03. BINARY SYSTEMS. GEOMETRY

STELLAR STRUCTURE

Equations of the hydrostatic equilibrium:

$$\frac{dP(r)}{dr} = -\frac{G M(r) \rho(r)}{r^2}$$

A.S. Eddington made two fundamental contributions

i) The source of stellar energy is thermonuclear
ii)

Mass Function for a Binary System

Tidal forces acting on the members of the binary system make the orbits to become circles after some time interval. Once both orbits are circular, Newton's second law is written

$$M_* a_* = M_* \frac{v_*^2}{r_*} = \frac{G M_D M_*}{r^2}$$

$$\frac{v_*^2}{r_*} = \frac{G M_D}{r^2}$$

where
- M_* : Mass of the visible star
- M_D : Mass of the invisible companion
- v_* : orbital speed of the star
- r_* : radius of the orbit of the star
- r : distance between M_* and M_D.

From the definition of the centre of mass we have

$$M_* r_* = M_D r_D$$

where r_D is the radius of the orbit of the invisible companion and $r = r_D + r_*$. Thus, we have

$$r = r_* \left[1 + \frac{r_D}{r_*} \right] = r_* \left[1 + \frac{M_*}{M_D} \right]$$

On the other hand, the orbital period of the system can be written

$$P = \frac{2\pi r_*}{v_*} .$$

Hence, using these equations we write Newton's second law as

$$r_* = \frac{G M_D}{v_*^2 \left[1 + \frac{M_*}{M_D}\right]^2}$$

and then

$$P = \frac{2\pi}{v_*} \frac{G M_D}{v_*^2 \left[1 + \frac{M_*}{M_D}\right]^2}$$

$$\frac{P v_*^3}{2\pi G} = \frac{M_D}{\left[1 + \frac{M_*}{M_D}\right]^2}$$

The velocity in the plane of the orbit, v_*, is related with the velocity in the line of sight V_* through

$$V_* = v_* \sin i$$

where i is the angle between the line of sight and the normal to the orbital plane.
Then, we obtain the mass function of the system as

$$f(M_D, M_*) = \frac{P V_*^3}{2\pi G} = \frac{M_D \sin^3 i}{\left[1 + \frac{M_*}{M_D}\right]^2}$$

Since

$$\frac{M_D}{\left[1 + \frac{M_*}{M_D}\right]^2} = \frac{M_D^3}{\left[M_* + M_D\right]^2}$$

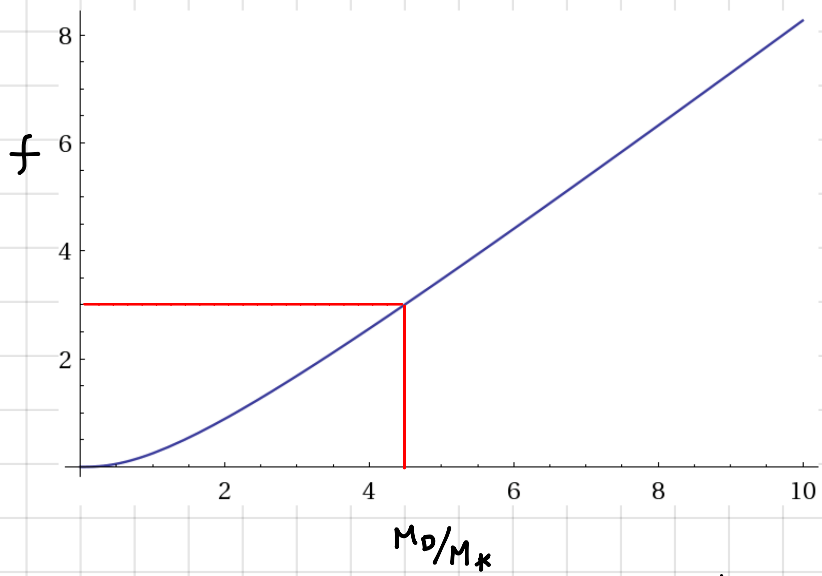
we can write the mass function as

$$f(M_D, M_*) = \frac{P V_*^3}{2\pi G} = \frac{M_D^3 \sin^3 i}{[M_* + M_D]^2}$$

Since $\sin i \leqslant 1$ and $\frac{M_D}{[M_* + M_D]} < M_D$ we conclude that

$$f(M_D, M_*) < M_D$$

and therefore f gives a lower limit to M_D.
Then, if $f(M_D, M_*) > 3 M_\odot$ we have a black hole candidate.



