In order to obtain the shadow of a rotating black hole, we begin with the radial equation of motion obtained from the Hamilton-Jacobi formulation,

$$\Sigma^2 \dot{r}^2 = \mathcal{R}$$

where $\mathcal{R}(r) = \left[(r^2+a^2)\mathcal{E} - a\ell_z\right]^2 - \Delta\left[(\ell_z - a\mathcal{E})^2 - \delta r^2 + \mathcal{C}\right]$

We can write this function, for photons ($\delta = 0$), as

$$R = \frac{\mathcal{R}}{\mathcal{E}^2} = \left[(r^2+a^2) - a\frac{\ell_z}{\mathcal{E}}\right]^2 - \Delta\left[\left(\frac{\ell_z}{\mathcal{E}} - a\right)^2 + \frac{\mathcal{C}}{\mathcal{E}^2}\right]$$

or, introducing the parameters $\lambda = \frac{\ell_z}{\mathcal{E}}$ and $q^2 = \frac{\mathcal{C}}{\mathcal{E}^2}$,

$$R = \frac{\mathcal{R}}{\mathcal{E}^2} = \left[r^2 + a^2 - a\lambda\right]^2 - \Delta\left[(\lambda - a)^2 + q^2\right]$$

$$R = r^4 + 2r^2 a(a-\lambda) + a^2(a-\lambda)^2 - (r^2 - 2Mr + a^2)\left[(a-\lambda)^2 + q^2\right]$$

$$R = r^4 + 2r^2 a(a-\lambda) + \cancel{a^2(a-\lambda)^2} - r^2(a-\lambda)^2 - r^2q^2 + 2Mr\left[(a-\lambda)^2 + q^2\right] \cancel{- a^2(a-\lambda)^2} - a^2q^2$$

$$R = r^4 + \left[2a(a-\lambda) - (a-\lambda)^2 - q^2\right]r^2 + 2Mr\left[(a-\lambda)^2 + q^2\right] - a^2q^2$$

$$R = r^4 + \left[\cancel{2}a^2 - \cancel{2a\lambda} - \cancel{a^2} + \cancel{2a\lambda} - \lambda^2 - q^2\right]r^2 + 2Mr\left[(a-\lambda)^2 + q^2\right] - a^2q^2$$

$$R = r^4 + \left[a^2 - \lambda^2 - q^2\right]r^2 + 2Mr\left[(a-\lambda)^2 + q^2\right] - a^2q^2$$

Motion is possible ($\dot{r}^2 \geq 0$) if $R \geq 0$. Hence, we identify two cases:

1. If R has NO roots for $r \geq r_+$, photons are captured by the BH.

2. If R has real roots for $r > r_+$, these are turning points for the photon's motion, which will be an open orbit.

The captured orbits and the open orbits are separated by an unstable orbit determined by the condition:

$$R(r_*) = \left.\frac{\partial R}{\partial r}\right|_{r_*} = 0 \quad \text{and} \quad \left.\frac{\partial^2 R}{\partial r^2}\right|_{r_*} \geq 0$$

These are

$$R = [r^2 + a^2 - a\lambda]^2 - (r^2 - 2Mr + a^2)[(\lambda - a)^2 + q^2] = 0 \qquad (\text{I})$$

$$\frac{\partial R}{\partial r} = 4r[r^2 + a^2 - a\lambda] - (2r - 2M)[(\lambda - a)^2 + q^2] = 0 \qquad (\text{II})$$

From (II) we have

$$[(\lambda - a)^2 + q^2] = \frac{2r[r^2 + a^2 - a\lambda]}{r - M} \qquad (\text{III})$$

Replacing in (I), we get

$$[r^2 + a^2 - a\lambda]^2 - (r^2 - 2Mr + a^2)\frac{2r[r^2 + a^2 - a\lambda]}{r - M} = 0$$

$$[r^2 + a^2 - a\lambda]^2 = (r^2 - 2Mr + a^2)\frac{2r[r^2 + a^2 - a\lambda]}{r - M}$$

Assuming that $r^2 + a^2 - a\lambda \neq 0$

$$r^2 + a^2 - a\lambda = \frac{2r(r^2 - 2Mr + a^2)}{r - M} \qquad (\text{IV})$$

$$a\lambda = r^2 + a^2 - \frac{2r(r^2 - 2Mr + a^2)}{r - M}$$

$$\boxed{\lambda_c = \frac{(r_*^2 + a^2)(r_* - M) - 2r_*(r_*^2 - 2Mr_* + a^2)}{a(r_* - M)}}$$

$$\lambda_c = \frac{r_*^3 - Mr_*^2 + a^2 r_* - Ma^2 - 2r_*^3 + 4Mr_*^2 - 2a^2 r_*}{a(r_* - M)}$$

$$\lambda_c = \frac{-r_*^3 + 3Mr_*^2 - a^2 r_* - Ma^2}{a(r_* - M)}$$

$$\boxed{\lambda_c = \frac{M(r_*^2 - a^2) - r_*(r_*^2 - 2Mr_* + a^2)}{a(r_* - M)}}$$

