05. CYLINDRICAL ACCRETION. DETAILED DESCRIPTION.

MATHEMATICAL DESCRIPTION OF ACCRETION

- ACCRETION RADIUS

The characteristic length scale in an accretion process is the "accretion radius" or "gravitational capture radius",

R_{acc} : distance at which kinetic and gravitational potential energies are equal.

$$\frac{1}{2}\left(c_s^2 + v_{rel}^2\right) = \frac{GM}{R_{acc}}$$

or

$$R_{acc} = \frac{GM}{c_s^2 + v_{rel}^2}$$

- HYDRODYNAMIC EQUATIONS (NON-RELATIVISTIC)

ρ: Density
T: Temperature
$\vec{v}$: velocity of the gas
P : Pressure
$\vec{g}$: Acceleration due to gravity

* Continuity Equation (Conservation of Mass)

$$\frac{\partial \rho}{\partial t} + \vec{\nabla} \cdot (\rho \vec{v}) = 0$$

* Conservation of Momentum (Ignoring radiation pressure)

$$\rho\left[\frac{\partial \vec{v}}{\partial t}+(\vec{v}\cdot\vec{\nabla})\vec{v}\right]=-\vec{\nabla}P+\rho\vec{g}+\vec{\nabla}\cdot\vec{\sigma}$$

where

$\sigma_{ij}=2\eta\tau_{ij}$ Viscosity Stress Tensor

$$\tau_{ij}=\frac{1}{2}\left(\frac{\partial v_i}{\partial x_j}+\frac{\partial v_j}{\partial x_i}-\frac{2}{3}\frac{\partial v_k}{\partial x_k}\delta_{ij}\right)$$

η : dynamic viscosity

$\eta=\rho\nu$ ν : kinematic viscosity.

If $\nu=0$: no-viscosity $\rightarrow$ ideal fluid $\rightarrow$ Euler Equation
If $\nu\neq 0$: viscous fluid $\rightarrow$ Navier-Stokes Equation

- If the self-gravity of the fluid is negligible,

$$\vec{g}=-\vec{\nabla}\Phi=\vec{\nabla}\left(\frac{GM}{r}\right)=-\frac{GM}{r^2}\hat{r}$$

- If η is a constant, the momentum equation becomes

$$\rho\left[\frac{\partial \vec{v}}{\partial t}+(\vec{v}\cdot\vec{\nabla})\vec{v}\right]=-\vec{\nabla}P+\rho\vec{g}+\eta\nabla^2\vec{v}+\frac{1}{3}\eta\vec{\nabla}(\vec{\nabla}\cdot\vec{v})$$

* Equation of State

$P=P(\rho)$

- Polytropic Equation $P\propto\rho^{\gamma}$

* Energy balance in the flow:

- Change in kinetic energy
- Change in the internal energy
- Flux of heat

$$\rho \frac{d\varepsilon}{dt} = -P\vec{\nabla}\cdot\vec{v} + 2\eta\left[S_{ij}S_{ij} - \frac{1}{3}(\vec{\nabla}\cdot\vec{v})^2\right] + Q$$

ε: internal energy per unit mass of the fluid
Q: Net heat exchanged by the element of fluid per unit time per unit volume.

$$S_{ij} = \frac{1}{2}\left(\frac{\partial v_i}{\partial x^j} - \frac{\partial v_j}{\partial x^i}\right) = \tau_{ij} - \frac{1}{3}(\vec{\nabla}\cdot\vec{v})\delta_{ij}$$

Then, $$q^+ = 2\eta\left[S_{ij}S_{ij} - \frac{1}{3}(\vec{\nabla}\cdot\vec{v})^2\right]$$

is the rate of energy dissipation per unit volume due to the work done by viscous forces.

