Equations of Motion

$$\frac{DP^\mu}{D\tau} = -\frac{1}{2} R^\mu{}_{\nu\rho\sigma} u^\nu S^{\rho\sigma}$$

$$\frac{DS^{\mu\nu}}{D\tau} = P^\mu u^\nu - P^\nu u^\mu$$

$$S^{01} = -\frac{P_3}{P_0} S^{31}$$

$$S^{03} = -\frac{P_1}{P_0} S^{13} = \frac{P_1}{P_0} S^{31}$$

$$S^{31} = \frac{s P_0}{m\sqrt{\Sigma}}$$

$$S^{01} = -\frac{s P_3}{m\sqrt{\Sigma}}$$

$$S^{03} = \frac{s P_1}{m\sqrt{\Sigma}}$$

Taking $\mu=0$; $\nu=1$ in the equation of motion for $S^{\mu\nu}$ gives

$$\frac{DS^{01}}{D\tau} = P^0 u^1 - P^1 u^0$$

$$\frac{D}{D\tau}\left(-\frac{sP_3}{m\sqrt{\Sigma}}\right) = P^0 u^1 - P^1 u^0$$

In the equatorial plane $\Sigma = r^2 + a^2\cos^2\theta \rightarrow r^2$:

$$-\frac{s}{m}\frac{D}{D\tau}\left(\frac{P_3}{r}\right) = P^0 u^1 - P^1 u^0$$

$$-\frac{s}{mr}\frac{DP_3}{D\tau} + \frac{sP_3}{mr^2}\frac{Dr}{D\tau} = P^0 u^1 - P^1 u^0$$

$$-\frac{s}{mr} g_{3\mu}\frac{DP^\mu}{D\tau} + \frac{sP_3}{mr^2}\frac{dx^\mu}{d\tau}\nabla_\mu r = P^0 u^1 - P^1 u^0$$

For scalar functions, the covariant derivative becomes a partial one:

$$\nabla_\mu r = \partial_\mu r = \delta^1_\mu$$

Then

$$-\frac{s}{mr} g_{3\mu}\frac{DP^\mu}{D\tau} + \frac{sP_3}{mr^2}\frac{dx^\mu}{d\tau}\delta^1_\mu = P^0 u^1 - P^1 u^0$$

Using the first of the equations of motion:

$$-\frac{s}{mr} g_{3\mu}\left(-\frac{1}{2} R^\mu{}_{\nu\rho\sigma} u^\nu S^{\rho\sigma}\right) + \frac{sP_3}{mr^2}\frac{dx^\mu}{d\tau}\delta^1_\mu = P^0 u^1 - P^1 u^0$$

$$\frac{1}{2}\frac{s}{mr} g_{3\mu} R^\mu{}_{\nu\rho\sigma} u^\nu S^{\rho\sigma} + \frac{sP_3}{mr^2}\frac{dx^\mu}{d\tau}\delta^1_\mu = P^0 u^1 - P^1 u^0$$

Remember that $u^\mu = \dfrac{dx^\mu}{d\tau}$

and in particular $u^0 = \dfrac{dx^0}{d\tau} = \dfrac{dt}{d\tau}$

Then, we have that

$$\frac{u^\mu}{u^0} = \frac{dx^\mu}{dt} \equiv \dot{x}^\mu$$

Using this result, we write:

$$\frac{1}{2}\frac{s}{mr} g_{3\mu} R^\mu{}_{\nu\rho\sigma} \frac{dx^\nu}{dt} S^{\rho\sigma} + \frac{sP_3}{mr^2}\frac{dx^\mu}{dt}\delta^1_\mu = P^0\frac{dr}{dt} - P^1$$

$$\frac{1}{2}\frac{s}{mr} g_{3\mu} R^\mu{}_{\nu\rho\sigma} \dot{x}^\nu S^{\rho\sigma} + \frac{sP_3}{mr^2}\dot{x}^\mu\delta^1_\mu = P^0\dot{r} - P^1$$

$$\boxed{\frac{1}{2}\frac{s}{mr} g_{3\mu} R^\mu{}_{\nu\rho\sigma} \dot{x}^\nu S^{\rho\sigma} + \frac{sP_3}{mr^2}\dot{r} = P^0\dot{r} - P^1}$$

