

# Ray Trace

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June 12, 2018



# Chapter 1

## Initial Conditions

In order to calculate the image of a black hole we will trace back the path of photons from a point in the observer's image plane, located at a distant point, onto the emission structure around the black hole. The image plane is considered as a grid and the received photons will have a momentum vector orthogonal to this plane.

### 1.1 Coordinate Systems

We will take Cartesian coordinates  $(X, Y, Z)$  at the image plane of the observer and Cartesian coordinates  $(x, y, z)$  centered at the black hole. These two systems are related by a rotation and a translation. As seen in Figure XXX we may define a rotation between the system  $(X, Y, Z)$  and an intermediate system  $(X', Y', Z')$  as

$$\begin{pmatrix} X' \\ Y' \\ Z' \end{pmatrix} = \begin{pmatrix} \cos \alpha & 0 & -\sin \alpha \\ 0 & 1 & 0 \\ \sin \alpha & 0 & \cos \alpha \end{pmatrix} \begin{pmatrix} X \\ Y \\ Z \end{pmatrix} \quad (1.1)$$

If the observer is located at a distance  $D$  from the black hole as in Figure XXX, we introduce the spatial translation from the intermediate system  $(X', Y', Z')$  to the system  $(x, y, z)$ ,

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} X' \\ Y' \\ Z' \end{pmatrix} + \begin{pmatrix} D \cos \alpha \\ 0 \\ D \sin \alpha \end{pmatrix} \quad (1.2)$$

The composition of these two transformations give

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} \cos \alpha & 0 & -\sin \alpha \\ 0 & 1 & 0 \\ \sin \alpha & 0 & \cos \alpha \end{pmatrix} \begin{pmatrix} X \\ Y \\ Z \end{pmatrix} + \begin{pmatrix} D \cos \alpha \\ 0 \\ D \sin \alpha \end{pmatrix} \quad (1.3)$$

or more explicitly,

$$\begin{cases} x &= X \cos \alpha - Z \sin \alpha + D \cos \alpha \\ y &= Y \\ z &= X \sin \alpha + Z \cos \alpha + D \sin \alpha \end{cases} \quad (1.4)$$

Since the angle  $\alpha$  is related with the inclination angle by  $\alpha = \frac{\pi}{2} - i$  we replace the trigonometric functions in the coordinate transformations by

$$\begin{cases} x &= X \sin i - Z \cos i + D \sin i \\ y &= Y \\ z &= X \cos i + Z \sin i + D \cos i \end{cases} \quad (1.5)$$

or better

$$\begin{cases} x &= (X + D) \sin i - Z \cos i \\ y &= Y \\ z &= (X + D) \cos i + Z \sin i \end{cases} \quad (1.6)$$

Since the black hole metric will be given in spherical coordinates, we introduce the relation between the Cartesian system  $(x, y, z)$  and the coordinates  $(r, \theta, \phi)$  by

$$\begin{cases} r &= \sqrt{x^2 + y^2 + z^2} \\ \theta &= \arccos\left(\frac{z}{r}\right) \\ \phi &= \arctan\left(\frac{y}{x}\right) \end{cases} \quad (1.7)$$

Hence, the complete coordinate transformation needed to describe the initial conditions of the photon is

$$\begin{cases} r &= \sqrt{(X + D)^2 + Y^2 + Z^2} \\ \theta &= \arccos\left(\frac{(X + D) \cos i + Z \sin i}{\sqrt{(X + D)^2 + Y^2 + Z^2}}\right) \\ \phi &= \arctan\left(\frac{Y}{(X + D) \sin i - Z \cos i}\right) \end{cases} \quad (1.8)$$

### 1.1.1 Initial Position of a photon

Consider a photon registered at the observer's image plane at the coordinates  $(X, Y, Z) = (0, \alpha, \beta)$  at the time  $t_0 = 0$ . The spherical coordinates of this photon as seen from the black hole are given by the equation (1.8) as

$$\begin{cases} t_0 &= 0 \\ r_0 &= \sqrt{D^2 + \alpha^2 + \beta^2} \\ \theta_0 &= \arccos\left(\frac{D \cos i + \beta \sin i}{\sqrt{D^2 + \alpha^2 + \beta^2}}\right) \\ \phi_0 &= \arctan\left(\frac{\alpha}{D \sin i - \beta \cos i}\right) \end{cases} \quad (1.9)$$

## 1.2 Momentum of a Photon

Photons received by the distant observer will be considered to arrive perpendicular to the image plane. According to the orientation of the coordinates  $(X, Y, Z)$  in Figure XXX, the Cartesian components of the incident photon are given by the 4-momentum vector

$$\tilde{k}_0^\nu = (\tilde{k}_0^t, \tilde{k}_0^X, \tilde{k}_0^Y, \tilde{k}_0^Z) = (K_0, K_0, 0, 0). \quad (1.10)$$

