03. SPHERICAL ACCRETION. DETAILED DESCRIPTION.

Mathematical Description of Accretion

- Accretion Radius

The characteristic length scale in an accretion process is the "accretion radius" or "gravitational capture radius",

R_{acc} : distance at which kinetic and gravitational potential energies are equal.

$$\frac{1}{2}\left(c_s^2 + v_{rel}^2\right) = \frac{GM}{R_{acc}}$$

or $$\boxed{R_{acc} = \frac{GM}{c_s^2 + v_{rel}^2}}$$

- Hydrodynamic Equations (Non-Relativistic)

ρ: Density
T: Temperature
$\vec{v}$: velocity of the gas
P : Pressure
$\vec{g}$: Acceleration due to gravity

* Continuity Equation (Conservation of Mass)

$$\boxed{\frac{\partial \rho}{\partial t} + \vec{\nabla} \cdot (\rho \vec{v}) = 0}$$

* Conservation of Momentum (Ignoring radiation pressure)

$$\rho \left[\frac{\partial \vec{v}}{\partial t} + (\vec{v} \cdot \vec{\nabla}) \vec{v} \right] = - \vec{\nabla} P + \rho \vec{g} + \vec{\nabla} \cdot \vec{\sigma}$$

where

$\sigma_{ij} = 2 \eta \tau_{ij}$ Viscosity Stress Tensor

$$\tau_{ij} = \frac{1}{2} \left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i} - \frac{2}{3} \frac{\partial v_k}{\partial x_k} \delta_{ij} \right)$$

η : dynamic viscosity

$\eta = \rho \nu$ ν : kinematic viscosity.

If $\nu = 0$: no-viscosity $\rightarrow$ ideal fluid $\rightarrow$ Euler Equation
If $\nu \neq 0$: viscous fluid $\rightarrow$ Navier-Stokes Equation

- If the self-gravity of the fluid is negligible,

$$\vec{g} = - \vec{\nabla} \Phi = \vec{\nabla} \left(\frac{GM}{r} \right) = - \frac{GM}{r^2} \hat{r}$$

- If η is a constant, the momentum equation becomes

$$\rho \left[\frac{\partial \vec{v}}{\partial t} + (\vec{v} \cdot \vec{\nabla}) \vec{v} \right] = - \vec{\nabla} P + \rho \vec{g} + \eta \nabla^2 \vec{v} + \frac{1}{3} \eta \vec{\nabla} (\vec{\nabla} \cdot \vec{v})$$

* Equation of State

$P = P(\rho)$

- Polytropic Equation $P \propto \rho^{\gamma}$

* Energy balance in the flow:

- Change in kinetic energy
- Change in the internal energy
- Flux of heat

$$\rho \frac{d\varepsilon}{dt} = - P \vec{\nabla} \cdot \vec{v} + 2\eta \left[S_{ij} S_{ij} - \frac{1}{3} (\vec{\nabla} \cdot \vec{v})^2 \right] + Q$$

ε: internal energy per unit mass of the fluid
Q: Net heat exchanged by the element of fluid per unit time per unit volume.

$$S_{ij} = \frac{1}{2} \left(\frac{\partial v_i}{\partial x^j} - \frac{\partial v_j}{\partial x^i} \right) = \tau_{ij} - \frac{1}{3} (\vec{\nabla} \cdot \vec{v}) \delta_{ij}$$

Then, $$q^+ = 2\eta \left[S_{ij} S_{ij} - \frac{1}{3} (\vec{\nabla} \cdot \vec{v})^2 \right]$$

is the rate of energy dissipation per unit volume due to the work done by viscous forces.

Note:/ The total or "material" derivative is

$$\frac{df(t, \vec{r})}{dt} = \frac{\partial f}{\partial t} + \underbrace{(\vec{v} \cdot \vec{\nabla}) f}_{\text{Convective derivative.}}$$

* The temperature is assumed to fit the perfect gas law,

$$T = \frac{\mu m_H}{K} \frac{P}{\rho}$$

Hidrogen mass $m_H \approx m_p$

Mean molecular weight : $\mu = 1$ for neutral H

$\mu = 1/2$ fully ionized H.

Boltzmann constant: K

- Mass Accretion Rate

Mass per unit time captured by the gravitating center.

$$\dot{M} = \sigma_G \, \rho \, v_{rel}$$

σ_G : cross section of gravitational capture

For a gas of collisionless non-relativistic particles the σ_G in a Schwarzschild BH is [3], [4]

$$\sigma_G^{(collisionless)} = 4\pi \left(\frac{c}{v_\infty} \right)^2 r_s^2$$

$v_\infty \ll c$: velocity of the particles relative to M at infinity.

