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DISK-ACCRETION ONTO A BLACK HOLE. I. TIME-AVERAGED STRUCTURE OF ACCRETION DISK*†

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ABSTRACT

An analysis is given of the time-averaged structure of a disk of material accreting onto a black hole. The analysis is valid even if the disk is highly dynamical. It assumes only that the hole is stationary and axially symmetric (e.g., that it is a Kerr hole); that the disk lies in the equatorial plane of the hole, with its material moving in nearly geodesic circular orbits; that the disk is thin; and that radial heat transport is negligible compared with heat losses through the surface of the disk. The most important result of the analysis is an explicit, algebraic expression for the radial dependence of the time-averaged energy flux emitted from the disk's surface, $F(r)$.

Subject headings: binaries — black holes

I. INTRODUCTION

It now seems probable that some compact X-ray sources are binary systems consisting of a normal star that dumps material onto a companion black hole. The most popular current models for the mass transfer (Pringle and Rees 1972; Shakura and Sunyaev 1973; review in Novikov and Thorne 1973) presume that the transferred material forms a thin disk around the hole. Viscous stresses (magnetic and/or turbulent) transfer angular momentum outward through the disk, thereby allowing the material to spiral gradually inward. The viscous stresses, working against the disk's differential rotation, heat the disk, causing it to emit a large flux of X-rays.

Disk accretion onto a black hole may also occur in the nuclei of galaxies (Lynden-Bell 1969; Lynden-Bell and Rees 1971; review in Novikov and Thorne 1973). In this case the hole is envisaged as supermassive ($M \sim 10^7$ to 10^{11} $M_\odot$), and the accreting material is interstellar gas and magnetic fields. The models predict significant radiation in the ultraviolet, optical, infrared, and radio regions of the spectrum, but not much X-rays.

Thus far all models for disk accretion onto black holes have been steady-state models, or at least quasi–steady-state. In this paper we ask the question: How much can be learned about the time-averaged behavior of a highly dynamical accreting disk by application of the laws of conservation of rest mass, angular momentum, and energy? (Of course, our results are also applicable to steady-state disks and quasi–steady-state disks.) We shall find that the conservation laws yield an explicit algebraic expression for the time-averaged energy flux, $F(r)$, emitted by the disk's surface, as a function of radius, and also an explicit algebraic expression for the time-averaged torque in the disk, $W_\varphi{}^r(r)$.

The precise assumptions that underlie these expressions are spelled out in § II; the expressions for $F(r)$ and $W_\varphi{}^r(r)$ are presented in § III*a* and are derived in §§ III*b* and III*c*; and some implications of these expressions for steady-state disk models are spelled out in § IV. Throughout we shall use the notation of Novikov and Thorne (1973) and of Misner, Thorne, and Wheeler (1973)—including units with $c = G = k = 1$ (k = Boltzmann constant) —except where typography limitations force changes. The main change is the use of a dagger ($E^\dagger$ and $L^\dagger$), where previous usage would be a tilde ($\tilde{E}$ and $\tilde{L}$).

II. ASSUMPTIONS AND NOTATION FOR THE ANALYSIS

In analyzing the time-averaged structure of the accretion disk, we make the following assumptions and use the following notation.

i) *Assumption*: The black hole has an external spacetime geometry in which the disk, with negligible self-gravity, resides. The external geometry is stationary, axially symmetric, asymptotically flat, and reflection-symmetric in an equatorial plane. (At the end of the analysis, and only there, we shall specialize our formulae to the Kerr geometry.)

* This paper and its companion (Thorne 1974) were cited in previous writings (e.g., in Misner, Thorne, and Wheeler 1973 ["MTW"] and in Novikov and Thorne 1973) as "K. S. Thorne, Black-Hole Models for Compact X-ray Sources, *Ap. J.*, in preparation (1973)." The research reported in these papers was completed in late 1972; it was reported by KST at the Texas Symposium on Relativistic Astrophysics in New York City, 1972 December 21, and at a variety of subsequent meetings in 1973; but it was not written up for publication until this late date (1973 November) because of a complete preoccupation with the proofs of MTW.