The components of the 4-momentum in the spherical coordinate system  $(r, \theta, \phi)$  are obtained using the transformation law

$$k^\mu = \frac{\partial x^\mu}{\partial \tilde{x}^\nu} \tilde{k}^\nu. \quad (1.11)$$

The relevant terms in the transformation matrix are obtained from equation (1.8) as

$$\left. \frac{\partial r}{\partial X} \right|_{(0, \alpha, \beta)} = \frac{X + D}{\sqrt{(X + D)^2 + Y^2 + Z^2}} \Big|_{(0, \alpha, \beta)} = \frac{D}{\sqrt{D^2 + \alpha^2 + \beta^2}} = \frac{D}{r_0} \quad (1.12)$$

$$\left. \frac{\partial \theta}{\partial X} \right|_{(0, \alpha, \beta)} = -\frac{1}{\sqrt{r^2 - [(X + D) \cos i + Z \sin i]^2}} \left[ \cos i - \frac{(X + D)^2 \cos i + Z(X + D) \sin i}{r^2} \right] \Big|_{(0, \alpha, \beta)}$$

$$\left. \frac{\partial \theta}{\partial X} \right|_{(0, \alpha, \beta)} = -\frac{1}{\sqrt{r_0^2 - [D \cos i + \beta \sin i]^2}} \left[ \cos i - \frac{D^2 \cos i + D\beta \sin i}{r_0^2} \right] \quad (1.13)$$

$$\left. \frac{\partial \theta}{\partial X} \right|_{(0, \alpha, \beta)} = -\frac{1}{\sqrt{\alpha^2 + [D \sin i - \beta \cos i]^2}} \left[ \cos i - \frac{D^2 \cos i + D\beta \sin i}{r_0^2} \right] \quad (1.14)$$

$$\begin{aligned} \left. \frac{\partial \phi}{\partial X} \right|_{(0, \alpha, \beta)} &= -\frac{Y \sin i}{[(X + D) \sin i - Z \cos i]^2 + Y^2} \Big|_{(0, \alpha, \beta)} \\ \left. \frac{\partial \phi}{\partial X} \right|_{(0, \alpha, \beta)} &= -\frac{\alpha \sin i}{\alpha^2 + [D \sin i - \beta \cos i]^2} \end{aligned} \quad (1.15)$$

Hence, the spatial components of the initial 4-momentum in spherical coordinates are

$$\begin{cases} k_0^r &= \frac{D}{r_0} K_0 \\ k_0^\theta &= -\frac{1}{\sqrt{\alpha^2 + [D \sin i - \beta \cos i]^2}} \left[ \cos i - \frac{D^2 \cos i + D\beta \sin i}{r_0^2} \right] K_0 \\ k_0^\phi &= -\frac{\alpha \sin i}{\alpha^2 + [D \sin i - \beta \cos i]^2} K_0 \end{cases} \quad (1.16)$$

The temporal component of the initial 4-momentum of the photon is obtained from the condition

$$k^\mu k_\mu = \eta_{\mu\nu} k^\mu k^\nu = 0, \quad (1.17)$$

which gives

$$k_0^t = \sqrt{(k_0^r)^2 + r_0^2 (k_0^\theta)^2 + r_0^2 \sin^2 \theta_0 (k_0^\phi)^2}. \quad (1.18)$$

### 1.3 Complete set of Initial Conditions

The complete set of initial conditions are given by equations (1.8), (1.16) and (1.18),

$$t_0 = 0 \quad (1.19)$$

$$r_0 = \sqrt{D^2 + \alpha^2 + \beta^2} \quad (1.20)$$

$$\theta_0 = \arccos \left( \frac{D \cos i + \beta \sin i}{\sqrt{D^2 + \alpha^2 + \beta^2}} \right) \quad (1.21)$$

$$\phi_0 = \arctan \left( \frac{\alpha}{D \sin i - \beta \cos i} \right) \quad (1.22)$$

$$k_0^t = \sqrt{(k_0^r)^2 + r_0^2 (k_0^\theta)^2 + r_0^2 \sin^2 \theta_0 (k_0^\phi)^2} \quad (1.23)$$

$$k_0^r = \frac{D}{r_0} K_0 \quad (1.24)$$

$$k_0^\theta = -\frac{1}{\sqrt{\alpha^2 + [D \sin i - \beta \cos i]^2}} \left[ \cos i - \frac{D^2 \cos i + D\beta \sin i}{r_0^2} \right] K_0 \quad (1.25)$$

$$k_0^\phi = -\frac{\alpha \sin i}{\alpha^2 + [D \sin i - \beta \cos i]^2} K_0 \quad (1.26)$$

