# Ray Trace

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# Chapter 1

# Initial Conditions

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In order to calculate the image of a black hole we will trace back the path of photons from a point in the observer's image plane, located at a distant point, onto the emission structure around the black hole. The image plane is considered as a grid and the received photons will have a momentum vector orthogonal to this plane.

## 1.1 Coordinate Systems

We will take Cartesian coordinates $(X, Y, Z)$ at the image plane of the observer and Cartesian coordinates $(x, y, z)$ centered at the black hole. These two systems are related by a rotation and a translation. As seen in Figure 1.1 we may define a rotation between the system $(X, Y, Z)$ and an intermediate system $(X', Y', Z')$ as

$$\begin{pmatrix} X' \\ Y' \\ Z' \end{pmatrix} = \begin{pmatrix} \cos\alpha & 0 & -\sin\alpha \\ 0 & 1 & 0 \\ \sin\alpha & 0 & \cos\alpha \end{pmatrix} \begin{pmatrix} X \\ Y \\ Z \end{pmatrix} \tag{1.1}$$

If the observer is located at a distance $D$ from the black hole as in Figure 1.1, we introduce the spatial translation from the intermediate system $(X', Y', Z')$ to the system $(x, y, z)$,

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} X' \\ Y' \\ Z' \end{pmatrix} + \begin{pmatrix} D\cos\alpha \\ 0 \\ D\sin\alpha \end{pmatrix} \tag{1.2}$$

The composition of these two transformations give

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} \cos\alpha & 0 & -\sin\alpha \\ 0 & 1 & 0 \\ \sin\alpha & 0 & \cos\alpha \end{pmatrix} \begin{pmatrix} X \\ Y \\ Z \end{pmatrix} + \begin{pmatrix} D\cos\alpha \\ 0 \\ D\sin\alpha \end{pmatrix} \tag{1.3}$$

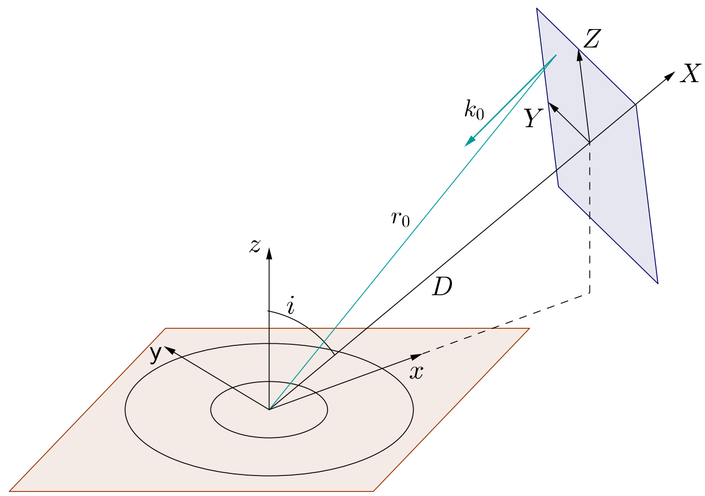


Figure 1.1: Coordinate Transformation for the initial conditions of the photon

or more explicitly,

$$\begin{cases} x &= X\cos\alpha - Z\sin\alpha + D\cos\alpha \\ y &= Y \\ z &= X\sin\alpha + Z\cos\alpha + D\sin\alpha \end{cases} \tag{1.4}$$

Since the angle $\alpha$ is related with the inclination angle by $\alpha = \frac{\pi}{2} - i$ we replace the trigonometric functions in the coordinate transformations by

$$\begin{cases} x &= X\sin i - Z\cos i + D\sin i \\ y &= Y \\ z &= X\cos i + Z\sin i + D\cos i \end{cases} \tag{1.5}$$

or better

$$\begin{cases} x &= (X + D)\sin i - Z\cos i \\ y &= Y \\ z &= (X + D)\cos i + Z\sin i \end{cases} \tag{1.6}$$

