \arctan\left(\frac{y}{x}\right). \end{cases} \tag{1.7}$$

Hence, the complete transformations needed to describe the initial conditions of the photon in spherical coordinates are

$$\begin{cases} r &= \sqrt{(X+D)^2+Y^2+Z^2} \\ \theta &= \arccos\left(\frac{(X+D)\cos i + Z\sin i}{\sqrt{(X+D)^2+Y^2+Z^2}}\right) \\ \phi &= \arctan\left(\frac{Y}{(X+D)\sin i - Z\cos i}\right). \end{cases} \tag{1.8}$$

1.1.2 Momentum of a Photon

The Cartesian components of a general incident photon are given by a 4-momentum vector

$$\tilde{k}_0^{\nu} = \left(\tilde{k}_0^t, \tilde{k}_0^X, \tilde{k}_0^Y, \tilde{k}_0^Z\right). \tag{1.9}$$

The components of the 4-momentum in the spherical coordinate system (r, θ, ϕ) are obtained using the transformation law

$$k^{\mu} = \frac{\partial x^{\mu}}{\partial \tilde{x}^{\nu}} \tilde{k}^{\nu}. \tag{1.10}$$

1.2 Image Plane Screen

The observer's image plane is considered as a grid located at a distant point and we will consider photons received there with a momentum vector which is orthogonal to the plane. The algorithm will evolve the trajectories back in time onto the emission structure around the black hole.

1.2.1 Initial Position of a photon

Consider a photon registered at the observer's image plane at the coordinates $(X, Y, Z) = (0, \alpha, \beta)$ at the time $t_0 = 0$. The spherical coordinates of this photon as seen from the black hole are given by the equation (1.8) as

$$\begin{cases} t_0 &= 0 \\ r_0 &= \sqrt{D^2+\alpha^2+\beta^2} \\ \theta_0 &= \arccos\left(\frac{D\cos i + \beta\sin i}{\sqrt{D^2+\alpha^2+\beta^2}}\right) \\ \phi_0 &= \arctan\left(\frac{\alpha}{D\sin i - \beta\cos i}\right) \end{cases} \tag{1.11}$$

1.2.2 Momentum of a Photon

Photons received by the distant observer will be considered to arrive perpendicular to the image plane. According to the orientation of the coordinates (X, Y, Z) in Figure 1.1, the Cartesian components of the incident photon are given by

$$\tilde{k}_0^\nu = \left(\tilde{k}_0^t, \tilde{k}_0^X, \tilde{k}_0^Y, \tilde{k}_0^Z\right) = (K_0, K_0, 0, 0)\,. \tag{1.12}$$

The components of the 4-momentum in the spherical coordinate system are obtained using Eq. (1.10). Given the perpendicular components of the incident photon, the relevant terms in the transformation matrix are obtained from equation (1.8) as

$$\left.\frac{\partial r}{\partial X}\right|_{(0,\alpha,\beta)} = \left.\frac{X+D}{\sqrt{(X+D)^2+Y^2+Z^2}}\right|_{(0,\alpha,\beta)} = \frac{D}{\sqrt{D^2+\alpha^2+\beta^2}} = \frac{D}{r_0} \tag{1.13}$$

$$\left.\frac{\partial \theta}{\partial X}\right|_{(0,\alpha,\beta)} = -\frac{1}{\sqrt{r^2-[(X+D)\cos i + Z\sin i]^2}}\left.\left[\cos i - \frac{(X+D)^2\cos i + Z(X+D)\sin i}{r^2}\right]\right|_{(0,\alpha,\beta)}$$

$$\left.\frac{\partial \theta}{\partial X}\right|_{(0,\alpha,\beta)} = -\frac{1}{\sqrt{r_0^2-[D\cos i + \beta\sin i]^2}}\left[\cos i - \frac{D^2\cos i + D\beta\sin i}{r_0^2}\right] \tag{1.14}$$

$$\begin{aligned} \left.\frac{\partial \phi}{\partial X}\right|_{(0,\alpha,\beta)} &= -\left.\frac{Y\sin i}{[(X+D)\sin i - Z\cos i]^2+Y^2}\right|_{(0,\alpha,\beta)} \\ \left.\frac{\partial \phi}{\partial X}\right|_{(0,\alpha,\beta)} &= -\frac{\alpha\sin i}{\alpha^2+[D\sin i - \beta\cos i]^2} \end{aligned} \tag{1.15}$$

Hence, the spatial components of the initial 4-momentum in spherical coordinates are

$$\begin{cases} k_0^r &= \frac{D}{r_0}K_0 \\ k_0^\theta &= -\frac{1}{\sqrt{\alpha^2+[D\sin i-\beta\cos i]^2}}\left[\cos i - \frac{D^2\cos i + D\beta\sin i}{r_0^2}\right]K_0 \\ k_0^\phi &= -\frac{\alpha\sin i}{\alpha^2+[D\sin i-\beta\cos i]^2}K_0 \end{cases} \tag{1.16}$$