#### 1.3.1 Covariant Components of the 4-Momentum Vector

In order to solve the Hamiltonian equations of motion for the photons, we will need the initial components of the 4-momentum vector in their covariant form. Therefore we need to apply the spacetime metric  $g_{\mu\nu}$  as

$$k_\mu = g_{\mu\nu} k^\nu. \quad (1.27)$$

For a general stationary and axially symmetric spacetime described with the line element

$$ds^2 = g_{tt} dt^2 + 2g_{t\phi} dt d\phi + g_{rr} dr^2 + g_{\theta\theta} d\theta^2 + g_{\phi\phi} d\phi^2 \quad (1.28)$$

the covariant components of the 4-momentum are

$$\begin{cases} k_t &= g_{tt}k^t + g_{t\phi}k^\phi \\ k_r &= g_{rr}k^r \\ k_\theta &= g_{\theta\theta}k^\theta \\ k_\phi &= g_{t\phi}k^t + g_{\phi\phi}k^\phi \end{cases} \quad (1.29)$$





## Chapter 2

# Geodesic Equations

Consider a general line element in Boyer-Lindquist coordinates,

$$ds^2 = - \left( 1 - \frac{2Mr}{\Sigma} \right) dt^2 - \frac{4Mar \sin^2 \theta}{\Sigma} dt d\phi \\ + \frac{\Sigma}{\Delta} dr^2 + \Sigma d\theta^2 + \sin^2 \theta \left( r^2 + a^2 + \frac{2Ma^2 r \sin^2 \theta}{\Sigma} \right) d\phi^2,$$

$$\Sigma = \Sigma(r, \theta) \\ \Delta = \Delta(r).$$

The equations of motion of a particle in this spacetime are

$$\dot{t} = \frac{1}{2\Delta\Sigma} \frac{\partial \Xi}{\partial E} \\ \dot{r} = \frac{\Delta}{\Sigma} p_r \\ \dot{\theta} = \frac{p_\theta}{\Sigma} \\ \dot{\phi} = - \frac{1}{2\Delta\Sigma} \frac{\partial \Xi}{\partial L}$$

$$\dot{p}_t = 0$$

$$\dot{p}_r = -\frac{\partial}{\partial r} \left( \frac{\Delta}{2\Sigma} \right) p_r^2 - \frac{\partial}{\partial r} \left( \frac{1}{2\Sigma} \right) p_\theta^2 + \frac{\partial}{\partial r} \left( \frac{\Xi}{2\Delta\Sigma} \right)$$

$$\dot{p}_\theta = -\frac{\partial}{\partial \theta} \left( \frac{\Delta}{2\Sigma} \right) p_r^2 - \frac{\partial}{\partial \theta} \left( \frac{1}{2\Sigma} \right) p_\theta^2 + \frac{\partial}{\partial \theta} \left( \frac{\Xi}{2\Delta\Sigma} \right)$$

$$\dot{p}_\phi = 0$$

$$\Xi = R + \Delta\Theta$$

$$R = P^2 - \Delta[r^2 + (L - aE)^2 + Q]$$

$$P = E(r^2 + a^2) - aL$$

$$\Theta = Q - \cos^2 \theta \left[ a^2(1 - E^2) + \frac{L^2}{\sin^2 \theta} \right]$$

## Chapter 3

# The Minkowski Spacetime

. The Minkowski spacetime is given by the line element

$$ds^2 = -dt^2 + dx^2 + dy^2 + dz^2 \quad (3.1)$$

or in spherical coordinates

$$ds^2 = -dt^2 + dr^2 + r^2 d\theta^2 + r^2 \sin^2 \theta d\phi^2. \quad (3.2)$$

When compared with the standard form of the Kerr's line element in Boyer-Lindquist coordinates,

$$\begin{aligned} ds^2 = & - \left( 1 - \frac{2Mr}{\Sigma} \right) dt^2 - \frac{4Mar \sin^2 \theta}{\Sigma} dt d\phi \\ & + \frac{\Sigma}{\Delta} dr^2 + \Sigma d\theta^2 + \sin^2 \theta \left( r^2 + a^2 + \frac{2Ma^2 r \sin^2 \theta}{\Sigma} \right) d\phi^2, \end{aligned} \quad (3.3)$$

