Ray Trace

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Chapter 1

Initial Conditions

In order to calculate the image of a black hole we will trace back the path of photons from a point in the observer's image plane, located at a distant point, onto the emission structure around the black hole. The image plane is considered as a grid and the received photons will have a momentum vector orthogonal to this plane.

1.1   Coordinate Systems

We will take Cartesian coordinates (X, Y, Z) at the image plane of the observer and Cartesian coordinates (x, y, z) centered at the black hole. These two systems are related by a rotation and a translation. As seen in Figure 1.1 we may define a rotation between the system (X, Y, Z) and an intermediate system (X', Y', Z') as

$$\begin{pmatrix} X' \\ Y' \\ Z' \end{pmatrix} = \begin{pmatrix} \cos\alpha & 0 & -\sin\alpha \\ 0 & 1 & 0 \\ \sin\alpha & 0 & \cos\alpha \end{pmatrix} \begin{pmatrix} X \\ Y \\ Z \end{pmatrix} \tag{1.1}$$

If the observer is located at a distance D from the black hole as in Figure 1.1, we introduce the spatial translation from the intermediate system (X', Y', Z') to the system (x, y, z),

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} X' \\ Y' \\ Z' \end{pmatrix} + \begin{pmatrix} D\cos\alpha \\ 0 \\ D\sin\alpha \end{pmatrix} \tag{1.2}$$

The composition of these two transformations give

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} \cos\alpha & 0 & -\sin\alpha \\ 0 & 1 & 0 \\ \sin\alpha & 0 & \cos\alpha \end{pmatrix} \begin{pmatrix} X \\ Y \\ Z \end{pmatrix} + \begin{pmatrix} D\cos\alpha \\ 0 \\ D\sin\alpha \end{pmatrix} \tag{1.3}$$

or more explicitly,
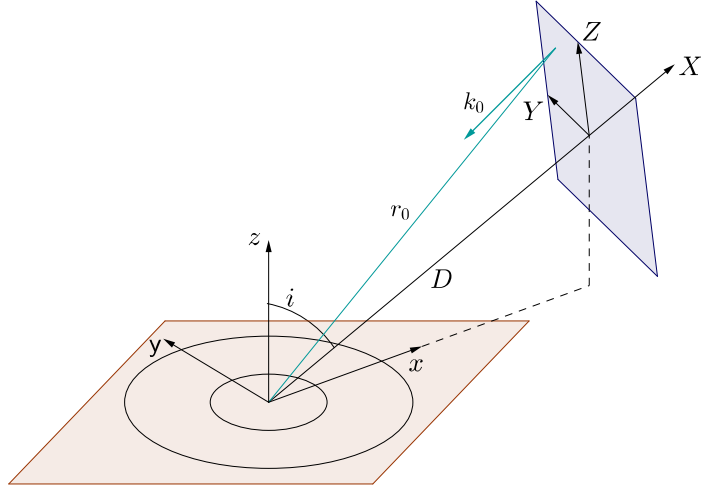
Figure 1.1: Coordinate Transformation for the initial conditions of the photon

$$\begin{cases} x & = X\cos\alpha - Z\sin\alpha + D\cos\alpha \\ y & = Y \\ z & = X\sin\alpha + Z\cos\alpha + D\sin\alpha \end{cases} \tag{1.4}$$

Since the angle α is related with the inclination angle by $\alpha = \frac{\pi}{2} - i$ we replace the trigonometric functions in the coordinate transformations by

$$\begin{cases} x & = X\sin i - Z\cos i + D\sin i \\ y & = Y \\ z & = X\cos i + Z\sin i + D\cos i \end{cases} \tag{1.5}$$

or better

$$\begin{cases} x & = (X+D)\sin i - Z\cos i \\ y & = Y \\ z & = (X+D)\cos i + Z\sin i \end{cases} \tag{1.6}$$

Since the black hole metric will be given in spherical coordinates, we introduce the relation between the Cartesian system (x, y, z) and the coordinates (r, θ, ϕ) by

$$\begin{cases} r & = \sqrt{x^2 + y^2 + z^2} \\ \theta & = \arccos\left(\frac{z}{r}\right) \\ \phi & = \arctan\left(\frac{y}{x}\right) \end{cases} \tag{1.7}$$

