Accion y Lagrangiana para una Particula Libre

$$S = \int_{\tau_1}^{\tau_2} L \, d\tau \qquad L = L[x^\mu(\lambda), \dot{x}^\mu(\lambda) ; \lambda] \qquad \dot{x}^\mu = \frac{dx^\mu}{d\lambda}$$

$$\delta S = \delta \int_{\tau_1}^{\tau_2} L \, d\tau = 0 \longrightarrow \delta L = 0$$

$$\downarrow$$

$$\frac{d}{d\lambda}\left(\frac{\partial L}{\partial \dot{x}^\alpha}\right) = \frac{\partial L}{\partial x^\alpha}$$

Para una partícula libre con masa propia m_0, se tiene

$$L = \frac{1}{2} m_0 \dot{x}^2 = \frac{1}{2} m_0 \, g_{\mu\nu} \dot{x}^\mu \dot{x}^\nu$$

Ya que m_0 es una constante, es posible trabajar con el Lagrangiano equivalente

$$\mathcal{L} = \frac{L}{m_0} = \frac{1}{2} g_{\mu\nu} \dot{x}^\mu \dot{x}^\nu$$

Para las ecuaciones de Euler-Lagrange se tiene

$$\frac{\partial \mathcal{L}}{\partial \dot{x}^\alpha} = g_{\alpha\nu} \dot{x}^\nu$$

$$\frac{\partial \mathcal{L}}{\partial x^\alpha} = \frac{1}{2} \partial_\alpha g_{\mu\nu} \dot{x}^\mu \dot{x}^\nu$$

De esta forma,

$$\frac{d}{d\lambda}\left(g_{\alpha\nu} \dot{x}^\nu \right) = \frac{1}{2} \partial_\alpha g_{\mu\nu} \dot{x}^\mu \dot{x}^\nu$$

$$g_{\alpha\nu} \ddot{x}^\nu + \frac{dx^\mu}{d\lambda} \partial_\mu g_{\alpha\nu} \dot{x}^\nu = \frac{1}{2} \partial_\alpha g_{\mu\nu} \dot{x}^\mu \dot{x}^\nu$$

$$g_{\alpha\nu} \ddot{x}^\nu = - \partial_\mu g_{\alpha\nu} \dot{x}^\mu \dot{x}^\nu + \frac{1}{2} \partial_\alpha g_{\mu\nu} \dot{x}^\mu \dot{x}^\nu$$

$$g_{\alpha\nu} \ddot{x}^\nu = -\frac{1}{2} \partial_\mu g_{\alpha\nu} \dot{x}^\mu \dot{x}^\nu - \frac{1}{2} \partial_\mu g_{\alpha\nu} \dot{x}^\mu \dot{x}^\nu + \frac{1}{2} \partial_\alpha g_{\mu\nu} \dot{x}^\mu \dot{x}^\nu$$

$$g_{\alpha\nu}\ddot{x}^{\nu} = -\frac{1}{2}\partial_{\nu} g_{\alpha\mu}\dot{x}^{\mu}\dot{x}^{\nu} - \frac{1}{2}\partial_{\mu} g_{\alpha\nu}\dot{x}^{\mu}\dot{x}^{\nu} + \frac{1}{2}\partial_{\alpha} g_{\mu\nu}\dot{x}^{\mu}\dot{x}^{\nu}$$

$$g_{\alpha\nu}\ddot{x}^{\nu} = -\frac{1}{2}\left[\partial_{\mu} g_{\alpha\nu} + \partial_{\nu} g_{\alpha\mu} - \partial_{\alpha} g_{\mu\nu}\right]\dot{x}^{\mu}\dot{x}^{\nu}$$

$$\ddot{x}^{\alpha} = -\frac{1}{2} g^{\alpha\sigma}\left[\partial_{\mu} g_{\sigma\nu} + \partial_{\nu} g_{\sigma\mu} - \partial_{\sigma} g_{\mu\nu}\right]\dot{x}^{\mu}\dot{x}^{\nu}$$

$$\ddot{x}^{\alpha} = -\Gamma^{\alpha}_{\mu\nu}\dot{x}^{\mu}\dot{x}^{\nu}$$

$$\boxed{\ddot{x}^{\alpha} + \Gamma^{\alpha}_{\mu\nu}\dot{x}^{\mu}\dot{x}^{\nu} = 0}$$

$$\Gamma^{\alpha}_{\mu\nu} = \frac{1}{2} g^{\alpha\sigma}\left[\partial_{\mu} g_{\sigma\nu} + \partial_{\nu} g_{\sigma\mu} - \partial_{\sigma} g_{\mu\nu}\right]$$