# Formulacion de Hamilton-Jacobi

Cantidades conservadas (Métrica de Kerr)

$P_0 = -\mathcal{E}$ : energia
$P_1 = l_z$ : momento angular
$m_0\mathcal{H} = \frac{1}{2}m_0c^2\delta$ : energía propia (masa propia) de la particula.

Para iniciar con el tratamiento de Hamilton-Jacobi introducimos la función principal de Hamilton, $S = S(x^\mu, P_\mu, \tau)$, a través de

$$P_\mu = \frac{\partial S}{\partial x^\mu}$$ con $P_\mu$ el momento canónicamente conjugado a $x^\mu$

De esta forma, la ecuación de Hamilton-Jacobi para el movimiento geodesico de la partícula es

$$2\frac{\partial S}{\partial \tau} = g^{\mu\nu}\frac{\partial S}{\partial x^\mu}\frac{\partial S}{\partial x^\nu}$$

La métrica de Kerr tiene las componentes

$$ds^2 = -\left(1 - \frac{2Mr}{\Sigma}\right)dt^2 - \frac{4aMr\sin^2\theta}{\Sigma}dt\,d\phi + \frac{\Sigma}{\Delta}dr^2 + \Sigma\,d\theta^2 + \left(r^2 + a^2 + \frac{2a^2Mr\sin^2\theta}{\Sigma}\right)\sin^2\theta\,d\phi^2$$

con $\Sigma = r^2 + a^2\cos^2\theta$
$\Delta = r^2 - 2Mr + a^2$

La métrica inversa viene dada por

$$\left(\frac{\partial}{\partial s}\right)^2 = -\frac{A}{\Sigma\Delta}\left(\frac{\partial}{\partial t}\right)^2 - \frac{4aMr}{\Sigma\Delta}\left(\frac{\partial}{\partial t}\right)\left(\frac{\partial}{\partial \phi}\right) + \frac{\Delta}{\Sigma}\left(\frac{\partial}{\partial r}\right)^2 + \frac{1}{\Sigma}\left(\frac{\partial}{\partial \theta}\right)^2 + \frac{\Delta - a^2\sin^2\theta}{\Sigma\Delta\sin^2\theta}\left(\frac{\partial}{\partial \phi}\right)^2$$

donde $A = (r^2+a^2)^2 - a^2\Delta\sin^2\theta$

Asi, la ecuación de H-J será

$$2\frac{\partial S}{\partial \tau} = g^{00}(\partial_0 S)^2 + g^{03}(\partial_0 S)(\partial_3 S) + g^{11}(\partial_1 S)^2 + g^{22}(\partial_2 S)^2 + g^{33}(\partial_3 S)^2$$

Debido a la existencia de las cantidades conservadas, podemos separar la función $S$ en la forma

$$S = \frac{1}{2}\delta\tau - \mathcal{E}t + l_z\phi + S_r(r) + S_\theta(\theta)$$

Al reemplazar en la ecuación de H-J se obtiene

$$\delta = g^{00}(-\mathcal{E})^2 + 2g^{03}(-\mathcal{E})(\ell_z) + g^{11}\left(\frac{dS_r}{dr}\right)^2 + g^{22}\left(\frac{dS_\theta}{d\theta}\right)^2 + g^{33}(\ell_z)^2$$

$$\boxed{\delta = g^{00}\mathcal{E}^2 - 2g^{03}\mathcal{E}\ell_z + g^{11}\left(\frac{dS_r}{dr}\right)^2 + g^{22}\left(\frac{dS_\theta}{d\theta}\right)^2 + g^{33}\ell_z^2}$$

Reemplazando las componentes de la métrica de Kerr,

$$\delta = -\frac{A}{\Sigma\Delta}\mathcal{E}^2 + \frac{4aMr}{\Sigma\Delta}\mathcal{E}\ell_z + \frac{\Delta}{\Sigma}\left(\frac{dS_r}{dr}\right)^2 + \frac{1}{\Sigma}\left(\frac{dS_\theta}{d\theta}\right)^2 + \frac{\Delta - a^2\sin^2\theta}{\Sigma\Delta\sin^2\theta}\ell_z^2$$

$$\delta\Sigma = -\frac{A}{\Delta}\mathcal{E}^2 + \frac{4aMr}{\Delta}\mathcal{E}\ell_z + \frac{\Delta - a^2\sin^2\theta}{\Delta\sin^2\theta}\ell_z^2 + \Delta\left(\frac{dS_r}{dr}\right)^2 + \left(\frac{dS_\theta}{d\theta}\right)^2$$