Hence, the complete coordinate transformation needed to describe the initial conditions of the photon is

$$
\begin{cases}
r & = \sqrt{(X+D)^2 + Y^2 + Z^2} \\
\theta & = \arccos\left(\frac{(X+D)\cos i + Z \sin i}{\sqrt{(X+D)^2+Y^2+Z^2}} \right) \\
\phi & = \arctan\left(\frac{Y}{(X+D)\sin i - Z \cos i} \right)
\end{cases}
\tag{1.8}
$$

1.2   Momentum of a Photon

The Cartesian components of the incident photon are given by the 4-momentum vector

$$
\tilde{k}_0^\nu = \left(\tilde{k}_0^t, \tilde{k}_0^X, \tilde{k}_0^Y, \tilde{k}_0^Z \right)
\tag{1.9}
$$

The components of the 4-momentum in the spherical coordinate system (r, θ, ϕ) are obtained using the transformation law

$$
k^\mu = \frac{\partial x^\mu}{\partial \tilde{x}^\nu} \tilde{k}^\nu.
\tag{1.10}
$$

1.3   Image Plane Screen

1.3.1   Initial Position of a photon

Consider a photon registered at the observer's image plane at the coordinates $(X, Y, Z) = (0, \alpha, \beta)$ at the time $t_0 = 0$. The spherical coordinates of this photon as seen from the black hole are given by the equation (1.8) as

$$
\begin{cases}
t_0 & = 0 \\
r_0 & = \sqrt{D^2 + \alpha^2 + \beta^2} \\
\theta_0 & = \arccos\left(\frac{D\cos i + \beta \sin i}{\sqrt{D^2+\alpha^2+\beta^2}} \right) \\
\phi_0 & = \arctan\left(\frac{\alpha}{D \sin i - \beta \cos i} \right)
\end{cases}
\tag{1.11}
$$

1.3.2   Momentum of a Photon

Photons received by the distant observer will be considered to arrive perpendicular to the image plane. According to the orientation of the coordinates (X, Y, Z) in Figure 1.1, the Cartesian components of the incident photon are given by

$$
\tilde{k}_0^\nu = \left(\tilde{k}_0^t, \tilde{k}_0^X, \tilde{k}_0^Y, \tilde{k}_0^Z \right) = (K_0, K_0, 0, 0).
\tag{1.12}
$$

The components of the 4-momentum in the spherical coordinate system are obtained using Eq. (1.10). Given the perpendicular components of the incident photon, the relevant terms in the transformation matrix are obtained from equation (1.8) as

$$\left.\frac{\partial r}{\partial X}\right|_{(0,\alpha,\beta)} = \left.\frac{X+D}{\sqrt{(X+D)^2+Y^2+Z^2}}\right|_{(0,\alpha,\beta)} = \frac{D}{\sqrt{D^2+\alpha^2+\beta^2}} = \frac{D}{r_0} \quad (1.13)$$

$$\left.\frac{\partial \theta}{\partial X}\right|_{(0,\alpha,\beta)} = -\frac{1}{\sqrt{r^2-[(X+D)\cos i+Z\sin i]^2}}\left[\cos i - \frac{(X+D)^2\cos i+Z(X+D)\sin i}{r^2}\right]\Bigg|_{(0,\alpha,\beta)}$$

$$\left.\frac{\partial \theta}{\partial X}\right|_{(0,\alpha,\beta)} = -\frac{1}{\sqrt{r_0^2-[D\cos i+\beta\sin i]^2}}\left[\cos i - \frac{D^2\cos i+D\beta\sin i}{r_0^2}\right] \quad (1.14)$$

$$\left.\frac{\partial \phi}{\partial X}\right|_{(0,\alpha,\beta)} = -\frac{Y\sin i}{[(X+D)\sin i-Z\cos i]^2+Y^2}\Bigg|_{(0,\alpha,\beta)}$$

$$\left.\frac{\partial \phi}{\partial X}\right|_{(0,\alpha,\beta)} = -\frac{\alpha\sin i}{\alpha^2+[D\sin i-\beta\cos i]^2} \quad (1.15)$$

