BLACK HOLES

OBSERVATORIO
ASTRONÓMICO
NACIONAL

Classical Black Holes

16. Geodesics around a Rotating Black Hole

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Outline for Part 1

Particle Motion around a Black Hole

Lagrangian Formulation

Lagrangian Formulation

Stationary and axis-symmetric spacetime

$$ds^2 = g_{tt}dt^2 + 2g_{t\varphi}dtd\varphi + g_{rr}dr^2 + g_{\theta\theta}d\theta^2 + g_{\varphi\varphi}d\varphi^2$$

Lagrangian for a particle moving in a spacetime defined by the metric $g_{\mu\nu}$

$$\mathscr{L} = \frac{1}{2}g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu$$

$$\dot{x}^\mu = \frac{dx^\mu}{d\lambda}$$

Lagrangian Formulation

$$\mathscr{L} = \frac{1}{2} g_{\mu\nu} \dot{x}^{\mu} \dot{x}^{\nu} = \frac{1}{2} \left(\frac{ds}{d\lambda} \right)^2$$

$$\mathscr{L} = \frac{1}{2} \left[g_{tt} \dot{t}^2 + 2 g_{t\varphi} \dot{t} \dot{\varphi} + g_{rr} \dot{r}^2 + g_{\theta\theta} \dot{\theta}^2 + g_{\varphi\varphi} \dot{\varphi}^2 \right] = \frac{1}{2} \delta$$

$$\delta = \begin{cases} 0 & \text{for photons} \\ -1 & \text{for massive particles} \end{cases}$$

Conserved Quantities

Conserved Quantities

$$\mathscr{L} = \frac{1}{2}\left[g_{tt}\dot{t}^2 + 2g_{t\varphi}\dot{t}\dot{\varphi} + g_{rr}\dot{r}^2 + g_{\theta\theta}\dot{\theta}^2 + g_{\varphi\varphi}\dot{\varphi}^2\right] = \frac{1}{2}\delta$$

$$p_t = \frac{\partial \mathscr{L}}{\partial \dot{t}} = \left[g_{tt}\dot{t} + g_{t\varphi}\dot{\varphi}\right] = -\varepsilon$$

$$p_\varphi = \frac{\partial \mathscr{L}}{\partial \dot{\varphi}} = \left[g_{t\varphi}\dot{t} + g_{\varphi\varphi}\dot{\varphi}\right] = \ell_z$$

Conserved Quantities

$$\varepsilon = \frac{E}{m_0} = -g_{tt}\dot{t} - g_{t\varphi}\dot{\varphi}$$
$$\ell_z = \frac{L_z}{m_0} = g_{t\varphi}\dot{t} + g_{\varphi\varphi}\dot{\varphi}$$

Conserved Quantities

$$\dot{t} = \frac{\varepsilon g_{\varphi\varphi} + \ell_z g_{t\varphi}}{g_{t\varphi}^2 - g_{tt} g_{\varphi\varphi}}$$

$$\dot{\varphi} = -\frac{\varepsilon g_{t\varphi} + \ell_z g_{tt}}{g_{t\varphi}^2 - g_{tt} g_{\varphi\varphi}}$$

Effective Potential

Effective Potential

$$\mathscr{L} = \frac{1}{2}\left[g_{tt}\dot{t}^2 + 2g_{t\varphi}\dot{t}\dot{\varphi} + g_{rr}\dot{r}^2 + g_{\theta\theta}\dot{\theta}^2 + g_{\varphi\varphi}\dot{\varphi}^2\right] = \frac{1}{2}\delta$$

$$g_{rr}\dot{r}^2 + g_{\theta\theta}\dot{\theta}^2 = V_{eff}(r,\theta)$$

$$V_{eff}(r,\theta) = \frac{\varepsilon^2 g_{\varphi\varphi} + 2\varepsilon\ell_z g_{t\varphi} + \ell_z^2 g_{tt}}{g_{t\varphi}^2 - g_{tt}g_{\varphi\varphi}} + \delta$$

Equations of Motion

Equations of Motion

$$\frac{d}{d\lambda}\left(\frac{\partial \mathscr{L}}{\partial \dot{x}^\alpha}\right) = \frac{\partial \mathscr{L}}{\partial x^\alpha}$$

$$\frac{d}{d\lambda}\left(g_{\mu\alpha}\dot{x}^\mu\right) = \frac{1}{2}\frac{\partial g_{\mu\nu}}{\partial x^\alpha}\dot{x}^\mu\dot{x}^\nu$$

Radial Equation of Motion

$$\frac{d}{d\lambda}\left(g_{\mu\alpha}\dot{x}^{\mu}\right) = \frac{1}{2}\frac{\partial g_{\mu\nu}}{\partial x^{\alpha}}\dot{x}^{\mu}\dot{x}^{\nu}$$

$$\frac{d}{d\lambda}\left(g_{\mu r}\dot{x}^{\mu}\right) = \frac{1}{2}\frac{\partial g_{\mu\nu}}{\partial r}\dot{x}^{\mu}\dot{x}^{\nu}$$

$$\frac{d}{d\lambda}(g_{rr}\dot{r}) = \frac{1}{2}\left[\partial_r g_{tt}\dot{t}^2 + 2\partial_r g_{t\varphi}\dot{t}\dot{\varphi} + \partial_r g_{rr}\dot{r}^2 + \partial_r g_{\theta\theta}\dot{\theta}^2 + \partial_r g_{\varphi\varphi}\dot{\varphi}^2\right]$$

θ-Equation of Motion

$$\frac{d}{d\lambda}\left(g_{\mu\alpha}\dot{x}^{\mu}\right) = \frac{1}{2}\frac{\partial g_{\mu\nu}}{\partial x^{\alpha}}\dot{x}^{\mu}\dot{x}^{\nu}$$

$$\frac{d}{d\lambda}\left(g_{\mu\theta}\dot{x}^{\mu}\right) = \frac{1}{2}\frac{\partial g_{\mu\nu}}{\partial \theta}\dot{x}^{\mu}\dot{x}^{\nu}$$

$$\frac{d}{d\lambda}\left(g_{\theta\theta}\dot{\theta}\right) = \frac{1}{2}\left[\partial_{\theta}g_{tt}\dot{t}^{2} + 2\partial_{\theta}g_{t\varphi}\dot{t}\dot{\varphi} + \partial_{\theta}g_{rr}\dot{r}^{2} + \partial_{\theta}g_{\theta\theta}\dot{\theta}^{2} + \partial_{\theta}g_{\varphi\varphi}\dot{\varphi}^{2}\right]$$