Taking $\mu=0$; $\nu=3$ in the equation of motion for $S^{\mu\nu}$ gives

$$\frac{DS^{03}}{D\tau} = P^0u^3 - P^3u^0$$

$$\frac{D}{D\tau}\left(\frac{sP_1}{m\sqrt{\Sigma}}\right) = P^0u^3 - P^3u^0$$

In the equatorial plane $\Sigma = r^2 + a^2\cos^2\theta \rightarrow r^2$:

$$\frac{s}{m}\frac{D}{D\tau}\left(\frac{P_1}{r}\right) = P^0u^3 - P^3u^0$$

$$\frac{s}{mr}\frac{DP_1}{D\tau} - \frac{sP_1}{mr^2}\frac{Dr}{D\tau} = P^0u^3 - P^3u^0$$

$$\frac{s}{mr} g_{1\mu}\frac{DP^\mu}{D\tau} - \frac{sP_1}{mr^2}\frac{dx^\mu}{d\tau}\nabla_\mu r = P^0u^3 - P^3u^0$$

$$\frac{s}{mr} g_{1\mu}\frac{DP^\mu}{D\tau} - \frac{sP_1}{mr^2}\frac{dx^\mu}{d\tau}\delta^1_\mu = P^0u^3 - P^3u^0$$

$$-\frac{s}{mr} g_{1\mu}\left(-\frac{1}{2} R^{\mu}{}_{\nu\rho\sigma} u^{\nu} S^{\rho\sigma}\right) - \frac{sP_1}{mr^2} \frac{dx^{\mu}}{d\tau} \delta^{1}_{\mu} = P^0 u^3 - P^3 u^0$$

$$-\frac{1}{2} \frac{s}{mr} g_{1\mu} R^{\mu}{}_{\nu\rho\sigma} u^{\nu} S^{\rho\sigma} - \frac{sP_1}{mr^2} \frac{dx^{\mu}}{d\tau} \delta^{1}_{\mu} = P^0 u^3 - P^3 u^0$$

$$-\frac{1}{2} \frac{s}{mr} g_{1\mu} R^{\mu}{}_{\nu\rho\sigma} \frac{dx^{\nu}}{dt} S^{\rho\sigma} - \frac{sP_1}{mr^2} \frac{dx^{\mu}}{dt} \delta^{1}_{\mu} = P^0 \frac{d\varphi}{dt} - P^3$$

$$-\frac{1}{2} \frac{s}{mr} g_{1\mu} R^{\mu}{}_{\nu\rho\sigma} \dot{x}^{\nu} S^{\rho\sigma} - \frac{sP_1}{mr^2} \dot{x}^{\mu} \delta^{1}_{\mu} = P^0 \dot{\varphi} - P^3$$

$$\boxed{-\frac{1}{2} \frac{s}{mr} g_{1\mu} R^{\mu}{}_{\nu\rho\sigma} \dot{x}^{\nu} S^{\rho\sigma} - \frac{sP_1}{mr^2} \dot{r} = P^0 \dot{\varphi} - P^3}$$

Equations of Motion for $\dot{r}$ and $\dot{\varphi}$

$$\frac{1}{2}\frac{s}{mr} g_{3\mu} R^{\mu}{}_{\nu\rho\sigma} \dot{x}^{\nu} S^{\rho\sigma} + \frac{sP_3}{mr^2}\dot{r} = P^0 \dot{r} - P^1$$

Note that

$$\begin{aligned}
g_{3\mu} R^{\mu}{}_{\nu\rho\sigma} \dot{x}^{\nu} S^{\rho\sigma} &= R_{3\nu\rho\sigma} \dot{x}^{\nu} S^{\rho\sigma} \\
&= R_{30\rho\sigma} \dot{x}^{0} S^{\rho\sigma} + R_{31\rho\sigma} \dot{x}^{1} S^{\rho\sigma} \\
&= R_{30\rho\sigma} S^{\rho\sigma} + R_{31\rho\sigma} \dot{r} S^{\rho\sigma} \\
&= 2R_{3001} S^{01} + 2R_{3003} S^{03} + 2R_{3013} S^{13} \\
&\quad + 2\dot{r}\left[R_{3101} S^{01} + R_{3103} S^{03} + R_{3113} S^{13}\right]
\end{aligned}$$

However, $R_{3001} = 0$

$R_{3013} = 0$

$R_{3103} = 0$

$$g_{3\mu} R^{\mu}{}_{\nu\rho\sigma} \dot{x}^{\nu} S^{\rho\sigma} = 2R_{3003} S^{03} + 2\dot{r}\left[R_{3101} S^{01} + R_{3113} S^{13}\right]$$

