Ray Trace

E. Larrañaga

June 15, 2018

Chapter 1

Initial Conditions

.

In order to calculate the image of a black hole we will trace back the path of photons from a point in the observer's image plane, located at a distant point, onto the emission structure around the black hole. The image plane is considered as a grid and the received photons will have a momentum vector orthogonal to this plane.

1.1 Coordinate Systems

We will take Cartesian coordinates (X, Y, Z) at the image plane of the observer and Cartesian coordinates (x, y, z) centered at the black hole. These two systems are related by a rotation and a translation. As seen in Figure XXX we may define a rotation between the system (X, Y, Z) and an intermediate system (X', Y', Z') as

$$\begin{pmatrix} X' \\ Y' \\ Z' \end{pmatrix} = \begin{pmatrix} \cos\alpha & 0 & -\sin\alpha \\ 0 & 1 & 0 \\ \sin\alpha & 0 & \cos\alpha \end{pmatrix} \begin{pmatrix} X \\ Y \\ Z \end{pmatrix} \tag{1.1}$$

If the observer is located at a distance D from the black hole as in Figure XXX, we introduce the spatial translation from the intermediate system (X', Y', Z') to the system (x, y, z),

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} X' \\ Y' \\ Z' \end{pmatrix} + \begin{pmatrix} D\cos\alpha \\ 0 \\ D\sin\alpha \end{pmatrix} \tag{1.2}$$

The composition of these two transformations give

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} \cos\alpha & 0 & -\sin\alpha \\ 0 & 1 & 0 \\ \sin\alpha & 0 & \cos\alpha \end{pmatrix} \begin{pmatrix} X \\ Y \\ Z \end{pmatrix} + \begin{pmatrix} D\cos\alpha \\ 0 \\ D\sin\alpha \end{pmatrix} \tag{1.3}$$

or more explicitly,

$$\begin{cases} x &= X\cos\alpha - Z\sin\alpha + D\cos\alpha \\ y &= Y \\ z &= X\sin\alpha + Z\cos\alpha + D\sin\alpha \end{cases} \tag{1.4}$$

Since the angle α is related with the inclination angle by $\alpha = \frac{\pi}{2} - i$ we replace the trigonometric functions in the coordinate transformations by

$$\begin{cases} x &= X\sin i - Z\cos i + D\sin i \\ y &= Y \\ z &= X\cos i + Z\sin i + D\cos i \end{cases} \tag{1.5}$$

or better

$$\begin{cases} x &= (X + D)\sin i - Z\cos i \\ y &= Y \\ z &= (X + D)\cos i + Z\sin i \end{cases} \tag{1.6}$$

Since the black hole metric will be given in spherical coordinates, we introduce the relation between the Cartesian system (x, y, z) and the coordinates (r, θ, ϕ) by

$$\begin{cases} r &= \sqrt{x^2 + y^2 + z^2} \\ \theta &= \arccos\left(\frac{z}{r}\right) \\ \phi &= \arctan\left(\frac{y}{x}\right) \end{cases} \tag{1.7}$$

Hence, the complete coordinate transformation needed to describe the initial conditions of the photon is

$$\begin{cases} r &= \sqrt{(X + D)^2 + Y^2 + Z^2} \\ \theta &= \arccos\left(\frac{(X+D)\cos i + Z\sin i}{\sqrt{(X+D)^2 + Y^2 + Z^2}}\right) \\ \phi &= \arctan\left(\frac{Y}{(X+D)\sin i - Z\cos i}\right) \end{cases} \tag{1.8}$$

1.1.1 Initial Position of a photon

Consider a photon registered at the observer's image plane at the coordinates $(X, Y, Z) = (0, \alpha, \beta)$ at the time $t_0 = 0$. The spherical coordinates of this photon as seen from the black hole are given by the equation (1.8) as

$$\begin{cases} t_0 &= 0 \\ r_0 &= \sqrt{D^2 + \alpha^2 + \beta^2} \\ \theta_0 &= \arccos\left(\frac{D\cos i + \beta\sin i}{\sqrt{D^2 + \alpha^2 + \beta^2}}\right) \\ \phi_0 &= \arctan\left(\frac{\alpha}{D\sin i - \beta\cos i}\right) \end{cases} \tag{1.9}$$

