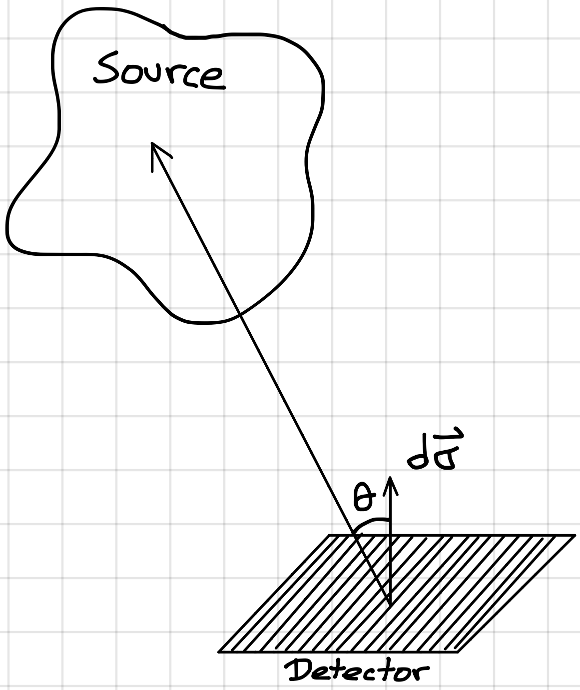
01. Astrophysics Basics

BRIGHTNESS: # of photons falling in the detector per unit area per unit time per unit solid angle.
(does not depend on distance)

FLUX: Total # of photons falling in the detector per unit area per unit time.
(decreace when increasing distance)



$d\sigma$: infinitesimal surface area of the detector

θ: Angle between the normal to $d\sigma$ and a "ray" of radiation.

$d\Omega$: Infinitesimal solid angle measured at the detector.

dE_ν: Energy flowing through $d\sigma$ in time dt in the freq. range ν to $\nu + d\nu$ within the solid angle $d\Omega$ on a ray which points an angle θ with respect to the surface's normal.

$$dE_\nu = I_\nu \cos\theta \, d\sigma \, d\Omega \, dt \, d\nu$$

I_ν: Specific Intensity or Spectral Brightness

$$[I_\nu] = \frac{\text{erg}}{\text{s}\cdot\text{cm}^2\cdot\text{sr}\cdot\text{Hz}}$$

Since power is $dP = \frac{dE_\nu}{dt}$

$$dP = I_\nu \cos\theta \, d\sigma \, d\Omega \, d\nu$$

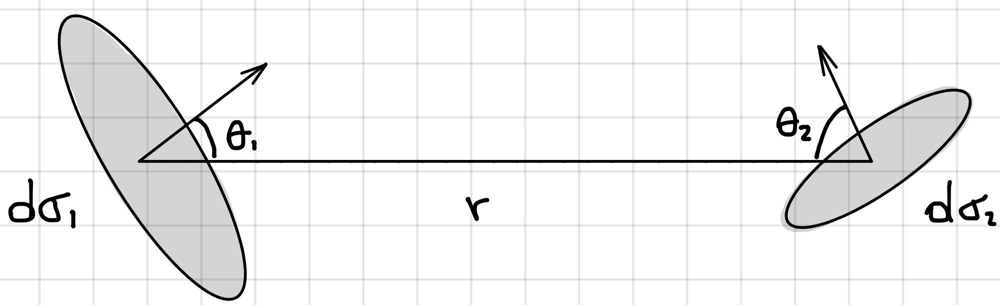
or

$$I_\nu = \frac{dP}{\cos\theta \, d\sigma \, d\Omega \, d\nu}$$

THEOREM/

Specific intensity I_ν is conserved along any ray in empty space.

Proof/: Consider the geometry



$d\Omega_1$: Solid angle subtended by $d\sigma_2$ seen from the center of $d\sigma_1$.

