Equations of Motion of the Spinning Particle

$$\frac{d}{d\tau}\left(W\dot{t} + \frac{qQ}{4\pi m r}\right) + \frac{hs}{2mr^2}\left(W''r - W'\right)\dot{r} = 0 \qquad (56)$$

$$W(r) = 1 - \frac{2M}{r} + \frac{Q^2}{r^2} \qquad (32) \quad \text{Reissner-Nordström}$$

$$W' = \frac{2M}{r^2} - \frac{2Q^2}{r^3}$$

$$W'' = -\frac{4M}{r^3} + \frac{6Q^2}{r^4}$$

$$\frac{d}{d\tau}\left(W\dot{t} + \frac{qQ}{4\pi m r}\right) + \frac{hs}{2mr^2}\left[\left(-\frac{4M}{r^3} + \frac{6Q^2}{r^4}\right)r - \left(\frac{2M}{r^2} - \frac{2Q^2}{r^3}\right)\right]\frac{dr}{d\tau} = 0$$

$$d\left(W\dot{t} + \frac{qQ}{4\pi m r}\right) = -\frac{hs}{2m}\left[-\frac{4M}{r^4} + \frac{6Q^2}{r^5} - \frac{2M}{r^4} + \frac{2Q^2}{r^5}\right]dr$$

$$\int d\left(W\dot{t} + \frac{qQ}{4\pi m r}\right) = -\frac{hs}{2m}\int\left[-\frac{6M}{r^4} + \frac{8Q^2}{r^5}\right]dr$$

$$W\dot{t} + \frac{qQ}{4\pi m r} = -\frac{hs}{2m}\left[\frac{6M}{3r^3} - \frac{8Q^2}{4r^4}\right] + K$$

$$\dot{t} = \frac{K}{W} - \frac{qQ}{4\pi m W r} - \frac{hs}{2mW}\left(\frac{2M}{r^3} - 2\frac{Q^2}{r^4}\right)$$

$$\dot{t} = \frac{K}{W} - \frac{qQ}{4\pi m W r} - \frac{hs}{mW}\left(\frac{M}{r^3} - \frac{Q^2}{r^4}\right)$$

$$(1+x)^\alpha = 1 + \alpha x + \frac{1}{2}\alpha(\alpha-1)x^2 + \dots$$

$$W^{-1}\left(\frac{M}{r^3} - \frac{Q^2}{r^4}\right) = \left(1 - \frac{2M}{r} + \frac{Q^2}{r^2}\right)^{-1}\left(\frac{M}{r^3} - \frac{Q^2}{r^4}\right)$$

$$\approx \left[1 - \left(-\frac{2M}{r} + \frac{Q^2}{r^2}\right) - \frac{1}{2}(-1-1)\left(-\frac{2M}{r} + \frac{Q^2}{r^2}\right)^2\right]\left(\frac{M}{r^3} - \frac{Q^2}{r^4}\right)$$

$$\approx \left[1 + \frac{2M}{r} - \frac{Q^2}{r^2} + \left(\frac{4M^2}{r^2} - \frac{4MQ^2}{r^3} + \frac{Q^4}{r^4}\right)\right]\left(\frac{M}{r^3} - \frac{Q^2}{r^4}\right)$$

$$\approx \frac{M}{r^3} - \frac{Q^2}{r^4} + \frac{2M^2}{r^4} - \frac{2MQ^2}{r^5} - \frac{MQ^2}{r^5} + \frac{4M^3}{r^5} + O\left(\frac{1}{r^6}\right)$$

$$\approx \frac{M}{r^3} + \frac{2M^2 - Q^2}{r^4} + \frac{4M^3 - 3MQ^2}{r^5} + O\left(\frac{1}{r^6}\right)$$

$$\dot{t} = -\frac{E}{W} - \frac{qQ}{4\pi m W r} - \frac{hs}{m}\left(\frac{M}{r^3} + \frac{2M^2 - Q^2}{r^4} + \frac{4M^3 - 3MQ^2}{r^5}\right) + O\left(\frac{1}{r^6}\right) \qquad (58)\ *$$

We identified $K = -E$ (Energy per unit mass) because taking $s = q = 0 \longrightarrow \dot{t} = -\frac{E}{W}$

$$\frac{d\dot{r}}{d\tau} + w' - \frac{W h^2}{r^3} + \frac{W s h}{2 m r^2}\left(w'' r - w'\right)\dot{t} - \frac{4QW}{4\pi m r^2}\dot{t} = 0 \qquad (57) \quad *$$

$$\frac{d\dot{r}}{d\tau} + \frac{2M}{r^2} - \frac{2Q^2}{r^3} - \frac{h^2}{r^3}\left(1 - \frac{2M}{r} + \frac{Q^2}{r^2}\right)$$

$$+ \frac{sh}{2mr^2}\left(1 - \frac{2M}{r} + \frac{Q^2}{r^2}\right)\left[\left(-\frac{4M}{r^3} + \frac{6Q^2}{r^4}\right)r - \left(\frac{2M}{r^2} - \frac{2Q^2}{r^3}\right)\right]\dot{t} - \frac{4QW}{4\pi m r^2}\dot{t} = 0$$

$$\frac{d\dot{r}}{d\tau} + \frac{2M}{r^2} - \frac{2Q^2}{r^3} - \frac{h^2}{r^3} + \frac{2Mh^2}{r^4} - \frac{Q^2h^2}{r^5} + \frac{sh}{2m}\left(1 - \frac{2M}{r} + \frac{Q^2}{r^2}\right)\left[-\frac{4M}{r^4} + \frac{6Q^2}{r^5} - \frac{2M}{r^4} + \frac{2Q^2}{r^5}\right]\dot{t}$$

