**BLACK HOLES**

OBSERVATORIO
ASTRONÓMICO
NACIONAL

# Classical Black Holes

## 06. Astrophysics of Supermassive Black Holes

**Edward Larrañaga**

# Outline for Part 1

# Supermassive Black Holes

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In the Milky Way, the central black hole, called Sagittarius $A^*$ has a mass $\sim 4 \times 10^6 M_\odot$ .

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- Dark Matter collapse
- Primordial fluctuations

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After that, Galaxies formed around these early black holes.

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The gas accumulated in the center producea very massive central object producing a black hole with a mass of $\sim 10^4 M_\odot$ or higher.

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If fragmentation occurs, star formation in a collapsing cloud of gas produces a compact nuclear stellar cluster.

Collisions of stars in the cluster are frequent and they can produce a black hole of $\sim 10^2 - 10^4 M_\odot$

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No heating in the collapse until the rate of annihilation of dark particles becomes significant.

This process create an object called a *Dark Matter Star* for which the annihilation radiation supports the gravitational collapse.

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When the dark matter is completely annihilated, the object will collapse into a massive black hole.

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At the regions where the density fluctuations are large enough, the gravitational force may overcome the pressure, giving as a result a complete collapse.

This process create primordial black holes with masses up to $\sim 1000 M_{\odot}$ .

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However, accretion from diffuse matter is to slow to grow an IMBH on the Hubble time. Thus, they may form in dense clusters.

They can also be created at the center of stellar clusters by merger of different kinds of objects, or they can be a supermassive black hole in the process of growth.

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Thus, the conclusion from these observations is that there are intermediate-mass black holes or there are some stellar black holes having non-isotropical accretion luminosity exceeding the Eddington limit.

# Outline for Part 2

# Spherically Symmetric Collapse

- Spherical Symmetry:

$$ds^2 = -e^{2\alpha(t,r)}dt^2 + e^{2\beta(t,r)}dr^2 + R^2(t,r)d\Omega^2$$

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$$ds^2 = -e^{2\alpha(t,r)}dt^2 + e^{2\beta(t,r)}dr^2 + R^2(t,r)d\Omega^2$$

- Suppose that the collapsing body can be described as a perfect fluid.
  Then, we consider that coordinates $t$ and $r$ are attached to every collapsing particle (co-moving coordinates).

## Spherically Symmetric Collapse

- In the co-moving frame, the 4-velocity of the fluid is just

$$u^{\mu} = \left(e^{-\alpha}, 0, 0, 0\right)$$

and thus $u^2 = -1$.

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- The energy-momentum tensor in the co-moving frame is

$$T^{\mu}_{\nu} = diag(\rho, P, P, P)$$

# Spherically Symmetric Collapse

The Einstein equations for this metric give

$$
\begin{aligned}
G^t_t = 8\pi T^t_t \quad &\Rightarrow \quad \frac{F'}{R^2 R'} = 8\pi\rho \\
G^r_r = 8\pi T^r_r \quad &\Rightarrow \quad \frac{\dot{F}}{R^2 \dot{R}} = -8\pi P \\
G^t_r = 0 \quad &\Rightarrow \quad \dot{R}' - \dot{R}\alpha' - \dot{\beta} R' = 0
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$$F = R\left(1 - e^{-2\beta} R'^2 + e^{-2\alpha} \dot{R}^2\right)$$

Misner-Sharp mass

## Spherically Symmetric Collapse

The Misner-Sharp mass

$$F = R\left(1 - e^{-2\beta} R'^2 + e^{-2\alpha} \dot{R}^2\right)$$

is defined by the relation

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* Note that $n^\mu = \partial^\mu R$ is normal to the surfaces $R = \mathrm{constant}$.
Thus, when $1 - \frac{F}{R} = 0$, the corresponding surface is null.

# Spherically Symmetric Collapse

From $(t, t)$-component of the Field equations we obtain the Misner-Sharp mass as

$$G_t^t = 8\pi T_t^t \Rightarrow \frac{F'}{R^2 R'} = 8\pi\rho$$

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$$F(r) = \int_0^r F' d\tilde{r} = 8\pi \int_0^r \rho R^2 R' d\tilde{r} = 2M(r)$$

# Spherically Symmetric Collapse

Finally, the conservation of the energy-momentum tensor gives the equation

$$\nabla_\mu T^\mu_\nu = 0 \Rightarrow \alpha' = -\frac{P'}{\rho + P}$$

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Then

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$$\alpha = \alpha(t)$$

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If the boundary of the cloud of dust is located at the co-moving coordinate $r = r_b$, we have $F(r_b) = 2M$ with $M$ the Schwarzschild's mass of the exterior metric.