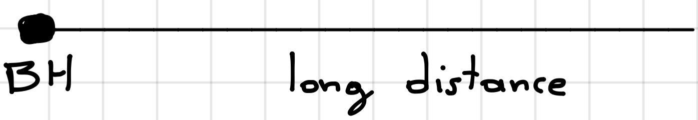
For a fluid, we have

$$\sigma_G^{(Fluid)} \approx \pi \, r_g^2$$

$$r_g = \frac{GM}{c^2} = \frac{r_s}{2}$$

BONDY ACCRETION [5]

First model of accretion. Smooth time-steady accretion with spherical symmetry.



gas moves very slowly

$\downarrow$

description in terms of Fluid Mechanics eqs.

We assume

* No viscosity : $\eta = 0$
* No angular momentum
* No e.m. fields

We are looking for:

- Steady accretion rate $\dot{M}$ on to the BH in terms of ρ_∞ and T_∞ (density and temperature in the gas far from the BH).
- How big is the region of the gas influenced by the gravity of the BH.
- Local velocity of the gas and local speed of sound.

Spherical Symmetry and steady state:

$\rho = \rho(r)$ $\qquad\qquad$ $\vec{v} = v(r)\,\hat{r}$

Then,

$$\frac{\partial \rho}{\partial t} + \vec{\nabla}\cdot(\rho\vec{v}) = 0$$

$$\vec{\nabla}\cdot(\rho\vec{v}) = 0$$

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left[r^2 \rho v\right] = 0$$

Note: In spherical coordinates

$$\vec{\nabla}\cdot\vec{F} = \frac{1}{r^2}\partial_r(r^2 F_r) + \frac{1}{r\sin\theta}\partial_\theta(\sin\theta\, F_\theta) + \frac{1}{r\sin\theta}\partial_\phi F_\phi$$

Integrating this equation,

$$r^2 \rho v = \text{Const.} = \frac{C}{4\pi}$$

$$4\pi r^2 \rho v = C$$

Since $\quad \rho = \frac{dM}{dV} = \frac{dM}{4\pi r^2 dr} \quad$ and $\quad v = \frac{dr}{dt}$

we identify the integration constant as the mass accretion rate:

$$\dot{M} = 4\pi r^2 \rho v$$

The Conservation of Momentum equation becomes

$$\rho \left[\frac{\partial \vec{v}}{\partial t} + (\vec{v} \cdot \vec{\nabla})\vec{v} \right] = -\vec{\nabla}P + \rho \vec{g}$$

$$\rho \left[\frac{\partial v}{\partial r} + (\vec{v} \cdot \vec{\nabla}) v \right] = -\frac{\partial P}{\partial r} - \rho \frac{GM}{r^2}$$

$$\rho \frac{dv}{dt} = -\frac{\partial P}{\partial r} - \rho \frac{GM}{r^2}$$

$$v \frac{dv}{dr} + \frac{1}{\rho} \frac{\partial P}{\partial r} + \frac{GM}{r^2} = 0$$

Finally, Bondy (1952) propose an equation of state in the form of a simple adiabatic law,

$$P \propto \rho^{\gamma}$$

$1 \leq \gamma \leq \frac{5}{3}$

$\gamma = 1$: Isothermic Flow
$\gamma = 5/3$: Adiabatic Flow

The complete set of equations describing Bondi accretion is

$$\begin{cases} \dot{M} = 4\pi r^2 \rho v & (\mathrm{I}) \\ v \dfrac{dv}{dr} + \dfrac{1}{\rho} \dfrac{\partial P}{\partial r} + \dfrac{GM}{r^2} = 0 & (\mathrm{II}) \\ P \propto \rho^{\gamma} & (\mathrm{III}) \end{cases}$$

From the polytropic equation we have

$$\frac{P}{\rho^{\gamma}} = \frac{P_\infty}{\rho_\infty^{\gamma}} \longleftarrow \text{values at infinity}$$

$$P = P_\infty \left(\frac{\rho}{\rho_\infty}\right)^{\gamma}$$

The local speed of sound is given by

$$C_s^2 = \gamma \frac{P}{\rho} = \gamma \frac{P_\infty}{\rho}\left(\frac{\rho}{\rho_\infty}\right)^{\gamma} = \gamma \frac{P_\infty}{\rho_\infty}\left(\frac{\rho}{\rho_\infty}\right)^{\gamma-1} = C_\infty^2 \left(\frac{\rho}{\rho_\infty}\right)^{\gamma-1}$$

where

$C_\infty = \sqrt{\dfrac{\gamma P_\infty}{\rho_\infty}}$: speed of sound in the gas at infinity.

