BLACK HOLES

OBSERVATORIO
ASTRONÓMICO
NACIONAL

Classical Black Holes

06. Rotating Black Holes

Edward Larrañaga

Outline for Part 1

Kerr’s Solution

Boyer-Lindquist coordinates: (t, r, θ, φ)

$$ds^2 = -\frac{\Delta - a^2 \sin^2\theta}{\varrho} dt^2 - \left(\frac{r^2 + a^2 - \Delta}{\varrho}\right) 2a \sin^2\theta dt d\varphi$$

$$+ \frac{\varrho}{\Delta} dr^2 + \varrho d\theta^2 + \left(\frac{\left(r^2 + a^2\right)^2 - \Delta a^2 \sin^2\theta}{\varrho}\right) \sin^2\theta d\varphi^2.$$

$$\varrho = r^2 + a^2 \cos^2\theta$$

$$\Delta = r^2 - 2Mr + a^2$$

The Kerr-Newman Family

$$
\begin{aligned}
ds^2 = &-\frac{\Delta - a^2 \sin^2\theta}{\varrho} dt^2 - \left(\frac{r^2 + a^2 - \Delta}{\varrho}\right) 2a \sin^2\theta dt d\varphi \\
&+ \frac{\varrho}{\Delta} dr^2 + \varrho d\theta^2 + \left(\frac{\left(r^2 + a^2\right)^2 - \Delta a^2 \sin^2\theta}{\varrho}\right) \sin^2\theta d\varphi^2
\end{aligned}
$$

$$
\begin{aligned}
\varrho &= r^2 + a^2 \cos^2\theta \\
\Delta &= r^2 - 2Mr + a^2 + e^2
\end{aligned}
$$

The Kerr-Newman Family

$$a = \frac{J}{M}$$

$$e = \sqrt{Q^2 + P^2}$$

Q: Electric charge

P: Magnetic monopole charge

The electromagnetic 4-potential is

$$A = \frac{Qr\left[dt - a\sin^2\theta d\varphi\right] - P\cos\theta\left[adt - \left(r^2 + a^2\right)d\varphi\right]}{\varrho}$$

Kerr's Solution

$$
\begin{aligned}
ds^2 = &-\frac{\Delta - a^2\sin^2\theta}{\varrho}dt^2 - \left(\frac{r^2+a^2-\Delta}{\varrho}\right)2a\sin^2\theta dt d\varphi \\
&+ \frac{\varrho}{\Delta}dr^2 + \varrho d\theta^2 + \left(\frac{\left(r^2+a^2\right)^2 - \Delta a^2\sin^2\theta}{\varrho}\right)\sin^2\theta d\varphi^2.
\end{aligned}
$$

$$\varrho = r^2 + a^2\cos^2\theta$$

$$\Delta = r^2 - 2Mr + a^2$$

$$a = \frac{J}{M}$$

Killing Vectors

$$\xi = \frac{\partial}{\partial t}$$

Asymptotically timelike vector

$$\zeta = \frac{\partial}{\partial \varphi}$$

Asymptotically spacelike vector

Kerr Black Hole Singularities

$$\varrho = r^2 + a^2 \cos^2 \theta = 0$$

$$\Delta = r^2 - 2Mr + a^2 = 0$$

Kerr Black Hole Singularities

$$\varrho = r^2 + a^2 \cos^2\theta = 0$$

$$r = 0 \quad , \quad \theta = \frac{\pi}{2}$$

This is an *essential singularity* as is probed by the Kretschmann scalar,

$$K = \frac{48M^2}{\varrho^6}\left[r^6 - 15a^2r^4\cos^2\theta + 15a^4r^2\cos^4\theta - a^6\cos^6\theta\right]$$

