Lagrangian Formulation non-spinning Particle

Following Bażański (1989)(1977), Wanas et. al. (1995), Wanas (1997) and Bakry et. al. (2021)

$$L = m\, g_{\mu\nu}\, u^\mu \frac{D\Psi^\nu}{D\tau} - q\, g_{\alpha\beta}\, \Psi^\beta F^\alpha{}_\mu u^\mu \qquad (1)$$

$u^\mu = \dot{x}^\mu = \frac{dx^\mu}{d\tau}$: timelike unit tangent vector ?

Ψ^μ: deviation vector associated with the pat

$$\frac{D}{D\tau} = \frac{dx^\mu}{d\tau} \nabla_\mu = u^\mu \nabla_\mu$$

Hence, we write:

$$L = m\, g_{\mu\nu}\, u^\mu u^\alpha \nabla_\alpha \Psi^\nu - q\, g_{\alpha\beta}\, \Psi^\beta F^\alpha{}_\mu u^\mu$$
$$L = m\, g_{\mu\nu}\, u^\mu u^\alpha (\partial_\alpha \Psi^\nu + \Gamma^\nu_{\alpha\beta} \Psi^\beta) - q\, g_{\alpha\beta}\, \Psi^\beta F^\alpha{}_\mu u^\mu$$
$$L = m\, g_{\mu\nu}\, u^\mu u^\alpha \partial_\alpha \Psi^\nu + m\, g_{\mu\nu}\, u^\mu u^\alpha \Gamma^\nu_{\alpha\beta} \Psi^\beta - q\, g_{\alpha\beta}\, \Psi^\beta F^\alpha{}_\mu u^\mu$$
$$L = m\, g_{\mu\nu}\, u^\mu \dot{\Psi}^\nu + m\, g_{\mu\nu}\, u^\mu u^\alpha \Gamma^\nu_{\alpha\beta} \Psi^\beta - q\, g_{\alpha\beta}\, \Psi^\beta F^\alpha{}_\mu u^\mu \qquad (2)$$

Equation of Motion

Momentum:

$$P_\alpha \equiv \frac{\partial L}{\partial \dot{\Psi}^\alpha} = m\, g_{\alpha\nu} u^\nu = m\, u_\alpha$$

Euler-Lagrange equations

$$\frac{d}{d\tau}\left(\frac{\partial L}{\partial \dot{\Psi}^\alpha}\right) = \frac{\partial L}{\partial \Psi^\alpha}$$

$$\frac{\partial L}{\partial \Psi^\alpha} = m\, g_{\mu\nu}\, u^\mu u^\sigma \Gamma^\nu_{\sigma\alpha} - q\, g_{\nu\alpha}\, F^\nu{}_\mu u^\mu$$
$$\frac{\partial L}{\partial \Psi^\alpha} = m\, g_{\mu\nu}\, u^\mu u^\sigma \Gamma^\nu_{\sigma\alpha} - q\, F_{\mu\alpha} u^\mu \qquad (4)$$

Then

$$\frac{d}{d\tau}(m u_\alpha) = m\, g_{\mu\nu}\, u^\mu u^\sigma \Gamma^\nu_{\sigma\alpha} - q\, F_{\mu\alpha} u^\mu$$

$$\frac{du_\alpha}{d\tau} - u^\sigma \Gamma^\nu_{\sigma\alpha} u_\nu = -\frac{q}{m} F_{\mu\alpha} u^\mu$$

$$\frac{dx^\sigma}{d\tau}\partial_\sigma u_\alpha - u^\sigma \Gamma^\nu_{\sigma\alpha} u_\nu = -\frac{q}{m} F_{\mu\alpha} u^\mu$$

$$\frac{Du_\alpha}{D\tau} = -\frac{q}{m} F_{\mu\alpha} u^\mu \qquad (6)$$

This can also be written as

$$\frac{Du^\alpha}{D\tau} = -\frac{q}{m} F_\mu{}^\alpha u^\mu \qquad (7)$$

$$\frac{du^\alpha}{d\tau} + \Gamma^\alpha_{\beta\gamma} u^\beta u^\gamma = -\frac{q}{m} F_\mu{}^\alpha u^\mu \qquad (8)$$

