BLACK HOLES

OBSERVATORIO
ASTRONÓMICO
NACIONAL

Classical Black Holes

08. Accretion

Edward Larrañaga

Outline for Part 1

Accretion Basics

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Process of matter falling into the potential well of a gravitating object.

Accretion Regimes

1. Spherical Accretion
2. Cylindrical Accretion
3. Accretion Disk
4. Two-Stream Accretion

Spherical Accretion

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  v_{rel}: relative velocity between accretor and matter
- $v_{rel} \ll c_s$
- If the accretor is a BH,

$$v_{rel} = v$$

  v: velocity of accreting matter (i.e. the BH doesn't move!)

Cylindrical Accretion

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- $v_{rel} \geq c_s$

Accretion Disk

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- Matter spirals down into the accretor

Two-Stream Accretion

- Quasi-spherically symmetric inflow coexist with an accretion disk

Spherical Accretion

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To avoid the disintegration of the accretion structure, the outward force due to radiation pressure must be counterbalanced by the gravitational force.

Radiation Pressure

Outward energy flux at distance r from the center

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Then, the outwards momentum flux (or pressure) is

$$P_{rad} = \frac{F}{c} = \frac{L}{4\pi r^2 c}$$

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The radiation force on a single electron is

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$$\sigma_e = \frac{8\pi}{3}\left(\frac{e^2}{m_e c^2}\right)^2 = 6.65 \times 10^{-25}\ \mathrm{cm}^2$$

Interaction with protons is negligible because it is lower by a factor of $\left(\frac{m_p}{m_e}\right)^2 \sim 3 \times 10^6$

Gravitational Force

The gravitational force between the central object M and one electron-proton pair is

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$$\vec{f}_g = -\frac{GM(m_e + m_p)}{r^2}\hat{r} \sim -\frac{GMm_p}{r^2}\hat{r}$$

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Eddington Lumionisty: Maximum luminosity of a source M powered by spherical accretion.

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$M_E = 8 \times 10^5 L_{44} M_\odot$: *Eddington's Mass*. Minimum mass for a given luminosity

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Black holes with $M \sim 10^5 - 10^9 M_\odot$

Estimation of the Central Mass from its Luminosity

Name	z	L_{bol}		M_{BH}	ref.	Type
3C 120	0.033	45.34	I	7.42	1	SY1
3C 390.3	0.056	44.88	I	8.55	1	SY1
Akn 120	0.032	44.91	I	8.27	1	SY1
F 9	0.047	45.23	F	7.91	1	SY1
IC 4329A	0.016	44.78	I	6.77	1	SY1
Mrk 79	0.022	44.57	I	7.86	1	SY1
Mrk 110	0.035	44.71	F	6.82	1	SY1
Mrk 335	0.026	44.69	I	6.69	1	SY1
Mrk 509	0.034	45.03	I	7.86	1	SY1
Mrk 590	0.026	44.63	I	7.20	1	SY1
Mrk 817	0.032	44.99	I	7.60	1	SY1
NGC 3227	0.004	43.86	I	7.64	1	SY1
NGC 3516	0.009	44.29	I	7.36	3	SY1
NGC 3783	0.010	44.41	I	6.94	2	SY1
NGC 4051	0.002	43.56	I	6.13	1	SY1
NGC 4151	0.003	43.73	I	7.13	1	SY1
NGC 4593	0.009	44.09	I	6.91	3	SY1
NGC 5548	0.017	44.83	I	8.03	1	SY1
NGC 7469	0.016	45.28	I	6.84	1	SY1
PG 0026+129	0.142	45.39	I	7.58	1	RQQ
PG 0052+251	0.155	45.93	F	8.41	1	RQQ
PG 0804+761	0.100	45.93	F	8.24	1	RQQ
PG 0844+349	0.064	45.36	F	7.38	1	RQQ
PG 0953+414	0.239	46.16	F	8.24	1	RQQ
PG 1211+143	0.085	45.81	F	7.49	1	RQQ
PG 1229+204	0.064	45.01	I	8.56	1	RQQ
PG 1307+085	0.155	45.83	F	7.90	1	RQQ
PG 1351+640	0.087	45.50	I	8.48	1	RQQ
PG 1411+442	0.089	45.58	F	7.57	1	RQQ
PG 1426+015	0.086	45.19	I	7.92	1	RQQ
PG 1613+658	0.129	45.66	I	8.62	1	RQQ
PG 1617+175	0.114	45.52	F	7.88	1	RQQ
PG 1700+518	0.292	46.56	F	8.31	1	RQQ
PG 2130+099	0.061	45.47	I	7.74	1	RQQ
PG 1226+023	0.158	47.35	I	7.22	1	RLQ
PG 1704+608	0.371	46.33	I	8.23	1	RLQ

[a] Column (1) Name, (2) redshift, (3) log of the bolometric luminosity (ergs s^{-1}), (4) method for bolometric luminosity estimation (I: flux integration; F: SED fitting), (5) black hole mass estimate from reverberation mapping (for Kaspi et al. (2000) sample, where black hole mass is log mean of rms FWHM and mean FWHM mass, in solar masses), (6) reference for black hole mass estimation, and (7) AGN type.

