BLACK HOLES

OBSERVATORIO
ASTRONÓMICO
NACIONAL

Classical Black Holes

06. Astrophysics of Supermassive Black Holes

Edward Larrañaga

Outline for Part 1

1. Supermassive Black Holes in the Astrophysical Scenario
   - 1.1 Supermassive Black Holes
   - 1.2 Origin of Supermassive Black Holes
   - 1.3 Stellar Collapse
2. Mathematical Description of the Collapse
   - 2.1 Spherically Symmetric Collapse
   - 2.2 Dust Collapse
   - 2.3 Homogeneous Dust Collapse
   - 2.4 Inhomogeneous Dust Collapse

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In the Milky Way, the central black hole, called Sagittarius A^* has a mass $\sim 4 \times 10^6 M_\odot$.

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After that, Galaxies formed around these early black holes.

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The gas accumulated in the center producea very massive central object producing a black hole with a mass of $\sim 10^4 M_\odot$ or higher.

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Collisions of stars in the cluster are frequent and they can produce a black hole of $\sim 10^2 - 10^4 M_\odot$

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No heating in the collapse until the rate of annihilation of dark particles becomes significant.

This process create an object called a *Dark Matter Star* for which the annihilation radiation supports the gravitational collapse.

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When the dark matter is completely annihilated, the object will collapse into a massive black hole.

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This process create primordial black holes with masses up to $\sim 1000 M_\odot$.

Physical Aspects of the Stellar Death

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For massive stars $(M > 5M_{\odot})$,

H $\longrightarrow$ He $\longrightarrow$ ... $\longrightarrow$ C $\longrightarrow$ ... $\longrightarrow$ Fe

White Dwarf. Degenerate Gas of Electrons

Equation of State for the degenerate gas of electrons

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$$\frac{M^{4/3}}{r^5} \propto \frac{GM^2}{r^5}$$

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$$M_C \approx 1.4 M_\odot$$

Mass of a completely degenerated star.

Neutron Stars. Degenerate Gas of Neutrons

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Baade and Zwicky (1934)
"...supernovae represent the transitions from ordinary stars into neutron stars which in their final stages consist of extremely closely packed neutrons."

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Oppenheimer and Volkoff (1939)

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"...when all thermonuclear sources of energy are exhausted a sufficiently heavy star will collapse. This contraction will continue indefinitely till the radius of the star approaches asymptotically its gravitational radius. Light from the surface of the star will be progressively reddened and can escape over a progressively narrower range of angles till eventually the star tends to close itself off from any communication with a distant observer".

Black Holes

If the collapsing core is too massive to be supported by the degenerate pressure of neutrons, there is no known mechanism capable of finding a new equilibrium configuration, and the body should undergo a complete collapse.

In this case, the final product is a black hole.

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Spherically Symmetric Collapse

- Spherical Symmetry:

$$ds^2 = -e^{2\alpha(t,r)}dt^2 + e^{2\beta(t,r)}dr^2 + R^2(t,r)d\Omega^2$$

Spherically Symmetric Collapse

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- Suppose that the collapsing body can be described as a perfect fluid.
  Then, we consider that coordinates t and r are attached to every collapsing particle (co-moving coordinates).

Spherically Symmetric Collapse

- In the co-moving frame, the 4-velocity of the fluid is just

$$u^\mu = \left(e^{-\alpha}, 0, 0, 0\right)$$

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- The energy-momentum tensor in the co-moving frame is

$$T^\mu_\nu = diag(\rho, P, P, P)$$

Spherically Symmetric Collapse

The Einstein equations for this metric give

$$\begin{aligned}
G^t_t = 8\pi T^t_t \quad &\Rightarrow \quad \frac{F'}{R^2 R'} = 8\pi\rho \\
G^r_r = 8\pi T^r_r \quad &\Rightarrow \quad \frac{\dot{F}}{R^2 \dot{R}} = -8\pi P \\
G^t_r = 0 \quad &\Rightarrow \quad \dot{R}' - \dot{R}\alpha' - \dot{\beta} R' = 0
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$$F = R\left(1 - e^{-2\beta} R'^2 + e^{-2\alpha}\dot{R}^2\right)$$

Misner-Sharp mass

Spherically Symmetric Collapse

The Misner-Sharp mass

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is defined by the relation

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* Note that $n^{\mu} = \partial^{\mu}R$ is normal to the surfaces $R = \mathrm{constant}$.
Thus, when $1 - \frac{F}{R} = 0$, the corresponding surface is null.

