Mathisson - Papapetrou - Dixon (MPD) Equations

$$\frac{DP^\alpha}{D\tau} = \frac{dP^\alpha}{d\tau} + \Gamma^\alpha_{\lambda\eta} u^\lambda P^\eta = -\frac{1}{2} R^\alpha{}_{\nu\rho\sigma} S^{\rho\sigma} u^\nu - q F^\alpha{}_\mu u^\mu$$

$$\frac{dP^\alpha}{d\tau} + \Gamma^\alpha_{\lambda\eta} u^\lambda P^\eta + S^{\rho\sigma}\left(\partial_\rho \Gamma^\alpha_{\nu\sigma} + \Gamma^\epsilon_{\nu\sigma}\Gamma^\alpha_{\epsilon\rho}\right) u^\nu + qF^\alpha{}_\mu u^\mu = 0 \qquad (23)$$

$$\frac{DS^{\mu\nu}}{D\tau} = P^\mu u^\nu - P^\nu u^\mu = 0$$

$$P^\alpha = m u^\alpha$$

Spherically Symmetric Static Spacetime

$$ds^2 = -W(r)\, dt^2 + \frac{dr^2}{W(r)} + r^2 d\Omega^2$$

$$\Gamma^0_{01} = \frac{W'}{2W} \qquad \Gamma^1_{00} = \frac{1}{2} W W' \qquad \Gamma^1_{11} = -\frac{W'}{2W}$$

$$\Gamma^1_{22} = -rW \qquad \Gamma^1_{33} = -r\sin^2\theta\, W \qquad (34)$$

$$\Gamma^2_{12} = \Gamma^3_{13} = \frac{1}{r}$$

$$\Gamma^2_{33} = -\sin\theta\cos\theta \qquad \Gamma^3_{23} = \cot\theta$$

$$F^0{}_1 = -\frac{Q}{4\pi r^2 W}$$

$$F^1{}_0 = -\frac{QW}{4\pi r^2}$$

Equations of Motion

$$\frac{dp^\alpha}{d\tau} + \Gamma^\alpha_{\lambda\eta}\, u^\lambda p^\eta + S^{\rho\sigma}\left(\partial_\rho \Gamma^\alpha_{\nu\sigma} + \Gamma^\epsilon_{\nu\sigma}\Gamma^\alpha_{\epsilon\rho}\right) u^\nu + qF^\alpha{}_\mu u^\mu = 0$$

$\alpha = 0$

$$\frac{dp^0}{d\tau} + \Gamma^0_{\lambda\eta}\, u^\lambda p^\eta + S^{\rho\sigma}\left(\partial_\rho \Gamma^0_{\nu\sigma} + \Gamma^\epsilon_{\nu\sigma}\Gamma^0_{\epsilon\rho}\right) u^\nu + qF^0{}_\mu u^\mu = 0$$

$$\frac{dp^0}{d\tau} + 2m\,\Gamma^0_{01} u^0 u^1 + \partial_\rho \Gamma^0_{01} u^0 S^{\rho 1} + \partial_\rho \Gamma^0_{10} u^1 S^{\rho 0} + \Gamma^0_{\nu\sigma}\Gamma^0_{01} u^\nu S^{1\sigma} + \Gamma^1_{\nu\sigma}\Gamma^0_{10} u^\nu S^{0\sigma}$$
$$+ qF^0{}_1 u^1 = 0$$

$$\frac{dp^0}{d\tau} + 2m\,\Gamma^0_{01} u^0 u^1 + \partial_1 \Gamma^0_{01} u^0 \cancel{S^{11}} + \partial_1 \Gamma^0_{10} u^1 S^{10} + \Gamma^0_{01}\Gamma^0_{01} u^0 \cancel{S^{11}} + \Gamma^0_{10}\Gamma^0_{01} u^1 S^{10}$$
$$+ \Gamma^1_{00}\Gamma^0_{10} u^0 \cancel{S^{00}} + \Gamma^1_{11}\Gamma^0_{10} u^1 S^{01} + \Gamma^1_{22}\Gamma^0_{10} u^2 \cancel{S^{02}} + \Gamma^1_{33}\Gamma^0_{10} u^3 S^{03} + qF^0{}_1 u^1 = 0$$