Since the black hole metric will be given in spherical coordinates, we introduce the relation between the Cartesian system $(x, y, z)$ and the coordinates $(r, \theta, \phi)$ by

$$\begin{cases} r &= \sqrt{x^2 + y^2 + z^2} \\ \theta &= \arccos\left(\frac{z}{r}\right) \\ \phi &= \arctan\left(\frac{y}{x}\right) \end{cases} \tag{1.7}$$

Hence, the complete coordinate transformation needed to describe the initial conditions of the photon is

$$\begin{cases} r &= \sqrt{(X+D)^2+Y^2+Z^2} \\ \theta &= \arccos\left(\frac{(X+D)\cos i + Z\sin i}{\sqrt{(X+D)^2+Y^2+Z^2}}\right) \\ \phi &= \arctan\left(\frac{Y}{(X+D)\sin i - Z\cos i}\right) \end{cases} \tag{1.8}$$

### 1.1.1 Initial Position of a photon

Consider a photon registered at the observer's image plane at the coordinates $(X, Y, Z) = (0, \alpha, \beta)$ at the time $t_0 = 0$. The spherical coordinates of this photon as seen from the black hole are given by the equation (1.8) as

$$\begin{cases} t_0 &= 0 \\ r_0 &= \sqrt{D^2+\alpha^2+\beta^2} \\ \theta_0 &= \arccos\left(\frac{D\cos i + \beta\sin i}{\sqrt{D^2+\alpha^2+\beta^2}}\right) \\ \phi_0 &= \arctan\left(\frac{\alpha}{D\sin i - \beta\cos i}\right) \end{cases} \tag{1.9}$$

## 1.2 Momentum of a Photon

Photons received by the distant observer will be considered to arrive perpendicular to the image plane. According to the orientation of the coordinates $(X, Y, Z)$ in Figure 1.1, the Cartesian components of the incident photon are given by the 4-momentum vector

$$\tilde{k}_0^\nu = \left(\tilde{k}_0^t, \tilde{k}_0^X, \tilde{k}_0^Y, \tilde{k}_0^Z\right) = (K_0, K_0, 0, 0)\,. \tag{1.10}$$

The components of the 4-momentum in the spherical coordinate system $(r, \theta, \phi)$ are obtained using the transformation law

$$k^\mu = \frac{\partial x^\mu}{\partial \tilde{x}^\nu}\tilde{k}^\nu. \tag{1.11}$$

The relevant terms in the transformation matrix are obtained from equation (1.8) as

$$\left.\frac{\partial r}{\partial X}\right|_{(0,\alpha,\beta)} = \left.\frac{X+D}{\sqrt{(X+D)^2+Y^2+Z^2}}\right|_{(0,\alpha,\beta)} = \frac{D}{\sqrt{D^2+\alpha^2+\beta^2}} = \frac{D}{r_0} \tag{1.12}$$

$$\left.\frac{\partial \theta}{\partial X}\right|_{(0,\alpha,\beta)} = -\frac{1}{\sqrt{r^2-[(X+D)\cos i + Z\sin i]^2}}\left[\cos i - \frac{(X+D)^2\cos i + Z(X+D)\sin i}{r^2}\right]\Bigg|_{(0,\alpha,\beta)}$$

$$\left.\frac{\partial\theta}{\partial X}\right|_{(0,\alpha,\beta)} = -\frac{1}{\sqrt{r_0^2 - [D\cos i + \beta\sin i]^2}}\left[\cos i - \frac{D^2\cos i + D\beta\sin i}{r_0^2}\right] \quad (1.13)$$

$$\left.\frac{\partial\theta}{\partial X}\right|_{(0,\alpha,\beta)} = -\frac{1}{\sqrt{\alpha^2 + [D\sin i - \beta\cos i]^2}}\left[\cos i - \frac{D^2\cos i + D\beta\sin i}{r_0^2}\right] \quad (1.14)$$

$$\begin{aligned}\left.\frac{\partial\phi}{\partial X}\right|_{(0,\alpha,\beta)} &= -\left.\frac{Y\sin i}{[(X+D)\sin i - Z\cos i]^2 + Y^2}\right|_{(0,\alpha,\beta)} \\ \left.\frac{\partial\phi}{\partial X}\right|_{(0,\alpha,\beta)} &= -\frac{\alpha\sin i}{\alpha^2 + [D\sin i - \beta\cos i]^2} \end{aligned} \quad (1.15)$$

