We begin with the MPD equation

$$\dot{S}^{\mu\sigma} = p^\mu v^\sigma - p^\sigma v^\mu$$

Contracting with p_σ we obtain

$$p_\sigma \dot{S}^{\mu\sigma} = p^\mu p_\sigma v^\sigma - p_\sigma p^\sigma v^\mu$$

Since $p_\sigma p^\sigma = -\mu^2$ and $p_\sigma v^\sigma = -m$ we have

$$p_\sigma \dot{S}^{\mu\sigma} = -m p^\mu + \mu^2 v^\mu$$

$$\mu^2 v^\mu - m p^\mu = p_\sigma \dot{S}^{\mu\sigma} \qquad (\mathrm{I})$$

From the TD-SSC,

$$p_\sigma S^{\mu\sigma} = 0$$

we have

$$\dot{p}_\sigma S^{\mu\sigma} + p_\sigma \dot{S}^{\mu\sigma} = 0$$

$$p_\sigma \dot{S}^{\mu\sigma} = -\dot{p}_\sigma S^{\mu\sigma}$$

Replacing in (I) gives

$$\mu^2 v^\mu - m p^\mu = -\dot{p}_\sigma S^{\mu\sigma} \qquad (\mathrm{II})$$

Now, we replace the MPD equation

$$\dot{p}_\sigma = -\frac{1}{2} R_{\sigma\alpha\beta\gamma} v^\alpha S^{\beta\gamma}$$

into equation (II) to obtain

$$\mu^2 v^\mu - m p^\mu = \frac{1}{2} R_{\sigma\alpha\beta\gamma} v^\alpha S^{\beta\gamma} S^{\mu\sigma}$$

$$\mu^2 v^\mu - m p^\mu = \frac{1}{2} S^{\mu\sigma} R_{\sigma\alpha\beta\gamma} v^\alpha S^{\beta\gamma} \qquad (\mathrm{III})$$

From equation (II) we also have

$$\mu^2 v^\alpha - m p^\alpha = - \dot{p}_\sigma S^{\alpha\sigma}$$

$$v^\alpha = \frac{1}{\mu^2}\left(m p^\alpha - \dot{p}_\epsilon S^{\alpha\epsilon}\right)$$

Replacing this in the right hand side of equation (III) gives

$$\mu^2 v^\mu - m p^\mu = \frac{1}{2} S^{\mu\sigma} R_{\sigma\alpha\beta\gamma} \frac{1}{\mu^2}\left(m p^\alpha - \dot{p}_\epsilon S^{\alpha\epsilon}\right) S^{\beta\gamma}$$

$$\mu^2 v^\mu - m p^\mu = \frac{m}{2\mu^2} S^{\mu\sigma} R_{\sigma\alpha\beta\gamma} p^\alpha S^{\beta\gamma} \underbrace{- \frac{1}{2\mu^2} S^{\mu\sigma} R_{\sigma\alpha\beta\gamma} \dot{p}_\epsilon S^{\alpha\epsilon} S^{\beta\gamma}}_{A} \quad (\mathrm{IV})$$

Consider the following factors in A :

$$R_{\sigma\alpha\beta\gamma} S^{\mu\sigma} S^{\alpha\epsilon} = R_{\sigma\alpha\beta\gamma} S^{\mu[\sigma} S^{\alpha]\epsilon}$$

and due to the anti-symmetry of $S^{\mu\sigma}$,

$$R_{\sigma\alpha\beta\gamma} S^{\mu\sigma} S^{\alpha\epsilon} = \frac{1}{2} R_{\sigma\alpha\beta\gamma} S^{\alpha\sigma} S^{\mu\epsilon}$$

Taking this back into (IV) gives

$$\mu^2 v^\mu - m p^\mu = \frac{m}{2\mu^2} S^{\mu\sigma} R_{\sigma\alpha\beta\gamma} p^\alpha S^{\beta\gamma} - \frac{1}{2\mu^2}\frac{1}{2} R_{\sigma\alpha\beta\gamma} S^{\alpha\sigma} S^{\mu\epsilon} \dot{p}_\epsilon S^{\beta\gamma}$$

$$\mu^2 v^\mu - m p^\mu = \frac{m}{2\mu^2} S^{\mu\sigma} R_{\sigma\alpha\beta\gamma} p^\alpha S^{\beta\gamma} - \frac{1}{4\mu^2} R_{\sigma\alpha\beta\gamma} S^{\alpha\sigma} S^{\mu\epsilon} \dot{p}_\epsilon S^{\beta\gamma}$$