The temporal component of the initial 4-momentum of the photon is obtained from the condition

$$k^\mu k_\mu = \eta_{\mu\nu}k^\mu k^\nu = 0, \tag{1.17}$$

which gives

$$k_0^t = \sqrt{(k_0^r)^2 + r_0^2(k_0^\theta)^2 + r_0^2\sin^2\theta_0(k_0^\phi)^2}. \tag{1.18}$$

1.2.3 Complete Set of Initial Conditions

The complete set of initial conditions are given by equations (1.8), (1.16) and (1.18),

$$
\begin{aligned}
t_0 &= 0 && (1.19)\\
r_0 &= \sqrt{D^2+\alpha^2+\beta^2} && (1.20)\\
\theta_0 &= \arccos\left(\frac{D\cos i+\beta\sin i}{\sqrt{D^2+\alpha^2+\beta^2}}\right) && (1.21)\\
\phi_0 &= \arctan\left(\frac{\alpha}{D\sin i-\beta\cos i}\right) && (1.22)\\
k_0^t &= \sqrt{(k_0^r)^2+r_0^2(k_0^\theta)^2+r_0^2\sin^2\theta_0(k_0^\phi)^2} && (1.23)\\
k_0^r &= \frac{D}{r_0}K_0 && (1.24)\\
k_0^\theta &= -\frac{1}{\sqrt{\alpha^2+[D\sin i-\beta\cos i]^2}}\left[\cos i-\frac{D^2\cos i+D\beta\sin i}{r_0^2}\right]K_0 && (1.25)\\
k_0^\phi &= -\frac{\alpha\sin i}{\alpha^2+[D\sin i-\beta\cos i]^2}K_0 && (1.26)
\end{aligned}
$$

1.3 Point Camera

Another type of screen considered in this project is a point camera. Photons arrive to this point receptor from different directions. Therefore, in this case the initial position of all photon in the image is the same (point camera) while the components of the initial 4-momentum vector distinguishes each photon.

1.3.1 Initial Position of a photon

Since all the photons arrive to one point, the initial position condition for any of the photons is just $(X,Y,Z)=(0,0,0)$ at the time $t_0=0$. The corresponding spherical coordinates as seen from the black hole are given by the equations (1.8),

$$
\begin{cases}
t_0 &= 0\\
r_0 &= D\\
\theta_0 &= i\\
\phi_0 &= 0
\end{cases}
\tag{1.27}
$$

1.3.2 Momentum of a Photon

Photons arrive to the point camera from different directions. Using the angular coordinates α and β to describe the direction of each photon, the cartesian components of the 4-momentum vector can be written as

$$\tilde{k}_0^\nu = \left(\tilde{k}_0^t, \tilde{k}_0^X, \tilde{k}_0^Y, \tilde{k}_0^Z\right) = (K_0, K_0 \cos\beta \cos\alpha, K_0 \cos\beta \sin\alpha, K_0 \sin\beta) \,. \quad (1.28)$$

The components of the 4-momentum in the spherical coordinate system are obtained using Eq. (1.10). The relevant terms in the transformation matrix are obtained from equation (1.8) evaluating at the position of the point camera, giving

$$\left.\frac{\partial r}{\partial X}\right|_{(0,0,0)} = \left.\frac{X+D}{\sqrt{(X+D)^2+Y^2+Z^2}}\right|_{(0,0,0)} = \frac{D}{\sqrt{D^2}} = 1 \quad (1.29)$$

$$\left.\frac{\partial r}{\partial Y}\right|_{(0,0,0)} = \left.\frac{Y}{\sqrt{(X+D)^2+Y^2+Z^2}}\right|_{(0,0,0)} = 0 \quad (1.30)$$

$$\left.\frac{\partial r}{\partial Z}\right|_{(0,0,0)} = \left.\frac{Z}{\sqrt{(X+D)^2+Y^2+Z^2}}\right|_{(0,0,0)} = 0 \quad (1.31)$$

$$\begin{aligned}
\left.\frac{\partial \theta}{\partial X}\right|_{(0,0,0)} &= -\frac{1}{\sqrt{r^2-[(X+D)\cos i + Z\sin i]^2}}\left.\left[\cos i - \frac{(X+D)^2\cos i + Z(X+D)\sin i}{r^2}\right]\right|_{(0,0,0)} \\
\left.\frac{\partial \theta}{\partial X}\right|_{(0,0,0)} &= -\frac{1}{\sqrt{D^2-[D\cos i]^2}}\left[\cos i - \frac{D^2\cos i}{D^2}\right] = 0
\end{aligned} \quad (1.32)$$

$$\begin{aligned}
\left.\frac{\partial \theta}{\partial Y}\right|_{(0,0,0)} &= -\frac{1}{\sqrt{r^2-[(X+D)\cos i + Z\sin i]^2}}\left.\left[-Y\frac{(X+D)\cos i + Z\sin i}{r^2}\right]\right|_{(0,0,0)} \\
\left.\frac{\partial \theta}{\partial Y}\right|_{(0,0,0)} &= 0
\end{aligned} \quad (1.33)$$