1.2 Momentum of a Photon

Photons received by the distant observer will be considered to arrive perpendicular to the image plane. According to the orientation of the coordinates (X, Y, Z) in Figure XXX, the Cartesian components of the incident photon are given by the 4-momentum vector

$$\tilde{k}_0^{\nu} = \left(\tilde{k}_0^t, \tilde{k}_0^X, \tilde{k}_0^Y, \tilde{k}_0^Z\right) = (K_0, K_0, 0, 0)\,. \tag{1.10}$$

The components of the 4-momentum in the spherical coordinate system (r, θ, ϕ) are obtained using the transformation law

$$k^{\mu} = \frac{\partial x^{\mu}}{\partial \tilde{x}^{\nu}} \tilde{k}^{\nu}. \tag{1.11}$$

The relevant terms in the transformation matrix are obtained from equation (1.8) as

$$\left.\frac{\partial r}{\partial X}\right|_{(0,\alpha,\beta)} = \left.\frac{X+D}{\sqrt{(X+D)^2+Y^2+Z^2}}\right|_{(0,\alpha,\beta)} = \frac{D}{\sqrt{D^2+\alpha^2+\beta^2}} = \frac{D}{r_0} \tag{1.12}$$

$$\left.\frac{\partial \theta}{\partial X}\right|_{(0,\alpha,\beta)} = -\frac{1}{\sqrt{r^2-[(X+D)\cos i + Z\sin i]^2}}\left.\left[\cos i - \frac{(X+D)^2\cos i + Z(X+D)\sin i}{r^2}\right]\right|_{(0,\alpha,\beta)}$$

$$\left.\frac{\partial \theta}{\partial X}\right|_{(0,\alpha,\beta)} = -\frac{1}{\sqrt{r_0^2-[D\cos i + \beta\sin i]^2}}\left[\cos i - \frac{D^2\cos i + D\beta\sin i}{r_0^2}\right] \tag{1.13}$$

$$\left.\frac{\partial \theta}{\partial X}\right|_{(0,\alpha,\beta)} = -\frac{1}{\sqrt{\alpha^2+[D\sin i - \beta\cos i]^2}}\left[\cos i - \frac{D^2\cos i + D\beta\sin i}{r_0^2}\right] \tag{1.14}$$

$$\left.\frac{\partial \phi}{\partial X}\right|_{(0,\alpha,\beta)} = -\left.\frac{Y\sin i}{[(X+D)\sin i - Z\cos i]^2+Y^2}\right|_{(0,\alpha,\beta)}$$

$$\left.\frac{\partial \phi}{\partial X}\right|_{(0,\alpha,\beta)} = -\frac{\alpha\sin i}{\alpha^2+[D\sin i - \beta\cos i]^2} \tag{1.15}$$

Hence, the spatial components of the initial 4-momentum in spherical coordinates are

$$\begin{cases} k_0^r &= \frac{D}{r_0}K_0 \\ k_0^{\theta} &= -\frac{1}{\sqrt{\alpha^2+[D\sin i-\beta\cos i]^2}}\left[\cos i - \frac{D^2\cos i + D\beta\sin i}{r_0^2}\right]K_0 \\ k_0^{\phi} &= -\frac{\alpha\sin i}{\alpha^2+[D\sin i-\beta\cos i]^2}K_0 \end{cases} \tag{1.16}$$

The temporal component of the initial 4-momentum of the photon is obtained from the condition

$$k^\mu k_\mu = \eta_{\mu\nu} k^\mu k^\nu = 0, \tag{1.17}$$

which gives

$$k_0^t = \sqrt{(k_0^r)^2 + r_0^2 (k_0^\theta)^2 + r_0^2 \sin^2\theta_0 (k_0^\phi)^2}. \tag{1.18}$$

1.3 Complete set of Initial Conditions

The complete set of initial conditions are given by equations (1.8), (1.16) and (1.18),

$$t_0 = 0 \tag{1.19}$$

$$r_0 = \sqrt{D^2 + \alpha^2 + \beta^2} \tag{1.20}$$

$$\theta_0 = \arccos\left(\frac{D\cos i + \beta \sin i}{\sqrt{D^2 + \alpha^2 + \beta^2}}\right) \tag{1.21}$$

$$\phi_0 = \arctan\left(\frac{\alpha}{D\sin i - \beta\cos i}\right) \tag{1.22}$$

$$k_0^t = \sqrt{(k_0^r)^2 + r_0^2 (k_0^\theta)^2 + r_0^2 \sin^2\theta_0 (k_0^\phi)^2} \tag{1.23}$$

$$k_0^r = \frac{D}{r_0} K_0 \tag{1.24}$$

$$k_0^\theta = -\frac{1}{\sqrt{\alpha^2 + [D\sin i - \beta\cos i]^2}}\left[\cos i - \frac{D^2\cos i + D\beta\sin i}{r_0^2}\right] K_0 \tag{1.25}$$

$$k_0^\phi = -\frac{\alpha \sin i}{\alpha^2 + [D\sin i - \beta \cos i]^2} K_0 \tag{1.26}$$

