BLACK HOLES

OBSERVATORIO
ASTRONÓMICO
NACIONAL

Classical Black Holes

07. Geodesics around a Rotating Black Hole

Edward Larrañaga

Outline for Part 1

Particle Motion around a Black Hole

Lagrangian Formulation

Lagrangian Formulation

Stationary and axis-symmetric spacetime

$$ds^2 = g_{tt}dt^2 + 2g_{t\varphi}dtd\varphi + g_{rr}dr^2 + g_{\theta\theta}d\theta^2 + g_{\varphi\varphi}d\varphi^2$$

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$$\dot{x}^\mu = \frac{dx^\mu}{d\lambda}$$

Lagrangian Formulation

$$\mathscr{L} = \frac{1}{2} g_{\mu\nu} \dot{x}^{\mu} \dot{x}^{\nu} = \frac{1}{2} \left(\frac{ds}{d\lambda} \right)^2$$

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$$\delta = \begin{cases} 0 & \text{for photons} \\ -1 & \text{for massive particles} \end{cases}$$

Conserved Quantities

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$$\mathscr{L} = \frac{1}{2}\left[g_{tt}\dot{t}^2 + 2g_{t\varphi}\dot{t}\dot{\varphi} + g_{rr}\dot{r}^2 + g_{\theta\theta}\dot{\theta}^2 + g_{\varphi\varphi}\dot{\varphi}^2\right] = \frac{1}{2}\delta$$

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$$p_t = \frac{\partial\mathscr{L}}{\partial\dot{t}} = \left[g_{tt}\dot{t} + g_{t\varphi}\dot{\varphi}\right] = -\varepsilon$$
$$p_\varphi = \frac{\partial\mathscr{L}}{\partial\dot{\varphi}} = \left[g_{t\varphi}\dot{t} + g_{\varphi\varphi}\dot{\varphi}\right] = \ell_z$$

Conserved Quantities

$$\varepsilon = \frac{E}{m_0} = -g_{tt}\dot{t} - g_{t\varphi}\dot{\varphi}$$

$$\ell_z = \frac{L_z}{m_0} = g_{t\varphi}\dot{t} + g_{\varphi\varphi}\dot{\varphi}$$

Conserved Quantities

$$\dot{t} = \frac{\varepsilon g_{\varphi\varphi} + \ell_z g_{t\varphi}}{g_{t\varphi}^2 - g_{tt} g_{\varphi\varphi}}$$

$$\dot{\varphi} = -\frac{\varepsilon g_{t\varphi} + \ell_z g_{tt}}{g_{t\varphi}^2 - g_{tt} g_{\varphi\varphi}}$$

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$$V_{eff}(r,\theta) = \frac{\varepsilon^2 g_{\varphi\varphi} + 2\varepsilon\ell_z g_{t\varphi} + \ell_z^2 g_{tt}}{g_{t\varphi}^2 - g_{tt}g_{\varphi\varphi}} + \delta$$

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$$\frac{d}{d\lambda}\left(\frac{\partial \mathscr{L}}{\partial \dot{x}^{\alpha}}\right) = \frac{\partial \mathscr{L}}{\partial x^{\alpha}}$$

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Equatorial Motion

Effective Potential in the Equatorial Plane

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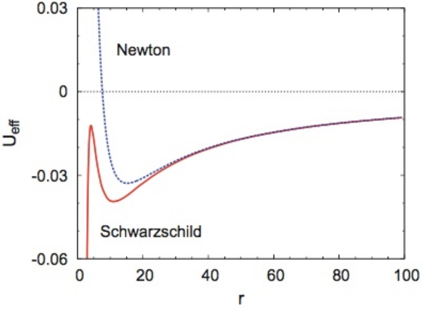
$-\frac{M\ell_z^2}{r^3}$: Relativistic Contribution. Responsible for the ISCO

Schwarzschild Effective Potential in the Equatorial Plane

Fig. 3.1 Effective potential $U_{\mathrm{eff}}(r)$ for a test-particle moving in the gravitational field of a Schwarzschild *black hole* (*red solid curve*) and of a point-like mass in Newtonian gravity (*blue dashed curve*). Here $L_z = 3.9\,M$ and $M = 1$. See the text for more details