$$\begin{aligned}
\left.\frac{\partial \theta}{\partial Z}\right|_{(0,0,0)} &= -\frac{1}{\sqrt{r^2-[(X+D)\cos i + Z\sin i]^2}}\left.\left[\sin i - Z\frac{(X+D)\cos i + Z\sin i}{r^2}\right]\right|_{(0,0,0)} \\
\left.\frac{\partial \theta}{\partial Z}\right|_{(0,0,0)} &= -\frac{\sin i}{\sqrt{D^2-[D\cos i]^2}} = -\frac{1}{D}
\end{aligned} \quad (1.34)$$

$$\begin{aligned}
\left.\frac{\partial \phi}{\partial X}\right|_{(0,0,0)} &= -\left.\frac{Y\sin i}{[(X+D)\sin i - Z\cos i]^2 + Y^2}\right|_{(0,0,0)} \\
\left.\frac{\partial \phi}{\partial X}\right|_{(0,0,0)} &= 0
\end{aligned} \quad (1.35)$$

$$\begin{aligned}
\left.\frac{\partial \phi}{\partial Y}\right|_{(0,0,0)} &= \left.\frac{[(X+D)\sin i - Z\cos i]}{[(X+D)\sin i - Z\cos i]^2 + Y^2}\right|_{(0,0,0)} \\
\left.\frac{\partial \phi}{\partial Y}\right|_{(0,0,0)} &= \frac{D\sin i}{[D\sin i]^2} = \frac{1}{D\sin i}
\end{aligned} \quad (1.36)$$

$$\left.\frac{\partial\phi}{\partial Z}\right|_{(0,0,0)} = \left.\frac{Y\cos i}{[(X+D)\sin i - Z\cos i]^2 + Y^2}\right|_{(0,0,0)}$$

$$\left.\frac{\partial\phi}{\partial Z}\right|_{(0,0,0)} = 0 \tag{1.37}$$

Using this components of the transformation matrix, the spatial components of the initial 4-momentum in spherical coordinates are

$$\begin{cases} k_0^r &= \tilde{k}_0^X = K_0\cos\beta\cos\alpha \\ k_0^\theta &= \left.\frac{\partial\theta}{\partial Z}\right|_{(0,0,0)}\tilde{k}_0^Z = -\frac{K_0\sin\beta}{D} \\ k_0^\phi &= \left.\frac{\partial\phi}{\partial Y}\right|_{(0,0,0)}\tilde{k}_0^Y = \frac{K_0\cos\beta\sin\alpha}{D\sin i} \end{cases} \tag{1.38}$$

The temporal component of the initial 4-momentum of the photon is obtained from the condition

$$k^\mu k_\mu = \eta_{\mu\nu}k^\mu k^\nu = 0, \tag{1.39}$$

which gives

$$k_0^t = \sqrt{(k_0^r)^2 + r_0^2(k_0^\theta)^2 + r_0^2\sin^2\theta_0(k_0^\phi)^2}. \tag{1.40}$$

1.4 Covariant Components of the 4-Momentum Vector

In order to solve the Hamiltonian equations of motion for the photons, we will need the initial components of the 4-momentum vector in their covariant form. Therefore we need to apply the spacetime metric $g_{\mu\nu}$ as

$$k_\mu = g_{\mu\nu}k^\nu. \tag{1.41}$$

For a general stationary and axially symmetric spacetime described with the line element

$$ds^2 = g_{tt}dt^2 + 2g_{t\phi}dtd\phi + g_{rr}dr^2 + g_{\theta\theta}d\theta^2 + g_{\phi\phi}d\phi^2 \tag{1.42}$$

the covariant components of the 4-momentum are

$$\begin{cases} k_t &= g_{tt}k^t + g_{t\phi}k^\phi \\ k_r &= g_{rr}k^r \\ k_\theta &= g_{\theta\theta}k^\theta \\ k_\phi &= g_{t\phi}k^t + g_{\phi\phi}k^\phi \end{cases} \tag{1.43}$$

Chapter 2

Geodesic Equations

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Consider a general line element of a general stationary and axially symmetric spacetime described by the line element

$$ds^2 = g_{tt}dt^2 + 2g_{t\phi}dtd\phi + g_{rr}dr^2 + g_{\theta\theta}d\theta^2 + g_{\phi\phi}d\phi^2. \tag{2.1}$$