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Notation: In and near the equatorial plane we introduce coordinates t ('time"), r ("radius"), z ("height above equatorial plane"), φ ("azimuthal angle"), with respect to which the metric reads

$$ds^2 = -e^{2\nu}dt^2 + e^{2\psi}(d\varphi - \omega dt)^2 + e^{2\mu}dr^2 + dz^2\,; \tag{1a}$$

$$\nabla^2 = -e^{-2\nu}\left(\frac{\partial}{\partial t} + \omega\frac{\partial}{\partial\varphi}\right)^2 + e^{-2\psi}\left(\frac{\partial}{\partial\varphi}\right)^2 + e^{-2\mu}\left(\frac{\partial}{\partial r}\right)^2 + \left(\frac{\partial}{\partial z}\right)^2; \tag{1b}$$

$$\nu,\ \psi,\ \mu,\ \omega \text{ are functions of } r \text{ only}\,; \tag{1c}$$

$$\text{as } r\to\infty\,, \quad \psi = \ln r + O(1/r)\,, \quad \nu \sim \mu \sim O(1/r)\,, \quad \omega \sim O(1/r^3)\,. \tag{1d}$$

Note that

$$(-g)^{1/2} \equiv (-\det\|g_{\alpha\beta}\|)^{1/2} = e^{\nu+\psi+\mu}\,. \tag{1e}$$

As one moves out of the equatorial plane, the metric coefficients acquire corrections of order $(z/r)^2$. (All such corrections will be ignored in this paper.) For proofs of the existence of such a coordinate system see Papapetrou (1966), Kundt and Trümper (1966), Carter (1969, 1970).

ii) *Assumption*: The central plane of the disk lies in the equatorial plane of the black hole.

iii) *Assumption*: The disk is thin; i.e., at radius r its thickness $\Delta z = 2h$ is always much less than r. This permits us to use the metric in its near-equatorial-plane form (1), with ν, ψ, μ, ω independent of z.

iv) *Assumption*: There exists a time interval Δt which (a) is small enough that during Δt the external geometry of the hole changes negligibly; but (b) is large enough that, for any radius r of interest, the total mass that flows inward across r during Δt is large compared with the typical mass contained between r and $2r$. *Notation*: by $\langle\ \rangle$ we denote an average over angle $\Delta\varphi = 2\pi$ and over time Δt:

$$\langle\Psi(z, r)\rangle \equiv (2\pi\Delta t)^{-1}\int_0^{\Delta t}\int_0^{2\pi}\Psi(t, r, z, \varphi)d\varphi dt\,. \tag{2}$$

If Ψ is a tensor field, it is to be Lie-dragged along $\partial/\partial t$ and $\partial/\partial\varphi$ during the averaging process. Equivalently, its components in the t, r, z, φ coordinate system are to be averaged.

v) *Notation*: The "local rest frame" of the baryons at an event $\mathscr{P}_0$ (the frame in which there is no net spatial baryon flux) has a 4-velocity $\boldsymbol{u}^{\text{inst}}(\mathscr{P}_0)$ ("inst" means "instantaneous"). When mass-averaged over φ and Δt and height, this 4-velocity is denoted

$$\boldsymbol{u}(r) \equiv (1/\Sigma)\int_{-H}^{+H}\langle\rho_0\boldsymbol{u}^{\text{inst}}\rangle dz\,. \tag{3}$$

Here ρ_0 is the density of rest mass (number density of baryons n multiplied by a standard constant, mean rest mass per baryon) as measured in the instantaneous local rest frame; Σ is the time-averaged surface density,

$$\Sigma(r) \equiv \int_{-H}^{+H}\langle\rho_0\rangle dz\,; \tag{4}$$

and H is the maximum half-thickness of the disk during the time Δt,

$$H \equiv \max_{\Delta t}\,(h)\,. \tag{5}$$

Without making any assumptions about the types of stress-energy present (magnetic fields, viscous stresses, etc.), we algebraically decompose the stress-energy tensor $\boldsymbol{T}$ with respect to the 4-velocity field $\boldsymbol{u}$:

$$\boldsymbol{T} = \rho_0(1 + \Pi)\boldsymbol{u}\otimes\boldsymbol{u} + \boldsymbol{t} + \boldsymbol{u}\otimes\boldsymbol{q} + \boldsymbol{q}\otimes\boldsymbol{u}\,, \tag{6a}$$

$$\Pi = \text{"specific internal energy,"} \tag{6b}$$

$$\boldsymbol{t} = \text{"stress tensor in averaged rest frame" is a second-rank, symmetric tensor orthogonal to } \boldsymbol{u},\ \boldsymbol{t}\cdot\boldsymbol{u} = \boldsymbol{u}\cdot\boldsymbol{t} = 0\,, \tag{6c}$$

$$\boldsymbol{q} = \text{"energy-flow vector" is a 4-vector orthogonal to } \boldsymbol{u},\ \boldsymbol{q}\cdot\boldsymbol{u} = 0\,. \tag{6d}$$