Geodesic Deviation Equation

$$L = m\, g_{\mu\nu} u^\mu \frac{D\Psi^\nu}{D\tau} - q\, g_{\alpha\beta} \Psi^\beta F^\alpha{}_\mu u^\mu \qquad (1)$$

$$L = m\, u^\mu \frac{D\Psi_\mu}{D\tau} - q\, \Psi_\alpha F^\alpha{}_\mu u^\mu$$

$$L = m\, u^\mu u^\alpha \nabla_\alpha \Psi_\mu - q\, \Psi_\alpha F^\alpha{}_\mu u^\mu$$

$$L = m\, u^\mu u^\alpha (\partial_\alpha \Psi_\mu - \Gamma^\nu_{\alpha\mu}\Psi_\nu) - q\, \Psi_\alpha F^\alpha{}_\mu u^\mu$$

$$L = m\, u^\mu u^\alpha \partial_\alpha \Psi_\mu - m\, u^\mu u^\alpha \Gamma^\nu_{\alpha\mu}\Psi_\nu - q\, \Psi_\alpha F^\alpha{}_\mu u^\mu$$

$$L = m\, u^\mu \dot{\Psi}_\mu - m\, u^\mu u^\alpha \Gamma^\nu_{\alpha\mu}\Psi_\nu - q\, \Psi_\alpha F^\alpha{}_\mu u^\mu \qquad (9)$$

Euler-Lagrange equation

$$\frac{d}{d\tau}\left(\frac{\partial L}{\partial \dot{x}^\alpha}\right) = \frac{\partial L}{\partial x^\alpha}$$

$$\frac{\partial L}{\partial \dot{x}^\alpha} = \frac{\partial L}{\partial u^\alpha} = m\dot{\Psi}_\alpha - 2m u^\mu \Gamma^\nu_{\alpha\mu}\Psi_\nu - q\,\Psi_\nu F^\nu{}_\alpha$$

$$\frac{\partial L}{\partial \dot{x}^\alpha} = m\dot{\Psi}_\alpha - m u^\mu \Gamma^\nu_{\alpha\mu}\Psi_\nu - m u^\mu \Gamma^\nu_{\alpha\mu}\Psi_\nu - q\,\Psi_\nu F^\nu{}_\alpha$$

$$\frac{\partial L}{\partial \dot{x}^\alpha} = m\left[\dot{\Psi}_\alpha - u^\mu \Gamma^\nu_{\alpha\mu}\Psi_\nu\right] - m u^\mu \Gamma^\nu_{\alpha\mu}\Psi_\nu - q\,\Psi_\nu F^\nu{}_\alpha$$

$$\frac{\partial L}{\partial \dot{x}^\alpha} = m\frac{D\Psi_\alpha}{D\tau} - m u^\mu \Gamma^\nu_{\alpha\mu}\Psi_\nu - q\,\Psi_\nu F^\nu{}_\alpha \qquad (10)$$

$$\frac{d}{d\tau}\left(\frac{\partial L}{\partial \dot{x}^\alpha}\right) = m\frac{d}{d\tau}\left(\frac{D\Psi_\alpha}{D\tau}\right) - m u^\mu u^\beta \partial_\beta \Gamma^\nu_{\alpha\mu}\Psi_\nu - m u^\mu \Gamma^\nu_{\alpha\mu}\dot{\Psi}_\nu - m\frac{du^\mu}{d\tau}\Gamma^\nu_{\alpha\mu}\Psi_\nu$$
$$- q\,\dot{\Psi}_\nu F^\nu{}_\alpha - q\,\Psi_\nu u^\beta \partial_\beta F^\nu{}_\alpha \qquad (11)$$

$$\frac{\partial L}{\partial x^\alpha} = - m u^\mu u^\beta \psi_\nu \partial_\alpha \Gamma^\nu_{\beta\mu} - q \psi_\beta u^\mu \partial_\alpha F^\beta{}_\mu \qquad (12)$$

Hence, the Euler-Lagrange equation is

$$m \frac{d}{d\tau}\left(\frac{D\psi_\alpha}{D\tau}\right) - m u^\mu u^\beta \partial_\beta \Gamma^\nu_{\alpha\mu} \psi_\nu - m u^\mu \Gamma^\nu_{\alpha\mu} \dot{\psi}_\nu - m \frac{du^\mu}{d\tau} \Gamma^\nu_{\alpha\mu} \psi_\nu$$
$$- q \dot{\psi}_\nu F^\nu{}_\alpha - q \psi_\nu u^\beta \partial_\beta F^\nu{}_\alpha = - m u^\mu u^\beta \psi_\nu \partial_\alpha \Gamma^\nu_{\beta\mu} - q \psi_\beta u^\mu \partial_\alpha F^\beta{}_\mu$$