Hence, the spatial components of the initial 4-momentum in spherical coordinates are

$$\begin{cases} k_0^r &= \frac{D}{r_0}K_0 \\ k_0^\theta &= -\frac{1}{\sqrt{\alpha^2+[D\sin i-\beta\cos i]^2}}\left[\cos i - \frac{D^2\cos i+D\beta\sin i}{r_0^2}\right]K_0 \\ k_0^\phi &= -\frac{\alpha\sin i}{\alpha^2+[D\sin i-\beta\cos i]^2}K_0 \end{cases} \quad (1.16)$$

The temporal component of the initial 4-momentum of the photon is obtained from the condition

$$k^\mu k_\mu = \eta_{\mu\nu}k^\mu k^\nu = 0, \quad (1.17)$$

which gives

$$k_0^t = \sqrt{(k_0^r)^2 + r_0^2(k_0^\theta)^2 + r_0^2\sin^2\theta_0(k_0^\phi)^2}. \quad (1.18)$$

## 1.3 Complete set of Initial Conditions

The complete set of initial conditions are given by equations (1.8), (1.16) and (1.18),

$$t_0 = 0 \tag{1.19}$$
$$r_0 = \sqrt{D^2 + \alpha^2 + \beta^2} \tag{1.20}$$
$$\theta_0 = \arccos\left(\frac{D\cos i + \beta \sin i}{\sqrt{D^2 + \alpha^2 + \beta^2}}\right) \tag{1.21}$$
$$\phi_0 = \arctan\left(\frac{\alpha}{D\sin i - \beta \cos i}\right) \tag{1.22}$$
$$k_0^t = \sqrt{(k_0^r)^2 + r_0^2(k_0^\theta)^2 + r_0^2 \sin^2\theta_0 (k_0^\phi)^2} \tag{1.23}$$
$$k_0^r = \frac{D}{r_0} K_0 \tag{1.24}$$
$$k_0^\theta = -\frac{1}{\sqrt{\alpha^2 + [D\sin i - \beta\cos i]^2}}\left[\cos i - \frac{D^2\cos i + D\beta \sin i}{r_0^2}\right] K_0 \tag{1.25}$$
$$k_0^\phi = -\frac{\alpha \sin i}{\alpha^2 + [D\sin i - \beta\cos i]^2} K_0 \tag{1.26}$$

### 1.3.1 Covariant Components of the 4-Momentum Vector

In order to solve the Hamiltonian equations of motion for the photons, we will need the initial components of the 4-momentum vector in their covariant form. Therefore we need to apply the spactime metric $g_{\mu\nu}$ as

$$k_\mu = g_{\mu\nu} k^\nu. \tag{1.27}$$

For a general stationary and axially symmetric spacetime described with the line element

$$ds^2 = g_{tt}dt^2 + 2g_{t\phi}dtd\phi + g_{rr}dr^2 + g_{\theta\theta}d\theta^2 + g_{\phi\phi}d\phi^2 \tag{1.28}$$

the covariant components of the 4-momentum are

$$\begin{cases} k_t &= g_{tt}k^t + g_{t\phi}k^\phi \\ k_r &= g_{rr}k^r \\ k_\theta &= g_{\theta\theta}k^\theta \\ k_\phi &= g_{t\phi}k^t + g_{\phi\phi}k^\phi \end{cases} \tag{1.29}$$

# Chapter 2

# Geodesic Equations

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Consider a general line element of a general stationary and axially symmetric spacetime described by the line element

$$ds^2 = g_{tt}dt^2 + 2g_{t\phi}dtd\phi + g_{rr}dr^2 + g_{\theta\theta}d\theta^2 + g_{\phi\phi}d\phi^2. \tag{2.1}$$

