04. SPHERICAL ACCRETION. ALTERNATE DESCRIPTION.

Mathematical Description of Accretion

- Accretion Radius

The characteristic length scale in an accretion process is the "accretion radius" or "gravitational capture radius",

R_{acc} : distance at which kinetic and gravitational potential energies are equal.

$$\frac{1}{2}\left(c_s^2 + v_{rel}^2\right) = \frac{GM}{R_{acc}}$$

or

$$R_{acc} = \frac{GM}{c_s^2 + v_{rel}^2}$$

- Hydrodynamic Equations (Non-Relativistic)

ρ: Density
T: Temperature
$\vec{v}$: velocity of the gas
P : Pressure
$\vec{g}$: Acceleration due to gravity

* Continuity Equation (Conservation of Mass)

$$\frac{\partial \rho}{\partial t} + \vec{\nabla} \cdot (\rho \vec{v}) = 0$$

* Conservation of Momentum (Ignoring radiation pressure)

$$\rho \left[\frac{\partial \vec{v}}{\partial t} + (\vec{v} \cdot \vec{\nabla})\vec{v} \right] = -\vec{\nabla}P + \rho\vec{g} + \vec{\nabla}\cdot\vec{\sigma}$$

where

$$\sigma_{ij} = 2\eta \tau_{ij} \qquad \text{Viscosity Stress Tensor}$$

$$\tau_{ij} = \frac{1}{2}\left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i} - \frac{2}{3}\frac{\partial v_k}{\partial x_k}\delta_{ij} \right)$$

η: dynamic viscosity

$\eta = \rho\nu$ $\qquad$ ν: kinematic viscosity.

If $\nu = 0$: no-viscosity $\rightarrow$ ideal fluid $\rightarrow$ Euler Equation
If $\nu \neq 0$: viscous fluid $\rightarrow$ Navier-Stokes Equation

- If the self-gravity of the fluid is negligible,

$$\vec{g} = -\vec{\nabla}\Phi = \vec{\nabla}\left(\frac{GM}{r}\right) = -\frac{GM}{r^2}\hat{r}$$

- If η is a constant, the momentum equation becomes

$$\rho \left[\frac{\partial \vec{v}}{\partial t} + (\vec{v} \cdot \vec{\nabla})\vec{v} \right] = -\vec{\nabla}P + \rho\vec{g} + \eta\nabla^2\vec{v} + \frac{1}{3}\eta\vec{\nabla}(\vec{\nabla}\cdot\vec{v})$$

* Equation of State

$P = P(\rho)$

- Polytropic Equation $\qquad P \propto \rho^{\gamma}$

* Energy balance in the flow:

- Change in kinetic energy
- Change in the internal energy
- Flux of heat

$$\rho \frac{d\varepsilon}{dt} = -P\vec{\nabla}\cdot\vec{v} + 2\eta\left[S_{ij}S_{ij} - \frac{1}{3}(\vec{\nabla}\cdot\vec{v})^2\right] + Q$$

ε: internal energy per unit mass of the fluid
Q: Net heat exchanged by the element of fluid per unit time per unit volume.

$$S_{ij} = \frac{1}{2}\left(\frac{\partial v_i}{\partial x^j} - \frac{\partial v_j}{\partial x^i}\right) = \tau_{ij} - \frac{1}{3}(\vec{\nabla}\cdot\vec{v})\delta_{ij}$$

Then, $$q^+ = 2\eta\left[S_{ij}S_{ij} - \frac{1}{3}(\vec{\nabla}\cdot\vec{v})^2\right]$$

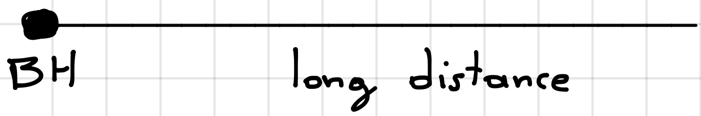
is the rate of energy dissipation per unit volume due to the work done by viscous forces.

Note:/ The total or "material" derivative is

$$\frac{df(t,\vec{r})}{dt} = \frac{\partial f}{\partial t} + \underbrace{(\vec{v}\cdot\vec{\nabla})f}_{\text{Convective derivative.}}$$

Bondy Accretion [5]

First model of accretion. Smooth time-steady accretion with spherical symmetry.



gas moves very slowly

$\downarrow$

description in terms of Fluid Mechanics eqs.

