MPD EQUATIONS

$$\frac{DP^\mu}{D\tau} = -\frac{1}{2} R^\mu{}_{\nu\rho\sigma} V^\nu S^{\rho\sigma}$$

$$\frac{DS^{\mu\nu}}{D\tau} = P^\mu V^\nu - P^\nu V^\mu$$

$$V^\mu = \frac{dx^\mu}{d\tau} \qquad V^\mu V_\mu = -1$$

TD-SSC : $\quad P_\mu S^{\mu\nu} = 0$

$$P_\mu P^\mu = -\mu^2$$

$$P_\mu V^\mu = -m$$

We begin with the MPD equation

$$\dot{S}^{\mu\sigma} = P^\mu V^\sigma - P^\sigma V^\mu$$

Contracting with P_σ we obtain

$$P_\sigma \dot{S}^{\mu\sigma} = P^\mu P_\sigma V^\sigma - P_\sigma P^\sigma V^\mu$$

Since $P_\sigma P^\sigma = -\mu^2$ and $P_\sigma V^\sigma = -m$ we have

$$P_\sigma \dot{S}^{\mu\sigma} = -m P^\mu + \mu^2 V^\mu$$

$$\mu^2 V^\mu - m P^\mu = P_\sigma \dot{S}^{\mu\sigma} \qquad \text{(I)}$$

From the TD-SSC,

$$P_\sigma S^{\mu\sigma} = 0$$

we have

$$\dot{P}_\sigma S^{\mu\sigma} + P_\sigma \dot{S}^{\mu\sigma} = 0$$

$$P_\sigma \dot{S}^{\mu\sigma} = -\dot{P}_\sigma S^{\mu\sigma}$$

Replacing in (I) gives

$$\mu^2 v^\mu - m p^\mu = -\dot{P}_\sigma S^{\mu\sigma} \qquad (\mathrm{II})$$

Now, we replace the MPD equation

$$\dot{P}_\sigma = -\frac{1}{2} R_{\sigma\alpha\beta\gamma} v^\alpha S^{\beta\gamma}$$

into equation (II) to obtain

$$\mu^2 v^\mu - m p^\mu = \frac{1}{2} R_{\sigma\alpha\beta\gamma} v^\alpha S^{\beta\gamma} S^{\mu\sigma}$$

$$\mu^2 v^\mu - m p^\mu = \frac{1}{2} S^{\mu\sigma} R_{\sigma\alpha\beta\gamma} v^\alpha S^{\beta\gamma} \qquad (\mathrm{III})$$

From equation (II) we also have

$$\mu^2 v^\alpha - m p^\alpha = -\dot{P}_\sigma S^{\alpha\sigma}$$

$$v^\alpha = \frac{1}{\mu^2}\left(m p^\alpha - \dot{P}_\epsilon S^{\alpha\epsilon}\right)$$

Replacing this in the right hand side of equation (III) gives

$$\mu^2 v^\mu - m p^\mu = \frac{1}{2} S^{\mu\sigma} R_{\sigma\alpha\beta\gamma} \frac{1}{\mu^2}\left(m p^\alpha - \dot{P}_\epsilon S^{\alpha\epsilon}\right) S^{\beta\gamma}$$

$$\mu^2 v^\mu - m p^\mu = \frac{m}{2\mu^2} S^{\mu\sigma} R_{\sigma\alpha\beta\gamma} p^\alpha S^{\beta\gamma} - \underbrace{\frac{1}{2\mu^2} S^{\mu\sigma} R_{\sigma\alpha\beta\gamma} \dot{p}_\epsilon S^{\alpha\epsilon} S^{\beta\gamma}}_{A} \quad (\mathrm{IV})$$

Consider the following factors in A :

$$R_{\sigma\alpha\beta\gamma} S^{\mu\sigma} S^{\alpha\epsilon} = R_{\sigma\alpha\beta\gamma} S^{\mu[\sigma} S^{\alpha]\epsilon}$$

and due to the anti-symmetry of $S^{\mu\sigma}$,

$$R_{\sigma\alpha\beta\gamma} S^{\mu\sigma} S^{\alpha\epsilon} = \frac{1}{2} R_{\sigma\alpha\beta\gamma} S^{\alpha\sigma} S^{\mu\epsilon}$$

