Funcion de Mundo de Synge

Si se dan dos puntos en el e-t, (x_1, x_2), y existe una única geodésica que los conecta, se puede definir una función uni-valuada denominada función de mundo. Esta se define como un medio de la distancia geodésica entre los puntos.

En el e-t plano las geodésicas son lineas rectas y con ello se tiene

$$\sigma(x_1, x_2) = \frac{1}{2}(x_1 - x_2)^\alpha (x_1 - x_2)_\alpha$$

Las primeras derivadas de σ corresponden a la separación entre los puntos,

$$\frac{\partial \sigma}{\partial x_1^\mu} = (x_1 - x_2)_\mu \qquad \frac{\partial \sigma}{\partial x_2^\nu} = -(x_1 - x_2)_\nu$$

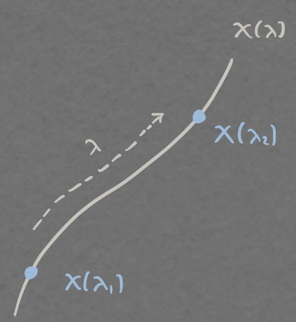
Las segundas derivadas son funciones delta:

$$\frac{\partial^2 \sigma}{\partial x_1^\nu \partial x_1^\mu} = \delta_{\mu\nu} \qquad \frac{\partial^2 \sigma}{\partial x_1^\nu \partial x_2^\mu} = -\delta_{\mu\nu}$$

$$\frac{\partial^2 \sigma}{\partial x_2^\nu \partial x_1^\mu} = -\delta_{\mu\nu} \qquad \frac{\partial^2 \sigma}{\partial x_2^\nu \partial x_2^\mu} = \delta_{\mu\nu}$$

Para generalizar este concepto a e-t curvos, consideramos la forma paramétrica de las geodésicas

$x = x(\lambda)$ $\qquad$ λ: parámetro afín



Los puntos x_1 y x_2 corresponden a

$$x_1 = x(\lambda_1)$$
$$x_2 = x(\lambda_2)$$

La función de mundo se define como

$$\boxed{\sigma(x_1, x_2) \equiv \frac{1}{2}(\lambda_2 - \lambda_1) \int_{\lambda_1}^{\lambda_2} g_{\alpha\beta}(x(\lambda))\, \dot{x}^\alpha \dot{x}^\beta \, d\lambda}$$

donde $\dot{x}^\alpha = \frac{dx^\alpha}{d\lambda}$

El integrando en esta función es

$$g_{\alpha\beta}(x(\lambda))\ \dot{x}^\alpha \dot{x}^\beta = g_{\alpha\beta}(x(\lambda))\ \frac{dx^\alpha}{d\lambda}\frac{dx^\beta}{d\lambda} = \left(\frac{ds}{d\lambda}\right)^2$$

con $ds^2 = g_{\alpha\beta}(x(\lambda))\, dx^\alpha dx^\beta$ el elemento de linea a lo largo de la geodésica.

Para obtener las derivadas de la función de mundo se considera el cambio en σ cuando uno de los puntos se desplaza.
Por ejemplo, al desplazar $x_1 \longrightarrow x_1 + \delta x_1$ manteniendo x_2 fijo se tiene

$$\delta\sigma = \frac{1}{2}(\lambda_2-\lambda_1)\ \delta\int_{\lambda_1}^{\lambda_2} g_{\alpha\beta}\, \dot{x}^\alpha \dot{x}^\beta\, d\lambda$$

$$\delta\sigma = \frac{1}{2}(\lambda_2-\lambda_1)\int_{\lambda_1}^{\lambda_2}\left[2 g_{\alpha\beta}\, \dot{x}^\alpha\, \delta\dot{x}^\beta + \partial_\gamma g_{\alpha\beta}\ \dot{x}^\alpha \dot{x}^\beta \delta x^\gamma\right] d\lambda$$

$$\delta\sigma = (\lambda_2-\lambda_1)\int_{\lambda_1}^{\lambda_2}\left[g_{\alpha\beta}\, \dot{x}^\alpha\, \delta\dot{x}^\beta + \frac{1}{2}\partial_\gamma g_{\alpha\beta}\ \dot{x}^\alpha \dot{x}^\beta \delta x^\gamma\right] d\lambda$$