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* If $P \neq 0$ the exterior metric must be a non-vacuum Vaidya spacetime.

# Dust Collapse

$\alpha = \alpha(t)$: One can re-define the time coordinate such that

$$e^{\alpha(t)} dt \Rightarrow dt$$

and therefore $g_{tt} = -1$

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Introducing $f(r) = e^{2h(r)} - 1$ we have

$$e^{2\beta} = \frac{R'^2}{1+f}$$

# Dust Collapse

$$ds^2 = -dt^2 + \frac{R'^2(t,r)}{1+f(r)}dr^2 + R^2(t,r)d\Omega^2$$

Lemaitre-Tolman-Bondi Metric

# Dust Collapse

Kretschmann Scalar:

$$K = 12\frac{F'^2}{R^4 R'^2} - 32\frac{FF'}{R^5 R'} + 48\frac{F^2}{R^6}$$

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Divergence at $R = 0$

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One can re-scale to set $R(t, r)$ to equal the co-moving radius $r$ at the time $t = 0$, i.e. we impose $R(0, r) = r$ and introduce a scale factor $a$ such that

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The collapse also impose the condition $\dot{a} < 0$

# Dust Collapse

One of the field equations,

$$\frac{F'}{R^2 R'} = 8\pi\rho$$

implies that a regular value of the density at $t = 0$ is obtained if the Misner-Sharp mass has the form

$$F(r) = r^3 m(r)$$

where $m(r)$ is a (sufficiently) regular function in the region $[0, r_b]$.

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$$\frac{3m + rm'}{a^2(a + ra')} = 8\pi\rho$$

# Dust Collapse

It is usual to consider $m(r)$ as a polynomial around $r = 0$,

$$m(r) = \sum_{k=0}^{\infty} m_k r^k$$

where $m_k$ are constants

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$$\dot{a} = -\sqrt{\frac{m}{a} + \frac{f}{r^2}}$$

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Evaluating at the time $t = 0$ we have $a(0, r) = 1$ and thus

$$\dot{a}(0, r) = -\sqrt{m + \frac{f}{r^2}}$$

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$$\dot{a}(0, r) = -\sqrt{m + \frac{f}{r^2}}$$

Note that the function $f$ defines the initial velocity of the particles in the cloud. It is usual to write this function as an expansion around $r = 0$ as

$$f(r) = r^2 b(r)$$

with

$$b(r) = \sum_{k=0}^{\infty} b_k r^k$$

# Homogeneous Dust Collapse

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Simplest Model of Gravitational Collapse. Oppenheimer-Snyder (1939)

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which is accomplished by choosing $b = b_0$
Then

$$\dot{a} = -\sqrt{\frac{m_0}{a} + b_0}$$

# Homogeneous Dust Collapse

The Lemaitre-Tolman-Bondi line element becomes

$$ds^2 = -dt^2 + \frac{R'^2(t,r)}{1+f(r)}dr^2 + R^2(t,r)d\Omega^2$$

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$$ds^2 = -dt^2 + a^2(t)\left[\frac{dr^2}{1+b_0 r^2} + r^2 d\Omega^2\right]$$

## Homogeneous Dust Collapse

$$ds^2 = -dt^2 + a^2(t)\left[\frac{dr^2}{1 + b_0 r^2} + r^2 d\Omega^2\right]$$

For $b_0 = 0$ this line element becomes

$$ds^2 = -dt^2 + a^2(t)\left[dr^2 + r^2 d\Omega^2\right]$$

and describes the counterpart of a flat spacetime, in which the collapse is marginally bound (i.e. the falling particles have vanishing velocity at infinity).