$$\delta\Sigma = -\frac{(r^2+a^2)^2}{\Delta}\mathcal{E}^2 + a^2\sin^2\theta\,\mathcal{E}^2 + \frac{4aMr}{\Delta}\mathcal{E}\ell_z + \frac{\ell_z^2}{\sin^2\theta} - \frac{a^2}{\Delta}\ell_z^2 + \Delta\left(\frac{dS_r}{dr}\right)^2 + \left(\frac{dS_\theta}{d\theta}\right)^2$$

$$\delta\Sigma = -\frac{(r^2+a^2)^2}{\Delta}\mathcal{E}^2 - \frac{a^2}{\Delta}\ell_z^2 + \frac{2(r^2+a^2)a\mathcal{E}\ell_z}{\Delta} - \frac{2(r^2+a^2)a\mathcal{E}\ell_z}{\Delta} + a^2\sin^2\theta\,\mathcal{E}^2 + \frac{4aMr}{\Delta}\mathcal{E}\ell_z + \frac{\ell_z^2}{\sin^2\theta}$$
$$+ \Delta\left(\frac{dS_r}{dr}\right)^2 + \left(\frac{dS_\theta}{d\theta}\right)^2$$

$$\delta\Sigma = -\frac{1}{\Delta}\left[(r^2+a^2)\mathcal{E} - a\ell_z\right]^2 - \frac{2(r^2+a^2)a\mathcal{E}\ell_z}{\Delta} + \frac{4aMr}{\Delta}\mathcal{E}\ell_z + a^2\sin^2\theta\,\mathcal{E}^2 + \frac{\ell_z^2}{\sin^2\theta} + \Delta\left(\frac{dS_r}{dr}\right)^2 + \left(\frac{dS_\theta}{d\theta}\right)^2$$

$$\delta\Sigma = -\frac{1}{\Delta}\left[(r^2+a^2)\mathcal{E} - a\ell_z\right]^2 - 2\frac{\left[r^2+a^2-2Mr\right]}{\Delta}a\mathcal{E}\ell_z + a^2\sin^2\theta\,\mathcal{E}^2 + \frac{\ell_z^2}{\sin^2\theta} + \Delta\left(\frac{dS_r}{dr}\right)^2 + \left(\frac{dS_\theta}{d\theta}\right)^2$$

$$\delta\Sigma = -\frac{1}{\Delta}\left[(r^2+a^2)\mathcal{E} - a\ell_z\right]^2 - 2a\mathcal{E}\ell_z + a^2\sin^2\theta\,\mathcal{E}^2 + \frac{\ell_z^2}{\sin^2\theta} + \Delta\left(\frac{dS_r}{dr}\right)^2 + \left(\frac{dS_\theta}{d\theta}\right)^2$$

$$\delta\Sigma = -\frac{1}{\Delta}\left[(r^2+a^2)\mathcal{E} - a\ell_z\right]^2 - 2a\mathcal{E}\ell_z + a^2(1-\cos^2\theta)\mathcal{E}^2 + \frac{\ell_z^2}{\sin^2\theta}(\sin^2\theta + \cos^2\theta) + \Delta\left(\frac{dS_r}{dr}\right)^2 + \left(\frac{dS_\theta}{d\theta}\right)^2$$

$$\delta\Sigma = -\frac{1}{\Delta}\left[(r^2+a^2)\mathcal{E} - a\ell_z\right]^2 - 2a\mathcal{E}\ell_z + a^2\mathcal{E}^2 + \ell_z^2 - a^2\mathcal{E}^2\cos^2\theta + \frac{\ell_z^2}{\sin^2\theta}\cos^2\theta + \Delta\left(\frac{dS_r}{dr}\right)^2 + \left(\frac{dS_\theta}{d\theta}\right)^2$$

$$\delta\Sigma = -\frac{1}{\Delta}\left[(r^2+a^2)\mathcal{E} - a\ell_z\right]^2 + (\ell_z - a\mathcal{E})^2 + \left(\frac{\ell_z^2}{\sin^2\theta} - a^2\mathcal{E}^2\right)\cos^2\theta + \Delta\left(\frac{dS_r}{dr}\right)^2 + \left(\frac{dS_\theta}{d\theta}\right)^2$$

$$\delta r^2 + \delta a^2\cos^2\theta = -\frac{1}{\Delta}\left[(r^2+a^2)\mathcal{E} - a\ell_z\right]^2 + (\ell_z - a\mathcal{E})^2 + \left(\frac{\ell_z^2}{\sin^2\theta} - a^2\mathcal{E}^2\right)\cos^2\theta + \Delta\left(\frac{dS_r}{dr}\right)^2 + \left(\frac{dS_\theta}{d\theta}\right)^2$$