References. — (1) Kaspi et al. (2000), (2) Onken & Peterson (2002), (3) Ho (1999).

J.-H. Woo, C. M. Urry. *AGN Black Hole Masses and Bolometric Luminosities*. *Astrophys. J.* 579 (2002) 530-544

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η: Efficiency of the process

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$$\eta \sim 0.1 - 0.2$$

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Using this point, one obtains an efficiency of $\eta \sim 0.1$

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$\dot{M}_E$: Maximum possible accretion rate for a mass $M_8 = \frac{M}{10^8 M_\odot}$.

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2. $\dot{M}_E$ can be exceeded with non-spherical models.

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For $L \sim L_E$, the BH grows exponentially on time scales of the order $\sim 10^8\ \mathrm{yr}$.

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$\bar{\nu}$: frequency of a typical (average) photon

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$$T_{eff} = 1.01 \times 10^6 M_8^{-1/4} \left(\frac{\dot{M}}{\dot{M}_E}\right)^{1/4} \left(\frac{r}{r_S}\right)^{-3/4}$$

compactness

One way to estimate the compactness of a source is using the luminosity and the effective surface temperature.

$$r_{BB} = \sqrt{\frac{L}{4\pi\sigma T_{eff}^4}}$$

Example

Consider a system in our galaxy with $L = 10^{37}\ \mathrm{erg} \cdot \mathrm{s}^{-1}$

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Eddington limit, the mass of the central object must be

$$M \geq \frac{L}{1.26 \times 10^{38}} M_{\odot} \sim \frac{10^{37}}{10^{38}} M_{\odot} \sim 0.1 M_{\odot}$$

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If the radiation is in the optical-UV,

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Typical size of a Star!

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https://rechneronline.de/spectrum

Example

If the radiation is in soft X-rays at 1 keV,
$\nu_{max} \sim 10^{17}\mathrm{Hz}$

$$T_{eff} \sim T_c = \frac{10^{17}}{5.88 \times 10^{10}} \sim 10^7\mathrm{K}$$

$$r_{bb} \sim 10^6\mathrm{cm} \sim 10\mathrm{km}$$

Typical size of a neutron star or a BH! https://rechneronline.de/spectrum

Virial Temperature

T_{th}: Temperature reached by the accreted material if all the gravitational energy is transformed into thermal energy.

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$$T_{th} = \frac{GMm_p}{3k_B r} = T_{vir}$$

Temperatures

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$$T_{rad} \sim T_{eff}$$

In general,

$$T_{eff} \lesssim T_{rad} \lesssim T_{th}$$

Outline for Part 2

Hamilton-Jacobi Formulation

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$$\mathscr{L} = \frac{1}{2} g_{\mu\nu} \dot{x}^{\mu} \dot{x}^{\nu}$$

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Hamilton-Jacobi Formulation

Hamilton's principal function

$$S = S\left(x^{\mu}; \lambda\right)$$

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Hamilton-Jacobi Equation

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$$p_{\mu} = \frac{\partial S}{\partial x^{\mu}}$$

Hamilton-Jacobi Equation

$$\frac{1}{2} g^{\mu\nu} \frac{\partial S}{\partial x^{\mu}} \frac{\partial S}{\partial x^{\nu}} - \frac{\partial S}{\partial \lambda} = 0$$

Kerr's Solution

Boyer-Lindquist coordinates: (t, r, θ, φ)

$$ds^2 = -\frac{\Delta - a^2 \sin^2\theta}{\varrho} dt^2 - \left(\frac{r^2 + a^2 - \Delta}{\varrho}\right) 2a \sin^2\theta dt d\varphi$$
$$+ \frac{\varrho}{\Delta} dr^2 + \varrho d\theta^2 + \left(\frac{\left(r^2 + a^2\right)^2 - \Delta a^2 \sin^2\theta}{\varrho}\right) \sin^2\theta d\varphi^2.$$

