SHADOW OF A STATIC BLACK HOLE

When a black hole is surrounded either by an optically thin emitting medium or by a Novikov-Thorne accretion disk, the image will have a dark area over a brighter background (called the 'shadow' of the black hole).

Consider the physical system including the black hole and the image plane of distant observer. If a photon with a 3-momentum vector orthogonal to the image plane is fired inside the shadow region, it will be captured by the black hole. On the other hand, a photon that is fired outside that region, will be scattered back to infinity.

The 'photon capture sphere' is the surface (in general it may not be a sphere) separating scattered and captured photons. Typically, this surface is the photon sphere but its image in the observer's plane will be larger due to light bending.

Shadow for a general spherically symmetric spacetime

$$ds^2 = -A(r)\,dt^2 + B(r)\,dr^2 + r^2 d\Omega^2$$

$$\lim_{r\to\infty} A(r) = \lim_{r\to\infty} B(r) = 1$$

The apparent size of the central object, as seen by the distance observer, will be determined by the photon impact parameter that separates captured and scattered photons.
Due to the spherical symmetry, we restrict the calculations to the plan $\theta = \frac{\pi}{2}$.
The conserved quantities in this system are

$$\mathcal{E} = A(r)\,\dot{t}$$
$$l_z = r^2 \dot{\phi}$$

Using these quantities in the photon Lagrangian gives

$$L = \frac{1}{2} g_{\mu\nu} \dot{x}^\mu \dot{x}^\nu = 0$$

$$-A(r)\dot{t}^2 + B(r)\,\dot{r}^2 + r^2\dot{\phi}^2 = 0$$

$$\dot{r}^2 = \frac{1}{B(r)}\left(A(r)\,\dot{t}^2 - r^2\dot{\phi}^2\right)$$

$$\dot{r}^2 = \frac{1}{B(r)}\left[\frac{\mathcal{E}^2}{A(r)} - \frac{\ell_z^2}{r^2}\right]$$

When the photon reaches a turning point ($\dot{r}=0$) before crossing the horizon, it will come back to the infinity. The turning point is given by the condition

$$\frac{\mathcal{E}^2}{A(r)} - \frac{\ell_z^2}{r^2} = 0$$

$$b \equiv \frac{\ell_z}{\mathcal{E}} = \frac{r}{\sqrt{A(r)}} \qquad \text{Impact parameter}$$

We will find the minimum value for the impact parameter, b_c, which is given by the condition

$$\left.\frac{db}{dr}\right|_{r_c} = \frac{1}{\sqrt{A(r_c)}} - \frac{1}{2} r_c \left[A(r_c)\right]^{-3/2} A'(r_c) \equiv 0$$

$$A(r_c) - \frac{1}{2} r_c A'(r_c) = 0$$

For Schwarzschild spacetime, $\quad A(r) = 1 - \frac{2M}{r}$

$$A'(r) = \frac{2M}{r^2}$$

Hence,

$$1 - \frac{2M}{r_c} - \frac{1}{2} r_c \left(\frac{2M}{r_c^2}\right) = 0$$

$$1 - \frac{2M}{r_c} - \frac{M}{r_c} = 0$$

$$r_c - 3M = 0$$

$$\longrightarrow \quad r_c = 3M = r_{ps}$$

Then, the minimum impact parameter (size of the shadow) is

$$b_c = \frac{r_c}{\sqrt{1-\frac{2M}{r_c}}} = \frac{3M}{\sqrt{1-\frac{2}{3}}} = \frac{3M}{\sqrt{\frac{1}{3}}}$$

$$b_c = b_{shadow} = 3\sqrt{3}\,M \approx 5.196\ M$$