Taking this back into (IV) gives

$$\mu^2 v^\mu - m p^\mu = \frac{m}{2\mu^2} S^{\mu\sigma} R_{\sigma\alpha\beta\gamma} p^\alpha S^{\beta\gamma} - \frac{1}{2\mu^2} \frac{1}{2} R_{\sigma\alpha\beta\gamma} S^{\alpha\sigma} S^{\mu\epsilon} \dot{p}_\epsilon S^{\beta\gamma}$$

$$\mu^2 v^\mu - m p^\mu = \frac{m}{2\mu^2} S^{\mu\sigma} R_{\sigma\alpha\beta\gamma} p^\alpha S^{\beta\gamma} - \frac{1}{4\mu^2} R_{\sigma\alpha\beta\gamma} S^{\alpha\sigma} S^{\mu\epsilon} \dot{p}_\epsilon S^{\beta\gamma}$$

$$4\mu^2 (\mu^2 v^\mu - m p^\mu) = 2m\, S^{\mu\sigma} R_{\sigma\alpha\beta\gamma} p^\alpha S^{\beta\gamma} - R_{\sigma\alpha\beta\gamma} S^{\alpha\sigma} S^{\mu\epsilon} \dot{p}_\epsilon S^{\beta\gamma}$$

$$4\mu^2 (\mu^2 v^\mu - m p^\mu) + R_{\sigma\alpha\beta\gamma} S^{\alpha\sigma} S^{\mu\epsilon} \dot{p}_\epsilon S^{\beta\gamma} = 2m\, S^{\mu\sigma} R_{\sigma\alpha\beta\gamma} p^\alpha S^{\beta\gamma}$$

Replacing equation (II) in the first term,

$$4\mu^2 (-\dot{p}_\epsilon S^{\mu\epsilon}) + R_{\sigma\alpha\beta\gamma} S^{\alpha\sigma} S^{\mu\epsilon} \dot{p}_\epsilon S^{\beta\gamma} = 2m\, S^{\mu\sigma} R_{\sigma\alpha\beta\gamma} p^\alpha S^{\beta\gamma}$$

$$-(4\mu^2 - R_{\sigma\alpha\beta\gamma} S^{\alpha\sigma} S^{\beta\gamma}) S^{\mu\epsilon} \dot{p}_\epsilon = 2m\, S^{\mu\sigma} R_{\sigma\alpha\beta\gamma} p^\alpha S^{\beta\gamma}$$

$$-4\mu^2 \left(1 - \frac{1}{4\mu^2} R_{\sigma\alpha\beta\gamma} S^{\alpha\sigma} S^{\beta\gamma}\right) S^{\mu\epsilon} \dot{p}_\epsilon = 2m\, S^{\mu\sigma} R_{\sigma\alpha\beta\gamma} p^\alpha S^{\beta\gamma}$$

Using the antisymmetry $R_{\sigma\alpha\beta\gamma} = -R_{\alpha\sigma\beta\gamma}$,

$$-4\mu^2\left(1+\frac{1}{4\mu^2}R_{\alpha\sigma\beta\gamma}S^{\alpha\sigma}S^{\beta\gamma}\right)S^{\mu\epsilon}\dot{P}_\epsilon = 2m\, S^{\mu\sigma}R_{\sigma\alpha\beta\gamma}P^\alpha S^{\beta\gamma}$$

$$-S^{\mu\epsilon}\dot{P}_\epsilon = \frac{2m\, S^{\mu\sigma}R_{\sigma\alpha\beta\gamma}P^\alpha S^{\beta\gamma}}{4\mu^2\left(1+\frac{1}{4\mu^2}R_{\alpha\sigma\beta\gamma}S^{\alpha\sigma}S^{\beta\gamma}\right)}$$

$$-S^{\mu\epsilon}\dot{P}_\epsilon = \frac{m\, S^{\mu\sigma}R_{\sigma\alpha\beta\gamma}P^\alpha S^{\beta\gamma}}{2\mu^2\left(1+\frac{1}{4\mu^2}R_{\alpha\sigma\beta\gamma}S^{\alpha\sigma}S^{\beta\gamma}\right)}$$

$$-S^{\mu\epsilon}\dot{P}_\epsilon = \frac{1}{2\mu^2\Delta}\, m\, S^{\mu\sigma}R_{\sigma\alpha\beta\gamma}P^\alpha S^{\beta\gamma} \qquad (\mathrm{V})$$

where we defined

$$\boxed{\Delta = \left(1+\frac{1}{4\mu^2}R_{\alpha\sigma\beta\gamma}S^{\alpha\sigma}S^{\beta\gamma}\right)}$$