This gives the equation of motion

$$g_{3\mu} R^{\mu}{}_{\nu\rho\sigma} \dot{x}^{\nu} S^{\rho\sigma}$$

$$P^0 \dot{r} - P^1 = \frac{sP_3}{mr^2}\dot{r} + \frac{1}{2}\frac{s}{mr}\left[2R_{3003} S^{03} + 2\dot{r}\left(R_{3101} S^{01} + R_{3113} S^{13}\right)\right]$$

$$P^0 \dot{r} - P^1 = \frac{sP_3}{mr^2}\dot{r} + \frac{s}{mr} R_{3003} S^{03} + \frac{s}{mr}\left(R_{3101} S^{01} + R_{3113} S^{13}\right)\dot{r}$$

$$\dot r\left[P^0 - \frac{sP_3}{mr^2} - \frac{s}{mr}\left(R_{3101}S^{01} + R_{3113}S^{13}\right)\right] = P^1 + \frac{s}{mr}R_{3003}S^{03}$$

$$\dot r\left[P^0 - \frac{sP_3}{mr^2} - \frac{s}{mr}\left(-R_{3101}\frac{sP_3}{mr} - R_{3113}\frac{sP_0}{mr}\right)\right] = P^1 + \frac{s}{mr}R_{3003}\frac{sP_1}{mr}$$

$$\dot r\left[P^0 - \frac{sP_3}{mr^2} + \frac{s^2}{m^2r^2}\left(R_{3101}P_3 + R_{3113}P_0\right)\right] = P^1 + \frac{s^2}{m^2r^2}R_{3003}P_1$$

$$\dot r\left[P^0 - \frac{sP_3}{mr^2} + \frac{s^2}{m^2r^2}\left(R_{3101}P_3 + R_{3113}P_0\right)\right] = P^1\left[1 + \frac{s^2}{m^2r^2}R_{3003}\,g_{11}\right]$$

$$\boxed{\dot r = P^1\left[1 + \frac{s^2}{m^2r^2}R_{3003}\,g_{11}\right]\left[P^0 - \frac{sP_3}{mr^2} + \frac{s^2}{m^2r^2}\left(R_{3101}P_3 + R_{3113}P_0\right)\right]^{-1}}$$

$$-\frac{1}{2}\frac{s}{mr}g_{1\mu}R^{\mu}{}_{\nu\rho\sigma}\dot x^{\nu}S^{\rho\sigma} - \frac{sP_1}{mr^2}\dot r = P^0\dot\varphi - P^3$$

In this case,

$$\begin{aligned} g_{1\mu}R^{\mu}{}_{\nu\rho\sigma}\dot x^{\nu}S^{\rho\sigma} &= R_{1\nu\rho\sigma}\dot x^{\nu}S^{\rho\sigma} \\ &= R_{10\rho\sigma}\dot x^0 S^{\rho\sigma} + R_{11\rho\sigma}\dot x^1 S^{\rho\sigma} + R_{13\rho\sigma}\dot x^3 S^{\rho\sigma} \\ &= 2R_{1001}S^{01} + 2R_{1003}S^{03} + 2R_{1013}S^{13} \\ &\quad + 2\dot\varphi\left[R_{1301}S^{01} + R_{1303}S^{03} + R_{1313}S^{13}\right] \end{aligned}$$

$$g_{1\mu}R^{\mu}{}_{\nu\rho\sigma}\dot x^{\nu}S^{\rho\sigma} = 2R_{1001}S^{01} + 2R_{1013}S^{13} + 2\dot\varphi\left[R_{1301}S^{01} + R_{1313}S^{13}\right]$$

$$P^0\dot\varphi - P^3 = -\frac{sP_1}{mr^2}\dot r - \frac{1}{2}\frac{s}{mr}\left[2R_{1001}S^{01} + 2R_{1013}S^{13} + 2\dot\varphi\left(R_{1301}S^{01} + R_{1313}S^{13}\right)\right]$$

$$\dot{\varphi}\left[P^0 + \frac{s}{mr}\left(R_{1301}S^{01} + R_{1313}S^{13}\right)\right] = P^3 - \frac{s}{mr}\left(R_{1001}S^{01} + R_{1013}S^{13}\right) - \frac{sP_1}{mr^2}\dot{r}$$

$$\dot{\varphi}\left[P^0 - \frac{s^2}{m^2r^2}\left(R_{1301}P_3 + R_{1313}P_0\right)\right] = P^3 + \frac{s^2}{m^2r^2}\left(R_{1001}P_3 + R_{1013}P_0\right) - \frac{sP_1}{mr^2}\dot{r}$$

$$\boxed{\dot{\varphi} = \left[P^3 + \frac{s^2}{m^2r^2}\left(R_{1001}P_3 + R_{1013}P_0\right) - \frac{sP_1}{mr^2}\dot{r}\right]\left[P^0 - \frac{s^2}{m^2r^2}\left(R_{1301}P_3 + R_{1313}P_0\right)\right]^{-1}}$$