The Lagrangian describing the geodesic motion of a test-particle in this spacetime is

$$\mathcal{L} = \frac{1}{2}g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu, \tag{2.2}$$

where $\dot{x}^\mu = \frac{dx^\mu}{d\lambda}$ and λ is an affine parameter of the geodesic. The momentum From the stationary and axially symmetric conditions of spacetime, we conclude that the metric components are independent of the coordinates t and ϕ. Therefore, there are two constants of motion: the specific energy at infinity, E, and the axial component of the specific angular momentum at infinity, L_z. These are defined as

$$-E = p_t = \frac{\partial\mathcal{L}}{\partial\dot{t}} = g_{tt}\dot{t} + g_{t\phi}\dot{\phi} \tag{2.3}$$

$$L_z = p_\phi = \frac{\partial\mathcal{L}}{\partial\dot{\phi}} = g_{t\phi}\dot{t} + g_{\phi\phi}\dot{\phi}. \tag{2.4}$$

in Boyer-Lindquist coordinates,

$$ds^2 = -\left(1 - \frac{2Mr}{\Sigma}\right)dt^2 - -\frac{4Mar\sin^2\theta}{\Sigma}dtd\phi \tag{2.5}$$

$$+\frac{\Sigma}{\Delta}dr^2 + \Sigma d\theta^2 + \sin^2\theta\left(r^2 + a^2 + \frac{2Ma^2r\sin^2\theta}{\Sigma}\right)d\phi^2, \tag{2.6}$$

$$\Sigma = \Sigma(r,\theta) \tag{2.7}$$

$$\Delta = \Delta(r). \tag{2.8}$$

The equations of motion of a particle in this spacetime are

$$\begin{aligned}\dot{t} &= \frac{1}{2\Delta\Sigma}\frac{\partial\Xi}{\partial E}\\ \dot{r} &= \frac{\Delta}{\Sigma}p_r\\ \dot{\theta} &= \frac{p_\theta}{\Sigma}\\ \dot{\phi} &= -\frac{1}{2\Delta\Sigma}\frac{\partial\Xi}{\partial L}\end{aligned}$$

$$\begin{aligned}\dot{p}_t &= 0\\ \dot{p}_r &= -\frac{\partial}{\partial r}\left(\frac{\Delta}{2\Sigma}\right)p_r^2 - \frac{\partial}{\partial r}\left(\frac{1}{2\Sigma}\right)p_\theta^2 + \frac{\partial}{\partial r}\left(\frac{\Xi}{2\Delta\Sigma}\right)\\ \dot{p}_\theta &= -\frac{\partial}{\partial\theta}\left(\frac{\Delta}{2\Sigma}\right)p_r^2 - \frac{\partial}{\partial\theta}\left(\frac{1}{2\Sigma}\right)p_\theta^2 + \frac{\partial}{\partial\theta}\left(\frac{\Xi}{2\Delta\Sigma}\right)\\ \dot{p}_\phi &= 0\end{aligned}$$

$$\begin{aligned}\Xi &= R + \Delta\Theta\\ R &= W^2 - \Delta\left[r^2 + (L - aE)^2 + Q\right]\\ W &= E(r^2 + a^2) - aL\\ \Theta &= Q - \cos^2\theta\left[a^2(1 - E^2) + \frac{L^2}{\sin^2\theta}\right]\end{aligned}$$

2.1 Minkowski Spacetime

. The Minkowski spacetime is given by the line element

$$ds^2 = -dt^2 + dx^2 + dy^2 + dz^2 \tag{2.9}$$

or in spherical coordinates

$$ds^2 = -dt^2 + dr^2 + r^2 d\theta^2 + r^2\sin^2\theta d\phi^2. \tag{2.10}$$

When compared with the standard form of the Kerr's line element in Boyer-Lindquist coordinates,

$$\begin{aligned}ds^2 = &-\left(1 - \frac{2Mr}{\Sigma}\right)dt^2 - \frac{4Mar\sin^2\theta}{\Sigma}dtd\phi\\ &+ \frac{\Sigma}{\Delta}dr^2 + \Sigma d\theta^2 + \sin^2\theta\left(r^2 + a^2 + \frac{2Ma^2r\sin^2\theta}{\Sigma}\right)d\phi^2,\end{aligned} \tag{2.11}$$

Minkowski's metric is obtained by taking $a = 0$, $M = 0$ and

$$\Sigma = r^2 \tag{2.12}$$
$$\Delta = r^2. \tag{2.13}$$

Therefore, the potentials in the description of a particle moving in this spacetime reduce to

$$R = E^2 r^4 - r^2 \left[r^2 + L^2 + Q\right] \tag{2.14}$$
$$\Theta = Q - \frac{\cos^2\theta}{\sin^2\theta} L^2 \tag{2.15}$$

Then, we have the function

$$\Xi = E^2 r^4 - r^2 \left[r^2 + L^2 + Q\right] + r^2 \left[Q - \frac{\cos^2\theta}{\sin^2\theta} L^2\right]$$
$$\Xi = E^2 r^4 - r^4 - L^2 r^2 - r^2 \frac{\cos^2\theta}{\sin^2\theta} L^2 \tag{2.16}$$