$$m\frac{du^0}{d\tau} + 2m\,\Gamma^0_{01} u^0 u^1 + \partial_1 \Gamma^0_{10} u^1 S^{10} + \Gamma^0_{10}\Gamma^0_{01} u^1 S^{10} + \Gamma^1_{11}\Gamma^0_{10} u^1 S^{01}$$
$$+ \Gamma^1_{33}\Gamma^0_{10} u^3 S^{03} + qF^0{}_1 u^1 = 0$$

$$m\frac{d\dot t}{d\tau} + 2m\frac{W'}{2W} u^0 u^1 + \partial_r\left(\frac{W'}{2W}\right) u^1 S^{10} + \left(\frac{W'}{2W}\right)^2 u^1 S^{10} - \frac{W'}{2W}\frac{W'}{2W} u^1 S^{01}$$
$$- rW\frac{W'}{2W} u^3 S^{03} + qF^0{}_1 u^1 = 0$$

$$m\frac{d\dot t}{d\tau} + \frac{mW'}{W} u^0 u^1 - \frac{1}{2}\left(\frac{W''W - W'W'}{W^2}\right) u^1 S^{01} - \left(\frac{W'}{2W}\right)^2 u^1 S^{01} - \left(\frac{W'}{2W}\right)^2 u^1 S^{01}$$
$$- \frac{1}{2} rW' u^3 S^{03} + qF^0{}_1 u^1 = 0$$

$$m\frac{d\dot t}{d\tau} + \frac{mW'}{W} u^0 u^1 - \frac{1}{2}\left(\frac{W''W - \overline{W'W'}}{W^2}\right) u^1 S^{01} - \underline{\frac{1}{2}\left(\frac{W'}{W}\right)^2 u^1 S^{01}} - \frac{1}{2} rW' u^3 S^{03} + qF^0{}_1 u^1 = 0$$

$$m\frac{d\dot{t}}{d\tau} + m\frac{W'}{W}u^0u^1 - \frac{1}{2}\frac{W''}{W}u^1S^{01} - \frac{1}{2}rW'u^3S^{03} + qF^0{}_1u^1 = 0$$

$$m\frac{d\dot{t}}{d\tau} + m\frac{W'}{W}\dot{t}\dot{r} - \frac{1}{2}\frac{W''}{W}\dot{r}(-rs\dot{\phi}) - \frac{1}{2}rW'\dot{\phi}\left(\frac{s\dot{r}}{Wr}\right) + qF^0{}_1\dot{r} = 0$$

$$mW\frac{d\dot{t}}{d\tau} + mW'\dot{t}\dot{r} + \frac{1}{2}W''rs\dot{r}\dot{\phi} - \frac{1}{2}W'\dot{r}\dot{\phi}s + qF^0{}_1W\dot{r} = 0$$

$$m\frac{d}{d\tau}(W\dot{t}) + \frac{1}{2}s(W''r - W')\dot{r}\dot{\phi} + qF^0{}_1W\dot{r} = 0$$

$$m\frac{d}{d\tau}(W\dot{t}) + \frac{1}{2}s(W''r - W')\dot{r}\dot{\phi} - \frac{qQ}{4\pi r^2}\dot{r} = 0$$

$$m\frac{d}{d\tau}\left[W\dot{t} + \frac{qQ}{4\pi m r}\dot{r}\right] + \frac{1}{2m}s(W''r - W')\dot{r}\dot{\phi} = 0 \qquad (51) \qquad *$$

Equations of Motion

$$\frac{dP^\alpha}{d\tau} + \Gamma^\alpha_{\lambda\eta} u^\lambda P^\eta = -\frac{1}{2} R^\alpha{}_{\nu\rho\sigma} S^{\rho\sigma} u^\nu - q F^\alpha{}_\mu u^\mu$$