$d\Omega_2$: Solid angle subtended by $d\sigma_1$ seen from the center of $d\sigma_2$.

$$d\Omega_1 = \frac{\cos\theta_2}{r^2}\, d\sigma_2 \qquad\qquad d\Omega_2 = \frac{\cos\theta_1}{r^2}\, d\sigma_1$$

dW_1: Energy flowing through $d\sigma_1$ in the solid angle $d\Omega_1$

$$dW_1 = (I_\nu)_1 \cos\theta_1\, d\Omega_1\, d\sigma_1\, d\nu$$

$$dW_1 = (I_\nu)_1 \cos\theta_1 \left(\frac{\cos\theta_2\, d\sigma_2}{r^2}\right) d\sigma_1\, d\nu$$

Similarly,

$$dW_2 = (I_\nu)_2 \cos\theta_2\, d\Omega_2\, d\sigma_2\, d\nu$$

$$dW_2 = (I_\nu)_2 \cos\theta_2 \left(\frac{\cos\theta_1\, d\sigma_1}{r^2}\right) d\sigma_2\, d\nu$$

Since radiation energy is conserved in free space,

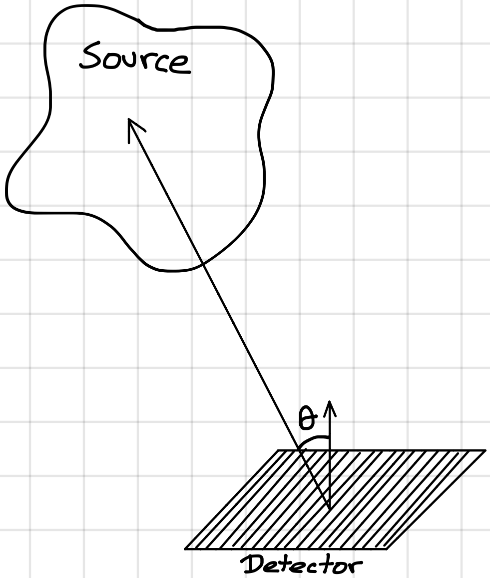
$$dW_1 = dW_2$$

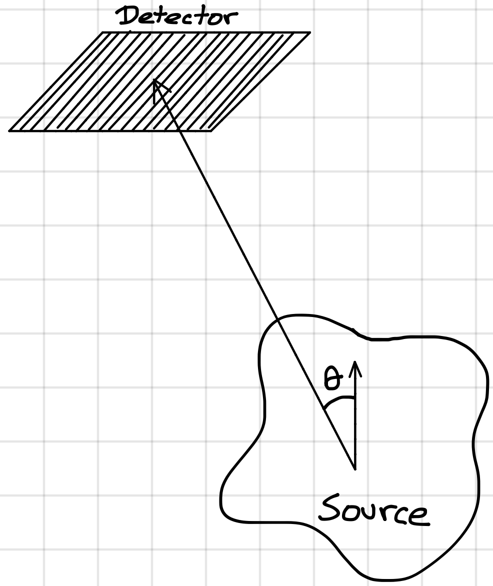
and therefore

$$(I_\nu)_1 = (I_\nu)_2$$

I_ν is independent of distance

I_ν is the same at the source and at the detector.





$$dW_\nu = I_\nu \cos\theta \, d\Omega \, d\sigma \, d\nu$$

$\uparrow$ Received or Emitted

I : Total Brightness or Total Intensity

$$I = \int_0^\infty I_\nu \, d\nu$$

The specific energy density per unit solid angle $u_\nu(\Omega)$ is defined as

$$u_\nu(\Omega) = \frac{I_\nu}{c}$$

Integrating over all solid angles we obtain the specific energy density u_ν,

$$u_\nu = \int \frac{I_\nu}{c}\, d\Omega$$

which can be written as

$$u_\nu = \frac{4\pi}{c} J_\nu$$

where $$J_\nu = \frac{1}{4\pi} \int I_\nu \, d\Omega$$

is known as the mean intensity. Isotropic radiation has

$$J_\nu = I_\nu .$$

Finally, the total radiation energy density is obtained by integrating over all the frequencies;

$$u = \frac{4\pi}{c} \int_0^\infty J_\nu \, d\nu$$

If the source is discrete (i.e it subtends a well defined solid angle) $\rightarrow$ we define the <u>Flux Density</u>