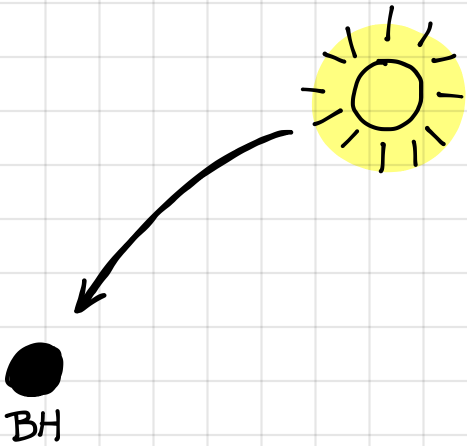
Since we can write

$$V_* = v_s \sin i = \frac{2\pi}{P} r_* \sin i,$$

the mass function becomes

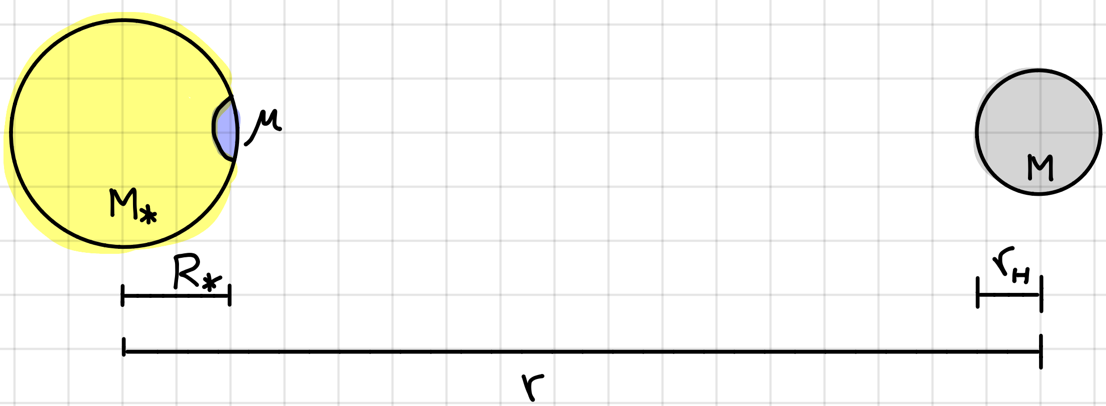
$$f(M_D, M_*) = \frac{4\pi^2 (r_* \sin i)^3}{G P^2} = \frac{M_D^3 \sin^3 i}{[M_* + M_D]^2}$$

ACCRETION ONTO A SCHWARZSCHILD BLACK HOLE



swallowed as a whole
or
disrupted by tidal forces?

ROCHE LIMIT FOR RIGID STARS



The Tidal force on the mass element μ of the star is calculated as

$$\Delta f_T = \frac{GM\mu}{(r-R_*)^2} - \frac{GM\mu}{r^2}$$

$$\Delta f_T = GM\mu\left[\frac{r^2-(r-R_*)^2}{r^2(r-R_*)^2}\right]$$

$$\Delta f_T = GM\mu\left[\frac{2rR_* - R_*^2}{r^2(r-R_*)^2}\right]$$

If $r \gg r_*$ we approximate

$$\Delta f_T \simeq GM\mu \left[\frac{2rR_*}{r^4} \right]$$

$$\Delta f_T \simeq \frac{2GM\mu R_*}{r^3}$$

The Roche limit is given at the point $r = r_R$ where the Tidal force and the gravitational force balance each other

$$\frac{GM_*\mu}{R_*^2} = \frac{2GM\mu R_*}{r_R^3}$$

$$r_R = \left(\frac{2M}{M_*} \right)^{1/3} R_*$$

$$r_R = 1.26 \left(\frac{M}{M_*} \right)^{1/3} R_*$$

$\Rightarrow$ For a fluid star, the Roche limit is corrected to

$$r_R = 2.43 \left(\frac{M}{M_*} \right)^{1/3} R_*$$

due to the deformation.

ROCHE LIMIT FOR A RIGID STAR INCLUDING THE ORBITAL MOTION

Assumptions

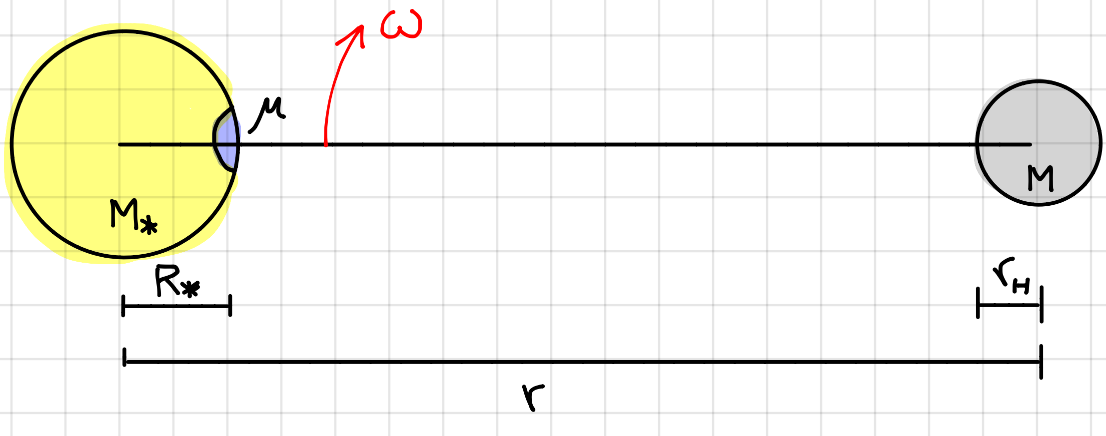
* The stars moves in a circular orbit
* Its motion is synchronous ; i.e. its angular speed w.r.t. its center of mass ω is equal to its angular speed w.r.t. the binary system baricenter.