$$\frac{d}{d\tau}\left(\frac{D\psi_\alpha}{D\tau}\right) - u^\mu u^\beta \partial_\beta \Gamma^\nu_{\alpha\mu} \psi_\nu - u^\mu \Gamma^\nu_{\alpha\mu} \dot{\psi}_\nu - \frac{du^\mu}{d\tau} \Gamma^\nu_{\alpha\mu} \psi_\nu$$
$$- \frac{q}{m} \dot{\psi}_\nu F^\nu{}_\alpha - \frac{q}{m} \psi_\nu u^\beta \partial_\beta F^\nu{}_\alpha = - u^\mu u^\beta \psi_\nu \partial_\alpha \Gamma^\nu_{\beta\mu} - \frac{q}{m} \psi_\beta u^\mu \partial_\alpha F^\beta{}_\mu$$

$$\frac{d}{d\tau}\left(\frac{D\psi_\alpha}{D\tau}\right) - u^\mu \Gamma^\nu_{\alpha\mu} \dot{\psi}_\nu = \frac{du^\mu}{d\tau} \Gamma^\nu_{\alpha\mu} \psi_\nu + u^\mu u^\beta \partial_\beta \Gamma^\nu_{\alpha\mu} \psi_\nu - u^\mu u^\beta \psi_\nu \partial_\alpha \Gamma^\nu_{\beta\mu}$$
$$+ \frac{q}{m} \dot{\psi}_\nu F^\nu{}_\alpha + \frac{q}{m} \psi_\nu u^\beta \partial_\beta F^\nu{}_\alpha - \frac{q}{m} \psi_\beta u^\mu \partial_\alpha F^\beta{}_\mu$$

$$\underline{\frac{d}{d\tau}\left(\frac{D\psi_\alpha}{D\tau}\right) - u^\mu \Gamma^\nu_{\alpha\mu} \dot{\psi}_\nu} = \frac{du^\mu}{d\tau} \Gamma^\nu_{\alpha\mu} \psi_\nu + u^\mu u^\beta \psi_\nu \left[\partial_\beta \Gamma^\nu_{\alpha\mu} - \partial_\alpha \Gamma^\nu_{\beta\mu}\right]$$
$$+ \frac{q}{m} \dot{\psi}_\nu F^\nu{}_\alpha + \frac{q}{m} \psi_\nu u^\beta \left[\partial_\beta F^\nu{}_\alpha - \partial_\alpha F^\nu{}_\beta\right]$$

- Since

$$\frac{D^2 \psi_\alpha}{D\tau^2} = \frac{D}{D\tau}\left(\frac{D\psi_\alpha}{D\tau}\right) = \frac{d}{d\tau}\left(\frac{D\psi_\alpha}{D\tau}\right) - u^\nu \Gamma^\mu_{\alpha\nu} \frac{D\psi_\mu}{D\tau}$$

$$\frac{D^2 \psi_\alpha}{D\tau^2} = \frac{d}{d\tau}\left(\frac{D\psi_\alpha}{D\tau}\right) - u^\nu \Gamma^\mu_{\alpha\nu} \frac{d\psi_\mu}{d\tau} + u^\nu \Gamma^\mu_{\alpha\nu} u^\beta \Gamma^\sigma_{\beta\mu} \psi_\sigma$$

$$\frac{D^2 \psi_\alpha}{D\tau^2} - \Gamma^\mu_{\alpha\nu} \Gamma^\sigma_{\beta\mu} u^\nu u^\beta \psi_\sigma = \frac{d}{d\tau}\left(\frac{D\psi_\alpha}{D\tau}\right) - u^\nu \Gamma^\mu_{\alpha\nu} \dot{\psi}_\mu$$

we get

$$\frac{D^2 \psi_\alpha}{D\tau^2} - \Gamma^\mu_{\alpha\nu} \Gamma^\sigma_{\beta\mu} u^\nu u^\beta \psi_\sigma = \frac{du^\mu}{d\tau} \Gamma^\nu_{\alpha\mu} \psi_\nu + u^\mu u^\beta \psi_\nu \left[\partial_\beta \Gamma^\nu_{\alpha\mu} - \partial_\alpha \Gamma^\nu_{\beta\mu}\right]$$
$$+ \underline{\frac{q}{m} \dot{\psi}_\nu F^\nu{}_\alpha} + \frac{q}{m} \psi_\nu u^\beta \left[\partial_\beta F^\nu{}_\alpha - \partial_\alpha F^\nu{}_\beta\right]$$