We use units in which $c = G = k$ (Boltzmann constant) $= 1$.

vi) *Assumption*: When mass-averaged over φ, Δt, and height, the baryons move very nearly in equatorial, circular, geodesic orbits about the black hole. Thus,

$$\boldsymbol{u}(r) \simeq \boldsymbol{w}(r) \equiv \text{(four-velocity for a circular geodesic orbit in the equatorial plane)}\,. \tag{7a}$$

Such orbits have specific energy-at-infinity $E^{\dagger}$, specific angular momentum $L^{\dagger}$, and angular velocity Ω given by

$$E^{\dagger}(r) \equiv -w_t(r)\,, \qquad L^{\dagger}(r) \equiv w_{\varphi}(r)\,, \qquad \Omega(r) \equiv w^{\varphi}/w^t\,. \tag{7b}$$

Consequence of above assumption.—Physically, the mean motion can be nearly geodesic only if radial pressure forces are negligible compared with the gravitational pull of the hole:

$$\text{(radial accelerations due to pressure gradients)} \sim \left|\frac{t_{rr,r}}{\rho_0}\right| \sim \left|\left(\frac{t_{rr}}{\rho_0}\right)_{,r}\right|$$
$$\ll \text{(gravitational acceleration of hole)} \sim |E^{\dagger}{}_{,r}| = |(1 - E^{\dagger})_{,r}|\,.$$

Integrating this inequality and using the relation (valid for any astrophysical material)

$$\text{(internal energy density)} \equiv \rho_0 \Pi \sim |t_{rr}|\,,$$

we see that

$$\Pi \ll 1 - E^{\dagger}\,. \tag{8}$$

We call this the "condition of negligible specific heat." It says that the internal energy is negligible compared with the gravitational potential energy. In other words, as the material of the disk spirals slowly inward, releasing gravitational energy, a negligible amount of the energy released is stored internally. Almost all energy is transported away or radiated away. In terms of temperatures, condition (8) says

$$\Pi \sim T/m_p \sim T/10^{13} K \ll 1 - E^{\dagger} \sim M/r\,,$$

where M is the mass of the hole and m_p is the mass of a proton.

vii) *Assumption*: Heat flow within the disk is negligible, except in the vertical direction; i.e.,

$$\langle \boldsymbol{q}(r, z)\rangle \approx \langle q^z(r, z)\rangle(\partial/\partial z)\,. \tag{9a}$$

(This is a reasonable assumption in view of the thinness of the disk.)

viii) *Assumption*: The only time-averaged stress-energy that reaches out of the faces of the disk is that carried by photons. (This assumption is meant to rule out gravitational waves as well as extended magnetic fields. If magnetic fields bulge out of the disk, but do not extend to heights $|z| \sim r$, then one can "redefine them into the disk" by making the "official" disk thickness, $2H$, large enough to enclose them.) Moreover, essentially all the stress-energy carried off is borne by photons of wavelength $\lambda \ll M$ = (size of hole).[1] (This allows one to neglect coherent superposition of the radiation reaction in adjacent [different r] regions of the disk, and to neglect "black-hole superradiance effects.") In addition—and as a corollary of (9a)—the photons emitted from the disk's surface are emitted, on the average, vertically as seen in the mean local rest frame of the orbiting gas. This, together with our neglect of (typically nonvertical) reimpinging radiation—see below—means that

$$\langle t_{\varphi}{}^z\rangle = \langle t_t{}^z\rangle = \langle t_r{}^z\rangle = \langle q_{\varphi}\rangle = \langle q_t\rangle = \langle q_r\rangle = 0 \qquad \text{at } z = \pm H\,. \tag{9b}$$

ix) *Assumption*: One can neglect energy and momentum transport from one region of the disk to another by photons emitted from the disk's surface. (This assumption is *not very reasonable*; heating of the outer regions by X-rays from the inner regions may be rather important—see Shakura and Sunyaev 1973. And in the inner regions, $M \lesssim r \lesssim 10M$, intense gravitational fields may pull a non-negligible fraction of the emitted photons back onto the disk. The effects of this are currently being studied by Cunningham 1974 and by Polnarev 1974.)