1.3.1 Covariant Components of the 4-Momentum Vector

In order to solve the Hamiltonian equations of motion for the photons, we will need the initial components of the 4-momentum vector in their covariant form. Therefore we need to apply the spactime metric $g_{\mu\nu}$ as

$$k_\mu = g_{\mu\nu} k^\nu. \tag{1.27}$$

For a general stationary and axially symmetric spacetime described with the line element

$$ds^2 = g_{tt}dt^2 + 2g_{t\phi}dtd\phi + g_{rr}dr^2 + g_{\theta\theta}d\theta^2 + g_{\phi\phi}d\phi^2 \tag{1.28}$$

the covariant components of the 4-momentum are

$$\begin{cases} k_t &= g_{tt}k^t + g_{t\phi}k^\phi \\ k_r &= g_{rr}k^r \\ k_\theta &= g_{\theta\theta}k^\theta \\ k_\phi &= g_{t\phi}k^t + g_{\phi\phi}k^\phi \end{cases} \tag{1.29}$$

Chapter 2

Geodesic Equations

.
Consider a general line element of a general stationary and axially symmetric spacetime described by the line element

$$ds^2 = g_{tt}dt^2 + 2g_{t\phi}dtd\phi + g_{rr}dr^2 + g_{\theta\theta}d\theta^2 + g_{\phi\phi}d\phi^2. \tag{2.1}$$

The Lagrangian describing the geodesic motion of a test-particle in this spacetime is

$$\mathcal{L} = \frac{1}{2}g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu, \tag{2.2}$$

where $\dot{x}^\mu = \frac{dx^\mu}{d\lambda}$ and λ is an affine parameter of the geodesic. The momentum From the stationary and axially symmetric conditions of spacetime, we conclude that the metric components are independent of the coordinates t and ϕ. Therefore, there are two constants of motion: the specific energy at infinity, E, and the axial component of the specific angular momentum at infinity, L_z. These are defined as

$$-E = p_t = \frac{\partial \mathcal{L}}{\partial \dot{t}} = g_{tt}\dot{t} + g_{t\phi}\dot{\phi} \tag{2.3}$$

$$L_z = p_\phi = \frac{\partial \mathcal{L}}{\partial \dot{\phi}} = g_{t\phi}\dot{t} + g_{\phi\phi}\dot{\phi}. \tag{2.4}$$

in Boyer-Lindquist coordinates,

$$\begin{aligned} ds^2 = &-\left(1 - \frac{2Mr}{\Sigma}\right)dt^2 - -\frac{4Mar\sin^2\theta}{\Sigma}dtd\phi \\ &+ \frac{\Sigma}{\Delta}dr^2 + \Sigma d\theta^2 + \sin^2\theta\left(r^2 + a^2 + \frac{2Ma^2r\sin^2\theta}{\Sigma}\right)d\phi^2, \end{aligned}$$

$$\begin{aligned} \Sigma &= \Sigma(r,\theta) \\ \Delta &= \Delta(r). \end{aligned}$$

The equations of motion of a particle in this spacetime are

$$
\begin{aligned}
\dot{t} &= \frac{1}{2\Delta\Sigma}\frac{\partial\Xi}{\partial E} \\
\dot{r} &= \frac{\Delta}{\Sigma}p_r \\
\dot{\theta} &= \frac{p_\theta}{\Sigma} \\
\dot{\phi} &= -\frac{1}{2\Delta\Sigma}\frac{\partial\Xi}{\partial L}
\end{aligned}
$$