Replacing in (III);

$$(\lambda_c - a)^2 + q_c^2 = \frac{2r_*[r_*^2 + a^2 - a\lambda_c]}{r_* - M}$$

$$q_c^2 = \frac{2r_*[r_*^2 + a^2 - a\lambda_c]}{r_* - M} - (\lambda_c - a)^2$$

$$q_c^2 = \frac{2r_*[r_*^2 + a^2 - a\lambda_c]}{r_* - M} - \left[\frac{(r_*^2+a^2)(r_*-M) - 2r_*(r_*^2 - 2Mr_* + a^2)}{a(r_*-M)} - a\right]^2$$

$$q_c^2 = \frac{2r_*[r_*^2 + a^2 - a\lambda_c]}{r_* - M} - \left[\frac{(r_*^2+a^2)(r_*-M) - 2r_*(r_*^2 - 2Mr_* + a^2) - a^2(r_*-M)}{a(r_*-M)}\right]^2$$

$$q_c^2 = \frac{2r_*[r_*^2 + a^2 - a\lambda_c]}{r_* - M} - \left[\frac{r_*^2(r_*-M) - 2r_*(r_*^2 - 2Mr_* + a^2)}{a(r_*-M)}\right]^2$$

$$q_c^2 = \frac{2r_*[r_*^2 + a^2 - a\lambda_c]}{r_* - M} - \frac{r_*^2(r_*^2 - Mr_* - 2r_*^2 + 4Mr_* - 2a^2)^2}{a^2(r_*-M)^2}$$

$$q_c^2 = \frac{2r_*[r_*^2 + a^2 - a\lambda_c]}{r_* - M} - \frac{r_*^2(-r_*^2 + 3Mr_* - 2a^2)^2}{a^2(r_*-M)^2}$$

$$q_c^2 = \frac{2r_*[r_*^2 + a^2 - a\lambda_c]}{r_* - M} - \frac{r_*^2(r_*^2 - 3Mr_* + 2a^2)^2}{a^2(r_*-M)^2}$$

using (IV)

$$q_c^2 = \frac{2r_*}{r_* - M}\,\frac{2r_*(r_*^2 - 2Mr_* + a^2)}{r_* - M} - \frac{r_*^2(r_*^2 - 3Mr_* + 2a^2)^2}{a^2(r_*-M)^2}$$

$$q_c^2 = \frac{4r_*^2(r_*^2 - 2Mr_* + a^2)}{(r_* - M)^2} - \frac{r_*^2(r_*^2 - 3Mr_* + 2a^2)^2}{a^2(r_*-M)^2}$$

$$q_c^2 = \frac{4r_*^2(r_*^2 - 2Mr_* + a^2)a^2 - r_*^2(r_*^2 - 3Mr_* + 2a^2)^2}{a^2(r_*-M)^2}$$

$$q_c^2 = \frac{r_*^2\left[4(r_*^2 - 2Mr_* + a^2)a^2 - (r_*^2 - 3Mr_* + 2a^2)^2\right]}{a^2(r_*-M)^2}$$

$$q_c^2 = \frac{r_*^2\left[4(r_*^2 - 2Mr_* + a^2)a^2 - (r_*^4 + 9M^2r_*^2 + 4a^4 - 6Mr_*^3 + 4a^2r_*^2 - 12Mr_*a^2)\right]}{a^2(r_* - M)^2}$$

$$q_c^2 = \frac{r_*^2\left[4Ma^2r_* - (r_*^4 + 9M^2r_*^2 - 6Mr_*^3)\right]}{a^2(r_* - M)^2}$$

$$\boxed{q_c^2 = \frac{r_*^3\left[4Ma^2 - r_*(r_* - 3M)^2\right]}{a^2(r_* - M)^2}}$$

One final condition is given by

$$\left.\frac{\partial^2 R}{\partial r^2}\right|_{r_*} = 4[r_*^2 + a^2 - a\lambda_c] + 4r_*[2r_*] - 2r_*[(\lambda_c - a)^2 + q_c^2] \geq 0$$

$$2r_*^2 + 2a^2 - 2a\lambda_c + 4r_*^2 - (\lambda_c - a)^2 r_* - q_c^2 r_* \geq 0 \qquad (\mathrm{V})$$

The Shadow of the Black Hole

The shadow of the BH is defined in the image plane using the coordinates

$$\alpha = \frac{\lambda_c}{\sin z}$$

$$\beta = \sqrt{q_c^2 - \lambda_c^2\cot^2 z + a^2\cos^2 z}$$

using r_* as a parameter taking values obeying equation (V)