From Kepler's third law, we have

$$\omega^2 = \frac{G(M_* + M)}{r^3}$$

but, if $M \gg M_*$ we approximate

$$\omega^2 \simeq \frac{GM}{r^3}$$



The tidal gravitational force is again

$$\Delta f_T \simeq \frac{2GM\mu R_*}{r^3}$$

However, now we include the "tidal" centripetal force,

$$\Delta f_c = \mu \omega^2 (r - R_*) - \mu \omega^2 r$$

$$\Delta f_c = -\mu \omega^2 R_*$$

← the minus sign indicates that the centripetal force at r is greater than the force at $(r - R_*)$

This time, the equilibrium equation reads

$$\frac{2GM\mu R_*}{r_R^3} = \frac{GM_*\mu}{R_*^2} - \mu \omega^2 R_*$$

$$\frac{2GM\mu R_*}{r_R^3} + \mu \omega^2 R_* = \frac{GM_*\mu}{R_*^2}$$

$$\frac{2GM\mu R_*}{r_R^3} + \mu \frac{GM}{r_R^3} R_* = \frac{GM_*\mu}{R_*^2}$$

$$\frac{3GM\mu R_*}{r_R^3} = \frac{GM_*\mu}{R_*^2}$$

From which the Roche limit is

$$r_R^3 = 3 \frac{M}{M_*} R_*^3$$

$$r_R = 1.44 \left(\frac{M}{M_*}\right)^{1/3} R_*$$

Roche Limit for a Fluid Star

Assumptions

* Perfect fluid
* Density ρ_* and volume V_*
* The stars moves in a circular orbit
* Its motion is synchronous; i.e. its angular speed w.r.t. its center of mass ω is equal to its angular speed w.r.t. the binary system baricenter.

From Kepler's third law, we have

$$\omega^2 = \frac{G(M_* + M)}{r^3}$$

but, if $M \gg M_*$ we approximate

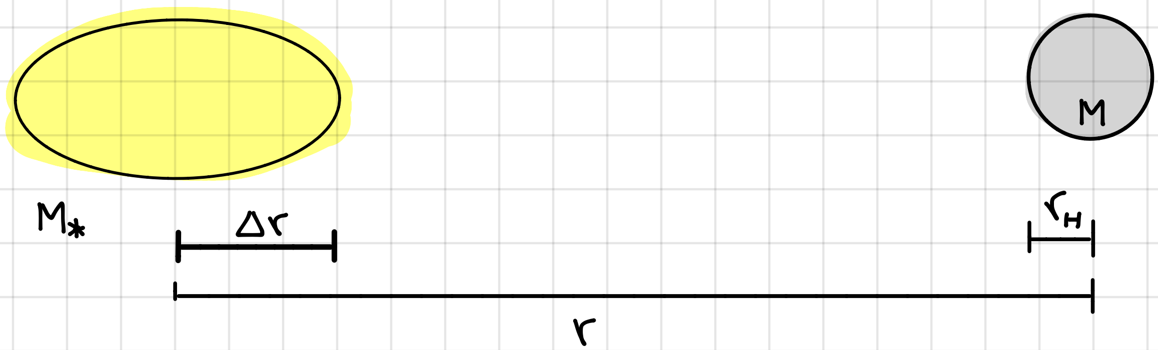
$$\omega^2 \simeq \frac{GM}{r^3}$$

Synchronous motion implies that the fluid doesn't move and the problem becomes a static one. Thus, viscosity and friction of the fluid are not important. In fact the relevant forces for the analysis are

- Gravitational force due to M
- Centrifugal force in the rotating frame
- Self-gravitational force of the star

All these forces can be derived from a potential and the surface of the fluid star is an equipotential (if not, the forces will act on the fluid producing its motion until it reaches a static configuration).

Since the orbit is circular, the gravitation force due to M is exactly contrarested by the orbital centrifugal force. Hence we will work just with two forces: the tidal force and the rotational centrifugal force.



The self-gravitational force on the star's material depends on the distance. For example, particles at the surface of the star on the line joining the star with the black hole are located at distance Δr from the star's center of mass.

As in the rigid case discussed above, the total tidal force (including the centipetal term) is proportional to the distance Δr and therefore the related potential will be proportional to $(\Delta r)^2$;

$$\Phi_T = -\frac{3GM}{2r^3}(\Delta r)^2 .$$

The geometry of the surface of the star is given by the condition

$$\Phi = \Phi_T + \Phi_* = \text{Constant}$$

Since gravitational force produced by M depends only on the radial distance, we assume that the shape of the star is an axially symmetric solid of revolution.