* No viscosity : $\eta = 0$

Spherical Symmetry

$$\rho = \frac{M}{\frac{4}{3}\pi r^3} \qquad \vec{v} = v\,\hat{r}$$

$$\frac{\partial \rho}{\partial t} = \frac{\dot{M}}{\frac{4}{3}\pi r^3} \quad \text{and} \quad \vec{\nabla}\cdot(\rho\vec{v}) = \frac{\partial}{\partial r}\left[\rho v\right]$$

$$\vec{\nabla}\cdot(\rho\vec{v}) = -\frac{Mv}{\frac{4}{3}\pi r^6}\,3r^2$$

$$\vec{\nabla}\cdot(\rho\vec{v}) = -\frac{9Mv}{4\pi r^4}$$

Then, the continuity equation is

$$\frac{\partial \rho}{\partial t} + \vec{\nabla} \cdot (\rho \vec{v}) = 0$$

$$\frac{\dot{M}}{\frac{4}{3}\pi r^3} - \frac{9Mv}{4\pi r^4} = 0$$

$$\dot{M} = \frac{3Mv}{r} = \frac{3v}{r} \rho \frac{4\pi}{3} r^3$$

$$\boxed{\dot{M} = 4\pi r^2 \rho v}$$

In the time-steady flow we have

$$(\vec{v} \cdot \vec{\nabla}) \vec{v} = v \frac{\partial}{\partial r} \vec{v} = v \frac{\partial v}{\partial r} \hat{r}$$

$$\vec{g} = -\frac{GM}{r^2} \hat{r}$$

Hence the force equation has only the radial component

$$\rho \frac{\partial \vec{v}}{\partial t} + \rho (\vec{v} \cdot \vec{\nabla}) \vec{v} = -\vec{\nabla} P + \rho \vec{g}$$

$$\rho \frac{\partial v}{\partial t} + \rho (\vec{v} \cdot \vec{\nabla}) v = -\frac{\partial P}{\partial r} + \rho g$$

$$\frac{\partial v}{\partial t} + v \frac{\partial v}{\partial r} = -\frac{1}{\rho} \frac{\partial P}{\partial r} - \frac{GM}{r^2}$$

or integrating

$$\underbrace{\frac{1}{2}v^2}_{} + \int_{P_\infty}^{P} \frac{dP}{\rho} - \frac{GM}{r} = \underbrace{\text{Constant}}_{}$$

v^2 : it describes inflows & outflows!!

at $r \to \infty$; $\begin{cases} \frac{GM}{r} \to 0 \\ v \sim 0 \end{cases}$

and thus constant = 0

$$\boxed{\frac{1}{2}v^2 + \int_{P_\infty}^{P} \frac{dP}{\rho} - \frac{GM}{r} = 0}$$

Finally, Bondy (1952) propose an equation of state in the form of a simple adiabatic law,

$$\boxed{P \propto \rho^\gamma}$$

$1 \leq \gamma \leq \frac{5}{3}$

$\gamma = 1$: Isothermic Flow
$\gamma = 5/3$: Adiabatic Flow

The complete set of equations describing Bondi accretion is

$$\begin{cases} \dot{M} = 4\pi r^2 \rho v & \text{(I)} \\ \frac{1}{2}v^2 + \int_{P_\infty}^{P} \frac{dP}{\rho} - \frac{GM}{r} = 0 & \text{(II)} \\ P \propto \rho^\gamma & \text{(III)} \end{cases}$$

From the polytropic equation we have

$$\frac{P}{\rho^\gamma} = \frac{P_\infty}{\rho_\infty^\gamma} \quad \longleftarrow \text{ values at infinity}$$

$$P = P_\infty \left(\frac{\rho}{\rho_\infty}\right)^\gamma$$

The local speed of sound is given by

$$c_s^2 = \gamma \frac{P}{\rho} = \gamma \frac{P_\infty}{\rho}\left(\frac{\rho}{\rho_\infty}\right)^\gamma = \gamma \frac{P_\infty}{\rho_\infty}\left(\frac{\rho}{\rho_\infty}\right)^{\gamma-1} = c_\infty^2 \left(\frac{\rho}{\rho_\infty}\right)^{\gamma-1}$$

where

$$c_\infty = \sqrt{\frac{\gamma P_\infty}{\rho_\infty}}$$: speed of sound in the gas at infinity.