$\alpha = 0$

$$\frac{dP^0}{d\tau} + \Gamma^0_{\lambda\eta} u^\lambda P^\eta = -\frac{1}{2} R^0{}_{\nu\rho\sigma} S^{\rho\sigma} u^\nu - q F^0{}_\mu u^\mu$$

$$R^0{}_{101} = -\frac{W''}{2W} = -R^0{}_{110}$$

$$R^0{}_{202} = -\frac{1}{2} r W' = -R^0{}_{220}$$

$$R^0{}_{303} = -\frac{1}{2} r \sin^2\theta \, W' = -R^0{}_{330}$$

$$\frac{dP^0}{d\tau} + 2m\Gamma^0_{01} u^0 u^1 = -\frac{1}{2} R^0{}_{101} u^1 S^{01} - \frac{1}{2} R^0{}_{110} u^1 S^{10} - \frac{1}{2} R^0{}_{303} u^3 S^{03} - \frac{1}{2} R^0{}_{330} u^3 S^{30} - q F^0{}_1 u^1$$

$$\frac{dP^0}{d\tau} + 2m\Gamma^0_{01} u^0 u^1 = -R^0{}_{101} u^1 S^{01} - R^0{}_{303} u^3 S^{03} - q F^0{}_1 u^1$$

$$m\frac{d\dot{t}}{d\tau} + 2m\frac{W'}{2W} \dot{t}\dot{r} = -\left(-\frac{W''}{2W}\right)\dot{r}\left(-rs\dot{\phi}\right) - \left(-\frac{1}{2} r \sin^2\theta \, W'\right)\dot{\phi}\left(\frac{\dot{r}s}{rW}\right) - qF^0{}_1 \dot{r}$$

$$m\frac{d\dot{t}}{d\tau} + m\frac{W'}{W}\dot{t}\dot{r} + \frac{1}{2}\frac{W''}{W} rs\dot{r}\dot{\phi} - \frac{1}{2}\frac{W'}{W} s\sin^2\theta \, \dot{r}\dot{\phi} + qF^0{}_1 \dot{r} = 0$$

$$mW\frac{d\dot{t}}{d\tau} + mW'\dot{t}\dot{r} + \frac{1}{2} W'' rs\dot{r}\dot{\phi} - \frac{1}{2} W' s\sin^2\theta \, \dot{r}\dot{\phi} + qF^0{}_1 W\dot{r} = 0$$

$$m\frac{d}{d\tau}(W\dot{t}) + \frac{1}{2} s\dot{r}\dot{\phi}\left(W''r - W'\sin^2\theta\right) + qF^0{}_1 W\dot{r} = 0$$

Using the field

$$F^0{}_1 = -\frac{Q}{4\pi r^2 W}$$

and $\theta = \frac{\pi}{2}$ gives

$$m\frac{d}{d\tau}(W\dot{t}) + \frac{1}{2} s\dot{r}\dot{\phi}\left(W''r - W'\sin^2\theta\right) + q\left(-\frac{Q}{4\pi r^2 W}\right) W\dot{r} = 0$$

$$m\frac{d}{d\tau}(W\dot{t}) + \frac{1}{2} s\dot{r}\dot{\phi}\left(W''r - W'\sin^2\theta\right) + \left(-\frac{qQ}{4\pi r^2}\right)\dot{r} = 0$$

$$\frac{d}{d\tau}\left(W\dot{t} + \frac{qQ}{4\pi m r}\right) + \frac{1}{2m} s\dot{r}\dot{\phi}\left(W''r - W'\right) = 0 \qquad (51) \; *$$

$\alpha = 1$

$$\frac{dP^1}{d\tau} + \Gamma^1_{\lambda\eta} u^\lambda P^\eta = -\frac{1}{2} R^1{}_{\nu\rho\sigma} S^{\rho\sigma} u^\nu - q F^1{}_\mu u^\mu$$

$$R^1{}_{001} = -\frac{1}{2} W W'' = -R^1{}_{010}$$

$$R^1{}_{212} = -\frac{1}{2} r W' = -R^1{}_{221}$$

$$R^1{}_{313} = -\frac{1}{2} r\sin^2\theta\, W' = -R^1{}_{331}$$

$$\frac{dP^1}{d\tau} + m\Gamma^1_{00} u^0u^0 + m\Gamma^1_{11} u^1u^1 + m\Gamma^1_{22} u^2u^2 + m\Gamma^1_{33} u^3u^3 = -R^1{}_{001} u^0 S^{01} - R^1{}_{313} u^3 S^{13} - qF^1{}_0 u^0$$