Solving Poisson equation

$$\nabla^2 \Phi = 4\pi G \rho$$

for an ellipsoid gives the potential at its surface

$$\Phi_s = \Phi_o + \pi G \rho_* f(\epsilon) (\Delta r)^2$$

where Φ_o is a constant, ϵ is the excentricity of the ellipsoid and

$$f(\epsilon) = \frac{1-\epsilon^2}{\epsilon^3} \left[(3-\epsilon^2) \tanh^{-1}\epsilon - 3\epsilon \right].$$

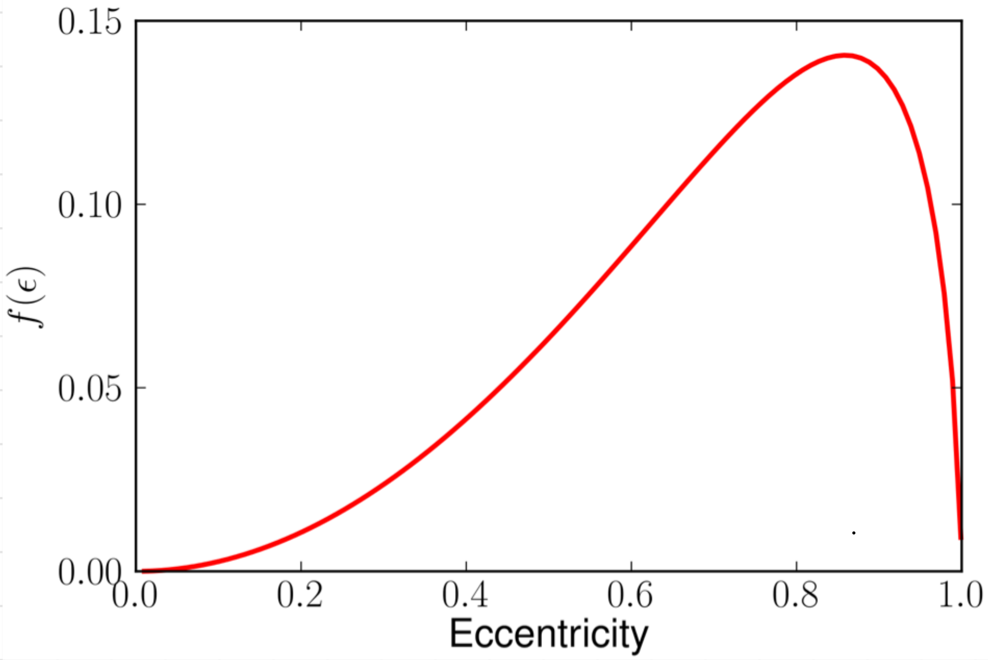
In order to obtain the shape of the star, we equal the ellipsoid potential with Φ_T (up to the constant Φ_o);

$$\frac{3GM}{2r^3} (\Delta r)^2 = \pi G \rho_* f(\epsilon) (\Delta r)^2$$

$$\frac{3GM}{2r^3} = \pi G \rho_* f(\epsilon)$$

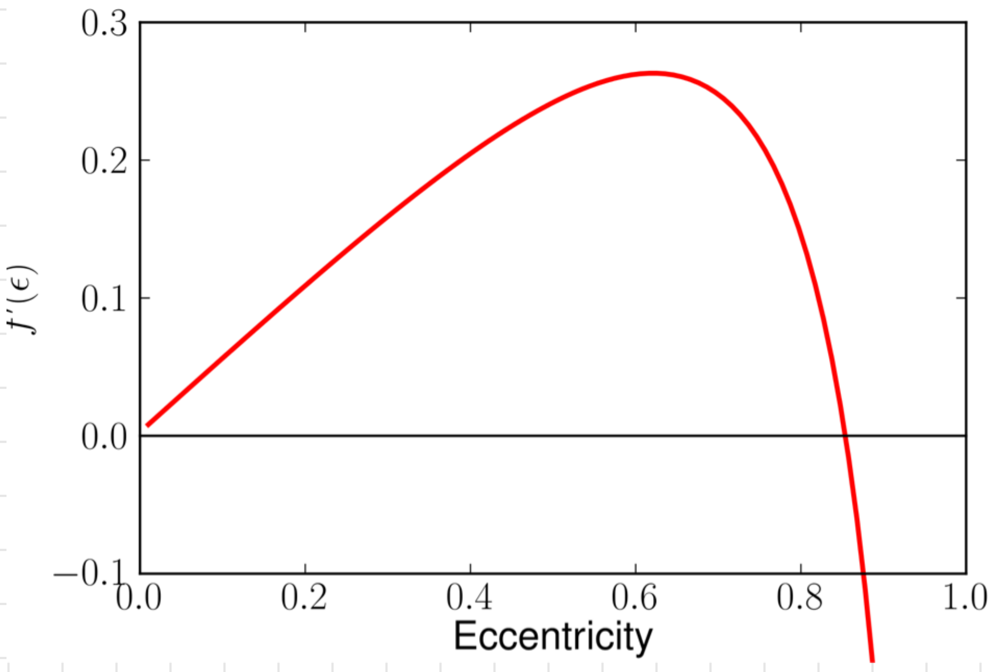
$$\frac{3M}{2\pi \rho_* r^3} = f(\epsilon)$$

The behavior of function $f(\epsilon)$ is



and in order to solve numerically the equation to obtain ϵ, we consider the derivative

$$0 = \frac{df}{d\epsilon}$$



The extreme values of $f(\epsilon)$ correspond to the stable equilibrium shapes of the ellipsoid. Numerically the maximum is obtained at

$$\epsilon_{max} \approx 0{,}86$$

which corresponds to an ellipsoid with axes satisfying a ratio 1:1.95. The corresponding value of $f(\epsilon)$ is

$$f_{max} = f(\epsilon_{max}) = 0.14$$

This value gives the equation for the Roche limit

$$\frac{3M}{2\pi \rho_* r_R^3} = f_{max}$$

$$r_R^3 = \frac{3M}{2\pi f_{max} \rho_*}$$

The density of the star can be writtes as

$$\rho_* = \frac{M_*}{V_*}$$

where the volume of the spheroid is

$$V_* = \frac{4\pi}{3} abc = \frac{4}{3} \pi a b^2$$

and the relation between the axis a and b is given by

$$b^2 = (1-\epsilon^2) a^2$$

Hence

$$V_* = \frac{4}{3}\pi a^3 (1-\epsilon^2).$$

We will introduce a Star's radius R_* such that

$$V_* = \frac{4}{3}\pi R_*^3$$

i.e.