III. TIME-AVERAGED RADIAL DISK STRUCTURE

a) Summary of Results

By combining the assumptions of § II with the laws of conservation of rest mass, angular momentum, and energy (§ III*c*, below) one can derive three important equations for the time-averaged radial structure of the disk. These are equations for three quantities:

$$\dot{M}_0 \equiv \frac{dM_0}{dt} \equiv \begin{pmatrix}\text{radius-independent, time-averaged rate [rate measured}\\ \text{in terms of group-theoretically defined coordinate}\\ \text{time } t\text{] at which rest mass flows inward through disk}\end{pmatrix}, \tag{10a}$$

[1] We thank Douglas M. Eardley for pointing out to us the need for this assumption.

$$F(r) \equiv \langle q^z(r, z = H)\rangle = \langle -q^z(r, z = -H)\rangle$$

$$= \begin{pmatrix}\text{time-averaged flux of radiant energy [energy per unit proper}\\ \text{time } \tau \text{ per unit proper area } A] \text{ flowing out of upper face}\\ \text{of disk, as measured by an observer on the upper face who}\\ \text{orbits with the time-averaged motion of the disk's matter}\end{pmatrix}$$

$$= (\text{time-averaged flux flowing out of lower face})\,, \tag{10b}$$

$$W_\varphi{}^r(r) \equiv \int_{-H}^{+H} \langle t_\varphi{}^r\rangle dz$$

$$= (g^{rr})^{1/2} \times \begin{pmatrix}\text{time-averaged torque per unit circumference acting across}\\ \text{a cylinder at radius } r, \text{ due to the stresses in the disk}\end{pmatrix}. \tag{10c}$$

The equations derived are

$$\dot{M}_0 = -2\pi e^{\nu+\psi+\mu}\Sigma u^r\,, \tag{11a}$$

$$F(r) = (\dot{M}_0/4\pi)e^{-(\nu+\psi+\mu)}f\,, \tag{11b}$$

$$W_\varphi{}^r = (\dot{M}_0/2\pi)e^{-(\nu+\psi+\mu)}[(E^\dagger - \Omega L^\dagger)/(-\Omega_{,r})]f\,. \tag{11c}$$

Here ν, ψ, μ are the metric coefficients (functions of r) of equation (1); $E^\dagger$, $L^\dagger$, Ω are the specific energy-at-infinity specific angular momentum, and angular velocity (functions of r) defined in equations (7a, b); Σ and u^r are the surface density and radial velocity (unknown functions of r) defined in equations (3) and (4); and f is the function of radius

$$f \equiv -\Omega_{,r}(E^\dagger - \Omega L^\dagger)^{-2}\int_{r_{\rm ms}}^{r} (E^\dagger - \Omega L^\dagger)L^\dagger{}_{,r}dr$$

$$= -(w^t{}_{,r}/w_\varphi)\int_{r_{\rm ms}}^{r} (w_{\varphi,r}/w^t)dr\,. \tag{12}$$

Here $r_{\rm ms}$ ("ms" = "marginally stable") is the radius of the innermost stable circular geodesic orbit:

$$r_{\rm ms} = (\text{radius at which } dE^\dagger/dr = dL^\dagger/dr = 0)\,. \tag{13}$$

The derivation of equations (11) will be presented in §§ III*b* and III*c*.

When specialized to the Kerr metric, the above functions have the following forms. We express them in terms of

M = mass of black hole,

a = specific angular momentum of hole ($a > 0$ if disk orbits in same direction as hole rotates; $a < 0$ if it orbits in opposite direction),

$a_* \equiv a/M$ (note: $-1 \le a_* \le +1$),

$x \equiv (r/M)^{1/2}$ = (dimensionless radial coordinate),

$x_0 = (r_{\rm ms}/M)^{1/2}$,

$x_1, x_2, x_3 \equiv$ the three roots of $x^3 - 3x + 2a_* = 0$; in particular,

$$x_1 = 2\cos\left(\tfrac{1}{3}\cos^{-1} a_* - \pi/3\right),$$

$$x_2 = 2\cos\left(\tfrac{1}{3}\cos^{-1} a_* + \pi/3\right),$$

$$x_3 = -2\cos\left(\tfrac{1}{3}\cos^{-1} a_*\right).$$

$$\mathscr{A} = 1 + a_*^2x^{-4} + 2a_*^2x^{-6}\,, \qquad \mathscr{B} = 1 + a_*x^{-3}\,,$$

$$\mathscr{C} = 1 - 3x^{-2} + 2a_*x^{-3}\,, \qquad \mathscr{D} = 1 - 2x^{-2} + a_*^2x^{-4}\,,$$

$$\mathscr{E} = 1 + 4a_*^2x^{-4} - 4a_*^2x^{-6} + 3a_*^4x^{-8}\,, \qquad \mathscr{F} = 1 - 2a_*x^{-3} + a_*^2x^{-4}\,,$$