Spherically Symmetric Collapse

From (t, t)-component of the Field equations we obtain the Misner-Sharp mass as

$$G^t_t = 8\pi T^t_t \Rightarrow \frac{F'}{R^2 R'} = 8\pi\rho$$

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$$F(r) = \int_0^r F' d\tilde{r} = 8\pi \int_0^r \rho R^2 R' d\tilde{r} = 2M(r)$$

Spherically Symmetric Collapse

Finally, the conservation of the energy-momentum tensor gives the equation

$$\nabla_\mu T^\mu_\nu = 0 \Rightarrow \alpha' = -\frac{P'}{\rho + P}$$

Dust Collapse

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Then

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Dust Collapse

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* If $P \neq 0$ the exterior metric must be a non-vacuum Vaidya spacetime.

Dust Collapse

$\alpha = \alpha(t)$: One can re-define the time coordinate such that

$$e^{\alpha(t)}dt \Rightarrow dt$$

and therefore $g_{tt} = -1$

Dust Collapse

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Introducing $f(r) = e^{2h(r)} - 1$ we have

$$e^{2\beta} = \frac{R'^2}{1 + f}$$

Dust Collapse

$$ds^2 = -dt^2 + \frac{R'^2(t,r)}{1+f(r)}dr^2 + R^2(t,r)d\Omega^2$$

Lemaitre-Tolman-Bondi Metric

Dust Collapse

Kretschmann Scalar:

$$K = 12\frac{F'^2}{R^4R'^2} - 32\frac{FF'}{R^5R'} + 48\frac{F^2}{R^6}$$

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Divergence at $R = 0$

Dust Collapse

One can re-scale to set $R(t, r)$ to equal the co-moving radius r at the time $t = 0$, i.e. we impose $R(0, r) = r$ and introduce a scale factor a such that

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The collapse also impose the condition $\dot{a} < 0$

Dust Collapse

One of the field equations,

$$\frac{F'}{R^2 R'} = 8\pi\rho$$

implies that a regular value of the density at $t = 0$ is obtained if the Misner-Sharp mass has the form

$$F(r) = r^3 m(r)$$

where $m(r)$ is a (sufficiently) regular function in the region $[0, r_b]$.

Dust Collapse

We have $F' = 3r^2m + r^3m'$ and therefore

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$$\frac{3r^2m + r^3m'}{(ra)^2(a + ra')} = 8\pi\rho$$

$$\frac{3m + rm'}{a^2(a + ra')} = 8\pi\rho$$

Dust Collapse

It is usual to consider $m(r)$ as a polynomial around $r = 0$,

$$m(r) = \sum_{k=0}^{\infty} m_k r^k$$

where m_k are constants

Dust Collapse

From the definition of the Misner-Sharp mass we have

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$$\dot{a} = -\sqrt{\frac{m}{a} + \frac{f}{r^2}}$$

Dust Collapse

Evaluating at the time $t = 0$ we have $a(0, r) = 1$ and thus

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$$\dot{a}(0, r) = -\sqrt{m + \frac{f}{r^2}}$$

Note that the function f defines the initial velocity of the particles in the cloud. It is usual to write this function as an expansion around $r = 0$ as

$$f(r) = r^2 b(r)$$

with

$$b(r) = \sum_{k=0}^{\infty} b_k r^k$$

Homogeneous Dust Collapse

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Simplest Model of Gravitational Collapse. Oppenheimer-Snyder (1939)

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Homogeneous: $\rho = \rho(t)$

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This implies that
$m' = 0 \longrightarrow m = m_0$

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which is accomplished by choosing $b = b_0$
Then

$$\dot{a} = -\sqrt{\frac{m_0}{a} + b_0}$$

Homogeneous Dust Collapse

The Lemaitre-Tolman-Bondi line element becomes

$$ds^2 = -dt^2 + \frac{R'^2(t,r)}{1+f(r)}dr^2 + R^2(t,r)d\Omega^2$$

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$$ds^2 = -dt^2 + a^2(t)\left[\frac{dr^2}{1+b_0 r^2} + r^2 d\Omega^2\right]$$

Homogeneous Dust Collapse

$$ds^2 = -dt^2 + a^2(t)\left[\frac{dr^2}{1 + b_0 r^2} + r^2 d\Omega^2\right]$$

For $b_0 = 0$ this line element becomes

$$ds^2 = -dt^2 + a^2(t)\left[dr^2 + r^2 d\Omega^2\right]$$

and describes the counterpart of a flat spacetime, in which the collapse is marginally bound (i.e. the falling particles have vanishing velocity at infinity).