$$R_*^3 = a^3(1-\epsilon^2) \longrightarrow R_* = \sqrt[3]{1-\epsilon^2}\, a.$$

In our particular case, $\epsilon = 0.86$ so

$$R_* \simeq 0.64\, a$$

Using this radius, we can write

$$r_R^3 = \frac{2}{f_{max}}\left(\frac{M}{M_*}\right) R_*^3$$

$$r_R^3 = 14.29\left(\frac{M}{M_*}\right) R_*^3$$

$$r_R = 2.43\left(\frac{M}{M_*}\right)^{1/3} R_*$$ Roche limit for a fluid star.

* Taking $f_{max} = 1 \rightarrow$ Rigid Star
$f_{max} = \frac{2}{3} \longrightarrow$ Rigid Star with orbital motion

In terms of the density $\rho_* = \frac{3M_*}{4\pi R_*^3}$ we have

$$r_R^3 = \frac{3M}{2\pi f_{max}\, \rho_*}$$

Introducing Sun's density

$$\rho_\odot = \frac{M_\odot}{\frac{4}{3}\pi R_\odot^3}$$

we write

$$r_R^3 = \frac{3M}{2\pi f_{max}\, \rho_*} \times \frac{\rho_\odot}{M_\odot} \frac{4\pi}{3} R_\odot^3$$

$$r_R^3 = \frac{2}{f_{max}} \left(\frac{M}{M_\odot}\right)\left(\frac{\rho_*}{\rho_\odot}\right)^{-1} R_\odot^3$$

Knowing that the Schwarzschild's radius of the Sun is

$$R_{\odot s} = \frac{2GM_\odot}{c^2} \sim 1\,\text{km} = 10^5\,\text{cm}$$

and $R_\odot \sim 700.000$ km $\sim 7\times10^5$ km $\sim 7\times10^5\, R_{\odot s}$

then

$$\frac{R_\odot c^2}{2GM_\odot} = \frac{R_\odot}{R_{\odot s}} \sim 7\times10^5$$

and we have

$$r_R^3 = \frac{2}{f_{max}}\left(\frac{M}{M_\odot}\right)\left(\frac{\rho_*}{\rho_\odot}\right)^{-1}\left(7\times10^5\right)^3\left(\frac{2GM_\odot}{c^2}\right)^3$$

$$r_R^3 = \frac{2}{f_{max}}\times(7\times10^5)^3\left(\frac{\rho_*}{\rho_\odot}\right)^{-1}\left(\frac{M_\odot}{M}\right)^2\left(\frac{2GM}{c^2}\right)^3$$

$$r_R^3 = \frac{2}{f_{max}}\times(7\times10^5)^3\left(\frac{\rho_*}{\rho_\odot}\right)^{-1}\left(\frac{M}{M_\odot}\right)^{-2} r_H^3$$

$$r_R = \frac{\sqrt[3]{2}\times(7\times10^5)}{\sqrt[3]{f_{max}}}\left(\frac{\rho_*}{\rho_\odot}\right)^{-1/3}\left(\frac{M}{M_\odot}\right)^{-2/3} r_H$$

$$\frac{r_R}{r_H} = \frac{8.82}{\sqrt[3]{f_{max}}}\times10^5\left(\frac{\rho_*}{\rho_\odot}\right)^{-1/3}\left(\frac{M}{M_\odot}\right)^{-2/3}$$

$$\frac{r_R}{r_H} = \frac{8.82}{\sqrt[3]{f_{max}}}\times10^5\left(\frac{\rho_*}{\rho_\odot}\right)^{-1/3}\left(\frac{M}{M_\odot\times10^8}\right)^{-2/3}\left(10^8\right)^{-2/3}$$

$$\frac{r_R}{r_H} = \frac{8.82}{\sqrt[3]{f_{max}}}\times10^5\times10^{-16/3}\left(\frac{\rho_*}{\rho_\odot}\right)^{-1/3} M_8^{-2/3}$$

$$\frac{r_R}{r_H} = \frac{4.09}{\sqrt[3]{f_{max}}}\left(\frac{\rho_*}{\rho_\odot}\right)^{-1/3} M_8^{-2/3}$$

For a rigid star we take $f_{max}=1$ and then

$$\frac{r_R}{r_H} = 4.09\left(\frac{\rho_*}{\rho_\odot}\right)^{-1/3} M_8^{-2/3}$$

For a rigid star with orbital motion we take $f_{max} = 2/3$ and then

$$\frac{r_R}{r_H} = 4.68 \left(\frac{\rho_*}{\rho_\odot}\right)^{-1/3} M_8^{-2/3}$$

