$$R = (r^2+a^2-a\xi)^2 - \Delta J$$

$$\Delta = r^2 - 2mr + a^2$$
$$J = \eta + (a-\xi)^2$$

$$R = (r^2+a^2-a\xi)^2 - (r^2-2mr+a^2)\left[\eta+(a-\xi)^2\right]$$

$$R = r^4 + a^4 + a^2\xi^2 + 2r^2a^2 - 2r^2a\xi - 2a^3\xi - (r^2+a^2)\left[\eta+(a-\xi)^2\right] + 2mr\left[\eta+(a-\xi)^2\right]$$

$$R = r^4 + r^2\left[2a^2 - 2a\xi - \{\eta+(a-\xi)^2\}\right] + 2mr\left[\eta+(a-\xi)^2\right] + a^4 + a^2\xi^2 - 2a^3\xi$$
$$-a^2\left[\eta+(a-\xi)^2\right]$$

$$R = r^4 + r^2\left[2a^2 - 2a\xi - \eta - (a^2+\xi^2-2a\xi)\right] + 2mr\left[\eta+(a-\xi)^2\right] + a^4 - 2a^3\xi + a^2\xi^2$$
$$-a^2\eta - a^2(a^2+\xi^2-2a\xi)$$

$$R = r^4 + r^2\left[2a^2 - \eta - (a^2+\xi^2)\right] + 2mr\left[\eta+(a-\xi)^2\right] - a^2\eta$$

$$R = r^4 + (a^2-\xi^2-\eta)r^2 + 2m\left[\eta+(a-\xi)^2\right]r - a^2\eta$$

Effective Potential

$$\mathcal{R} = r^4 + (a^2 - \xi^2 - \eta)r^2 + 2m[\eta + (\xi - a)^2]r - a^2\eta = 0$$

$$\frac{\partial \mathcal{R}}{\partial r} = 4r^3 + 2r(a^2 - \xi^2 - \eta) + 2m[\eta + (\xi - a)^2] + 2m'r[\eta + (\xi - a)^2] = 0$$

$$m' = \frac{dm}{dr}$$

$$4r^3 + 2(a^2 - \xi^2 - \eta)r + 2m[\eta + (\xi - a)^2]f = 0$$

$$f = 1 + \frac{m'}{m}r$$

$$\begin{cases} r^4 + (a^2 - \xi^2 - \eta)r^2 + 2m[\eta + (\xi - a)^2]r - a^2\eta = 0 & (\mathrm{I}) \\ 4r^3 + 2(a^2 - \xi^2 - \eta)r + 2m[\eta + (\xi - a)^2]f = 0 & (\mathrm{II}) \end{cases}$$

Multiplying (II) by r:

$$4r^4 + 2(a^2 - \xi^2 - \eta)r^2 + 2mrf[\eta + (\xi - a)^2] = 0$$

Solving,

$$2mrf\left[\eta+(\xi-a)^2\right]=-4r^4-2(a^2-\xi^2-\eta)r^2$$

$$2mr\left[\eta+(\xi-a)^2\right]=-\frac{4r^4}{f}-2\frac{(a^2-\xi^2-\eta)r^2}{f}$$

and replacing in (I),

$$r^4+(a^2-\xi^2-\eta)r^2-\frac{4r^4}{f}-2\frac{(a^2-\xi^2-\eta)}{f}r^2-a^2\eta=0$$

$$r^4\left(1-\frac{4}{f}\right)+(a^2-\xi^2-\eta)r^2\left(1-\frac{2}{f}\right)-a^2\eta=0$$

$$r^4\left(1-\frac{4}{f}\right)+(a^2-\xi^2)r^2\left(1-\frac{2}{f}\right)-\eta\left[a^2+r^2\left(1-\frac{2}{f}\right)\right]=0$$

$$\eta=\frac{r^4\left(1-\frac{4}{f}\right)+r^2(a^2-\xi^2)\left(1-\frac{2}{f}\right)}{a^2+r^2\left(1-\frac{2}{f}\right)}$$

$$\eta=\frac{r^4(f-4)+r^2(a^2-\xi^2)(f-2)}{fa^2+r^2(f-2)}$$

Replacing η in (II):

$$2r^3 + r\left[a^2 - \xi^2 - \frac{r^4(f-4) + r^2(a^2-\xi^2)(f-2)}{fa^2 + r^2(f-2)}\right] + m\left[\frac{r^4(f-4) + r^2(a^2-\xi^2)(f-2)}{fa^2 + r^2(f-2)} + (\xi - a)^2\right] f = 0$$