From the equation of continuity

$$\frac{1}{r^2}\frac{d}{dr}\left(r^2 \rho v\right) = 0$$

$$2r\rho v + r^2 \rho \frac{dv}{dr} + r^2 v \frac{d\rho}{dr} = 0$$

$$2r\rho v + \frac{r^2 \rho}{2v}\frac{d(v^2)}{dr} + r^2 v \frac{d\rho}{dr} = 0$$

$$\frac{1}{\rho}\frac{d\rho}{dr} = -\frac{2}{r} - \frac{1}{2v^2}\frac{d(v^2)}{dr}$$

Note that from the polytropic equation $P = K\rho^{\gamma}$,

$$\frac{dP}{dr} = \frac{d(K\rho^{\gamma})}{dr} = K\gamma\rho^{\gamma-1}\frac{d\rho}{dr} = \frac{\gamma P}{\rho}\frac{d\rho}{dr}$$

or better $\frac{1}{\rho}\frac{d\rho}{dr} = \frac{1}{\gamma P}\frac{dP}{dr}$ and thus

$$\frac{1}{\rho}\frac{d\rho}{dr} = \frac{1}{\gamma P}\frac{dP}{dr} = -\frac{2}{r} - \frac{1}{2v^2}\frac{d(v^2)}{dr}$$

From the definition of c_s we have $\gamma P = \rho c_s^2$

and then

$$\frac{1}{\rho c_s^2}\frac{dP}{dr} = -\frac{2}{r} - \frac{1}{2v^2}\frac{d(v^2)}{dr}$$

$$\frac{1}{\rho}\frac{dP}{dr} = -\frac{2c_s^2}{r} - \frac{c_s^2}{2v^2}\frac{d(v^2)}{dr}$$

Equation (II) can be written

$$v\frac{dv}{dr} + \frac{1}{\rho}\frac{\partial P}{\partial r} + \frac{GM}{r^2} = 0$$

$$\frac{1}{2}\frac{d}{dr}(v^2) + \frac{1}{\rho}\frac{\partial P}{\partial r} = -\frac{GM}{r^2}$$

$$\frac{1}{2}\frac{d}{dr}(v^2) - \frac{2c_s^2}{r} - \frac{c_s^2}{2v^2}\frac{d(v^2)}{dr} = -\frac{GM}{r^2}$$

$$\frac{1}{2}\left[1 - \frac{c_s^2}{v^2}\right]\frac{d(v^2)}{dr} = -\frac{GM}{r^2}\left[1 - \frac{2rc_s^2}{GM}\right]$$

Solutions of this equation can be sorted in six families. In order to understand the possibilities, note that c_s is a function of r. Hence:

$$\left[1 - \frac{2c_s r}{GM}\right] < 0$$ for large r because $c_s(\infty) = c_\infty$ is finite.

$$\Rightarrow \quad -\frac{GM}{r^2}\left[1 - \frac{2c_s r}{GM}\right] > 0$$ for large r

However, $\frac{d}{dr}(v^2) < 0$ for large r because in the asymptotic region the gas must be at rest.

Therefore, in order to have a positive r.h.s we need that

$v^2 < c_s^2$ for large r (i.e. the motion of the gas is subsonic)

When the gas is approaching the B.H., the radius r decreases and hence the term $\left[1 - \frac{2c_s r}{GM}\right]$ increses until, eventually it vanishes at

$$r_s = \frac{GM}{2c_s^2(r_s)}$$ Sonic Radius

At this point the left hand side must be zero, so

$v^2(r_s) = c_s^2(r_s)$ or $\left.\frac{d}{dr}(v^2)\right|_{r=r_s} = 0$.

For $r < r_s$, the sign of the r.h.s inverses and then from the l.h.s we conclude that

$v^2 > c_s^2$ for $r < r_s$ (i.e. the motion of the gas is supersonic).

With this information we may classify six types of solutions according to the behavior at the sonic radius and at the boundary ($r \to \infty$ or $r \to 0$).

Type 1 : $v^2(r_s) = c_s^2(r_s)$
$v^2 \longrightarrow 0$ for $r \to \infty$

Type 2 : $v^2(r_s) = c_s^2(r_s)$
$v^2 \longrightarrow 0$ for $r \to 0$

Type 3 : $\left.\dfrac{d(v^2)}{dr}\right|_{r=r_s} = 0$ and $v^2(r) < c_s^2(r)$ everywhere

Type 4 : $\left.\dfrac{d(v^2)}{dr}\right|_{r=r_s} = 0$ and $v^2(r) > c_s^2(r)$ everywhere

Type 5 : $\left.\dfrac{d(v^2)}{dr}\right|_{v^2=c_s^2(r_s)} \to \infty$ and $r > r_s$

Type 6 : $\left.\dfrac{d(v^2)}{dr}\right|_{v^2=c_s^2(r_s)} \to \infty$ and $r < r_s$

