TULCZYJEW SPIN SUPPLEMENTARY CONDITION

We'll use the Tulczyjew spin-supplementary condition,

$P_\mu S^{\mu\nu} = 0$

to obtain:

$$P_0 S^{0\alpha} + P_1 S^{1\alpha} + \underbrace{P_2}_{=0} S^{2\alpha} + P_3 S^{3\alpha} = 0$$

(Motion is restricted to equatorial plane!)

$$P_0 S^{0\alpha} + P_1 S^{1\alpha} + P_3 S^{3\alpha} = 0$$

For $\alpha = 1$

$$P_0 S^{01} + P_1 \underline{S^{11}} + P_3 S^{31} = 0$$

$$\boxed{S^{01} = -\frac{P_3}{P_0} S^{31}}$$

For $\alpha = 3$

$$P_0 S^{03} + P_1 S^{13} + P_3 \underline{S^{33}} = 0$$

$$\boxed{S^{03} = -\frac{P_1}{P_0} S^{13} = \frac{P_1}{P_0} S^{31}}$$

Now, consider the definition of the spin angular momentum

$$S^2 = \frac{1}{2} S^{\mu\nu} S_{\mu\nu}$$

$$2S^2 = g_{\mu\alpha} g_{\nu\beta} S^{\mu\nu} S^{\alpha\beta}$$

Remember that the non-vanishing components of the spin tensor are S^{01}, S^{03} and S^{13}

$$2S^2 = g_{0\alpha} g_{\nu\beta} S^{0\nu} S^{\alpha\beta} + g_{1\alpha} g_{\nu\beta} S^{1\nu} S^{\alpha\beta} + g_{3\alpha} g_{\nu\beta} S^{3\nu} S^{\alpha\beta}$$

$$2s^2 = g_{0\alpha}\, g_{1\beta}\, S^{01} S^{\alpha\beta} + g_{0\alpha}\, g_{3\beta}\, S^{03} S^{\alpha\beta}$$
$$+ g_{1\alpha}\, g_{0\beta}\, S^{10} S^{\alpha\beta} + g_{1\alpha}\, g_{3\beta}\, S^{13} S^{\alpha\beta}$$
$$+ g_{3\alpha}\, g_{0\beta}\, S^{30} S^{\alpha\beta} + g_{3\alpha}\, g_{1\beta}\, S^{31} S^{\alpha\beta}$$

$$2s^2 = g_{0\alpha}\, g_{11}\, S^{01} S^{\alpha 1} + g_{00}\, g_{3\beta}\, S^{03} S^{0\beta} + g_{03}\, g_{3\beta}\, S^{03} S^{3\beta}$$
$$+ g_{11}\, g_{0\beta}\, S^{10} S^{1\beta} + g_{11}\, g_{3\beta}\, S^{13} S^{1\beta}$$
$$+ g_{3\alpha}\, g_{00}\, S^{30} S^{\alpha 0} + g_{3\alpha}\, g_{03}\, S^{30} S^{\alpha 3} + g_{3\alpha}\, g_{11}\, S^{31} S^{\alpha 1}$$

$$2s^2 = g_{00}\, g_{11}\, S^{01} S^{01} + g_{03}\, g_{11}\, S^{01} S^{31}$$
$$+ g_{00}\, g_{33}\, S^{03} S^{03} + g_{03}\, g_{30}\, S^{03} S^{30}$$
$$+ g_{11}\, g_{00}\, S^{10} S^{10} + g_{11}\, g_{03}\, S^{10} S^{13}$$
$$+ g_{11}\, g_{30}\, S^{13} S^{10} + g_{11}\, g_{33}\, S^{13} S^{13}$$
$$+ g_{33}\, g_{00}\, S^{30} S^{30} + g_{30}\, g_{03}\, S^{30} S^{03}$$
$$+ g_{30}\, g_{11}\, S^{31} S^{01} + g_{33}\, g_{11}\, S^{31} S^{31}$$

$$2s^2 = 2g_{00}\, g_{11}\, S^{01} S^{01} + 2g_{00}\, g_{33}\, S^{03} S^{03} + 2g_{11} g_{33}\, S^{13} S^{13}$$
$$+ 2g_{03}\, g_{03}\, S^{03} S^{30} + 4g_{03}\, g_{11}\, S^{01} S^{31}$$

$$s^2 = g_{00}\, g_{11}\, S^{01} S^{01} + g_{00}\, g_{33}\, S^{03} S^{03} + g_{11} g_{33}\, S^{13} S^{13}$$
$$+ g_{03}\, g_{03}\, S^{03} S^{30} + 2g_{03}\, g_{11}\, S^{01} S^{31}$$

$$s^2 = g_{00}\, g_{11} (S^{01})^2 + g_{00} g_{33} (S^{03})^2 + g_{11} g_{33} (S^{13})^2 - g_{03}\, g_{03} (S^{03})^2$$
$$+ 2g_{03}\, g_{11}\, S^{01} S^{31}$$