The Lagrangian describing the geodesic motion of a test-particle in this spacetime is

$$\mathcal{L} = \frac{1}{2}g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu, \tag{2.2}$$

where $\dot{x}^\mu = \frac{dx^\mu}{d\lambda}$ and $\lambda$ is an affine parameter of the geodesic. The momentum From the stationary and axially symmetric conditions of spacetime, we conclude that the metric components are independent of the coordinates $t$ and $\phi$. Therefore, there are two constants of motion: the specific energy at infinity, $E$, and the axial component of the specific angular momentum at infinity, $L_z$. These are defined as

$$-E = p_t = \frac{\partial\mathcal{L}}{\partial\dot{t}} = g_{tt}\dot{t} + g_{t\phi}\dot{\phi} \tag{2.3}$$

$$L_z = p_\phi = \frac{\partial\mathcal{L}}{\partial\dot{\phi}} = g_{t\phi}\dot{t} + g_{\phi\phi}\dot{\phi}. \tag{2.4}$$

in Boyer-Lindquist coordinates,

$$\begin{aligned} ds^2 = &-\left(1 - \frac{2Mr}{\Sigma}\right)dt^2 - -\frac{4Mar\sin^2\theta}{\Sigma}dtd\phi \\ &+ \frac{\Sigma}{\Delta}dr^2 + \Sigma d\theta^2 + \sin^2\theta\left(r^2 + a^2 + \frac{2Ma^2r\sin^2\theta}{\Sigma}\right)d\phi^2, \end{aligned}$$

$$\begin{aligned} \Sigma &= \Sigma(r,\theta) \\ \Delta &= \Delta(r). \end{aligned}$$

The equations of motion of a particle in this spacetime are

$$\begin{aligned}
\dot{t} &= \frac{1}{2\Delta\Sigma}\frac{\partial\Xi}{\partial E} \\
\dot{r} &= \frac{\Delta}{\Sigma}p_r \\
\dot{\theta} &= \frac{p_\theta}{\Sigma} \\
\dot{\phi} &= -\frac{1}{2\Delta\Sigma}\frac{\partial\Xi}{\partial L}
\end{aligned}$$

$$\begin{aligned}
\dot{p}_t &= 0 \\
\dot{p}_r &= -\frac{\partial}{\partial r}\left(\frac{\Delta}{2\Sigma}\right)p_r^2 - \frac{\partial}{\partial r}\left(\frac{1}{2\Sigma}\right)p_\theta^2 + \frac{\partial}{\partial r}\left(\frac{\Xi}{2\Delta\Sigma}\right) \\
\dot{p}_\theta &= -\frac{\partial}{\partial \theta}\left(\frac{\Delta}{2\Sigma}\right)p_r^2 - \frac{\partial}{\partial \theta}\left(\frac{1}{2\Sigma}\right)p_\theta^2 + \frac{\partial}{\partial \theta}\left(\frac{\Xi}{2\Delta\Sigma}\right) \\
\dot{p}_\phi &= 0
\end{aligned}$$

$$\begin{aligned}
\Xi &= R + \Delta\Theta \\
R &= P^2 - \Delta\left[r^2 + (L - aE)^2 + Q\right] \\
P &= E(r^2 + a^2) - aL \\
\Theta &= Q - \cos^2\theta\left[a^2(1 - E^2) + \frac{L^2}{\sin^2\theta}\right]
\end{aligned}$$

# Chapter 3

# The Minkowski Spacetime

. The Minkowski spacetime is given by the line element

$$ds^2 = -dt^2 + dx^2 + dy^2 + dz^2 \tag{3.1}$$

or in spherical coordinates

$$ds^2 = -dt^2 + dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2. \tag{3.2}$$