Replacing (V) into equation (II),

$$\mu^2 v^\mu - mP^\mu = -\dot{P}_\epsilon S^{\mu\epsilon}$$

$$\mu^2 v^\mu - mP^\mu = \frac{1}{2\mu^2\Delta}\, m\, S^{\mu\sigma}R_{\sigma\alpha\beta\gamma}P^\alpha S^{\beta\gamma}$$

$$\mu^2 v^\mu = mP^\mu + \frac{1}{2\mu^2\Delta}\, m\, S^{\mu\sigma}R_{\sigma\alpha\beta\gamma}P^\alpha S^{\beta\gamma}$$

$$v^\mu = \frac{mP^\mu}{\mu^2} + \frac{1}{2\mu^2\Delta}\frac{m}{\mu^2} S^{\mu\sigma}R_{\sigma\alpha\beta\gamma}P^\alpha S^{\beta\gamma}$$

Defining the 'normalized' momentum $u^\mu = \frac{p^\mu}{\mu}$ gives

$$v^\mu = \frac{m u^\mu}{\mu} + \frac{1}{2\mu^2\Delta}\frac{m}{\mu} S^{\mu\sigma} R_{\sigma\alpha\beta\gamma} u^\alpha S^{\beta\gamma}$$

$$v^\mu = \frac{m}{\mu}\left(u^\mu + \frac{1}{2\mu^2\Delta} S^{\mu\sigma} R_{\sigma\alpha\beta\gamma} u^\alpha S^{\beta\gamma}\right)$$

and defining $N = \frac{m}{\mu}$ we have

$$\boxed{v^\mu = N\left(u^\mu + \frac{1}{2\mu^2\Delta} S^{\mu\sigma} R_{\sigma\alpha\beta\gamma} u^\alpha S^{\beta\gamma}\right)}$$

The factor N is obtained by using the normalization conditions

$$v_\mu v^\mu = -1 \qquad \text{and} \qquad u_\mu u^\mu = -1$$

This is:

$$v_\mu v^\mu = N\left(u^\mu + \frac{1}{2\mu^2\Delta} S^{\mu\sigma} R_{\sigma\alpha\beta\gamma} u^\alpha S^{\beta\gamma}\right) \cdot N\left(u_\mu + \frac{1}{2\mu^2\Delta} S_\mu{}^\tau R_{\tau\rho\varphi\lambda} u^\rho S^{\varphi\lambda}\right)$$

$$v_\mu v^\mu = N^2\Big(u_\mu u^\mu + \frac{1}{2\mu^2\Delta} u_\mu S^{\mu\sigma} R_{\sigma\alpha\beta\gamma} u^\alpha S^{\beta\gamma} + \frac{1}{2\mu^2\Delta} u^\mu S_\mu{}^\tau R_{\tau\rho\varphi\lambda} u^\rho S^{\varphi\lambda} + \frac{1}{2\mu^2\Delta} S^{\mu\sigma} R_{\sigma\alpha\beta\gamma} u^\alpha S^{\beta\gamma} \frac{1}{2\mu^2\Delta} S_\mu{}^\tau R_{\tau\rho\varphi\lambda} u^\rho S^{\varphi\lambda}\Big)$$

Due to the TD-SSC; $\quad P_\mu S^{\mu\sigma} = 0$

$$\downarrow$$

$$u_\mu S^{\mu\sigma} = 0$$

Then:

$$V_\mu V^\mu = N^2 \left(u_\mu u^\mu + \frac{1}{2\mu^2\Delta} S^{\mu\sigma} R_{\sigma\alpha\beta\gamma} u^\alpha S^{\beta\gamma} \frac{1}{2\mu^2\Delta} S_\mu{}^\tau R_{\tau\rho\varphi\lambda} u^\rho S^{\varphi\lambda} \right)$$

$$V_\mu V^\mu = N^2 \left(-1 + \frac{1}{4\mu^4\Delta^2} S^{\mu\sigma} R_{\sigma\alpha\beta\gamma} u^\alpha S^{\beta\gamma} S_{\mu\tau} R^{\tau\rho\varphi\lambda} u_\rho S_{\varphi\lambda} \right) \equiv -1$$

$$N^2 \equiv - \frac{1}{\left(-1 + \frac{1}{4\mu^4\Delta^2} S^{\mu\sigma} R_{\sigma\alpha\beta\gamma} u^\alpha S^{\beta\gamma} S_{\mu\tau} R^{\tau\rho\varphi\lambda} u_\rho S_{\varphi\lambda} \right)}$$

$$\boxed{N \equiv \frac{1}{\left(1 - \frac{1}{4\mu^4\Delta^2} S^{\mu\sigma} R_{\sigma\alpha\beta\gamma} u^\alpha S^{\beta\gamma} S_{\mu\tau} R^{\tau\rho\varphi\lambda} u_\rho S_{\varphi\lambda} \right)^{1/2}}}$$