Kerr's Solution

Boyer-Lindquist coordinates: (t, r, θ, φ)

$$ds^2 = -\frac{\Delta - a^2 \sin^2\theta}{\varrho} dt^2 - \left(\frac{r^2 + a^2 - \Delta}{\varrho} \right) 2a \sin^2\theta dt d\varphi$$

$$+ \frac{\varrho}{\Delta} dr^2 + \varrho d\theta^2 + \left(\frac{\left(r^2 + a^2\right)^2 - \Delta a^2 \sin^2\theta}{\varrho} \right) \sin^2\theta d\varphi^2.$$

$$\varrho = r^2 + a^2 \cos^2\theta$$

$$\Delta = r^2 - 2Mr + a^2$$

θ-Equation of Motion

$$\partial_\theta g_{\mu\nu} \propto \sin\theta$$

$$\frac{d}{d\lambda}\left(g_{\theta\theta}\dot{\theta}\right) = \frac{1}{2}\left[\partial_\theta g_{tt}\dot{t}^2 + 2\partial_\theta g_{t\varphi}\dot{t}\dot{\varphi} + \partial_\theta g_{rr}\dot{r}^2 + \partial_\theta g_{\theta\theta}\dot{\theta}^2 + \partial_\theta g_{\varphi\varphi}\dot{\varphi}^2\right]$$

Hence we have a particular solution of this equation as

$$\begin{cases} \theta = & \frac{\pi}{2} \\ \dot{\theta} = & 0. \end{cases}$$

Equatorial Motion

Effective Potential in the Equatorial Plane

Considering equatorial motion, $\theta = \frac{\pi}{2}$ and $\dot{\theta} = 0$,

$$g_{rr}\dot{r}^2 = V_{eff}(r)$$

$$V_{eff}(r) = \frac{\varepsilon^2 g_{\varphi\varphi} + 2\varepsilon\ell_z g_{t\varphi} + \ell_z^2 g_{tt}}{g_{t\varphi}^2 - g_{tt}g_{\varphi\varphi}} + \delta$$

Schwarzschild Effective Potential in the Equatorial Plane

Considering massive particles, $\delta = -1$, moving in the equatorial plane, $\theta = \frac{\pi}{2}$, around Schwarzschild's geometry,

$$\frac{1}{2}\dot{r}^2 = \frac{\varepsilon^2 - 1}{2} - U_{eff}(r)$$

$$U_{eff}(r) = -\frac{M}{r} + \frac{\ell_z^2}{2r^2} - \frac{M\ell_z^2}{r^3}$$

Schwarzschild Effective Potential in the Equatorial Plane

$$U_{eff}(r) = -\frac{M}{r} + \frac{\ell_z^2}{2r^2} - \frac{M\ell_z^2}{r^3}$$

$-\frac{M}{r}$: Newtonian Potential

$\frac{\ell_z^2}{2r^2}$: Centrifugal Potential (repulsive)

Schwarzschild Effective Potential in the Equatorial Plane

$$U_{eff}(r) = -\frac{M}{r} + \frac{\ell_z^2}{2r^2} - \frac{M\ell_z^2}{r^3}$$

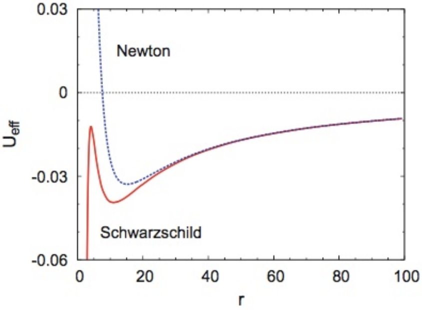
$-\frac{M}{r}$: Newtonian Potential

$\frac{\ell_z^2}{2r^2}$: Centrifugal Potential (repulsive)

$-\frac{M\ell_z^2}{r^3}$: Relativistic Contribution. Responsible for the ISCO

Schwarzschild Effective Potential in the Equatorial Plane

Fig. 3.1 Effective potential $U_{\rm eff}(r)$ for a test-particle moving in the gravitational field of a Schwarzschild *black hole* (*red solid curve*) and of a point-like mass in Newtonian gravity (*blue dashed curve*). Here $L_z = 3.9\ M$ and $M = 1$. See the text for more details



Equatorial Circular Orbits

Circular Motion in the Equatorial Plane

Circular motion is obtained by the conditions

$$\dot{r} = \dot{\theta} = 0$$

which imply

$$\begin{cases} V_{eff} = 0 \\ \partial_r V_{eff} = 0 \end{cases}$$

Equation of a Circular Motion in the Equatorial Plane

Another way to calculate the equation of a circular motion in the equatorial plane is taking

$$\frac{d}{d\lambda}\left(g_{rr}\dot{r}\right) = \frac{1}{2}\left[\partial_r g_{tt}\dot{t}^2 + 2\partial_r g_{t\varphi}\dot{t}\dot{\varphi} + \partial_r g_{rr}\dot{r}^2 + \partial_r g_{\theta\theta}\dot{\theta}^2 + \partial_r g_{\varphi\varphi}\dot{\varphi}^2\right]$$

Considering equatorial motion, $\theta = \frac{\pi}{2}$, in circular orbits, $\ddot{r} = \dot{r} = \dot{\theta} = 0$,

$$\partial_r g_{tt}\dot{t}^2 + 2\partial_r g_{t\varphi}\dot{t}\dot{\varphi} + \partial_r g_{\varphi\varphi}\dot{\varphi}^2 = 0$$

Angular Velocity of a Particle in Circular Motion in the Equatorial Plane

$$\partial_r g_{tt}\dot{t}^2 + 2\partial_r g_{t\varphi}\dot{t}\dot{\varphi} + \partial_r g_{\varphi\varphi}\dot{\varphi}^2 = 0$$

$$\partial_r g_{\varphi\varphi}\left(\frac{\dot{\varphi}}{\dot{t}}\right)^2 + 2\partial_r g_{t\varphi}\left(\frac{\dot{\varphi}}{\dot{t}}\right) + \partial_r g_{tt} = 0$$

$$\Omega = \frac{\dot{\varphi}}{\dot{t}} = \frac{-\partial_r g_{t\varphi} \pm \sqrt{\left(\partial_r g_{t\varphi}\right)^2 - \left(\partial_r g_{tt}\right)\left(\partial_r g_{\varphi\varphi}\right)}}{\partial_r g_{\varphi\varphi}}$$