Kerr Black Hole Singularities

$$\Delta = r^2 - 2Mr + a^2 = 0$$

$$\Delta = (r - r_+)(r - r_-) = 0$$

$$r_\pm = M \pm \sqrt{M^2 - a^2}$$

These are *coordinate singularities*

Eddington-Finkelstein Coordinates

$$(t, r, \theta, \phi) \longrightarrow (v, r, \theta, \chi)$$

$$dv = dt + \frac{\left(r^2 + a^2\right)}{\Delta} dr$$

$$d\chi = d\varphi + \frac{a}{\Delta} dr$$

$$ds^2 = -\frac{\Delta - a^2 \sin^2\theta}{\varrho} dv^2 + 2dvdr - \frac{2a\sin^2\theta\left(r^2 + a^2 - \Delta\right)}{\varrho} dvd\chi$$
$$-2a\sin^2\theta drd\chi + \varrho d\theta^2 + \frac{\left(r^2+a^2\right)^2 - \Delta a^2 \sin^2\theta}{\varrho}\sin^2\theta d\chi^2$$

Kerr Black Hole Singularities

$$r_\pm = M \pm \sqrt{M^2 - a^2}$$

Case I: $M < a$

- Both roots $r_\pm$ are complex
- Δ has no real zeros
- There are no coordinate singularities
- The essential singularity $\varrho = 0$ exists

Kerr-Schild's Coordinates

Case I: $M < a$

Kerr-Schild's coordinates: $\left(\tilde{t}, x, y, z\right)$

$$
\begin{aligned}
\tilde{t} &= \int \left[dt + \frac{r^2 + a^2}{\Delta} dr \right] - r \\
x + iy &= (r + ia) \sin\theta e^{i \int \left[d\varphi + \frac{a}{\Delta} dr \right]} \\
z &= r \cos\theta
\end{aligned}
$$

Kerr-Schild's Coordinates

Case I: $M < a$

$$ds^2 = -d\tilde{t}^2 + dx^2 + dy^2 + dz^2 + \frac{2Mr^3}{r^4 + a^2z^2}\left[\frac{r\left(xdx + ydy\right) - a\left(xdx - ydy\right)}{r^2 + a^2} + \frac{zdz}{r} + d\tilde{t}^2\right]^2$$

$M = 0$: Kerr's becomes Minkowski's space.

Kerr-Schild's Coordinates

Case I: $M < a$

$r =$ constant $\neq 0$ gives

$$\frac{x^2}{r^2} + \frac{y^2}{r^2} + \left(1 + \frac{a^2}{r^2}\right)\frac{z^2}{r^2} = 1 + \frac{a^2}{r^2}$$

- Ellipsoids with foci at $x = \pm a$
- These ellipsoids degenerate into the disk $\left\{z = 0, x^2 + y^2 \leq a^2\right\}$ for $r = 0$

The Essential Singularity in Kerr-Schild's Coordinates

Case I: $M < a$

Essential singularity of Kerr's metric: $\varrho = 0$

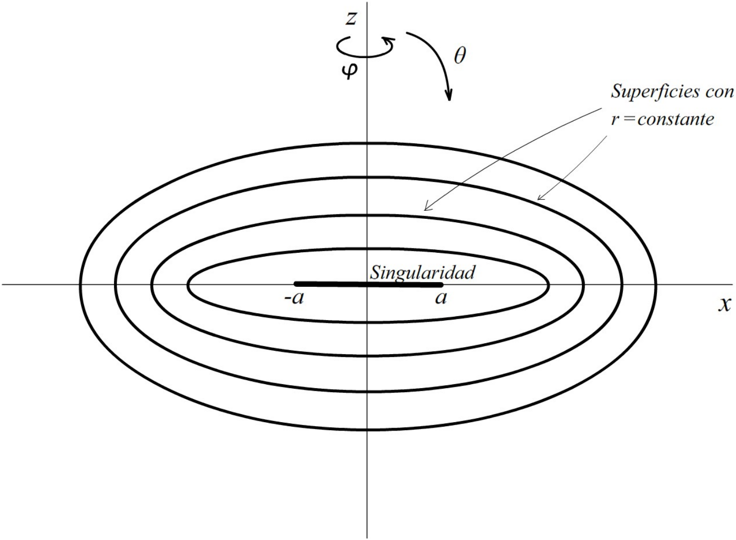
$$r = 0 \quad , \quad \theta = \frac{\pi}{2}$$

$$x^2 + y^2 = a^2$$

Ring with radius a centered at the origin

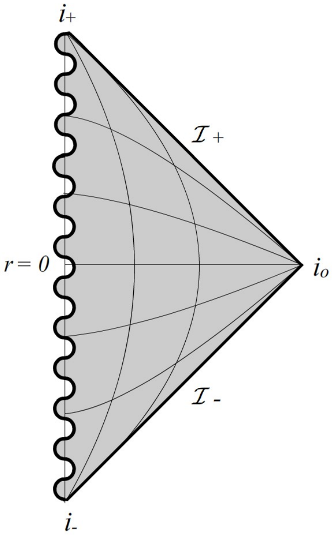
Surfaces of $r = \mathrm{constant}$ in Kerr's metric