El primer término en la integral es

$$\int_{\lambda_1}^{\lambda_2} g_{\alpha\beta}\, \dot{x}^\alpha\, \delta\dot{x}^\beta\, d\lambda = \int_{\lambda_1}^{\lambda_2} g_{\alpha\beta}\, \dot{x}^\alpha\, \frac{d(\delta x^\beta)}{d\lambda}\, d\lambda$$

$$u = g_{\alpha\beta}\, \dot{x}^\alpha \qquad du = \frac{d}{d\lambda}(g_{\alpha\beta}\dot{x}^\alpha)\, d\lambda = g_{\alpha\beta}\, \ddot{x}^\alpha\, d\lambda + \partial_\gamma g_{\alpha\beta}\, \dot{x}^\alpha \dot{x}^\gamma\, d\lambda$$

$$dv = \frac{d(\delta x^\beta)}{d\lambda}\, d\lambda \qquad v = \delta x^\beta$$

$$\int_{\lambda_1}^{\lambda_2} g_{\alpha\beta}\, \dot{x}^\alpha\, \delta\dot{x}^\beta\, d\lambda = \left[g_{\alpha\beta}\, \dot{x}^\alpha \delta x^\beta\right]_{\lambda_1}^{\lambda_2} - \int_{\lambda_1}^{\lambda_2} \delta x^\beta\left[g_{\alpha\beta}\, \ddot{x}^\alpha + \partial_\gamma g_{\alpha\beta}\, \dot{x}^\alpha \dot{x}^\gamma\right] d\lambda$$

Reemplazando este resultado,

$$\delta\sigma = (\lambda_2-\lambda_1)\left[g_{\alpha\beta}\, \dot{x}^\alpha \delta x^\beta\right]_{\lambda_1}^{\lambda_2}$$

$$- (\lambda_2-\lambda_1)\int_{\lambda_1}^{\lambda_2}\left[g_{\alpha\beta}\, \ddot{x}^\alpha\, \delta x^\beta + \partial_\gamma g_{\alpha\beta}\, \dot{x}^\alpha \dot{x}^\gamma \delta x^\beta - \frac{1}{2}\partial_\gamma g_{\alpha\beta}\ \dot{x}^\alpha \dot{x}^\beta \delta x^\gamma\right] d\lambda$$

Nótese que los dos últimos términos en la integral son

$$\partial_\gamma g_{\alpha\beta}\,\dot{x}^\alpha\dot{x}^\gamma\delta x^\beta-\frac{1}{2}\partial_\gamma g_{\alpha\beta}\,\dot{x}^\alpha\dot{x}^\beta\delta x^\gamma=\partial_\gamma g_{\alpha\beta}\,\dot{x}^\alpha\dot{x}^\gamma\delta x^\beta-\frac{1}{2}\partial_\beta g_{\alpha\gamma}\,\dot{x}^\alpha\dot{x}^\gamma\delta x^\beta$$

$$\partial_\gamma g_{\alpha\beta}\,\dot{x}^\alpha\dot{x}^\gamma\delta x^\beta-\frac{1}{2}\partial_\gamma g_{\alpha\beta}\,\dot{x}^\alpha\dot{x}^\beta\delta x^\gamma=\left[\partial_\gamma g_{\alpha\beta}\,\dot{x}^\alpha\dot{x}^\gamma-\frac{1}{2}\partial_\beta g_{\alpha\gamma}\,\dot{x}^\alpha\dot{x}^\gamma\right]\delta x^\beta$$

$$\partial_\gamma g_{\alpha\beta}\,\dot{x}^\alpha\dot{x}^\gamma\delta x^\beta-\frac{1}{2}\partial_\gamma g_{\alpha\beta}\,\dot{x}^\alpha\dot{x}^\beta\delta x^\gamma=\frac{1}{2}\left[\partial_\gamma g_{\alpha\beta}\,\dot{x}^\alpha\dot{x}^\gamma+\partial_\gamma g_{\alpha\beta}\,\dot{x}^\alpha\dot{x}^\gamma-\partial_\beta g_{\alpha\gamma}\,\dot{x}^\alpha\dot{x}^\gamma\right]\delta x^\beta$$