$$2r^3(fa^2 + r^2(f-2)) + r(a^2-\xi^2)(fa^2 + r^2(f-2)) - r\left[r^4(f-4) + r^2(a^2-\xi^2)(f-2)\right]$$
$$+ mf\left[r^4(f-4) + r^2(a^2-\xi^2)(f-2) + (\xi-a)^2(fa^2 + r^2(f-2))\right] = 0$$

$$(a^2-\xi^2)\left[r(fa^2 + r^2(f-2)) + mfr^2(f-2) - r^3(f-2)\right] + (\xi-a)^2 mf(fa^2 + r^2(f-2)) =$$
$$r^5(f-4) - 2r^3(fa^2 + r^2(f-2)) - r^4 mf(f-4)$$

$$(a^2-\xi^2)\left[ra^2 f + r^2 mf(f-2)\right] + \underbrace{(\xi-a)^2}_{(\xi^2+a^2-2a\xi)} mf\left[a^2 f + r^2(f-2)\right] = -r^5 f - 2r^3 a^2 f - r^4 mf(f-4)$$

$$(a^2-\xi^2)[ra^2f + r^2mf(f-2)] + (\xi^2+a^2-2a\xi)mf[a^2f+r^2(f-2)] = -r^5f - 2r^3a^2f - r^4mf(f-4)$$

$$\xi^2[mf(a^2f+r^2(f-2)) - ra^2f - r^2mf(f-2)] - \xi 2amf[a^2f+r^2(f-2)]$$
$$+a^2[mf(a^2f+r^2(f-2)) + ra^2f + r^2mf(f-2)] = -r^5f - 2r^3a^2f - r^4mf(f-4)$$

$$\xi^2[a^2mf^2 + r^2mf^2 - 2r^2mf - ra^2f - r^2mf^2 + 2r^2mf] - \xi 2amf[a^2f+r^2(f-2)]$$
$$+a^2[a^2mf^2 + r^2mf^2 - 2r^2mf + ra^2f + r^2mf^2 - 2r^2mf] = -r^5f - r^4mf(f-4) - 2r^3a^2f$$

$$\xi^2a^2f(mf-r) - \xi 2amf[a^2f+r^2(f-2)] + a^2[a^2mf^2 + 2r^2mf^2 - 4r^2mf + ra^2f] =$$
$$-r^5f - r^4mf(f-4) - 2r^3a^2f$$

$$\xi^2a^2(mf-r) - \xi 2am[a^2f+r^2(f-2)] + a^2[a^2mf + 2r^2m(f-2) + ra^2] = -r^5 - r^4m(f-4) - 2r^3a^2$$

$$a^2(mf-r)\xi^2 - 2am[a^2f+r^2(f-2)]\xi + r^5 + r^4m(f-4) + 2r^3a^2 + 2r^2a^2m(f-2) + ra^4 + a^4mf = 0$$

$$a^2(mf-r)\xi^2 - 2am\left[a^2 f + r^2(f-2)\right]\xi + r^5 + r^4 m(f-4) + 2r^3a^2 + 2r^2a^2m(f-2) + ra^4 + a^4mf = 0$$

$$\underbrace{a^2(r-mf)}_{A}\xi^2 \underbrace{-2am\left[(2-f)r^2 - a^2 f\right]}_{B}\xi \underbrace{-\left[r^5+r^4m(f-4)+2r^3a^2+2r^2a^2m(f-2)+ra^4+a^4mf\right]}_{C} = 0$$

$$\xi = \frac{-B \pm \sqrt{B^2-4AC}}{2A}$$

$$B^2 = 4a^2m^2\left[(2-f)r^2 - a^2 f\right]^2 = 4a^2m^2\left[(2-f)^2 r^4 + a^4f^2 - 2(2-f)r^2a^2 f\right]$$

$$B^2 = 4a^2\left[r^4m^2(f^2+4-4f) + 2r^2a^2m^2f(f-2) + a^4m^2f^2\right]$$

$$-4AC = 4a^2(r-mf)\left[r^5+r^4m(f-4)+2r^3a^2+2r^2a^2m(f-2)+ra^4+a^4mf\right]$$

$$-4AC = 4a^2\left[r^6 + r^5m(f-4) + 2r^4a^2 + 2r^3a^2m(f-2) + r^2a^4 + ra^4mf - r^5mf - r^4m^2f(f-4) - 2r^3a^2mf - 2r^2a^2m^2f(f-2) - ra^4mf - a^4m^2f^2\right]$$