We define $r_s = \frac{GM}{c_s^2}$ as the radius at which the escape velocity is c_s and we define also the new variables

$$y = \frac{v}{c_s} \qquad x = \frac{r}{r_s} \qquad z = \frac{\rho}{\rho_\infty}$$

Hence, equation (I) becomes

$$\dot{M} = 4\pi \frac{r^2}{r_\infty^2} \frac{\rho}{\rho_\infty} \frac{v}{c_\infty} c_\infty \rho_\infty r_\infty^2$$

$$\frac{\dot{M}}{4\pi c_\infty \rho_\infty r_\infty^2} = x^2 y z$$

$$\lambda = x^2 y z$$

where we defined the variable

$$\lambda = \frac{\dot{M}}{4\pi r_\infty^2 c_\infty \rho_\infty} = \frac{\dot{M}\, c_\infty^4}{4\pi (GM)^2 c_\infty \rho_\infty}$$

$$\lambda = \frac{\dot{M}\, c_\infty^3}{4\pi (GM)^2 \rho_\infty}$$

In order to write eq. (II) we need

$$\frac{GM}{r} = \frac{GM}{r_\infty}\frac{r_\infty}{r} = \frac{c_\infty^2}{X}$$

$$\frac{1}{2}v^2 = \frac{1}{2}\frac{v^2}{c_\infty^2}c_\infty^2 = \frac{1}{2}y^2 c_\infty^2$$

Now, since $P \propto \rho^\gamma$ we write

$$\frac{P}{\rho^\gamma} = \frac{P_\infty}{\rho_\infty^\gamma}$$

$$dP = \left(\frac{P_\infty}{\rho_\infty^\gamma}\right)\gamma \rho^{\gamma-1} d\rho$$

and then

$$\int_{P_\infty}^{P} \frac{dP}{\rho} = \left(\frac{P_\infty}{\rho_\infty^\gamma}\right)\gamma \int_{\rho_\infty}^{\rho_\infty\left(\frac{P}{P_\infty}\right)^{1/\gamma}} \rho^{\gamma-2}\, d\rho$$

$$\int_{P_\infty}^{P} \frac{dP}{\rho} = \left(\frac{P_\infty}{\rho_\infty^\gamma}\right)\gamma \left[\frac{\rho^{\gamma-1}}{\gamma-1}\right]_{\rho_\infty}^{\rho_\infty\left(\frac{P}{P_\infty}\right)^{1/\gamma}}$$

$$\int_{P_\infty}^{P} \frac{dP}{\rho} = \left(\frac{P_\infty}{\rho_\infty^\gamma}\right)\frac{\gamma}{\gamma-1}\left\{\left[\rho_\infty\left(\frac{P}{P_\infty}\right)^{1/\gamma}\right]^{\gamma-1} - \left[\rho_\infty\right]^{\gamma-1}\right\}$$

$$\int_{P_\infty}^{P} \frac{dP}{\rho} = \left(\frac{P_\infty}{\rho_\infty^\gamma}\right)\frac{\gamma}{\gamma-1}\ \rho_\infty^{\gamma-1}\left[\left(\frac{P^{1/\gamma}}{P_\infty^{1/\gamma}}\right)^{\gamma-1} - 1\right]$$

$$\int_{P_\infty}^{P} \frac{dP}{\rho} = \left(\frac{P_\infty}{\rho_\infty^\gamma}\right)\frac{\gamma}{\gamma-1}\ \rho_\infty^{\gamma-1}\left[\left(\frac{\rho}{\rho_\infty}\right)^{\gamma-1} - 1\right]$$

$$\int_{P_\infty}^{P} \frac{dP}{\rho} = \left(\frac{P_\infty}{\rho_\infty}\right)\frac{\gamma}{\gamma-1}\left[z^{\gamma-1} - 1\right]$$

$$\int_{P_\infty}^{P} \frac{dP}{\rho} = \frac{c_\infty^2}{\gamma-1}\left(z^{\gamma-1} - 1\right)$$