In order to write the equations of motion we need the derivatives

$$\frac{\partial \Xi}{\partial E} = 2 E r^4 \tag{2.17}$$
$$\frac{\partial \Xi}{\partial L} = -2 L r^2 - 2 r^2 \frac{\cos^2\theta}{\sin^2\theta} L \tag{2.18}$$

and also

$$\frac{\partial}{\partial r}\left(\frac{\Delta}{2\Sigma}\right) = \frac{\partial}{\partial r}\left(\frac{1}{2}\right) = 0 \tag{2.19}$$
$$\frac{\partial}{\partial r}\left(\frac{1}{2\Sigma}\right) = \frac{\partial}{\partial r}\left(\frac{1}{2r^2}\right) = -\frac{1}{r^3} \tag{2.20}$$
$$\begin{aligned}
\frac{\partial}{\partial r}\left(\frac{\Xi}{2\Delta\Sigma}\right) &= \frac{\partial}{\partial r}\left(\frac{E^2 r^4 - r^4 - L^2 r^2 - r^2 \frac{\cos^2\theta}{\sin^2\theta} L^2}{2r^4}\right) \\
&= \frac{1}{2}\frac{\partial}{\partial r}\left(E^2 - 1 - \frac{L^2}{r^2} - \frac{\cos^2\theta}{\sin^2\theta}\frac{L^2}{r^2}\right) \\
&= \frac{L^2}{r^3} + \frac{\cos^2\theta}{\sin^2\theta}\frac{L^2}{r^3}
\end{aligned} \tag{2.21}$$

$$\frac{\partial}{\partial\theta}\left(\frac{\Delta}{2\Sigma}\right) = \frac{\partial}{\partial r}\left(\frac{1}{2}\right) = 0 \tag{2.22}$$

$$\frac{\partial}{\partial\theta}\left(\frac{1}{2\Sigma}\right) = \frac{\partial}{\partial r}\left(\frac{1}{2r^2}\right) = 0 \tag{2.23}$$

$$\begin{aligned}\frac{\partial}{\partial\theta}\left(\frac{\Xi}{2\Delta\Sigma}\right) &= \frac{\partial}{\partial\theta}\left(\frac{E^2r^4 - r^4 - L^2r^2 - r^2\frac{\cos^2\theta}{\sin^2\theta}L^2}{2r^4}\right)\\ &= \frac{1}{2}\frac{\partial}{\partial\theta}\left(E^2 - 1 - \frac{L^2}{r^2} - \frac{\cos^2\theta}{\sin^2\theta}\frac{L^2}{r^2}\right)\\ &= \cot\theta\csc^2\theta\frac{L^2}{r^2}\end{aligned} \tag{2.24}$$

Using this expressions, the equations of motion of a particle in this spacetime are given by the Hamilton's equations

$$\begin{aligned}\dot{t} &= \frac{1}{2r^4}2Er^4 = E\\ \dot{r} &= p_r\\ \dot{\theta} &= \frac{p_\theta}{r^2}\\ \dot{\phi} &= -\frac{1}{2r^4}\left[-2Lr^2 - 2r^2\frac{\cos^2\theta}{\sin^2\theta}L\right] = \left[1 + \frac{\cos^2\theta}{\sin^2\theta}\right]\frac{L}{r^2} = \frac{L}{r^2\sin^2\theta}\end{aligned}$$

$$\begin{aligned}\dot{p}_t &= 0\\ \dot{p}_r &= \frac{p_\theta^2}{r^3} + \frac{L^2}{r^3} + \frac{\cos^2\theta}{\sin^2\theta}\frac{L^2}{r^3} = \frac{p_\theta^2}{r^3} + \frac{1}{\sin^2\theta}\frac{L^2}{r^3}\\ \dot{p}_\theta &= \frac{\cos\theta}{\sin^3\theta}\frac{L^2}{r^2}\\ \dot{p}_\phi &= 0\end{aligned}$$

2.2 Schwarzschild Spacetime

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The Schwarzschild spacetime is given by the line element

$$ds^2 = -\left(1 - \frac{2M}{r}\right)dt^2 + \left(1 - \frac{2M}{r}\right)^{-1}dr^2 + r^2d\theta^2 + r^2\sin^2\theta d\phi^2. \tag{2.25}$$

When compared with the standard form of the Kerr's line element in Boyer-Lindquist coordinates,

$$\begin{aligned} ds^2 = &-\left(1-\frac{2Mr}{\Sigma}\right)dt^2 - \frac{4Mar\sin^2\theta}{\Sigma}dtd\phi \\ &+\frac{\Sigma}{\Delta}dr^2 + \Sigma d\theta^2 + \sin^2\theta\left(r^2+a^2+\frac{2Ma^2r\sin^2\theta}{\Sigma}\right)d\phi^2, \end{aligned} \tag{2.26}$$

