BLACK HOLES

OBSERVATORIO
ASTRONÓMICO
NACIONAL

Classical Black Holes

01. General Relativity Review

Edward Larrañaga

Outline for Part 1

Newton's Law of Gravity

$$\vec{F} = -\frac{GMm}{r^2}\hat{r}$$

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$$G = 6.67 \times 10^{-11} \frac{\mathrm{N} \cdot \mathrm{m}^2}{\mathrm{kg}^2}$$

Poisson's Equation

$$\nabla^2 \Phi = 4\pi G \rho$$

The Special Theory of Relativity

Minkowskian Metric

$$ds^2 = \eta_{\mu\nu} dx^\mu dx^\nu$$

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$$\eta_{\mu\nu} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Motion of particles

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$$p^\mu = m_0 \frac{dx^\mu}{d\tau} = m_0 U^\mu$$

Variational Principle. Action for a free particle

$$S = \int_{\tau_1}^{\tau_2} \left[-\frac{1}{2} \eta_{\mu\nu} \dot{x}^{\mu} \dot{x}^{\nu} \right] d\tau$$

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$$\dot{x}^\mu = \frac{dx^\mu}{d\tau} = U^\mu$$

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Variational Principle. Geometric Interpretation

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The variational principle $\delta S = 0$ can be interpreted geometrically as a condition giving an extreme value of the length of the trajectory of the particle in spacetime (i.e. worldline as a geodesic trajectory).

The General Theory of Relativity

General Metric

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Considering general metrics is a result of the Equivalence Principle

Coordinate Transformations in Special Relativity

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$$\beta = \frac{v}{c}$$

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$$\beta = \frac{v}{c}$$

$$\gamma = \frac{1}{\sqrt{1-\beta^2}}$$

General Coordinate Transformations

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Considering general coordinate transformations is a result of the Principle of Relativity in the General Sense

General Coordinate Transformations

$$dx^{\mu'} = \frac{\partial x^{\mu'}}{\partial x^{\alpha}} dx^{\alpha}$$

Tensors

$$T^{\mu'\nu'} = \frac{\partial x^{\mu'}}{\partial x^{\alpha}} \frac{\partial x^{\nu'}}{\partial x^{\beta}} T^{\alpha\beta}$$

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Covariant Derivative

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$$\Gamma^\nu_{\mu\sigma} = \frac{1}{2} g^{\nu\alpha} \left[\partial_\mu g_{\alpha\sigma} + \partial_\sigma g_{\mu\alpha} - \partial_\alpha g_{\mu\sigma} \right]$$

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$$\nabla_\mu T^{\nu\sigma}_{\rho} = \partial_\mu T^{\nu\sigma}_{\rho} + \Gamma^\nu_{\mu\alpha} T^{\alpha\sigma}_{\rho} + \Gamma^\sigma_{\mu\alpha} T^{\nu\alpha}_{\rho} - \Gamma^\alpha_{\mu\rho} T^{\nu\sigma}_{\alpha}$$

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$$L = -\frac{1}{2} g_{\mu\nu} \dot{x}^\mu \dot{x}^\nu$$

Geodesics

Geodesics represent the trajectories of free particles.

$$\frac{d^2x^\mu}{d\tau^2} + \Gamma^\mu_{\alpha\beta}\frac{dx^\alpha}{d\tau}\frac{dx^\beta}{d\tau} = 0$$

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Killing Vectors

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$$\nabla_\mu k_\nu + \nabla_\nu k_\mu = 0$$

Killing's Equation

Killing Vectors

Choosing an appropriate coordinate system,

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Conserved Quantities associated with Killing Vectors

Given a particle moving with momentum p^μ in a spacetime with a Killing vector k^ν, the quantity

$$Q = k^\mu p_\mu$$

is conserved.

Killing Vectors

Example. Minkowski's Spacetime

$$ds^2 = -c^2 dt^2 + dx^2 + dy^2 + dz^2$$

10 Killing vectors. Maximally Symmetric

$$k^\mu = a^{\mu\nu} x_\nu + b^\mu$$

with $a^{\mu\nu} = -a^{\nu\mu}$

Killing Vectors

Example. Spherically Symmetric Space

$$ds^2 = dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2$$

3 Killing vectors.

$$\zeta^\mu = \frac{\partial}{\partial\phi}$$

Locally Measured Physical Quantities

Moving particle described by its momentum p^μ.

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Observer moving with velocity U^α.

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Moving particle described by its momentum p^μ.

Observer moving with velocity U^α.

The energy of the particle measured by this observer is calculated by the relation

$$E = -p^\mu U_\mu$$

Riemann Tensor

Riemann tensor is defined as

$$R^{\lambda}{}_{\mu\rho\nu} = \partial_{\rho}\Gamma^{\lambda}_{\nu\mu} - \partial_{\nu}\Gamma^{\lambda}_{\rho\mu} + \Gamma^{\lambda}_{\rho\alpha}\Gamma^{\alpha}_{\nu\mu} - \Gamma^{\lambda}_{\nu\alpha}\Gamma^{\alpha}_{\rho\mu},$$

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or with all of its indexes downstairs,

$$R_{\sigma\mu\rho\nu} = g_{\sigma\lambda} R^{\lambda}{}_{\mu\rho\nu} = g_{\sigma\lambda}\left[\partial_\rho \Gamma^{\lambda}_{\nu\mu} - \partial_\nu \Gamma^{\lambda}_{\rho\mu} + \Gamma^{\lambda}_{\rho\alpha}\Gamma^{\alpha}_{\nu\mu} - \Gamma^{\lambda}_{\nu\alpha}\Gamma^{\alpha}_{\rho\mu}\right].$$

Ricci Tensor

The Ricci tensor is obtained by contraction,

$$R_{\mu\nu} = g^{\sigma\rho} R_{\sigma\mu\rho\nu}.$$

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Ricci Curvature Scalar

The curvature scalar is the contraction of Ricci,

$$R = R^{\nu}_{\nu} = g^{\mu\nu} R_{\mu\nu}.$$

Einstein Tensor

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$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R$$

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$$\nabla_\nu G_{\mu\nu} = 0$$

Einstein Field Equations

Einstein field equations are

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 8\pi G T_{\mu\nu}$$

Energy-Momentum Tensor

- Provides a complete description of the energy and momentum for extended systems.
- Is a symmetric $(2, 0)$-tensor
- Its components include the energy density, pressure and stresses
- Usually it is defined as the “flux of 4-momentum p^ν across a surface of constant x^ν“
- $\nabla_\nu T^{\mu\nu} = 0$

Single Isolated Particle

Position

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Velocity

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Energy Momentum Tensor

$$T^{\alpha\beta} = \frac{E}{c^2} U^\alpha(\tau) U^\beta(\tau) \delta\left(\vec{r} - \vec{r}_p(\tau)\right)$$

Next Lecture

02. Black holes Introduction