When compared with the standard form of the Kerr's line element in Boyer-Lindquist coordinates,

$$\begin{aligned} ds^2 = &-\left(1 - \frac{2Mr}{\Sigma}\right) dt^2 - \frac{4Mar\sin^2\theta}{\Sigma} dt d\phi \\ &+ \frac{\Sigma}{\Delta} dr^2 + \Sigma d\theta^2 + \sin^2\theta \left(r^2 + a^2 + \frac{2Ma^2 r \sin^2\theta}{\Sigma}\right) d\phi^2, \end{aligned} \tag{3.3}$$

Minkowski's metric is obtained by taking $a = 0$, $M = 0$ and

$$\Sigma = r^2 \tag{3.4}$$

$$\Delta = r^2. \tag{3.5}$$

Therefore, the potentials in the description of a particle moving in this spacetime reduce to

$$R = E^2 r^4 - r^2 \left[r^2 + L^2 + Q\right] \tag{3.6}$$

$$\Theta = Q - \frac{\cos^2\theta}{\sin^2\theta} L^2 \tag{3.7}$$

Then, we have the function

$$\Xi = E^2 r^4 - r^2 \left[r^2 + L^2 + Q\right] + r^2 \left[Q - \frac{\cos^2\theta}{\sin^2\theta} L^2\right]$$

$$\Xi = E^2 r^4 - r^4 - L^2 r^2 - r^2 \frac{\cos^2\theta}{\sin^2\theta} L^2 \tag{3.8}$$

In order to write the equations of motion we need the derivatives

$$\frac{\partial \Xi}{\partial E} = 2Er^4 \tag{3.9}$$

$$\frac{\partial \Xi}{\partial L} = -2Lr^2 - 2r^2\frac{\cos^2\theta}{\sin^2\theta}L \tag{3.10}$$

and also

$$\frac{\partial}{\partial r}\left(\frac{\Delta}{2\Sigma}\right) = \frac{\partial}{\partial r}\left(\frac{1}{2}\right) = 0 \tag{3.11}$$

$$\frac{\partial}{\partial r}\left(\frac{1}{2\Sigma}\right) = \frac{\partial}{\partial r}\left(\frac{1}{2r^2}\right) = -\frac{1}{r^3} \tag{3.12}$$

$$\begin{aligned}\frac{\partial}{\partial r}\left(\frac{\Xi}{2\Delta\Sigma}\right) &= \frac{\partial}{\partial r}\left(\frac{E^2r^4 - r^4 - L^2r^2 - r^2\frac{\cos^2\theta}{\sin^2\theta}L^2}{2r^4}\right)\\ &= \frac{1}{2}\frac{\partial}{\partial r}\left(E^2 - 1 - \frac{L^2}{r^2} - \frac{\cos^2\theta}{\sin^2\theta}\frac{L^2}{r^2}\right)\\ &= \frac{L^2}{r^3} + \frac{\cos^2\theta}{\sin^2\theta}\frac{L^2}{r^3}\end{aligned} \tag{3.13}$$

$$\frac{\partial}{\partial \theta}\left(\frac{\Delta}{2\Sigma}\right) = \frac{\partial}{\partial r}\left(\frac{1}{2}\right) = 0 \tag{3.14}$$

$$\frac{\partial}{\partial \theta}\left(\frac{1}{2\Sigma}\right) = \frac{\partial}{\partial r}\left(\frac{1}{2r^2}\right) = 0 \tag{3.15}$$

$$\begin{aligned}\frac{\partial}{\partial \theta}\left(\frac{\Xi}{2\Delta\Sigma}\right) &= \frac{\partial}{\partial \theta}\left(\frac{E^2r^4 - r^4 - L^2r^2 - r^2\frac{\cos^2\theta}{\sin^2\theta}L^2}{2r^4}\right)\\ &= \frac{1}{2}\frac{\partial}{\partial \theta}\left(E^2 - 1 - \frac{L^2}{r^2} - \frac{\cos^2\theta}{\sin^2\theta}\frac{L^2}{r^2}\right)\\ &= \cot\theta\csc^2\theta\frac{L^2}{r^2}\end{aligned} \tag{3.16}$$