Schwarzschild's metric is obtained by taking $a = 0$ and

$$\Sigma = r^2 \tag{2.27}$$

$$\Delta = r^2 - 2Mr. \tag{2.28}$$

Therefore, the potentials in the description of a particle moving in this spacetime reduce to

$$R = E^2r^4 - (r^2-2Mr)\left[r^2+L^2+Q\right] \tag{2.29}$$

$$\Theta = Q - \frac{\cos^2\theta}{\sin^2\theta}L^2 \tag{2.30}$$

Then, we have the function

$$\begin{aligned} \Xi &= E^2r^4 - (r^2-2Mr)\left[r^2+L^2+Q\right] + (r^2-2Mr)\left[Q-\frac{\cos^2\theta}{\sin^2\theta}L^2\right] \\ \Xi &= E^2r^4 - (r^2-2Mr)(r^2+L^2) - (r^2-2Mr)\frac{\cos^2\theta}{\sin^2\theta}L^2 \\ \Xi &= E^2r^4 - r^4 - L^2r^2 + 2Mr^3 + 2ML^2r - r^2\frac{\cos^2\theta}{\sin^2\theta}L^2 + 2Mr\frac{\cos^2\theta}{\sin^2\theta}L^2 \end{aligned} \tag{2.31}$$

In order to write the equations of motion we need the derivatives

$$\frac{\partial\Xi}{\partial E} = 2Er^4 \tag{2.32}$$

$$\begin{aligned} \frac{\partial\Xi}{\partial L} &= -2(r^2-2Mr)L - 2(r^2-2Mr)\frac{\cos^2\theta}{\sin^2\theta}L \\ &= -2(r^2-2Mr)L\left[1+\frac{\cos^2\theta}{\sin^2\theta}\right] \\ &= -2(r^2-2Mr)L\csc^2\theta \end{aligned} \tag{2.33}$$

$$\tag{2.34}$$

and also

$$\frac{\partial}{\partial r}\left(\frac{\Delta}{2\Sigma}\right) = \frac{\partial}{\partial r}\left(\frac{1}{2} - \frac{M}{r}\right) = \frac{M}{r^2} \tag{2.35}$$

$$\frac{\partial}{\partial r}\left(\frac{1}{2\Sigma}\right) = \frac{\partial}{\partial r}\left(\frac{1}{2r^2}\right) = -\frac{1}{r^3} \tag{2.36}$$

$$\begin{aligned}
\frac{\partial}{\partial r}\left(\frac{\Xi}{2\Delta\Sigma}\right) &= \frac{\partial}{\partial r}\left(\frac{E^2 r^4 - (r^2 - 2Mr)(r^2 + L^2) - (r^2 - 2Mr)\frac{\cos^2\theta}{\sin^2\theta}L^2}{2r^2(r^2 - 2Mr)}\right) \\
&= \frac{1}{2}\frac{\partial}{\partial r}\left(E^2\left(1 - \frac{2M}{r}\right)^{-1} - 1 - \frac{L^2}{r^2} - \frac{\cos^2\theta}{\sin^2\theta}\frac{L^2}{r^2}\right) \\
&= -\frac{E^2}{(1 - \frac{2M}{r})^2}\frac{M}{r^2} + \frac{L^2}{r^3} + \frac{\cos^2\theta}{\sin^2\theta}\frac{L^2}{r^3} \\
&= -\frac{E^2 M}{(r - 2M)^2} + \frac{L^2}{r^3}\csc^2\theta
\end{aligned} \tag{2.37}$$

$$\frac{\partial}{\partial \theta}\left(\frac{\Delta}{2\Sigma}\right) = \frac{\partial}{\partial r}\left(\frac{1}{2} - \frac{M}{r}\right) = 0 \tag{2.38}$$

$$\frac{\partial}{\partial \theta}\left(\frac{1}{2\Sigma}\right) = \frac{\partial}{\partial r}\left(\frac{1}{2r^2}\right) = 0 \tag{2.39}$$

$$\begin{aligned}
\frac{\partial}{\partial \theta}\left(\frac{\Xi}{2\Delta\Sigma}\right) &= \frac{\partial}{\partial \theta}\left(\frac{E^2 r^4 - (r^2 - 2Mr)(r^2 + L^2) - (r^2 - 2Mr)\frac{\cos^2\theta}{\sin^2\theta}L^2}{2r^2(r^2 - 2Mr)}\right) \\
&= \frac{1}{2}\frac{\partial}{\partial \theta}\left(-\frac{\cos^2\theta}{\sin^2\theta}\frac{L^2}{r^2}\right) \\
&= \cot\theta\csc^2\theta\frac{L^2}{r^2}
\end{aligned} \tag{2.40}$$

