Conserved Quantities

The conserved charge, associated with a killing vector k is

$$C_k = P^{\mu} k_{\mu} + \frac{1}{2} S^{\mu\nu} \nabla_{\mu} k_{\nu}$$

Working in the tetrad,

$$C_k = P^{(\alpha)} k_{(\alpha)} + \frac{1}{2} S^{(\alpha)(\beta)} \nabla_{(\alpha)} k_{(\beta)}$$

$$C_k = \eta_{(\alpha)(\beta)} P^{(\alpha)} k^{(\beta)} + \frac{1}{2} S^{(\alpha)(\beta)} \nabla_{(\alpha)} k_{(\beta)}$$

* Energy

$$\xi = \frac{\partial}{\partial t} \longrightarrow \begin{aligned} \xi^{(0)} &= e^{(0)}{}_{0} = \sqrt{\frac{\Delta}{\Sigma}} \\ \xi^{(1)} &= e^{(1)}{}_{0} = 0 \\ \xi^{(2)} &= e^{(2)}{}_{0} = 0 \\ \xi^{(3)} &= e^{(3)}{}_{0} = -\frac{a \sin\theta}{\sqrt{\Sigma}} \end{aligned}$$

$$C_{\xi} = -E = \eta_{(\alpha)(\beta)} P^{(\alpha)} \xi^{(\beta)} + \frac{1}{2} S^{(\alpha)(\beta)} \nabla_{(\alpha)} \xi_{(\beta)}$$

$$-E = \eta_{(0)(0)} P^{(0)} \xi^{(0)} + \eta_{(3)(3)} P^{(3)} \xi^{(3)} + \frac{1}{2} S^{(\alpha)(\beta)} \nabla_{(\alpha)} \xi_{(\beta)}$$

$$-E = -P^{(0)} \xi^{(0)} + P^{(3)} \xi^{(3)} + \frac{1}{2} S^{(\alpha)(\beta)} \nabla_{(\alpha)} \xi_{(\beta)}$$

The non-vanishing components are $S^{(0)(1)}$, $S^{(0)(3)}$ and $S^{(1)(3)}$ and we have the Killing equation

$$\nabla_{\mu} k_{\nu} = -\nabla_{\nu} k_{\mu}$$

Then:

$$S^{(0)(1)}\nabla_{(0)}\xi_{(1)} + S^{(1)(0)}\nabla_{(1)}\xi_{(0)} = S^{(0)(1)}\nabla_{(0)}\xi_{(1)} - S^{(0)(1)}\nabla_{(1)}\xi_{(0)}$$
$$= S^{(0)(1)}\nabla_{(0)}\xi_{(1)} + S^{(0)(1)}\nabla_{(0)}\xi_{(1)}$$
$$= 2S^{(0)(1)}\nabla_{(0)}\xi_{(1)}$$

Hence

$$-E = -P^{(0)}\xi^{(0)} + P^{(3)}\xi^{(3)} + S^{(0)(1)}\nabla_{(0)}\xi_{(1)} + S^{(0)(3)}\nabla_{(0)}\xi_{(3)} + S^{(1)(3)}\nabla_{(1)}\xi_{(3)}$$

Since

$$S^{(0)(1)} = -sP^{(3)}$$
$$S^{(0)(3)} = sP^{(1)}$$
$$S^{(1)(3)} = sP^{(0)}$$

$$\boxed{-E = -P^{(0)}\xi^{(0)} + P^{(3)}\xi^{(3)} - sP^{(3)}\nabla_{(0)}\xi_{(1)} + sP^{(1)}\nabla_{(0)}\xi_{(3)} + sP^{(0)}\nabla_{(1)}\xi_{(3)}}$$

* ANGULAR MOMENTUM

$$\varphi^{(0)} = e^{(0)}_3 = -\sqrt{\frac{\Delta}{\Sigma}}\, a\sin^2\theta$$

$$\varphi = \frac{\partial}{\partial\phi} \longrightarrow \quad \varphi^{(1)} = e^{(1)}_3 = 0$$

$$\varphi^{(2)} = e^{(2)}_3 = 0$$

$$\varphi^{(3)} = e^{(3)}_3 = (r^2+a^2)\frac{\sin\theta}{\sqrt{\Sigma}}$$

$$C_\varphi = J = \eta_{(\alpha)(\beta)}P^{(\alpha)}\varphi^{(\beta)} + \frac{1}{2}S^{(\alpha)(\beta)}\nabla_{(\alpha)}\varphi_{(\beta)}$$

$$J = \eta_{(0)(0)}P^{(0)}\varphi^{(0)} + \eta_{(3)(3)}P^{(3)}\varphi^{(3)} + \frac{1}{2}S^{(\alpha)(\beta)}\nabla_{(\alpha)}\varphi_{(\beta)}$$

$$J = -P^{(0)}\varphi^{(0)} + P^{(3)}\varphi^{(3)} + \frac{1}{2}S^{(\alpha)(\beta)}\nabla_{(\alpha)}\varphi_{(\beta)}$$

Hence

$$J = -P^{(0)}\varphi^{(0)} + P^{(3)}\varphi^{(3)} + S^{(0)(1)}\nabla_{(0)}\varphi_{(1)} + S^{(0)(3)}\nabla_{(0)}\varphi_{(3)} + S^{(1)(3)}\nabla_{(1)}\varphi_{(3)}$$

$$\boxed{J = -P^{(0)}\varphi^{(0)} + P^{(3)}\varphi^{(3)} - sP^{(3)}\nabla_{(0)}\varphi_{(1)} + sP^{(1)}\nabla_{(0)}\varphi_{(3)} + sP^{(0)}\nabla_{(1)}\varphi_{(3)}}$$