Kerr Solution

Describes a rotating uncharged black hole in 4D (GR). Using the Boyer-Lindquist coordinates (t, r, θ, ϕ), the line element is

$$ds^2 = -\left(1 - \frac{2Mr}{\Sigma}\right) dt^2 - \frac{4aMr\sin^2\theta}{\Sigma}\, dt\, d\phi + \frac{\Sigma}{\Delta} dr^2 + \Sigma\, d\theta^2$$
$$+ \left(r^2 + a^2 + \frac{2a^2 Mr \sin^2\theta}{\Sigma}\right) \sin^2\theta\, d\phi^2$$

$\Delta = r^2 - 2Mr + a^2$

$\Sigma = r^2 + a^2 \cos^2\theta$

$a = \frac{J}{M}$

J: spin angular momentum of the BH

Essential Singularity

$$\Sigma = r^2 + a^2\cos^2\theta = 0 \longrightarrow r = 0 \;\&\; \theta = \frac{\pi}{2}$$

Kretschmann scalar:

$$\boxed{K = \frac{48M^2}{\Sigma^6}\left(r^6 - 15a^2r^4\cos^2\theta + 15a^4r^2\cos^2\theta - a^6\cos^6\theta\right)}$$

K diverges if $r = 0$ & $\theta = \frac{\pi}{2}$

$$\lim_{a \to 0} K = \frac{48M^2}{(r^2)^6} r^6 = \frac{48M^2}{r^6} \qquad \text{(Schwarzschild)}$$

Kerr-Schild Coordinates

$$(t', x, y, z) \longleftarrow (t, r, \theta, \phi)$$

$$x + iy = (r + ia)\sin\theta \exp\left[i\int d\phi + i\int \frac{a}{\Delta} dr\right]$$

$$z = r\cos\theta$$

$$t' = \int dt - \int \frac{r^2 + a^2}{\Delta} dr - r$$

Note that

$$(x+iy)(x-iy) = (r+ia)\sin\theta \, \exp\left[i\int d\phi + i\int \frac{a}{\Delta} dr\right] \cdot$$
$$(r-ia)\sin\theta \, \exp\left[-i\int d\phi - i\int \frac{a}{\Delta} dr\right]$$

$$x^2+y^2 = (r^2+a^2)\sin^2\theta$$

Since $z = r\cos\theta$

$$z^2 = r^2\cos^2\theta = r^2 - r^2\sin^2\theta$$

$$r^2\sin^2\theta = r^2 - z^2$$

Then

$$r^2(x^2+y^2) = (r^2+a^2)\, r^2\sin^2\theta$$

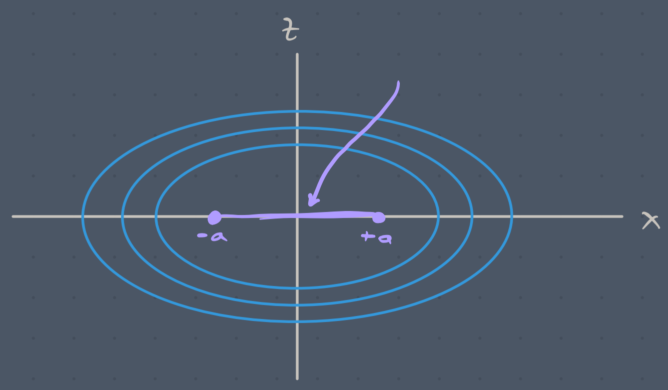
$$r^2(x^2+y^2) = (r^2+a^2)(r^2-z^2)$$

$$x^2+y^2 = \left(1+\frac{a^2}{r^2}\right)(r^2-z^2)$$

$$\frac{x^2}{r^2}+\frac{y^2}{r^2} = \left(1+\frac{a^2}{r^2}\right)\left(1-\frac{z^2}{r^2}\right)$$

$$\frac{x^2}{r^2}+\frac{y^2}{r^2}+\left(1+\frac{a^2}{r^2}\right)\frac{z^2}{r^2} = \left(1+\frac{a^2}{r^2}\right)$$

For $a=0$ → surfaces of $r=\text{constant}$ are spheres.
For $a\neq 0$ → surfaces of $r=\text{constant}$ are ellipsoids with focci at $x^2+y^2=a^2$



The essential singularity $r=0$ & $\theta=\frac{\pi}{2}$ corresponds to

$$x^2+y^2 = (r^2+a^2)\sin^2\theta$$

$$x^2+y^2 = a^2 \qquad \leftarrow \text{Ring of radius } a.$$

Horizons

$$\Delta(r_\pm) = r_\pm^2 - 2Mr_\pm + a^2 = 0$$

$$\boxed{r_\pm = M \pm \sqrt{M^2-a^2}}$$

Case I: $M < |a|$

Naked singularity at $\Sigma = 0$ (i.e. $r=0$ & $\theta=\frac{\pi}{2}$)

This case is not allowed by the Cosmic Censorship Hypothesis.

Case II: $M > |a|$

r_+ : Event Horizon
r_- : Inner (Cauchy) Horizon

Case III: $M = |a|$

$r_+ = r_- = M$: Degenerate Horizon