Hence, equation (II) becomes

$$\frac{1}{2}v^2 + \int_{P_\infty}^{P}\frac{dP}{\rho} - \frac{GM}{r} = 0$$

$$\frac{1}{2}y^2 c_\infty^2 + \frac{c_\infty^2}{\gamma-1}\left(z^{\gamma-1}-1\right) - \frac{c_\infty^2}{x} = 0$$

$$\frac{1}{2}y^2 + \frac{\left(z^{\gamma-1}-1\right)}{\gamma-1} = \frac{1}{x}$$

$$\begin{cases} \lambda = x^2 y z & \text{(IV)} \\ \dfrac{1}{z} y^2 + \dfrac{(z^{\gamma-1} - 1)}{\gamma - 1} = \dfrac{1}{x} & \text{(V)} \end{cases}$$

In order to solve this set of equations we make the change of variable

$$\mu = \frac{y}{z^{\frac{\gamma-1}{2}}}$$

$$\mu = \frac{\frac{v}{c_\infty}}{\left(\frac{\rho}{\rho_\infty}\right)^{\frac{\gamma-1}{2}}} = \frac{v}{c_\infty \sqrt{\frac{\rho^\gamma}{\rho_\infty^\gamma} \frac{\rho_\infty}{\rho}}}$$

$$\mu = \frac{v}{\sqrt{\gamma \frac{p_\infty}{\rho_\infty}} \sqrt{\frac{p}{\rho} \frac{p_\infty}{\rho}}}$$

$$\mu = \frac{v}{\sqrt{\gamma \frac{p}{\rho}}} = \frac{v}{c_s}$$

$\downarrow$

μ is the local Mach number; i.e. the ratio of the bulk velocity v to the local sound speed $\sqrt{\gamma \frac{p}{\rho}} = c_s$

Then

$$y^2 = \mu^2 z^{\gamma-1}$$

and therefore eq. (V) writes

$$\frac{1}{z} \mu^2 z^{\gamma-1} + \frac{(z^{\gamma-1} - 1)}{\gamma - 1} = \frac{1}{x}$$

$$\mu^2 z^{\gamma-1} \left[\frac{1}{z} + \frac{1}{\mu^2 (\gamma - 1)} \right] - \frac{1}{\gamma - 1} = \frac{1}{x} \qquad \text{(VI)}$$

Similarly, eq. (IV) is

$$y = \frac{\lambda}{x^2 z} = \mu z^{\frac{\gamma-1}{2}}$$

$$z^2 z^{\gamma-1} = \frac{\lambda^2}{x^4 \mu^2}$$

$$z^{\gamma+1} = \frac{\lambda^2}{\mu^2 x^4}$$

$$z = \left(\frac{\lambda^2}{\mu^2 x^4}\right)^{\frac{1}{\gamma+1}}$$

Hence, replacing in (VI) we have

$$\mu^2\left(\frac{\lambda^2}{\mu^2 x^4}\right)^{\frac{\gamma-1}{\gamma+1}}\left[\frac{1}{2}+\frac{1}{\mu^2(\gamma-1)}\right]-\frac{1}{\gamma-1}=\frac{1}{x}$$

$$\mu^{2-2\left(\frac{\gamma-1}{\gamma+1}\right)}\left[\frac{1}{2}+\frac{1}{\mu^2(\gamma-1)}\right]=\left[\frac{1}{x}+\frac{1}{\gamma-1}\right]x^{4\left(\frac{\gamma-1}{\gamma+1}\right)}\lambda^{-2\left(\frac{\gamma-1}{\gamma+1}\right)}$$

$$2-2\left(\frac{\gamma-1}{\gamma+1}\right)=\frac{2(\gamma+1)-2(\gamma-1)}{\gamma+1}=\frac{2}{\gamma+1}$$

$$\underbrace{\mu^{\frac{2}{\gamma+1}}\left[\frac{1}{2}+\frac{1}{\mu^2(\gamma-1)}\right]}_{g(\mu)}=\underbrace{\left[\frac{1}{x}+\frac{1}{\gamma-1}\right]x^{4\left(\frac{\gamma-1}{\gamma+1}\right)}}_{f(x)}\lambda^{-2\left(\frac{\gamma-1}{\gamma+1}\right)}$$

$$g(\mu) = \lambda^{-2\left(\frac{\gamma-1}{\gamma+1}\right)} f(x)$$

If $1<\gamma<\frac{5}{3} \longrightarrow 5-3\gamma>0$

Then
$$\begin{cases} g(\mu) \text{ minimum}: \left.\dfrac{dg}{d\mu}\right|_{\mu=\mu_m}=0 \rightarrow \mu_m=1 \\ f(x) \text{ minimum}: \left.\dfrac{df}{dx}\right|_{x=x_m}=0 \longrightarrow x_m=\dfrac{5-3\gamma}{4} \end{cases}$$