Note:/ The total or "material" derivative is

$$\frac{df(t,\vec{r})}{dt} = \frac{\partial f}{\partial t} + \underbrace{(\vec{v}\cdot\vec{\nabla})f}_{\text{Convective derivative.}}$$

* The temperature is assumed to fit the perfect gas law,

$$T = \frac{\mu m_H}{k} \frac{P}{\rho}$$

Hidrogen mass $m_H \approx m_p$
Mean molecular weight : $\mu = 1$ for neutral H
$\mu = 1/2$ fully ionized H.

Boltzmann constant: k

Bondy-Hoyle-Lyttleton Model [1],[2],[3]

Axisymmetric accretion. It also describes the accretion rate onto a moving BH.

- No analytical solution of the hydrodynamic eqs.
- Reference frame fixed to the accretor (point with mass M)
- Polar coordinates (r,θ) $r=0$ at the B.H.
- velocity of the flow at infinity v_∞
- Density of the flow at infinity ρ_∞
- No viscosity : $\eta = 0$
- Impact parameter : b
- Hoyle & Lyttleton approximation: Pressure of the gas is neglected.

  $\Rightarrow$ Equations of motion in Newtonian gravit. field!

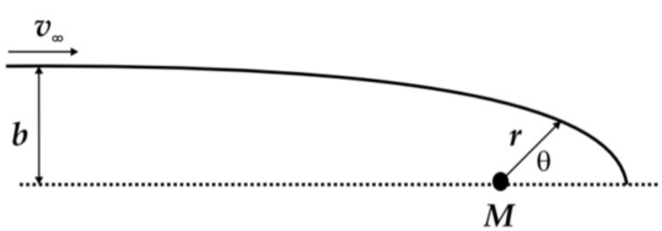


Fig. 4.2 Sketch of the geometry of a Bondi-Hoyle-Lyttleton axisymmetric accretion flow. The path of an element of fluid (in the ballistic approximation) is shown and the relevant parameters of the problem are indicated

$$\mathcal{L} = \frac{1}{2}m\left[\dot{r}^2 + r^2\dot{\theta}^2\right] + \frac{GMm}{r}$$

Conserved Energy

$$E = \frac{1}{2}m\left[\dot{r}^2 + r^2\dot{\theta}^2\right] - \frac{GMm}{r} = \frac{1}{2}mv_\infty^2$$

Conserved Angular Momentum

$$l = \frac{\partial \mathcal{L}}{\partial \dot{\theta}} = mr\dot{\theta} = mbv_\infty$$

Radial equation of motion

$$\frac{d}{dt}\left[\frac{\partial \mathcal{L}}{\partial \dot{r}}\right] = \frac{\partial \mathcal{L}}{\partial r}$$

$$\frac{d}{dt}\left[m\dot{r}\right] = mr\dot{\theta}^2 - \frac{GMm}{r^2}$$

Hence, the e.o.m. are

$$\frac{d^2r}{dt^2} - r\left(\frac{d\theta}{dt}\right)^2 = -\frac{GM}{r^2}$$

$$r^2\frac{d\theta}{dt} = bv_\infty$$

Equation of the trajectory:

$$\frac{dr}{dt} = \frac{d\theta}{dt}\frac{dr}{d\theta} = \frac{bv_\infty}{r^2}\frac{dr}{d\theta}$$

From energy conservation:

$$\frac{1}{2}\left[\left(\frac{bv_\infty}{r^2}\frac{dr}{d\theta}\right)^2 + r^2\left(\frac{bv_\infty}{r^2}\right)^2\right] - \frac{GM}{r} = \frac{1}{2}v_\infty^2$$

$$\frac{1}{r^4}\left(\frac{dr}{d\theta}\right)^2 + \frac{1}{r^2} - \frac{2GM}{r}\left(\frac{1}{bv_\infty}\right)^2 = v_\infty^2\left(\frac{1}{bv_\infty}\right)^2$$

$$\frac{1}{r^4}\left(\frac{dr}{d\theta}\right)^2 + \frac{1}{r^2} - \frac{2GM}{b^2v_\infty^2 r} = \frac{1}{b^2}$$