Spherically Symmetric Spacetime

$$ds^2 = g_{tt}\,dt^2 + g_{rr}\,dr^2 + r^2\left(d\theta^2 + \sin^2\theta\, d\varphi^2\right)$$

Conserved Quantities

Given the existence of a Killing vector K^μ, the corresponding conserved quantity is

$$C_K = P^\mu K_\mu + \frac{1}{2} S^{\mu\nu}\nabla_\mu k_\nu + \tilde{q}\, K^\mu \phi_\mu$$

I will include a 'MOG' type interaction term. (Just in case!)

For a general Stationary and Axisymmetric spacetime with a Reflection symmetry along the equatorial plane (SAR spacetime) we have two Killing vectors

$$\xi = \frac{\partial}{\partial t} \longrightarrow \xi^\mu = \delta^\mu_t$$

$$\zeta = \frac{\partial}{\partial \varphi} \longrightarrow \zeta^\mu = \delta^\mu_\varphi$$

- **Energy:**

$$C_{(t)} = \xi^\mu P_\mu + \frac{1}{2} S^{\mu\nu}\nabla_\mu \xi_\nu + \tilde{q}\,\xi^\mu \phi_\mu$$

$$C_{(t)} = \delta^\mu_t P_\mu + \frac{1}{2} S^{\mu\nu}\nabla_\mu \xi_\nu + \tilde{q}\,\delta^\mu_t \phi_\mu$$

Note that $\xi_\nu = g_{\nu\alpha}\xi^\alpha = g_{\nu\alpha}\delta^\alpha_t = g_{\nu t}$

$$\nabla_\mu \xi_\nu = \partial_\mu \xi_\nu - \Gamma^\alpha_{\mu\nu}\xi_\alpha$$

$$\nabla_\mu \xi_\nu = \partial_\mu g_{\nu t} - \Gamma^\alpha_{\mu\nu} g_{\alpha t}$$

$$S^{\mu\nu}\nabla_\mu \xi_\nu = S^{\mu\nu}\partial_\mu g_{\nu t} - \cancel{S^{\mu\nu}\Gamma^\alpha_{\mu\nu}}\, g_{\alpha t}$$

and therefore

$$C_{(t)} = P_t + \frac{1}{2} S^{\mu\nu} \partial_\mu g_{\nu t} + \tilde{q}\Phi \equiv -E$$

$$-E = P_t + \frac{1}{2} S^{\mu\nu} \partial_\mu g_{\nu t} + \tilde{q}\Phi$$

$$\boxed{E = -P_t - \frac{1}{2} S^{rt} \partial_r g_{tt} - \tilde{q}\Phi}$$

$$\Phi = \phi_t$$

- Angular Momentum

$$C_{(\varphi)} = \xi^\mu P_\mu + \frac{1}{2} S^{\mu\nu} \nabla_\mu \xi_\nu + \tilde{q}\,\xi^\mu \phi_\mu$$

$$C_{(\varphi)} = \delta^\mu_\varphi P_\mu + \frac{1}{2} S^{\mu\nu} \nabla_\mu \xi_\nu + \tilde{q}\,\delta^\mu_\varphi \phi_\mu$$

Now, $\xi_\nu = g_{\nu\alpha}\xi^\alpha = g_{\nu\alpha}\delta^\alpha_\varphi = g_{\nu\varphi}$

$$C_{(\varphi)} = P_\varphi + \frac{1}{2} S^{\mu\nu} \partial_\mu g_{\nu\varphi} \equiv J_z$$

$$J_z = P_\varphi + \frac{1}{2} S^{\mu\nu} \partial_\mu g_{\nu\varphi}$$

$$J_z = P_\varphi + \frac{1}{2}\left[S^{r\varphi} \partial_r g_{\varphi\varphi} + S^{\theta\varphi} \partial_\theta g_{\varphi\varphi} \right]$$

$$J_z = P_\varphi + \frac{1}{2}\left[S^{r\varphi}\, 2r\sin^2\theta + S^{\theta\varphi}\, 2r^2 \sin\theta\cos\theta \right]$$

$$\boxed{J_z = P_\varphi + r\left[S^{r\varphi} + S^{\theta\varphi}\, r\cot\theta \right]\sin^2\theta}$$

