



**BLACK HOLES**

OBSERVATORIO  
ASTRONÓMICO  
NACIONAL

# Classical Black Holes

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## 06. Astrophysics of Supermassive Black Holes

Edward Larrañaga

# Outline for Part 1

## Astrophysical Evidence of Black Holes existence

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- **Supermassive black holes.** At the center of galaxies with masses in the range  $M \sim 10^5 - 10^{10}M_{\odot}$
- **Intermediate-mass black holes.** Objects with a masses in the range  $M \sim 10^2 - 10^4M_{\odot}$ , filling the gap between the stellar-mass and the supermassive ones.

## Origin of Black Holes

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- When a star exhausts all its nuclear fuel, it shrinks to find a new equilibrium configuration.
- The degenerate pressure of electrons can, eventually, stop the collapse to produce a white dwarf.
- If the mass of the star beginning the collapse is greater than the Chandrasekhar's Limit, the degenerate pressure of electrons can not stop the collapse.

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- The final product in this stage is thought to be a *Black Hole*.

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- Stellar-dynamical stabilities.

## A First Model of Stellar Structure

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$k$ : Boltzmann's Constant

$\mu$ : mean molecular weight

$T$ : temperature

$\rho$ : mass density

$m_p$ : mass of the proton

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The total energy of the Sun will be released in just  $10^7$  years.

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1. The source of energy is thermonuclear.
2. Outward pressure of radiation should be included in the equations.

## A Better Model of Stellar Structure

Hydrostatic Equilibrium Equation

$$\frac{d}{dr} \left[ \frac{\rho k T}{\mu m_p} + \frac{1}{3} a T^4 \right] = - \frac{G M(r) \rho(r)}{r^2}$$

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Energy Equation

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$\ell$ : mean free path of the protons

$L$ : luminosity

$\varepsilon$ : energy generated per gram of material per unit time

# Physical Aspects of the Stellar Death

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Nuclear reactions  $\rightarrow$  thermal pressure  $\rightarrow$  support gravity

For massive stars ( $M > 5M_{\odot}$ ),

$H \rightarrow He \rightarrow \dots \rightarrow C \rightarrow \dots \rightarrow Fe$

# White Dwarf. Degenerate Gas of Electrons

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$$M_C \approx 1.4M_\odot$$

Mass of a completely degenerated star.

# Neutron Stars. Degenerate Gas of Neutrons

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"...supernovae represent the transitions from ordinary stars into neutron stars which in their final stages consist of extremely closely packed neutrons."

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“...when all thermonuclear sources of energy are exhausted a sufficiently heavy star will collapse. This contraction will continue indefinitely till the radius of the star approaches asymptotically its gravitational radius. Light from the surface of the star will be progressively reddened and can escape over a progressively narrower range of angles till eventually the star tends to close itself off from any communication with a distant observer”.

# Black Holes

If the collapsing core is too massive to be supported by the degenerate pressure of neutrons, there is no known mechanism capable of finding a new equilibrium configuration, and the body should undergo a complete collapse.

In this case, the final product is a black hole.

## Outline for Part 2

# Spherically Symmetric Collapse

- Spherical Symmetry:

$$ds^2 = -e^{2\alpha(t,r)} dt^2 + e^{2\beta(t,r)} dr^2 + R^2(t,r) d\Omega^2$$

# Spherically Symmetric Collapse

- Spherical Symmetry:

$$ds^2 = -e^{2\alpha(t,r)}dt^2 + e^{2\beta(t,r)}dr^2 + R^2(t,r)d\Omega^2$$

- Suppose that the collapsing body can be described as a perfect fluid.

Then, we consider that coordinates  $t$  and  $r$  are attached to every collapsing particle (co-moving coordinates).