Change of variable

$$u = \frac{1}{r} \quad \Rightarrow \quad \frac{du}{d\theta} = -\frac{1}{r^2}\frac{dr}{d\theta}$$

$$\left(\frac{du}{d\theta}\right)^2 = \frac{1}{r^4}\left(\frac{dr}{d\theta}\right)^2$$

$$\left(\frac{du}{d\theta}\right)^2 + u^2 - \frac{2GM}{b^2v_\infty^2}u = \frac{1}{b^2}$$

Differentiating:

$$\frac{du}{d\theta}\frac{d^2u}{d\theta^2} + u\frac{du}{d\theta} - \frac{GM}{b^2v_\infty^2}\frac{du}{d\theta} = 0$$

One solution: $\frac{du}{d\theta} = 0 \Rightarrow u(\theta) = \text{const.}$

$\Rightarrow r(\theta) = \text{const.}$ (Circular Orbit)

Other solution:

$$\frac{d^2u}{d\theta^2} + u = \frac{GM}{b^2 v_\infty^2}$$

$$u(\theta) = A\cos\theta + B\sin\theta + \frac{GM}{b^2 v_\infty^2}$$

or

$$r(\theta) = \left[A\cos\theta + B\sin\theta + \frac{GM}{b^2 v_\infty^2}\right]^{-1}$$

Integration constants A anb B are determined by the initial conditions, such that the solution is

$$r(\theta) = \left[\frac{GM}{b^2 v_\infty^2}\left(1 + \cos\theta\right) - \frac{1}{b}\sin\theta\right]^{-1}$$

The components of the velocity are easily obtained from the energy equation,

$$\frac{1}{2} m \left[\dot{r}^2 + r^2\dot{\theta}^2\right] - \frac{GMm}{r} = \frac{1}{2} m v_\infty^2$$

$$\left[\dot{r}^2 + \frac{b^2 v_\infty^2}{r^2}\right] - \frac{2GM}{r} = v_\infty^2$$

$$V_r = \dot{r} = \pm\sqrt{v_\infty^2 + \frac{2GM}{r} - \frac{b^2 v_\infty^2}{r^2}}$$

from which we choose the $-$ sign to describe accretion,

$$\boxed{V_r = \dot{r} = -\sqrt{V_\infty^2 + \frac{2GM}{r} - \frac{b^2 V_\infty^2}{r^2}}}$$

and using the angular momentum conservation,

$$V_\theta = r\dot{\theta} = r\frac{d\theta}{dt}$$

$$\boxed{V_\theta = \frac{V_\infty b}{r}}$$

Note that at $\theta = \pi$ we obtain

$$r(\pi) = \left[\frac{GM}{b^2 v_\infty^2}\left(1 + \cos\pi\right) - \frac{1}{b}\sin\pi\right]^{-1} \longrightarrow \infty$$

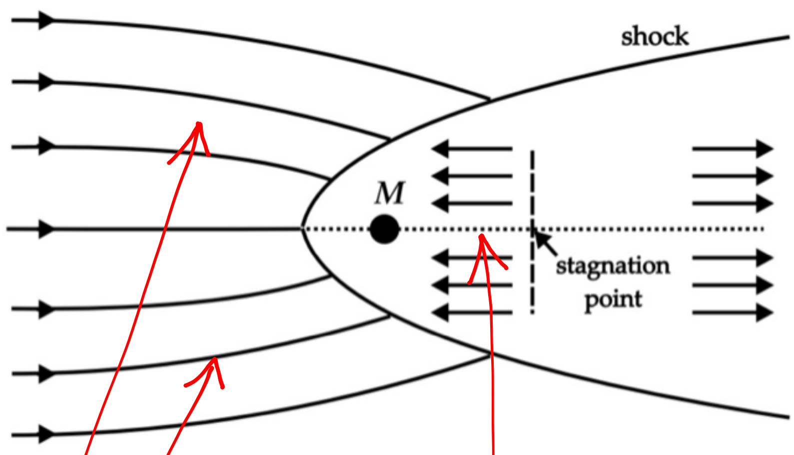
and $\begin{cases} V_r = -V_\infty \\ V_\theta = 0 \end{cases}$

