SPHERICALLY SYMMETRIC SPACETIEME

$$ds^2 = g_{tt}\,dt^2 + g_{rr}\,dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta\, d\phi^2$$

The Lagrangian for photons is

$$L = \frac{1}{2} g_{\mu\nu} \dot{x}^\mu \dot{x}^\nu = 0$$

$$\epsilon = -P_t = -\frac{\partial L}{\partial \dot{t}} = -g_{tt}\,\dot{t} \qquad \ell_z = P_\phi = \frac{\partial L}{\partial \dot{\phi}} = g_{\phi\phi}\,\dot{\phi} = r^2 \sin^2\theta\,\dot{\phi}$$

Radial equation of motion for equatorial photons

$$g_{tt}\,\dot{t}^2 + g_{rr}\,\dot{r}^2 + r^2\dot{\phi}^2 = 0$$

$$g_{rr}\,\dot{r}^2 = -g_{tt}\,\dot{t}^2 - r^2\dot{\phi}^2$$

$$g_{rr}\,\dot{r}^2 = -\frac{\epsilon^2}{g_{tt}} - \frac{\ell_z^2}{r^2}$$

$$\boxed{\dot{r} = \sqrt{-\frac{1}{g_{rr}}\left(\frac{\epsilon^2}{g_{tt}} + \frac{\ell_z^2}{r^2}\right)}}$$

$$\frac{dr}{d\phi} = \frac{\dot{r}}{\dot{\phi}} = \sqrt{-\frac{1}{g_{rr}}\left(\frac{\epsilon^2}{g_{tt}} + \frac{\ell_z^2}{r^2}\right)}\;\frac{r^2}{\ell_z}$$

$$\frac{dr}{d\phi} = r^2\sqrt{-\frac{1}{g_{rr}}\left(\frac{1}{g_{tt}}\frac{\epsilon^2}{\ell_z^2} + \frac{1}{r^2}\right)}$$

Using the impact parameter $b = \frac{\ell_z}{\epsilon}$

$$\boxed{\frac{dr}{d\phi} = r^2\sqrt{-\frac{1}{g_{rr}}\left(\frac{1}{b^2 g_{tt}} + \frac{1}{r^2}\right)}}$$

SPECIFIC INTENSITY WITH STATIC SPHERICAL ACCRETION

$$I(\nu_o) = \int g^3 \, j(\nu_e) \, dl_{prop} \qquad [\mathrm{erg\ s^{-1}\ cm^{-2}\ str^{-1}}]$$

ν_o : observed frequency

ν_e : emmited frequency

$g = \dfrac{\nu_o}{\nu_e}$ redshift factor

$$g = \sqrt{-g_{tt}}$$

dl_{prop} : proper lenght along the photon's trajectory

$$ds^2 = dl_{prop}^2 = g_{rr}\,dr^2 + r^2\,d\phi^2 \qquad (dt = d\theta = 0)$$

$$dl_{prop} = \sqrt{g_{rr} + r^2\left(\frac{d\phi}{dr}\right)^2}\,dr$$

$$dl_{prop}^2 = \left[g_{rr} - r^2\frac{g_{rr}}{r^4}\left(\frac{1}{b^2 g_{tt}} + \frac{1}{r^2}\right)^{-1}\right]dr^2$$

$$dl_{prop}^2 = \left[g_{rr} - \frac{g_{rr}}{r^2}\,\frac{b^2 g_{tt}\, r^2}{r^2 + b^2 g_{tt}}\right]dr^2$$

$$dl_{prop}^2 = g_{rr}\left[1 - \frac{b^2 g_{tt}}{r^2 + b^2 g_{tt}}\right]dr^2$$

$$\boxed{dl_{prop} = \sqrt{g_{rr}\left(1 - \frac{b^2 g_{tt}}{r^2 + b^2 g_{tt}}\right)}\,dr}$$

$j(\nu_e)$: specific emissivity measured in the static frame (emissivity per unit volume)

Static spherical accretion is an idealized description. In this case, and for monchromatic emission, we consider

$$\boxed{j(\nu_e) \propto \frac{1}{r^2}}$$

Using these relations, we have

$$\boxed{I(\nu_o) \propto \int \frac{(-g_{tt})^{3/2}}{r^2} \sqrt{g_{rr}\left(1 - \frac{b^2 g_{tt}}{r^2 + b^2 g_{tt}}\right)}\, dr}$$