$$\mathscr{G} = 1 - 2x^{-2} + a_*x^{-3}\,. \tag{14}$$

They are (cf. Bardeen, Press, and Teukolsky 1972)

$$e^{2\nu} = \mathscr{A}^{-1}\mathscr{D}\,, \qquad e^{2\psi} = M^2x^4\mathscr{A}\,, \qquad e^{2\mu} = \mathscr{D}^{-1}\,, \tag{15a, b, c}$$

$$e^{\nu+\mu+\psi} = r = Mx^2\,, \qquad \omega = 2a_*M^{-1}x^{-6}\mathscr{A}^{-1}\,, \qquad \Omega = M^{-1}x^{-3}\mathscr{B}^{-1}\,, \tag{15d, e, f}$$

$$E^\dagger = \mathscr{C}^{-1/2}\mathscr{G}\,, \qquad L^\dagger = Mx\mathscr{C}^{-1/2}\mathscr{F}\,, \tag{15g, h}$$

$$E^\dagger - \Omega L^\dagger = \mathscr{B}^{-1}\mathscr{C}^{1/2}\,, \qquad \Omega_{,r} = -\tfrac{3}{2}M^{-2}x^{-5}\mathscr{B}^{-2}\,; \tag{15i, j}$$

$$x_0 = \{3 + Z_2 - \operatorname{sgn}(a_*)[(3 - Z_1)(3 + Z_1 + 2Z_2)]^{1/2}\}^{1/2}\,, \qquad \text{where} \tag{15k}$$

$$Z_1 \equiv 1 + (1 - a_*^2)^{1/3}[(1 + a_*)^{1/3} + (1 - a_*)^{1/3}]\,, \qquad Z_2 \equiv (3a_*^2 + Z_1^2)^{1/2}\,; \tag{15l, m}$$

$$\begin{aligned} f = \frac{3}{2M}\frac{1}{x^2(x^3 - 3x + 2a_*)}\Bigg[x - x_0 - \tfrac{3}{2}a_*\ln\left(\frac{x}{x_0}\right) - \frac{3(x_1 - a_*)^2}{x_1(x_1 - x_2)(x_1 - x_3)}\ln\left(\frac{x - x_1}{x_0 - x_1}\right) \\ - \frac{3(x_2 - a_*)^2}{x_2(x_2 - x_1)(x_2 - x_3)}\ln\left(\frac{x - x_2}{x_0 - x_2}\right) - \frac{3(x_3 - a_*)^2}{x_3(x_3 - x_1)(x_3 - x_2)}\ln\left(\frac{x - x_3}{x_0 - x_3}\right)\Bigg]\,. \end{aligned} \tag{15n}$$

The rest of § III is a derivation of the radial-structure formulae (11), beginning (§ III*b*) with a crucial relation for geodesic orbits, and then turning (§ III*c*) to formulation and manipulation of the conservation laws.

b) The Energy–Angular-Momentum Relation for Circular Geodesic Orbits

Consider circular geodesic orbits in the equatorial plane of metric (1) with four-velocities $\boldsymbol{w}(r)$ having nonzero components given by equation (7b). By combining the geodesic equation and the normalization condition,

$$\nabla_{\boldsymbol{w}}\boldsymbol{w} = 0\,, \qquad \boldsymbol{w}\cdot\boldsymbol{w} = -1\,, \tag{16}$$

with the symmetry of the covariant derivative and the vanishing of the radial component w_r, one obtains

$$\begin{aligned} 0 &= [\nabla_{\boldsymbol{w}}\boldsymbol{w} - \tfrac{1}{2}\nabla(\boldsymbol{w}\cdot\boldsymbol{w})]\cdot(\partial/\partial r) \equiv w^\alpha w_{r;\alpha} - w^\alpha w_{\alpha;r} = w^\alpha(w_{r,\alpha} - w_{\alpha,r}) \\ &= -w^\alpha w_{\alpha,r} = -w^t w_{t,r} - w^\varphi w_{\varphi,r} = w^t(E^\dagger{}_{,r} - \Omega L^\dagger{}_{,r})\,. \end{aligned} \tag{17}$$

Hence, the circular geodesic orbits satisfy the fundamental relation

$$E^\dagger{}_{,r} = \Omega L^\dagger{}_{,r}\,. \tag{18}$$

This is a special case of the universal "energy–angular-momentum relation," $dE = \Omega dJ$ or

$$\text{(change of energy)} = \text{(angular velocity)}\cdot\text{(change of angular momentum)}\,, \tag{19}$$

which plays a fundamental role throughout astrophysics. (See, e.g., Appendix B of Ostriker and Gunn 1969; eq. [80] of Bardeen 1970; Hartle 1970; and § 10.7 of Zel'dovich and Novikov 1971.)

c) Formulation and Manipulation of the Conservation Laws

The radial structure of the disk is governed by three conservation laws: conservation of rest mass, of angular momentum, and of energy.