For a fluid star: $f_{max} = 0.14$ then

$$\frac{r_R}{r_H} = 7.88 \left(\frac{\rho_*}{\rho_\odot}\right)^{-1/3} M_8^{-2/3}$$

The tidal disruption of the star occurs if the Roche limit is outside the event horizon, i.e. if

$$\frac{r_R}{r_H} = 7.88 \left(\frac{\rho_*}{\rho_\odot}\right)^{-1/3} M_8^{-2/3} > 1$$

$$M_8^{2/3} < 7.88 \left(\frac{\rho_*}{\rho_\odot}\right)^{-1/3}$$

$$M_8 < 22.12 \left(\frac{\rho_*}{\rho_\odot}\right)^{-1/2}$$

$$M < 2{,}21 \left(\frac{\rho_*}{\rho_\odot}\right)^{-1/2} 10^9 M_\odot$$ Fluid Star

Similarly,

$$M < 8{,}27 \left(\frac{\rho_*}{\rho_\odot}\right)^{-1/2} 10^8 M_\odot$$ Rigid

$$M < 1{,}01 \left(\frac{\rho_*}{\rho_\odot}\right)^{-1/2} 10^9 M_\odot$$ Rigid with orbital motion

Sun's density is $\rho_\odot \approx 1.41 \frac{gr}{cm^3}$

Main sequence stars have densities $\rho_* \sim 1 \left(\frac{M_*}{M_\odot}\right)^{-2} \frac{gr}{cm^3}$.

From this information we conclude that

If $M > 10^9 M_\odot$ (Supermassive B.H.) the tidal radius for solar-type stars lies within the event horizon of the BH. $\longrightarrow$ No disruption !!!

Only BHs with $M \ll 10^9 M_\odot$ (or stars with a considerable less density than main sequence stars, e.g. red giants); produce (feel) tidal forces that effectively disrupt stars.

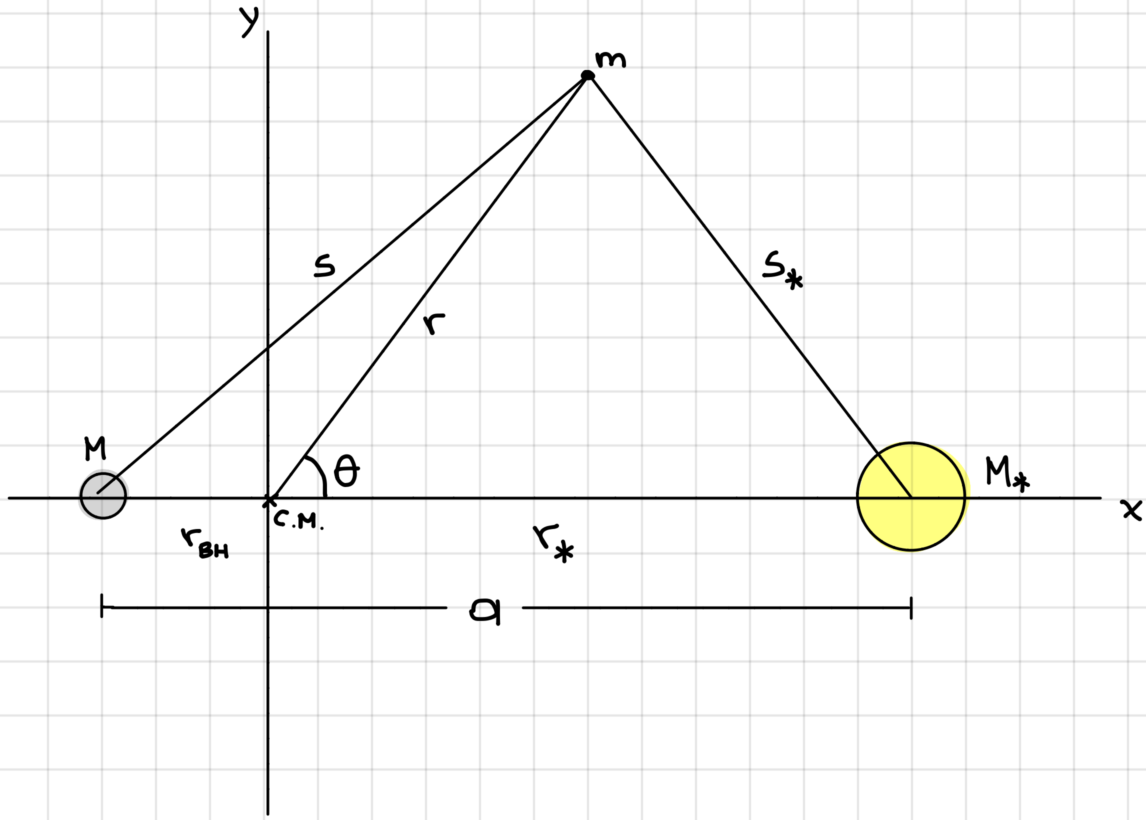
ROCHE LOBE

When the objects in a binary system, e.g. a star and a black hole, are close enough, the gravitational force can be so strong that the star may have its outer layers deformed into a teardrop shape.
As the star rotates and because the black hole creates the tidal bulge, it is forced to pulsate. These oscillations are the result of standing sound waves resonating in the star's interior with a nod at the center of the star (gases do not move) and an antinode at the star's surface. The thermodynamical proccesses at the interor provide some mechanism that can damp (or amplify) the oscillations. This behavior results in the dissipation of orbital and rotational energy until the system reaches the state of minimum energy for its constant angular momentum: synchronous rotation in circular orbits. In this state, the star always faces the black hole and the system rotates "rigidly" so no more energy can be lost.