$$
\begin{aligned}
\dot{p}_t &= 0 \\
\dot{p}_r &= -\frac{\partial}{\partial r}\left(\frac{\Delta}{2\Sigma}\right)p_r^2 - \frac{\partial}{\partial r}\left(\frac{1}{2\Sigma}\right)p_\theta^2 + \frac{\partial}{\partial r}\left(\frac{\Xi}{2\Delta\Sigma}\right) \\
\dot{p}_\theta &= -\frac{\partial}{\partial\theta}\left(\frac{\Delta}{2\Sigma}\right)p_r^2 - \frac{\partial}{\partial\theta}\left(\frac{1}{2\Sigma}\right)p_\theta^2 + \frac{\partial}{\partial\theta}\left(\frac{\Xi}{2\Delta\Sigma}\right) \\
\dot{p}_\phi &= 0
\end{aligned}
$$

$$
\begin{aligned}
\Xi &= R + \Delta\Theta \\
R &= P^2 - \Delta\left[r^2 + (L - aE)^2 + Q\right] \\
P &= E(r^2 + a^2) - aL \\
\Theta &= Q - \cos^2\theta\left[a^2(1 - E^2) + \frac{L^2}{\sin^2\theta}\right]
\end{aligned}
$$

Chapter 3

The Minkowski Spacetime

. The Minkowski spacetime is given by the line element

$$ds^2 = -dt^2 + dx^2 + dy^2 + dz^2 \tag{3.1}$$

or in spherical coordinates

$$ds^2 = -dt^2 + dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2. \tag{3.2}$$

When compared with the standard form of the Kerr's line element in Boyer-Lindquist coordinates,

$$\begin{aligned} ds^2 = &-\left(1 - \frac{2Mr}{\Sigma}\right) dt^2 - \frac{4Mar\sin^2\theta}{\Sigma} dt d\phi \\ &+ \frac{\Sigma}{\Delta} dr^2 + \Sigma d\theta^2 + \sin^2\theta \left(r^2 + a^2 + \frac{2Ma^2 r \sin^2\theta}{\Sigma}\right) d\phi^2, \end{aligned} \tag{3.3}$$

Minkowski's metric is obtained by taking $a = 0$, $M = 0$ and

$$\Sigma = r^2 \tag{3.4}$$

$$\Delta = r^2. \tag{3.5}$$

Therefore, the potentials in the description of a particle moving in this spacetime reduce to

$$R = E^2 r^4 - r^2 \left[r^2 + L^2 + Q\right] \tag{3.6}$$

$$\Theta = Q - \frac{\cos^2\theta}{\sin^2\theta} L^2 \tag{3.7}$$

Then, we have the function

$$\Xi = E^2 r^4 - r^2 \left[r^2 + L^2 + Q\right] + r^2 \left[Q - \frac{\cos^2\theta}{\sin^2\theta} L^2\right]$$

$$\Xi = E^2 r^4 - r^4 - L^2 r^2 - r^2 \frac{\cos^2\theta}{\sin^2\theta} L^2 \tag{3.8}$$

In order to write the equations of motion we need the derivatives

$$\frac{\partial \Xi}{\partial E} = 2Er^4 \tag{3.9}$$

$$\frac{\partial \Xi}{\partial L} = -2Lr^2 - 2r^2 \frac{\cos^2\theta}{\sin^2\theta} L \tag{3.10}$$

and also

$$\frac{\partial}{\partial r}\left(\frac{\Delta}{2\Sigma}\right) = \frac{\partial}{\partial r}\left(\frac{1}{2}\right) = 0 \tag{3.11}$$

$$\frac{\partial}{\partial r}\left(\frac{1}{2\Sigma}\right) = \frac{\partial}{\partial r}\left(\frac{1}{2r^2}\right) = -\frac{1}{r^3} \tag{3.12}$$

$$\begin{aligned}
\frac{\partial}{\partial r}\left(\frac{\Xi}{2\Delta\Sigma}\right) &= \frac{\partial}{\partial r}\left(\frac{E^2r^4 - r^4 - L^2r^2 - r^2\frac{\cos^2\theta}{\sin^2\theta}L^2}{2r^4}\right) \\
&= \frac{1}{2}\frac{\partial}{\partial r}\left(E^2 - 1 - \frac{L^2}{r^2} - \frac{\cos^2\theta}{\sin^2\theta}\frac{L^2}{r^2}\right) \\
&= \frac{L^2}{r^3} + \frac{\cos^2\theta}{\sin^2\theta}\frac{L^2}{r^3}
\end{aligned} \tag{3.13}$$

$$\frac{\partial}{\partial \theta}\left(\frac{\Delta}{2\Sigma}\right) = \frac{\partial}{\partial r}\left(\frac{1}{2}\right) = 0 \tag{3.14}$$