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$$\varrho = r^2 + a^2\cos^2\theta$$

$$\Delta = r^2 - 2Mr + a^2$$

Kerr's Solution

$$\left(\frac{\partial}{\partial s}\right)^2 = -\frac{A}{\varrho\Delta}\left(\frac{\partial}{\partial t}\right)^2 - \frac{4aMr}{\varrho\Delta}\left(\frac{\partial}{\partial t}\right)\left(\frac{\partial}{\partial \varphi}\right) + \frac{\Delta}{\varrho}\left(\frac{\partial}{\partial r}\right)^2 + \frac{1}{\varrho}\left(\frac{\partial}{\partial \theta}\right)^2 + \frac{\Delta - a^2\sin^2\theta}{\varrho\Delta\sin^2\theta}\left(\frac{\partial}{\partial \varphi}\right)^2$$

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$$A = (r^2 + a^2)^2 - a^2\Delta\sin^2\theta$$

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Hamilton-Jacobi Formulation

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Hamilton-Jacobi Formulation

Hamilton Principal Function

$$S = \frac{1}{2}\lambda\delta - \varepsilon t + \ell_z\varphi + S_r(\theta) + S_\theta(\theta)$$

Separation of the Hamilton-Jacobi Equation. Carter Constant

$$\Delta\left(\frac{dS_r}{dr}\right)^2 - \frac{1}{\Delta}\left[(r^2 + a^2)\varepsilon - a\ell_z\right]^2 + (\ell_z - a\varepsilon)^2 + \delta r^2 =$$
$$-\left(\frac{dS_\theta}{d\theta}\right)^2 - \left(\frac{\ell_z^2}{\sin^2\theta} - a^2\varepsilon^2 + \delta a^2\right)\cos^2\theta = \mathscr{C}$$

Hamilton-Jacobi Formulation

Separation of the Hamilton-Jacobi Equation. Carter Constant

$$\Delta\left(\frac{dS_r}{dr}\right)^2 = \frac{1}{\Delta}\left[(r^2+a^2)\varepsilon - a\ell_z\right]^2 - \left[\mathscr{C} + (\ell_z - a\varepsilon)^2 + \delta r^2\right]$$

$$\left(\frac{dS_\theta}{d\theta}\right)^2 = \mathscr{C} - \left(\frac{\ell_z^2}{\sin^2\theta} - a^2\varepsilon^2 + \delta a^2\right)\cos^2\theta$$

Hamilton-Jacobi Formulation

$$S_r = \int \frac{\sqrt{R(r')}}{\Delta} dr'$$

$$S_\theta = \int \sqrt{\Theta(\theta')} d\theta'$$

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$$R(r) = \left[(r^2 + a^2)\varepsilon - a\ell_z\right]^2 - \Delta\left[\mathscr{C} + (\ell_z - a\varepsilon)^2 + \delta r^2\right]$$
$$\Theta(\theta) = \mathscr{C} - \left[\frac{\ell_z^2}{\sin^2\theta} + a^2\left(\delta - \varepsilon^2\right)\right]\cos^2\theta$$

Carter Constant

$$\left(\frac{dS_\theta}{d\theta}\right)^2 = \mathscr{C} - \left(\frac{\ell_z^2}{\sin^2\theta} - a^2\varepsilon^2 + \delta a^2\right)\cos^2\theta$$

Carter Constant

$$\left(\frac{dS_\theta}{d\theta}\right)^2 = \mathscr{C} - \left(\frac{\ell_z^2}{\sin^2\theta} - a^2\varepsilon^2 + \delta a^2\right)\cos^2\theta$$

$$\mathscr{C} = p_\theta^2 + p_\varphi^2\cot^2\theta + a^2(\delta - \varepsilon^2)\cos^2\theta$$

Carter Constant

Schwarzschild:

$$\mathscr{C} = \left(p_\theta^2 + \frac{p_\varphi^2}{\sin^2\theta} \right) - p_\varphi^2 = \ell^2 - \ell_z^2$$

where $\ell = p_\theta^2 + \frac{p_\varphi^2}{\sin^2\theta}$ is the total angular momentum.

Carter Constant

Kerr:

Carter Constant

Kerr:

- $\mathscr{C}$ has not a direct physical interpretation.
- $\mathscr{C} = 0$ implies that the motion is in the equatorial plane.

Equations of Motion

Equations of Motion

Hamilton Canonical Equations

Equations of Motion

Hamilton Canonical Equations

$$\dot{x}^{\mu} = p^{\mu} = g^{\mu\nu} p_{\nu} = g^{\mu\nu} \frac{\partial S}{x^{\nu}}$$

Equations of Motion

$$\begin{aligned}
\varrho^2 \dot{r}^2 &= R \\
\varrho^2 \dot{\theta}^2 &= \Theta \\
\varrho \dot{\varphi} &= \frac{1}{\Delta}\left[2aMr\varepsilon + (\varrho - 2Mr)\frac{\ell_z}{\sin^2\theta}\right] \\
\varrho \dot{t} &= \frac{1}{\Delta}\left[A\varepsilon + 2aMr\ell_z\right]
\end{aligned}$$