Ahora es posible separar variables,

$$\Delta\left(\frac{dS_r}{dr}\right)^2 - \frac{1}{\Delta}\left[(r^2+a^2)\mathcal{E} - a\ell_z\right]^2 + (\ell_z - a\mathcal{E})^2 - \delta r^2 = -\left(\frac{dS_\theta}{d\theta}\right)^2 - \left(\frac{\ell_z^2}{\sin^2\theta} - a^2\mathcal{E}^2\right)\cos^2\theta + \delta a^2\cos^2\theta$$

La ecuación se separa introduciendo la constante $\mathcal{C}$. De esta forma se obtiene el sistema

$$\begin{cases} \Delta\left(\dfrac{dS_r}{dr}\right)^2 = \dfrac{1}{\Delta}\left[(r^2+a^2)\mathcal{E} - a\ell_z\right]^2 - (\ell_z - a\mathcal{E})^2 + \delta r^2 - \mathcal{C} \\ \left(\dfrac{dS_\theta}{d\theta}\right)^2 = \mathcal{C} - \left(\dfrac{\ell_z^2}{\sin^2\theta} - a^2\mathcal{E}^2 - \delta a^2\right)\cos^2\theta \end{cases}$$

Definiendo las funciones

$$R(r) = \left[(r^2+a^2)\mathcal{E} - a\ell_z\right]^2 - \Delta\left[(\ell_z - a\mathcal{E})^2 - \delta r^2 + \mathcal{C}\right]$$

$$\Theta(\theta) = \mathcal{C} - \left(\frac{\ell_z^2}{\sin^2\theta} - a^2\mathcal{E}^2 - \delta a^2\right)\cos^2\theta$$

El sistema se lleva a cuadraturas en la forma

$$\begin{cases} S_r(r) = \displaystyle\int dr\, \frac{\sqrt{R(r)}}{\Delta} \\ S_\theta(\theta) = \displaystyle\int d\theta\, \sqrt{\Theta} \end{cases}$$

y con ello, el problema queda formalmente solucionado. Para interpretar la constante de separación $\mathcal{C}$, nótese que

$$\left(\frac{dS_\theta}{d\theta}\right)^2 = \mathcal{C} - \left(\frac{\ell_z^2}{\sin^2\theta} - a^2\mathcal{E}^2 - \delta a^2\right)\cos^2\theta$$

$$P_\theta^2 = \mathcal{C} - \left(\frac{P_\phi^2}{\sin^2\theta} - a^2\mathcal{E}^2 - \delta a^2\right)\cos^2\theta$$

$$\boxed{\mathcal{C} = P_\theta^2 + P_\phi^2\cot^2\theta - a^2(\delta + \mathcal{E}^2)\cos^2\theta}$$

En el límite $a \to 0$ ; $\mathcal{C} = P_\theta^2 + P_\phi^2\cot^2\theta = \left(P_\theta^2 + \dfrac{P_\phi^2}{\sin^2\theta}\right) - P_\phi^2 \equiv \ell^2 - \ell_z^2$

con $\ell^2$ el Momento Angular Total.

## Ecuaciones de Movimiento

Las ecuaciones de movimiento para la partícula se obtienen al considerar las derivadas de $S$ con respecto a las constantes de movimiento $(\delta, \varepsilon, \ell, \mathcal{Q})$ e igualar a cero. De esta forma se tiene