- We also note that

$$\frac{q}{m}\frac{D\psi_\nu}{D\tau}F^\nu{}_\alpha = \frac{q}{m}\dot{\psi}_\nu F^\nu{}_\alpha - \frac{q}{m}\Gamma^\beta_{\nu\mu}u^\mu\psi_\beta F^\nu{}_\alpha$$

$$\frac{q}{m}\dot{\psi}_\nu F^\nu{}_\alpha = \frac{q}{m}\frac{D\psi_\nu}{D\tau}F^\nu{}_\alpha + \frac{q}{m}\Gamma^\beta_{\nu\mu}u^\mu\psi_\beta F^\nu{}_\alpha$$

Then,

$$\frac{D^2\psi_\alpha}{D\tau^2} - \Gamma^\mu_{\alpha\nu}\Gamma^\sigma_{\beta\mu}u^\nu u^\beta\psi_\sigma = \frac{du^\mu}{d\tau}\Gamma^\nu_{\alpha\mu}\psi_\nu + u^\mu u^\beta\psi_\nu\left[\partial_\beta\Gamma^\nu_{\alpha\mu} - \partial_\alpha\Gamma^\nu_{\beta\mu}\right] + \frac{q}{m}\frac{D\psi_\nu}{D\tau}F^\nu{}_\alpha + \frac{q}{m}\Gamma^\beta_{\nu\mu}u^\mu\psi_\beta F^\nu{}_\alpha + \frac{q}{m}\psi_\nu u^\beta\left[\partial_\beta F^\nu{}_\alpha - \partial_\alpha F^\nu{}_\beta\right]$$

$$\frac{D^2\psi_\alpha}{D\tau^2} - \Gamma^\mu_{\alpha\nu}\Gamma^\sigma_{\beta\mu}u^\nu u^\beta\psi_\sigma = \frac{du^\mu}{d\tau}\Gamma^\nu_{\alpha\mu}\psi_\nu + u^\mu u^\beta\psi_\nu\left[\partial_\beta\Gamma^\nu_{\alpha\mu} - \partial_\alpha\Gamma^\nu_{\beta\mu}\right] + \frac{q}{m}\frac{D\psi_\nu}{D\tau}F^\nu{}_\alpha + \frac{q}{m}\psi_\nu u^\beta\left[\Gamma^\nu_{\mu\beta}F^\mu{}_\alpha + \partial_\beta F^\nu{}_\alpha - \partial_\alpha F^\nu{}_\beta\right]$$

Using the equation of motion, we replace the derivative $\frac{du^\mu}{d\tau}$,

$$\frac{D^2\psi_\alpha}{D\tau^2} - \Gamma^\mu_{\alpha\nu}\Gamma^\sigma_{\beta\mu}u^\nu u^\beta\psi_\sigma = \left(-\frac{q}{m}F^\mu{}_\beta u^\beta - \Gamma^\mu_{\beta\gamma}u^\beta u^\gamma\right)\Gamma^\nu_{\alpha\mu}\psi_\nu + u^\mu u^\beta\psi_\nu\left[\partial_\beta\Gamma^\nu_{\alpha\mu} - \partial_\alpha\Gamma^\nu_{\beta\mu}\right] + \frac{q}{m}\frac{D\psi_\nu}{D\tau}F^\nu{}_\alpha + \frac{q}{m}\psi_\nu u^\beta\left[\Gamma^\nu_{\mu\beta}F^\mu{}_\alpha + \partial_\beta F^\nu{}_\alpha - \partial_\alpha F^\nu{}_\beta\right]$$

$$\frac{D^2\psi_\alpha}{D\tau^2} - \Gamma^\mu_{\alpha\nu}\Gamma^\sigma_{\beta\mu}u^\nu u^\beta\psi_\sigma = -\Gamma^\mu_{\beta\gamma}\Gamma^\nu_{\alpha\mu}u^\beta u^\gamma\psi_\nu + u^\mu u^\beta\psi_\nu\left[\partial_\beta\Gamma^\nu_{\alpha\mu} - \partial_\alpha\Gamma^\nu_{\beta\mu}\right] + \frac{q}{m}\frac{D\psi_\nu}{D\tau}F^\nu{}_\alpha - \frac{q}{m}\Gamma^\nu_{\alpha\mu}F^\mu{}_\beta u^\beta\psi_\nu + \frac{q}{m}\psi_\nu u^\beta\left[\Gamma^\nu_{\mu\beta}F^\mu{}_\alpha + \partial_\beta F^\nu{}_\alpha - \partial_\alpha F^\nu{}_\beta\right]$$