Minkowski's metric is obtained by taking  $a = 0$ ,  $M = 0$  and

$$\Sigma = r^2 \quad (3.4)$$

$$\Delta = r^2. \quad (3.5)$$

Therefore, the potentials in the description of a particle moving in this spacetime reduce to

$$R = E^2 r^4 - r^2 [r^2 + L^2 + Q] \quad (3.6)$$

$$\Theta = Q - \frac{\cos^2 \theta}{\sin^2 \theta} L^2 \quad (3.7)$$

Then, we have the function

$$\begin{aligned} \Xi &= E^2 r^4 - r^2 [r^2 + L^2 + Q] + r^2 \left[ Q - \frac{\cos^2 \theta}{\sin^2 \theta} L^2 \right] \\ \Xi &= E^2 r^4 - r^4 - L^2 r^2 - r^2 \frac{\cos^2 \theta}{\sin^2 \theta} L^2 \end{aligned} \quad (3.8)$$

In order to write the equations of motion we need the derivatives

$$\frac{\partial \Xi}{\partial E} = 2Er^4 \quad (3.9)$$

$$\frac{\partial \Xi}{\partial L} = -2Lr^2 - 2r^2 \frac{\cos^2 \theta}{\sin^2 \theta} L \quad (3.10)$$

and also

$$\frac{\partial}{\partial r} \left( \frac{\Delta}{2\Sigma} \right) = \frac{\partial}{\partial r} \left( \frac{1}{2} \right) = 0 \quad (3.11)$$

$$\frac{\partial}{\partial r} \left( \frac{1}{2\Sigma} \right) = \frac{\partial}{\partial r} \left( \frac{1}{2r^2} \right) = -\frac{1}{r^3} \quad (3.12)$$

$$\begin{aligned} \frac{\partial}{\partial r} \left( \frac{\Xi}{2\Delta\Sigma} \right) &= \frac{\partial}{\partial r} \left( \frac{E^2 r^4 - r^4 - L^2 r^2 - r^2 \frac{\cos^2 \theta}{\sin^2 \theta} L^2}{2r^4} \right) \\ &= \frac{1}{2} \frac{\partial}{\partial r} \left( E^2 - 1 - \frac{L^2}{r^2} - \frac{\cos^2 \theta}{\sin^2 \theta} \frac{L^2}{r^2} \right) \\ &= \frac{L^2}{r^3} + \frac{\cos^2 \theta}{\sin^2 \theta} \frac{L^2}{r^3} \end{aligned} \quad (3.13)$$

$$\frac{\partial}{\partial \theta} \left( \frac{\Delta}{2\Sigma} \right) = \frac{\partial}{\partial \theta} \left( \frac{1}{2} \right) = 0 \quad (3.14)$$

$$\frac{\partial}{\partial \theta} \left( \frac{1}{2\Sigma} \right) = \frac{\partial}{\partial \theta} \left( \frac{1}{2r^2} \right) = 0 \quad (3.15)$$

$$\begin{aligned} \frac{\partial}{\partial \theta} \left( \frac{\Xi}{2\Delta\Sigma} \right) &= \frac{\partial}{\partial \theta} \left( \frac{E^2 r^4 - r^4 - L^2 r^2 - r^2 \frac{\cos^2 \theta}{\sin^2 \theta} L^2}{2r^4} \right) \\ &= \frac{1}{2} \frac{\partial}{\partial \theta} \left( E^2 - 1 - \frac{L^2}{r^2} - \frac{\cos^2 \theta}{\sin^2 \theta} \frac{L^2}{r^2} \right) \\ &= \cot \theta \csc^2 \theta \frac{L^2}{r^2} \end{aligned} \quad (3.16)$$

Using these expressions, the equations of motion of a particle in this spacetime are given by the Hamilton's equations

$$\dot{t} = \frac{1}{2r^4} 2Er^4 = E$$

$$\dot{r} = p_r$$

$$\dot{\theta} = \frac{p_\theta}{r^2}$$

$$\dot{\phi} = -\frac{1}{2r^4} \left[ -2Lr^2 - 2r^2 \frac{\cos^2 \theta}{\sin^2 \theta} L \right] = \left[ 1 + \frac{\cos^2 \theta}{\sin^2 \theta} \right] \frac{L}{r^2} = \frac{L}{r^2 \sin^2 \theta}$$

$$\dot{p}_t = 0$$

$$\dot{p}_r = \frac{p_\theta^2}{r^3} + \frac{L^2}{r^3} + \frac{\cos^2 \theta}{\sin^2 \theta} \frac{L^2}{r^3} = \frac{p_\theta^2}{r^3} + \frac{1}{\sin^2 \theta} \frac{L^2}{r^3}$$

$$\dot{p}_\theta = \frac{\cos \theta}{\sin^3 \theta} \frac{L^2}{r^2}$$

$$\dot{p}_\phi = 0$$