$$\partial_\gamma g_{\alpha\beta}\,\dot{x}^\alpha\dot{x}^\gamma\delta x^\beta-\frac{1}{2}\partial_\gamma g_{\alpha\beta}\,\dot{x}^\alpha\dot{x}^\beta\delta x^\gamma=\frac{1}{2}\left[\partial_\gamma g_{\alpha\beta}\,\dot{x}^\alpha\dot{x}^\gamma+\partial_\alpha g_{\gamma\beta}\,\dot{x}^\gamma\dot{x}^\alpha-\partial_\beta g_{\alpha\gamma}\,\dot{x}^\alpha\dot{x}^\gamma\right]\delta x^\beta$$

$$\partial_\gamma g_{\alpha\beta}\,\dot{x}^\alpha\dot{x}^\gamma\delta x^\beta-\frac{1}{2}\partial_\gamma g_{\alpha\beta}\,\dot{x}^\alpha\dot{x}^\beta\delta x^\gamma=\frac{1}{2}\left[\partial_\gamma g_{\alpha\beta}+\partial_\alpha g_{\gamma\beta}-\partial_\beta g_{\alpha\gamma}\right]\dot{x}^\alpha\dot{x}^\gamma\delta x^\beta$$

$$\partial_\gamma g_{\alpha\beta}\,\dot{x}^\alpha\dot{x}^\gamma\delta x^\beta-\frac{1}{2}\partial_\gamma g_{\alpha\beta}\,\dot{x}^\alpha\dot{x}^\beta\delta x^\gamma=g_{\sigma\beta}\Gamma^\sigma_{\alpha\gamma}\,\dot{x}^\alpha\dot{x}^\gamma\delta x^\beta$$

donde

$$g_{\sigma\beta}\Gamma^\sigma_{\alpha\gamma}=\frac{1}{2}\left[\partial_\gamma g_{\alpha\beta}+\partial_\alpha g_{\gamma\beta}-\partial_\beta g_{\alpha\gamma}\right]$$

Así se tiene

$$\delta\sigma=(\lambda_2-\lambda_1)\left[g_{\alpha\beta}\,\dot{x}^\alpha\delta x^\beta\right]_{\lambda_1}^{\lambda_2}-(\lambda_2-\lambda_1)\int_{\lambda_1}^{\lambda_2}\left[g_{\alpha\beta}\,\ddot{x}^\alpha\delta x^\beta+g_{\sigma\beta}\Gamma^\sigma_{\alpha\gamma}\,\dot{x}^\alpha\dot{x}^\gamma\delta x^\beta\right]d\lambda$$

$$\delta\sigma=(\lambda_2-\lambda_1)\left[g_{\alpha\beta}\,\dot{x}^\alpha\delta x^\beta\right]_{\lambda_1}^{\lambda_2}-(\lambda_2-\lambda_1)\int_{\lambda_1}^{\lambda_2}\left[\ddot{x}^\sigma+\Gamma^\sigma_{\alpha\gamma}\,\dot{x}^\alpha\dot{x}^\gamma\right]g_{\sigma\beta}\delta x^\beta d\lambda$$

El integrando es cero debido a la ecuación de la geodésica,

$$\delta\sigma=(\lambda_2-\lambda_1)\left[g_{\alpha\beta}\,\dot{x}^\alpha\delta x^\beta\right]_{\lambda_1}^{\lambda_2}$$

Al evaluar esta expresión hay que recordar que el punto x_2 se deja fijo, es decir que

$$\delta x^\beta\big|_{\lambda_2}=0$$

De esta forma se tiene únicamente

$$\delta\sigma = -(\lambda_2-\lambda_1)\left[g_{\alpha\beta}\,\dot{x}^{\alpha}\,\delta x^{\beta}\right]\Big|_{\lambda_1}$$