$\mu_m=1$ is a "sonic point"; i.e. Mach number $=1$

$$x_m = \begin{cases} 0{,}5 & \text{for } \gamma=1 \\ \longrightarrow 0 & \text{for } \gamma\longrightarrow\frac{5}{3} \end{cases}$$

when γ is close to $\frac{5}{3}$ the radius of the minimum is closer to $x=0$

At the minimum:

$$g(1) = \lambda^{-2\left(\frac{\gamma-1}{\gamma+1}\right)} f(x_m)$$

$$\frac{1}{2}+\frac{1}{\gamma-1} = \lambda^{-2\left(\frac{\gamma-1}{\gamma+1}\right)}\left\{\frac{1}{\gamma-1}\left[\frac{5-3\gamma}{4}\right]^{4\left(\frac{\gamma-1}{\gamma+1}\right)}+\left[\frac{5-3\gamma}{4}\right]^{-\left(\frac{5-3\gamma}{\gamma+1}\right)}\right\}$$

$$\frac{\gamma+1}{2(\gamma-1)} = \lambda^{-2\left(\frac{\gamma-1}{\gamma+1}\right)}\left[\frac{5-3\gamma}{4}\right]^{-\left(\frac{5-3\gamma}{\gamma+1}\right)}\left\{1+\frac{1}{\gamma-1}\left[\frac{5-3\gamma}{4}\right]^{4\left(\frac{\gamma-1}{\gamma+1}\right)+\left(\frac{5-3\gamma}{\gamma+1}\right)}\right\}$$

$$4\left(\frac{\gamma-1}{\gamma+1}\right)+\left(\frac{5-3\gamma}{\gamma+1}\right)=\frac{4\gamma-4+5-3\gamma}{\gamma+1}=\frac{\gamma+1}{\gamma+1}=1$$

$$\frac{\gamma+1}{2(\gamma-1)} = \lambda^{-2\left(\frac{\gamma-1}{\gamma+1}\right)} \left[\frac{5-3\gamma}{4}\right]^{-\left(\frac{5-3\gamma}{\gamma+1}\right)} \left\{1 + \frac{1}{\gamma-1}\left[\frac{5-3\gamma}{4}\right]\right\}$$

$$\frac{\gamma+1}{2(\gamma-1)} = \lambda^{-2\left(\frac{\gamma-1}{\gamma+1}\right)} \left[\frac{5-3\gamma}{4}\right]^{-\left(\frac{5-3\gamma}{\gamma+1}\right)} \left\{\frac{4\gamma-4+5-3\gamma}{4(\gamma-1)}\right\}$$

$$\frac{\gamma+1}{2(\gamma-1)} = \lambda^{-2\left(\frac{\gamma-1}{\gamma+1}\right)} \left[\frac{5-3\gamma}{4}\right]^{-\left(\frac{5-3\gamma}{\gamma+1}\right)} \left\{\frac{\gamma+1}{4(\gamma-1)}\right\}$$

$$2 = \lambda^{-2\left(\frac{\gamma-1}{\gamma+1}\right)} \left[\frac{5-3\gamma}{4}\right]^{-\left(\frac{5-3\gamma}{\gamma+1}\right)}$$

$$\lambda(\gamma) = \left(\frac{1}{2}\right)^{\frac{\gamma+1}{2(\gamma-1)}} \left[\frac{5-3\gamma}{4}\right]^{-\left[\frac{5-3\gamma}{2(\gamma+1)}\right]}$$

Remembering that

$$\lambda = \frac{\dot{M}\, c_\infty^3}{4\pi (GM)^2 \rho_\infty}$$

we have

$$\dot{M} = 4\pi \lambda(\gamma) \frac{(GM)^2 \rho_\infty}{c_\infty^3}$$ Physical Accretion rate

The accretion rate depends on the equation of state.