F_ν : Spectral power received by the detector

$$dF_\nu = \frac{dP}{d\sigma\, d\nu} = I_\nu \cos\theta\, d\Omega$$

$$F_\nu = \int_{\text{source}} I_\nu(\theta,\varphi) \cos\theta\, d\Omega$$

If the source is small (angular size << 1 rad) then $\cos\theta \simeq 1$ and

$$F_\nu \simeq \int_{\text{source}} I_\nu(\theta,\varphi)\, d\Omega$$ (usual for astronomical sources)

$$[F_\nu] = \frac{erg}{cm^2 \cdot s \cdot Hz}$$

$$1 \text{ Jansky} = 1 \text{ Jy} = 10^{-26} \frac{W}{m^2\, Hz}$$

Flux density depends on the source distance

$$F_\nu \propto \int d\Omega \propto \frac{1}{r^2} \longrightarrow F_\nu \propto r^{-2}$$ (Inverse Square law)

The Flux density can also be writen as

$$\vec{F}_\nu = \int d\Omega \; \hat{n}(\Omega) \, I_\nu(\Omega)$$

showing the vectorial character of the radiation flux per unit frequency in the direction $\hat{n}$.

Since photons have

$$p^\mu_{ph} = \left(\frac{E}{c}, \vec{p}_{ph} \right) \qquad \text{and} \qquad p^2_{ph} = 0 = \frac{E^2}{c^2} - |\vec{p}_{ph}|^2 ,$$

the outward momentum flux or pressure along a ray at angle θ is

$$dP^{rad}_\nu = \frac{dF_\nu}{c} = \frac{I_\nu}{c} \cos\theta \, d\Omega = dp_\nu$$

To get the moment flux normal to $d\sigma$ we multiply by $\cos\theta$ an integrate

$$p_\nu = \frac{1}{c} \int I_\nu \cos^2\theta \, d\Omega$$

$$[p_\nu] = \frac{erg}{cm^2 \cdot s \cdot Hz} \cdot \frac{s}{cm} = \frac{dyn}{cm^2 \cdot Hz}$$

The total momentum flux (radiation presion) is

$$p = \int p_\nu \, d\nu = \frac{1}{c} \iint I_\nu \cos^2\theta \, d\Omega \, d\nu$$

$$[p] = \frac{dyn}{cm^2}$$

Example //

Consider a reflecting enclosure for photons containing isotropic radiation. Then

$$P = \int P_\nu \, d\nu = \frac{1}{c} \int \int I_\nu \cos^2\theta \, d\Omega \, d\nu$$

Since isotropy implies $I_\nu = J_\nu$; we have

$$P = \frac{1}{c} \int J_\nu \, d\nu \int \cos^2\theta \, d\Omega$$

$$P = \frac{1}{c} \int J_\nu \, d\nu \int_0^{\pi} \cos^2\theta \, \sin\theta \, d\theta \int_0^{2\pi} d\phi$$

$$P = \frac{1}{c} \int J_\nu \, d\nu \; 2\pi \; \frac{1}{3} \cos^3\theta \Big|_{\pi}^{0}$$

$$P = \frac{2\pi}{3c} \int J_\nu \, d\nu \; \left(1^3 - (-1)^3\right)$$

$$P = \frac{4\pi}{3c} \int J_\nu \, d\nu$$

$$P = \frac{u}{3}$$

The radiation presion of an isotropic radiation field is one third of its energy density. This will be important for the study of black body radiation.