The components of the angular momentum are given by the Killing vectors

$$\begin{cases} K_{(x)} = -\sin\varphi \dfrac{\partial}{\partial\theta} - \cot\theta\cos\varphi \dfrac{\partial}{\partial\varphi} \\ K_{(y)} = \cos\varphi \dfrac{\partial}{\partial\theta} - \cot\theta \sin\varphi \dfrac{\partial}{\partial\varphi} \\ K_{(z)} = \dfrac{\partial}{\partial\varphi} \equiv \zeta \end{cases}$$

Working now with $K_{(x)}$ and $K_{(y)}$ we obtain the angular momentum components J_x and J_y :

$$K^{\mu}_{(x)} = -\sin\varphi\, \delta^{\mu}_{\theta} - \cot\theta\cos\varphi\, \delta^{\mu}_{\varphi}$$

$$C_{(x)} = K^{\mu}_{(x)} P_{\mu} - \frac{1}{2} S^{\mu\nu} \nabla_{\mu} K_{\nu(x)} + \tilde{q} K^{\mu}_{(x)} \phi_{\mu}$$

$$C_{(x)} = -\sin\varphi\, \delta^{\mu}_{\theta} P_{\mu} - \cot\theta\cos\varphi\, \delta^{\mu}_{\varphi} P_{\mu} + \frac{1}{2} S^{\mu\nu} \nabla_{\mu} K_{\nu(x)} + \tilde{q} K^{\theta}_{(x)} \phi_{\theta} + \tilde{q} K^{\varphi}_{(x)} \phi_{\varphi}$$

$$C_{(x)} = -\sin\varphi\, P_{\theta} - \cot\theta\cos\varphi\, P_{\varphi} + \frac{1}{2} S^{\mu\nu} \partial_{\mu} K_{\nu(x)} - \frac{1}{2} S^{\mu\nu} \Gamma^{\alpha}_{\mu\nu} K_{\alpha(x)}$$

$$C_{(x)} = -\sin\varphi\, P_{\theta} - \cot\theta\cos\varphi\, P_{\varphi} + \frac{1}{2} S^{\mu\nu} \partial_{\mu} K_{\nu(x)} \equiv J_x$$

Now,

$$K_{\nu(y)} = g_{\nu\alpha} K^{\alpha}_{(y)} = -g_{\nu\alpha} \sin\varphi\, \delta^{\alpha}_{\theta} - g_{\nu\alpha} \cot\theta\cos\varphi\, \delta^{\alpha}_{\varphi}$$

$$K_{\nu(y)} = -g_{\nu\theta} \sin\varphi - g_{\nu\varphi} \cot\theta\cos\varphi$$

Then

$$J_x = -P_\theta \sin\varphi - P_\varphi \cot\theta \cos\varphi - \frac{1}{2} S^{\mu\nu} \partial_\mu (g_{\theta\nu} \sin\varphi) - \frac{1}{2} S^{\mu\nu} \partial_\mu (g_{\varphi\nu} \cot\theta \cos\varphi)$$

$$J_x = -P_\theta \sin\varphi - P_\varphi \cot\theta \cos\varphi - \frac{1}{2} S^{\mu\theta} \partial_\mu (g_{\theta\theta} \sin\varphi) - \frac{1}{2} S^{\mu\varphi} \partial_\mu (g_{\varphi\varphi} \cot\theta \cos\varphi)$$

$$\begin{aligned} J_x = & -P_\theta \sin\varphi - P_\varphi \cot\theta \cos\varphi - \frac{1}{2} S^{r\theta} \partial_r (g_{\theta\theta} \sin\varphi) - \frac{1}{2} S^{\varphi\theta} \partial_\varphi (g_{\theta\theta} \sin\varphi) \\ & - \frac{1}{2} S^{r\varphi} \partial_r (g_{\varphi\varphi} \cot\theta \cos\varphi) - \frac{1}{2} S^{\theta\varphi} \partial_\theta (g_{\varphi\varphi} \cot\theta \cos\varphi) \end{aligned}$$