$$-4AC = 4a^2\left[r^6 - 4r^5m + r^4\left(2a^2 - m^2f(f-4)\right) - 4r^3a^2m + r^2a^2\left(a^2 - 2m^2f(f-2)\right) - a^4m^2f^2\right]$$

$$-4AC = 4a^2\left[r^6 - 4r^5m + r^4\left(2a^2 - m^2f^2 + 4m^2f\right) - 4r^3a^2m + r^2a^2\left(a^2 - 2m^2f^2 + 4m^2f\right) - a^4m^2f^2\right]$$

$$\begin{cases} B^2 = 4a^2\left[r^4 m^2(f^2+4-4f) + 2r^2a^2m^2f(f-2) + a^4m^2f^2\right] \\ -4AC = 4a^2\left[r^6 - 4r^5m + r^4(2a^2 - m^2f^2 + 4m^2f) - 4r^3a^2m + r^2a^2(a^2 - 2m^2f^2 + 4m^2f) - a^4m^2f^2\right] \end{cases}$$

$$B^2 - 4AC = 4a^2\left[r^6 - 4r^5m + r^4(4m^2 + 2a^2) - 4r^3a^2m + r^2a^4\right]$$

$$B^2 - 4AC = 4a^2r^2\left[r^4 - 4r^3m + 2r^2(2m^2 + a^2) - 4ra^2m + a^4\right]$$

$$B^2 - 4AC = 4a^2r^2\left[r^4 - 4r^3m + 4r^2m^2 + 2r^2a^2 - 4ra^2m + a^4\right]$$

$$B^2 - 4AC = 4a^2r^2\left[r^2 - 2mr + a^2\right]^2$$

Thus;

$$\zeta = \frac{-B \pm \sqrt{B^2 - 4AC}}{2A}$$

$$\zeta = \frac{2am\left[(2-f)r^2 - a^2f\right] \pm 2ar\left[r^2 - 2mr + a^2\right]}{2a^2(r - mf)}$$

+ or − ?

$$\xi = \frac{m[(2-f)r^2 - a^2 f] \pm r[r^2 - 2mr + a^2]}{a(r-mf)}$$

Using this result we calculate η:

$$\eta = \frac{r^4(f-4) + r^2(a^2 - \xi^2)(f-2)}{fa^2 + r^2(f-2)}$$

$$\xi^2 = \left[\frac{m[(2-f)r^2 - a^2 f] \pm r[r^2 - 2mr + a^2]}{a(r-mf)}\right]^2$$

$$\xi^2 = \frac{[m[(2-f)r^2 - a^2 f] \pm r[r^2 - 2mr + a^2]]^2}{a^2(r-mf)^2}$$

$$a^2 - \xi^2 = \frac{a^4(r-mf)^2 - [m[(2-f)r^2 - a^2 f] \pm r[r^2 - 2mr + a^2]]^2}{a^2(r-mf)^2}$$

$$a^2-\xi^2=\frac{a^4(r-mf)^2-\left[m\left[(2-f)r^2-a^2f\right]\pm r\left[r^2-2mr+a^2\right]\right]^2}{a^2(r-mf)^2}$$

$$[\quad]=a^4(r^2-2rmf+m^2f^2)-\Big[m^2\left[(2-f)r^2-a^2f\right]^2+r^2\left[r^2-2mr+a^2\right]^2$$
$$\pm 2rm\left[(2-f)r^2-a^2f\right]\left[r^2-2mr+a^2\right]\Big]$$

$$[\quad]=a^4(r^2-2rmf+m^2f^2)-\Big\{m^2\left[(2-f)^2r^4+a^4f^2-2a^2r^2f(2-f)\right]$$
$$+r^2\left[r^4+4m^2r^2+a^4-4mr^3+2r^2a^2-4mra^2\right]$$
$$\pm 2rm\Big[(2-f)r^4-2(2-f)mr^3+(2-f)r^2a^2$$
$$-r^2a^2f+2rma^2f-a^4f\Big]\Big\}$$

$$[\quad]=a^4(r^2-2rmf+m^2f^2)-\Big\{r^6+r^5\left[-4m\pm 2m(2-f)\right]+r^4\left[m^2(2-f)^2+4m^2+2a^2\mp 4(2-f)m^2\right]$$
$$+r^3\left[-4ma^2\pm 2m(2-f)a^2\mp 2ma^2f\right]$$
$$+r^2\left[-2m^2a^2f(2-f)+a^4\pm 4m^2a^2f\right]\mp 2rma^4f+m^2a^4f^2\Big\}$$