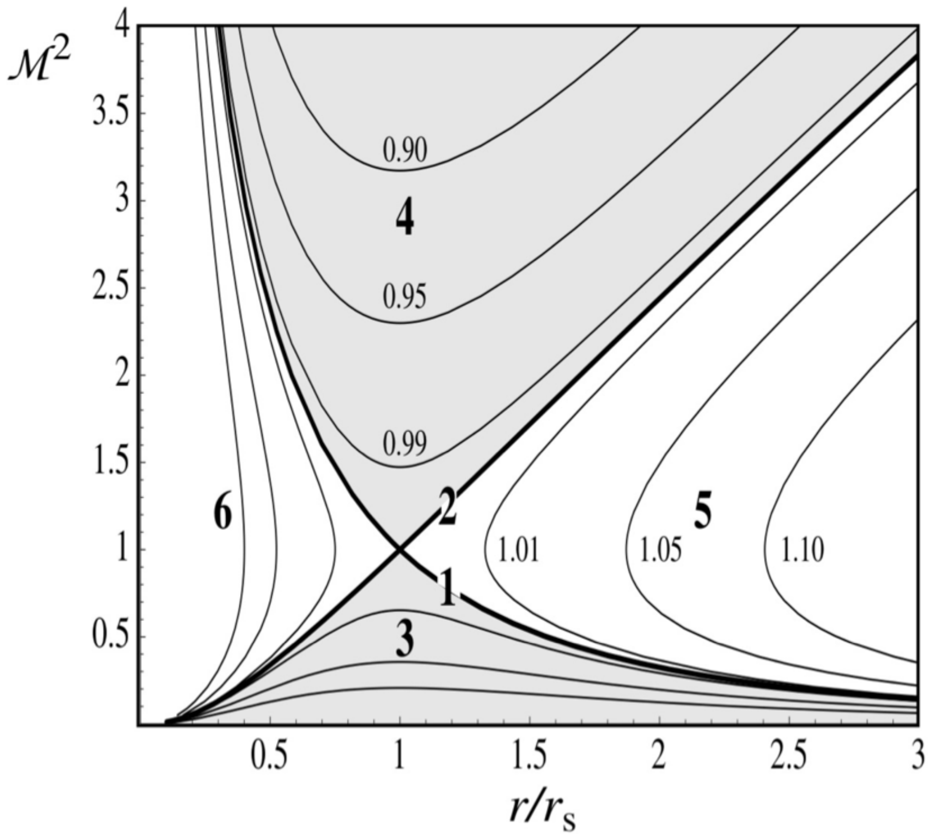


Fig. 2.1. Mach number squared $\mathcal{M}^2 = v^2(r)/c_s^2(r)$ as a function of radius r/r_s for spherically symmetrical adiabatic gas flows in the gravitational field of a star. For $v < 0$ these are accretion flows, while for $v > 0$ they are winds or 'breezes'. The two trans-sonic solutions **1**, **2** indicated by thick solid lines divide the remaining solutions into the families **3**–**6** described in the text (the case shown here is $\gamma = 4/3$, the integral curves are calculated and labelled as in Holzer & Axford (1970)).

Introducing the Mach number squared,

$$\mathcal{M}^2 = \left(\frac{v}{c_s}\right)^2$$

the behavior of the 6 types of solution is illustrated in the figure.

Type 5 and 6 do not cover all values of r and are double valued $\rightarrow$ not physically meaningful.

Type 2 and 4 are excluded because $v^2 \to \infty$ for $r \to \infty$

Solutions of Type 3 are subsonic everywhere $\rightarrow$ slow accretion flow that settles to equilibrium. However we expected the gas to be supersonic for $r < r_s$.

Thus we are interested only on Type 1 solutions which represent transonic accretion.

With this information we are ready to integrate equation (II)

$$v \frac{dv}{dr} + \frac{1}{\rho} \frac{\partial P}{\partial r} + \frac{GM}{r^2} = 0$$

Since $P = K \rho^{\gamma}$ $\Rightarrow$ $dP = K\gamma \rho^{\gamma-1} d\rho$

$$\Rightarrow \int \frac{1}{\rho} dP = K\gamma \int \frac{\rho^{\gamma-1}}{\rho} d\rho = K\gamma \int \rho^{\gamma-2} d\rho = \frac{K\gamma}{\gamma-1} \rho^{\gamma-1}$$

$$\int \frac{1}{\rho} dP = \frac{\gamma}{\gamma-1} \frac{K\rho^{\gamma}}{\rho} = \frac{\gamma P}{(\gamma-1)\rho} = \frac{c_s^2}{\gamma-1} \qquad (\gamma \neq 1)$$

or $$\int \frac{1}{\rho} dP = \int \frac{K}{\rho} d\rho = K \ln \rho \qquad (\gamma = 1)$$

but $c_s^2 = \gamma \frac{P}{\rho} = \frac{P}{\rho} = \frac{K\rho^{\gamma}}{\rho} = K$ thus

$$\int \frac{1}{\rho} dP = c_s^2 \ln \rho \qquad (\gamma = 1)$$

Therefore, for $\gamma \neq 1$ we have

$$\frac{v^2}{2} + \frac{c_s^2}{\gamma - 1} - \frac{GM}{r} = \text{Constant}$$

Type 1 solutions have $v \to \infty$ for $r \to \infty$
and we also have $\frac{GM}{r} \to 0$ for $r \to \infty$

then at $r \to \infty$

$$\frac{c_\infty^2}{\gamma - 1} = \text{Constant}$$

and hence

$$\frac{v^2}{2} + \frac{c_s^2}{\gamma - 1} - \frac{GM}{r} = \frac{c_\infty^2}{\gamma - 1}$$