L_ν : Spectral Luminosity : Total power per unit bandwith radiated by a source at frequency ν.

$$L_\nu = \int F_\nu \, dA$$

$$L_\nu = 4\pi r^2 F_\nu$$

$$[L_\nu] = \frac{erg}{s \cdot Hz}$$

Spectral luminosity does not depend on the distance observer-source; i.e. it is an intrinsec property of the source.

L_{bol} : Bolometric luminosity

$$L_{bol} = \int_0^\infty L_\nu \, d\nu$$

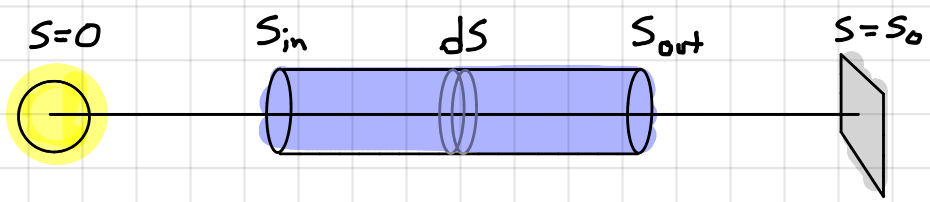
In free space the specific intensity I_ν of radiation is conserved along a ray:

$$\frac{dI_\nu}{ds} = 0$$

s: coordinate along the ray

$s=0$ at the source
$s=s_0$ at the observer.

RADIATIVE TRANSFER



① Absorption

The infinitesimal probability of a photon with freq. ν being absorbed in a thin slab is proportional to the intensity of radiation and to the thickness ds of the slab. Hence, we define the Linear absorption coefficient α_ν through

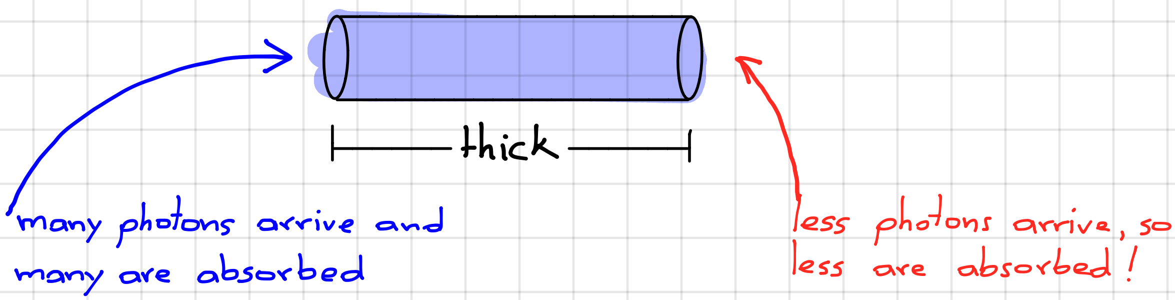
$$dI_\nu = -\alpha_\nu I_\nu \, ds$$

$$\frac{dI_\nu}{ds} = -\alpha_\nu I_\nu$$

$$[\alpha_\nu] = \frac{1}{cm}$$

α_ν is defined in differential terms so that α_ν is a constant along ds.

In a thick layer the absorption is NOT linear!



Since absorption must be proportional to the mass density ρ of the medium, it is usual to write

$$\alpha_\nu = \rho K_\nu$$

where K_ν is called the "mass absorption coefficient" or "opacity coefficient".

$$[K_\nu] = cm^2\, gr^{-1}$$

Integrating

$$\frac{dI_\nu}{ds} = -\alpha_\nu I_\nu$$

we obtain

$$\int_{S_{in}}^{S_{out}} \frac{dI_\nu}{I_\nu} = -\int_{S_{in}}^{S_{out}} \alpha_\nu(s)\, ds$$

$$I_\nu(s_{out}) = I_\nu(s_{in}) \exp\left[-\int_{s_{in}}^{s_{out}} \kappa_\nu(s)\, ds\right]$$

We define the <u>optical depth</u> or <u>opacity</u> as

$$\tau_\nu = \int_{s_{in}}^{s_{out}} \kappa_\nu(s)\, ds$$

and thus

$$I_\nu(s_{out}) = I_\nu(s_{in})\, e^{-\tau}$$

The intensity decreases exponentially along the line of sight.