Conserved Quantities for a Particle in Circular Motion

$$\mathscr{L} = \frac{1}{2}\left[g_{tt}\dot{t}^2 + 2g_{t\varphi}\dot{t}\dot{\varphi} + g_{rr}\dot{r}^2 + g_{\theta\theta}\dot{\theta}^2 + g_{\varphi\varphi}\dot{\varphi}^2\right] = \frac{1}{2}\delta$$

Considering $\dot{r} = \dot{\theta} = 0$,

$$\dot{t} = \sqrt{\frac{\delta}{g_{tt} + 2g_{t\varphi}\Omega + g_{\varphi\varphi}\Omega^2}}$$

Conserved Quantities for a Particle in Circular Motion

$$\varepsilon = -\left(g_{tt} + \Omega g_{t\varphi}\right)\sqrt{\frac{\delta}{g_{tt} + 2g_{t\varphi}\Omega + g_{\varphi\varphi}\Omega^2}}$$

$$\ell_z = -\left(g_{t\varphi} + \Omega g_{\varphi\varphi}\right)\sqrt{\frac{\delta}{g_{tt} + 2g_{t\varphi}\Omega + g_{\varphi\varphi}\Omega^2}}$$

Conserved Quantities for a Particle in Circular Motion in Kerr Spacetime

$$\varepsilon = \frac{r^{3/2} - 2Mr^{1/2} \pm aM^{1/2}}{r^{3/4}\sqrt{r^{3/2} - 3Mr^{1/2} \pm 2aM^{1/2}}}$$

$$\ell_z = \pm \frac{M^{1/2}\left(r^2 \mp 2aM^{1/2}r^{1/2} + a^2\right)}{r^{3/4}\sqrt{r^{3/2} - 3Mr^{1/2} \pm 2aM^{1/2}}}$$

$$\Omega_\pm = \pm \frac{M^{1/2}}{r^{3/2} \pm aM^{1/2}}$$

Upper sign: co-rotating orbit

Lower sign: counter-rotating orbit

Photon Motion

Photon Sphere

$$\varepsilon = -\left(g_{tt} + \Omega g_{t\varphi}\right)\sqrt{\frac{\delta}{g_{tt} + 2g_{t\varphi}\Omega + g_{\varphi\varphi}\Omega^2}}$$

For photons, $m_0 = 0$. Then $\varepsilon \to \infty$ and $\ell_z \to \infty$.

This occurs at the surface called *photon sphere*, with radius $r = r_{ps}$ such that

$$g_{tt} + 2g_{t\varphi}\Omega + g_{\varphi\varphi}\Omega^2 = 0$$

Photon Sphere

Scharzschild: $r_{ps} = 3M = \frac{3r_s}{2}$

Photon Sphere

Scharzschild: $r_{ps} = 3M = \frac{3r_s}{2}$

Kerr: $r_{ps} = 2M\left\{1 + \cos\left[\frac{2}{3}\cos^{-1}\left(\mp\frac{a}{M}\right)\right]\right\}$

Extreme Kerr $(M = a)$:

$$r_{ps} = \begin{cases} M & \text{co-rotating orbit} \\ 4M & \text{counter-rotating orbit} \end{cases}$$

Massive Particle Motion

Conserved Quantities for a Massive Particle in Circular Motion

Taking $\delta = -1$,

$$\varepsilon = -\left(g_{tt} + \Omega g_{t\phi}\right)\sqrt{-\frac{1}{g_{tt} + 2g_{t\varphi}\Omega + g_{\varphi\varphi}\Omega^2}}$$

$$\ell_z = -\left(g_{t\varphi} + \Omega g_{\varphi\varphi}\right)\sqrt{-\frac{1}{g_{tt} + 2g_{t\varphi}\Omega + g_{\varphi\phi}\Omega^2}}$$

Marginally Bound Orbit

- For $r > r_{ps}$ not all circular orbits are bound.
- Orbits with $\varepsilon \geq 1$ are unbound.

This means that giving an infinitesimal outward perturbation to a particle in this orbit, it will escape to infinity on an asymptotically hyperbolic (parabolic for the equal sign) trajectory.

The condition $\varepsilon = 1$ defines the marginally bound circular orbit radius, $r = r_{mb}$.

$$\varepsilon = -\left(g_{tt} + \Omega g_{t\varphi}\right)\sqrt{-\frac{1}{g_{tt} + 2g_{t\varphi}\Omega + g_{\varphi\varphi}\Omega^2}} = 1$$

Marginally Bound Orbit

Scharzschild: $r_{mb} = 4M$

Marginally Bound Orbit

Scharzschild: $r_{mb} = 4M$

Kerr: $r_{mb} = 2M \mp a + 2\sqrt{M}(M \mp a)$

Extreme Kerr $(M = a)$:

$$r_{mb} = \begin{cases} M & \text{co-rotating orbit} \\ 5.83M & \text{counter-rotating orbit} \end{cases}$$

Marginally Stable Orbit. ISCO

The Marginally Stable Orbit, a.k.a. the Innermost Stable Circular Orbit (ISCO), has a radius $r = r_{ISCO}$ defined by the conditions

$$\partial_r V_{eff} = 0$$
$$\partial_r^2 V_{eff} = 0$$

Circular orbits at $r < r_{ISCO}$ are unstable.