In differential form the law of rest-mass conservation reads

$$\nabla\cdot(\rho_0\boldsymbol{u}^{\text{inst}}) = 0\,. \tag{20}$$

We convert to a more useful integral conservation law by integrating over the 3-volume of the disk between radius r and $r + \Delta r$ and over time Δt, and by then using Gauss's theorem to convert to a surface integral:

$$\begin{aligned} 0 &= \int_{\mathscr{V}} \nabla\cdot(\rho_0\boldsymbol{u}^{\text{inst}})(-g)^{1/2}dt\,dr\,dz\,d\varphi = \int_{\partial\mathscr{V}} \rho_0\boldsymbol{u}^{\text{inst}}\cdot d^3\boldsymbol{\Sigma} \\ &= \left[\int_{-H}^{+H}\int_t^{t+\Delta t}\int_0^{2\pi} \rho_0 u^r{}_{\text{inst}}(-g)^{1/2}d\varphi\,dt\,dz\right]_r^{r+\Delta r} + [\text{total rest mass in the 3-volume}]_t^{t+\Delta t} \\ &= (\Delta t)(2\pi e^{\nu+\psi+\mu}\Sigma u^r)_{,r}\Delta r + 0\,. \end{aligned}$$

The second bracket, [ ], can be neglected compared with the first because of assumption (iv) of § II (mass in Δr negligible compared with mass that flows across r in time Δt). Physically the above equation says

$$\dot{M}_0 \equiv -2\pi e^{\nu+\psi+\mu}\Sigma u^r = \text{(time-averaged rate of accretion of rest mass) is independent of radius } r\,. \tag{21}$$

This is the first of our radial structure equations, equation (11a).

In differential form the law of angular-momentum conservation reads

$$\nabla\cdot\boldsymbol{J} = 0\,, \qquad \boldsymbol{J} \equiv \boldsymbol{T}\cdot\partial/\partial\varphi = \text{(density-flux 4-vector for angular momentum)}\,. \tag{22}$$

Again we convert to an integral conservation law by integrating over the 3-volume between radius r and $r + \Delta r$ and over time Δt, and by then using Gauss's theorem to convert to a surface integral. In the case of rest mass there was no flux across the upper and lower faces of the disk ($z = \pm H$), so the only contributions to the surface integral were at the outer and inner radii, $r + \Delta r$ and r, and at the hypersurfaces of constant time, $t + \Delta t$ and t. However, radiation pouring out of the disk produces an angular-momentum flux across the upper and lower faces, so in this case we get six terms in the surface integral:

$$\begin{aligned} 0 &= \int_{\mathscr{V}} \nabla\cdot\boldsymbol{J}(-g)^{1/2}dtdrdzd\varphi = \int_{\partial\mathscr{V}} \boldsymbol{J}\cdot d^3\boldsymbol{\Sigma} = \int_{\partial\mathscr{V}} T_\varphi{}^\alpha d^3\Sigma_\alpha \\ &= \left\{\int_{-H}^{+H}\int_t^{t+\Delta t}\int_0^{2\pi}[\rho_0(1+\Pi)u_\varphi u^r + t_\varphi{}^r + u_\varphi q^r + q_\varphi u^r](-g)^{1/2}d\varphi dt dz\right\}_r^{r+\Delta r} \\ &\quad + \left\{\int_r^{r+\Delta r}\int_t^{t+\Delta t}\int_0^{2\pi}[\rho_0(1+\Pi)u_\varphi u^z + t_\varphi{}^z + u_\varphi q^z + q_\varphi u^z](-g)^{1/2}d\varphi dt dr\right\}_{-H}^{+H} \\ &\quad + \{\text{total angular momentum in the 3-volume}\}_t^{t+\Delta t}\,. \end{aligned} \tag{23}$$

