BLACK HOLES

OBSERVATORIO
ASTRONÓMICO
NACIONAL

Classical Black Holes

10. Viscous Torques

Edward Larrañaga

Outline for Part 1

Viscous Torques

Accretion Disk

- If the only interaction between fluid elements is gravity, the angular momentum is conserved.

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- In the description of the aaccretion disk it is needed a force responsible for the redistribution of angular momentum.

Differential Rotation

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- Keplerian Velocity

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- Neighboring material at different radii moves with different velocity
- The disk is not a solid body. It moves with *differential rotation*.
- The thermal motion of the fluid molecules and the turbulent motion of the fluid produce *viscous stresses*.

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- It appears when there are internal distortions (usually local stresses that are proportional to the local rate of strain)
- Although the following is the simplest description, it can be used to describe other mechanisms of angular momentum transport such as magnetic loops that couple fluid elements at macroscopic distances across the disk.

Modeling Viscosity

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- $\tilde{v}$: typical speed in the accretion disk

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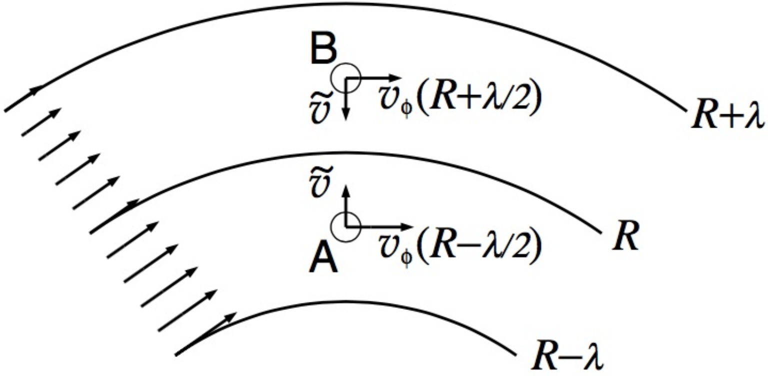
- As a first model, consider a uniform gas moving only in the tangential direction woth velocity $v_\phi(r)$.

Modeling Viscosity



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The only non-vanishing component of the stress tensor is the ϕ-component of the force per unit surface of the $r =$constant surface,

$$\sigma_{r\phi} = -\eta \left. \frac{\partial v_\phi}{\partial r} \right|_R$$

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η: dynamical viscosity

$v_\phi(r) = \Omega(r) r$

$\frac{\partial v_\phi}{\partial r}$: velocity gradient

Modeling Viscosity

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Turbulent motion:
λ: spatial scale (or characteristic wavelength) of the turbulence
$\tilde{v}$: typical velocity of the eddies

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H: Height of the disk

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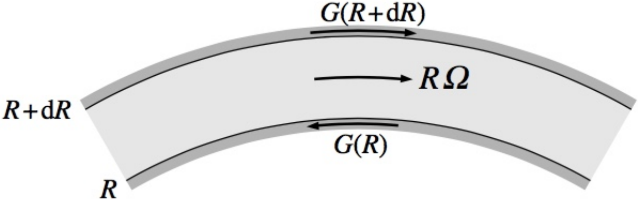
  $\longrightarrow$ The gas slowly spirals in!

Differential viscous torque



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and the associated power is

$$P = \frac{\partial G}{\partial r} \Omega dr = \left[\frac{\partial}{\partial r} (G(r)\Omega(r)) - G(r) \frac{\partial \Omega}{\partial r} \right] dr$$

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$$P = \left[\frac{\partial}{\partial r}\left(G(r)\Omega(r)\right) - G(r)\frac{\partial \Omega}{\partial r} \right] dr$$

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The zero corresponds to the rigid body, $\left(\frac{\partial\Omega}{\partial r} = 0\right)$

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Surface mass density: $\Sigma = \Sigma(t, r)$

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In the limit $dr \longrightarrow 0$ this gives the mass conservation equation