Case I: $M < a$



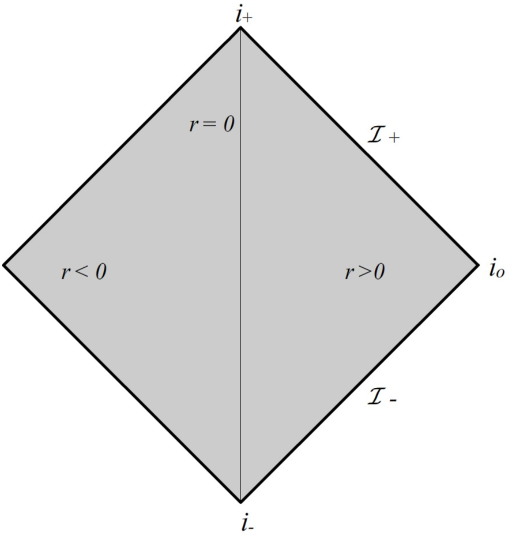
Carter-Penrose Diagram. $\theta = \frac{\pi}{2}$

Case I: $M < a$



Carter-Penrose Diagram. $\theta = 0$

Case I: $M < a$



Causal Structure near the Singularity

Case I: $M < a$

- The cosmic censorship hypotesis rules out the Case 1 of Kerr's metric
- Another reason to consider it as non-physical is the Causal structure near the essential singularity.

Causal Structure near the Singularity

Case I: $M < a$

$$\zeta = \frac{\partial}{\partial \varphi}$$

This vector field has closed orbits

$$\zeta^2 = g_{\mu\nu}\zeta^\mu\zeta^\nu = g_{\varphi\varphi}$$

$$\zeta^2 = \left(\frac{\left(r^2 + a^2\right)^2 - \Delta a^2 \sin^2\theta}{\varrho} \right) \sin^2\theta$$

Causal Structure near the Singularity

Case I: $M < a$

In the neighborhood of the ring singularity: $\frac{r}{a} = \delta << 1$ and $\theta = \frac{\pi}{2}$

$$\zeta^2 = a^2 + \frac{Ma}{\delta} + O\left(\delta^2\right)$$

For points near the singularity in the region with negative r, we have $\delta < 0$.

The Killing vector may have a negative magnitude, $\zeta^2 < 0$, i.e. it can be timelike.

Causal Structure near the Singularity

Case I: $M < a$

Since ξ has closed orbits, this fact permit the existence of closed timelike curves.

Violation of Causality!

Kerr Black Hole Singularities

$$r_{\pm} = M \pm \sqrt{M^2 - a^2}$$

Case II: $M > a$

- The essential ring singularity $\varrho = 0$ is dressed with the coordinate singularities at $r = r_\pm$
- The function Δ is positive for $r > r_+$ and $r < r_-$, but it is negative for $r_- < r < r_+$
- The double change in sign makes the singularity $r = 0$ timelike, just as in case I.

Killing Horizons

Case II: $M > a$

The hypersurfaces $r = r_\pm$ are Killing horizons of the Killing vector fields

$$\psi_\pm = \frac{\partial}{\partial v} + \left(\frac{a}{r_\pm^2 + a^2} \right) \frac{\partial}{\partial \chi}$$

Killing Horizons

Case II: $M > a$

$$\Phi_\pm = r - r_\pm$$

$\Phi_\pm = 0$ corresponds to the hypersurfaces $r = r_\pm$.

