To study the motion of a spinning test particle we begin by searching for those points in the r-θ plane where P^r and P^θ vanish (apsides). Since

$$S^{r\theta} = -\frac{P_\theta}{r} = \pm \frac{s\, P_t\, b \sin\alpha}{\sqrt{-g_{tt}(r_0)\, g_{rr}(r_0)\left(J^2 \sin^2\alpha + r_0^2 \mu^2\right)}} = 0$$

implies that the return points have $\alpha = 0$ or $\alpha = \pi$; i.e. the spin lies in the meridian plane.

Using $P^r = P^\theta = 0$ in the normalization condition of P^μ gives

$$g^{tt} P_t^2 + g^{\varphi\varphi} P_\varphi^2 = -\mu^2$$

$$\frac{P_t^2}{\mu^2 g_{tt}} + \frac{P_\varphi^2}{\mu^2 g_{\varphi\varphi}} = -1$$

and therefore we can choose

$$\left\{ \begin{aligned} \frac{P_t^2}{\mu^2 g_{tt}} &= -\cosh^2 X \\ \frac{P_\varphi^2}{\mu^2 g_{\varphi\varphi}} &= \sinh^2 X \end{aligned} \right. \longrightarrow \sinh^2 X - \cosh^2 X = -1$$

we chose a minus sign to obtain a positive energy below.

Hence:

$$P_t = -\mu \sqrt{-g_{tt}} \cosh X$$

$$P_\varphi = \mu \sqrt{g_{\varphi\varphi}} \sinh X = \mu r \sin\theta \sinh X$$

The function $X = X(r,\theta)$ may be obtained by replacing these results in the spin magnitude

$$s^2 = \frac{1}{2} S^{\mu\nu} S_{\mu\nu}$$

$$s^2 = S^{tr} S_{tr} + S^{t\theta} S_{t\theta} + S^{t\varphi} S_{t\varphi} + S^{r\theta} S_{r\theta} + S^{r\varphi} S_{r\varphi} + S^{\theta\varphi} S_{\theta\varphi}$$

$$s^2 = g_{tt} g_{rr} (S^{tr})^2 + g_{tt} r^2 (S^{t\theta})^2 + g_{tt} r^2 \sin^2\theta\, (S^{t\varphi})^2 + g_{rr} r^2 (S^{r\theta})^2$$

$$+ g_{rr} r^2 \sin^2\theta\, (S^{r\varphi})^2 + r^4 \sin^2\theta\, (S^{\theta\varphi})^2$$

Since we have

$$S^{\theta\varphi} = \frac{J}{r^2}\cot\theta$$

$$S^{r\theta} = -\frac{P_\theta}{r} = 0$$

$$S^{\varphi r} = \frac{1}{r}\left(-J + \frac{P_\varphi}{\sin^2\theta}\right)$$

$$S^{tr} = -\frac{1}{rP_t}\left(P_\theta^2 + \frac{P_\varphi^2}{\sin^2\theta} - JP_\varphi\right) = -\frac{1}{rP_t}\left(\frac{P_\varphi^2}{\sin^2\theta} - JP_\varphi\right)$$

$$S^{t\theta} = \frac{1}{rP_t}\left(P_r P_\theta + J\frac{P_\varphi}{r}\cot\theta\right) = \frac{J}{r^2}\frac{P_\varphi}{P_t}\cot\theta$$

$$S^{t\varphi} = -\frac{1}{rP_t}\left(JP_r - \frac{P_r P_\varphi}{\sin^2\theta} + \frac{JP_\theta}{r}\cot\theta\right) = 0$$

Then

$$S^2 = g_{tt}g_{rr}(S^{tr})^2 + g_{tt}r^2(S^{t\theta})^2 + g_{rr}r^2\sin^2\theta\,(S^{r\varphi})^2 + r^4\sin^2\theta\,(S^{\theta\varphi})^2$$

$$S^2 = g_{tt}g_{rr}\frac{1}{r^2P_t^2}\left(\frac{P_\varphi^2}{\sin^2\theta} - JP_\varphi\right)^2 + g_{tt}r^2\frac{J^2}{r^4}\frac{P_\varphi^2}{P_t^2}\cot^2\theta$$

$$+ g_{rr}r^2\sin^2\theta\,\frac{1}{r^2}\left(-J + \frac{P_\varphi}{\sin^2\theta}\right)^2 + r^4\sin^2\theta\,\frac{J^2}{r^4}\cot^2\theta$$