$$m\frac{d\dot{r}}{d\tau} + m\left(\frac{1}{2}WW'\right)\dot{t}^2 + m\left(-\frac{1}{2}\frac{W'}{W}\right)\dot{r}^2 + m\left(-rW\right)\cancel{\dot{\theta}^2} + m\left(-rW\sin^2\theta\right)\dot{\phi}^2$$

$$= -\left(-\frac{1}{2}WW''\right)\dot{t}\left(-r\dot{\phi}s\right) - \left(-\frac{1}{2} r\sin^2\theta\, W'\right)\dot{\phi}\left(\frac{W\dot{t}s}{r}\right) - qF^1{}_0\dot{t}$$

$$m\frac{d\dot r}{d\tau} + \frac{1}{2} m W W' \dot t^2 - \frac{1}{2} m \frac{W'}{W}\dot r^2 - m W r \sin^2\theta\, \dot\phi^2 + \frac{1}{2} W W'' r s\, \dot t \dot\phi$$

$$- \frac{1}{2} W W' s \sin^2\theta\, \dot t \dot \phi + q F^1{}_0 \dot t = 0$$

$$\frac{d\dot r}{d\tau} + \frac{1}{2} W W' \dot t^2 - \frac{1}{2}\frac{W'}{W}\dot r^2 - W r \sin^2\theta\, \dot\phi^2 + \frac{1}{2m} W s \left(W'' r - W' \sin^2\theta\right)\dot t \dot\phi + \frac{q}{m} F^1{}_0 \dot t = 0$$

Using the field

$$F^1{}_0 = -\frac{QW}{4\pi r^2}$$

and $\theta = \frac{\pi}{2}$ gives

$$\frac{d\dot r}{d\tau} + \frac{1}{2} W W' \dot t^2 - \frac{1}{2}\frac{W'}{W}\dot r^2 - W r\, \dot\phi^2 + \frac{1}{2m} W s \left(W'' r - W'\right)\dot t \dot\phi - \frac{qQW}{4\pi m r^2}\dot t = 0 \qquad (S2)$$

$\alpha = 2$

$$\frac{dP^2}{d\tau} + \Gamma^2_{\lambda\eta} u^\lambda P^\eta = -\frac{1}{2} R^2{}_{\nu\rho\sigma} S^{\rho\sigma} u^\nu - q F^2{}_\mu u^\mu$$

$$R^2{}_{002} = -\frac{1}{2}\frac{WW'}{r} = -R^2{}_{020}$$

$$R^2{}_{112} = \frac{1}{2}\frac{W'}{Wr} = -R^2{}_{121}$$

$$R^2{}_{323} = -\sin^2\theta\,(W-1) = -R^2{}_{332}$$

$$\frac{dP^2}{d\tau} + 2m\Gamma^2_{12} u^1 u^2 + m\Gamma^2_{33} u^3 u^3 = -q \cancel{F^2{}_\mu} u^\mu$$

$$\frac{dP^2}{d\tau} + 2m\Gamma^2_{12}u^1u^2 + m\Gamma^2_{33}u^3u^3 = 0$$

$$m\frac{d\dot\theta}{d\tau} + 2m\left(\frac{1}{r}\right)\dot r\cancel{\dot\theta} + m\left(-\sin\theta\;\cos\theta\right)\dot\phi^2 = 0$$

$$\frac{d\dot\theta}{d\tau} - \sin\theta\cancel{\cos\theta}\,\dot\phi^2 = 0$$

$$\frac{d\dot\theta}{d\tau} = 0 \qquad (53)$$

$\alpha = 3$

$$\frac{dP^3}{d\tau} + \Gamma^3_{\lambda\eta}u^\lambda P^\eta = -\frac{1}{2}R^3{}_{\nu\rho\sigma}S^{\rho\sigma}u^\nu - qF^3{}_\mu u^\mu$$

$$R^3{}_{003} = -\frac{1}{2}\frac{WW'}{r} = -R^3{}_{030}$$

$$R^3{}_{113} = \frac{1}{2}\frac{W'}{Wr} = -R^3{}_{131}$$

$$R^3{}_{223} = (W-1) = -R^3{}_{232}$$

$$\frac{dP^3}{d\tau} + 2m\Gamma^3_{13}u^1u^3 + 2\Gamma^3_{23}\cancel{u^2}u^3 = -R^3{}_{003}u^0S^{03} - R^3{}_{113}u^1S^{13} - q\cancel{F^3{}_\mu}u^\mu$$