Therefore, the ISCO radius corresponds to the inner edge of thin accretion disks (such as in the Novikov-Thorne model)

Marginally Stable Orbit. ISCO

Scharzschild: $r_{ISCO} = 6M$

Marginally Stable Orbit. ISCO

Scharzschild: $r_{ISCO} = 6M$

Kerr: $r_{ISCO} = 3M + Z_2 \mp \sqrt{(3M - Z_1)(3M + Z_1 + 2Z_2)}$

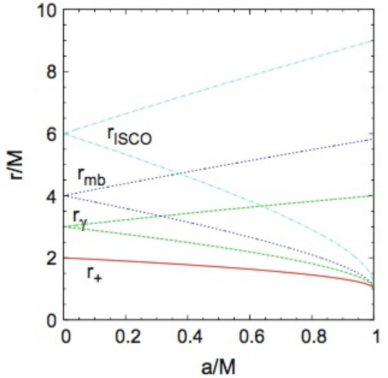
$Z_1 = M + (M^2 - a^2)^{1/3}[\,(M + a)^{1/3} + (M - a)^{1/3}]$
$Z_2 = \sqrt{3a^2 + Z_1^2}$

Extreme Kerr $(M = a)$:

$$r_{ISCO} = \begin{cases} M & \text{co-rotating orbit} \\ 9M & \text{counter-rotating orbit} \end{cases}$$

Important Radii for Circular Orbits in the Equatorial Plane of Kerr Spacetime

Fig. 3.4 Radial coordinates of the event horizon r_+, of the photon orbit r_γ, of the marginally bound circular orbit r_{mb}, and of the ISCO r_{ISCO} in the Kerr metric in Boyer–Lindquist coordinates as functions of a/M. For every radius, the upper curve refers to the counterrotating orbit, the lower curve to the corotating orbit



Outline for Part 2

Hamilton-Jacobi Formulation

Hamilton-Jacobi Formulation

$$\mathscr{L} = \frac{1}{2} g_{\mu\nu} \dot{x}^{\mu} \dot{x}^{\nu}$$

$$p_{\mu} = \frac{\partial \mathscr{L}}{\partial \dot{x}^{\mu}} = g_{\mu\nu} \dot{x}^{\nu}$$

$$\mathscr{H} = \frac{1}{2} g^{\mu\nu} p_{\mu} p_{\nu}$$

Hamilton-Jacobi Formulation

Hamilton's principal function

$$S = S\left(x^{\mu}; \lambda\right)$$

$$p_{\mu} = \frac{\partial S}{\partial x^{\mu}}$$

Hamilton-Jacobi Equation

$$\frac{1}{2} g^{\mu\nu} \frac{\partial S}{\partial x^{\mu}} \frac{\partial S}{\partial x^{\nu}} = \frac{\partial S}{\partial \lambda}$$

Kerr's Solution

Boyer-Lindquist coordinates: (t, r, θ, φ)

$$
\begin{aligned}
ds^2 = &-\frac{\Delta - a^2 \sin^2\theta}{\varrho} dt^2 - \left(\frac{r^2 + a^2 - \Delta}{\varrho}\right) 2a \sin^2\theta dt d\varphi \\
&+ \frac{\varrho}{\Delta} dr^2 + \varrho d\theta^2 + \left(\frac{\left(r^2 + a^2\right)^2 - \Delta a^2 \sin^2\theta}{\varrho}\right) \sin^2\theta d\varphi^2 .
\end{aligned}
$$

$$\varrho = r^2 + a^2 \cos^2\theta$$

$$\Delta = r^2 - 2Mr + a^2$$

Kerr's Solution

$$\left(\frac{\partial}{\partial s}\right)^2 = -\frac{A}{\varrho\Delta}\left(\frac{\partial}{\partial t}\right)^2 - \frac{4aMr}{\varrho\Delta}\left(\frac{\partial}{\partial t}\right)\left(\frac{\partial}{\partial\varphi}\right) + \frac{\Delta}{\varrho}\left(\frac{\partial}{\partial r}\right)^2 + \frac{1}{\varrho}\left(\frac{\partial}{\partial\theta}\right)^2 + \frac{\Delta - a^2\sin^2\theta}{\varrho\Delta\sin^2\theta}\left(\frac{\partial}{\partial\varphi}\right)^2$$

$$A = (r^2 + a^2)^2 - a^2\Delta\sin^2\theta$$

$$\varrho = r^2 + a^2\cos^2\theta$$

$$\Delta = r^2 - 2Mr + a^2$$

Hamilton-Jacobi Formulation

Hamilton-Jacobi Equation

$$2\frac{\partial S}{\partial \lambda} = g^{\mu\nu}\frac{\partial S}{\partial x^{\mu}}\frac{\partial S}{\partial x^{\nu}}$$

$$\begin{aligned}2\frac{\partial S}{\partial \lambda} =& -\frac{A}{\varrho\Delta}\left(\frac{\partial S}{\partial t}\right)^2 - \frac{4aMr}{\varrho\Delta}\left(\frac{\partial S}{\partial t}\right)\left(\frac{\partial S}{\partial \varphi}\right) + \frac{\Delta}{\varrho}\left(\frac{\partial S}{\partial r}\right)^2 \\ &+ \frac{1}{\varrho}\left(\frac{\partial S}{\partial \theta}\right)^2 + \frac{\Delta - a^2\sin^2\theta}{\varrho\Delta\sin^2\theta}\left(\frac{\partial S}{\partial \varphi}\right)^2\end{aligned}$$

Hamilton-Jacobi Formulation

Hamilton Principal Function

$$S = \frac{1}{2}\lambda\delta - \varepsilon t + \ell_z\varphi + S_r(\theta) + S_\theta(\theta)$$

Separation of the Hamilton-Jacobi Equation: **Carter Constant**

$$\Delta\left(\frac{dS_r}{dr}\right)^2 - \frac{1}{\Delta}\left[(r^2+a^2)\varepsilon - a\ell_z\right]^2 + (\ell_z - a\varepsilon)^2 + \delta r^2 =$$
$$-\left(\frac{dS_\theta}{d\theta}\right)^2 - \left(\frac{\ell_z^2}{\sin^2\theta} - a^2\varepsilon^2 + \delta a^2\right)\cos^2\theta = \mathscr{C}$$

Hamilton-Jacobi Formulation

Separation of the Hamilton-Jacobi Equation. Carter Constant

$$\begin{cases} \Delta \left(\frac{dS_r}{dr} \right)^2 & = \frac{1}{\Delta} \left[(r^2 + a^2)\varepsilon - a\ell_z \right]^2 - \left[\mathscr{C} + (\ell_z - a\varepsilon)^2 + \delta r^2 \right] \\ \left(\frac{dS_\theta}{d\theta} \right)^2 & = \mathscr{C} - \left(\frac{\ell_z^2}{\sin^2\theta} - a^2\varepsilon^2 + \delta a^2 \right) \cos^2\theta \end{cases}$$