## Spherically Symmetric Collapse

- In the co-moving frame, the 4-velocity of the fluid is just

$$u^\mu = (e^{-\alpha}, 0, 0, 0)$$

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- The energy-momentum tensor in the co-moving frame is

$$T^\mu_\nu = \text{diag}(\rho, P, P, P)$$

# Spherically Symmetric Collapse

The Einstein equations for this metric give

$$\begin{aligned} G_t^t = 8\pi T_t^t &\Rightarrow \frac{F'}{R^2 R'} = 8\pi\rho \\ G_r^r = 8\pi T_r^r &\Rightarrow \frac{F}{R^2 \dot{R}} = -8\pi P \\ G_r^t = 0 &\Rightarrow \dot{R}' - \dot{R}\alpha' - \dot{\beta}R' = 0 \end{aligned}$$



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$$F = R \left( 1 - e^{-2\beta} R'^2 + e^{-2\alpha} \dot{R}^2 \right)$$

Misner-Sharp mass

# Spherically Symmetric Collapse

The Misner-Sharp mass

$$F = R \left( 1 - e^{-2\beta} R'^2 + e^{-2\alpha} \dot{R}^2 \right)$$

is defined by the relation

$$1 - \frac{F}{R} = g_{\mu\nu} (\partial^\mu R) (\partial^\nu R)$$

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\* Note that  $n^\mu = \partial^\mu R$  is normal to the surfaces  $R = \text{constant}$ .  
Thus, when  $1 - \frac{F}{R} = 0$ , the corresponding surface is null.

# Spherically Symmetric Collapse

From  $(t, t)$ -component of the Field equations we obtain the Misner-Sharp mass as

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$$F(r) = \int_0^r F' d\tilde{r} = 8\pi \int_0^r \rho R^2 R' d\tilde{r} = 2M(r)$$

# Spherically Symmetric Collapse

Finally, the conservation of the energy-momentum tensor gives the equation

$$\nabla_{\mu} T^{\mu}_{\nu} = 0 \Rightarrow \alpha' = -\frac{p'}{\rho + p}$$

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Then

$$F = F(r)$$

$$\alpha = \alpha(t)$$

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\* If  $P \neq 0$  the exterior metric must be a non-vacuum Vaidya spacetime.

# Dust Collapse

$\alpha = \alpha(t)$ : One can re-define the time coordinate such that

$$e^{\alpha(t)} dt \Rightarrow dt$$

and therefore  $g_{tt} = -1$

## Dust Collapse

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Introducing  $f(r) = e^{2h(r)} - 1$  we have

$$e^{2\beta} = \frac{R'^2}{1+f}$$

# Dust Collapse

$$ds^2 = -dt^2 + \frac{R'^2(t, r)}{1 + f(r)} dr^2 + R^2(t, r) d\Omega^2$$

Lemaitre-Tolman-Bondi Metric

# Dust Collapse

Kretschmann Scalar:

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Divergence at  $R = 0$



## Dust Collapse

One can re-scale to set  $R(t, r)$  to equal the co-moving radius  $r$  at the time  $t = 0$ , i.e. we impose  $R(0, r) = r$  and introduce a scale factor  $a$  such that

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The collapse also impose the condition  $\dot{a} < 0$

# Dust Collapse

One of the field equations,

$$\frac{F'}{R^2 R'} = 8\pi\rho$$

implies that a regular value of the density at  $t = 0$  is obtained if the Misner-Sharp mass has the form

$$F(r) = r^3 m(r)$$

where  $m(r)$  is a (sufficiently) regular function in the region  $[0, r_b]$ .

## Dust Collapse

We have  $F' = 3r^2m + r^3m'$  and therefore

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$$\frac{3m + rm'}{a^2(a + ra')} = 8\pi\rho$$

# Dust Collapse

It is usual to consider  $m(r)$  as a polynomial around  $r = 0$ ,

$$m(r) = \sum_{k=0}^{\infty} m_k r^k$$

where  $m_k$  are constants



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Note that the function  $f$  defines the initial velocity of the particles in the cloud. It is usual to write this function as an expansion around  $r = 0$  as

$$f(r) = r^2 b(r)$$

with

$$b(r) = \sum_{k=0}^{\infty} b_k r^k$$

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This implies that

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Then

$$\dot{a} = -\sqrt{\frac{m_0}{a} + b_0}$$

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The Lemaitre-Tolman-Bondi line element becomes

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$$ds^2 = -dt^2 + \frac{a^2(t)}{1 + r^2 b_0} dr^2 + r^2 a^2(t) d\Omega^2$$

# Homogeneous Dust Collapse

The Lemaitre-Tolman-Bondi line element becomes

$$ds^2 = -dt^2 + \frac{R'^2(t, r)}{1 + f(r)} dr^2 + R^2(t, r) d\Omega^2$$

$$ds^2 = -dt^2 + \frac{a^2(t)}{1 + r^2 b_0} dr^2 + r^2 a^2(t) d\Omega^2$$

$$ds^2 = -dt^2 + a^2(t) \left[ \frac{dr^2}{1 + b_0 r^2} + r^2 d\Omega^2 \right]$$