Bernoulli Integral or Bernoulli Equation

At the sonic radius we have

$$r = r_s = \frac{GM}{2c_s^2(r_s)} \qquad c_s = c_s(r_s) = v(r_s)$$

$$\frac{c_s^2(r_s)}{2} + \frac{c_s^2(r_s)}{\gamma - 1} - \frac{GM\,(2c_s^2(r_s))}{GM} = \frac{c_\infty^2}{\gamma - 1}$$

$$c_s^2(r_s)\left[\frac{1}{\gamma - 1} - \frac{3}{2}\right] = \frac{c_\infty^2}{\gamma - 1}$$

$$c_s^2(r_s)\,\frac{5 - 3\gamma}{2(\gamma - 1)} = \frac{c_\infty^2}{\gamma - 1}$$

$$c_s^2(r_s) = \frac{2}{5-3\gamma} c_\infty^2$$

From the definion of the sound speed,

$$\frac{c_s^2(r_s)}{c_\infty^2} = \left(\frac{\rho(r_s)}{\rho_\infty}\right)^{\gamma-1}$$

and then

$$\left(\frac{\rho(r_s)}{\rho_\infty}\right)^{\gamma-1} = \frac{2}{5-3\gamma}$$

$$\rho(r_s) = \left[\frac{2}{5-3\gamma}\right]^{\frac{1}{\gamma-1}} \rho_\infty$$

* For $\gamma = \frac{5}{3} \Rightarrow \begin{cases} \rho(r_s) \to \infty \\ c_s(r_s) \to \infty \\ r_s \longrightarrow 0 \end{cases}$

The value of $\dot{M}$ at the sonic radius (where the gas becomes supersonic) is called the critical accretion rate,

$$\dot{M}_{crit} = 4\pi r_s^2 v(r_s) \rho(r_s)$$

$$\dot{M}_{crit} = 4\pi \frac{G^2M^2}{4 c_s^4(r_s)} c_s(r_s) \rho(r_s) = \pi \frac{G^2M^2}{c_s^3(r_s)} \rho(r_s)$$

$$\dot{M}_{crit} = \pi G^2M^2 \left(\frac{5-3\gamma}{2}\right)^{3/2} \frac{1}{c_\infty^3} \left(\frac{2}{5-3\gamma}\right)^{\frac{1}{\gamma-1}} \rho_\infty$$

$$\dot{M}_{crit} = \pi G^2M^2 \frac{\rho_\infty}{c_\infty^3} \left(\frac{2}{5-3\gamma}\right)^{\frac{5-3\gamma}{2(\gamma-1)}} \qquad (\gamma \neq 1)$$

From $\dot{M} = 4\pi r^2 \rho v$ we write

$$v = \frac{\dot{M}}{4\pi r^2 \rho}$$

or using the local sound speed, $c_s^2 = c_\infty^2 \left(\frac{\rho}{\rho_\infty}\right)^{\gamma-1}$

$$v(r) = \frac{\dot{M}}{4\pi r^2 \rho_\infty} \left(\frac{c_\infty}{c_s(r)}\right)^{\frac{2}{\gamma-1}}$$

Using this equation in Bernoulli's integral we obtain $c_s(r)$ and this gives $\rho(r)$ and $v(r)$.

For $\gamma=1$ (Isothermal flow) we have

$$\frac{v^2}{2} + c_s^2 \ln \rho - \frac{GM}{r} = \text{Constant}$$

Type 1 solutions have $v \to \infty$ for $r \to \infty$
and we also have $\frac{GM}{r} \to 0$ for $r \to \infty$

then at $r \to \infty$

$$c_\infty^2 \ln \rho_\infty = \text{Constant}$$

and hence

$$\frac{v^2}{2} + c_s^2 \ln \rho - \frac{GM}{r} = c_\infty^2 \ln \rho_\infty$$

This time the critical accretion rate is [5]

$$\dot{M}_{crit} = \pi G^2 M^2 e^{3/2} \frac{\rho_\infty}{c_\infty^3} \qquad (\gamma=1)$$

$$\dot{M}_{crit} = 4\pi G^2 M^2 \lambda(\gamma) \frac{\rho_\infty}{c_\infty^3}$$ Physical Accretion rate

$$\lambda(\gamma) = \begin{cases} \frac{1}{4}\left(\frac{2}{5-3\gamma}\right)^{\frac{5-3\gamma}{2(\gamma-1)}} & \gamma \neq 1 \\ e^{3/2} & \gamma = 1 \end{cases}$$

The accretion rate depends on the equation of state.