Hence, the spatial components of the initial 4-momentum in spherical coordinates are

$$\begin{cases} k_0^r &= \frac{D}{r_0}K_0 \\ k_0^\theta &= -\frac{1}{\sqrt{\alpha^2+[D\sin i-\beta\cos i]^2}}\left[\cos i - \frac{D^2\cos i+D\beta\sin i}{r_0^2}\right]K_0 \\ k_0^\phi &= -\frac{\alpha\sin i}{\alpha^2+[D\sin i-\beta\cos i]^2}K_0 \end{cases} \quad (1.16)$$

The temporal component of the initial 4-momentum of the photon is obtained from the condition

$$k^\mu k_\mu = \eta_{\mu\nu}k^\mu k^\nu = 0, \quad (1.17)$$

which gives

$$k_0^t = \sqrt{(k_0^r)^2+r_0^2(k_0^\theta)^2+r_0^2\sin^2\theta_0(k_0^\phi)^2}. \quad (1.18)$$

1.3.3   Complete set of Initial Conditions

The complete set of initial conditions are given by equations (1.8), (1.16) and (1.18),

$$t_0 = 0 \tag{1.19}$$

$$r_0 = \sqrt{D^2 + \alpha^2 + \beta^2} \tag{1.20}$$

$$\theta_0 = \arccos\left(\frac{D\cos i + \beta\sin i}{\sqrt{D^2 + \alpha^2 + \beta^2}}\right) \tag{1.21}$$

$$\phi_0 = \arctan\left(\frac{\alpha}{D\sin i - \beta\cos i}\right) \tag{1.22}$$

$$k_0^t = \sqrt{(k_0^r)^2 + r_0^2(k_0^\theta)^2 + r_0^2\sin^2\theta_0(k_0^\phi)^2} \tag{1.23}$$

$$k_0^r = \frac{D}{r_0}K_0 \tag{1.24}$$

$$k_0^\theta = -\frac{1}{\sqrt{\alpha^2 + [D\sin i - \beta\cos i]^2}}\left[\cos i - \frac{D^2\cos i + D\beta\sin i}{r_0^2}\right]K_0 \tag{1.25}$$

$$k_0^\phi = -\frac{\alpha\sin i}{\alpha^2 + [D\sin i - \beta\cos i]^2}K_0 \tag{1.26}$$

1.3.4   Covariant Components of the 4-Momentum Vector

In order to solve the Hamiltonian equations of motion for the photons, we will need the initial components of the 4-momentum vector in their covariant form. Therefore we need to apply the spacetime metric $g_{\mu\nu}$ as

$$k_\mu = g_{\mu\nu}k^\nu. \tag{1.27}$$

For a general stationary and axially symmetric spacetime described with the line element

$$ds^2 = g_{tt}dt^2 + 2g_{t\phi}dtd\phi + g_{rr}dr^2 + g_{\theta\theta}d\theta^2 + g_{\phi\phi}d\phi^2 \tag{1.28}$$

the covariant components of the 4-momentum are

$$\begin{cases} k_t & = g_{tt}k^t + g_{t\phi}k^\phi \\ k_r & = g_{rr}k^r \\ k_\theta & = g_{\theta\theta}k^\theta \\ k_\phi & = g_{t\phi}k^t + g_{\phi\phi}k^\phi \end{cases} \tag{1.29}$$

1.4   Point Camera

Another type of screen considered in this study is a point camera. Photons arrive to this point receptor from different directions. Therefore, in this case the initial position of all photon in the image is the same (point camera) while the components of the initial 4-momentum vector distinguishes each photon.

1.4.1   Initial Position of a photon

Since all the photons arrive to one point, the initial position condition for any of the photons is just $(X, Y, Z) = (0, 0, 0)$ at the time $t_0 = 0$. The corresponding spherical coordinates as seen from the black hole are given by the equations (1.8),

$$\begin{cases} t_0 & = 0 \\ r_0 & = D \\ \theta_0 & = i \\ \phi_0 & = 0 \end{cases} \tag{1.30}$$

