Schwarzschild's Limit

$$\begin{cases} G(r) = G_0 = 1 & \longrightarrow \quad G'(r) = 0 \\ a = 0 \end{cases}$$

$$K_1 := r^3 + a^2 r - as[r + M(G(r) - rG'(r))]$$

$$K_1 = r^3$$

$$K_2 := Ms(G(r) - rG'(r)) - ar$$

$$K_2 = Ms$$

$$X = K_1 E + K_2 J$$

$$X = r^3 E + Ms J$$

$$Y = J - (a-s)E$$

$$Y = sE + J$$

$$Z = r^3 - Ms^2(G(r) - rG'(r))$$

$$Z = r^3 - Ms^2$$

$$R = r^2[K_1^2 - \Delta r^2(a-s)^2]E^2 + r^2[2K_1K_2 J + 2\Delta r^2(a-s)J]E$$
$$+ [r^2 K_2^2 J^2 - \Delta r^4 J^2 - \Delta m^2 Z^2]$$

$$\Delta = r^2 + a^2 - 2Mr \equiv \Theta$$

$$rZ = r^4 - Ms^2 r \equiv \Xi$$

$$P^0 = \frac{1}{\Delta Z}\left[(r^2+a^2)X + ar\Delta Y\right]$$

$$P^0 = \frac{1}{\Delta Z}\left[(r^2+a^2)(r^3E + MsJ) + ar(r^2+a^2-2Mr)(sE+J)\right]$$

$$P^0 = \frac{1}{\Delta Z}\left[E\left(r^5 + r^3a^2 + ars(r^2+a^2-2Mr)\right) + J\left(Msr^2 + Ma^2s + ar(r^2+a^2-2Mr)\right)\right]$$

$$P^0 = \frac{1}{\Delta r Z}\left[r^2aJ2Mr - Er^6 - r^2a^2E(2Mr+r^2) + (J-3aE)Mr^3s\right.$$
$$\left. + a^2(aE-J)(-Mrs)\right]$$

$$P^0 = \frac{1}{\Delta r Z}\left[E\left(-r^6 - 2Mr^3a^2 - r^4a^2 - 3aMr^3s - Ma^3rs\right)\right.$$
$$\left. + J\left(2Mr^3a + Mr^3s + Ma^2rs\right)\right]$$

$$P^0 = \frac{1}{\Delta Z}\left[E\left(-r^5 - 2Mr^2a^2 - r^3a^2 - 3aMr^2s - Ma^3s\right)\right.$$
$$\left. + J\left(2Mr^2a + Mr^2s + Ma^2s\right)\right]$$

$$P^1 = \pm\frac{\sqrt{R}}{rZ}$$

$$P^2 = 0$$

$$P^3 = \frac{1}{\Delta Z}\left[aX + r\Delta Y\right]$$

$$\alpha = r^2\left[K_1^2 - \Delta r^2 (a-s)^2\right]$$

$$\alpha = r^2\left[(r^3)^2 - (r^2 - 2Mr)\, r^2 (-s)^2\right]$$

$$\alpha = r^2\left[r^6 - s^2 r^3 (r-2M)\right]$$

$$\alpha = r^5\left[r^3 - s^2 (r-2M)\right]$$

$$\beta = r^2\left[2K_1K_2\, j + 2\Delta r^2 (a-s)\, j\right]$$

$$\beta = r^2\left[2r^3 Ms\, j + 2(r^2 - 2Mr)\, r^2 (-s)\, j\right]$$

$$\beta = 2r^2 j\left[Msr^3 - s r^3 (r-2M)\right]$$

$$\beta = 2r^5 sj\left[M - (r-2M)\right]$$

$$\beta = 2r^5 sj\,(3M - r)$$

$$\gamma = r^2 j^2 (K_2^2 - r^2\Delta) - \Delta Z^2$$

$$\gamma = r^2 j^2\left[(Ms)^2 - r^2 (r^2 - 2Mr)\right] - (r^2 - 2Mr)(r^3 - Ms^2)^2$$

$$\gamma = r^2 j^2\left[M^2 s^2 - r^3 (r-2M)\right] - r(r-2M)(r^3 - Ms^2)^2$$

For $s = 0$:

$$\alpha = r^8$$

$$\beta = 0$$

$$\gamma = r^2 j^2\left[-r^3 (r-2M)\right] - r(r-2M)(r^3)^2$$

$$\gamma = -r^5 (r-2M)\, j^2 - r^7 (r-2M)$$

$$\gamma = -r^5 (r-2M)(j^2 + r^2)$$

$$V_{eff}(r) = \frac{-\beta + \sqrt{\beta^2 - 4\alpha\gamma}}{2\alpha}$$

$$V_{eff}(r) = \frac{-0 + \sqrt{0 + 4r^8 r^5 (r-2M)(j^2+r^2)}}{2r^8}$$

$$V_{eff}(r) = \frac{r^6}{r^8}\sqrt{r(r-2M)(j^2+r^2)}$$

$$V_{eff}(r) = \frac{1}{r^2}\sqrt{r(r-2M)(j^2+r^2)}$$

$$V_{eff}(r) = \sqrt{\frac{r}{r^4}(r-2M)(j^2+r^2)}$$

$$\boxed{V_{eff}(r) = \sqrt{\left(1-\frac{2M}{r}\right)\left(\frac{j^2}{r^2}+1\right)}}$$

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