In the first brace, { }, we can ignore Π (negligible specific heat; eq. [8]); and we can ignore $u_\varphi q^r$ and $q_\varphi u^r$ by comparison with the $u_\varphi q^z$ of the second brace (negligible heat transport along the plane of the disk; eq. [9a]). Hence, the first brace reduces to

$$\begin{aligned} \left\{\int_{-H}^{+H}(2\pi\Delta t)[\langle\rho_0\rangle u_\varphi u^r + \langle t_\varphi{}^r\rangle](-g)^{1/2}dz\right\}_r^{r+\Delta r} &= \{(2\pi\Delta t)[\Sigma L^\dagger u^r + W_\varphi{}^r]e^{\nu+\psi+\mu}\}_r^{r+\Delta r} \\ &= \Delta t[-\dot{M}_0 L^\dagger + 2\pi e^{\nu+\psi+\mu}W_\varphi{}^r]_{,r}\Delta r\,. \end{aligned} \tag{24}$$

Here we have used equations (4), (7), and (21), and the definition

$$W_\alpha{}^\beta \equiv \int_{-H}^{+H}\langle t_\alpha{}^\beta\rangle dz\,. \tag{25}$$

In the second brace { } of formula (23) the first and last terms vanish because $u^z = 0$, and the second term vanishes by equation (9b). Hence, the second brace becomes

$$\left\{\int_r^{r+\Delta r}(2\pi\Delta t)u_\varphi\langle q^z\rangle(-g)^{1/2}dr\right\}_{-H}^{+H} = 2\Delta t[2\pi e^{\nu+\psi+\mu}L^\dagger F]\Delta r\,, \tag{26}$$

where we have used equations (7a, b), plus definition (10b) of F. The third brace { } in formula (23) can be neglected compared with the first brace because of assumption (iv). Combining a zero value for the third brace with equation (26) for the second brace and equation (24) for the first brace, we obtain

$$[\dot{M}_0 L^\dagger - 2\pi e^{\nu+\psi+\mu}W_\varphi{}^r]_{,r} = 4\pi e^{\nu+\psi+\mu}FL^\dagger\,. \tag{27}$$

This is our final form for the law of angular momentum conservation. The first term represents the angular momentum carried by the rest mass of the disk; the second term is the angular momentum transported mechanically by torques in the disk (by viscous stresses, by turbulent stresses, by magnetic stresses, etc.); the third term is the angular momentum carried away from the disk's surface by radiation.

The differential form of the law of energy conservation is

$$\nabla\cdot\boldsymbol{E} = 0\,, \qquad \boldsymbol{E} \equiv -\boldsymbol{T}\cdot\partial/\partial t = \text{(density-flux 4-vector for energy-at-infinity)}\,. \tag{28}$$

By manipulating this conservation law in precisely the same manner as we manipulated the law of angular momentum conservation (22), we arrive at the time-averaged and volume-integrated conservation law

$$[\dot{M}_0 E^\dagger + 2\pi e^{\nu+\psi+\mu}W_t{}^r]_{,r} = 4\pi e^{\nu+\psi+\mu}FE^\dagger\,. \tag{29}$$

The second term can be rewritten in terms of $W_\varphi{}^r$ by use of the orthogonality relation $u^\alpha t_\alpha{}^\beta = 0$, which implies that $u^\alpha W_\alpha{}^\beta = 0$, or

$$W_t{}^r = -(u^\varphi/u^t)W_\varphi{}^r = -\Omega W_\varphi{}^r .$$

The result for equation (29) is

$$[\dot{M}_0 E^\dagger - 2\pi e^{\nu+\psi+\mu} W_\varphi{}^r \Omega]_{,r} = 4\pi e^{\nu+\psi+\mu} F E^\dagger . \tag{30}$$

d) Integration of the Conservation Laws

Equations (27) and (30) can be integrated to obtain the emitted flux F and the torque per unit circumference $W_\varphi{}^r$. This is done as follows:

i) Change variables to

$$f \equiv 4\pi e^{\nu+\psi+\mu} F/\dot{M}_0 , \qquad w \equiv 2\pi e^{\nu+\psi+\mu} W_\varphi{}^r/\dot{M}_0 . \tag{31a, b}$$

ii) In terms of these variables the conservation laws (27) and (30) become

$$(L^\dagger - w)_{,r} = fL^\dagger , \qquad (E^\dagger - \Omega w)_{,r} = fE^\dagger . \tag{32a, b}$$

iii) Multiply (32a) by Ω, subtract from (32b), and use the "energy–angular-momentum relation" (18) to obtain the algebraic relation

$$w = [(E^\dagger - \Omega L^\dagger)/(-\Omega_{,r})]f . \tag{33}$$

iv) Insert this expression for w into (32a), and integrate the resulting first-order differential equation for f, making use of (18). The result is