We will call ω the angular velocity of the system,

$$\omega = \frac{v_*}{r_*} = \frac{v_{BH}}{r_{BH}}$$

Working in a corotating frame centered at the center of mass of the system. At this frame, the two objects are at rest, with their gravitational force balanced by the centrifugal force.



From the definition of center of mass we have

$$r_{BH} + r_* = a$$

$$M r_{BH} = M_* r_* .$$

In the corotating frame, a third object with negligible mass m feels a centrifugal force

$$\vec{f}_c = m\omega^2 r\,\hat{r}$$

and this force can be obtained from the potential

$$U_c = -\frac{1}{2} m\omega^2 r^2 .$$

The gravitational potential that m feels is

$$U_g = -\frac{GMm}{s} - \frac{GM_* m}{s_*}$$

Hence, the total potential for the particle can be written

$$\Phi = \frac{U}{m} = \frac{U_g + U_c}{m}$$

$$\Phi = -\frac{GM}{s} - \frac{GM_*}{s_*} - \frac{1}{2}\omega^2 r^2$$

From the law of cosines,

$$s^2 = r^2 + r_{BH}^2 + 2 r r_{BH} \cos\theta$$

$$s_*^2 = r^2 + r_*^2 - 2 r r_* \cos\theta$$

and from Kepler's third law,

$$\omega^2 = \left(\frac{2\pi}{P}\right)^2 = \frac{G(M+M_*)}{a^3} = \frac{G(M+M_*)}{r_{BH} + r_*}$$

LAGRANGE POINTS

With the choosen geometry (M & M_* lying along the x-axis) we can write in Cartesian coordinates,

$$r^2 = x^2 + y^2 + z^2$$

$$s^2 = (x + r_{BH})^2 + y^2 + z^2$$

$$s_*^2 = (r_* - x)^2 + y^2 + z^2$$

where $r_{BH} = \dfrac{M_* a}{M + M_*}$ and $r_* = a - r_{BH}$.

Taking $y = z = 0$, the potential along the x-axis is

$$\Phi_x = \Phi(x,0,0) = -\frac{GM}{(x + r_{BH})} - \frac{GM_*}{(x_* - x)} - \frac{1}{2}\omega^2 x^2$$

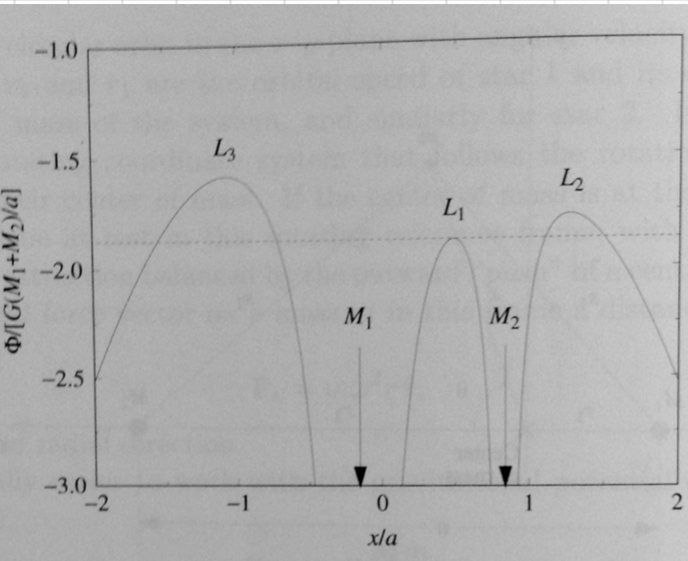


Figure 17.2 The effective gravitational potential Φ for two stars of mass $M_1 = 0.85\ M_\odot$, $M_2 = 0.17\ M_\odot$ on the x-axis. The stars are separated by a distance $a = 5 \times 10^{10}$ cm $= 0.718\ R_\odot$, with their center of mass located at the origin. The x-axis is in units of a, and Φ is expressed in units of $G(M_1 + M_2)/a = 2.71 \times 10^{15}$ ergs g^{-1}. (In fact, the figure is the same for any M_2/M_1=0.2.) The dashed line is the value of Φ at the inner Lagrangian point. If the total energy per gram of a particle exceeds this value of Φ, it can flow through the inner Lagrangian point between the two stars.

The Lagrange points are the unstable equilibrium points given by the condition

$$\frac{\partial \Phi_x}{\partial x} = 0$$

Along the x-axis there are 3 Lagrange points labeled by L_1, L_2, L_3

Considering also the condition

$$\frac{\partial \Phi}{\partial y} = 0$$

we obtain two more points labeled as L_4 and L_5.

