BLACK HOLES

OBSERVATORIO
ASTRONÓMICO
NACIONAL

Classical Black Holes

05. Astrophysics of Black Holes

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Outline for Part 1

Astrophysical Evidence of Black Holes existence

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- **Supermassive black holes**. At the center of galaxies with masses in the range $M \sim 10^5 - 10^{10} M_\odot$
- **Intermediate-mass black holes**. Objects with a masses in the range $M \sim 10^2 - 10^4 M_\odot$, filling the gap between the stellar-mass and the supermassive ones.

Origin of Black Holes

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- The degenerate pressure of electrons can, eventually, stop the collapse to produce a white dwarf.
- If the mass of the star beginning the collapse is greater than the Chandrasekhar's Limit, the degenerate pressure of electrons can not stop the collapse.

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- The final product in this stage is thought to be a *Black Hole*.

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- Stellar-dynamical stabilities.

A First Model of Stellar Structure

XIX Century Model by Lane, Kelvin and Helmholtz.

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k: Boltzmann's Constant

μ: mean molecular weight

T: temperature

ρ: mass density

m_p: mass of the proton

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The total energy of the Sun will be released in just 10^7 years.

A Better Model of Stellar Structure

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A Better Model of Stellar Structure

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1. The source of energy is thermonuclear.
2. Outward pressure of radiation should be included in the equations.

A Better Model of Stellar Structure

Hydrostatic Equilibrium Equation

$$\frac{d}{dr}\left[\frac{\rho kT}{\mu m_p} + \frac{1}{3}aT^4\right] = -\frac{GM(r)\rho(r)}{r^2}$$

A Better Model of Stellar Structure

Hydrostatic Equilibrium Equation

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Energy Transport Equation

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ℓ: mean free path of the protons

L: luminosity

ε: energy generated per gram of material per unit time

Physical Aspects of the Stellar Death

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For massive stars $(M > 5M_\odot)$,
H $\rightarrow$ He $\rightarrow$ ... $\rightarrow$ C $\rightarrow$ ... $\rightarrow$ Fe

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Equation of State for the degenerate gas of electrons

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Using $\frac{dP(r)}{dr} = -\frac{GM(r)\rho(r)}{r^2}$ we obtain

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$$M_C \approx 1.4 M_\odot$$

Mass of a completely degenerated star.

Neutron Stars. Degenerate Gas of Neutrons

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"...supernovae represent the transitions from ordinary stars into neutron stars which in their final stages consist of extremely closely packed neutrons."

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Neutron Stars. Degenerate Gas of Neutrons

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"...when all thermonuclear sources of energy are exhausted a sufficiently heavy star will collapse. This contraction will continue indefinitely till the radius of the star approaches asymptotically its gravitational radius. Light from the surface of the star will be progressively reddened and can escape over a progressively narrower range of angles till eventually the star tends to close itself off from any communication with a distant observer".

Black Holes

If the collapsing core is too massive to be supported by the degenerate pressure of neutrons, there is no known mechanism capable of finding a new equilibrium configuration, and the body should undergo a complete collapse.

In this case, the final product is a black hole.

Outline for Part 2

Spherically Symmetric Collapse

- Spherical Symmetry:

$$ds^2 = -e^{2\alpha(t,r)}dt^2 + e^{2\beta(t,r)}dr^2 + R^2(t,r)d\Omega^2$$

Spherically Symmetric Collapse

- Spherical Symmetry:

$$ds^2 = -e^{2\alpha(t,r)}dt^2 + e^{2\beta(t,r)}dr^2 + R^2(t,r)d\Omega^2$$

- Suppose that the collapsing body can be described as a perfect fluid.
  Then, we consider that coordinates t and r are attached to every collapsing particle (co-moving coordinates).

Spherically Symmetric Collapse

- In the co-moving frame, the 4-velocity of the fluid is just

$$u^\mu = \left(e^{-\alpha}, 0, 0, 0\right)$$

  and thus $u^2 = -1$.

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- The energy-momentum tensor in the co-moving frame is

$$T^\mu_\nu = diag(\rho, P, P, P)$$

Spherically Symmetric Collapse

The Einstein equations for this metric give

$$
\begin{aligned}
G^t_t = 8\pi T^t_t \quad &\Rightarrow \quad \frac{F'}{R^2 R'} = 8\pi\rho \\
G^r_r = 8\pi T^r_r \quad &\Rightarrow \quad \frac{\dot{F}}{R^2 \dot{R}} = -8\pi P \\
G^t_r = 0 \quad &\Rightarrow \quad \dot{R}' - \dot{R}\alpha' - \dot{\beta}R' = 0
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$$F = R\left(1 - e^{-2\beta} R'^2 + e^{-2\alpha} \dot{R}^2\right)$$

Misner-Sharp mass

Spherically Symmetric Collapse

The Misner-Sharp mass

$$F = R\left(1 - e^{-2\beta} R'^2 + e^{-2\alpha} \dot{R}^2\right)$$

is defined by the relation

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* Note that $n^{\mu} = \partial^{\mu} R$ is normal to the surfaces $R = \mathrm{constant}.$
Thus, when $1 - \frac{F}{R} = 0$, the corresponding surface is null.