$$4\mu^2\left(\mu^2 v^\mu - m p^\mu\right) = 2m\, S^{\mu\sigma} R_{\sigma\alpha\beta\gamma} p^\alpha S^{\beta\gamma} - R_{\sigma\alpha\beta\gamma} S^{\alpha\sigma} S^{\mu\epsilon} \dot{p}_\epsilon S^{\beta\gamma}$$

$$4\mu^2\left(\mu^2 v^\mu - m p^\mu\right) + R_{\sigma\alpha\beta\gamma} S^{\alpha\sigma} S^{\mu\epsilon} \dot{p}_\epsilon S^{\beta\gamma} = 2m\, S^{\mu\sigma} R_{\sigma\alpha\beta\gamma} p^\alpha S^{\beta\gamma}$$

Replacing equation (II) in the first term,

$$4\mu^2(-\dot{P}_\epsilon S^{\mu\epsilon}) + R_{\sigma\alpha\beta\gamma} S^{\alpha\sigma} S^{\mu\epsilon} \dot{P}_\epsilon S^{\beta\gamma} = 2m\, S^{\mu\sigma} R_{\sigma\alpha\beta\gamma} P^\alpha S^{\beta\gamma}$$

$$-(4\mu^2 - R_{\sigma\alpha\beta\gamma} S^{\alpha\sigma} S^{\beta\gamma}) S^{\mu\epsilon} \dot{P}_\epsilon = 2m\, S^{\mu\sigma} R_{\sigma\alpha\beta\gamma} P^\alpha S^{\beta\gamma}$$

$$-4\mu^2\left(1 - \frac{1}{4\mu^2} R_{\sigma\alpha\beta\gamma} S^{\alpha\sigma} S^{\beta\gamma}\right) S^{\mu\epsilon} \dot{P}_\epsilon = 2m\, S^{\mu\sigma} R_{\sigma\alpha\beta\gamma} P^\alpha S^{\beta\gamma}$$

Using the antisymmetry $R_{\sigma\alpha\beta\gamma} = -R_{\alpha\sigma\beta\gamma}$,

$$-4\mu^2\left(1 + \frac{1}{4\mu^2} R_{\alpha\sigma\beta\gamma} S^{\alpha\sigma} S^{\beta\gamma}\right) S^{\mu\epsilon} \dot{P}_\epsilon = 2m\, S^{\mu\sigma} R_{\sigma\alpha\beta\gamma} P^\alpha S^{\beta\gamma}$$

$$-S^{\mu\epsilon} \dot{P}_\epsilon = \frac{2m\, S^{\mu\sigma} R_{\sigma\alpha\beta\gamma} P^\alpha S^{\beta\gamma}}{4\mu^2\left(1 + \frac{1}{4\mu^2} R_{\alpha\sigma\beta\gamma} S^{\alpha\sigma} S^{\beta\gamma}\right)}$$

$$-S^{\mu\epsilon} \dot{P}_\epsilon = \frac{m\, S^{\mu\sigma} R_{\sigma\alpha\beta\gamma} P^\alpha S^{\beta\gamma}}{2\mu^2\left(1 + \frac{1}{4\mu^2} R_{\alpha\sigma\beta\gamma} S^{\alpha\sigma} S^{\beta\gamma}\right)}$$

$$-S^{\mu\epsilon} \dot{P}_\epsilon = \frac{1}{2\mu^2\Delta} m\, S^{\mu\sigma} R_{\sigma\alpha\beta\gamma} P^\alpha S^{\beta\gamma} \qquad (\mathrm{V})$$

where we defined

$$\boxed{\Delta = \left(1 + \frac{1}{4\mu^2} R_{\alpha\sigma\beta\gamma} S^{\alpha\sigma} S^{\beta\gamma}\right)}$$

Replacing (V) into equation (II),

$$\mu^2 v^\mu - m p^\mu = - \dot{p}_\epsilon S^{\mu\epsilon}$$

$$\mu^2 v^\mu - m p^\mu = \frac{1}{2\mu^2 \Delta} m S^{\mu\sigma} R_{\sigma\alpha\beta\gamma} p^\alpha S^{\beta\gamma}$$

$$\mu^2 v^\mu = m p^\mu + \frac{1}{2\mu^2 \Delta} m S^{\mu\sigma} R_{\sigma\alpha\beta\gamma} p^\alpha S^{\beta\gamma}$$

$$v^\mu = \frac{m p^\mu}{\mu^2} + \frac{1}{2\mu^2 \Delta} \frac{m}{\mu^2} S^{\mu\sigma} R_{\sigma\alpha\beta\gamma} p^\alpha S^{\beta\gamma}$$