As γ increases from $1 \longrightarrow \frac{5}{3}$;
λ falls from 1.12 to 0.25

$\gamma = \frac{5}{3}$: Particles with <u>no</u> internal degrees of freedom
In this case $r_s \to 0$; the sonic point goes to the origin.
The accretion rate falls

$\gamma = 1$: Nearly isothermal
Sonic point at relatively large distance
Larger accretion rate

ASTROPHYSICAL PLASMAS: Adiabatic with $\gamma = \frac{5}{3}$
(no heating or cooling)

Often, heating and/or cooling is important $\longrightarrow$
the equation of state is integrated from an energy condition!!

From Bernoulli's equation we have

$$\frac{v^2}{2} + \frac{c_s^2}{\gamma - 1} - \frac{GM}{r} = \frac{c_\infty^2}{\gamma - 1}$$

$$c_s^2 = c_\infty^2 + \left[\frac{GM}{r} - \frac{v^2}{2}\right](\gamma - 1)$$

$$c_s^2 = c_\infty^2 + \left[\frac{GM}{r} - \frac{v^2}{2}\right](\gamma - 1)$$

$$\left(\frac{c_s}{c_\infty}\right)^2 = 1 + \left[\frac{GM}{r} - \frac{v^2}{2}\right]\frac{(\gamma - 1)}{c_\infty^2}$$

$$\left(\frac{c_\infty}{c_s(r)}\right)^{\frac{2}{\gamma-1}} = \left\{1 + \left[\frac{GM}{r} - \frac{v^2}{2}\right]\frac{(\gamma - 1)}{c_\infty^2}\right\}^{-\frac{1}{\gamma-1}}$$

and thus,

$$v(r) = \frac{\dot{M}}{4\pi r^2 \rho_\infty}\left(\frac{c_\infty}{c_s(r)}\right)^{\frac{2}{\gamma-1}}$$

$$v = \frac{\dot{M}}{4\pi r^2 \rho_\infty}\left\{1 + \left[\frac{GM}{r} - \frac{v^2}{2}\right]\frac{(\gamma - 1)}{c_\infty^2}\right\}^{-\frac{1}{\gamma-1}}$$

Introducing the accretion radius $r_{acc} = \dfrac{2GM}{c_\infty^2}$

$$v = \frac{\dot{M}\, c_\infty^4}{16\pi G^2 M^2 \rho_\infty}\left(\frac{r_{acc}}{r}\right)^2\left\{1 + \left[2c_\infty^2\left(\frac{r_{acc}}{r}\right) - \frac{v^2}{2}\right]\frac{(\gamma - 1)}{c_\infty^2}\right\}^{-\frac{1}{\gamma-1}}$$

$$v = \frac{c_\infty}{16}\left(\frac{2}{5-3\gamma}\right)^{\frac{5-3\gamma}{2(\gamma-1)}}\left(\frac{r_{acc}}{r}\right)^2\left\{1+\left[2c_\infty^2\left(\frac{r_{acc}}{r}\right)-\frac{v^2}{2}\right]\frac{(\gamma-1)}{c_\infty^2}\right\}^{-\frac{1}{\gamma-1}}$$

In the limit $r \gg r_{acc}$ we have $v \to 0$ and from above equation the approximation

$$v \approx \frac{c_\infty}{16}\left(\frac{2}{5-3\gamma}\right)^{\frac{5-3\gamma}{2(\gamma-1)}}\left(\frac{r_{acc}}{r}\right)^2\left[1-\frac{1}{2}\frac{r_{acc}}{r}\right] \approx 0$$

From Bernoulli's equation

$$\left(\frac{c_s}{c_\infty}\right)^2 = 1+\left[\frac{GM}{r}-\frac{v^2}{2}\right]\frac{(\gamma-1)}{c_\infty^2}$$

but

$$\frac{v^2}{2} \approx \frac{c_\infty^2}{16^2}\left(\frac{2}{5-3\gamma}\right)^{\frac{5-3\gamma}{(\gamma-1)}}\left(\frac{r_{acc}}{r}\right)^4\left[1-\frac{1}{2}\frac{r_{acc}}{r}\right]^2 \cong 0$$

hence

$$\left(\frac{c_s}{c_\infty}\right)^2 \approx 1+\left[\frac{GM}{r}\right]\frac{(\gamma-1)}{c_\infty^2} \approx 1+\frac{1}{2}\frac{r_{acc}}{r}(\gamma-1)$$

$$c_s^2 \approx c_\infty^2\left[1+\left(\frac{\gamma-1}{2}\right)\frac{r_{acc}}{r}\right]$$

$$c_s \approx c_\infty\left[1-\left(\frac{\gamma-1}{4}\right)\frac{r_{acc}}{r}\right] \approx c_\infty$$

Finally, the density profile can be approximated as

$$\rho = \rho_\infty \left(\frac{c_s}{c_\infty}\right)^{\frac{2}{\gamma-1}} \approx \rho_\infty \left[1 + \left(\frac{\gamma-1}{4}\right)\frac{r_{acc}}{r}\right]^{\frac{2}{\gamma-1}}$$

$$\rho \approx \rho_\infty \left(1 + \frac{1}{2}\frac{r_{acc}}{r}\right) \approx \rho_\infty$$