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\begin{aligned}
R(r) &= \left[(r^2 + a^2)\varepsilon - a\ell_z\right]^2 - \Delta\left[\mathscr{C} + (\ell_z - a\varepsilon)^2 + \delta r^2\right] \\
\Theta(\theta) &= \mathscr{C} - \left[\frac{\ell_z^2}{\sin^2\theta} + a^2\left(\delta - \varepsilon^2\right)\right]\cos^2\theta \\
A &= (r^2 + a^2)^2 - a^2\Delta\sin^2\theta \\
\varrho &= r^2 + a^2\cos^2\theta \\
\Delta &= r^2 - 2Mr + a^2
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$$

Imaging a Black Hole

Image Plane of a Distant Observer

Image Plane of a Distant Observer

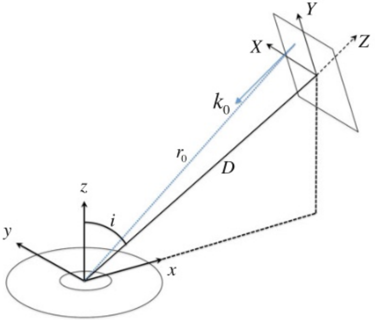
- A distant observer receives the electromagnetic radiation from the accretion disk, around the black hole.

Image Plane of a Distant Observer

- A distant observer receives the electromagnetic radiation from the accretion disk, around the black hole.
- It is usual to define a plane of observation and consider the photons with momentum orthogonal to the plane. These photons' trajectories are integrated backwards in time to find the position of the emission point in the disk.

Image Plane of a Distant Observer

Fig. 3.5 The Cartesian coordinates (x, y, z) are centered at the black hole, while the Cartesian coordinates (X, Y, Z) are for the image plane of the distant observer, who is located at the distant D from the *black hole* and with the inclination angle i. From [1]



Coordinate Systems

- (X, Y, Z): Cartesian coordinates in the image plane

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- (X, Y, Z): Cartesian coordinates in the image plane
- (x, y, z): Cartesian coordinates centered at the black hole.
- i: Inclination angle of the observer with respect to the z direction.
- D: Distance observer-black hole.

Coordinate transformations

$$x = D \sin i - Y \cos i + Z \sin i$$
$$y = X$$
$$z = D \cos i + Y \sin i + Z \cos i$$

$$r = \sqrt{x^2 + y^2 + z^2}$$
$$\theta = \arccos\left(\frac{z}{r}\right)$$
$$\varphi = \arctan\left(\frac{z}{r}\right)$$

Photon Trajectory Tracing

Photon Trajectory Tracing

Consider a photon received at $(X_0, Y_0, 0)$ with 3-momentum $\mathbf{k}_0 = -k_0\hat{Z}$, i.e. perpendicular to the observer plane.

The initial conditions for the position of the photon (to trace back the trajectory), as seen from the black hole and in spherical coordinates, are

$$t_0 = 0$$

$$r_0 = \sqrt{X_0^2 + Y_0^2 + D^2}$$

$$\theta_0 = \arccos\left(\frac{Y_0 \sin i + D\cos i}{r_0}\right)$$

$$\varphi_0 = \arctan\left(\frac{X_0}{D\sin i - Y_0 \cos i}\right)$$

Photon Trajectory Tracing

The initial conditions for the 4-momentum of the photon k^μ (to trace back the trajectory), as seen from the black hole and in spherical coordinates, are calculated with the transformation law,

$$k^\mu = \frac{\partial x^\mu}{\partial \bar{x}^\alpha} \bar{k}^\alpha$$

Photon Trajectory Tracing

The initial conditions for the 4-momentum of the photon k^μ (to trace back the trajectory), as seen from the black hole and in spherical coordinates, are calculated with the transformation law,

$$k^\mu = \frac{\partial x^\mu}{\partial \bar{x}^\alpha} \bar{k}^\alpha$$

$$k_0^r = -\frac{D}{r} k_0$$

$$k_0^\theta = \frac{\cos i - (Y_0 \sin i + D \cos i)\frac{D}{r_0^2}}{\sqrt{X_0^2 + (D \sin i - Y_0 \cos i)^2}} k_0$$

$$k_0^\varphi = \frac{X_0 \sin i}{X_0^2 + (D \sin i - Y_0 \cos i)^2} k_0$$

Photon Trajectory Tracing

The k_0^t component of the initial 4-momentum is calculated by the condition $g_{\mu\nu}g^{\mu}k'^{\nu} = 0$,

$$k_0^t = \sqrt{(k_0^r)^2 + r_0^2(k_0^\theta)^2 + r_0^2 \sin^2\theta_0 (k_0^\varphi)^2}$$

Photon Trajectory Tracing

Given the initial conditions for position and momentum, it is possible to trace the trajectory of any photon in the observer plane back to the accretion disk.