$$\begin{aligned} J_x = & -P_\theta \sin\varphi - P_\varphi \cot\theta \cos\varphi - \frac{1}{2} S^{r\theta} \sin\varphi \, \partial_r g_{\theta\theta} - \frac{1}{2} S^{\varphi\theta} g_{\theta\theta} \, \partial_\varphi \sin\varphi \\ & - \frac{1}{2} S^{r\varphi} \cot\theta \cos\varphi \, \partial_r g_{\varphi\varphi} - \frac{1}{2} S^{\theta\varphi} \cos\varphi \, \partial_\theta (g_{\varphi\varphi} \cot\theta) \end{aligned}$$

$$\begin{aligned} J_x = & -P_\theta \sin\varphi - P_\varphi \cot\theta \cos\varphi - \frac{1}{2} S^{r\theta} \sin\varphi \, 2r - \frac{1}{2} S^{\varphi\theta} g_{\theta\theta} \cos\varphi \\ & - \frac{1}{2} S^{r\varphi} \cot\theta \cos\varphi \, 2r \sin^2\theta - \frac{1}{2} S^{\theta\varphi} \cos\varphi \, r^2 \partial_\theta (\sin^2\theta \cot\theta) \end{aligned}$$

$$\begin{aligned} J_x = & -P_\theta \sin\varphi - P_\varphi \cot\theta \cos\varphi - r S^{r\theta} \sin\varphi - \frac{r^2}{2} S^{\varphi\theta} \cos\varphi \\ & - r S^{r\varphi} \cos\varphi \sin\theta \cos\theta - \frac{r^2}{2} S^{\theta\varphi} \cos\varphi \, \partial_\theta (\sin\theta \cos\theta) \end{aligned}$$

$$\begin{aligned} J_x = & -P_\theta \sin\varphi - P_\varphi \cot\theta \cos\varphi - r S^{r\theta} \sin\varphi + \frac{r^2}{2} S^{\theta\varphi} \cos\varphi \\ & - r S^{r\varphi} \cos\varphi \sin\theta \cos\theta - \frac{r^2}{2} S^{\theta\varphi} \cos\varphi (\cos^2\theta - \sin^2\theta) \end{aligned}$$

$$\begin{aligned} J_x = & -P_\theta \sin\varphi - P_\varphi \cot\theta \cos\varphi - r S^{r\theta} \sin\varphi + \frac{r^2}{2} S^{\theta\varphi} \cos\varphi \\ & - r S^{r\varphi} \cos\varphi \sin\theta \cos\theta - \frac{r^2}{2} S^{\theta\varphi} \cos\varphi (1 - 2\sin^2\theta) \end{aligned}$$

$$\begin{aligned} J_x = & -P_\theta \sin\varphi - P_\varphi \cot\theta \cos\varphi - r S^{r\theta} \sin\varphi \\ & + r S^{\varphi r} \cos\varphi \sin\theta \cos\theta + r^2 S^{\theta\varphi} \cos\varphi \sin^2\theta \end{aligned}$$

Similarly,

$$K^{\mu}_{(y)} = \cos\varphi\, \delta^{\mu}_{\theta} - \cot\theta \sin\varphi\, \delta^{\mu}_{\varphi}$$

$$C_{(y)} = K^{\mu}_{(y)} P_{\mu} - \frac{1}{2} S^{\mu\nu} \nabla_{\mu} K_{\nu(y)} + \tilde{q} K^{\mu}_{(y)} \phi_{\mu}$$

$$C_{(y)} = \cos\varphi\, \delta^{\mu}_{\theta} P_{\mu} - \cot\theta \sin\varphi\, \delta^{\mu}_{\varphi} P_{\mu} + \frac{1}{2} S^{\mu\nu} \nabla_{\mu} k_{\nu(y)} + \tilde{q} K^{\theta}_{(y)} \phi_{\theta} + \tilde{q} K^{\varphi}_{(y)} \phi_{\varphi}$$

$$C_{(y)} = \cos\varphi\, P_{\theta} - \cot\theta \sin\varphi\, P_{\varphi} + \frac{1}{2} S^{\mu\nu} \partial_{\mu} k_{\nu(y)} - \frac{1}{2} S^{\mu\nu} \Gamma^{\alpha}_{\mu\nu} K_{\alpha(y)}$$

$$C_{(y)} = \cos\varphi\, P_{\theta} - \cot\theta \sin\varphi\, P_{\varphi} + \frac{1}{2} S^{\mu\nu} \partial_{\mu} k_{\nu(y)} \equiv J_y$$