Defining the 'normalized' momentum $u^\mu = \frac{p^\mu}{\mu}$ gives

$$v^\mu = \frac{m u^\mu}{\mu} + \frac{1}{2\mu^2 \Delta} \frac{m}{\mu} S^{\mu\sigma} R_{\sigma\alpha\beta\gamma} u^\alpha S^{\beta\gamma}$$

$$v^\mu = \frac{m}{\mu} \left(u^\mu + \frac{1}{2\mu^2 \Delta} S^{\mu\sigma} R_{\sigma\alpha\beta\gamma} u^\alpha S^{\beta\gamma} \right)$$

and defining $N = \frac{m}{\mu}$ we have

$$\boxed{v^\mu = N \left(u^\mu + \frac{1}{2\mu^2 \Delta} S^{\mu\sigma} R_{\sigma\alpha\beta\gamma} u^\alpha S^{\beta\gamma} \right)}$$

The factor N is obtained by using the normalization conditions

$$v_\mu v^\mu = -1 \qquad \text{and} \qquad u_\mu u^\mu = -1$$

This is:

$$V_\mu V^\mu = N\left(u^\mu + \frac{1}{2\mu^2\Delta} S^{\mu\sigma} R_{\sigma\alpha\beta\gamma} u^\alpha S^{\beta\gamma}\right) \cdot$$

$$N\left(u_\mu + \frac{1}{2\mu^2\Delta} S_\mu{}^\tau R_{\tau\rho\varphi\lambda} u^\rho S^{\varphi\lambda}\right)$$

$$V_\mu V^\mu = N^2\Big(u_\mu u^\mu + \frac{1}{2\mu^2\Delta} u_\mu S^{\mu\sigma} R_{\sigma\alpha\beta\gamma} u^\alpha S^{\beta\gamma}$$

$$+ \frac{1}{2\mu^2\Delta} u^\mu S_\mu{}^\tau R_{\tau\rho\varphi\lambda} u^\rho S^{\varphi\lambda}$$

$$+ \frac{1}{2\mu^2\Delta} S^{\mu\sigma} R_{\sigma\alpha\beta\gamma} u^\alpha S^{\beta\gamma} \frac{1}{2\mu^2\Delta} S_\mu{}^\tau R_{\tau\rho\varphi\lambda} u^\rho S^{\varphi\lambda}\Big)$$

Due to the TD-SSC; $\quad P_\mu S^{\mu\sigma} = 0$

$$\downarrow$$

$$u_\mu S^{\mu\sigma} = 0$$

Then:

$$V_\mu V^\mu = N^2\left(u_\mu u^\mu + \frac{1}{2\mu^2\Delta} S^{\mu\sigma} R_{\sigma\alpha\beta\gamma} u^\alpha S^{\beta\gamma} \frac{1}{2\mu^2\Delta} S_\mu{}^\tau R_{\tau\rho\varphi\lambda} u^\rho S^{\varphi\lambda}\right)$$

$$V_\mu V^\mu = N^2\left(-1 + \frac{1}{4\mu^4\Delta^2} S^{\mu\sigma} R_{\sigma\alpha\beta\gamma} u^\alpha S^{\beta\gamma} S_{\mu\tau} R^{\tau\rho\varphi\lambda} u_\rho S_{\varphi\lambda}\right) \equiv -1$$

$$N^2 \equiv - \frac{1}{\left(-1 + \frac{1}{4\mu^4\Delta^2} S^{\mu\sigma} R_{\sigma\alpha\beta\gamma} u^\alpha S^{\beta\gamma} S_{\mu\tau} R^{\tau\rho\varphi\lambda} u_\rho S_{\varphi\lambda}\right)}$$

$$\boxed{N \equiv \frac{1}{\left(1 - \frac{1}{4\mu^4\Delta^2} S^{\mu\sigma} R_{\sigma\alpha\beta\gamma} u^\alpha S^{\beta\gamma} S_{\mu\tau} R^{\tau\rho\varphi\lambda} u_\rho S_{\varphi\lambda}\right)^{1/2}}}$$