Using the Tulczyjew relation, we obtain

$$s^2 = g_{00}\, g_{11} \left(-\frac{P_3}{P_0} S^{31}\right)^2 + g_{00}\, g_{33} \left(\frac{P_1}{P_0} S^{31}\right)^2 + g_{11}\, g_{33} \left(-S^{31}\right)^2$$

$$- g_{03}\, g_{03} \left(\frac{P_1}{P_0} S^{31}\right)^2 + 2 g_{03}\, g_{11} \left(-\frac{P_3}{P_0} S^{31}\right) S^{31}$$

$$s^2 = \frac{(S^{31})^2}{P_0^2}\left[g_{00}\, g_{11} P_3^2 + g_{00}\, g_{33} P_1^2 + g_{11}\, g_{33} P_0^2 - g_{03}\, g_{03} P_1^2 - 2 g_{03}\, g_{11} P_3 P_0 \right]$$

$$s^2 P_0^2 = (S^{31})^2 \left[g_{11}\, g_{33} P_0^2 + (g_{00}\, g_{33} - g_{03}\, g_{03}) P_1^2 - 2 g_{03}\, g_{11} P_0 P_3 + g_{00}\, g_{11} P_3^2 \right]$$

$$g_{\mu\nu} = \begin{pmatrix} g_{00} & 0 & 0 & g_{03} \\ 0 & g_{11} & 0 & 0 \\ 0 & 0 & g_{22} & 0 \\ g_{03} & 0 & 0 & g_{33} \end{pmatrix}$$

$$g = \det g_{\mu\nu} = g_{00}\, g_{11}\, g_{22}\, g_{33} - g_{03}\, g_{11}\, g_{22}\, g_{03}$$

Inverse:

$$g^{\mu\nu} = \frac{1}{g} \begin{pmatrix} g_{11}\, g_{22}\, g_{33} & 0 & 0 & -g_{11}\, g_{22}\, g_{03} \\ 0 & g/g_{11} & 0 & 0 \\ 0 & 0 & g/g_{22} & 0 \\ -g_{11}\, g_{22}\, g_{03} & 0 & 0 & g_{00}\, g_{11}\, g_{22} \end{pmatrix}$$

$$g^{00} \frac{g}{g_{22}} = g_{11}\, g_{33}$$

$$g^{11} \frac{g}{g_{22}} = g_{00}\, g_{33} - g_{03}^2$$

$$g^{03} \frac{g}{g_{22}} = - g_{11}\, g_{03}$$

$$g^{33} \frac{g}{g_{22}} = g_{00}\, g_{11}$$

Replacing these relations:

$$s^2 P_0^2 = (S^{31})^2 \left[g^{00} P_0^2 + g^{11} P_1^2 + 2 g^{03} P_0 P_3 + g^{33} P_3^2 \right] \frac{g}{g_{22}}$$

Since $p^2 = g^{\mu\nu} P_\mu P_\nu = -m^2$; we have

$$s^2 P_0^2 = -m^2 (S^{31})^2 \frac{g}{g_{22}}$$

$$(S^{31})^2 = \frac{s^2 P_0^2}{m^2} \frac{g_{22}}{(-g)}$$

$$\boxed{S^{31} = \sqrt{\frac{g_{22}}{-g}} \frac{s P_0}{m}}$$

For the Improved Metric,

$g = -\Sigma^2$

$g_{22} = \Sigma$

Then:

$$S^{31} = \sqrt{\frac{\Sigma}{\Sigma^2}} \frac{s P_0}{m}$$

$$\boxed{S^{31} = \frac{s P_0}{m\sqrt{\Sigma}}}$$

$$S^{01} = -\frac{P_3}{P_0} S^{31} \qquad S^{03} = \frac{P_1}{P_0} S^{31}$$

$$\boxed{S^{01} = -\frac{s P_3}{m\sqrt{\Sigma}}} \qquad \boxed{S^{03} = \frac{s P_1}{m\sqrt{\Sigma}}}$$