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In the limit $dr \to 0$ this gives the momentum conservation equation

$$r\frac{\partial}{\partial t}\left(\Sigma r^2 \Omega\right) + \frac{\partial}{\partial r}\left(r\Sigma v_r r^2 \Omega\right) = \frac{1}{2\pi}\frac{\partial G}{\partial r}$$

Radial Structure Evolution

Equations determining the radial structure of the accretion disk:

$$r\frac{\partial \Sigma}{\partial t} + \frac{\partial}{\partial r}(r\Sigma v_r) = 0$$

$$r\frac{\partial}{\partial t}\left(\Sigma r^2 \Omega\right) + \frac{\partial}{\partial r}\left(r\Sigma v_r r^2 \Omega\right) = \frac{1}{2\pi}\frac{\partial G}{\partial r}$$

$$G(r,t) = 2\pi r \nu \Sigma r^2 \frac{\partial \Omega}{\partial r}$$

Equation for the kinematical viscosity ν

Radial Structure Evolution

Combining the equations determining the radial structure of the accretion disk and assuming that $\frac{\partial \Omega}{\partial t} = 0$, we obtain

$$r \Sigma v_r \frac{\partial}{\partial r} \left(r^2 \Omega \right) = \frac{1}{2\pi} \frac{\partial G}{\partial r}$$

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Combining the equations determining the radial structure of the accretion disk and assuming that $\frac{\partial \Omega}{\partial t} = 0$, we obtain

$$r\Sigma v_r \frac{\partial}{\partial r}\left(r^2\Omega\right) = \frac{1}{2\pi}\frac{\partial G}{\partial r}$$

Eliminating the velocity, v_r, we have

$$r\frac{\partial \Sigma}{\partial t} = -\frac{\partial}{\partial r}\left[\frac{1}{2\pi \frac{\partial (r^2\Omega)}{\partial r}}\frac{\partial G}{\partial r}\right]$$

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The radial velocity becomes

$$v_r = -\frac{3}{\Sigma r^{1/2}}\frac{\partial}{\partial r}\left[\nu \Sigma r^{1/2}\right]$$

Example of Viscosity

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$$\frac{\partial}{\partial t}\left(\Sigma r^{1/2}\right) = \frac{3\nu}{r}\left(r^{1/2}\frac{\partial}{\partial r}\right)^2\left(\Sigma r^{1/2}\right)$$

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T: Exponential function

S: Bessel function

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The solution of this equation considering the initial distribution

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$$\Sigma(x, \tau) = \frac{m}{\pi r_0^2}\tau^{-1}x^{-1/4}\exp\left[-\frac{1 + x^2}{\tau}\right]I_{1/4}\left(\frac{2x}{\tau}\right)$$

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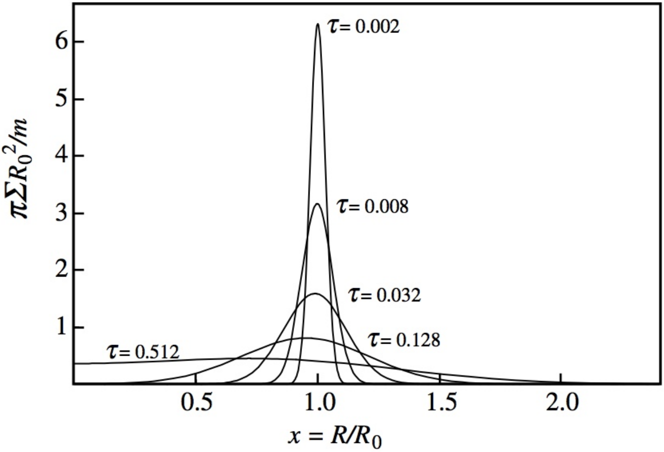
$I_{1/4}(z)$: Modified Bessel function
$x = \frac{r}{r_0}$
$\tau = \frac{12\nu t}{r_0^2}$