Equatorial Circular Orbits

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which imply

$$\begin{cases} V_{eff} = 0 \\ \partial_r V_{eff} = 0 \end{cases}$$

Equation of a Circular Motion in the Equatorial Plane

Another way to calculate the equation of a circular motion in the equatorial plane is taking

$$\frac{d}{d\lambda}\left(g_{rr}\dot{r}\right) = \frac{1}{2}\left[\partial_r g_{tt}\dot{t}^2 + 2\partial_r g_{t\varphi}\dot{t}\dot{\varphi} + \partial_r g_{rr}\dot{r}^2 + \partial_r g_{\theta\theta}\dot{\theta}^2 + \partial_r g_{\varphi\varphi}\dot{\varphi}^2\right]$$

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$$\partial_r g_{tt}\dot{t}^2 + 2\partial_r g_{t\varphi}\dot{t}\dot{\varphi} + \partial_r g_{\varphi\varphi}\dot{\varphi}^2 = 0$$

Angular Velocity of a Particle in Circular Motion in the Equatorial Plane

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$$\Omega = \frac{\dot{\varphi}}{\dot{t}} = \frac{-\partial_r g_{t\varphi} \pm \sqrt{\left(\partial_r g_{t\varphi}\right)^2 - \left(\partial_r g_{tt}\right)\left(\partial_r g_{\varphi\varphi}\right)}}{\partial_r g_{\varphi\varphi}}$$

Conserved Quantities for a Particle in Circular Motion

$$\mathscr{L} = \frac{1}{2}\left[g_{tt}\dot{t}^2 + 2g_{t\varphi}\dot{t}\dot{\varphi} + g_{rr}\dot{r}^2 + g_{\theta\theta}\dot{\theta}^2 + g_{\varphi\varphi}\dot{\varphi}^2\right] = \frac{1}{2}\delta$$

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Considering $\dot{r} = \dot{\theta} = 0$,

$$\dot{t} = \sqrt{\frac{\delta}{g_{tt} + 2g_{t\varphi}\Omega + g_{\varphi\varphi}\Omega^2}}$$

Conserved Quantities for a Particle in Circular Motion

$$\varepsilon = -\left(g_{tt} + \Omega g_{t\varphi}\right)\sqrt{\frac{\delta}{g_{tt} + 2g_{t\varphi}\Omega + g_{\varphi\varphi}\Omega^2}}$$

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Quantities for a Particle in Circular Motion in Kerr Spacetime

$$\varepsilon = \frac{r^{3/2} - 2Mr^{1/2} \pm aM^{1/2}}{r^{3/4}\sqrt{r^{3/2} - 3Mr^{1/2} \pm 2aM^{1/2}}}$$

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Upper sign: co-rotating orbit

Lower sign: counter-rotating orbit

Photon Sphere

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This occurs at the surface called *phton sphere*, with radius $r = r_{ps}$ such that

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Extreme Kerr $(M = a)$:

$$r_{ps} = \begin{cases} M & \text{co-rotating orbit} \\ 4M & \text{counter-rotating orbit} \end{cases}$$

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The condition $\varepsilon = 1$ defines the marginally bound circular orbit radius, $r = r_{mb}$.

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$$\varepsilon = -\left(g_{tt} + \Omega g_{t\varphi}\right)\sqrt{-\frac{1}{g_{tt} + 2g_{t\varphi}\Omega + g_{\varphi\varphi}\Omega^2}} = 1$$

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Extreme Kerr $\left(M = a\right)$:

$$r_{mb} = \begin{cases} M & \text{co-rotating orbit} \\ 5.83M & \text{counter-rotating orbit} \end{cases}$$

Marginally Stable Orbit. ISCO

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Therefore, the ISCO radius corresponds to the inner edge of thin accretion disks (such as in the Novikov-Thorne model)

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$Z_1 = M + (M^2 - a^2)^{1/3}[(M + a)^{1/3} + (M - a)^{1/3}]$

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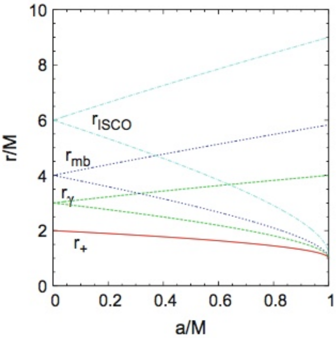
Extreme Kerr $(M = a)$:

$$r_{ISCO} = \begin{cases} M & \text{co-rotating orbit} \\ 9M & \text{counter-rotating orbit} \end{cases}$$