- $$\frac{\partial S}{\partial \mathcal{Q}} = \frac{\partial S_r}{\partial \mathcal{Q}} + \frac{\partial S_\theta}{\partial \mathcal{Q}} = 0$$

$$\frac{\partial}{\partial \mathcal{Q}} \int \frac{\sqrt{R}}{\Delta}\, dr + \frac{\partial}{\partial \mathcal{Q}} \int \sqrt{\Theta}\, d\theta = 0$$

$$\frac{1}{2} \int \frac{-\Delta}{\sqrt{R}\,\Delta}\, dr + \frac{1}{2} \int \frac{1}{\sqrt{\Theta}}\, d\theta = 0$$

$$\int \frac{dr}{\sqrt{R}} = \int \frac{d\theta}{\sqrt{\Theta}}$$

- $$\frac{\partial S}{\partial \delta} = \frac{\tau}{2} + \frac{\partial}{\partial \delta} \int \frac{\sqrt{R}}{\Delta}\, dr + \frac{\partial}{\partial \delta} \int d\theta\, \sqrt{\Theta} = 0$$

$$\frac{\tau}{2} + \int \frac{\Delta r^2}{2\sqrt{R}\,\Delta}\, dr + \int \frac{a^2 \cos^2\theta}{2\sqrt{\Theta}}\, d\theta = 0$$

$$\tau = -\int \frac{r^2}{\sqrt{R}}\, dr - a^2 \int \frac{\cos^2\theta}{\sqrt{\Theta}}\, d\theta$$

- $$\frac{\partial S}{\partial \varepsilon} = -t + \frac{\partial}{\partial \varepsilon} \int \frac{\sqrt{R}}{\Delta}\, dr + \frac{\partial}{\partial \varepsilon} \int d\theta\, \sqrt{\Theta} = 0$$

$$-t + \frac{1}{2} \int \frac{dr}{\sqrt{R}\,\Delta} \left\{ 2\left[(r^2+a^2)\varepsilon - a\ell_z\right](r^2+a^2) + 2a\Delta(\ell_z - a\varepsilon) \right\} + \int \frac{d\theta}{2\sqrt{\Theta}} (2a^2 \varepsilon \cos^2\theta) = 0$$

$$t = \int \frac{dr}{\sqrt{R}\,\Delta} \left\{ \left[(r^2+a^2)\varepsilon - a\ell_z\right](r^2+a^2) + a\Delta(\ell_z - a\varepsilon) \right\} + a^2 \int \frac{d\theta \cos^2\theta\, \varepsilon}{\sqrt{\Theta}}$$

$$t = \int \frac{dr}{\sqrt{R}\,\Delta} \left\{ \left[(r^2+a^2)\varepsilon - a\ell_z\right](r^2+a^2) + a\Delta(\ell_z - a\varepsilon) \right\} - \varepsilon \int \frac{r^2}{\sqrt{R}}\, dr - \tau\varepsilon$$

$$t = \int \frac{dr}{\sqrt{R}\,\Delta} \left\{ \left[(r^2+a^2)\varepsilon - a\ell_z\right](r^2+a^2) - r^2\Delta\varepsilon + a\Delta(\ell_z - a\varepsilon) \right\} - \tau\varepsilon$$

$$t = \int \frac{dr}{\sqrt{R}\,\Delta} \left\{ \left[(r^2+a^2)\varepsilon - a\ell_z\right](r^2+a^2) - r^2(r^2+a^2-2Mr)\varepsilon + a(r^2+a^2-2Mr)(\ell_z - a\varepsilon) \right\} - \tau\varepsilon$$

$$t = \int \frac{dr}{\sqrt{R}\,\Delta} \left\{ (r^2+a^2)\mathcal{E}a^2 - a l_z (r^2+a^2) + r^2(2Mr)\mathcal{E} + a(r^2+a^2-2Mr) l_z - a^2(r^2+a^2-2Mr)\mathcal{E} \right\} - \tau \mathcal{E}$$

$$t = \int \frac{dr}{\sqrt{R}\,\Delta} \left\{ r^2(2Mr)\mathcal{E} - 2Mra\, l_z + 2Mra^2\, \mathcal{E} \right\} - \tau \mathcal{E}$$

$$t = 2M \int \frac{dr}{\sqrt{R}\,\Delta} \left[ r^2\mathcal{E} - a(l_z - a\mathcal{E}) \right] r - \tau \mathcal{E}$$