Normal vectors:

$$\mathbf{n}_\pm = N_\pm g^{\mu\nu} \partial_\mu \Phi \partial_\nu$$

$$\mathbf{n}_\pm = N_\pm \left[g^{rr} \partial_r + g^{rv} \partial_v + g^{r\chi} \partial_\chi \right]$$

Killing Horizons

Case II: $M > a$

$$g^{vv} = \frac{a^2 \sin^2\theta}{\varrho} \qquad g^{vr} = \frac{a^2+r^2}{\varrho}$$

$$g^{v\chi} = \frac{a}{\varrho} \qquad g^{r\chi} = \frac{a}{\varrho}$$

$$g^{rr} = \frac{\Delta}{\varrho} \qquad g^{\theta\theta} = \frac{1}{\varrho}$$

$$g^{\chi\chi} = \frac{\csc^2\theta}{\varrho}$$

$$\mathbf{n}_\pm = \frac{N_\pm}{\varrho}\left[\Delta\partial_r + \left(a^2 + r^2\right)\partial_v + a\partial_\chi\right]$$

Killing Horizons

Case II: $M > a$

Magnitude of the normal vector

$$\begin{aligned}\mathbf{n}_{\pm}^2 &= \tfrac{N_{\pm}^2}{\varrho^2} \quad \Bigg[-\left(\frac{\Delta - a^2\sin^2\theta}{\varrho}\right)\left(a^2+r^2\right)^2 + \frac{(r^2+a^2)^2 - \Delta a^2 \sin^2\theta}{\varrho} a^2\sin^2\theta \\ &\qquad +2\Delta\left(a^2+r^2\right) - \frac{2a^2\sin^2\theta\,(r^2+a^2-\Delta)}{\varrho}\left(a^2+r^2\right) - 2a^2\Delta\sin^2\theta\Bigg]\end{aligned}$$

At $r = r_{\pm}$ we have

$$\mathbf{n}_{\pm}^2\Big|_{r=r_{\pm}} = 0$$

i.e. these are null hypersurfaces.

Killing Horizons

Case II: $M > a$

Evaluating the normal vector at $r = r_\pm$ gives

$$\mathbf{n}_\pm\big|_{r=r_\pm} = N_\pm \left(\frac{a^2 + r_\pm^2}{r_\pm^2 + a^2 \cos^2\theta} \right) \psi_\pm$$

Surface Gravity

Case II: $M > a$

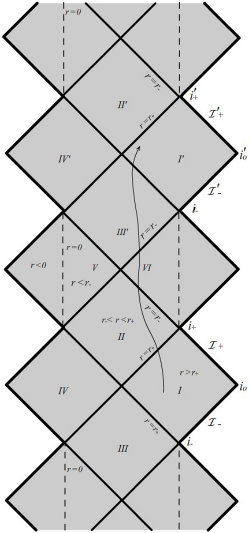
$$n^{\sigma}\nabla_{\sigma}n^{\mu}\big|_{\mathcal{N}} = 0$$

$$\xi^{\sigma}\nabla_{\sigma}\xi^{\mu}\big|_{\mathcal{N}} = \kappa\xi^{\mu}$$

$$\kappa_{\pm} = \frac{r_{\pm} - r_{\mp}}{2\left(a^2 + r_{\pm}^2\right)}$$

Carter-Penrose Diagram. $\theta = \frac{\pi}{2}$ and $\theta = 0$

Case II: $M > a$



Outline for Part 2

Angular Velocity of the Black Hole

Case II: $M > a$

The Killing vector ψ_+ in Boyer-Lindquist's coordinates is

$$\psi_+ = \frac{\partial}{\partial t} + \Omega \frac{\partial}{\partial \varphi}$$

where

$$\Omega = \frac{a}{a^2 + r_+^2}$$

Angular Velocity of the Black Hole

Case II: $M > a$

$$\psi_+^\mu \partial_\mu [\varphi - \Omega t] = 0$$

Orbits of this Killing vector ψ_+:

$$\varphi - \Omega t = \text{constante}$$

$$\varphi = \Omega t + \text{constante}$$

Angular Velocity of the Black Hole

Case II: $M > a$

Particles moving in orbits of ψ_+ are rotating with the angular velocity Ω with respect to asymptotic observers at rest.