$d\tau_\nu = \kappa_\nu ds \longrightarrow$

If $\tau_\nu \ll 1$ optically thin (transparent)
If $\tau_\nu \gg 1$ optically thick (opaque)

$\tau > 0 \to$ absorber
$\tau < 0 \to$ emitter ?

$\tau = 0$: No loss in $I_\nu \to \quad I_\nu(s_{out}) = I_\nu(s_{in})$

II Emission

The monochromatic spontaneous emission coefficient j_ν is defined as the energy emitted per unit time per unit frequency per unit volume and per unit solid angle,

$$dE_\nu = j_\nu \, d\nu \, dt \, d\Omega \, d\sigma \, ds$$

Hence the change in intensity due to spontaneous emission is

$$j_\nu = \frac{dI_\nu}{ds} \qquad [j_\nu] = \frac{erg}{cm^3 \, Hz \cdot s \cdot Sr}$$

Sometimes it is usefull to introduce the angle integrated "emissivity" ϵ_ν throug the relation

$$j_\nu = \frac{\rho \epsilon_\nu}{4\pi}$$

$$[\epsilon_\nu] = \frac{erg}{s \cdot gr \cdot Hz}$$

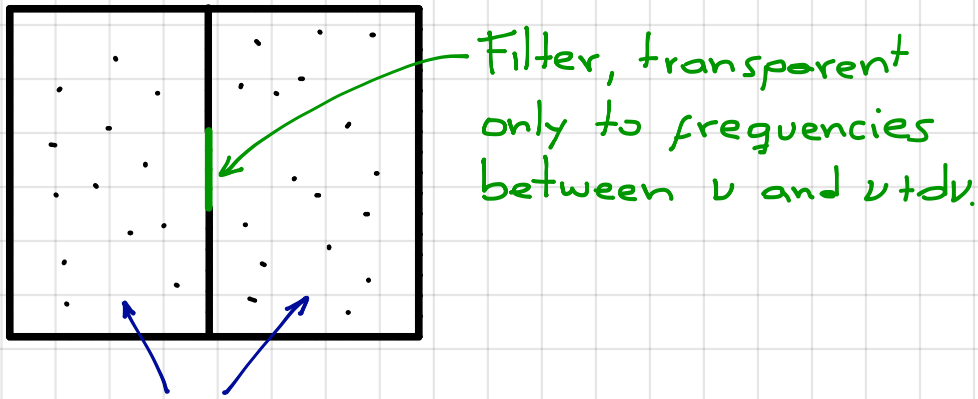
EQUATION OF RADIATIVE TRANSFER

$$\frac{dI_\nu}{ds} = -\kappa_\nu I_\nu + \epsilon_\nu$$

In thermodynamical equilibrium (TE), κ_ν and ϵ_ν are <u>not</u> independent.

In T.E. matter and radiation are in equilibrium at the same temperature T.

Radiation at equilibrium temperature T is isotropic with blackbody spectrum.



Radiation moving from one cavity into the other carries no net power (else, one cavity cools down and the other heats up !!)

Then, at T.E. at temperature T; we have

$$\frac{dI_\nu}{ds} = 0 \qquad \text{and} \qquad I_\nu = B_\nu(T)$$

only depends on T.

$$\frac{dI_\nu}{ds} = 0 = -k_\nu B_\nu(T) + \epsilon_\nu$$

$$\boxed{B_\nu(T) = \frac{\epsilon_\nu(T)}{k_\nu(T)}} \qquad \text{Kirchoff's Law}$$

Kirchoff's Law is defined in T.E. but it is also applicable in local themodynamical equilibrium (LTE)

L.T.E $\longrightarrow$ when the radiating/absorbing material is in thermal equilibrium even if it is not in equilibrium with the radiation field.