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For $b_0 = 0$ the scale factor is

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$$\sqrt{a}da = -\sqrt{m_0}dt$$

$$\frac{2}{3}a^{3/2} - \frac{2}{3}a^{3/2}(0) = -\sqrt{m_0}t$$

# Homogeneous Dust Collapse

The value of the initial condition was supposed as $a(0) = 1$. Thus

$$a(t) = \left[1 - \frac{3\sqrt{m_0}}{2}t\right]^{2/3}$$

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The value of the initial condition was supposed as $a(0) = 1$. Thus

$$a(t) = \left[1 - \frac{3\sqrt{m_0}}{2}t\right]^{2/3}$$

Therefore, the singularity $R = 0$ occurs at the time $t_s$ for which $a(t_s) = 0$. This is

$$t_s = \frac{2}{3\sqrt{m_0}}$$

## Homogeneous Dust Collapse

The curve $t_H(r)$ describing the time at which the shell $r$ crosses the horizon can be obtained from the condition

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$$a(t_H) = m_0 r^2$$

# Homogeneous Dust Collapse

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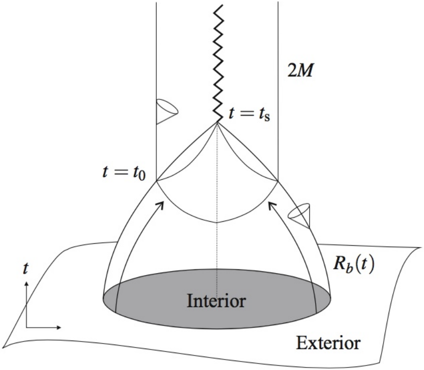
$$t_H(r) = \frac{2}{3\sqrt{m_0}} - \frac{2}{3} m_0 r^3$$

# Gravitational Collapse of a Homogeneous Cloud of Dust

*Eddington-Finkelstein Diagram*



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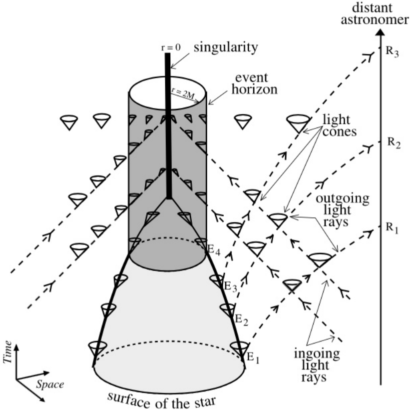

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$$\rho = \rho(t, r)$$
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$$b = b(r)$$
$$a = a(t, r)$$

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The simplest form for the function $m(r)$ is

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The condition that $\rho$ must be a decreasing function of $r$ implies that $m_2 < 0$.

* $m_1 = 0$ in order to have a non divergent density.

# Inhomogeneous Dust Collapse

The equation

$$\dot{R}^2 = \frac{F}{R} + f$$

gives this time

$$ra(\dot{t}, r) = -\sqrt{\frac{mr^2}{ra} + br^2}$$

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gives this time

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# Inhomogeneous Dust Collapse

In the marginally bound case, $b = 0$, it gives

$$\dot{a}(t, r) = -\sqrt{\frac{m(r)}{a}}$$

$$\sqrt{a(t, r)}da = -\sqrt{m(r)}dt$$

$$\frac{2}{3}a^{3/2}(t, r) - \frac{2}{3}a^{3/2}(0, r) = -\sqrt{m(r)}t$$

## Inhomogeneous Dust Collapse

The value of the initial condition was supposed as $a(0) = 1$. Thus

$$a(t,r) = \left[1 - \frac{3\sqrt{m(r)}}{2} t\right]^{2/3}$$

## Inhomogeneous Dust Collapse

The value of the initial condition was supposed as $a(0) = 1$. Thus

$$a(t, r) = \left[1 - \frac{3\sqrt{m(r)}}{2} t\right]^{2/3}$$

This equation shows how each shell of the distribution (each $r$) collapses with a different scale factor and with a different velocity.

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# Inhomogeneous Dust Collapse

Using $m(r) = m_0 + m_2 r^2$ gives the functions

$$t_s(r) = \frac{2}{3\sqrt{m_0 + m_2 r^2}}$$

$$t_H(r) = \frac{2}{3\sqrt{m_0 + m_2 r^2}} - \frac{2}{3} r^3 \left(m_0 + m_2 r^2\right)$$

# Gravitational Collapse of a Inhomogeneous Cloud of Dust

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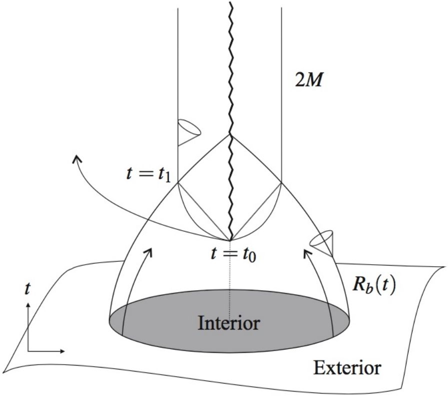

## Next Lecture

06. Black Holes Astrophysics