Important Radii for Circular Orbits in the Equatorial Plane of Kerr Spacetime

Fig. 3.4 Radial coordinates of the event horizon r_+, of the photon orbit r_γ, of the marginally bound circular orbit $r_{\rm mb}$, and of the ISCO $r_{\rm ISCO}$ in the Kerr metric in Boyer–Lindquist coordinates as functions of a/M. For every radius, the upper curve refers to the counterrotating orbit, the lower curve to the corotating orbit



Outline for Part 2

Hamilton-Jacobi Formulation

Hamilton-Jacobi Formulation

$$\mathscr{L} = \frac{1}{2} g_{\mu\nu} \dot{x}^{\mu} \dot{x}^{\nu}$$

Hamilton-Jacobi Formulation

$$\mathscr{L} = \frac{1}{2} g_{\mu\nu} \dot{x}^{\mu} \dot{x}^{\nu}$$

$$p_{\mu} = \frac{\partial \mathscr{L}}{\partial \dot{x}^{\mu}} = g_{\mu\nu} \dot{x}^{\nu}$$

Hamilton-Jacobi Formulation

$$\mathscr{L} = \frac{1}{2} g_{\mu\nu} \dot{x}^{\mu} \dot{x}^{\nu}$$

$$p_{\mu} = \frac{\partial \mathscr{L}}{\partial \dot{x}^{\mu}} = g_{\mu\nu} \dot{x}^{\nu}$$

$$\mathscr{H} = \frac{1}{2} g^{\mu\nu} p_{\mu} p_{\nu}$$

Hamilton-Jacobi Formulation

Hamilton's principal function

$$S = S\left(x^{\mu}; \lambda\right)$$

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Hamilton-Jacobi Equation

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Hamilton's principal function

$$S = S\left(x^{\mu}; \lambda\right)$$

$$p_{\mu} = \frac{\partial S}{\partial x^{\mu}}$$

Hamilton-Jacobi Equation

$$\frac{1}{2} g^{\mu\nu} \frac{\partial S}{\partial x^{\mu}} \frac{\partial S}{\partial x^{\nu}} - \frac{\partial S}{\partial \lambda} = 0$$

Kerr's Solution

Boyer-Lindquist coordinates: (t, r, θ, φ)

$$ds^2 = -\frac{\Delta - a^2 \sin^2\theta}{\varrho} dt^2 - \left(\frac{r^2 + a^2 - \Delta}{\varrho} \right) 2a \sin^2\theta dt d\varphi$$

$$+ \frac{\varrho}{\Delta} dr^2 + \varrho d\theta^2 + \left(\frac{\left(r^2 + a^2\right)^2 - \Delta a^2 \sin^2\theta}{\varrho} \right) \sin^2\theta d\varphi^2.$$

Kerr's Solution

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$$\varrho = r^2 + a^2 \cos^2\theta$$

$$\Delta = r^2 - 2Mr + a^2$$

Kerr's Solution

$$\left(\frac{\partial}{\partial s}\right)^2 = -\frac{A}{\varrho\Delta}\left(\frac{\partial}{\partial t}\right)^2 - \frac{4aMr}{\varrho\Delta}\left(\frac{\partial}{\partial t}\right)\left(\frac{\partial}{\partial \varphi}\right) + \frac{\Delta}{\varrho}\left(\frac{\partial}{\partial r}\right)^2 + \frac{1}{\varrho}\left(\frac{\partial}{\partial \theta}\right)^2 + \frac{\Delta - a^2\sin^2\theta}{\varrho\Delta\sin^2\theta}\left(\frac{\partial}{\partial \varphi}\right)^2$$

Kerr's Solution

$$\left(\frac{\partial}{\partial s}\right)^2 = -\frac{A}{\varrho\Delta}\left(\frac{\partial}{\partial t}\right)^2 - \frac{4aMr}{\varrho\Delta}\left(\frac{\partial}{\partial t}\right)\left(\frac{\partial}{\partial \varphi}\right) + \frac{\Delta}{\varrho}\left(\frac{\partial}{\partial r}\right)^2 + \frac{1}{\varrho}\left(\frac{\partial}{\partial \theta}\right)^2 + \frac{\Delta - a^2\sin^2\theta}{\varrho\Delta\sin^2\theta}\left(\frac{\partial}{\partial \varphi}\right)^2$$

$$A = (r^2 + a^2)^2 - a^2\Delta\sin^2\theta$$

$$\varrho = r^2 + a^2\cos^2\theta$$

$$\Delta = r^2 - 2Mr + a^2$$

Hamilton-Jacobi Formulation

Hamilton-Jacobi Equation

$$2\frac{\partial S}{\partial \lambda} = g^{\mu\nu}\frac{\partial S}{\partial x^{\mu}}\frac{\partial S}{\partial x^{\nu}}$$