Blackbody Radiation

<u>Thermodynamics</u> Consider the first law applied to an isotropic radiation field in a blackbody enclosure with a piston;

$$dU = TdS - pdV$$

where $U = uV$

$p = \frac{1}{3}u$

$I_\nu = J_\nu$

S is the entropy and u only depends on T.

Then, we have

$$TdS = dU + pdV = d(uV) + \frac{1}{3}u\,dV$$

$$TdS = u\,dV + V\,du + \frac{1}{3}u\,dV$$

$$TdS = \frac{4}{3}u\,dV + V\frac{du}{dT}dT$$

$$dS = \frac{4}{3}\frac{u}{T}dV + \frac{V}{T}\frac{du}{dT}dT$$

Since dS is an exact differential, then

$$\frac{\partial S}{\partial V} = \frac{4}{3}\frac{u}{T} \qquad \frac{\partial S}{\partial T} = \frac{V}{T}\frac{du}{dT}$$

and then

$$\frac{\partial^2 S}{\partial T \partial V} = \frac{4}{3}\frac{1}{T}\frac{du}{dT} - \frac{4}{3}\frac{u}{T^2} = \frac{1}{T}\frac{du}{dT} = \frac{\partial^2 S}{\partial V \partial T}$$

$$\frac{4}{3}\frac{1}{T}\frac{du}{dT} - \frac{1}{T}\frac{du}{dT} = \frac{4}{3}\frac{u}{T^2}$$

$$\frac{1}{3}\frac{du}{dT} = \frac{4}{3}\frac{u}{T}$$

$$\frac{du}{u} = 4\frac{dT}{T}$$

$$\ln u = 4\ln T + \ln a \qquad a: \text{const.}$$

$$u = aT^4$$ Stefan-Boltzmann Law

On the other hand,

$$u = \frac{4\pi}{c}\int I_\nu \, d\nu = \frac{4\pi}{c}\int B_\nu(T)\, d\nu$$

$$u = \frac{4\pi}{c} B(T) = aT^4$$

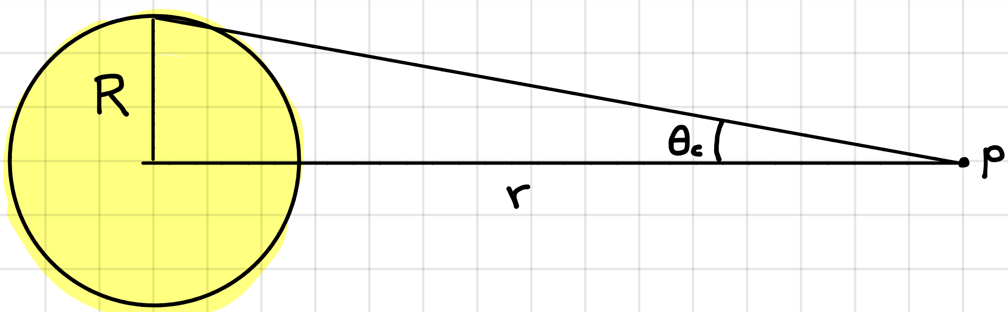
$$B(T) = \frac{ac}{4\pi} T^4$$

Consider a sphere of radius R with uniform brightness, i.e. $I = B = \text{constant}$. The flux density is given by the integral

$$F = \int I \cos\theta \, d\Omega = B \int_0^{2\pi} d\phi \int_0^{\theta_c} \sin\theta \cos\theta \, d\theta$$

where θ_c is such that

$\sin\theta_c \simeq \dfrac{R}{r}$ if $r \gg R$ (see figure)