$$\frac{(E^\dagger - \Omega L^\dagger)^2}{-\Omega_{,r}} f = \int (E^\dagger - \Omega L^\dagger) L^\dagger{}_{,r} dr + \text{const.}$$

v) To fix the constant of integration, use the following physical fact: When the accreting material reaches the innermost stable circular orbit, $r = r_{\rm ms}$, it drops out of the disk and falls directly down the hole. Hence, just inside $r = r_{\rm ms}$ there is negligible material to "torque up" the material just outside $r = r_{\rm ms}$—which means that the torque $W_\varphi{}^r$, and hence w, must vanish at $r = r_{\rm ms}$.[2] To make $w(t_{\rm ms})$ vanish we must choose our constant of integration such that

$$\frac{(E^\dagger - \Omega L^\dagger)^2}{-\Omega_{,r}} f = \int_{r_{\rm ms}}^{r} (E^\dagger - \Omega L^\dagger) L^\dagger{}_{,r} dr .$$

vi) Bring this result into the following alternative forms by integration by parts and use of the energy–angular-momentum relation (18):

$$\begin{aligned} f &= -\Omega_{,r}(E^\dagger - \Omega L^\dagger)^{-2} \int_{r_{\rm ms}}^{r} (E^\dagger - \Omega L^\dagger) L^\dagger{}_{,r} dr \\ &= -\Omega_{,r}(E^\dagger - \Omega L^\dagger)^{-2}\left[E^\dagger L^\dagger - E^\dagger{}_{\rm ms} L^\dagger{}_{\rm ms} - 2\int_{r_{\rm ms}}^{r} L^\dagger E^\dagger{}_{,r} dr\right] \\ &= -\Omega_{,r}(E^\dagger - \Omega L^\dagger)^{-2}\left[-E^\dagger L^\dagger + E^\dagger{}_{\rm ms} L^\dagger{}_{\rm ms} + 2\int_{r_{\rm ms}}^{r} E^\dagger L^\dagger{}_{,r} dr\right] . \end{aligned} \tag{34}$$

Equation (34) for f and equations (31), (33) for F and $W_\varphi{}^r$ are the radial-structure equations (12) and (11b, c) quoted in § III*a*.

IV. STEADY-STATE DISK MODELS

Steady-state relativistic models for the accretion disk around a Kerr black hole have been built by Novikov and Thorne (1973). These models are patterned after Newtonian models by Shakura and Sunyaev. They include details of vertical structure (vertical force balance; vertical energy transport; etc.), as well as details of radial structure.

[2] It is conceivable that the disk material might contain extremely strong magnetic fields, and that these fields might transport a torque from the infalling material at $r < r_{\rm ms}$ to the disk at $r \geq r_{\rm ms}$. In this case the boundary condition at $r_{\rm ms}$ would be modified, and the solution for f would be changed. It seems to us unlikely that the changes would be substantial, except very near $r_{\rm ms}$ (i.e., at $r - r_{\rm ms} \lesssim 0.1 r_{\rm ms}$). But when constructing explicit disk models, one should examine this possibility carefully.

All of the quantities appearing in the Novikov-Thorne models (§§ 5.9 and 5.10 of their paper) are expressed as explicit, algebraic functions of radius, except one: the function $\mathscr{Q}(r)$. The results of this paper allow one to also express $\mathscr{Q}$ as an explicit algebraic function of r—or, equivalently, of $x = (r/M)^{1/2}$. Direct comparison of equations (5.6.14b) and (5.4.1b, c) of Novikov-Thorne with equations (11b) and (15d, n) of this paper shows that

$$\mathscr{Q} = \frac{1 + a_* x^{-3}}{(1 - 3x^{-2} + 2a_* x^{-3})^{1/2}} \frac{1}{x} \left[x - x_0 - \tfrac{3}{2} a_* \ln\left(\frac{x}{x_0}\right) - \frac{3(x_1 - a_*)^2}{x_1(x_1 - x_2)(x_1 - x_3)} \ln\left(\frac{x - x_1}{x_0 - x_1}\right) - \frac{3(x_2 - a_*)^2}{x_2(x_2 - x_1)(x_2 - x_3)} \ln\left(\frac{x - x_2}{x_0 - x_2}\right) - \frac{3(x_3 - a_*)^2}{x_3(x_3 - x_1)(x_3 - x_2)} \ln\left(\frac{x - x_3}{x_0 - x_3}\right) \right]. \tag{35}$$

(NOTE.—In equation [5.4.1h] of Novikov and Thorne there is an error, pointed out to us by Chris Cunningham: the sign in the exponential should be plus rather than minus.)

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