$$\frac{\partial}{\partial \theta}\left(\frac{1}{2\Sigma}\right) = \frac{\partial}{\partial r}\left(\frac{1}{2r^2}\right) = 0 \tag{3.15}$$

$$\begin{aligned}
\frac{\partial}{\partial \theta}\left(\frac{\Xi}{2\Delta\Sigma}\right) &= \frac{\partial}{\partial \theta}\left(\frac{E^2r^4 - r^4 - L^2r^2 - r^2\frac{\cos^2\theta}{\sin^2\theta}L^2}{2r^4}\right) \\
&= \frac{1}{2}\frac{\partial}{\partial \theta}\left(E^2 - 1 - \frac{L^2}{r^2} - \frac{\cos^2\theta}{\sin^2\theta}\frac{L^2}{r^2}\right) \\
&= \cot\theta\csc^2\theta\frac{L^2}{r^2}
\end{aligned} \tag{3.16}$$

Using this expressions, the equations of motion of a particle in this spacetime are given by the Hamilton's equations

$$\begin{aligned}
\dot{t} &= \frac{1}{2r^4}2Er^4 = E \\
\dot{r} &= p_r \\
\dot{\theta} &= \frac{p_\theta}{r^2} \\
\dot{\phi} &= -\frac{1}{2r^4}\left[-2Lr^2 - 2r^2\frac{\cos^2\theta}{\sin^2\theta}L\right] = \left[1 + \frac{\cos^2\theta}{\sin^2\theta}\right]\frac{L}{r^2} = \frac{L}{r^2\sin^2\theta}
\end{aligned}$$

$$\begin{aligned}
\dot{p}_t &= 0 \\
\dot{p}_r &= \frac{p_\theta^2}{r^3} + \frac{L^2}{r^3} + \frac{\cos^2\theta}{\sin^2\theta}\frac{L^2}{r^3} = \frac{p_\theta^2}{r^3} + \frac{1}{\sin^2\theta}\frac{L^2}{r^3} \\
\dot{p}_\theta &= \frac{\cos\theta}{\sin^3\theta}\frac{L^2}{r^2} \\
\dot{p}_\phi &= 0
\end{aligned}$$

Chapter 4

The Schwarzschild Spacetime

.

The Schwarzschild spacetime is given by the line element

$$ds^2 = -\left(1 - \frac{2M}{r}\right) dt^2 + \left(1 - \frac{2M}{r}\right)^{-1} dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2. \tag{4.1}$$

When compared with the standard form of the Kerr's line element in Boyer-Lindquist coordinates,

$$\begin{aligned} ds^2 = &-\left(1 - \frac{2Mr}{\Sigma}\right) dt^2 - \frac{4Mar\sin^2\theta}{\Sigma} dt d\phi \\ &+ \frac{\Sigma}{\Delta} dr^2 + \Sigma d\theta^2 + \sin^2\theta \left(r^2 + a^2 + \frac{2Ma^2 r \sin^2\theta}{\Sigma}\right) d\phi^2, \end{aligned} \tag{4.2}$$

Schwarzschild's metric is obtained by taking $a = 0$ and

$$\Sigma = r^2 \tag{4.3}$$

$$\Delta = r^2 - 2Mr. \tag{4.4}$$

Therefore, the potentials in the description of a particle moving in this spacetime reduce to

$$R = E^2 r^4 - (r^2 - 2Mr)\left[r^2 + L^2 + Q\right] \tag{4.5}$$

$$\Theta = Q - \frac{\cos^2\theta}{\sin^2\theta} L^2 \tag{4.6}$$

Then, we have the function

$$\begin{aligned}
\Xi &= E^2r^4 - (r^2-2Mr)\left[r^2+L^2+Q\right] + (r^2-2Mr)\left[Q - \frac{\cos^2\theta}{\sin^2\theta}L^2\right] \\
\Xi &= E^2r^4 - (r^2-2Mr)(r^2+L^2) - (r^2-2Mr)\frac{\cos^2\theta}{\sin^2\theta}L^2 \\
\Xi &= E^2r^4 - r^4 - L^2r^2 + 2Mr^3 + 2ML^2r - r^2\frac{\cos^2\theta}{\sin^2\theta}L^2 + 2Mr\frac{\cos^2\theta}{\sin^2\theta}L^2 \qquad (4.7)
\end{aligned}$$