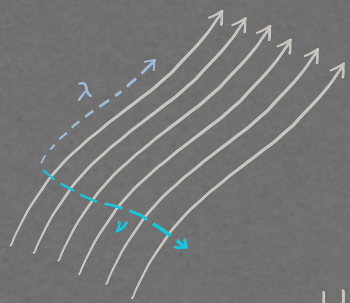
De donde se obtiene

$$\frac{\partial\sigma}{\partial x_1^{\beta}} = -(\lambda_2-\lambda_1)\,g_{\alpha\beta}\,\dot{x}^{\alpha}$$

y de forma completamente similar

$$\frac{\partial\sigma}{\partial x_2^{\beta}} = (\lambda_2-\lambda_1)\,g_{\alpha\beta}\,\dot{x}^{\alpha}$$

Ecuacion de Jacobi



Familia de geodesicas

$x(\lambda;\nu)$

λ: Parametro afin a lo largo de cada geodésica
ν: Etiqueta para cada geodésica de la familia

$$U^\alpha = \frac{dx^\alpha}{d\lambda} = \dot{x}^\alpha \quad : \text{Vector tangente}$$

$$V^\alpha = \frac{dx^\alpha}{d\nu} \quad : \text{Vector de desviación}$$

Ecuación de desvio geodésico

$$\frac{\partial^2 V^\alpha}{\partial \lambda^2} + R^\alpha{}_{\beta\gamma\delta} U^\beta U^\gamma V^\delta = 0$$

Tomando los puntos $x_0 = x(\lambda=0;\nu)$
$x_1 = x(\lambda;\nu)$

y utilizando la construcción de la función de mundo descrita arriba,

$$\sigma = \sigma(x_0, x_1) = \sigma(x(0;\nu), x(\lambda;\nu))$$

se tiene

$$\frac{\partial\sigma}{\partial x_1^\beta} = -(\lambda_1-\lambda_0)\, g_{\alpha\beta}\, \dot{x}_0^\alpha$$
$$\frac{\partial\sigma}{\partial x_1^\beta} = -(\lambda-0)\, g_{\alpha\beta}\, U^\alpha(0,\nu)$$
$$\frac{\partial\sigma}{\partial x_1^\beta} = -\lambda\, g_{\alpha\beta}\, U^\alpha(0,\nu)$$

Ya que para funciones escalares $\nabla_\beta \sigma = \partial_\beta \sigma$ se puede escribir

$$\nabla_\beta \sigma = -\lambda\, g_{\alpha\beta}\, U^\alpha(0,\nu)$$

$$\nabla^\alpha \sigma = -\lambda U^\alpha(0,\nu) = -\lambda U_0$$

Derivando ahora con respecto a ν

$$\frac{\partial}{\partial \nu} \nabla^\alpha \sigma(x_0, x_1) = \frac{\partial x_0^\beta}{\partial \nu} \nabla_\beta \nabla^\alpha \sigma + \frac{\partial x_1^\rho}{\partial \nu} \nabla_\rho \nabla^\alpha \sigma = V_0^\beta \nabla_\beta \nabla^\alpha \sigma + V_1^\rho \nabla_\rho \nabla^\alpha \sigma = -\lambda \frac{\partial U_0^\alpha}{\partial \nu}$$

Ya que $\frac{\partial U^\alpha}{\partial \nu} = \frac{\partial^2 x^\alpha}{\partial \nu \partial \lambda} = \frac{\partial^2 x^\alpha}{\partial \lambda \partial \nu} = \frac{\partial V^\alpha}{\partial \lambda}$ se tiene

$$V_0^\beta \nabla_\beta \nabla^\alpha \sigma + V_1^\rho \nabla_\rho \nabla^\alpha \sigma = -\lambda \frac{\partial V_0^\alpha}{\partial \lambda}$$

De aqui se puede despejar

$$V_1^\rho \nabla_\rho \nabla^\alpha \sigma = -V_0^\beta \nabla_\beta \nabla^\alpha \sigma - \lambda \frac{\partial V_0^\alpha}{\partial \lambda}$$

$$V_1^\rho = -(\nabla_\rho \nabla^\alpha \sigma)^{-1} (\nabla_\beta \nabla^\alpha \sigma) V_0^\beta - \lambda (\nabla_\rho \nabla^\alpha \sigma)^{-1} \frac{\partial V_0^\alpha}{\partial \lambda}$$