1.4.2   Momentum of a Photon

Photons arrive to the point camera from different directions. Using the angular coordinates α and β to describe the direction of each photon, the cartesian components of the 4-momentum vector can be written as

$$\tilde{k}_0^\nu = \left(\tilde{k}_0^t, \tilde{k}_0^X, \tilde{k}_0^Y, \tilde{k}_0^Z \right) = (K_0, K_0 \cos\beta \cos\alpha, K_0 \cos\beta \sin\alpha, K_0 \sin\beta). \tag{1.31}$$

The components of the 4-momentum in the spherical coordinate system are obtained using Eq. (1.10). The relevant terms in the transformation matrix are obtained from equation (1.8) evaluating at the position of the point camera, giving

$$\left. \frac{\partial r}{\partial X} \right|_{(0,0,0)} = \left. \frac{X + D}{\sqrt{(X+D)^2 + Y^2 + Z^2}} \right|_{(0,0,0)} = \frac{D}{\sqrt{D^2}} = 1 \tag{1.32}$$

$$\left. \frac{\partial r}{\partial Y} \right|_{(0,0,0)} = \left. \frac{Y}{\sqrt{(X+D)^2 + Y^2 + Z^2}} \right|_{(0,0,0)} = 0 \tag{1.33}$$

$$\left. \frac{\partial r}{\partial Z} \right|_{(0,0,0)} = \left. \frac{Z}{\sqrt{(X+D)^2 + Y^2 + Z^2}} \right|_{(0,0,0)} = 0 \tag{1.34}$$

$$\left. \frac{\partial \theta}{\partial X} \right|_{(0,0,0)} = -\frac{1}{\sqrt{r^2 - [(X+D)\cos i + Z\sin i]^2}} \left[\cos i - \frac{(X+D)^2 \cos i + Z(X+D)\sin i}{r^2} \right]\Bigg|_{(0,0,0)}$$

$$\left. \frac{\partial \theta}{\partial X} \right|_{(0,0,0)} = -\frac{1}{\sqrt{D^2 - [D\cos i]^2}} \left[\cos i - \frac{D^2 \cos i}{D^2} \right] = 0 \tag{1.35}$$

$$\left. \frac{\partial \theta}{\partial Y} \right|_{(0,0,0)} = -\frac{1}{\sqrt{r^2 - [(X+D)\cos i + Z\sin i]^2}} \left[-Y \frac{(X+D)\cos i + Z\sin i}{r^2} \right]\Bigg|_{(0,0,0)}$$

$$\left. \frac{\partial \theta}{\partial Y} \right|_{(0,0,0)} = 0 \tag{1.36}$$

$$\frac{\partial\theta}{\partial Z}\bigg|_{(0,0,0)} = -\frac{1}{\sqrt{r^2 - [(X+D)\cos i + Z\sin i]^2}}\left[\sin i - Z\frac{(X+D)\cos i + Z\sin i}{r^2}\right]\bigg|_{(0,0,0)}$$

$$\frac{\partial\theta}{\partial Z}\bigg|_{(0,0,0)} = -\frac{\sin i}{\sqrt{D^2 - [D\cos i]^2}} = -\frac{1}{D} \tag{1.37}$$

$$\frac{\partial\phi}{\partial X}\bigg|_{(0,0,0)} = -\frac{Y\sin i}{[(X+D)\sin i - Z\cos i]^2 + Y^2}\bigg|_{(0,0,0)}$$

$$\frac{\partial\phi}{\partial X}\bigg|_{(0,0,0)} = 0 \tag{1.38}$$

$$\frac{\partial\phi}{\partial Y}\bigg|_{(0,0,0)} = \frac{[(X+D)\sin i - Z\cos i]}{[(X+D)\sin i - Z\cos i]^2 + Y^2}\bigg|_{(0,0,0)}$$

$$\frac{\partial\phi}{\partial Y}\bigg|_{(0,0,0)} = \frac{D\sin i}{[D\sin i]^2} = \frac{1}{D\sin i} \tag{1.39}$$

$$\frac{\partial\phi}{\partial Z}\bigg|_{(0,0,0)} = \frac{Y\cos i}{[(X+D)\sin i - Z\cos i]^2 + Y^2}\bigg|_{(0,0,0)}$$

$$\frac{\partial\phi}{\partial Z}\bigg|_{(0,0,0)} = 0 \tag{1.40}$$

Using this components of the transformation matrix, the spatial components of the initial 4-momentum in spherical coordinates are