$$S^2 = g_{tt}g_{rr}\frac{P_\varphi^2}{r^2P_t^2}\left(\frac{P_\varphi}{\sin^2\theta} - J\right)^2 + g_{tt}\frac{J^2}{r^2}\frac{P_\varphi^2}{P_t^2}\cot^2\theta$$

$$+ g_{rr}\sin^2\theta\left(\frac{P_\varphi}{\sin^2\theta} - J\right)^2 + J^2\cos^2\theta$$

$$s^2 = g_{tt}\, g_{rr} \frac{1}{r^2}\left(\frac{p_\varphi}{\sin^2\theta} - J\right)^2\left(-\frac{r^2\sin^2\theta}{g_{tt}}\right)\tanh^2 X + g_{tt}\frac{J^2}{r^2}\cot^2\theta\left(-\frac{r^2\sin^2\theta}{g_{tt}}\right)\tanh^2 X$$
$$+ g_{rr}\sin^2\theta\left(\frac{p_\varphi}{\sin^2\theta} - J\right)^2 + J^2\cos^2\theta$$

$$s^2 = -g_{rr}\left(\frac{p_\varphi}{\sin^2\theta} - J\right)^2\sin^2\theta\tanh^2 X - J^2\cos^2\theta\tanh^2 X$$
$$+ g_{rr}\sin^2\theta\left(\frac{p_\varphi}{\sin^2\theta} - J\right)^2 + J^2\cos^2\theta$$

$$s^2 = -g_{rr}\left(\frac{\mu r\sinh X}{\sin\theta} - J\right)^2\sin^2\theta\tanh^2 X - J^2\cos^2\theta\tanh^2 X$$
$$+ g_{rr}\sin^2\theta\left(\frac{\mu r\sinh X}{\sin\theta} - J\right)^2 + J^2\cos^2\theta$$

$$s^2 = -g_{rr}(\mu r\sinh X - J\sin\theta)^2\tanh^2 X - J^2\cos^2\theta\tanh^2 X$$
$$+ g_{rr}(\mu r\sinh X - J\sin\theta)^2 + J^2\cos^2\theta$$

$$s^2 = g_{rr}(\mu r\sinh X - J\sin\theta)^2(1-\tanh^2 X) + J^2\cos^2\theta(1-\tanh^2 X)$$

$$1 - \tanh^2 X = 1 - \frac{\sinh^2 X}{\cosh^2 X} = \frac{\cosh^2 X - \sinh^2 X}{\cosh^2 X} = \frac{1}{\cosh^2 X}$$

$$s^2 = \frac{g_{rr}}{\cosh^2 X}(\mu r\sinh X - J\sin\theta)^2 + J^2\frac{\cos^2\theta}{\cosh^2 X}$$

$$s^2\cosh^2 X = g_{rr}(\mu r\sinh X - J\sin\theta)^2 + J^2\cos^2\theta$$

$$s^2\cosh^2 X = g_{rr}(\mu^2 r^2\sinh^2 X + J^2\sin^2\theta - 2\mu r J\sin\theta\sinh X) + J^2\cos^2\theta$$

$$s^2(\sinh^2 x + 1) = g_{rr}\left(\mu^2 r^2 \sinh^2 X + J^2 \sin^2\theta - 2\mu r J \sin\theta \sinh X\right) + J^2 \cos^2\theta$$

$$s^2 g^{rr} \sinh^2 X + g^{rr} s^2 = \mu^2 r^2 \sinh^2 X + J^2 \sin^2\theta - 2\mu r J \sin\theta \sinh X + g^{rr} J^2 \cos^2\theta$$

$$(\mu^2 r^2 - g^{rr} s^2) \sinh^2 X - 2\mu r J \sin\theta \sinh X + J^2 \sin^2\theta + g^{rr} J^2 \cos^2\theta - g^{rr} s^2 = 0$$

$$(\mu^2 r^2 - g^{rr} s^2) \sinh^2 X - 2\mu r J \sin\theta \sinh X + J^2 \sin^2\theta + g^{rr} J^2 (1 - \sin^2\theta) - g^{rr} s^2 = 0$$

$$\boxed{(\mu^2 r^2 - g^{rr} s^2) \sinh^2 X - 2\mu r J \sin\theta \sinh X + (1 - g^{rr}) J^2 \sin^2\theta + g^{rr} (J^2 - s^2) = 0}$$

This is a quadratic equation to obtain two possible functions:

$\sinh X_{(\pm)}$ and then : $\cosh X_{(\pm)}$

$$\sinh X_{(\pm)} = \frac{2\mu r J \sin\theta \pm \sqrt{(2\mu r J \sin\theta)^2 - 4(\mu^2 r^2 - g^{rr} s^2)\left[(1 - g^{rr}) J^2 \sin^2\theta + g^{rr}(J^2 - s^2)\right]}}{2(\mu^2 r^2 - g^{rr} s^2)}$$

$$\boxed{\sinh X_{(\pm)} = \frac{\mu r J \sin\theta \pm \sqrt{\mu^2 r^2 J^2 \sin^2\theta - (\mu^2 r^2 - g^{rr} s^2)\left[(1 - g^{rr}) J^2 \sin^2\theta + g^{rr}(J^2 - s^2)\right]}}{(\mu^2 r^2 - g^{rr} s^2)}}$$

The energy equation gives:

$$E = -P_t - \frac{1}{2rP_t}\left(P_\theta^2 + \frac{P_\varphi^2}{\sin^2\theta} - JP_\varphi\right)\partial_r g_{tt} - \tilde{q}\,\Phi$$

$$E = -P_t - \frac{1}{2rP_t}\left(\frac{P_\varphi^2}{\sin^2\theta} - JP_\varphi\right)\partial_r g_{tt} - \tilde{q}\,\Phi$$

$$E = -P_t - \frac{P_\varphi}{2rP_t}\left(\frac{P_\varphi}{\sin^2\theta} - J\right)\partial_r g_{tt} - \tilde{q}\,\Phi$$

$$E = \mu\sqrt{-g_{tt}}\cosh\chi_{(\pm)} + \frac{1}{2r}\,\frac{r\sin\theta}{\sqrt{-g_{tt}}}\tanh\chi_{(\pm)}\left(\frac{\mu r\sinh\chi_{(\pm)}}{\sin\theta} - J\right)\partial_r g_{tt} - \tilde{q}\,\Phi$$

$$E = \mu\sqrt{-g_{tt}}\cosh\chi_{(\pm)} + \frac{1}{2}\,\frac{\tanh\chi_{(\pm)}}{\sqrt{-g_{tt}}}\left(\mu r\sinh\chi_{(\pm)} - J\sin\theta\right)\partial_r g_{tt} - \tilde{q}\,\Phi$$

$$E = \mu\left[\sqrt{-g_{tt}}\cosh\chi_{(\pm)} + \frac{1}{2}\,\frac{\tanh\chi_{(\pm)}}{\sqrt{-g_{tt}}}\left(r\sinh\chi_{(\pm)} - \frac{J}{\mu}\sin\theta\right)\partial_r g_{tt} - \frac{\tilde{q}}{\mu}\,\Phi\right]$$

$$\boxed{e = \sqrt{-g_{tt}}\cosh\chi_{(\pm)} + \frac{1}{2}\,\frac{\tanh\chi_{(\pm)}}{\sqrt{-g_{tt}}}\left(r\sinh\chi_{(\pm)} - j\sin\theta\right)\partial_r g_{tt} - q\,\Phi}$$

$$e = \frac{E}{\mu} \qquad j = \frac{J}{\mu} \qquad q = \frac{\tilde{q}}{\mu}$$

The solutions $(\pm)$ correspond to the direction of the spin.
If the z-component of the spin is parallel to the total angular momentum, $S^{\phi r} < 0 \rightarrow \frac{\pi}{2} < \alpha < \frac{3\pi}{2}$, we take $V_{(-)}$

If the z-component of the spin is antiparallel to the total angular momentum, $S^{\phi r} > 0 \rightarrow -\frac{\pi}{2} < \alpha < \frac{\pi}{2}$, we take $V_{(+)}$

It is important to note that $S^{\phi r}$ does not change its signature during the evolution

When $s=0$ and $\theta=\frac{\pi}{2}$ $\Rightarrow$ $V_{eff}(r)$ $\rightarrow$ spinless particle around the BH in the equatorial plane.

Motion occurs when $E^2 > V_{eff}^2$

From the plots:

$V_{(-)} \leq V_{(+)}$

$\max(V_{(-)})$ occurs at a radius smaller that the corresponding to $\max(V_{(+)})$. $\longrightarrow$ particles under $V_{(-)}$ move closer to the event horizon

$\rightarrow$ because spin-orbit interaction is repulsive if the spin is parallel to the orbital angular momentum, balancing gravity.

Since we are interested in chaotic behavior and this is present in a strong field scenario, we will consider smaller orbits; i.e. we will only study the effective potential $V_{(-)} \equiv V(r,\theta;J,s)$

Given J and s $\longrightarrow$ Plot of $V(r,\theta;J,s)$ in the (r,θ)-plane