Como un caso particular, si se toma $V^\rho = \xi^\rho$: vector de killing, se tendrá

$$\xi^\rho(x_1) = -(\nabla_\rho \nabla^\alpha \sigma)^{-1} (\nabla_\beta \nabla^\alpha \sigma)\, \xi^\beta(x_0) - \lambda (\nabla_\rho \nabla^\alpha \sigma)^{-1} \left.\frac{\partial \xi^\alpha}{\partial \lambda}\right|_{x_0}$$

$$\xi^\rho(x_1) = -(\nabla_\rho \nabla^\alpha \sigma)^{-1} (\nabla_\beta \nabla^\alpha \sigma)\, \xi^\beta(x_0) - \lambda (\nabla_\rho \nabla^\alpha \sigma)^{-1} \left.\frac{\partial \xi^\alpha}{\partial \lambda}\right|_{x_0}$$

Definiendo

$$H^\rho{}_\alpha \equiv -(\nabla_\rho \nabla^\alpha \sigma)^{-1}$$

$$K^\rho{}_\beta \equiv -(\nabla_\rho \nabla^\alpha \sigma)^{-1} (\nabla_\beta \nabla^\alpha \sigma) = H^\rho{}_\alpha (\nabla_\beta \nabla^\alpha \sigma)$$

se tiene

$$\xi^\rho(x_1) = K^\rho{}_\beta\, \xi^\beta(x_0) - \lambda H^\rho{}_\alpha \left.\frac{\partial \xi^\alpha}{\partial \lambda}\right|_{x_0}$$

ó

$$\xi_\alpha(x_1) = K_\alpha{}^\beta\, \xi_\beta(x_0) - \lambda H_\alpha{}^\beta \left.\frac{\partial \xi_\beta}{\partial \lambda}\right|_{x_0}$$

Haciendo $\dfrac{\partial \xi^\beta}{\partial\lambda} = \dot{x}^\gamma \nabla_\gamma \xi^\beta$

$$\frac{\partial \xi_\beta}{\partial\lambda} = U^\gamma \nabla_\gamma \xi_\beta = U^\gamma \nabla_{[\gamma}\xi_{\beta]}$$

Por las propiedades de los vectores de Killing $\nabla_{(\mu}\xi_{\nu)} = 0$

$$\frac{\partial \xi_\beta}{\partial\lambda} = U^\gamma \nabla_{[\gamma}\xi_{\beta]}$$

$$\frac{\partial \xi_\beta}{\partial\lambda} = -\frac{1}{\lambda}\nabla^\gamma\sigma\, \nabla_{[\gamma}\xi_{\beta]}$$

se tiene

$$\boxed{\xi_\alpha(x_1) = K_\alpha{}^\beta \xi_\beta(x_0) + H_\alpha{}^\beta \nabla^\gamma\sigma\, \nabla_{[\gamma}\xi_{\beta]}(x_0)}$$

Ecuación de Jacobi

ESPACIO PLANO: Cuando el e-t es plano se tiene $\nabla_\mu\nabla_\nu\sigma = \delta_{\mu\nu}$ y por ello

$$\lim_{x_1\to x_0} H^\rho{}_\alpha = \lim_{x_1\to x_0} K^\rho{}_\alpha = \delta^\rho_\alpha$$

y así

$$\xi_\alpha(x_1) = \delta_\alpha{}^\beta \xi_\beta(x_0) + \delta_\alpha{}^\beta \nabla^\gamma\sigma\, \nabla_{[\gamma}\xi_{\beta]}(x_0)$$

$$\xi_\alpha(x_1) = \xi_\alpha(x_0) + \nabla^\gamma\sigma\, \nabla_{[\gamma}\xi_{\alpha]}(x_0)$$

Referencias

[1] I. Burak Ilhan. "Dynamics of Extended Objects in General Relativity". arXiv: 0911.3645 [gr-qc]

[2] J. L. Synge. "Relativity: The General Theory". North-Holland Pub. Co. (1960)