$$\begin{cases} k_0^r &= \tilde{k}_0^X = K_0\cos\beta\cos\alpha \\ k_0^\theta &= \frac{\partial\theta}{\partial Z}\big|_{(0,0,0)}\,\tilde{k}_0^Z = -\frac{K_0\sin\beta}{D} \\ k_0^\phi &= \frac{\partial\phi}{\partial Y}\big|_{(0,0,0)}\,\tilde{k}_0^Y = \frac{K_0\cos\beta\sin\alpha}{D\sin i} \end{cases} \tag{1.41}$$

The temporal component of the initial 4-momentum of the photon is obtained from the condition

$$k^\mu k_\mu = \eta_{\mu\nu}k^\mu k^\nu = 0, \tag{1.42}$$

which gives

$$k_0^t = \sqrt{(k_0^r)^2 + r_0^2(k_0^\theta)^2 + r_0^2\sin^2\theta_0(k_0^\phi)^2}. \tag{1.43}$$

Chapter 2

Geodesic Equations

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Consider a general line element of a general stationary and axially symmetric spacetime described by the line element

$$ds^2 = g_{tt}dt^2 + 2g_{t\phi}dtd\phi + g_{rr}dr^2 + g_{\theta\theta}d\theta^2 + g_{\phi\phi}d\phi^2. \tag{2.1}$$

The Lagrangian describing the geodesic motion of a test-particle in this spacetime is

$$\mathcal{L} = \frac{1}{2}g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu, \tag{2.2}$$

where $\dot{x}^\mu = \frac{dx^\mu}{d\lambda}$ and λ is an affine parameter of the geodesic. The momentum From the stationary and axially symmetric conditions of spacetime, we conclude that the metric components are independent of the coordinates t and ϕ. Therefore, there are two constants of motion: the specific energy at infinity, E, and the axial component of the specific angular momentum at infinity, L_z. These are defined as

$$-E = p_t = \frac{\partial \mathcal{L}}{\partial \dot{t}} = g_{tt}\dot{t} + g_{t\phi}\dot{\phi} \tag{2.3}$$

$$L_z = p_\phi = \frac{\partial \mathcal{L}}{\partial \dot{\phi}} = g_{t\phi}\dot{t} + g_{\phi\phi}\dot{\phi}. \tag{2.4}$$

in Boyer-Lindquist coordinates,

$$ds^2 = -\left(1 - \frac{2Mr}{\Sigma}\right)dt^2 - -\frac{4Mar\sin^2\theta}{\Sigma}dtd\phi$$

$$+ \frac{\Sigma}{\Delta}dr^2 + \Sigma d\theta^2 + \sin^2\theta\left(r^2 + a^2 + \frac{2Ma^2r\sin^2\theta}{\Sigma}\right)d\phi^2,$$

$$\Sigma = \Sigma(r,\theta)$$
$$\Delta = \Delta(r).$$

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The equations of motion of a particle in this spacetime are

$$\dot{t} = \frac{1}{2\Delta\Sigma}\frac{\partial\Xi}{\partial E}$$
$$\dot{r} = \frac{\Delta}{\Sigma}p_r$$
$$\dot{\theta} = \frac{p_\theta}{\Sigma}$$
$$\dot{\phi} = -\frac{1}{2\Delta\Sigma}\frac{\partial\Xi}{\partial L}$$

$$\dot{p}_t = 0$$
$$\dot{p}_r = -\frac{\partial}{\partial r}\left(\frac{\Delta}{2\Sigma}\right)p_r^2 - \frac{\partial}{\partial r}\left(\frac{1}{2\Sigma}\right)p_\theta^2 + \frac{\partial}{\partial r}\left(\frac{\Xi}{2\Delta\Sigma}\right)$$
$$\dot{p}_\theta = -\frac{\partial}{\partial\theta}\left(\frac{\Delta}{2\Sigma}\right)p_r^2 - \frac{\partial}{\partial\theta}\left(\frac{1}{2\Sigma}\right)p_\theta^2 + \frac{\partial}{\partial\theta}\left(\frac{\Xi}{2\Delta\Sigma}\right)$$
$$\dot{p}_\phi = 0$$

$$\Xi = R + \Delta\Theta$$
$$R = P^2 - \Delta\left[r^2 + (L - aE)^2 + Q\right]$$
$$P = E(r^2 + a^2) - aL$$
$$\Theta = Q - \cos^2\theta\left[a^2(1 - E^2) + \frac{L^2}{\sin^2\theta}\right]$$