In order to write the equations of motion we need the derivatives

$$\begin{aligned}
\frac{\partial\Xi}{\partial E} &= 2Er^4 \qquad (4.8)\\
\frac{\partial\Xi}{\partial L} &= -2(r^2-2Mr)L - 2(r^2-2Mr)\frac{\cos^2\theta}{\sin^2\theta}L \\
&= -2(r^2-2Mr)L\left[1+\frac{\cos^2\theta}{\sin^2\theta}\right] \\
&= -2(r^2-2Mr)L\csc^2\theta \qquad (4.9)\\
& \qquad (4.10)
\end{aligned}$$

and also

$$\begin{aligned}
\frac{\partial}{\partial r}\left(\frac{\Delta}{2\Sigma}\right) &= \frac{\partial}{\partial r}\left(\frac{1}{2}-\frac{M}{r}\right) = \frac{M}{r^2} \qquad (4.11)\\
\frac{\partial}{\partial r}\left(\frac{1}{2\Sigma}\right) &= \frac{\partial}{\partial r}\left(\frac{1}{2r^2}\right) = -\frac{1}{r^3} \qquad (4.12)\\
\frac{\partial}{\partial r}\left(\frac{\Xi}{2\Delta\Sigma}\right) &= \frac{\partial}{\partial r}\left(\frac{E^2r^4-(r^2-2Mr)(r^2+L^2)-(r^2-2Mr)\frac{\cos^2\theta}{\sin^2\theta}L^2}{2r^2(r^2-2Mr)}\right) \\
&= \frac{1}{2}\frac{\partial}{\partial r}\left(E^2\left(1-\frac{2M}{r}\right)^{-1} - 1 - \frac{L^2}{r^2} - \frac{\cos^2\theta}{\sin^2\theta}\frac{L^2}{r^2}\right) \\
&= -\frac{E^2}{(1-\frac{2M}{r})^2}\frac{M}{r^2} + \frac{L^2}{r^3} + \frac{\cos^2\theta}{\sin^2\theta}\frac{L^2}{r^3} \\
&= -\frac{E^2M}{(r-2M)^2} + \frac{L^2}{r^3}\csc^2\theta \qquad (4.13)
\end{aligned}$$

$$\frac{\partial}{\partial\theta}\left(\frac{\Delta}{2\Sigma}\right) = \frac{\partial}{\partial r}\left(\frac{1}{2} - \frac{M}{r}\right) = 0 \tag{4.14}$$

$$\frac{\partial}{\partial\theta}\left(\frac{1}{2\Sigma}\right) = \frac{\partial}{\partial r}\left(\frac{1}{2r^2}\right) = 0 \tag{4.15}$$

$$\begin{aligned}\frac{\partial}{\partial\theta}\left(\frac{\Xi}{2\Delta\Sigma}\right) &= \frac{\partial}{\partial\theta}\left(\frac{E^2r^4 - (r^2-2Mr)(r^2+L^2) - (r^2-2Mr)\frac{\cos^2\theta}{\sin^2\theta}L^2}{2r^2(r^2-2Mr)}\right) \\ &= \frac{1}{2}\frac{\partial}{\partial\theta}\left(-\frac{\cos^2\theta}{\sin^2\theta}\frac{L^2}{r^2}\right) \\ &= \cot\theta\csc^2\theta\frac{L^2}{r^2} \end{aligned} \tag{4.16}$$

Using this expressions, the equations of motion of a particle in this spacetime are given by the Hamilton's equations

$$\begin{aligned}\dot{t} &= \frac{1}{2r^2(r^2-2Mr)}2Er^4 = \frac{Er^2}{(r^2-2Mr)} \\ \dot{r} &= p_r\left(1 - \frac{2M}{r}\right) \\ \dot{\theta} &= \frac{p_\theta}{r^2} \\ \dot{\phi} &= -\frac{1}{2r^2(r^2-2Mr)}\left[-2(r^2-2Mr)L\csc^2\theta\right] = \frac{L}{r^2\sin^2\theta}\end{aligned}$$

$$\begin{aligned}\dot{p}_t &= 0 \\ \dot{p}_r &= -\frac{M}{r^2}p_r^2 + \frac{p_\theta^2}{r^3} - \frac{E^2M}{(r-2M)^2} + \frac{L^2}{r^3\sin^2\theta} \\ \dot{p}_\theta &= \frac{\cos\theta}{\sin^3\theta}\frac{L^2}{r^2} \\ \dot{p}_\phi &= 0\end{aligned}$$