Hamilton-Jacobi Formulation

$$S_r = \int \frac{\sqrt{R(r')}}{\Delta} dr'$$

$$S_\theta = \int \sqrt{\Theta(\theta')} d\theta'$$

$$R(r) = \left[(r^2 + a^2)\varepsilon - a\ell_z\right]^2 - \Delta\left[\mathscr{C} + (\ell_z - a\varepsilon)^2 + \delta r^2\right]$$

$$\Theta(\theta) = \mathscr{C} - \left[\frac{\ell_z^2}{\sin^2\theta} + a^2\left(\delta - \varepsilon^2\right)\right]\cos^2\theta$$

Meaning of the Carter Constant

$$\left(\frac{dS_\theta}{d\theta}\right)^2 = \mathscr{C} - \left(\frac{\ell_z^2}{\sin^2\theta} - a^2\varepsilon^2 + \delta a^2\right)\cos^2\theta$$

$$\mathscr{C} = p_\theta^2 + p_\varphi^2\cot^2\theta + a^2(\delta - \varepsilon^2)\cos^2\theta$$

Carter Constant

Schwarzschild:

$$\mathscr{C} = \left(p_\theta^2 + \frac{p_\varphi^2}{\sin^2\theta} \right) - p_\varphi^2 = \ell^2 - \ell_z^2$$

where $\ell = p_\theta^2 + \frac{p_\varphi^2}{\sin^2\theta}$ is the total angular momentum.

Carter Constant

Kerr:

- $\mathscr{C}$ has not a direct physical interpretation.
- $\mathscr{C} = 0$ implies that the motion is in the equatorial plane.

Equations of Motion

Equations of Motion

Hamilton Canonical Equations

$$\dot{x}^\mu = p^\mu = g^{\mu\nu} p_\nu = g^{\mu\nu} \frac{\partial S}{x^\nu}$$

Equations of Motion

$$\begin{cases} \varrho^2 \dot{r}^2 & = R \\ \varrho^2 \dot{\theta}^2 & = \Theta \\ \varrho \dot{\varphi} & = \frac{1}{\Delta} \left[2aMr\varepsilon + (\varrho - 2Mr) \frac{\ell_z}{\sin^2\theta} \right] \\ \varrho \dot{t} & = \frac{1}{\Delta} \left[A\varepsilon + 2aMr\ell_z \right] \end{cases}$$

$$\begin{cases} R(r) & = \left[(r^2 + a^2)\varepsilon - a\ell_z \right]^2 - \Delta \left[\mathscr{C} + (\ell_z - a\varepsilon)^2 + \delta r^2 \right] \\ \Theta(\theta) & = \mathscr{C} - \left[\frac{\ell_z^2}{\sin^2\theta} + a^2 (\delta - \varepsilon^2) \right] \cos^2\theta \end{cases}$$

$$A = (r^2 + a^2)^2 - a^2 \Delta \sin^2\theta$$

$$\varrho = r^2 + a^2 \cos^2\theta$$

$$\Delta = r^2 - 2Mr + a^2$$

Imaging a Black Hole

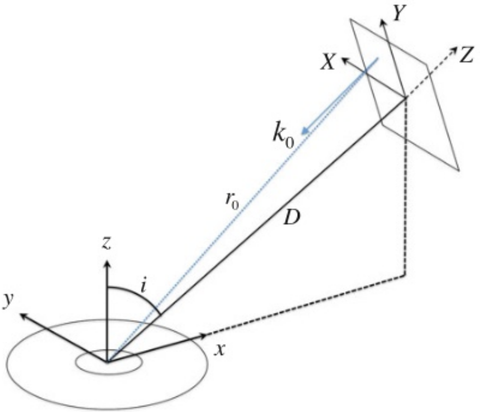
Image Plane of a Distant Observer

Image Plane of a Distant Observer

- A distant observer receives the electromagnetic radiation from the accretion disk, around the black hole.
- It is usual to define a plane of observation and consider the photons with momentum orthogonal to the plane. These photons' trajectories are integrated backwards in time to find the position of the emission point in the disk.

Image Plane of a Distant Observer

Fig. 3.5 The Cartesian coordinates (x, y, z) are centered at the black hole, while the Cartesian coordinates (X, Y, Z) are for the image plane of the distant observer, who is located at the distant D from the *black hole* and with the inclination angle i. From [1]



Coordinate Systems

- (X, Y, Z): Cartesian coordinates in the image plane
- (x, y, z): Cartesian coordinates centered at the black hole.
- i: Inclination angle of the observer with respect to the z direction.
- D: Distance observer-black hole.

Coordinate transformations

$$\begin{cases} x &= D\sin i - Y\cos i + Z\sin i \\ y &= X \\ z &= D\cos i + Y\sin i + Z\cos i \end{cases}$$

$$\begin{cases} r &= \sqrt{x^2 + y^2 + z^2} \\ \theta &= \arccos\left(\frac{z}{r}\right) \\ \varphi &= \arctan\left(\frac{z}{r}\right) \end{cases}$$

Photon Trajectory Tracing

Photon Trajectory Tracing

Consider a photon received at $(X_0, Y_0, 0)$ with 3-momentum $\mathbf{k}_0 = -k_0\hat{Z}$, i.e. perpendicular to the observer plane.
The initial conditions for the position of the photon (to trace back the trajectory), as seen from the black hole and in spherical coordinates, are

$$
\begin{aligned}
t_0 &= 0 \\
r_0 &= \sqrt{X_0^2 + Y_0^2 + D^2} \\
\theta_0 &= \arccos\left(\frac{Y_0 \sin i + D \cos i}{r_0}\right) \\
\varphi_0 &= \arctan\left(\frac{X_0}{D \sin i - Y_0 \cos i}\right)
\end{aligned}
$$

Photon Trajectory Tracing

The initial conditions for the 4-momentum of the photon k^μ (to trace back the trajectory), as seen from the black hole and in spherical coordinates, are calculated with the transformation law,

$$k^\mu = \frac{\partial x^\mu}{\partial \bar{x}^\alpha}\bar{k}^\alpha$$

$$k_0^r = -\frac{D}{r}k_0$$

$$k_0^\theta = \frac{\cos i - (Y_0 \sin i + D\cos i)\frac{D}{r_0^2}}{\sqrt{X_0^2 + (D\sin i - Y_0\cos i)^2}}k_0$$

$$k_0^\varphi = \frac{X_0 \sin i}{X_0^2 + (D\sin i - Y_0\cos i)^2}k_0$$

Photon Trajectory Tracing

The k_0^t component of the initial 4-momentum is calculated by the condition $g_{\mu\nu} g^{\mu} k'^{\nu} = 0$,

$$k_0^t = \sqrt{(k_0^r)^2 + r_0^2 (k_0^\theta)^2 + r_0^2 \sin^2 \theta_0 (k_0^\varphi)^2}$$

Photon Trajectory Tracing

Given the initial conditions for position and momentum, it is possible to trace the trajectory of any photon in the observer plane back to the accretion disk.