$$m\frac{d\dot\phi}{d\tau} + 2m\left(\frac{1}{r}\right)\dot r\dot\phi + \left(-\frac{1}{2}\frac{WW'}{r}\right)\dot t\left(\frac{\dot r s}{Wr}\right) + \left(\frac{1}{2}\frac{W'}{Wr}\right)\dot r\left(\frac{Ws\dot t}{r}\right) = 0$$

$$m\frac{d\dot\phi}{d\tau} + 2m\frac{\dot r\dot\phi}{r} \underline{- \frac{1}{2}W'\frac{s}{r^2}\dot t\dot r} + \underline{\frac{1}{2}W'\frac{s}{r^2}\dot t\dot r} = 0$$

$$\frac{d\dot\phi}{d\tau} + 2\frac{\dot r}{r}\dot\phi = 0$$

$$r^2\frac{d\dot\phi}{d\tau} + 2r\dot r\dot\phi = 0$$

$$\frac{d}{d\tau}\left(r^2\dot{\phi}\right) = 0 \qquad (54)$$

$$\downarrow$$

$$\dot{\phi} = \frac{h}{r^2} \qquad (55)$$

$$h = \text{const.}$$

Using the constant angular momentum,

$$\frac{d}{d\tau}\left(W\dot{t} + \frac{qQ}{4\pi m r}\right) + \frac{1}{2m} s\dot{r}\dot{\phi}\left(W''r - W'\sin^2\theta\right) = 0 \qquad (51)$$

$$\frac{d}{d\tau}\left(W\dot{t} + \frac{qQ}{4\pi m r}\right) + \frac{hs}{2mr^2}\left(W''r - W'\sin^2\theta\right)\dot{r} = 0 \qquad (56)$$

$$\frac{d\dot{r}}{d\tau} + \frac{1}{2}WW'\dot{t}^2 - \frac{1}{2}\frac{W'}{W}\dot{r}^2 - Wr\dot{\phi}^2 + \frac{1}{2m}Ws\left(W''r - W'\right)\dot{t}\dot{\phi} - \frac{qQW}{4\pi m r^2}\dot{t} = 0 \qquad (52)$$

$$\frac{d\dot{r}}{d\tau} + \frac{1}{2}WW'\dot{t}^2 - \frac{1}{2}\frac{W'}{W}\dot{r}^2 - \frac{Wh^2}{r^3} + \frac{Wsh}{2mr^2}\left(W''r - W'\right)\dot{t} - \frac{qQW}{4\pi m r^2}\dot{t} = 0$$

From the line element (and $c=1$);

$$ds^2 = -W\,dt^2 + \frac{dr^2}{W} + r^2 d\Omega^2 \equiv -d\tau^2$$

$$\rightarrow \dot{t}^2 = \left(\frac{dt}{d\tau}\right)^2 = -\left(\frac{dt}{ds}\right)^2 = -\left(-\frac{1}{W}\right) = \frac{1}{W}$$

$$\rightarrow \dot{r}^2 = \left(\frac{dr}{d\tau}\right)^2 = -\left(\frac{dr}{ds}\right)^2 = -\left(W\right) = -W$$

Then

$$\frac{d\dot{r}}{d\tau} + \frac{1}{2}ww'\frac{1}{w} - \frac{1}{2}\frac{w'}{w}(-w) - \frac{wh^2}{r^3} + \frac{wsh}{2mr^2}(w''r - w')\dot{t} - \frac{4Qw}{4\pi m r^2}\dot{t} = 0$$

$$\frac{d\dot{r}}{d\tau} + \frac{1}{2}w' + \frac{1}{2}w' - \frac{wh^2}{r^3} + \frac{wsh}{2mr^2}(w''r - w')\dot{t} - \frac{4Qw}{4\pi m r^2}\dot{t} = 0$$

$$\frac{d\dot{r}}{d\tau} + w' - \frac{wh^2}{r^3} + \frac{wsh}{2mr^2}(w''r - w')\dot{t} - \frac{4Qw}{4\pi m r^2}\dot{t} = 0 \qquad (57) \quad *$$