- $$\frac{\partial S}{\partial l_z} = \phi + \frac{\partial}{\partial l_z} \int \frac{\sqrt{R}}{\Delta}\, dr + \frac{\partial}{\partial l_z} \int \sqrt{\Theta}\, d\theta = 0$$

$$\phi + \frac{1}{2} \int \frac{dr}{\sqrt{R}\,\Delta} \left\{ -2a\left[(r^2+a^2)\mathcal{E} - a l_z\right] - 2\Delta(l_z - a\mathcal{E}) \right\} + \frac{1}{2} \int \frac{1}{\sqrt{\Theta}} \left( -\frac{2 l_z \cos^2\theta}{\sin^2\theta} \right) d\theta = 0$$

$$\phi - \int \frac{dr}{\sqrt{R}\,\Delta} \left\{ a\left[(r^2+a^2)\mathcal{E} - a l_z\right] + \Delta(l_z - a\mathcal{E}) \right\} - \int \frac{l_z}{\sqrt{\Theta}} \frac{\cos^2\theta}{\sin^2\theta}\, d\theta = 0$$

$$\phi = a \int \frac{dr}{\sqrt{R}\,\Delta} \left[(r^2+a^2)\mathcal{E} - a l_z\right] + (l_z - a\mathcal{E}) \int \frac{dr}{\sqrt{R}} + \int \frac{l_z}{\sqrt{\Theta}} \frac{\cos^2\theta}{\sin^2\theta}\, d\theta$$

Usando la relación $\int \frac{dr}{\sqrt{R}} = \int \frac{d\theta}{\sqrt{\Theta}}$ encontrada arriba, se tiene

$$\phi = a \int \frac{dr}{\sqrt{R}\,\Delta} \left[(r^2+a^2)\mathcal{E} - a l_z\right] + (l_z - a\mathcal{E}) \int \frac{d\theta}{\sqrt{\Theta}} + \int \frac{l_z}{\sqrt{\Theta}} \frac{\cos^2\theta}{\sin^2\theta}\, d\theta$$

$$\phi = a \int \frac{dr}{\sqrt{R}\,\Delta} \left[(r^2+a^2)\mathcal{E} - a l_z\right] + \int \frac{d\theta}{\sqrt{\Theta}} \left( l_z \frac{\cos^2\theta}{\sin^2\theta} + (l_z - a\mathcal{E}) \right)$$

$$\phi = a \int \frac{dr}{\sqrt{R}\,\Delta} \left[(r^2+a^2)\mathcal{E} - a l_z\right] + \int \frac{d\theta}{\sqrt{\Theta}} \left\{ l_z \left( 1 + \frac{\cos^2\theta}{\sin^2\theta} \right) - a\mathcal{E} \right\}$$

$$\phi = a \int \frac{dr}{\sqrt{R}\,\Delta} \left[(r^2+a^2)\mathcal{E} - a l_z\right] + \int \frac{d\theta}{\sqrt{\Theta}} \left( \frac{l_z}{\sin^2\theta} - a\mathcal{E} \right)$$

## Ecuaciones Diferenciales de Movimiento

Ya que $\dot{x}^\mu = p^\mu = g^{\mu\nu} p_\nu$ se tiene

$$\dot{x}^0 = \dot{t} = g^{00} p_0 + g^{03} p_3$$

$$\dot{t} = -\frac{A}{\Sigma\Delta}(-\varepsilon) - \frac{2aMr}{\Sigma\Delta}(\ell_z)$$

$$\Sigma \dot{t} = \frac{1}{\Delta}\left[A\varepsilon - 2aMr\,\ell_z\right]$$

$$\dot{x}^1 = \dot{r} = g^{11} p_1 = \frac{\Delta}{\Sigma}\frac{\partial S}{\partial r}$$

$$\dot{r} = \frac{\Delta}{\Sigma}\frac{\sqrt{R}}{\Delta}$$

$$\dot{r} = \frac{\sqrt{R}}{\Sigma}$$

$$\Sigma^2 \dot{r}^2 = R$$

$$\dot{x}^2 = \dot{\theta} = g^{22} p_2 = \frac{1}{\Sigma}\frac{\partial S}{\partial \theta} = \frac{\sqrt{\Theta}}{\Sigma}$$

$$\Sigma^2 \dot{\theta}^2 = \Theta$$

$$\dot{x}^3 = \dot{\phi} = g^{30} p_0 + g^{33} p_3$$

$$\dot{\phi} = -\frac{2aMr}{\Sigma\Delta}(-\varepsilon) + \frac{\Delta - a^2\sin^2\theta}{\Sigma\Delta\sin^2\theta}\ell_z$$

$$\Sigma\dot{\phi} = \frac{1}{\Delta}\left[2aMr\,\varepsilon + \frac{\Delta - a^2\sin^2\theta}{\sin^2\theta}\ell_z\right]$$

$$\Sigma\dot{\phi} = \frac{1}{\Delta}\left[2aMr\,\varepsilon + (\Sigma - 2Mr)\frac{\ell_z}{\sin^2\theta}\right]$$