Non-Coordinate Basis

Tetrads

Introduce a non-coordinate basis or orthonormal tetrad,

$$\mathbf{E}_{(a)} = E^{\mu}_{(a)}\partial_{\mu}$$
$$\mathbf{E}^{(a)} = E^{(a)}_{\mu}dx^{\mu},$$

subject to the conditions

$$\eta_{(a)(b)} = E^{\mu}_{(a)}E^{\nu}_{(b)}g_{\mu\nu}$$
$$\eta^{(a)(b)} = E^{(a)}_{\mu}E^{(b)}_{\nu}g\mu\nu$$

and $det|E_{(a)^{\mu}}| > 0$ (to preserve the orientation).

Tetrads

Components of a vector in the orthonormal tetrad basis,

$$V^{(a)} = E^{(a)}_{\mu} V^{\mu}$$
$$V_{(a)} = E^{\mu}_{(a)} V_{\mu}$$

General Metric

Consider a general stationary, axisymmetric, asymptotically flat metric

$$ds^2 = -e^{2\alpha(r)}dt^2 + e^{2\beta(r)}dr^2 + e^{2\gamma(r,\theta)}d\theta^2 + e^{2\epsilon(r,\theta)}(d\varphi - \omega dt)^2$$

We identify the *locally non-rotating observers* as those whose world-lines have

$$\begin{cases} r &= \text{constant} \\ \theta &= \text{constant} \\ \varphi &= \omega t + \text{constant} \end{cases}$$

General Metric

The non-coordinate basis of the locally non-rotating observers is given by the tetrad

$$\begin{cases} E^{\mu}_{(t)} & = \left(e^{-\beta}, 0, 0, \omega e^{-\beta}\right) \\ E^{\mu}_{(r)} & = \left(0, e^{-\alpha}, 0, 0\right) \\ E^{\mu}_{(\theta)} & = \left(0, 0, e^{-\gamma}, 0\right) \\ E^{\mu}_{(\varphi)} & = \left(0, 0, 0, e^{-\epsilon}\right) \end{cases}$$

General Metric

and the dual basis,

$$\begin{cases} E_\mu^{(t)} & = \left(e^\beta, 0, 0, 0\right) \\ E_\mu^{(r)} & = (0, e^\alpha, 0, 0) \\ E_\mu^{(\theta)} & = (0, 0, e^\gamma, 0) \\ E_\mu^{(\varphi)} & = (-\omega e^\epsilon, 0, 0, e^\epsilon) \end{cases}$$

Kerr Metric

For the particular case of the Kerr metric the tetrad describing the non-coordinate basis of the locally non-rotating observers is

$$
\begin{cases}
E^{\mu}_{(t)} &= \left(\sqrt{\frac{A}{\varrho\Delta}}, 0, 0, \frac{2Mar}{\sqrt{A\varrho\Delta}}\right) \\
E^{\mu}_{(r)} &= \left(0, \sqrt{\frac{\Delta}{\varrho}}, 0, 0\right) \\
E^{\mu}_{(\theta)} &= \left(0, 0, \frac{1}{\sqrt{\varrho}}, 0\right) \\
E^{\mu}_{(\varphi)} &= \left(0, 0, 0, \frac{1}{\sin\theta}\sqrt{\frac{\varrho}{A}}\right)
\end{cases}
$$

Kerr Metric

and the dual basis is

$$\begin{cases} E^{(t)}_{\mu} & = \left(\sqrt{\frac{\varrho\Delta}{A}}, 0, 0, 0\right) \\ E^{(r)}_{\mu} & = \left(0, \sqrt{\frac{\varrho}{\Delta}}, 0, 0\right) \\ E^{(\theta)}_{\mu} & = \left(0, 0, \sqrt{\varrho}, 0\right) \\ E^{(\varphi)}_{\mu} & = \left(-\frac{2Mar\sin\theta}{\sqrt{\varrho A}}, 0, 0, \sqrt{\frac{A}{\varrho}}\sin\theta\right) \end{cases}$$

Momentum Components in the Non-Coordinate Basis

The momentum components of a particle moving in Kerr's spacetime are

$$p_\mu = \frac{\partial S}{\partial x^\mu}$$

$$\begin{cases} p_t &= -\varepsilon \\ p_r &= \frac{\sqrt{R}}{\Delta} \\ p_\theta &= \sqrt{\Theta} \\ p_\varphi &= \ell_z \end{cases}$$

Momentum Components in the Non-Coordinate Basis

In the non-coordinate basis, the momentum components of a particle are

$$p^{(a)} = E^{(a)}_{\mu} p^{\mu} = \eta^{(a)(b)} E^{\mu}_{(b)} p_{\mu}$$

$$\begin{cases} p^{(t)} &= -E^{\mu}_{(t)} p_{\mu} \\ p^{(r)} &= E^{\mu}_{(r)} p_{\mu} \\ p^{(\theta)} &= E^{\mu}_{(\theta)} p_{\mu} \\ p^{(\varphi)} &= E^{\mu}_{(\varphi)} p_{\mu} \end{cases}$$

Celestial Coordinates

Celestial Coordinates

The position of a photon in the image plane of the distant observer is given by the coordinates

$$\begin{cases} X_0 & = \alpha = \lim_{r\to\infty} \left(\frac{rp^{(\varphi)}}{p^{(t)}} \right) \\ Y_0 & = \beta = \lim_{r\to\infty} \left(\frac{rp^{(\theta)}}{p^{(t)}} \right) \end{cases}$$

Celestial Coordinates

$$\begin{cases} \alpha &= -\xi \csc i \\ \beta &= \pm\sqrt{\eta + a^2 \cos^2 i - \xi^2 \cot^2 i} \end{cases}$$

$$\begin{cases} \xi &= \frac{\ell_z}{\varepsilon} \\ \eta &= \frac{\mathscr{C}}{\varepsilon^2} \end{cases}$$