Chapter 3

The Minkowski Spacetime

. The Minkowski spacetime is given by the line element

$$ds^2 = -dt^2 + dx^2 + dy^2 + dz^2 \tag{3.1}$$

or in spherical coordinates

$$ds^2 = -dt^2 + dr^2 + r^2 d\theta^2 + r^2 \sin^2 \theta d\phi^2. \tag{3.2}$$

When compared with the standard form of the Kerr's line element in Boyer-Lindquist coordinates,

$$ds^2 = -\left(1 - \frac{2Mr}{\Sigma}\right) dt^2 - \frac{4Mar\sin^2\theta}{\Sigma} dtd\phi$$
$$+ \frac{\Sigma}{\Delta} dr^2 + \Sigma d\theta^2 + \sin^2\theta \left(r^2 + a^2 + \frac{2Ma^2r\sin^2\theta}{\Sigma}\right) d\phi^2, \tag{3.3}$$

Minkowski's metric is obtained by taking $a = 0$, $M = 0$ and

$$\Sigma = r^2 \tag{3.4}$$
$$\Delta = r^2. \tag{3.5}$$

Therefore, the potentials in the description of a particle moving in this spacetime reduce to

$$R = E^2 r^4 - r^2 \left[r^2 + L^2 + Q\right] \tag{3.6}$$
$$\Theta = Q - \frac{\cos^2\theta}{\sin^2\theta} L^2 \tag{3.7}$$

Then, we have the function

$$\Xi = E^2 r^4 - r^2 \left[r^2 + L^2 + Q\right] + r^2 \left[Q - \frac{\cos^2\theta}{\sin^2\theta} L^2\right]$$
$$\Xi = E^2 r^4 - r^4 - L^2 r^2 - r^2 \frac{\cos^2\theta}{\sin^2\theta} L^2 \tag{3.8}$$

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In order to write the equations of motion we need the derivatives

$$\frac{\partial \Xi}{\partial E} = 2Er^4 \tag{3.9}$$

$$\frac{\partial \Xi}{\partial L} = -2Lr^2 - 2r^2 \frac{\cos^2 \theta}{\sin^2 \theta} L \tag{3.10}$$

and also

$$\frac{\partial}{\partial r}\left(\frac{\Delta}{2\Sigma}\right) = \frac{\partial}{\partial r}\left(\frac{1}{2}\right) = 0 \tag{3.11}$$

$$\frac{\partial}{\partial r}\left(\frac{1}{2\Sigma}\right) = \frac{\partial}{\partial r}\left(\frac{1}{2r^2}\right) = -\frac{1}{r^3} \tag{3.12}$$

$$\frac{\partial}{\partial r}\left(\frac{\Xi}{2\Delta\Sigma}\right) = \frac{\partial}{\partial r}\left(\frac{E^2 r^4 - r^4 - L^2 r^2 - r^2 \frac{\cos^2 \theta}{\sin^2 \theta} L^2}{2r^4}\right)$$

$$= \frac{1}{2}\frac{\partial}{\partial r}\left(E^2 - 1 - \frac{L^2}{r^2} - \frac{\cos^2 \theta}{\sin^2 \theta}\frac{L^2}{r^2}\right)$$

$$= \frac{L^2}{r^3} + \frac{\cos^2 \theta}{\sin^2 \theta}\frac{L^2}{r^3} \tag{3.13}$$

$$\frac{\partial}{\partial \theta}\left(\frac{\Delta}{2\Sigma}\right) = \frac{\partial}{\partial r}\left(\frac{1}{2}\right) = 0 \tag{3.14}$$

$$\frac{\partial}{\partial \theta}\left(\frac{1}{2\Sigma}\right) = \frac{\partial}{\partial r}\left(\frac{1}{2r^2}\right) = 0 \tag{3.15}$$