Now,

$$K_{\nu(y)} = g_{\nu\alpha} K^{\alpha}_{(y)} = g_{\nu\alpha} \cos\varphi\, \delta^{\alpha}_{\theta} - g_{\nu\alpha} \cot\theta \sin\varphi\, \delta^{\alpha}_{\varphi}$$

$$K_{\nu(y)} = g_{\nu\theta} \cos\varphi - g_{\nu\varphi} \cot\theta \sin\varphi$$

Then

$$J_y = P_{\theta} \cos\varphi - P_{\varphi} \cot\theta \sin\varphi + \frac{1}{2} S^{\mu\nu} \partial_{\mu} (g_{\theta\nu} \cos\varphi) - \frac{1}{2} S^{\mu\nu} \partial_{\mu} (g_{\varphi\nu} \cot\theta \sin\varphi)$$

$$J_y = P_{\theta} \cos\varphi - P_{\varphi} \cot\theta \sin\varphi + \frac{1}{2} S^{\mu\theta} \partial_{\mu} (g_{\theta\theta} \cos\varphi) - \frac{1}{2} S^{\mu\varphi} \partial_{\mu} (g_{\varphi\varphi} \cot\theta \sin\varphi)$$

$$J_y = P_{\theta} \cos\varphi - P_{\varphi} \cot\theta \sin\varphi + \frac{1}{2} S^{r\theta} \partial_{r} (g_{\theta\theta} \cos\varphi) + \frac{1}{2} S^{\varphi\theta} \partial_{\varphi} (g_{\theta\theta} \cos\varphi)$$
$$- \frac{1}{2} S^{r\varphi} \partial_{r} (g_{\varphi\varphi} \cot\theta \sin\varphi) - \frac{1}{2} S^{\theta\varphi} \partial_{\theta} (g_{\varphi\varphi} \cot\theta \sin\varphi)$$

$$J_y = P_{\theta} \cos\varphi - P_{\varphi} \cot\theta \sin\varphi + \frac{1}{2} S^{r\theta} \cos\varphi\, \partial_{r} g_{\theta\theta} + \frac{1}{2} S^{\varphi\theta} g_{\theta\theta}\, \partial_{\varphi} \cos\varphi$$
$$- \frac{1}{2} S^{r\varphi} \cot\theta \sin\varphi\, \partial_{r} g_{\varphi\varphi} - \frac{1}{2} S^{\theta\varphi} \sin\varphi\, \partial_{\theta} (g_{\varphi\varphi} \cot\theta)$$

$$J_y = P_\theta \cos\varphi - P_\varphi \cot\theta \sin\varphi + \frac{1}{2} S^{r\theta} \cos\varphi \, 2r - \frac{1}{2} S^{\varphi\theta} g_{\theta\theta} \sin\varphi$$
$$- \frac{1}{2} S^{r\varphi} \cot\theta \sin\varphi \, 2r \sin^2\theta - \frac{1}{2} S^{\theta\varphi} \sin\varphi \, r^2 \partial_\theta (\sin^2\theta \cot\theta)$$

$$J_y = P_\theta \cos\varphi - P_\varphi \cot\theta \sin\varphi + r S^{r\theta} \cos\varphi - \frac{r^2}{2} S^{\varphi\theta} \sin\varphi$$
$$- r S^{r\varphi} \sin\varphi \sin\theta \cos\theta - \frac{r^2}{2} S^{\theta\varphi} \sin\varphi \, \partial_\theta (\sin\theta \cos\theta)$$

$$J_y = P_\theta \cos\varphi - P_\varphi \cot\theta \sin\varphi + r S^{r\theta} \cos\varphi + \frac{r^2}{2} S^{\theta\varphi} \sin\varphi$$
$$- r S^{r\varphi} \sin\varphi \sin\theta \cos\theta - \frac{r^2}{2} S^{\theta\varphi} \sin\varphi \, (\cos^2\theta - \sin^2\theta)$$

$$J_y = P_\theta \cos\varphi - P_\varphi \cot\theta \sin\varphi + r S^{r\theta} \cos\varphi + \frac{r^2}{2} S^{\theta\varphi} \sin\varphi$$
$$- r S^{r\varphi} \sin\varphi \sin\theta \cos\theta - \frac{r^2}{2} S^{\theta\varphi} \sin\varphi \, (1 - 2\sin^2\theta)$$

$$\boxed{J_y = P_\theta \cos\varphi - P_\varphi \cot\theta \sin\varphi + r S^{r\theta} \cos\varphi + r S^{\varphi r} \sin\varphi \sin\theta \cos\theta + r^2 S^{\theta\varphi} \sin\varphi \sin^2\theta}$$