Using this expressions, the equations of motion of a particle in this spacetime are given by the Hamilton's equations

$$\begin{aligned}
\dot{t} &= \frac{1}{2r^2(r^2 - 2Mr)}2Er^4 = \frac{Er^2}{(r^2 - 2Mr)} \\
\dot{r} &= p_r\left(1 - \frac{2M}{r}\right) \\
\dot{\theta} &= \frac{p_\theta}{r^2} \\
\dot{\phi} &= -\frac{1}{2r^2(r^2 - 2Mr)}\left[-2(r^2 - 2Mr)L\csc^2\theta\right] = \frac{L}{r^2\sin^2\theta}
\end{aligned}$$

$$\begin{aligned}
\dot{p}_t &= 0 \\
\dot{p}_r &= -\frac{M}{r^2}p_r^2 + \frac{p_\theta^2}{r^3} - \frac{E^2 M}{(r-2M)^2} + \frac{L^2}{r^3 \sin^2\theta} \\
\dot{p}_\theta &= \frac{\cos\theta}{\sin^3\theta}\frac{L^2}{r^2} \\
\dot{p}_\phi &= 0
\end{aligned}$$

2.3 Kerr Spacetime

.

The Kerr spacetime in Boyer-Lindquist coordinates is given by the line element

$$ds^2 = -\left(1 - \frac{2Mr}{\Sigma}\right)dt^2 - -\frac{4Mar\sin^2\theta}{\Sigma}dtd\phi \tag{2.41}$$

$$+\frac{\Sigma}{\Delta}dr^2 + \Sigma d\theta^2 + \sin^2\theta\left(r^2 + a^2 + \frac{2Ma^2 r\sin^2\theta}{\Sigma}\right)d\phi^2, \tag{2.42}$$

where

$$\Sigma = r^2 + a^2\cos\theta \tag{2.43}$$

$$\Delta = r^2 - 2Mr + a^2. \tag{2.44}$$

Therefore, the potentials in the description of a particle moving in this spacetime reduce to

$$R = W^2 - (r^2 - 2Mr + a^2)\left[r^2 + (L - aE)^2 + Q\right] \tag{2.45}$$

$$W = E(r^2 + a^2) - aL \tag{2.46}$$

$$\Theta = Q - \cos^2\theta\left[a^2(1 - E^2) + \frac{L^2}{\sin^2\theta}\right] \tag{2.47}$$

Then, we have the function

$$\begin{aligned}
\Xi =& R + \Delta\Theta \\
\Xi =& W^2 - \Delta\left[r^2 + (L - aE)^2 + Q\right] + \Delta Q - \Delta\cos^2\theta\left[a^2(1 - E^2) + \frac{L^2}{\sin^2\theta}\right] \\
\Xi =& W^2 - \Delta\left[r^2 + (L - aE)^2 + a^2(1 - E^2)\cos^2\theta + \frac{\cos^2\theta}{\sin^2\theta}L^2\right] \qquad (2.48)
\end{aligned}$$

In order to write the equations of motion we need the derivatives

$$\begin{aligned}\frac{\partial \Xi}{\partial E} &= 2W(r^2+a^2) + 2a\Delta(L-aE) + 2Ea^2\Delta\cos^2\theta \\ &= 2W(r^2+a^2) + 2a\Delta\left(L-aE+aE\cos^2\theta\right) \\ &= 2W(r^2+a^2) + 2a\Delta\left(L-aE\sin^2\theta\right) \end{aligned} \tag{2.49}$$

$$\begin{aligned}\frac{\partial \Xi}{\partial L} &= -2aW - 2\Delta(L-aE) - 2L\Delta\frac{\cos^2\theta}{\sin^2\theta} \\ &= -2aW + 2aE\Delta - 2L\Delta\left[1+\frac{\cos^2\theta}{\sin^2\theta}\right] \\ &= -2aW + 2aE\Delta - 2L\Delta\csc^2\theta. \end{aligned} \tag{2.50}$$

Since

$$\frac{\partial \Delta}{\partial r} = 2(r-M) \tag{2.51}$$

and

$$\frac{\partial \Sigma}{\partial r} = 2r, \tag{2.52}$$

we obtain the derivatives

$$\frac{\partial}{\partial r}\left(\frac{\Delta}{2\Sigma}\right) = \frac{1}{2\Sigma}\frac{\partial \Delta}{\partial r} - \frac{\Delta}{2\Sigma^2}\frac{\partial \Sigma}{\partial r} = \frac{r-M}{\Sigma} - \frac{r\Delta}{\Sigma^2} \tag{2.53}$$

$$\frac{\partial}{\partial r}\left(\frac{1}{2\Sigma}\right) = -\frac{1}{2\Sigma^2}\frac{\partial \Sigma}{\partial r} = -\frac{r}{\Sigma^2} \tag{2.54}$$

$$\begin{aligned}\frac{\partial}{\partial r}\left(\frac{\Xi}{2\Delta\Sigma}\right) &= \frac{1}{2\Delta\Sigma}\frac{\partial \Xi}{\partial r} - \frac{\Xi}{2\Delta^2\Sigma}\frac{\partial \Delta}{\partial r} - \frac{\Xi}{2\Delta\Sigma^2}\frac{\partial \Sigma}{\partial r} \\ &= \frac{1}{2\Delta\Sigma}\frac{\partial \Xi}{\partial r} - \frac{\Xi(r-M)}{\Delta^2\Sigma} - \frac{\Xi r}{\Delta\Sigma^2} \end{aligned} \tag{2.55}$$

where

$$\frac{\partial \Xi}{\partial r} = 4rEW - 2(r-M)\left[r^2 + (L-aE)^2 + a^2(1-E^2)\cos^2\theta + \frac{\cos^2\theta}{\sin^2\theta}L^2\right] - 2r\Delta. \tag{2.56}$$