Non-Coordinate Basis

Tetrads

Introduce a non-coordinate basis or orthonormal tetrad,

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Introduce a non-coordinate basis or orthonormal tetrad,

$$\mathbf{E}_{(a)} = E^{\mu}_{(a)} \partial_{\mu}$$
$$\mathbf{E}^{(a)} = E^{(a)}_{\mu} dx^{\mu},$$

subject to the conditions

$$\eta_{(a)(b)} = E^{\mu}_{(a)} E^{\nu}_{(b)} g_{\mu\nu}$$
$$\eta^{(a)(b)} = E^{(a)}_{\mu} E^{(b)}_{\nu} g\mu\nu$$

and $det|E_{(a)^{\mu}}| > 0$ (to preserve the orientation).

Tetrads

Components of a vector in the orthonormal tetrad basis,

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Components of a vector in the orthonormal tetrad basis,

$$V^{(a)} = E^{(a)}_{\mu} V^{\mu}$$

$$V_{(a)} = E^{\mu}_{(a)} V_{\mu}$$

General Metric

Consider a general stationary, axisymmetric, asymptotically flat metric

$$ds^2 = -e^{2\alpha(r)}dt^2 + e^{2\beta(r)}dr^2 + e^{2\gamma(r,\theta)}d\theta^2 + e^{2\epsilon(r,\theta)}(d\varphi - \omega dt)^2$$

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We identify the *locally non-rotating observers* as those whose world-lines have

$$r = \text{constant}$$

$$\theta = \text{constant}$$

$$\varphi = \omega t + \text{constant}$$

General Metric

The non-coordinate basis of the locally non-rotating observers is given by the tetrad

$$E^{\mu}_{(t)} = \left(e^{-\beta}, 0, 0, \omega e^{-\beta}\right)$$
$$E^{\mu}_{(r)} = \left(0, e^{-\alpha}, 0, 0\right)$$
$$E^{\mu}_{(\theta)} = \left(0, 0, e^{-\gamma}, 0\right)$$
$$E^{\mu}_{(\varphi)} = \left(0, 0, 0, e^{-\epsilon}\right)$$

General Metric

and the dual basis,

$$
\begin{aligned}
E_{\mu}^{(t)} &= \left(e^{\beta}, 0, 0, 0\right) \\
E_{\mu}^{(r)} &= (0, e^{\alpha}, 0, 0) \\
E_{\mu}^{(\theta)} &= (0, 0, e^{\gamma}, 0) \\
E_{\mu}^{(\varphi)} &= (-\omega e^{\epsilon}, 0, 0, e^{\epsilon})
\end{aligned}
$$

Kerr Metric

For the particular case of the Kerr metric the tetrad describing the non-coordinate basis of the locally non-rotating observers is

$$E^{\mu}_{(t)} = \left(\sqrt{\frac{A}{\varrho\Delta}}, 0, 0, \frac{2Mar}{\sqrt{A\varrho\Delta}} \right)$$

$$E^{\mu}_{(r)} = \left(0, \sqrt{\frac{\Delta}{\varrho}}, 0, 0 \right)$$

$$E^{\mu}_{(\theta)} = \left(0, 0, \frac{1}{\sqrt{\varrho}}, 0 \right)$$

$$E^{\mu}_{(\varphi)} = \left(0, 0, 0, \frac{1}{\sin\theta}\sqrt{\frac{\varrho}{A}} \right)$$

Kerr Metric

and the dual basis is

$$E_\mu^{(t)} = \left(\sqrt{\frac{\varrho\Delta}{A}}, 0, 0, 0 \right)$$

$$E_\mu^{(r)} = \left(0, \sqrt{\frac{\varrho}{\Delta}}, 0, 0 \right)$$

$$E_\mu^{(\theta)} = \left(0, 0, \sqrt{\varrho}, 0 \right)$$

$$E_\mu^{(\varphi)} = \left(-\frac{2Mar\sin\theta}{\sqrt{\varrho A}}, 0, 0, \sqrt{\frac{A}{\varrho}}\sin\theta \right)$$