For $r \gg r_{acc}$

$$v \approx \frac{c_\infty}{16}\left(\frac{2}{5-3\gamma}\right)^{\frac{5-3\gamma}{2(\gamma-1)}}\left(\frac{r_{acc}}{r}\right)^2\left[1 - \frac{1}{2}\frac{r_{acc}}{r}\right] \approx 0$$

$$c_s \approx c_\infty \left[1 - \left(\frac{\gamma-1}{4}\right)\frac{r_{acc}}{r}\right] \approx c_\infty$$

$$\rho \approx \rho_\infty \left(1 + \frac{1}{2}\frac{r_{acc}}{r}\right) \approx \rho_\infty$$

Note/: The accretion radius can be interpreted as that where thermal and gravitational energy of the gas are comparable:

$$\frac{\rho\, c_s^2(r_{acc})}{2} \sim \frac{\rho G M}{r_{acc}}$$

For $r \ll r_s$ the flow is supersonic $v \gg c_s$.
From Bernoulli's equation,

$$\frac{v^2}{2} \approx \frac{GM}{r}$$

$$\boxed{v \approx \sqrt{\frac{2GM}{r}} = v_{ff}}$$: Free fall velocity

$v \sim v_{ff}$ because the underlying layers do not affect the gas motion.
From the continuity equation,

$$\frac{d}{dr}(r^2 \rho v) = 0 \quad \Rightarrow \quad r^2 \rho v = \text{const.} = r_s^2 \rho(r_s)\, v(r_s)$$

$$\rho = \frac{r_s^2}{r^2} \frac{\rho(r_s)\, c_s}{v}$$

$$\rho \approx \frac{r_s^2}{r^2} \frac{\rho(r_s)\, c_s}{\sqrt{2GM}} \sqrt{r} = \rho(r_s) \frac{r_s^2}{r^{3/2}} \frac{c_s}{\sqrt{2GM}} = \rho(r_s) \frac{r_s^2}{r^{3/2}} \frac{1}{r_s^{1/2}}$$

$$\boxed{\rho \approx \rho(r_s) \left(\frac{r_s}{r}\right)^{3/2}}$$

Considering an ideal gas behavior in this region, we have the temperature

$$T = \frac{\mu m_H}{k} \frac{P}{\rho} = \frac{\mu m_H}{k} \frac{P(r_s)}{\rho} \left(\frac{\rho}{\rho(r_s)}\right)^{\gamma}$$

$$T = \frac{\mu m_H}{k} \frac{P(r_s)}{\rho(r_s)} \frac{\rho^{\gamma-1}}{\rho^{\gamma-1}(r_s)} = T(r_s) \frac{\rho^{\gamma-1}}{\rho^{\gamma-1}(r_s)}$$

$$\boxed{T = T(r_s)\left(\frac{r_s}{r}\right)^{\frac{3}{2}(\gamma-1)}}$$

*Revisar [6] eq. 4.39

If this temperature is high enough, the flow will radiate and cool. Using the second law,

$$d\varepsilon = dQ - PdV$$

where $V = \frac{1}{\rho}$: specific volume (i.e. per unit mass)

dQ : heat exchanged per unit mass

$$d\varepsilon = dQ - \left(-\frac{k\rho}{\mu m_H}\frac{T}{\rho^2}d\rho\right)$$

$$d\varepsilon = dQ + \frac{k}{\mu m_H}\frac{T}{\rho}d\rho$$

For a monoatomic gas and due to the equipartition of energy (eq. 2.4 of [7]).

$$\varepsilon = \frac{3}{2}\frac{kT}{\mu m_H} \longrightarrow d\varepsilon = \frac{3}{2}\frac{k}{\mu m_H}dT$$

Takin this into account and introducing a term due to Bremsstrahlung (free-free) radiation

$$\frac{dQ}{dt} = -\alpha_{ff}T^{1/2}\rho$$

$\alpha_{ff} \simeq 5\times10^{20}$ erg cm^3 g^{-2} s^{-1} K$^{-1/2}$ for H.

we write

$$\frac{3}{2}\frac{K}{\mu m_H}\frac{dT}{dt} = \frac{K}{\mu m_H}\frac{T}{\rho}\frac{d\rho}{dt} - \underbrace{\alpha_{ff}\, T^{1/2}\rho}_{\text{Bremsstrahlung}} + \underbrace{\frac{dQ}{dt}}_{\substack{\text{other}\\ \text{Raddiative}\\ \text{losses.}}}$$