Hamilton-Jacobi Formulation

Hamilton-Jacobi Equation

$$2\frac{\partial S}{\partial \lambda} = g^{\mu\nu}\frac{\partial S}{\partial x^\mu}\frac{\partial S}{\partial x^\nu}$$

$$\begin{aligned} 2\frac{\partial S}{\partial \lambda} = &-\frac{A}{\varrho\Delta}\left(\frac{\partial S}{\partial t}\right)^2 - \frac{4aMr}{\varrho\Delta}\left(\frac{\partial S}{\partial t}\right)\left(\frac{\partial S}{\partial \varphi}\right) + \frac{\Delta}{\varrho}\left(\frac{\partial S}{\partial r}\right)^2 \\ &+ \frac{1}{\varrho}\left(\frac{\partial S}{\partial \theta}\right)^2 + \frac{\Delta - a^2\sin^2\theta}{\varrho\Delta\sin^2\theta}\left(\frac{\partial S}{\partial \varphi}\right)^2 \end{aligned}$$

Hamilton-Jacobi Formulation

Hamilton Principal Function

$$S = \frac{1}{2}\lambda\delta - \varepsilon t + \ell_z\varphi + S_r(\theta) + S_\theta(\theta)$$

Separation of the Hamilton-Jacobi Equation. Carter Constant

$$\Delta\left(\frac{dS_r}{dr}\right)^2 - \frac{1}{\Delta}\left[(r^2+a^2)\varepsilon - a\ell_z\right]^2 + (\ell_z - a\varepsilon)^2 + \delta r^2 =$$
$$-\left(\frac{dS_\theta}{d\theta}\right)^2 - \left(\frac{\ell_z^2}{\sin^2\theta} - a^2\varepsilon^2 + \delta a^2\right)\cos^2\theta = \mathscr{C}$$

Hamilton-Jacobi Formulation

Separation of the Hamilton-Jacobi Equation. Carter Constant

$$\Delta\left(\frac{dS_r}{dr}\right)^2 = \frac{1}{\Delta}\left[(r^2+a^2)\varepsilon - a\ell_z\right]^2 - \left[\mathscr{C} + (\ell_z - a\varepsilon)^2 + \delta r^2\right]$$

$$\left(\frac{dS_\theta}{d\theta}\right)^2 = \mathscr{C} - \left(\frac{\ell_z^2}{\sin^2\theta} - a^2\varepsilon^2 + \delta a^2\right)\cos^2\theta$$

Hamilton-Jacobi Formulation

$$S_r = \int \frac{\sqrt{R(r')}}{\Delta} dr'$$

$$S_\theta = \int \sqrt{\Theta(\theta')} d\theta'$$

Hamilton-Jacobi Formulation

$$S_r = \int \frac{\sqrt{R(r')}}{\Delta} dr'$$

$$S_\theta = \int \sqrt{\Theta(\theta')} d\theta'$$

$$R(r) = \left[(r^2 + a^2)\varepsilon - a\ell_z\right]^2 - \Delta\left[\mathscr{C} + (\ell_z - a\varepsilon)^2 + \delta r^2\right]$$

$$\Theta(\theta) = \mathscr{C} - \left[\frac{\ell_z^2}{\sin^2\theta} + a^2\left(\delta - \varepsilon^2\right)\right]\cos^2\theta$$

Carter Constant

$$\left(\frac{dS_\theta}{d\theta}\right)^2 = \mathscr{C} - \left(\frac{\ell_z^2}{\sin^2\theta} - a^2\varepsilon^2 + \delta a^2\right)\cos^2\theta$$

Carter Constant

$$\left(\frac{dS_\theta}{d\theta}\right)^2 = \mathscr{C} - \left(\frac{\ell_z^2}{\sin^2\theta} - a^2\varepsilon^2 + \delta a^2\right)\cos^2\theta$$

$$\mathscr{C} = p_\theta^2 + p_\varphi^2 \cot^2\theta + a^2(\delta - \varepsilon^2)\cos^2\theta$$

Carter Constant

Schwarzschild:

$$\mathscr{C} = \left(p_\theta^2 + \frac{p_\varphi^2}{\sin^2\theta} \right) - p_\varphi^2 = \ell^2 - \ell_z^2$$

where $\ell = p_\theta^2 + \frac{p_\varphi^2}{\sin^2\theta}$ is the total angular momentum.