$$[\quad] = a^4(r^2-2rmf+m^2f^2)-\{r^6+r^5[-4m\pm 4m\mp 2mf]$$
$$+r^4[8m^2-4m^2f+m^2f^2+2a^2\mp 8m^2\pm 4m^2f]$$
$$+r^3[-4ma^2\pm 4ma^2\mp 4ma^2f]$$
$$+r^2[-4m^2a^2f+2m^2a^2f^2+a^4\pm 4m^2a^2f]$$
$$\mp 2rma^4f+m^2a^4f^2\}$$

$$[\quad] = -r^6+r^5[4m\mp 4m\pm 2mf]-r^4[8m^2\mp 8m^2-4m^2f\pm 4m^2f+m^2f^2+2a^2]$$
$$+r^3[4ma^2\mp 4ma^2\pm 4ma^2f]+r^2[4m^2a^2f-2m^2a^2f^2\mp 4m^2a^2f]$$
$$-2rma^4f(1\mp 1)$$

Now we have to analise each sign separetely;

1) For the upper sign:

$$[+] = -r^6+r^5[+2mf]-r^4[+m^2f^2+2a^2]+r^3[+4ma^2f]+r^2[-2m^2a^2f^2]$$

$$\boxed{[+] = -r^6+2r^5mf-r^4[m^2f^2+2a^2]+4r^3ma^2f-2r^2m^2a^2f^2}$$

For the lower sign;

$$[-] = -r^6 + r^5[4m + 4m - 2mf] - r^4[8m^2 + 8m^2 - 4m^2f - 4m^2f + m^2f^2 + 2a^2]$$
$$+ r^3[4ma^2 + 4ma^2 - 4ma^2f] + r^2[4m^2a^2f - 2m^2a^2f^2 + 4m^2a^2f]$$
$$- 2rma^4f(1+1)$$

$$[-] = -r^6 + r^5[8m - 2mf] - r^4[16m^2 - 8m^2f + m^2f^2 + 2a^2]$$
$$+ r^3[8ma^2 - 4ma^2f] + r^2[8m^2a^2f - 2m^2a^2f^2] - 4rma^4f$$

$$\boxed{\begin{aligned}[-] = &-r^6 + 2r^5m(4-f) - r^4[16m^2 - 8m^2f + m^2f^2 + 2a^2] \\ &+ 4r^3ma^2(2-f) + 2r^2m^2a^2f(4-f) - 4rma^4f\end{aligned}}$$

$$\eta = \frac{r^4(f-4) + r^2(a^2 - \xi^2)(f-2)}{fa^2 + r^2(f-2)}$$

$$\eta = \frac{r^4(f-4) + r^2(f-2)\dfrac{[\pm]}{a^2(r-mf)^2}}{fa^2 + r^2(f-2)}$$

$$\eta = \frac{r^4(f-4)a^2(r-mf)^2 + r^2(f-2)[\pm]}{a^2(r-mf)^2[fa^2 + r^2(f-2)]}$$

$$\eta = \frac{r^2}{a^2(r-mf)^2[fa^2 + r^2(f-2)]}\left[r^2(f-4)a^2(r-mf)^2 + (f-2)[\pm]\right]$$

$$r^2(f-4)a^2(r-mf)^2 = r^2a^2(f-4)(r^2 - 2rmf + m^2f^2)$$

$$r^2(f-4)a^2(r-mf)^2 = (f-4)a^2(r^4 - 2r^3mf + r^2m^2f^2)$$

$$\eta = \frac{r^2}{a^2(r-mf)^2[fa^2 + r^2(f-2)]}\left[(f-4)a^2(r^4 - 2r^3mf + r^2m^2f^2) + (f-2)[\pm]\right]$$

$$\eta = \frac{r^2}{a^2(r-mf)^2\left[fa^2+r^2(f-2)\right]}\left[(f-4)a^2(r^4-2r^3mf+r^2m^2f^2)+(f-2)[\pm]\right]$$