$$\frac{\partial}{\partial \theta}\left(\frac{\Xi}{2\Delta\Sigma}\right) = \frac{\partial}{\partial \theta}\left(\frac{E^2 r^4 - r^4 - L^2 r^2 - r^2 \frac{\cos^2 \theta}{\sin^2 \theta} L^2}{2r^4}\right)$$

$$= \frac{1}{2}\frac{\partial}{\partial \theta}\left(E^2 - 1 - \frac{L^2}{r^2} - \frac{\cos^2 \theta}{\sin^2 \theta}\frac{L^2}{r^2}\right)$$

$$= \cot \theta \csc^2 \theta \frac{L^2}{r^2} \tag{3.16}$$

Using this expressions, the equations of motion of a particle in this spacetime are given by the Hamilton's equations

$$\dot{t} = \frac{1}{2r^4} 2Er^4 = E$$

$$\dot{r} = p_r$$

$$\dot{\theta} = \frac{p_\theta}{r^2}$$

$$\dot{\phi} = -\frac{1}{2r^4}\left[-2Lr^2 - 2r^2 \frac{\cos^2 \theta}{\sin^2 \theta} L\right] = \left[1 + \frac{\cos^2 \theta}{\sin^2 \theta}\right]\frac{L}{r^2} = \frac{L}{r^2 \sin^2 \theta}$$

$$\dot{p}_t = 0$$

$$\dot{p}_r = \frac{p_\theta^2}{r^3} + \frac{L^2}{r^3} + \frac{\cos^2\theta}{\sin^2\theta}\frac{L^2}{r^3} = \frac{p_\theta^2}{r^3} + \frac{1}{\sin^2\theta}\frac{L^2}{r^3}$$

$$\dot{p}_\theta = \frac{\cos\theta}{\sin^3\theta}\frac{L^2}{r^2}$$

$$\dot{p}_\phi = 0$$

Chapter 4

The Schwarzschild Spacetime

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The Schwarzschild spacetime is given by the line element

$$ds^2 = -\left(1 - \frac{2M}{r}\right)dt^2 + \left(1 - \frac{2M}{r}\right)^{-1}dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2. \qquad (4.1)$$

When compared with the standard form of the Kerr's line element in Boyer-Lindquist coordinates,

$$ds^2 = -\left(1 - \frac{2Mr}{\Sigma}\right)dt^2 - \frac{4Mar\sin^2\theta}{\Sigma}dtd\phi$$
$$+ \frac{\Sigma}{\Delta}dr^2 + \Sigma d\theta^2 + \sin^2\theta\left(r^2 + a^2 + \frac{2Ma^2r\sin^2\theta}{\Sigma}\right)d\phi^2, \qquad (4.2)$$

Schwarzschild's metric is obtained by taking $a = 0$ and

$$\Sigma = r^2 \qquad (4.3)$$
$$\Delta = r^2 - 2Mr. \qquad (4.4)$$

Therefore, the potentials in the description of a particle moving in this spacetime reduce to

$$R = E^2 r^4 - (r^2 - 2Mr)\left[r^2 + L^2 + Q\right] \qquad (4.5)$$
$$\Theta = Q - \frac{\cos^2\theta}{\sin^2\theta}L^2 \qquad (4.6)$$

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Then, we have the function

$$\Xi = E^2 r^4 - (r^2 - 2Mr)\left[r^2 + L^2 + Q\right] + (r^2 - 2Mr)\left[Q - \frac{\cos^2\theta}{\sin^2\theta}L^2\right]$$

$$\Xi = E^2 r^4 - (r^2 - 2Mr)(r^2 + L^2) - (r^2 - 2Mr)\frac{\cos^2\theta}{\sin^2\theta}L^2$$

$$\Xi = E^2 r^4 - r^4 - L^2 r^2 + 2Mr^3 + 2ML^2 r - r^2\frac{\cos^2\theta}{\sin^2\theta}L^2 + 2Mr\frac{\cos^2\theta}{\sin^2\theta}L^2 \quad (4.7)$$