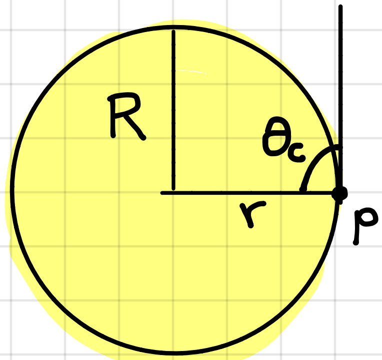


Then

$$F = B \, 2\pi \, \frac{\sin^2\theta}{2} \Bigg|_0^{\theta_c} = \pi B \sin^2\theta_c$$

When $r \gg R$ we have $F = \pi B \left(\dfrac{R}{r}\right)^2$

On the other hand, if $r = R$, the integral has to be done between $\theta = 0$ and $\theta = \frac{\pi}{2}$;



Hence,

$$F = B\, 2\pi \left. \frac{\sin^2\theta}{2} \right|_0^{\frac{\pi}{2}} = \pi B$$

Since the emergent flux from an isotropically emitting surface (as the blackbody) satisfies

$$B(T) = \frac{ac}{4\pi} T^4$$

we have

$$F = \frac{ac}{4} T^4$$

$$\boxed{F = \sigma T^4}$$

Planck's Law

$$B_\nu(T) = \frac{2h\nu^3}{c^2\left[\exp\left(\frac{h\nu}{kT}\right) - 1\right]}$$

Spectral brightness of a blackbody radiation

Consider the integral

$$F = \pi \int_0^\infty B_\nu(T)\, d\nu = \pi \int_0^\infty \frac{2h\nu^3}{c^2\left[\exp\left(\frac{h\nu}{kT}\right) - 1\right]}\, d\nu$$

Making $x = \frac{h\nu}{kT} \longrightarrow d\nu = \frac{kT}{h} dx$

$$F = \pi \left(\frac{kT}{h}\right)^4 \frac{2h}{c^2} \int_0^\infty \frac{x^3\, dx}{e^x - 1}$$

$$F = \frac{2\pi k^4 T^4}{c^2 h^3} \frac{\pi^4}{15}$$

$$F = \left(\frac{2\pi^5 k^4}{15 c^2 h^3}\right) T^4 = \sigma T^4$$

where

$$\sigma = \frac{ac}{\pi} = \frac{2\pi^5 k^4}{15 c^2 h^3} = 5{,}6 \times 10^{-5} \frac{\text{erg}}{\text{cm}^2\ \text{s}\ K^4}$$

Stefan-Boltzmann constant

Properties of Planck's law

① In the low frequency limit $h\nu \ll kT$ we have

$$\exp\left(\frac{h\nu}{KT}\right) - 1 \approx \frac{h\nu}{KT} + O\left[\left(\frac{h\nu}{KT}\right)^2\right]$$

Thus

$$B_\nu(T) \approx \frac{2h\nu^3}{c^2\left(\frac{h\nu}{KT}\right)}$$

$$B_\nu = \frac{2kT\nu^2}{c^2}$$

Rayleigh-Jeans Law

* B_ν diverges at high ν (ultraviolet catastrophe)

$$B(T) = \int_0^\infty \frac{2KT\nu^2}{c^2}\, d\nu \longrightarrow \infty$$

② In the high frequency limit $h\nu \gg kT$ we have

$$\exp\left(\frac{h\nu}{KT}\right) - 1 \approx \exp\left(\frac{h\nu}{KT}\right)$$

Thus

$$B_\nu(T) \approx \frac{2h\nu^3}{c^2}\exp\left(-\frac{h\nu}{KT}\right)$$

Wien Law

* This corresponds to the right hand rapid decrease in blackbody brightness plot.