Black Hole's Shadow

Effective Potential for Photons

$$\begin{cases} R(r) &= \left[(r^2+a^2)\varepsilon - a\ell_z\right]^2 - \Delta\left[\mathscr{C} + (\ell_z - a\varepsilon)^2 + \delta r^2\right] \\ \Theta(\theta) &= \mathscr{C} - \left[\frac{\ell_z^2}{\sin^2\theta} + a^2\left(\delta - \varepsilon^2\right)\right]\cos^2\theta \end{cases}$$

Consider these expressions for photons, $\delta = 0$, and using the quantities

$$\begin{cases} \xi &= \frac{\ell_z}{\varepsilon} \\ \eta &= \frac{\mathscr{C}}{\varepsilon^2} \end{cases}$$

Effective Potential for Photons

$$\begin{cases} R(r) & = \left[r^2 + a^2 - a\xi\right]^2 \varepsilon^2 - \Delta\left[\eta + (\xi - a)^2\right]\varepsilon^2 \\ \Theta(\theta) & = \eta\varepsilon^2 - \left[\frac{\xi^2}{\sin^2\theta} - a^2\right]\varepsilon^2 \cos^2\theta \end{cases}$$

Effective Potential for Photons

$$\begin{cases} \mathscr{R}(r) & = \frac{R(r)}{\varepsilon^2} = \left[r^2 + a^2 - a\xi\right]^2 - \Delta\left[\eta + (\xi - a)^2\right] \\ \vartheta(\theta) & = \frac{\Theta(\theta)}{\varepsilon^2} = \left[\eta + (\xi - a)^2\right] - \left[a\sin\theta - \xi\csc\theta\right]^2 \end{cases}$$

$$\Delta = r^2 - 2Mr + a^2$$

Equations of Motion for Photons

$$\begin{cases} \varrho^2 \dot{r}^2 & = \mathscr{R} \\ \varrho^2 \dot{\theta}^2 & = \vartheta \\ \varrho \dot{\varphi} & = \frac{1}{\Delta} \left[2aMr + \frac{\xi(\varrho - 2Mr)}{\sin^2 \theta} \right] \\ \varrho \dot{t} & = \frac{1}{\Delta} \left[A + 2aMr\xi \right] \end{cases}$$

Photon Sphere

Circular motion of photons: $\dot{r} = 0$

$$\begin{cases} \mathscr{R} & = 0 \\ \partial_r \mathscr{R} & = 0 \end{cases}$$

Photon Sphere

Solving for ξ and η we obtain these quantities for the circular orbit as functions of the parameter r,

$$\begin{cases} \xi_c & = \frac{M(r^2-a^2)-r\Delta}{a(r-M)} \\ \eta_c & = \frac{r^3\left[4M\Delta-r(r-M)^2\right]}{a^2(r-M)^2} \end{cases}$$

Photon Sphere

There are three possible cases regarding the stability of circular orbits of photons

1. If $\partial_r^2 \mathscr{R} > 0$: Stable circular orbits
2. If $\partial_r^2 \mathscr{R} < 0$: Unstable circular orbits. The photon straddles the boundary between two regions: $\partial_r \mathscr{R} = 0$; if perturbed one way it falls into the horizon, if perturbed the other way it flies outwards.
3. If $\partial_r^2 \mathscr{R} = 0$: Marginally stable circular orbit (Photon Sphere). $r = r_{ps}$.

Black Hole's Shadow

$$\begin{cases} \alpha &= -\xi_c \csc i \\ \beta &= \sqrt{\eta_c + a^2 \cos^2 i - \xi_c^2 \cot^2 i} \end{cases}$$

Black Hole's Shadow

$$\begin{cases} \alpha & = -\xi_c \csc i \\ \beta & = \sqrt{\eta_c + a^2 \cos^2 i - \xi_c^2 \cot^2 i} \end{cases}$$

$$\begin{cases} \xi_c & = \frac{M(r^2 - a^2) - r\Delta}{a(r-M)} \\ \eta_c & = \frac{r^3\left[4M\Delta - r(r-M)^2\right]}{a^2(r-M)^2} \end{cases}$$

Black Hole's Shadow

Schwarzschild's Black Hole:

$$\alpha^2 + \beta^2 = R^2_{shadow}$$

$$R_{shadow} = \sqrt{27M^2}$$

Shadow of Sagittarius A*

$$M_{SgrA*} = 4 \times 10^6 M_\odot$$

$$r_H = 2M = \frac{2GM}{c^2}$$
$$r_H = 1.2 \times 10^{10} \text{ m.} = 1.2 \times 10^7 \text{ km.}$$
$$r_H = 0.1 \text{ AU}$$

$$R_{shadow} = \sqrt{27M^2} = 3\sqrt{3}M$$
$$R_{shadow} = 3\sqrt{3}\frac{GM}{c^2} = 3 \times 10^{10} \text{ m.}$$
$$R_{shadow} = 3 \times 10^7 \text{ km.} = 0.2 \text{ AU}$$

Shadow of Sagittarius A*

$R_{shadow} = 0.2\ \mathrm{AU} = 9.7 \times 10^{-10}\ \mathrm{kpc}$

$D = 8\ \mathrm{kpc}$

$\theta_{shadow} = \frac{R_{shadow}}{D}$
$\theta_{shadow} = 1.2 \times 10^{-11}\ \mathrm{rad}$
$\theta_{shadow} = 2.5\ \mu\text{arcsec}$

Shadow of Sagittarius A*

Very Long Baseline Interferometry (VLBI)

$\theta_{res} \sim \frac{\lambda}{d}$

$\lambda \sim 1 \ \mathrm{mm}$
$d \sim 10^3 \ \mathrm{km}$

$\theta_{res} \sim 10 \ \mu\mathrm{arcsec}$

Next Lecture

08. Accretion

Alternative derivation of the Celestial Coordinates

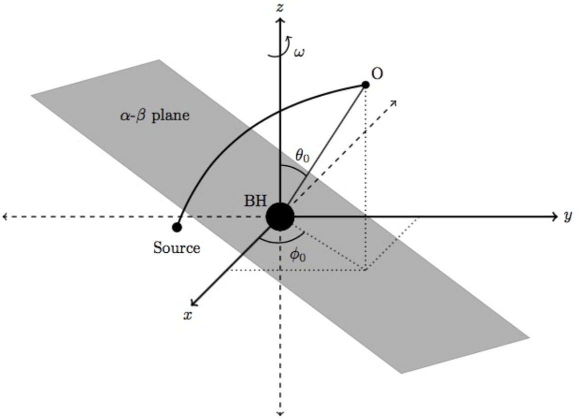
Celestial Coordinates

- A distant observer receives the electromagnetic radiation from the surroundings of the black hole.
- Let the observer be at $r \to \infty$ with inclination angle i (between the rotation axis of the black hole and the observer's line of sight).
- The celestial coordinates (α, β) are the apparent angular distances of the image on the celestial sphere measured by the observer.