As γ increases from $1 \longrightarrow \frac{5}{3}$;
λ falls from 1.12 to 0.25

$\gamma = \frac{5}{3}$: Particles with <u>no</u> internal degrees of freedom
In this case $x_m \rightarrow 0$; the sonic point goes to the origin.
The accretion rate falls

$\gamma = 1$: Nearly isothermal
Sonic point at relatively large distance
Larger accretion rate

ASTROPHYSICAL PLASMAS: Adiabatic with $\gamma = \frac{5}{3}$
(no heating or cooling)

Often, heating and/or cooling is important $\longrightarrow$
the equation of state is integrated from an energy condition!!

THE EDDINGTON LIMIT IN DETAIL

$$\vec{F}_\nu = \int d\Omega \, \hat{n}(\Omega) \, I_\nu(\Omega)$$

Radiation flux per unit frequency travelling in the direction $\hat{n}$

If $K(\nu)$ is the total opacity of the gas per unit mass as function of ν (including scattering + absorption) the radiation force on the gas is

$$\vec{f}_{rad} = \frac{1}{c} \int d\nu \, \vec{F}_\nu \, K(\nu) \, \rho$$

In general, $K(\nu)$ can be very complicated. However, for a <u>completely ionized</u> gas, the opacity is described for all photons with energies below ~ 100 KeV, as the "Thomson Opacity"

In the ionized gas we have p^+ and e^-. However, the scattering cross section for p^+ is a factor $\left(\frac{m_e}{m_p}\right)^2$ smaller than that of e^-.

$\left(\frac{m_e}{m_p}\right) \sim 5 \times 10^{-4}$

Hence we will consider only the electron scattering.

When free e^- have thermal motions with

$$kT \ll m_e c^2$$

or $$T \ll 5.9 \times 10^9 \text{ K}$$

the cross section for scattering of photons off electrons is frequency independent.

Then
$$\vec{f}_{rad} = \frac{\kappa_e \rho}{c} \int d\nu \, \vec{F}_\nu$$

For a spherically symmetric distribution, the luminosity per unit frequency L_ν is related with the flux by
$$\vec{F}_\nu = \frac{L_\nu}{4\pi r^2} \hat{r}$$

and then
$$f_{rad} = \frac{\kappa_e \rho}{c} \int d\nu \frac{L_\nu}{4\pi r^2} \quad \text{in the radial direction}$$

$$f_{rad} = \frac{\kappa_e \rho}{4\pi r^2 c} L \qquad L = \int L_\nu d\nu$$

The Thomson opacity per unit mass of the gas is

$$\kappa_e = n_e \frac{\sigma_e}{\rho}$$

n_e: density of electrons i.e
$$n_e = \frac{\# e^-}{cm^3}$$

σ_e: Thomson cross-section of the electrons

$$\sigma_e = \frac{8\pi}{3}\left(\frac{e^2}{m_e c^2}\right)^2 = 0.665 \times 10^{-24} \, cm^2$$

Thus the force due to radiation is

$$f_{rad} = \frac{\sigma_e L}{4\pi c r^2} \qquad\qquad f_G = \frac{GMm}{r^2}$$

This force depends on $\frac{1}{r^2}$ as gravity does!

Here $m = m_p + m_e \approx m_p$

Therefore, there is a limiting luminosity that can be contrarested by gravity; given by

$$\frac{\sigma_e L_E}{4\pi c r^2} = \frac{GMm_p}{r^2}$$

$$L_E = \frac{4\pi c G M m_p}{\sigma_e} \simeq 1.51 \times 10^{38} \frac{M}{M_\odot} \ \mathrm{erg\ s^{-1}}$$

Eddington's luminosity

L_E is the critical luminosity beyond which the radiation force overpower gravity.

In AGNs, is observed	$L \sim 10^{43} - 10^{47}$ erg s^{-1}
Therefore we infer	$M \sim 10^5 - 10^9 M_\odot$

Referencias

[1] Bradley Peterson. An Introduction to Active Galactic Nuclei. Cambidge University Press. (1997)

[2] https://rechneronline.de/spectrum/