$$P' = \pm \frac{\sqrt{R}}{r Z}$$

$$X = \left[r^3 + a^2 r + as\left[r + M\left(G(r) - rG'(r)\right)\right]\right]E - \left[ar + Ms\left(G(r) - rG'(r)\right)\right]J$$

$$Y = J - (a+s)E$$

$$Z = r^3 - Ms^2\left(G(r) - rG'(r)\right)$$

$$R = r^2 X^2 - \Delta\left(r^4 Y^2 + m^2 Z^2\right)$$

$$X = \left[r^3 + a^2 r + as\left[r + M\left(G(r) - rG'(r)\right)\right]\right]E - \left[ar + Ms\left(G(r) - rG'(r)\right)\right]J$$

$$X = k_1 E + k_2 J$$

$$k_1 := r^3 + a^2 r + as\left[r + M\left(G(r) - rG'(r)\right)\right]$$

$$k_2 := -\left[ar + Ms\left(G(r) - rG'(r)\right)\right]$$

$$R = r^2 X^2 - \Delta\left(r^4 Y^2 + m^2 Z^2\right)$$

$$R = r^2\left[k_1^2 E^2 + 2k_1 k_2 EJ + k_2^2 J^2\right] - \Delta\left[r^4\left[J^2 - 2(a+s)EJ + (a+s)^2 E^2\right] + m^2 Z^2\right]$$

$$R = r^2\left[k_1^2 - \Delta r^2 (a+s)^2\right]E^2 + r^2\left[2k_1 k_2 J + 2\Delta r^2 (a+s) J\right]E + \left[r^2 k_2^2 J^2 - \Delta r^4 J^2 - \Delta m^2 Z^2\right]$$

$$P' = \pm \frac{\sqrt{R}}{rZ}$$

$$(P')^2 = \frac{R}{r^2 Z^2}$$

$$(P')^2 = \frac{m^2}{r^2 Z^2}\left[\alpha \mathcal{E}^2 + \beta \mathcal{E} + \gamma\right]$$

$$\alpha = r^2\left[K_1^2 - \Delta r^2 (a+s)^2\right]$$

$$\beta = 2jr^2\left[K_1 K_2 + \Delta r^2 (a+s)\right]$$

$$\gamma = r^2 j^2 (K_2^2 - r^2 \Delta) - \Delta Z^2$$

$$\mathcal{E} = \frac{E}{m} \qquad j = \frac{J}{m}$$

$$K_1 = r^3 + a^2 r + as\left[r + M\left(G(r) - rG'(r)\right)\right]$$

$$K_2 = -\left[ar + Ms\left(G(r) - rG'(r)\right)\right]$$

$$Z = r^3 - Ms^2\left(G(r) - rG'(r)\right)$$

$$(P')^2 = \frac{m^2 \alpha}{r^2 Z^2}\left(\mathcal{E} - \frac{-\beta + \sqrt{\beta^2 - 4\alpha\gamma}}{2\alpha}\right)\left(\mathcal{E} + \frac{\beta + \sqrt{\beta^2 - 4\alpha\gamma}}{2\alpha}\right)$$

$$\boxed{V_{eff}(r) = \frac{-\beta + \sqrt{\beta^2 - 4\alpha\gamma}}{2\alpha}}$$

KERR'S LIMIT

We take $G(r) = G_0 = 1$ and compare with equations (2.26) - (2.27) in [Saijo et. al. 1998]

* Quantities with ~ are from [Saijo et. al 1998]

$$\alpha = r^2 \left[K_1^2 - \Delta r^2 (a+s)^2 \right]$$

$$\alpha = r^2 \left[(r^3 + a^2 r + asr + asM)^2 - \Delta r^2 (a+s)^2 \right]$$

$$\alpha = r^2 \left\{ r^2 \left[r^2 + a^2 + as + as\frac{M}{r} \right]^2 - r^2 \Delta (a+s)^2 \right]$$

$$\alpha = r^4 \left[\left(r^2 + a^2 + \frac{as}{r}(r+M) \right)^2 - \Delta (a+s)^2 \right]$$

$$\alpha = r^4 \tilde{\alpha}$$

$$\beta = 2jr^2 \left[K_1 K_2 + \Delta r^2 (a+s) \right]$$

$$\beta = 2jr^2 \left[(r^3 + a^2 r + asr + asM)(-ar - Ms) + \Delta r^2 (a+s) \right]$$

$$\beta = 2jr^2 \left[-r^2 \left(r^2 + a^2 + as + as\frac{M}{r} \right) \left(a + \frac{Ms}{r} \right) + r^2 \Delta (a+s) \right]$$

$$\beta = -2jr^4 \left[\left(a + \frac{Ms}{r} \right) \left(r^2 + a^2 + \frac{as}{r}(r+M) \right) - \Delta (a+s) \right]$$

$$\beta = -2r^4 \tilde{\beta}$$

$$\gamma = r^2 j^2 (K_2^2 - r^2\Delta) - \Delta Z^2$$

$$\gamma = r^2 j^2 \left[(-ar - Ms)^2 - r^2\Delta\right] - \Delta (r^3 - Ms^2)^2$$

$$\gamma = r^2 j^2 \left[r^2\left(a + \frac{Ms}{r}\right)^2 - r^2\Delta\right] - \Delta r^6\left(1 - \frac{Ms^2}{r^3}\right)^2$$

$$\gamma = r^4 \left\{ \left(a + \frac{Ms}{r}\right)^2 j^2 - \Delta\left[j^2 + r^2\left(1 - \frac{Ms^2}{r^3}\right)^2\right]\right\}$$

$$\gamma = r^4 \tilde{\gamma}$$

$$V_{eff} = \frac{-\beta + \sqrt{\beta^2 - 4\alpha\gamma}}{2\alpha}$$

$$V_{eff} = \frac{2r^4\tilde{\beta} + \sqrt{4r^8\tilde{\beta} - 4r^4\tilde{\alpha}r^4\tilde{\gamma}}}{2r^4\tilde{\alpha}}$$

$$V_{eff} = \frac{\tilde{\beta} + \sqrt{\tilde{\beta} - \tilde{\alpha}\tilde{\gamma}}}{\tilde{\alpha}} = \tilde{V}_{eff}$$