## Homogeneous Dust Collapse

$$ds^2 = -dt^2 + a^2(t) \left[ \frac{dr^2}{1 + b_0 r^2} + r^2 d\Omega^2 \right]$$

For  $b_0 = 0$  this line element becomes

$$ds^2 = -dt^2 + a^2(t) [dr^2 + r^2 d\Omega^2]$$

and describes the counterpart of a flat spacetime, in which the collapse is marginally bound (i.e. the falling particles have vanishing velocity at infinity).

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$$\frac{2}{3}a^{3/2} - \frac{2}{3}a^{3/2}(0) = -\sqrt{m_0}t$$

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The value of the initial condition was supposed as  $a(0) = 1$ . Thus

$$a(t) = \left[ 1 - \frac{3\sqrt{m_0}}{2}t \right]^{2/3}$$



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Therefore, the singularity  $R = 0$  occurs at the time  $t_s$  for which  $a(t_s) = 0$ . This is

$$t_s = \frac{2}{3\sqrt{m_0}}$$

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The curve  $t_H(r)$  describing the time at which the shell  $r$  crosses the horizon can be obtained from the condition

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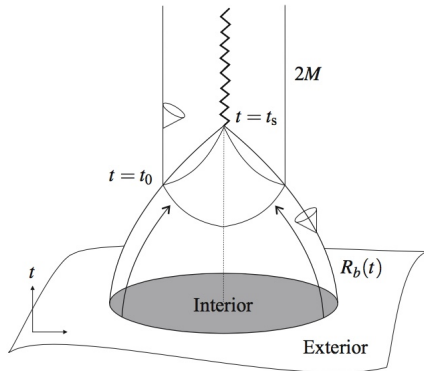
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$$t_H(r) = \frac{2}{3\sqrt{m_0}} - \frac{2}{3} m_0 r^3$$

# Gravitational Collapse of a Homogeneous Cloud of Dust

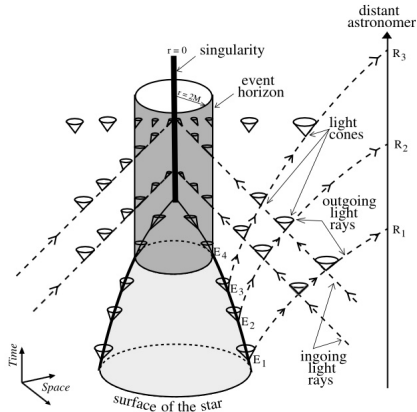
## *Eddington-Finkelstein Diagram*





# Gravitational Collapse of a Homogeneous Cloud of Dust

## *Eddington-Finkelstein Diagram*



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$$\rho = \rho(t, r)$$

$$m = m(r)$$

$$b = b(r)$$

$$a = a(t, r)$$

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\*  $m_1 = 0$  in order to have a non divergent density.



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gives this time

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This equation shows how each shell of the distribution (each  $r$ ) collapses with a different scale factor and with a different velocity.



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## Inhomogeneous Dust Collapse

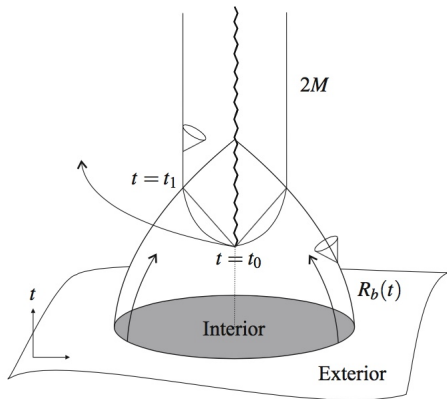
Using  $m(r) = m_0 + m_2 r^2$  gives the functions

$$t_s(r) = \frac{2}{3\sqrt{m_0 + m_2 r^2}}$$

$$t_H(r) = \frac{2}{3\sqrt{m_0 + m_2 r^2}} - \frac{2}{3} r^3 (m_0 + m_2 r^2)$$

# Gravitational Collapse of a Inhomogeneous Cloud of Dust

## *Eddington-Finkelstein Diagram*



Next Lecture

## 06. Black Holes Astrophysics