Since the background is spherically symmetric, it is possible to choose the z-axis in the direction of the total angular momentum. Then

$$(J_x, J_y, J_z) = (0, 0, J) \qquad J > 0$$

Hence, we have

$$J = P_\varphi + r \left[S^{r\varphi} + S^{\theta\varphi} r \cot\theta \right] \sin^2\theta \qquad (\mathrm{I})$$

$$0 = -P_\theta \sin\varphi - P_\varphi \cot\theta \cos\varphi + r^2 S^{\theta\varphi} \sin^2\theta \cos\varphi + r S^{\varphi r} \sin\theta \cos\theta \cos\varphi - r S^{r\theta} \sin\varphi \qquad (\mathrm{II})$$

$$0 = P_\theta \cos\varphi - P_\varphi \cot\theta \sin\varphi + r^2 S^{\theta\varphi} \sin^2\theta \sin\varphi + r S^{\varphi r} \sin\theta \cos\theta \sin\varphi + r S^{r\theta} \cos\varphi \qquad (\mathrm{III})$$

Solving with MATHEMATICA, we obtain

$$S^{\theta\phi} = \frac{J}{r^2}\cot\theta$$

$$S^{r\theta} = -\frac{P_\theta}{r}$$

$$S^{\varphi r} = \frac{1}{r}\left(-J + \frac{P_\varphi}{\sin^2\theta}\right)$$

Tulczyjew spin supplementary condition

We'll use the Tulczyjew spin-supplementary condition, $P_\mu S^{\mu\alpha} = 0$ to obtain:

$$P_t S^{t\alpha} + P_r S^{r\alpha} + P_\theta S^{\theta\alpha} + P_\varphi S^{\varphi\alpha} = 0$$

For $\alpha = t$

$$P_r S^{rt} + P_\theta S^{\theta t} + P_\varphi S^{\varphi t} = 0$$

For $\alpha = \theta$

$$P_t S^{t\theta} + P_r S^{r\theta} + P_\varphi S^{\varphi\theta} = 0$$

For $\alpha = r$

$$P_t S^{tr} + P_\theta S^{\theta r} + P_\varphi S^{\varphi r} = 0$$

For $\alpha = \varphi$

$$P_t S^{t\varphi} + P_r S^{r\varphi} + P_\theta S^{\theta\varphi} = 0$$

$$\begin{cases} P_r S^{tr} + P_\theta S^{t\theta} + P_\varphi S^{t\varphi} = 0 \\ P_t S^{tr} - P_\theta S^{r\theta} + P_\varphi S^{\varphi r} = 0 \\ P_t S^{t\theta} + P_r S^{r\theta} - P_\varphi S^{\theta\varphi} = 0 \\ P_t S^{t\varphi} - P_r S^{\varphi r} + P_\theta S^{\theta\varphi} = 0 \end{cases}$$

Solving this system with MATHEMATICA gives:

$$S^{tr} = -\frac{1}{r P_t}\left(P_\theta^2 + \frac{P_\varphi^2}{\sin^2\theta} - J P_\varphi\right)$$

$$S^{t\theta} = \frac{1}{r P_t}\left(P_r P_\theta + J \frac{P_\varphi}{r} \cot\theta\right)$$

$$S^{t\varphi} = -\frac{1}{r P_t}\left(J P_r - \frac{P_r P_\varphi}{\sin^2\theta} + \frac{J P_\theta}{r} \cot\theta\right)$$

Using these expressions in the energy relation, we easily obtain

$$E = -P_t - \frac{1}{2} S^{rt} \partial_r g_{tt} - \tilde{q}\Phi$$

$$E = -P_t - \frac{1}{2}\frac{1}{r P_t}\left(P_\theta^2 + \frac{P_\varphi^2}{\sin^2\theta} - J P_\varphi\right)\partial_r g_{tt} - \tilde{q}\Phi$$

$$\boxed{E = -P_t - \frac{1}{2 r P_t}\left(P_\theta^2 + \frac{P_\varphi^2}{\sin^2\theta} - J P_\varphi\right)\partial_r g_{tt} - \tilde{q}\Phi}$$