Using $dr = v\,dt$ we write

$$\frac{3}{2}\frac{Kv}{\mu m_H}\frac{dT}{dr} = \frac{Kv}{\mu m_H}\frac{T}{\rho}\frac{d\rho}{dr} - \alpha_{ff}\, T^{1/2}\rho + v\frac{dQ}{dr}$$

$$\frac{dT}{dr} = \frac{2}{3}\frac{T}{\rho}\frac{d\rho}{dr} - \frac{2\mu m_H}{3Kv}\alpha_{ff}\, T^{1/2}\rho + \frac{2\mu m_H}{3K}\frac{dQ}{dr}$$

Considering $v \approx \sqrt{\frac{2GM}{r}}$ and $\rho = \rho(r_s)\left(\frac{r_s}{r}\right)^{3/2}$

$$\frac{d\rho}{dt} = \rho(r_s)\frac{3}{2}\left(\frac{r_s}{r}\right)^{1/2}\left(-\frac{r_s}{r^2}\right)$$

$$\frac{2}{3}\frac{T}{\rho}\frac{d\rho}{dt} = -\frac{T\rho(r_s)}{\rho}\left(\frac{r_s}{r}\right)^{3/2}\frac{1}{r}$$

$$\frac{2}{3}\frac{T}{\rho}\frac{d\rho}{dt} = -\frac{T}{r}$$

Thus

$$\frac{dT}{dr} = -\frac{T}{r} - \frac{2\mu m_H}{3K}\sqrt{\frac{r}{2GM}}\,\alpha_{ff}\, T^{1/2}\rho(r_s)\left(\frac{r_s}{r}\right)^{3/2} + \frac{2\mu m_H}{3K}\frac{dQ}{dr}$$

$$\frac{dT}{dr} = -\frac{T}{r} - \alpha_{ff}\,\rho(r_s)\sqrt{\frac{r_s}{2GM}}\,\frac{T^{1/2}}{r}\left(\frac{2\mu m_H}{3K} r_s\right) + \frac{2\mu m_H}{3K}\frac{dQ}{dr}$$

When there are no additional radiation losses besides Bremsstrahlung, we have

$$\frac{dT}{dr} = -\frac{T}{r} - \frac{1}{2}K\frac{T^{1/2}}{r}$$

where $\frac{1}{2}K = \alpha_{ff}\, P(r_s)\sqrt{\frac{r_s}{2GM}}\left(\frac{2\mu m_H}{3K} r_s\right)$

Note that $\frac{dT}{dr} < 0$ so the temperature of the flow decreases as the flow approaches the BH.
This is called a "cooling flow".

The Bremsstrahlung term gives

$$\frac{2\,dT}{K T^{1/2}} = -\frac{dr}{r}$$

$$\left.\frac{T^{1/2}}{K}\right|_{T}^{T_\infty} = -\ln r\Big|_{r}^{r_G}$$

where we assume that $T(r_G) = T_\infty$,

$$T = \left[K \ln\left|\frac{r}{r_G}\right| + T_\infty^{1/2}\right]^2 \qquad \text{see [6] (4.43)}$$

Considering the two terms in the r.h.s. we obtain

$$\frac{dT}{T + \frac{1}{2} K T^{1/2}} = - \frac{dr}{r}$$

$$2 \ln \left[K + 2\sqrt{T} \right] \Big|_{T}^{T_\infty} = - \ln r \Big|_{r}^{r_G}$$

$$\left[K + 2\sqrt{T} \right] \Big|_{T}^{T_\infty} = r^{-1/2} \Big|_{r}^{r_G}$$

$$2 \left(\sqrt{T_\infty} - \sqrt{T} \right) = \frac{1}{\sqrt{r_G}} - \frac{1}{\sqrt{r}}$$

$$\sqrt{T} - \frac{1}{2\sqrt{r}} = \sqrt{T_\infty} - \frac{1}{2\sqrt{r_G}} = \text{Constant} = C$$

$$T = \left[C + \frac{1}{2\sqrt{r}} \right]^2$$

The radial free fall time is

$$t_{ff} \approx \frac{r}{v_{ff}} \propto \frac{r}{r^{-1/2}} \propto r^{3/2}$$

while the cooling time is

$$t_{cool} \approx \frac{\frac{3}{2}\frac{kT}{\mu m_p}}{\alpha_{ff}\, \rho\, T^{1/2}} \propto \frac{\sqrt{T}}{\rho} \propto \frac{r^{-1/2}}{r^{-3/2}} \propto r$$

Referencias

[1] Bradley Peterson. An Introduction to Active Galactic Nuclei. Cambidge University Press. (1997)

[2] https://rechneronline.de/spectrum/

[3] S.L. Shapiro & S.A. Teukolsky. "Black Holes, White Dwarfs and Neutron Stars: The Physics of Compact Objects". (1983)

[4] V. Frolov and I. Novikov. "Black Hole Physics: Basic Concepts and New Developments" (1998)

[5] H. Bondi. Mon. Not. R. Astron. Soc. 112, 95 (1952)

[6] G. Romero & G. Vila. "Introduction to BH Astrophysics". (2014)

[7] J. Frank, A. King & D. Raine. "Accretion Power in Astrophysics". (2002)