Spherically Symmetric Collapse

From (t, t)-component of the Field equations we obtain the Misner-Sharp mass as

$$G^t_t = 8\pi T^t_t \Rightarrow \frac{F'}{R^2 R'} = 8\pi\rho$$

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$$F(r) = \int_0^r F' d\tilde{r} = 8\pi \int_0^r \rho R^2 R' d\tilde{r} = 2M(r)$$

Spherically Symmetric Collapse

Finally, the conservation of the energy-momentum tensor gives the equation

$$\nabla_\mu T^\mu_\nu = 0 \Rightarrow \alpha' = -\frac{P'}{\rho + P}$$

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Then

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$$\alpha = \alpha(t)$$

Dust Collapse

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* If $P \neq 0$ the exterior metric must be a non-vacuum Vaidya spacetime.

Dust Collapse

$\alpha = \alpha(t)$: One can re-define the time coordinate such that

$$e^{\alpha(t)} dt \Rightarrow dt$$

and therefore $g_{tt} = -1$

Dust Collapse

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Introducing $f(r) = e^{2h(r)} - 1$ we have

$$e^{2\beta} = \frac{R'^2}{1+f}$$

Dust Collapse

$$ds^2 = -dt^2 + \frac{R'^2(t,r)}{1+f(r)}dr^2 + R^2(t,r)d\Omega^2$$

Lemaitre-Tolman-Bondi Metric

Dust Collapse

Kretschmann Scalar:

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Divergence at $R = 0$

Dust Collapse

One can re-scale to set $R(t, r)$ to equal the co-moving radius r at the time $t = 0$, i.e. we impose $R(0, r) = r$ and introduce a scale factor a such that

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$a(0, r) = 1$

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The collapse also impose the condition $\dot{a}<0$

Dust Collapse

One of the field equations,

$$\frac{F'}{R^2 R'} = 8\pi\rho$$

implies that a regular value of the density at $t = 0$ is obtained if the Misner-Sharp mass has the form

$$F(r) = r^3 m(r)$$

where $m(r)$ is a (sufficiently) regular function in the region $[\,0, r_b\,]$.

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$$\frac{3r^2m + r^3m'}{(ra)^2(a + ra')} = 8\pi\rho$$

$$\frac{3m + rm'}{a^2(a + ra')} = 8\pi\rho$$

Dust Collapse

It is usual to consider $m(r)$ as a polynomial around $r = 0$,

$$m(r) = \sum_{k=0}^{\infty} m_k r^k$$

where m_k are constants

Dust Collapse

From the definition of the Misner-Sharp mass we have

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$$\dot{a} = -\sqrt{\frac{m}{a} + \frac{f}{r^2}}$$

Dust Collapse

Evaluating at the time $t = 0$ we have $a(0, r) = 1$ and thus

$$\dot{a}(0, r) = -\sqrt{m + \frac{f}{r^2}}$$

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Note that the function f defines the initial velocity of the particles in the cloud. It is usual to write this function as an expansion around $r = 0$ as

$$f(r) = r^2 b(r)$$

with

$$b(r) = \sum_{k=0}^{\infty} b_k r^k$$

Homogeneous Dust Collapse

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Simplest Model of Gravitational Collapse. Oppenheimer-Snyder (1939)

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This implies that
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which is accomplished by choosing $b = b_0$
Then

$$\dot{a} = -\sqrt{\frac{m_0}{a} + b_0}$$

Homogeneous Dust Collapse

The Lemaitre-Tolman-Bondi line element becomes

$$ds^2 = -dt^2 + \frac{R'^2(t,r)}{1+f(r)}dr^2 + R^2(t,r)d\Omega^2$$

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$$ds^2 = -dt^2 + a^2(t)\left[\frac{dr^2}{1+b_0 r^2} + r^2 d\Omega^2\right]$$

Homogeneous Dust Collapse

$$ds^2 = -dt^2 + a^2(t)\left[\frac{dr^2}{1 + b_0 r^2} + r^2 d\Omega^2\right]$$

For $b_0 = 0$ this line element becomes

$$ds^2 = -dt^2 + a^2(t)\left[dr^2 + r^2 d\Omega^2\right]$$

and describes the counterpart of a flat spacetime, in which the collapse is marginally bound (i.e. the falling particles have vanishing velocity at infinity).