$$\Omega = \frac{a}{a^2 + r_+^2}$$

$$\Omega = \frac{J}{2M\left(M^2 + \sqrt{M^4 - J^2}\right)}$$

The Ergosphere

Case II: $M > a$

Killing Vector $\xi = \frac{\partial}{\partial t}$

$$\xi^2 = g_{\mu\nu}\xi^\mu\xi^\nu = g_{tt} = -\left(\frac{\Delta - a^2\sin^2\theta}{\varrho}\right)$$

At $r = r_\pm$:

$$\xi^2\Big|_{r=r_\pm} = \frac{a^2\sin^2\theta}{r_\pm^2 + a^2\cos^2\theta} \neq 0$$

The Ergosphere

Case II: $M > a$

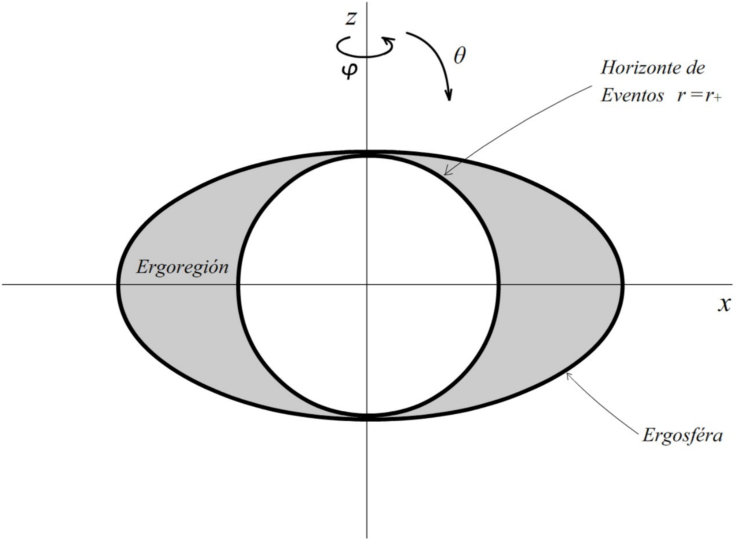
$$\xi^2 = g_{\mu\nu}\xi^\mu\xi^\nu = g_{tt} = -\left(\frac{\Delta - a^2\sin^2\theta}{\varrho}\right) = 0$$

$$r_e = M + \sqrt{M^2 - a^2\cos^2\theta}$$

Ergosphere

The Ergosphere

Case II: $M > a$



Dragging of inertial frames

Case II: $M > a$

Photons moving in the equatorial plane

$$ds^2 = 0 = g_{tt}dt^2 + 2g_{t\varphi}dtd\varphi + g_{\varphi\varphi}d\varphi^2$$

The velocity of the photons is

$$\frac{d\varphi}{dt} = -\frac{g_{t\varphi}}{g_{\varphi\varphi}} \pm \sqrt{\left(\frac{g_{t\varphi}}{g_{\varphi\varphi}}\right)^2 - \frac{g_{tt}}{g_{\varphi\varphi}}}$$

Dragging of inertial frames

Case II: $M > a$

The velocity of the photons at the ergosphere is

$$\left.\frac{d\varphi}{dt}\right|_{r=r_e} = \begin{cases} \frac{a}{Mr_e + a^2 \sin^2\theta} \\ 0 \end{cases}$$

Conserved Quantities for particle motion

Case II: M > a

Energy

$$E = -\xi^{\mu} p_{\mu} = -\xi^{t} p_{t}$$

$$E = -m \frac{dt}{d\tau} g_{tt} - m \frac{d\varphi}{d\tau} g_{t\varphi}$$

Angular Momentum

$$L = -\left[\frac{a \sin^2 \theta \, (r^2 + a^2 - \Delta)}{\varrho} \right] m \frac{dt}{d\tau} + \left[\frac{(r^2 + a^2) - \Delta a^2 \sin^2 \theta}{\varrho} \right] \sin^2 \theta m \frac{d\varphi}{d\tau}$$

Penrose's Process

Case II: $M > a$

Energy

$E = -\xi^{\mu} p_{\mu} > 0$: Outside the ergosphere

$E = -\xi^{\mu} p_{\mu} < 0$: Inside the ergosphere

Penrose's Process

Case II: $M > a$

Consider a system composed by two particles initially in the asymptotic region and moving towards the black hole.