Using this expressions, the equations of motion of a particle in this spacetime are given by the Hamilton's equations

$$\begin{aligned}\dot{t} &= \frac{1}{2r^4}2Er^4 = E\\ \dot{r} &= p_r\\ \dot{\theta} &= \frac{p_\theta}{r^2}\\ \dot{\phi} &= -\frac{1}{2r^4}\left[-2Lr^2 - 2r^2\frac{\cos^2\theta}{\sin^2\theta}L\right] = \left[1 + \frac{\cos^2\theta}{\sin^2\theta}\right]\frac{L}{r^2} = \frac{L}{r^2\sin^2\theta}\end{aligned}$$

$$\begin{aligned}
\dot{p}_t &= 0\\
\dot{p}_r &= \frac{p_\theta^2}{r^3} + \frac{L^2}{r^3} + \frac{\cos^2\theta}{\sin^2\theta}\frac{L^2}{r^3} = \frac{p_\theta^2}{r^3} + \frac{1}{\sin^2\theta}\frac{L^2}{r^3}\\
\dot{p}_\theta &= \frac{\cos\theta}{\sin^3\theta}\frac{L^2}{r^2}\\
\dot{p}_\phi &= 0
\end{aligned}$$

# Chapter 4

# The Schwarzschild Spacetime

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The Schwarzschild spacetime is given by the line element

$$ds^2 = -\left(1 - \frac{2M}{r}\right) dt^2 + \left(1 - \frac{2M}{r}\right)^{-1} dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2. \tag{4.1}$$

When compared with the standard form of the Kerr's line element in Boyer-Lindquist coordinates,

$$\begin{aligned} ds^2 = & -\left(1 - \frac{2Mr}{\Sigma}\right) dt^2 - \frac{4Mar\sin^2\theta}{\Sigma} dtd\phi \\ & + \frac{\Sigma}{\Delta} dr^2 + \Sigma d\theta^2 + \sin^2\theta \left(r^2 + a^2 + \frac{2Ma^2 r\sin^2\theta}{\Sigma}\right) d\phi^2, \end{aligned} \tag{4.2}$$

Schwarzschild's metric is obtained by taking $a = 0$ and

$$\Sigma = r^2 \tag{4.3}$$

$$\Delta = r^2 - 2Mr. \tag{4.4}$$

Therefore, the potentials in the description of a particle moving in this spacetime reduce to

$$R = E^2 r^4 - (r^2 - 2Mr)\left[r^2 + L^2 + Q\right] \tag{4.5}$$

$$\Theta = Q - \frac{\cos^2\theta}{\sin^2\theta} L^2 \tag{4.6}$$

Then, we have the function

$$
\begin{aligned}
\Xi &= E^2r^4 - (r^2 - 2Mr)\left[r^2 + L^2 + Q\right] + (r^2 - 2Mr)\left[Q - \frac{\cos^2\theta}{\sin^2\theta}L^2\right] \\
\Xi &= E^2r^4 - (r^2 - 2Mr)(r^2 + L^2) - (r^2 - 2Mr)\frac{\cos^2\theta}{\sin^2\theta}L^2 \\
\Xi &= E^2r^4 - r^4 - L^2r^2 + 2Mr^3 + 2ML^2r - r^2\frac{\cos^2\theta}{\sin^2\theta}L^2 + 2Mr\frac{\cos^2\theta}{\sin^2\theta}L^2 \quad (4.7)
\end{aligned}
$$