Formation of the Accretion Disk



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Outline for Part 2

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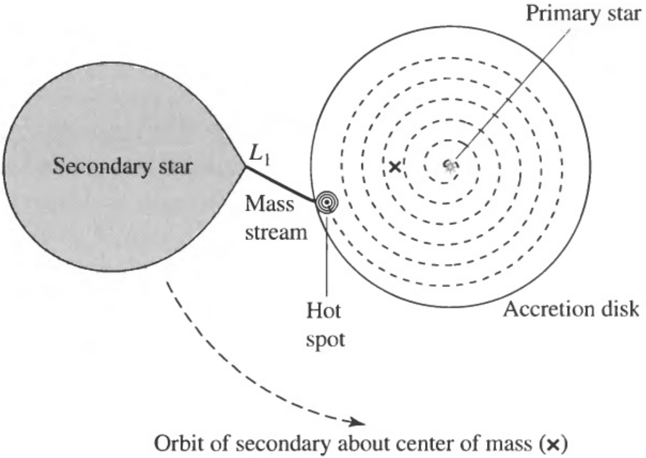
$v_{||}$: parallel component of the velocity with respect to the line of centers.
$v_{\perp}$: perpendicular component of the velocity with respect to the line of centers.

Formation of the Accretion Disk



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Using

$$b \approx a\left[0.500 - 0.227\log_1 0\left(q\right)\right]$$

$$a = 3.5 \times 10^{10}\left(\frac{M}{M_\odot}\right)^{1/3}(1+q)^{1/3}\,\tau_{hr}^{2/3}\ [\mathrm{cm}]$$

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$$a = 3.5 \times 10^{10} \left(\frac{M}{M_\odot}\right)^{1/3} (1+q)^{1/3}\, \tau_{hr}^{2/3}\ [\mathrm{cm}]$$

we get

$$v_\perp \gtrsim 305.6 \left(\frac{M}{M_\odot}\right)^{1/3} (1+q)^{1/3}\, \tau_{hr}^{-1/3}\ [\mathrm{km/s}]$$

Formation of the Accretion Disk

$$v_{\parallel} \lesssim c_s \lesssim 10 \ \mathrm{km/s}$$

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- The particle will describe an elliptical trajectory around the BH but the presence of the star will make this ellipse to precess slowly.

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- The radius of the resulting circular orbit is called *circularization radius*, r_{circ}.

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$$r_{circ} = (1+q)^{4/3}\left[0.500 - 0.227\log_{10} q\right]^4 \left(\frac{M}{M_\odot}\right)^{1/3} \tau_{days}^{2/3}\ [R_\odot]$$

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- In the absence of external torques, the only possible process is a *transfer of angular momentum* from inner regions outwards by internal torques.
- The redistribution of angular momentum makes particles in the outer parts move outwards (gaining angular momentum) and the particles in the inner particles spiral inwards.

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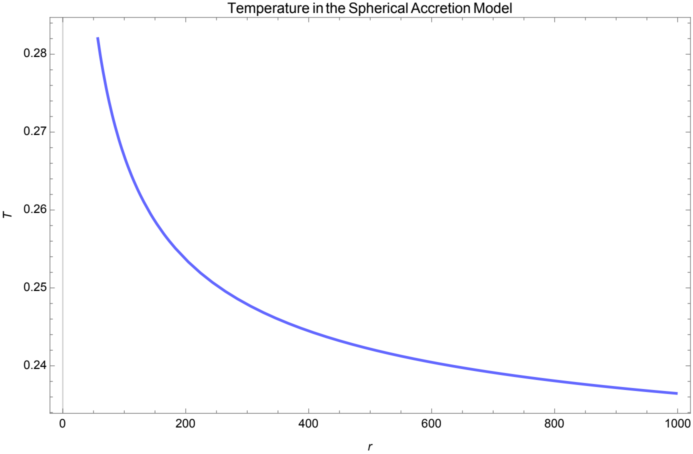
Temperature of the gas in the accretion structure



Next Lecture

10. Accretion Disks. Detailed Description