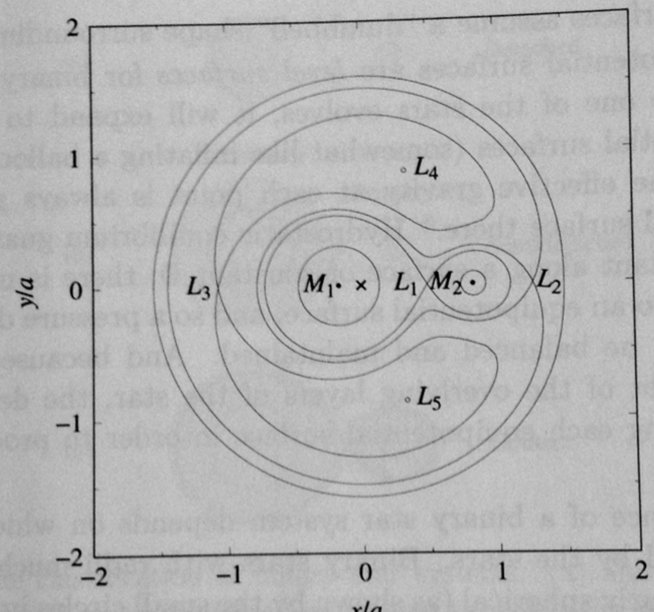


Figure 17.3 Equipotentials for $M_1 = 0.85\ \mathrm{M}_\odot$, $M_2 = 0.17\ \mathrm{M}_\odot$, and $a = 5 \times 10^{10}$ cm $= 0.718\ R_\odot$. The axes are in units of a, with the system's center of mass (the "×") at the origin. Starting at the top of the figure and moving down toward the center of mass, the values of Φ (in units of $G(M_1 + M_2)/a = 2.71 \times 10^{15}$ ergs g^{-1}) for the equipotential curves are $\Phi = -1.875, -1.768, -1.583, -1.583, -1.768$ (the "dumbbell"), -1.875 (the Roche lobe), and -3 (the spheres). L_4 and L_5 are local maxima, with $\Phi = -1.431$.

The distances from L_1 to M_{BH} and to M_* labeled b and b_* respectively can be approximated by the expressions

$$b \approx a\left[0.500 - 0.227 \log_{10}\left(\frac{M_*}{M}\right)\right]$$

$$b_* = a\left[0.500 + 0.227 \log_{10}\left(\frac{M_*}{M}\right)\right]$$

The equipotential surfaces in the Figure are level surfaces for the stars in the binary system. From its construction, equipotentials are always perpendicular to the effective gravity,

$$\vec{f} = -m\vec{\nabla}\Phi$$

i.e. equipotentials are surfaces with a constant value of the effective gravity magnitude.
Hydrostatic equilibrium guarantees that pressure is constant along equipotentials (because there is no component of force parallel to the equipotential). Since pressure is due to the weight of the overlying layers of material, then density has a constant value along the equipotential.

The appearance of a binary system depends on which equipotential surface is filled by the star.

* Detached Binary: Stars with radii much less than their separation. Stars are nearly spherical.

* Semidetached Binary: The star is so big that it fills the Roche lobe ("teardrop-shaped region).

* Contact Binary: In the case of two stars, if both of them fill their Roche lobes, they share a common atmosphere.

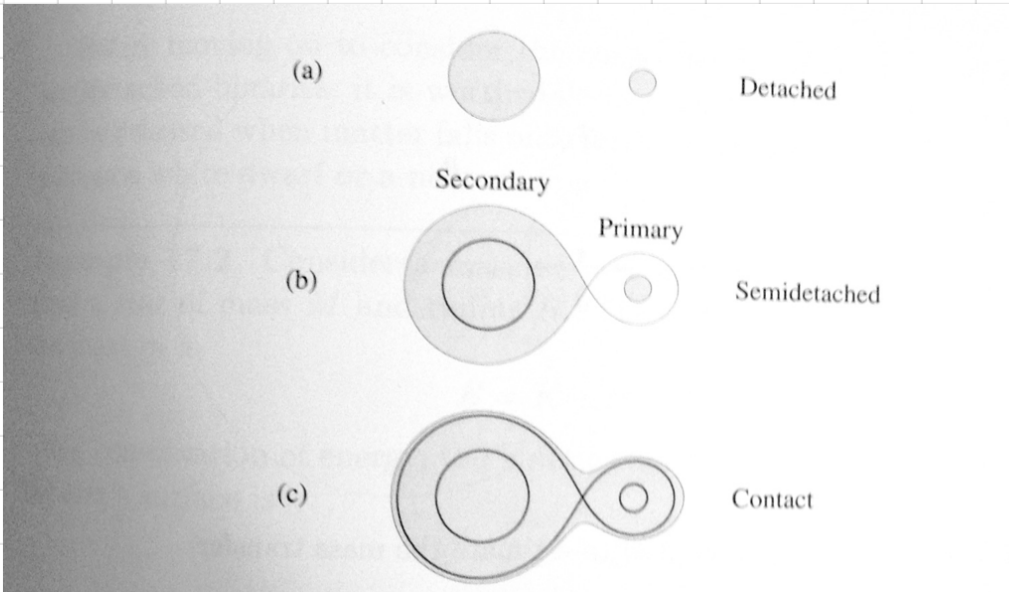


Figure 17.4 The classification of binary star systems. (a) Shows a detached system, (b) shows a semidetached system in which the secondary star has expanded to fill its Roche lobe, and (c) shows a contact binary.

Referencias

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