$$\frac{D^2\psi_\alpha}{D\tau^2} = \Gamma^\mu_{\alpha\nu}\Gamma^\sigma_{\beta\mu}u^\nu u^\beta\psi_\sigma - \Gamma^\mu_{\beta\gamma}\Gamma^\nu_{\alpha\mu}u^\beta u^\gamma\psi_\nu + u^\mu u^\beta\psi_\nu\left[\partial_\beta\Gamma^\nu_{\alpha\mu} - \partial_\alpha\Gamma^\nu_{\beta\mu}\right] + \frac{q}{m}\frac{D\psi_\nu}{D\tau}F^\nu{}_\alpha + \frac{q}{m}\psi_\nu u^\beta\left[\Gamma^\nu_{\mu\beta}F^\mu{}_\alpha - \Gamma^\nu_{\alpha\mu}F^\mu{}_\beta + \partial_\beta F^\nu{}_\alpha - \partial_\alpha F^\nu{}_\beta\right]$$

$$\frac{D^2\psi_\alpha}{D\tau^2} = \Gamma^\sigma_{\alpha\mu}\Gamma^\nu_{\beta\sigma}u^\mu u^\beta\psi_\nu - \Gamma^\sigma_{\beta\mu}\Gamma^\nu_{\alpha\sigma}u^\beta u^\mu\psi_\nu + u^\mu u^\beta\psi_\nu\left[\partial_\beta\Gamma^\nu_{\alpha\mu} - \partial_\alpha\Gamma^\nu_{\beta\mu}\right]$$
$$+ \frac{q}{m}\frac{D\psi_\nu}{D\tau}F^\nu{}_\alpha + \frac{q}{m}\psi_\nu u^\beta\left[\Gamma^\nu_{\mu\beta}F^\mu_\alpha - \Gamma^\nu_{\alpha\mu}F^\mu_\beta + \partial_\beta F^\nu{}_\alpha - \partial_\alpha F^\nu{}_\beta\right]$$

$$\frac{D^2\psi_\alpha}{D\tau^2} = u^\mu u^\beta\psi_\nu\left[\partial_\beta\Gamma^\nu_{\alpha\mu} - \partial_\alpha\Gamma^\nu_{\beta\mu} + \Gamma^\sigma_{\alpha\mu}\Gamma^\nu_{\beta\sigma} - \Gamma^\sigma_{\beta\mu}\Gamma^\nu_{\alpha\sigma}\right]$$
$$+ \frac{q}{m}\frac{D\psi_\nu}{D\tau}F^\nu{}_\alpha + \frac{q}{m}\psi_\nu u^\beta\left[\Gamma^\nu_{\mu\beta}F^\mu_\alpha - \Gamma^\nu_{\alpha\mu}F^\mu_\beta + \partial_\beta F^\nu{}_\alpha - \partial_\alpha F^\nu{}_\beta\right]$$

We recognize the Riemann tensor in the first term in the r.h.s,

$$R^\nu{}_{\mu\beta\alpha} = \partial_\beta\Gamma^\nu_{\alpha\mu} - \partial_\alpha\Gamma^\nu_{\beta\mu} + \Gamma^\sigma_{\alpha\mu}\Gamma^\nu_{\beta\sigma} - \Gamma^\sigma_{\beta\mu}\Gamma^\nu_{\alpha\sigma}$$

and then

$$\frac{D^2\psi_\alpha}{D\tau^2} = R^\nu{}_{\mu\beta\alpha}u^\mu u^\beta\psi_\nu + \frac{q}{m}\frac{D\psi_\nu}{D\tau}F^\nu{}_\alpha + \frac{q}{m}\psi_\nu u^\beta\left[\partial_\beta F^\nu{}_\alpha - \partial_\alpha F^\nu{}_\beta + \Gamma^\nu_{\mu\beta}F^\mu_\alpha - \Gamma^\nu_{\alpha\mu}F^\mu_\beta\right] \quad (14)$$