On the other hand, at $\theta = 0$ we have

$$r(0) = \left[\frac{GM}{b^2 v_\infty^2}\left(1 + \cos 0\right) - \frac{1}{b}\sin 0\right]^{-1} = \frac{b^2 V_\infty^2}{2GM}$$

with $\begin{cases} V_r = -V_\infty \\ V_\theta = \dfrac{2GM}{bV_\infty} \end{cases}$

Fig. 4.3 Sketch of the geometry of an axisymmetric accretion flow in the Bondi-Hoyle model. The *arrows* indicate the direction of the flow



As matter comes from both sides of the symmetry axis ...

... the density increases in this region!

$\Downarrow$

The pressure can not be neglected.

In order to simplify the description, Hoyle & Littleton [2] suppose that, due to collisions, particles at $\theta = 0$ have no tangential velocity, $v_\theta = 0$.
This assumption implies that particles on the symmetry axis have energy

$$E = \frac{1}{2} m v_\infty^2 - \frac{1}{2} m \left(\frac{2GM}{b v_\infty} \right)^2$$

$$\left. v_\theta^2 \right|_{\theta=0}$$

$$\boxed{E = \frac{1}{2} m v_\infty^2 - m \frac{2G^2M^2}{b^2 v_\infty^2}}$$

Particles with $E<0$ are captured by the BH.

$$E = \frac{1}{2} m v_\infty^2 - m \frac{2G^2M^2}{b^2 v_\infty^2} < 0$$

Hence, particles are captured if

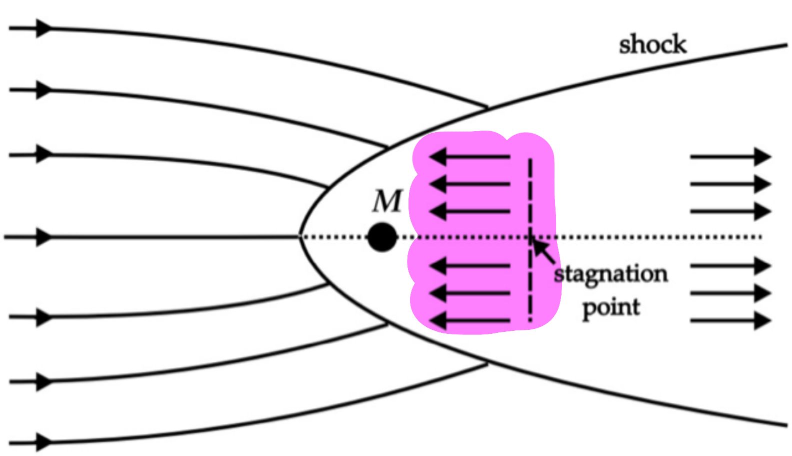
$$b < \frac{2GM}{v_\infty^2} := r_{HL}$$ (Hoyle-Littleton radius)

Thus, the accretion rate is the mass flux at infinity through a circle of area πr_{HL}^2 centered on the symmetry axis:

$$\dot{M}_{HL} = \pi r_{HL}^2 \, v_\infty \rho_\infty$$

$$\dot{M}_{HL} = \frac{4\pi G^2 M^2 \rho_\infty}{v_\infty^3}$$

Fig. 4.3 Sketch of the geometry of an axisymmetric accretion flow in the Bondi-Hoyle model. The *arrows* indicate the direction of the flow



In the H-L Model the accretion proceeds along the axis $\theta = 0$ (which has infinite density).
Bondy & Hoyle [3] changed the model so that accretion proceeds along an "accretion column.

Inside the accretion column the pressure is non negligible and the characteristic size of this column is determined by the balance of

Pressure from the inside fluid $\Longleftrightarrow$ Transverse momentum Flux of ingoing matter.

Thickness of the column $\rightarrow$ depends on ρ_∞
- for large densities, the column is a surface of discontinuity.