Celestial Coordinates

- The distant observer may set up a euclidean coordinate system (x, y, z) with the black hole at the origin and its rotation axis along z.
- Using the Boyer-Lindquist coordinates describing the black hole, the observer will be located at some coordinates $(r_0, \theta_0, \varphi_0)$, with r_0 very large.

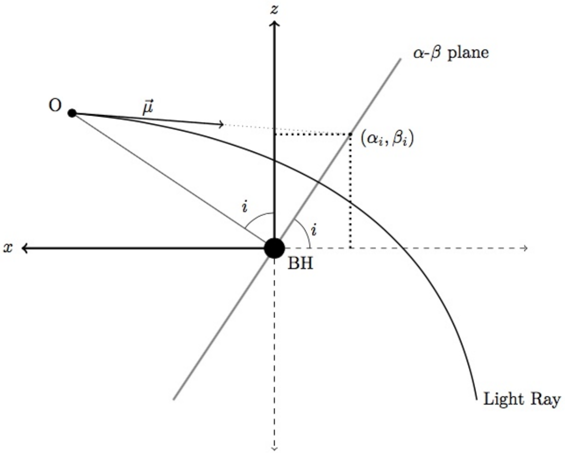
Celestial Coordinates



Celestial Coordinates

- Note that the inclination angle is precisely $\theta_0 = i$. Hence the position of the observer is (r_0, i, φ_0)
- Without loosing generality, we rotate the (x, y, z) system so that the angular position of the observer in the Boyer-Lindquist coordinates is $\varphi_0 = 0$.
- The position of the observer becomes $(r_0, i, 0)$
- Then, the observer lies in the $x - z$ plane while the y-axis lies in the (α, β) plane (remember that the observer plane is perpendicular to the line of sight).

Celestial Coordinates



Trajectory of a Light Ray

In the observer's reference frame, an incoming light ray trajectory may be decribed by a parametric curve

$$\begin{cases} X = X(r) \\ Y = Y(r) \\ Z = Z(r) \end{cases}$$

such that

$$r^2 = X^2(r) + Y^2(r) + Z^2(r)$$

Trajectory of a Light Ray

The tangent vector to this parametric curve at the observer's location is

$$\vec{\mu} = (\mu_1, \mu_2, \mu_3) = \left(\left.\frac{dX}{dr}\right|_{r_0}, \left.\frac{dY}{dr}\right|_{r_0}, \left.\frac{dZ}{dr}\right|_{r_0} \right)$$

Trajectory of a Light Ray

From the point of view of the observer, this tangent vector defines the trajectory of the photon as a straight line which intersects the observer's celestial plane at the coordinates (α_i, β_i) and can be written parametrically as

$$\frac{x - x_0}{\mu_1} = \frac{y - y_0}{\mu_2} = \frac{z - z_0}{\mu_3}$$

Trajectory of a Light Ray

(x_0, y_0, z_0) are the coordinates of the observer's position.
This can be written as

$$(x_0, y_0, z_0) = (r_0 \sin i, 0, r_0 \cos i)$$

The celestial coordinates (α_i, β_i) can be written as

$$(\alpha_i, \beta_i) = (x_i, y_i, z_i) = (-\beta_i \cos i, \alpha_i, \beta_i \sin i)$$

Trajectory of a Light Ray

$$\frac{x - x_0}{\mu_1} = \frac{y - y_0}{\mu_2} = \frac{z - z_0}{\mu_3}$$

$$\frac{-\beta_i \cos i - r_0 \sin i}{\mu_1} = \frac{\alpha_i}{\mu_2} = \frac{\beta_i \sin i - r_0 \cos i}{\mu_3}$$

Trajectory of a Light Ray

Using the transformation between cartesian and spherical coordinates,

$$\begin{cases} X(r) &= r\sin\theta\cos\varphi \\ Y(r) &= r\sin\theta\sin\varphi \\ Z(r) &= r\cos\theta, \end{cases}$$

we obtain the components of $\vec{\mu}$,

$$\begin{cases} \mu_1 &= \left.\frac{dX}{dr}\right|_{r_0} = \sin i + r_0\cos i \left.\frac{d\theta}{dr}\right|_{r_0} \\ \mu_2 &= \left.\frac{dY}{dr}\right|_{r_0} = r_0\sin i \left.\frac{d\varphi}{dr}\right|_{r_0} \\ \mu_3 &= \left.\frac{dZ}{dr}\right|_{r_0} = \cos i - r_0\sin i \left.\frac{d\theta}{dr}\right|_{r_0} \end{cases}$$

Trajectory of a Light Ray

$$\frac{-\beta_i \cos i - r_0 \sin i}{\mu_1} = \frac{\alpha_i}{\mu_2} = \frac{\beta_i \sin i - r_0 \cos i}{\mu_3}$$

$$\frac{-\beta_i \cos i - r_0 \sin i}{\sin i + r_0 \cos i \left.\frac{d\theta}{dr}\right|_{r_0}} = \frac{\alpha_i}{r_0 \sin i \left.\frac{d\varphi}{dr}\right|_{r_0}} = \frac{\beta_i \sin i - r_0 \cos i}{\cos i - r_0 \sin i \left.\frac{d\theta}{dr}\right|_{r_0}}$$

Celestial Coordinates

$$\begin{cases} \alpha_i & = \lim_{r_0 \to \infty} -r_0^2 \sin^2 \theta_0 \left. \frac{d\varphi}{dr} \right|_{r_0} \\ \beta_i & = \lim_{r_0 \to \infty} r_0^2 \left. \frac{d\theta}{dr} \right|_{r_0} \end{cases}$$

Celestial Coordinates

Using the equations of motion for the photons, we obtain

$$\begin{cases} \alpha_i &= -\xi \csc i \\ \beta_i &= \sqrt{\eta + a^2 \cos^2 i - \xi^2 \cot^2 i} \end{cases}$$