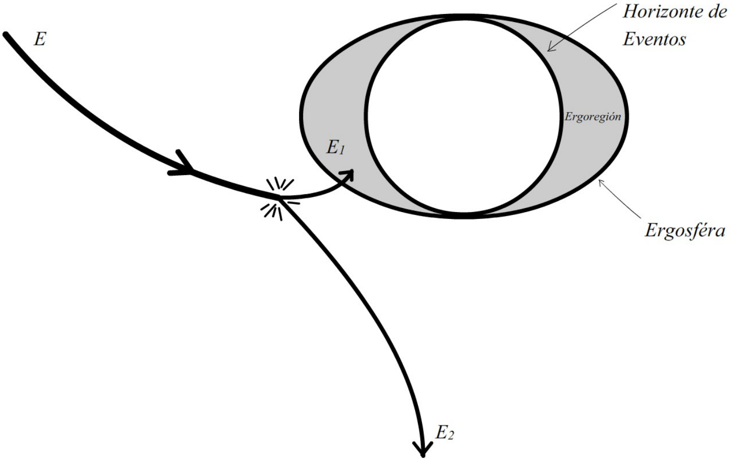
Let p^o_μ be the initial 4-momentum of the composed system.

The initial energy is positive and given by

$$E^o = -\xi^\mu p^o_\mu$$

Penrose’s Process

Case II: $M > a$



Penrose’s Process

Case II: $M > a$

When the system is near the ergosphere, the system splits such that one of the particles goes into the ergoregion while the other escapes to the infinity.

The conservation of the 4-momentum gives

$$p_\mu^o = p_\mu^1 + p_\mu^2$$

p_μ^1 is the 4-momentum of the particle that goes into the ergoregion and p_μ^2 is the 4-momentum of the particle that escapes.

Penrose's Process

Case II: $M > a$

Contracting this equation with ξ, we obtain

$$E^o = E^1 + E^2$$

The energy of the particle inside is negative, $E^1 < 0$

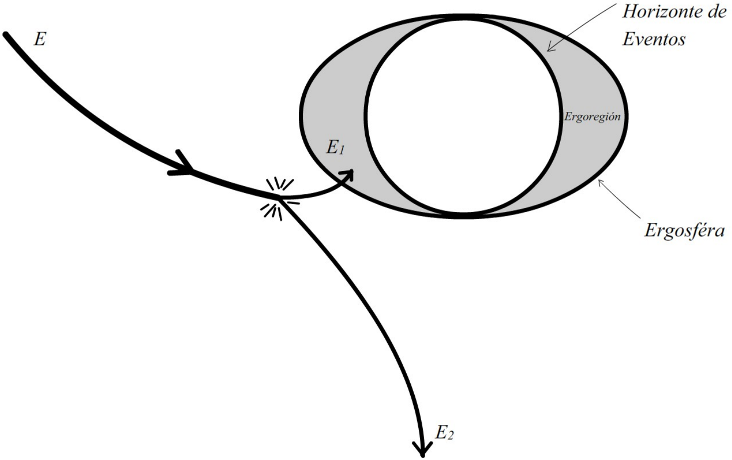
The energy of the particle that escapes is greater than the initial energy:

$$E^2 > E^o$$

i.e. the process extracts energy from the black hole.

Penrose's Process

Case II: $M > a$



Origin of the Extracted Energy

Case II: $M > a$

$$\psi_+ = \xi + \Omega\zeta$$

Is null (directed to the future) at the horizon $r = r_+$ and timelike outside this surface.

$\psi_+^\mu p_\mu$ is negative or null outside the horizon,

$$-\psi_+^\mu p_\mu \geq 0$$

Origin of the Extracted Energy

Case II: $M > a$

Replacing ψ_+ and contracting

$$E - \Omega L \geq 0$$

$$L \leq \frac{E}{\Omega}$$

Origin of the Extracted Energy

Case II: $M > a$

For particle 1 (going inside the ergosphere)

$$L^1 \leq \frac{E^1}{\Omega}$$

Since $E^1 < 0$, the angular momentum of this particle is negative,

$$L^1 < 0$$

The particle that goes inside diminishes the total angular momentum of the black hole, giving the energy extracted by the Penrose process.