In [3] it is assumed that

* the mass per unit length per unit time entering the column is equal to that in H-L model. Then

$$r = \frac{b^2 v_\infty^2}{2GM}$$

$$dr = \frac{2b\, v_\infty^2}{2GM}\, db$$

$$2\pi b \rho_\infty v_\infty\, db = 2\pi \rho_\infty \frac{GM}{v_\infty}\, dr \equiv A\, dr$$

$$A := 2\pi \rho_\infty \frac{GM}{v_\infty}$$

* The radial velocity of the matter entering the column is v_∞.

* The pressure gradient along the symmetry axis can be neglected compared with gravity

* The velocity v of the flow in the column region is uniform on any cross section and it is parallel to the axis.

Bondi & Hoyle [3] showed that the equation of conservation of mass and component of momentum along the axis are

$$\frac{d}{dr}(mv) = A$$

$$\frac{d}{dr}\left(mv^2\right) = Av_\infty - \frac{mMG}{r^2}$$

where mdr : mass per unit length on the axis.
The boundary conditions are

$v = v_\infty$ for $r \longrightarrow \infty$
$v = 0$ at $r = r_0$
v is bound for $r \neq 0$

Using these conditions, integration of the mass' conservation gives

$mv = A(r - r_0)$ $\leftarrow$ r_0 is an stagnation point ; $v(r_0) = 0$.

Note that $m > 0$; thus

for $r < r_0$ $\rightarrow$ flow inwards
for $r > r_0$ $\rightarrow$ flow outwards !

Only material entering in the region $r < r_0$ is accreted.
The accretion rate is approximately

$$\dot{M}_{BH} \approx A r_0 = \frac{2\pi \alpha G^2 M^2 \rho_\infty}{V_\infty^3}$$

Bondy-Hoyle

where $\alpha = r_0 \dfrac{V_\infty^2}{GM} > 0$

Any value of α gives a solution under the imposed assumptions. However, if we demand that $v(r)$ is monotonic, then $\alpha > 1 \longrightarrow r_0 > \dfrac{GM}{V_\infty^2}$

Since $\dot{M}_{crit}$ of spherical accretion for $\gamma = \frac{3}{2}$

$$\dot{M}_{crit} = \pi G^2 M^2 \frac{\rho_\infty}{C_\infty^3} \left(\frac{2}{5-3\gamma} \right)^{\frac{5-3\gamma}{2(\gamma-1)}} \qquad (\gamma \neq 1)$$

$$\dot{M}_{crit}\Big|_{\gamma=\frac{3}{2}} = \pi G^2 M^2 \frac{\rho_\infty}{C_\infty^3}$$

and $\dot{M}_{BH}$ are similar, Bondi [4] proposed an interpolated expression

$$\dot{M} \approx \frac{2\pi G^2 M^2 \rho_\infty}{(C_\infty^2 + V_\infty^2)^{3/2}}$$

which reduces to $\dot{M}_{crit}\Big|_{\gamma=\frac{3}{2}}$ for $C_\infty^2 \gg V_\infty^2$ and becomes $\dot{M}_{BH}\Big|_{\alpha=1}$ for $V_\infty^2 \gg C_\infty^2$.

REFERENCIAS

[1] G. Romero & G. Vila. "Introduction to BH Astrophysics". (2014)

[2] F. Hoyle & R.A. Littleton. Proc. Camb. Philos. Soc. 35, 405 (1939)

[3] H. Bondi & F. Hoyle. Mon. Not. R. Astron. Soc. 104, 273 (1944)

S.L. Shapiro & S.A. Teukolsky. "Black Holes, White Dwarfs and Neutron Stars: The Physics of Compact Objects". (1983)

[4] V. Frolov and I. Novikov. "Black Hole Physics: Basic Concepts and New Developments" (1998)

[5] H. Bondi. Mon. Not. R. Astron. Soc. 112, 95 (1952)

[6]

[7] J. Frank, A. King & D. Raine. "Accretion Power in Astrophysics". (2002)