In order to write the equations of motion we need the derivatives

$$
\begin{aligned}
\frac{\partial\Xi}{\partial E} &= 2Er^4 && (4.8) \\
\frac{\partial\Xi}{\partial L} &= -2(r^2 - 2Mr)L - 2(r^2 - 2Mr)\frac{\cos^2\theta}{\sin^2\theta}L \\
&= -2(r^2 - 2Mr)L\left[1 + \frac{\cos^2\theta}{\sin^2\theta}\right] \\
&= -2(r^2 - 2Mr)L\csc^2\theta && (4.9) \\
& && (4.10)
\end{aligned}
$$

and also

$$
\begin{aligned}
\frac{\partial}{\partial r}\left(\frac{\Delta}{2\Sigma}\right) &= \frac{\partial}{\partial r}\left(\frac{1}{2} - \frac{M}{r}\right) = \frac{M}{r^2} && (4.11) \\
\frac{\partial}{\partial r}\left(\frac{1}{2\Sigma}\right) &= \frac{\partial}{\partial r}\left(\frac{1}{2r^2}\right) = -\frac{1}{r^3} && (4.12) \\
\frac{\partial}{\partial r}\left(\frac{\Xi}{2\Delta\Sigma}\right) &= \frac{\partial}{\partial r}\left(\frac{E^2r^4 - (r^2 - 2Mr)(r^2 + L^2) - (r^2 - 2Mr)\frac{\cos^2\theta}{\sin^2\theta}L^2}{2r^2(r^2 - 2Mr)}\right) \\
&= \frac{1}{2}\frac{\partial}{\partial r}\left(E^2\left(1 - \frac{2M}{r}\right)^{-1} - 1 - \frac{L^2}{r^2} - \frac{\cos^2\theta}{\sin^2\theta}\frac{L^2}{r^2}\right) \\
&= -\frac{E^2}{(1 - \frac{2M}{r})^2}\frac{M}{r^2} + \frac{L^2}{r^3} + \frac{\cos^2\theta}{\sin^2\theta}\frac{L^2}{r^3} \\
&= -\frac{E^2M}{(r - 2M)^2} + \frac{L^2}{r^3}\csc^2\theta && (4.13)
\end{aligned}
$$

$$\frac{\partial}{\partial\theta}\left(\frac{\Delta}{2\Sigma}\right) = \frac{\partial}{\partial r}\left(\frac{1}{2} - \frac{M}{r}\right) = 0 \tag{4.14}$$

$$\frac{\partial}{\partial\theta}\left(\frac{1}{2\Sigma}\right) = \frac{\partial}{\partial r}\left(\frac{1}{2r^2}\right) = 0 \tag{4.15}$$

$$\begin{aligned}\frac{\partial}{\partial\theta}\left(\frac{\Xi}{2\Delta\Sigma}\right) &= \frac{\partial}{\partial\theta}\left(\frac{E^2r^4 - (r^2-2Mr)(r^2+L^2) - (r^2-2Mr)\frac{\cos^2\theta}{\sin^2\theta}L^2}{2r^2(r^2-2Mr)}\right)\\ &= \frac{1}{2}\frac{\partial}{\partial\theta}\left(-\frac{\cos^2\theta}{\sin^2\theta}\frac{L^2}{r^2}\right)\\ &= \cot\theta\csc^2\theta\frac{L^2}{r^2}\end{aligned} \tag{4.16}$$

Using this expressions, the equations of motion of a particle in this spacetime are given by the Hamilton's equations

$$\begin{aligned}\dot{t} &= \frac{1}{2r^2(r^2-2Mr)}2Er^4 = \frac{Er^2}{(r^2-2Mr)}\\ \dot{r} &= p_r\left(1-\frac{2M}{r}\right)\\ \dot{\theta} &= \frac{p_\theta}{r^2}\\ \dot{\phi} &= -\frac{1}{2r^2(r^2-2Mr)}\left[-2(r^2-2Mr)L\csc^2\theta\right] = \frac{L}{r^2\sin^2\theta}\end{aligned}$$

$$\begin{aligned}\dot{p}_t &= 0\\ \dot{p}_r &= -\frac{M}{r^2}p_r^2 + \frac{p_\theta^2}{r^3} - \frac{E^2M}{(r-2M)^2} + \frac{L^2}{r^3\sin^2\theta}\\ \dot{p}_\theta &= \frac{\cos\theta}{\sin^3\theta}\frac{L^2}{r^2}\\ \dot{p}_\phi &= 0\end{aligned}$$