For the lower sign:

$$[-] = (f-4)a^2(r^4-2r^3mf+r^2m^2f^2)+(f-2)\Big\{-r^6+2r^5m(4-f)-r^4\left[16m^2-8m^2f+m^2f^2+2a^2\right]$$
$$+4r^3ma^2(2-f)+2r^2m^2a^2f(4-f)-4rma^4f\Big\}$$

$$[-] = -r^6(f-2)+2r^5m(4-f)(f-2)-r^4\Big\{\left[16m^2-8m^2f+m^2f^2+2a^2\right](f-2)-a^2(f-4)\Big\}$$
$$+r^3a^2\left[-4m(2-f)^2-2mf(f-4)\right]$$
$$+r^2a^2m^2f\left[2(4-f)(f-2)+f(f-4)\right]-4rma^4f(f-2)$$

$$[-] = r\Big\{-r^5(f-2)+2r^4m(4-f)(f-2)-r^3\Big\{\left[16m^2-8m^2f+m^2f^2\right](f-2)+2a^2(f-2)-a^2(f-4)\Big\}$$
$$+r^2a^2\left[-4m(2-f)^2-2mf(f-4)\right]$$
$$+ra^2m^2f\left[2(4-f)(f-2)+f(f-4)\right]-4ma^4f(f-2)\Big\}$$

$$[-] = r\Big\{-r^5(f-2)+2r^4m(4-f)(f-2)-r^3\Big\{\left[16m^2-8m^2f+m^2f^2\right](f-2)+a^2f\Big\}$$
$$-2r^2a^2mf(f-4)$$
$$+ra^2m^2f\left[2(4-f)(f-2)+f(f-4)\right]+4ma^2(2-f)\left[fa^2+r^2(f-2)\right]\Big\}$$

$$[-] = r\Big\{-r\Big\{r^4(f-2) - 2r^3m(4-f)(f-2) + r^2\big\{[16m^2 - 8m^2f + m^2f^2](f-2) + a^2f\big\}$$
$$+2ra^2mf(f-4)$$
$$-a^2m^2f[2(4-f)(f-2) + f(f-4)]\Big\} + 4ma^2(2-f)[fa^2 + r^2(f-2)]\Big\}$$

$$\{\} = r^4(f-2) - 2r^3m(4-f)(f-2) + r^2a^2f + r^2m^2[16-8f+f^2](f-2)$$
$$+2ra^2mf(f-4) - a^2m^2f(4-f)[2(f-2)-f]\}$$

$$\{\} = r^4(f-2) - 2r^3m(4-f)(f-2) + r^2a^2f + r^2m^2(4-f)^2(f-2)$$
$$+2ra^2mf(f-4) - a^2m^2f(4-f)(f-4)\}$$

$$\{\ \} = r^2fa^2 + r^4(f-2) + (4-f)^2m^2fa^2 + r^2(4-f)^2m^2(f-2) - 2r(4-f)mfa^2 - 2r^3(4-f)m(f-2)$$

$$\{\ \} = [r^2 + (4-f)^2m^2 - 2r(4-f)m][fa^2 + r^2(f-2)]$$

$$\{\ \} = [r-(4-f)m]^2[fa^2 + r^2(f-2)]$$

Hence,

$$[-] = -r\left\{ r\{\ \} - 4ma^2(2-f)\left[fa^2 + r^2(f-2)\right]\right\}$$

$$[-] = -r\left\{ r\left[r-(4-f)m\right]^2\left[fa^2+r^2(f-2)\right] - 4ma^2(2-f)\left[fa^2 + r^2(f-2)\right]\right\}$$

$$[-] = -r\left\{ r\left[r-(4-f)m\right]^2 - 4ma^2(2-f)\right\}\left[fa^2 + r^2(f-2)\right]$$

and therefore:

$$\eta = \frac{-r^3}{a^2(r-mf)^2\left[fa^2+r^2(f-2)\right]}\left\{ r\left[r-(4-f)m\right]^2 - 4ma^2(2-f)\right\}\left[fa^2 + r^2(f-2)\right]$$

$$\eta = \frac{-r^3\left\{ r\left[r-(4-f)m\right]^2 - 4ma^2(2-f)\right\}}{a^2(r-mf)^2}$$

$$\eta = \frac{r^3\left\{ 4ma^2(2-f) - r\left[r-(4-f)m\right]^2\right\}}{a^2(r-mf)^2}$$

Parameters of the shadow

$$\xi = \frac{m\left[(2-f)r^2 - a^2 f\right] \pm r\left[r^2 - 2mr + a^2\right]}{a(r-mf)}$$

$$\eta = \frac{r^3\left\{4ma^2(2-f) - r\left[r-(4-f)m\right]^2\right\}}{a^2(r-mf)^2}$$

$$m' = \frac{dm}{dr}$$

$$f = 1 + \frac{m'}{m} r$$