(III) Monoticity with Temperature. The plot of $B_\nu(T_1)$ for $T_1 > T_2$ is completely above the curve of $B_\nu(T_2)$. To prove it consider the derivative

$$\frac{\partial B_\nu}{\partial T} = \frac{2h^2\nu^4}{c^2 K T^2} \frac{\exp\left(\frac{h\nu}{KT}\right)}{\left[\exp\left(\frac{h\nu}{KT}\right) - 1\right]^2} > 0$$

(IV) Wien Displacement Law. The frequency ν_{max} at which $B_\nu(T)$ has its maximum is given by the condition

$$\left.\frac{\partial B_\nu}{\partial \nu}\right|_{\nu_{max}} = 0.$$

This condition is

$$\frac{\partial B_\nu}{\partial \nu} = \frac{\partial}{\partial \nu}\left[\frac{2h\nu^3}{c^2\left[\exp\left(\frac{h\nu}{KT}\right) - 1\right]}\right]$$

$$\frac{\partial B_\nu}{\partial \nu} = \frac{6h\nu^2}{c^2\left[\exp\left(\frac{h\nu}{KT}\right) - 1\right]} - \frac{2h\nu^3 \exp\left(\frac{h\nu}{KT}\right)}{c^2\left[\exp\left(\frac{h\nu}{KT}\right) - 1\right]^2} \frac{h}{KT}$$

$$\frac{\partial B_\nu}{\partial \nu} = \frac{2h\nu^2}{c^2\left[\exp\left(\frac{h\nu}{KT}\right) - 1\right]^2}\left[3\left[\exp\left(\frac{h\nu}{KT}\right) - 1\right] - \frac{h\nu}{KT}\exp\left(\frac{h\nu}{KT}\right)\right]$$

$$\frac{\partial B_\nu}{\partial \nu} = \frac{2h\nu^3 \exp\left(\frac{h\nu}{KT}\right)}{c^2\left[\exp\left(\frac{h\nu}{KT}\right) - 1\right]^2}\left[3\left[1 - \exp\left(-\frac{h\nu}{KT}\right)\right] - \frac{h\nu}{KT}\right]$$

Then, condition

$$\left.\frac{\partial B_\nu}{\partial \nu}\right|_{\nu_{max}} = 0$$

corresponds to

$$3\left[1 - \exp\left(-\frac{h\nu_{max}}{KT}\right)\right] - \frac{h\nu_{max}}{KT} = 0$$

or defining $x = \dfrac{h\nu_{max}}{KT}$,

$$x = 3(1 - e^{-x}).$$

The solution of this equation is done numerically to obtain

$$x \approx 2.88$$

or

$$\frac{\nu_{max}}{T} = 2.88\,\frac{k}{h}$$

Using $k = 1.38 \times 10^{-16}$ erg $\cdot$ K^{-1}
$h = 6.625 \times 10^{-27}$ erg $\cdot$ s

gives

$$\frac{\nu_{max}}{T} = 5.88 \times 10^{10} \text{ Hz} \cdot K^{-1}$$

CHARACTERISTIC TEMPERATURES RELATED WITH PLANCK'S SPECTRUM

* Brightness Temperature

For a given frequency ν, we associate the "brightness temperature" T_B such that

$$I_\nu = B_\nu(T_B)$$

In radio-astronomy it is usual to use the limit $h\nu \ll kT$ (Rayleigh-Jeans Law) from which

$$I_\nu = \frac{2kT_B\nu^2}{c^2}$$

or

$$T_B = \frac{I_\nu c^2}{2k\nu^2}$$

* Color Temperature

By adjusting a blackbody spectrum, we can associate a "Color temperature" to an observed spectrum. Usually the adjust uses only the maximum and Wien displacement law,

$$T_c = \frac{\nu_{max}}{5.88 \times 10^{10}\ \mathrm{Hz} \cdot \mathrm{K}^{-1}}$$

* Effective Temperature

Using the total amount of radiation flux integrated over all frequencies, we define the "effective temperature", or "blackbody temperature",

$$F = \int I_\nu \cos\theta \, d\Omega \, d\nu = \sigma T_{eff}^4$$

$$T_{eff} = \sqrt[4]{\frac{F}{\sigma}}$$