Similarly, since

$$\frac{\partial \Delta}{\partial \theta} = 0 \tag{2.57}$$

and

$$\frac{\partial \Sigma}{\partial \theta} = -2a^2\cos\theta\sin\theta, \tag{2.58}$$

we have the derivatives

$$
\begin{aligned}
\frac{\partial}{\partial\theta}\left(\frac{\Delta}{2\Sigma}\right) &= -\frac{\Delta}{2\Sigma^2}\frac{\partial\Sigma}{\partial\theta} = \frac{\Delta}{\Sigma^2}a^2\cos\theta\sin\theta && (2.59)\\
\frac{\partial}{\partial\theta}\left(\frac{1}{2\Sigma}\right) &= -\frac{1}{2\Sigma^2}\frac{\partial\Sigma}{\partial\theta} = \frac{a^2\cos\theta\sin\theta}{\Sigma^2} && (2.60)\\
\frac{\partial}{\partial\theta}\left(\frac{\Xi}{2\Delta\Sigma}\right) &= \frac{1}{2\Delta\Sigma}\frac{\partial\Xi}{\partial\theta} - \frac{\Xi}{2\Delta\Sigma^2}\frac{\partial\Sigma}{\partial\theta}\\
&= \frac{1}{2\Delta\Sigma}\left[2\Delta a^2(1-E^2)\cos\theta\sin\theta + 2\Delta L^2\cot\theta\csc^2\theta\right] + \frac{\Xi}{\Delta\Sigma^2}a^2\cos\theta\sin\theta\\
&= \frac{1}{\Sigma}\left[a^2(1-E^2)\cos\theta\sin\theta + L^2\cot\theta\csc^2\theta\right] + \frac{\Xi}{\Delta\Sigma^2}a^2\cos\theta\sin\theta && (2.61)
\end{aligned}
$$

Using this expressions, the equations of motion of a particle in this spacetime are given by the Hamilton's equations

The equations of motion of a particle in this spacetime are

$$
\begin{aligned}
\dot{t} &= \frac{1}{2\Delta\Sigma}\frac{\partial\Xi}{\partial E}\\
\dot{r} &= \frac{\Delta}{\Sigma}p_r\\
\dot{\theta} &= \frac{p_\theta}{\Sigma}\\
\dot{\phi} &= -\frac{1}{2\Delta\Sigma}\frac{\partial\Xi}{\partial L}
\end{aligned}
$$

$$
\begin{aligned}
\dot{p}_t &= 0\\
\dot{p}_r &= -\frac{\partial}{\partial r}\left(\frac{\Delta}{2\Sigma}\right)p_r^2 - \frac{\partial}{\partial r}\left(\frac{1}{2\Sigma}\right)p_\theta^2 + \frac{\partial}{\partial r}\left(\frac{\Xi}{2\Delta\Sigma}\right)\\
\dot{p}_\theta &= -\frac{\partial}{\partial\theta}\left(\frac{\Delta}{2\Sigma}\right)p_r^2 - \frac{\partial}{\partial\theta}\left(\frac{1}{2\Sigma}\right)p_\theta^2 + \frac{\partial}{\partial\theta}\left(\frac{\Xi}{2\Delta\Sigma}\right)\\
\dot{p}_\phi &= 0
\end{aligned}
$$

$$
\begin{aligned}
\dot{t} &= \frac{1}{2r^2(r^2-2Mr)}2Er^4 = \frac{Er^2}{(r^2-2Mr)} \\
\dot{r} &= p_r\left(1-\frac{2M}{r}\right) \\
\dot{\theta} &= \frac{p_\theta}{r^2} \\
\dot{\phi} &= -\frac{1}{2r^2(r^2-2Mr)}\left[-2(r^2-2Mr)L\csc^2\theta\right] = \frac{L}{r^2\sin^2\theta}
\end{aligned}
$$

$$
\begin{aligned}
\dot{p}_t &= 0 \\
\dot{p}_r &= -\frac{M}{r^2}p_r^2 + \frac{p_\theta^2}{r^3} - \frac{E^2M}{(r-2M)^2} + \frac{L^2}{r^3\sin^2\theta} \\
\dot{p}_\theta &= \frac{\cos\theta}{\sin^3\theta}\frac{L^2}{r^2} \\
\dot{p}_\phi &= 0
\end{aligned}
$$