Specific Intensity with Infalling Spherical Accretion

In a more realistic scenario, the plasma particles are trapped by the black hole with the initial velocities that they have. This is the 'infalling spherical accretion'.
The redshift is now

$$g = \frac{K_\mu u_o^\mu}{K_\nu u_e^\nu}$$

K_μ : Momentum of the radiated photon
u_o^μ : velocity of the distant observer
u_e^μ : velocity of the accretion flow

For simplicity, we chose a stationary observer

$$u_o^\mu = (1,0,0,0)$$

For the velocity of the matter in the accretion flow, we chose

$$u_e^t = -\frac{1}{g_{tt}} \qquad \leftarrow \text{Then} \quad \mathcal{E} = -g_{tt}\dot{t} \equiv 1$$

$$u_e^\theta = u_e^\phi = 0 \qquad \leftarrow \text{radial motion}$$

Using the normalization condition $u_\mu u^\mu = -1$ gives

$$g_{\mu\nu} u_e^\mu u_e^\nu = g_{tt}(u_e^t)^2 + g_{rr}(u_e^r)^2 = -1$$

$$\frac{1}{g_{tt}} + g_{rr}(u_e^r)^2 = -1$$

$$(u_e^r)^2 = -\frac{1}{g_{rr}}\left(1 + \frac{1}{g_{tt}}\right)$$

$$u_e^r = -\sqrt{-\frac{1}{g_{rr}}\left(1+\frac{1}{g_{tt}}\right)}$$

where we've chosen a '−' sign to describe infalling particles.

Hence, the redshift factor is

$$g = \frac{K_t}{K_t u_e^t + K_r u_e^r}$$

$$g = \frac{1}{u_e^t + \frac{k_r}{k_t} u_e^r}$$

So, we only need the ratio $\frac{k_r}{k_t}$, which is found from

$$k_\mu k^\mu = 0$$

$$g^{\mu\nu} k_\mu k_\nu = g^{tt} (k_t)^2 + g^{rr} (k_r)^2 + g^{\phi\phi} (k_\phi)^2 = 0$$

$$g^{tt} + g^{rr} \left(\frac{k_r}{k_t}\right)^2 + g^{\phi\phi} \left(\frac{k_\phi}{k_t}\right)^2 = 0$$

Since $\frac{k_\phi}{k_t} = \frac{l_z}{-E} = -b$,

$$g^{tt} + g^{rr} \left(\frac{k_r}{k_t}\right)^2 + \frac{b^2}{r^2} = 0$$

$$\left(\frac{k_r}{k_t}\right)^2 = -\frac{1}{g^{rr}} \left(\frac{b^2}{r^2} + g^{tt}\right)$$

$$\frac{k_r}{k_t} = \pm \sqrt{-\frac{1}{g^{rr}} \left(\frac{b^2}{r^2} + g^{tt}\right)}$$

$+$: photon moving outwards
$-$: photon moving inwards

For a diagonal metric $g^{tt} = \frac{1}{g_{tt}}$ and $g^{rr} = \frac{1}{g_{rr}}$. Then

$$\frac{k_r}{k_t} = \pm \sqrt{-g_{rr} \left(\frac{b^2}{r^2} + \frac{1}{g_{tt}}\right)}$$

On the other hand, the proper distance along the photon's trajectory is

$$dl_{prop} = K_\mu u^\mu_e \, d\lambda = \frac{K_\mu u^\mu_o}{g} d\lambda = \frac{K_t u^t_o}{g} d\lambda = \frac{K_t}{g} d\lambda$$

$$dl_{prop} = \frac{K_t}{|K_r|} \frac{dr}{g}$$

In summary, we have the specific intensity for infalling spherical accretion as

$$\boxed{I(\nu_o) \propto \int_\gamma \frac{g^3}{r^2} \frac{K_t}{|K_r|} dr}$$

with

$$\frac{K_r}{K_t} = \pm \sqrt{-g_{rr}\left(\frac{b^2}{r^2} + \frac{1}{g_{tt}}\right)}$$

$$g = \frac{1}{u^t_e + \frac{K_r}{K_t} u^r_e}$$

$$u^t_e = -\frac{1}{g_{tt}}$$

$$u^r_e = -\sqrt{-\frac{1}{g_{rr}}\left(1 + \frac{1}{g_{tt}}\right)}$$

and

$$j(\nu_e) \propto \frac{1}{r^2}$$