Homogeneous Dust Collapse

For $b_0 = 0$ the scale factor is

$$\dot{a} = -\sqrt{\frac{m_0}{a} + b_0}$$

Homogeneous Dust Collapse

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$$\dot{a} = -\sqrt{\frac{m_0}{a} + b_0}$$

$$\frac{da}{dt} = -\sqrt{\frac{m_0}{a}}$$

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Homogeneous Dust Collapse

The value of the initial condition was supposed as $a(0) = 1$. Thus

$$a(t) = \left[1 - \frac{3\sqrt{m_0}}{2}t\right]^{2/3}$$

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Therefore, the singularity $R = 0$ occurs at the time t_s for which $a(t_s) = 0$. This is

$$t_s = \frac{2}{3\sqrt{m_0}}$$

Homogeneous Dust Collapse

The curve $t_H(r)$ describing the time at which the shell r crosses the horizon can be obtained from the condition

$$1-\frac{F}{R}=0$$

Homogeneous Dust Collapse

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Homogeneous Dust Collapse

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$$a(t_H) = m_0 r^2$$

Homogeneous Dust Collapse

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Homogeneous Dust Collapse

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Homogeneous Dust Collapse

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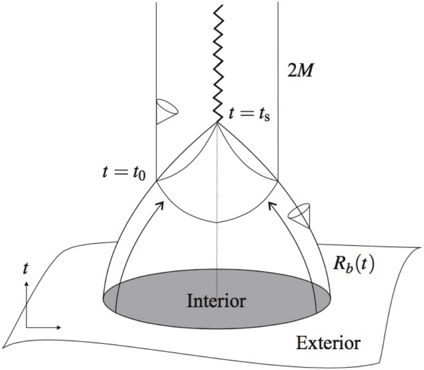
$$t_H(r) = \frac{2}{3\sqrt{m_0}} - \frac{2}{3} m_0 r^3$$

Gravitational Collapse of a Homogeneous Cloud of Dust

Eddington-Finkelstein Diagram



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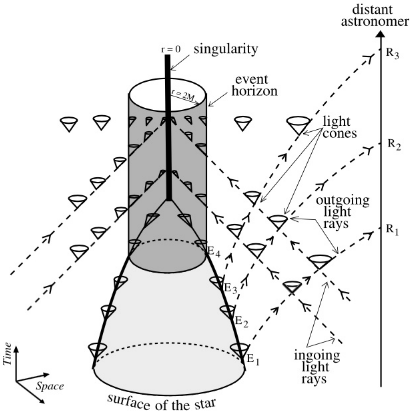

Inhomogeneous Dust Collapse

Inhomogeneous Dust Collapse

ρ depends on both t and r, and thus

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$$\rho = \rho(t, r)$$
$$m = m(r)$$
$$b = b(r)$$
$$a = a(t, r)$$

Inhomogeneous Dust Collapse

The simplest form for the function $m(r)$ is

$$m(r) = m_0 + m_2 r^2$$

Inhomogeneous Dust Collapse

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This function defines the density interms of the two parameter m_0 and m_2.
The condition that ρ must be a decreasing function of r implies that $m_2 < 0$.

* $m_1 = 0$ in order to have a non divergent density.

Inhomogeneous Dust Collapse

The equation

$$\dot{R}^2 = \frac{F}{R} + f$$

gives this time

$$ra(\dot{t}, r) = -\sqrt{\frac{mr^2}{ra} + br^2}$$

Inhomogeneous Dust Collapse

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$$\dot{R}^2 = \frac{F}{R} + f$$

gives this time

$$r\dot{a}(t, r) = -\sqrt{\frac{mr^2}{ra} + br^2}$$

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Inhomogeneous Dust Collapse

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$$a(\dot{t}, r) = -\sqrt{\frac{m(r)}{a}}$$

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Inhomogeneous Dust Collapse

In the marginally bound case, $b = 0$, it gives

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This equation shows how each shell of the distribution (each r) collapses with a different scale factor and with a different velocity.

Inhomogeneous Dust Collapse

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Inhomogeneous Dust Collapse

Using $m(r) = m_0 + m_2 r^2$ gives the functions

$$t_s(r) = \frac{2}{3\sqrt{m_0 + m_2 r^2}}$$

$$t_H(r) = \frac{2}{3\sqrt{m_0 + m_2 r^2}} - \frac{2}{3} r^3 \left(m_0 + m_2 r^2\right)$$

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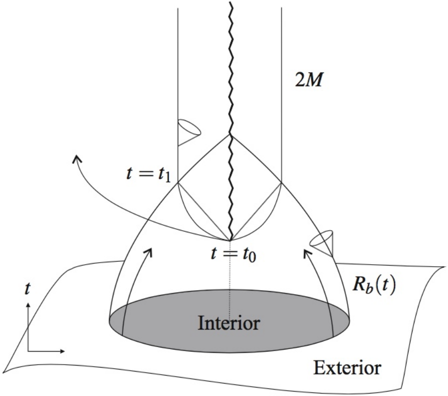

Next Lecture

06. Black Holes Astrophysics