Momentum Components in the Non-Coordinate Basis

The momentum components of a particle moving in Kerr's spacetime are

$$p_\mu = \frac{\partial S}{\partial x^\mu}$$

Momentum Components in the Non-Coordinate Basis

The momentum components of a particle moving in Kerr's spacetime are

$$p_\mu = \frac{\partial S}{\partial x^\mu}$$

$$p_t = -\varepsilon$$

$$p_r = \frac{\sqrt{R}}{\Delta}$$

$$p_\theta = \sqrt{\Theta}$$

$$p_\varphi = \ell_z$$

Momentum Components in the Non-Coordinate Basis

In the non-coordinate basis, the momentum components of a particle are

$$p^{(a)} = E^{(a)}_{\mu} p^{\mu} = \eta^{(a)(b)} E^{\mu}_{(b)} p_{\mu}$$

Momentum Components in the Non-Coordinate Basis

In the non-coordinate basis, the momentum components of a particle are

$$p^{(a)} = E^{(a)}_{\mu} p^{\mu} = \eta^{(a)(b)} E^{\mu}_{(b)} p_{\mu}$$

$$p^{(}t) = -E^{\mu}_{(t)} p_{\mu}$$

$$p^{r} = E^{\mu}_{(r)} p_{\mu}$$

$$p^{\theta} = E^{\mu}_{(\theta)} p_{\mu}$$

$$p^{\varphi} = E^{\mu}_{(\varphi)} p_{\mu}$$

Celestial Coordinates

Celestial Coordinates

The position of a photon in the image plane of the distant observer is given by the coordinates

$$\begin{cases} X_0 & = \alpha = \lim_{r\to\infty}\left(\frac{rp^{(\varphi)}}{p^{(t)}}\right) \\ Y_0 & = \beta = \lim_{r\to\infty}\left(\frac{rp^{(\theta)}}{p^{(t)}}\right) \end{cases}$$

Celestial Coordinates

$$\begin{cases} \alpha & = -\xi \csc i \\ \beta & = \pm\sqrt{\eta + a^2 \cos^2 i - \xi^2 \cot^2 i} \end{cases}$$

Celestial Coordinates

$$\begin{cases} \alpha &= -\xi \csc i \\ \beta &= \pm\sqrt{\eta + a^2 \cos^2 i - \xi^2 \cot^2 i} \end{cases}$$

$$\begin{cases} \xi &= \frac{\ell_z}{\varepsilon} \\ \eta &= \frac{\mathscr{C}}{\varepsilon^2} \end{cases}$$

Black Hole’s Shadow

Effective Potential for Photons

$$R(r) = \left[(r^2 + a^2)\varepsilon - a\ell_z\right]^2 - \Delta\left[\mathscr{C} + (\ell_z - a\varepsilon)^2 + \delta r^2\right]$$
$$\Theta(\theta) = \mathscr{C} - \left[\frac{\ell_z^2}{\sin^2\theta} + a^2\left(\delta - \varepsilon^2\right)\right]\cos^2\theta$$

Effective Potential for Photons

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$$\Theta(\theta) = \mathscr{C} - \left[\frac{\ell_z^2}{\sin^2\theta} + a^2\left(\delta - \varepsilon^2\right)\right]\cos^2\theta$$

Consider these expressions for photons, $\delta = 0$, and using the quantities

$$\begin{cases} \xi & = \frac{\ell_z}{\varepsilon} \\ \eta & = \frac{\mathscr{C}}{\varepsilon^2} \end{cases}$$

Effective Potential for Photons

$$R(r) = \left[r^2 + a^2 - a\xi\right]^2 \varepsilon^2 - \Delta\left[\eta + (\xi - a)^2\right]\varepsilon^2$$

$$\Theta(\theta) = \eta\varepsilon^2 - \left[\frac{\xi^2}{\sin^2\theta} - a^2\right]\varepsilon^2 \cos^2\theta$$

Effective Potential for Photons

$$\mathscr{R}(r) = \frac{R(r)}{\varepsilon^2} = \left[r^2 + a^2 - a\xi\right]^2 - \Delta\left[\eta + (\xi - a)^2\right]$$

$$\vartheta(\theta) = \frac{\Theta(\theta)}{\varepsilon^2} = \left[\eta + (\xi - a)^2\right] - \left[a\sin\theta - \xi\csc\theta\right]^2$$

Effective Potential for Photons

$$\mathscr{R}(r) = \frac{R(r)}{\varepsilon^2} = \left[r^2 + a^2 - a\xi\right]^2 - \Delta\left[\eta + (\xi - a)^2\right]$$

$$\vartheta(\theta) = \frac{\Theta(\theta)}{\varepsilon^2} = \left[\eta + (\xi - a)^2\right] - \left[a\sin\theta - \xi\csc\theta\right]^2$$

$$\Delta = r^2 - 2Mr + a^2$$

Equations of Motion for Photons

$$
\begin{aligned}
\varrho^2 \dot{r}^2 &= \mathscr{R} \\
\varrho^2 \dot{\theta}^2 &= \vartheta \\
\varrho \dot{\varphi} &= \frac{1}{\Delta}\left[2aMr + \frac{\xi(\varrho - 2Mr)}{\sin^2\theta}\right] \\
\varrho \dot{t} &= \frac{1}{\Delta}\left[A + 2aMr\xi\right]
\end{aligned}
$$