Homogeneous Dust Collapse

For $b_0 = 0$ the scale factor is

$$\dot{a} = -\sqrt{\frac{m_0}{a} + b_0}$$

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$$\frac{2}{3}a^{3/2} - \frac{2}{3}a^{3/2}(0) = -\sqrt{m_0}t$$

Homogeneous Dust Collapse

The value of the initial condition was supposed as $a(0) = 1$. Thus

$$a(t) = \left[1 - \frac{3\sqrt{m_0}}{2}t\right]^{2/3}$$

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Therefore, the singularity $R = 0$ occurs at the time t_s for which $a(t_s) = 0$. This is

$$t_s = \frac{2}{3\sqrt{m_0}}$$

Homogeneous Dust Collapse

The curve $t_H(r)$ describing the time at which the shell r crosses the horizon can be obtained from the condition

$$1 - \frac{F}{R} = 0$$

Homogeneous Dust Collapse

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Homogeneous Dust Collapse

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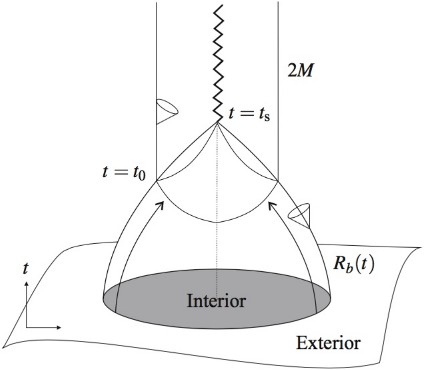
$$t_H(r) = \frac{2}{3\sqrt{m_0}} - \frac{2}{3} m_0 r^3$$

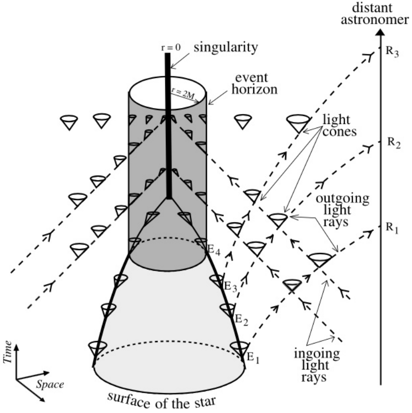
Gravitational Collapse of a Homogeneous Cloud of Dust

Eddington-Finkelstein Diagram



Gravitational Collapse of a Homogeneous Cloud of Dust

Eddington-Finkelstein Diagram



Inhomogeneous Dust Collapse

Inhomogeneous Dust Collapse

ρ depends on both t and r, and thus

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$$\rho = \rho(t, r)$$
$$m = m(r)$$
$$b = b(r)$$
$$a = a(t, r)$$

Inhomogeneous Dust Collapse

The simplest form for the function $m(r)$ is

$$m(r) = m_0 + m_2 r^2$$

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This function defines the density interms of the two parameter m_0 and m_2.
The condition that ρ must be a decreasing function of r implies that $m_2 < 0$.

* $m_1 = 0$ in order to have a non divergent density.

Inhomogeneous Dust Collapse

The equation

$$\dot{R}^2 = \frac{F}{R} + f$$

gives this time

$$ra(\dot{t}, r) = -\sqrt{\frac{mr^2}{ra} + br^2}$$

Inhomogeneous Dust Collapse

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$$\dot{R}^2 = \frac{F}{R} + f$$

gives this time

$$r\dot{a}(t, r) = -\sqrt{\frac{mr^2}{ra} + br^2}$$

$$\dot{a}(t, r) = -\sqrt{\frac{m}{a} + b}$$

Inhomogeneous Dust Collapse

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Inhomogeneous Dust Collapse

In the marginally bound case, $b = 0$, it gives

$$a(\dot{t}, r) = -\sqrt{\frac{m(r)}{a}}$$

$$\sqrt{a(t,r)}da = -\sqrt{m(r)}dt$$

$$\frac{2}{3}a^{3/2}(t,r) - \frac{2}{3}a^{3/2}(0,r) = -\sqrt{m(r)}t$$

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This equation shows how each shell of the distribution (each r) collapses with a different scale factor and with a different velocity.

Inhomogeneous Dust Collapse

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Inhomogeneous Dust Collapse

Using $m(r) = m_0 + m_2 r^2$ gives the functions

$$t_s(r) = \frac{2}{3\sqrt{m_0 + m_2 r^2}}$$

$$t_H(r) = \frac{2}{3\sqrt{m_0 + m_2 r^2}} - \frac{2}{3} r^3 \left(m_0 + m_2 r^2 \right)$$

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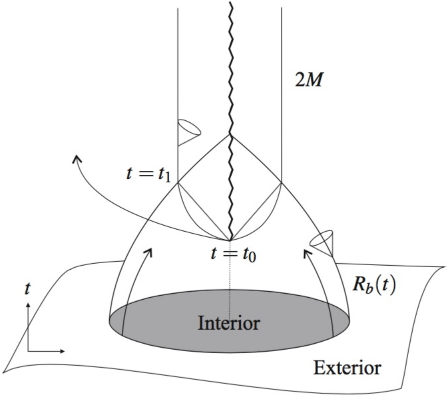

Next Lecture

06. Black Holes Astrophysics