In order to write the equations of motion we need the derivatives

$$\frac{\partial\Xi}{\partial E} = 2Er^4 \tag{4.8}$$

$$\frac{\partial\Xi}{\partial L} = -2(r^2 - 2Mr)L - 2(r^2 - 2Mr)\frac{\cos^2\theta}{\sin^2\theta}L$$

$$= -2(r^2 - 2Mr)L\left[1 + \frac{\cos^2\theta}{\sin^2\theta}\right]$$

$$= -2(r^2 - 2Mr)L\csc^2\theta \tag{4.9}$$

$$\tag{4.10}$$

and also

$$\frac{\partial}{\partial r}\left(\frac{\Delta}{2\Sigma}\right) = \frac{\partial}{\partial r}\left(\frac{1}{2} - \frac{M}{r}\right) = \frac{M}{r^2} \tag{4.11}$$

$$\frac{\partial}{\partial r}\left(\frac{1}{2\Sigma}\right) = \frac{\partial}{\partial r}\left(\frac{1}{2r^2}\right) = -\frac{1}{r^3} \tag{4.12}$$

$$\frac{\partial}{\partial r}\left(\frac{\Xi}{2\Delta\Sigma}\right) = \frac{\partial}{\partial r}\left(\frac{E^2 r^4 - (r^2 - 2Mr)(r^2 + L^2) - (r^2 - 2Mr)\frac{\cos^2\theta}{\sin^2\theta}L^2}{2r^2(r^2 - 2Mr)}\right)$$

$$= \frac{1}{2}\frac{\partial}{\partial r}\left(E^2\left(1 - \frac{2M}{r}\right)^{-1} - 1 - \frac{L^2}{r^2} - \frac{\cos^2\theta}{\sin^2\theta}\frac{L^2}{r^2}\right)$$

$$= -\frac{E^2}{(1 - \frac{2M}{r})^2}\frac{M}{r^2} + \frac{L^2}{r^3} + \frac{\cos^2\theta}{\sin^2\theta}\frac{L^2}{r^3}$$

$$= -\frac{E^2 M}{(r - 2M)^2} + \frac{L^2}{r^3}\csc^2\theta \tag{4.13}$$

$$\frac{\partial}{\partial\theta}\left(\frac{\Delta}{2\Sigma}\right)=\frac{\partial}{\partial r}\left(\frac{1}{2}-\frac{M}{r}\right)=0 \tag{4.14}$$

$$\frac{\partial}{\partial\theta}\left(\frac{1}{2\Sigma}\right)=\frac{\partial}{\partial r}\left(\frac{1}{2r^2}\right)=0 \tag{4.15}$$

$$\frac{\partial}{\partial\theta}\left(\frac{\Xi}{2\Delta\Sigma}\right)=\frac{\partial}{\partial\theta}\left(\frac{E^2r^4-(r^2-2Mr)(r^2+L^2)-(r^2-2Mr)\frac{\cos^2\theta}{\sin^2\theta}L^2}{2r^2(r^2-2Mr)}\right)$$

$$=\frac{1}{2}\frac{\partial}{\partial\theta}\left(-\frac{\cos^2\theta}{\sin^2\theta}\frac{L^2}{r^2}\right)$$

$$=\cot\theta\csc^2\theta\frac{L^2}{r^2} \tag{4.16}$$

Using this expressions, the equations of motion of a particle in this spacetime are given by the Hamilton's equations

$$\dot{t}=\frac{1}{2r^2(r^2-2Mr)}2Er^4=\frac{Er^2}{(r^2-2Mr)}$$

$$\dot{r}=p_r\left(1-\frac{2M}{r}\right)$$

$$\dot{\theta}=\frac{p_\theta}{r^2}$$

$$\dot{\phi}=-\frac{1}{2r^2(r^2-2Mr)}\left[-2(r^2-2Mr)L\csc^2\theta\right]=\frac{L}{r^2\sin^2\theta}$$

$$\dot{p}_t=0$$

$$\dot{p}_r=-\frac{M}{r^2}p_r^2+\frac{p_\theta^2}{r^3}-\frac{E^2M}{(r-2M)^2}+\frac{L^2}{r^3\sin^2\theta}$$

$$\dot{p}_\theta=\frac{\cos\theta}{\sin^3\theta}\frac{L^2}{r^2}$$

$$\dot{p}_\phi=0$$