Photon Sphere

Circular motion of photons: $\dot{r} = 0$

Photon Sphere

Circular motion of photons: $\dot{r} = 0$

$$\begin{cases} \mathscr{R} & = 0 \\ \partial_r \mathscr{R} & = 0 \end{cases}$$

Photon Sphere

Solving for ξ and η we obtain these quantities for the circular orbit as functions of the parameter r,

Photon Sphere

Solving for ξ and η we obtain these quantities for the circular orbit as functions of the parameter r,

$$\xi_c = \frac{M(r^2 - a^2) - r\Delta}{a(r - M)}$$

$$\eta_c = \frac{r^3\left[4M\Delta - r(r - M)^2\right]}{a^2(r - M)^2}$$

Photon Sphere

There are three possible cases regarding the stability of circular orbits of photons

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1. If $\partial_r^2 \mathscr{R} > 0$: Stable circular orbits

Photon Sphere

There are three possible cases regarding the stability of circular orbits of photons

1. If $\partial_r^2 \mathscr{R} > 0$: Stable circular orbits
2. If $\partial_r^2 \mathscr{R} < 0$: Unstable circular orbits. The photon straddles the boundary between two regions: $\partial_r \mathscr{R} = 0$; if perturbed one way it falls into the horizon, if perturbed the other way it flies outwards.

Photon Sphere

There are three possible cases regarding the stability of circular orbits of photons

1. If $\partial_r^2 \mathscr{R} > 0$: Stable circular orbits
2. If $\partial_r^2 \mathscr{R} < 0$: Unstable circular orbits. The photon straddles the boundary between two regions: $\partial_r \mathscr{R} = 0$; if perturbed one way it falls into the horizon, if perturbed the other way it flies outwards.
3. If $\partial_r^2 \mathscr{R} = 0$: Marginally stable circular orbit (Photon Sphere). $r = r_{ps}$.

Black Hole's Shadow

$$\alpha = -\xi_c \csc i$$

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$$\eta_c = \frac{r^3 \left[4M\Delta - r(r-M)^2\right]}{a^2(r-M)^2}$$

Black Hole's Shadow

Schwarzschild's Black Hole:

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Black Hole's Shadow

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Shadow of Sagittarius A*

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$D = 8 \text{ kpc}$

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Very Long Baseline Interferometry (VLBI)

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Next Lecture

08. Accretion

Alternative derivation of the Celestial Coordinates

Celestial Coordinates

- A distant observer receives the electromagnetic radiation from the surroundings of the black hole.

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- Let the observer be at $r \to \infty$ with inclination angle i (between the rotation axis of the black hole and the observer's line of sight).

Celestial Coordinates

- A distant observer receives the electromagnetic radiation from the surroundings of the black hole.
- Let the observer be at $r \longrightarrow \infty$ with inclination angle i (between the rotation axis of the black hole and the observer's line of sight).
- The celestial coordinates (α, β) are the apparent angular distances of the image on the celestial sphere measured by the observer.

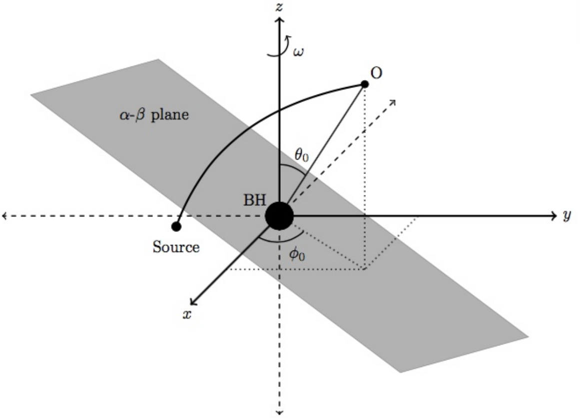
Celestial Coordinates

- The distant observer may set up a euclidean coordinate system (x, y, z) with the black hole at the origin and its rotation axis along z.

Celestial Coordinates

- The distant observer may set up a euclidean coordinate system (x, y, z) with the black hole at the origin and its rotation axis along z.
- Using the Boyer-Lindquist coordinates describing the black hole, the observer will be located at some coordinates $(r_0, \theta_0, \varphi_0)$, with r_0 very large.