Carter Constant

Kerr:

Carter Constant

Kerr:

- $\mathscr{C}$ has not a direct physical interpretation.
- $\mathscr{C} = 0$ implies that the motion is in the equatorial plane.

Equations of Motion

Equations of Motion

Hamilton Canonical Equations

Equations of Motion

Hamilton Canonical Equations

$$\dot{x}^{\mu} = p^{\mu} = g^{\mu\nu} p_{\nu} = g^{\mu\nu} \frac{\partial S}{x^{\nu}}$$

Equations of Motion

$$
\begin{aligned}
\varrho^2 \dot{r}^2 &= R \\
\varrho^2 \dot{\theta}^2 &= \Theta \\
\varrho \dot{\varphi} &= \frac{1}{\Delta}\left[2aMr\varepsilon + (\varrho - 2Mr)\frac{\ell_z}{\sin^2\theta}\right] \\
\varrho \dot{t} &= \frac{1}{\Delta}\left[A\varepsilon + 2aMr\ell_z\right]
\end{aligned}
$$

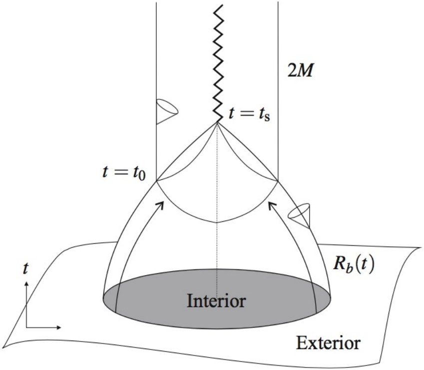
Equations of Motion

$$\varrho^2 \dot{r}^2 = R$$
$$\varrho^2 \dot{\theta}^2 = \Theta$$
$$\varrho \dot{\varphi} = \frac{1}{\Delta} \left[2aMr\varepsilon + (\varrho - 2Mr) \frac{\ell_z}{\sin^2 \theta} \right]$$
$$\varrho \dot{t} = \frac{1}{\Delta} \left[A\varepsilon + 2aMr\ell_z \right]$$

$$R(r) = \left[(r^2 + a^2)\varepsilon - a\ell_z \right]^2 - \Delta \left[\mathscr{C} + (\ell_z - a\varepsilon)^2 + \delta r^2 \right]$$
$$\Theta(\theta) = \mathscr{C} - \left[\frac{\ell_z^2}{\sin^2 \theta} + a^2 \left(\delta - \varepsilon^2 \right) \right] \cos^2 \theta$$
$$A = (r^2 + a^2)^2 - a^2 \Delta \sin^2 \theta$$
$$\varrho = r^2 + a^2 \cos^2 \theta$$
$$\Delta = r^2 - 2Mr + a^2$$

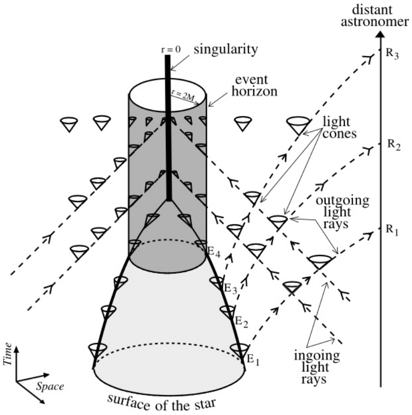
Gravitational Collapse of a Homogeneous Cloud of Dust

Eddington-Finkelstein Diagram



Gravitational Collapse of a Homogeneous Cloud of Dust

Eddington-Finkelstein Diagram



Inhomogeneous Dust Collapse

Inhomogeneous Dust Collapse

ρ depends on both t and r, and thus

Inhomogeneous Dust Collapse

ρ depends on both t and r, and thus

$$\rho = \rho(t, r)$$
$$m = m(r)$$
$$b = b(r)$$
$$a = a(t, r)$$

Next Lecture

06. Black Holes Astrophysics