Limit for the Extraction of Energy

Case II: $M > a$

Once we have extracted an amount of energy E^1, the black hole reaches, after some appropriate period of time, a new state of equilibrium in which its new mass is $M + \delta M$ and its new angular momentum is $J + \delta J$, where

$$\begin{aligned} \delta M &= E^1 \\ \delta J &= L^1 \end{aligned}$$

Limit for the Extraction of Energy

Case II: $M > a$

The relation between these quantities is as given above,

$$\delta J \leq \frac{\delta M}{\Omega}$$

or

$$\delta J \leq \frac{2M\left[M^2 + \sqrt{M^4 - J^2}\right]\delta M}{J}$$

Limit for the Extraction of Energy

Case II: $M > a$

$$\delta J \leq \frac{2M\left[M^2 + \sqrt{M^4 - J^2}\right]\delta M}{J}$$

$$\delta\left[M^2 + \sqrt{M^4 - J^2}\right] \geq 0$$

Area of the Event Horizon

Case II: $M > a$

Area of the Event Horizon

$$A_H = \int_0^{\pi} \int_0^{2\pi} \left[\sqrt{\left| g_{\theta\theta} g_{\varphi\varphi} \right|} \right]_{r=r_+} d\theta d\varphi$$

$$A_H = 8\pi \left[M^2 + \sqrt{M^4 - J^2} \right]$$

Limit for the Extraction of Energy

Case II: $M > a$

$$\delta \left[M^2 + \sqrt{M^4 - J^2} \right] \geq 0$$

$$\delta A_H \geq 0$$

Kerr Black Hole Singularities

$$r_\pm = M \pm \sqrt{M^2 - a^2}$$

Case III: $M = a$

- Extremal Kerr's metric: $M = a$
- $r_+ = r_- = M$

$$\Delta = (r - M)^2$$

Eddington-Finkelstein's coordinates
Case III: $M = a$

Eddington-Finkelstein's coordinates

$$\begin{aligned} dv &= dt + \frac{\left(r^2 + M^2\right)}{(r-M)^2} dr \\ d\chi &= d\varphi + \frac{M}{(r-M)^2} dr \end{aligned}$$

$$\begin{aligned} ds^2 = & -\frac{r^2 - 2Mr + M^2\cos^2\theta}{\varrho} dv^2 + 2dvdr - \frac{4M^2 r\sin^2\theta}{\varrho} dvd\chi \\ & -2M\sin^2\theta drd\chi + \varrho d\theta^2 + \frac{(r^2+M^2)^2 - (r-M)^2 M^2 \sin^2\theta}{\varrho}\sin^2\theta d\chi^2 \end{aligned}$$

Killing Vectors

Case III: $M = a$

$$\xi = \frac{\partial}{\partial v}$$

$$\zeta = \frac{\partial}{\partial \chi}$$

Killing Horizon

The hypersurface $r = M$ is a degenerate Killing horizon (i.e. with $\kappa = 0$) of the vector

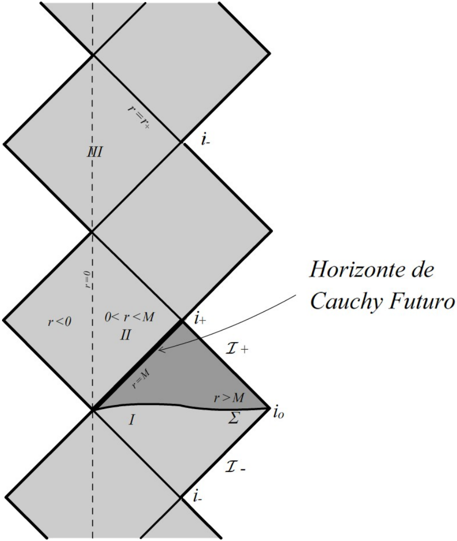
$$\psi = \xi + \Omega\zeta$$

Angular velocity:

$$\Omega = \frac{a}{2M^2} = \frac{1}{2M}$$

Carter-Penrose Diagram. $\theta = 0$ and $\theta = \frac{\pi}{2}$

Case III: $M = a$



Next Lecture

07. Black Holes Astrophysics