Celestial Coordinates



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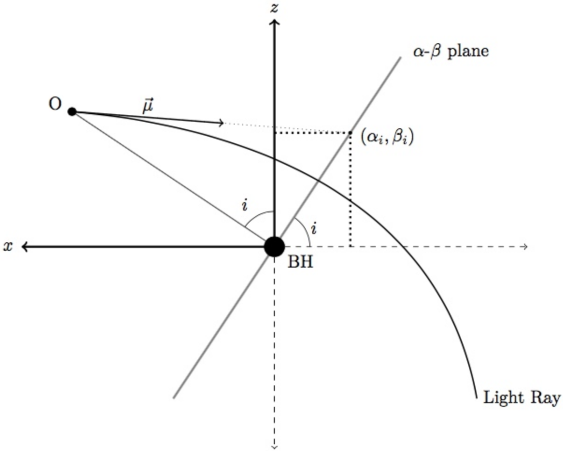
- Without loosing generality, we rotate the (x, y, z) system so that the angular position of the observer in the Boyer-Lindquist coordinates is $\varphi_0 = 0$.
- The position of the observer becomes $(r_0, i, 0)$
- Then, the observer lies in the $x - z$ plane while the y-axis lies in the (α, β) plane (remember that the observer plane is perpendicular to the line of sight).

Celestial Coordinates



Trajectory of a Light Ray

In the observer's reference frame, an incoming light ray trajectory may be decribed by a parametric curve

$$\begin{cases} X = X(r) \\ Y = Y(r) \\ Z = Z(r) \end{cases}$$

such that

$$r^2 = X^2(r) + Y^2(r) + Z^2(r)$$

Trajectory of a Light Ray

The tangent vector to this parametric curve at the observer's location is

$$\vec{\mu} = (\mu_1, \mu_2, \mu_3) = \left(\left.\frac{dX}{dr}\right|_{r_0}, \left.\frac{dY}{dr}\right|_{r_0}, \left.\frac{dZ}{dr}\right|_{r_0} \right)$$

Trajectory of a Light Ray

From the point of view of the observer, this tangent vector defines the trajectory of the photon as a straight line which intersects the observer's celestial plane at the coordinates (α_i, β_i) and can be written parametrically as

$$\frac{x - x_0}{\mu_1} = \frac{y - y_0}{\mu_2} = \frac{z - z_0}{\mu_3}$$

Trajectory of a Light Ray

(x_0, y_0, z_0) are the coordinates of the observer's position.

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The celestial coordinates (α_i, β_i) can be written as

$$(\alpha_i, \beta_i) = (x_i, y_i, z_i) = (-\beta_i \cos i, \alpha_i, \beta_i \sin i)$$

Trajectory of a Light Ray

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Trajectory of a Light Ray

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$$\frac{-\beta_i \cos i - r_0 \sin i}{\mu_1} = \frac{\alpha_i}{\mu_2} = \frac{\beta_i \sin i - r_0 \cos i}{\mu_3}$$

Trajectory of a Light Ray

Using the transformation between cartesian and spherical coordinates,

$$X(r) = r\sin\theta\cos\varphi$$
$$Y(r) = r\sin\theta\sin\varphi$$
$$Z(r) = r\cos\theta,$$

we obtain the components of $\vec{\mu}$,

$$\mu_1 = \left.\frac{dX}{dr}\right|_{r_0} = \sin i + r_0\cos i \left.\frac{d\theta}{dr}\right|_{r_0}$$
$$\mu_2 = \left.\frac{dY}{dr}\right|_{r_0} = r_0\sin i \left.\frac{d\varphi}{dr}\right|_{r_0}$$
$$\mu_3 = \left.\frac{dZ}{dr}\right|_{r_0} = \cos i - r_0\sin i \left.\frac{d\theta}{dr}\right|_{r_0}$$

Trajectory of a Light Ray

$$\frac{-\beta_i \cos i - r_0 \sin i}{\mu_1} = \frac{\alpha_i}{\mu_2} = \frac{\beta_i \sin i - r_0 \cos i}{\mu_3}$$

Trajectory of a Light Ray

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$$\frac{-\beta_i \cos i - r_0 \sin i}{\sin i + r_0 \cos i \left.\frac{d\theta}{dr}\right|_{r_0}} = \frac{\alpha_i}{r_0 \sin i \left.\frac{d\varphi}{dr}\right|_{r_0}} = \frac{\beta_i \sin i - r_0 \cos i}{\cos i - r_0 \sin i \left.\frac{d\theta}{dr}\right|_{r_0}}$$

Celestial Coordinates

$$\alpha_i = \lim_{r_0 \to \infty} -r_0^2 \sin^2\theta_0 \left.\frac{d\varphi}{dr}\right|_{r_0}$$

$$\beta_i = \lim_{r_0 \to \infty} r_0^2 \left.\frac{d\theta}{dr}\right|_{r_0}$$

Celestial Coordinates

Using the equations of motion for the photons, we obtain

$$\alpha_i = -\xi \csc i$$

$$\beta_i = \sqrt{\eta + a^2 \cos^2 i - \xi^2 \cot^2 i}$$