$$- \frac{4QW}{4\pi m r^2}\dot{t} = 0$$

$$\frac{d\dot{r}}{d\tau} + \frac{2M}{r^2} - \frac{2Q^2 + h^2}{r^3} + \frac{2Mh^2}{r^4} - \frac{Q^2h^2}{r^5} + \frac{sh}{2m}\left(1 - \frac{2M}{r} + \frac{Q^2}{r^2}\right)\left[-\frac{6M}{r^4} + \frac{8Q^2}{r^5}\right]\dot{t} - \frac{4QW}{4\pi m r^2}\dot{t} = 0$$

$$\frac{d\dot{r}}{d\tau} + \frac{2M}{r^2} - \frac{2Q^2 + h^2}{r^3} + \frac{2Mh^2}{r^4} - \frac{Q^2h^2}{r^5} + \frac{sh}{m}\left(1 - \frac{2M}{r} + \frac{Q^2}{r^2}\right)\left[-\frac{3M}{r^4} + \frac{4Q^2}{r^5}\right]\dot{t} - \frac{4QW}{4\pi m r^2}\dot{t} = 0$$

$$\frac{d\dot{r}}{d\tau} + \frac{2M}{r^2} - \frac{2Q^2 + h^2}{r^3} + \frac{2Mh^2}{r^4} - \frac{Q^2h^2}{r^5} + \frac{sh}{m}\left(-\frac{3M}{r^4} + \frac{4Q^2}{r^5} + \frac{6M^2}{r^5}\right)\dot{t} - \frac{4QW}{4\pi m r^2}\dot{t} = 0$$

$$\frac{d\dot{r}}{d\tau} + \frac{2M}{r^2} - \frac{2Q^2 + h^2}{r^3} + \frac{2Mh^2}{r^4} - \frac{Q^2h^2}{r^5} + \frac{sh}{m}\left(-\frac{3M}{r^4} + \frac{6M^2 + 4Q^2}{r^5}\right)\dot{t}$$

$$- \frac{4Q}{4\pi m r^2}\left(1 - \frac{2M}{r} + \frac{Q^2}{r^2}\right)\dot{t} = 0$$

$$\frac{d\dot{r}}{d\tau} + \frac{2M}{r^2} - \frac{2Q^2 + h^2}{r^3} + \frac{2Mh^2}{r^4} - \frac{Q^2h^2}{r^5} + \frac{sh}{m}\left(-\frac{3M}{r^4} + \frac{6M^2 + 4Q^2}{r^5}\right)\dot{t}$$

$$- \frac{4Q}{4\pi m}\left(\frac{1}{r^2} - \frac{2M}{r^3} + \frac{Q^2}{r^4}\right)\dot{t} = 0$$

$$\left(-\frac{3M}{r^4}+\frac{6M^2+4Q^2}{r^5}\right)\dot{t}=\left(-\frac{3M}{r^4}+\frac{6M^2+4Q^2}{r^5}\right)\cdot$$

$$\cdot\left[-\frac{E}{W}-\frac{qQ}{4\pi m W r}-\frac{hs}{m}\left(\frac{M}{r^3}+\frac{2M^2-Q^2}{r^4}+\frac{4M^3-3MQ^2}{r^5}\right)\right]$$

$$=-EW^{-1}\left(-\frac{3M}{r^4}+\frac{6M^2+4Q^2}{r^5}\right)-\frac{qQ}{4\pi m}\frac{W^{-1}}{r}\left(-\frac{3M}{r^4}+\frac{6M^2+4Q^2}{r^5}\right)+O\left(\frac{1}{r^6}\right)$$

$$=-EW^{-1}\left(-\frac{3M}{r^4}+\frac{6M^2+4Q^2}{r^5}\right)+\frac{3qQ}{4\pi m}W^{-1}\left(\frac{M}{r^5}\right)+O\left(\frac{1}{r^6}\right)$$

$$\begin{aligned}
W^{-1}&=\left(1-\frac{2M}{r}+\frac{Q^2}{r^2}\right)^{-1}\approx\left[1-\left(-\frac{2M}{r}+\frac{Q^2}{r^2}\right)-\frac{1}{2}(-1-1)\left(-\frac{2M}{r}+\frac{Q^2}{r^2}\right)^2\right]\\
&\approx\left[1+\frac{2M}{r}-\frac{Q^2}{r^2}+\left(\frac{4M^2}{r^2}-\frac{4MQ^2}{r^3}+\frac{Q^4}{r^4}\right)\right]\\
&\approx\left(1-\frac{2M}{r}+\frac{4M^2-Q^2}{r^2}-\frac{4MQ^2}{r^3}+\frac{Q^4}{r^4}\right)
\end{aligned}$$

$$\left(-\frac{3M}{r^4}+\frac{6M^2+4Q^2}{r^5}\right)\dot{t}=-E\left(1-\frac{2M}{r}+\frac{4M^2-Q^2}{r^2}-\frac{4MQ^2}{r^3}+\frac{Q^4}{r^4}\right)\left(-\frac{3M}{r^4}+\frac{6M^2+4Q^2}{r^5}\right)$$

$$+\frac{3qQ}{4\pi m}\left(1-\frac{2M}{r}+\frac{4M^2-Q^2}{r^2}-\frac{4MQ^2}{r^3}+\frac{Q^4}{r^4}\right)\left(\frac{M}{r^5}\right)+O\left(\frac{1}{r^6}\right)$$

$$\left(-\frac{3M}{r^4}+\frac{6M^2+4Q^2}{r^5}\right)\dot{t}=-E\left(-\frac{3M}{r^4}+\frac{6M^2+4Q^2}{r^5}+\frac{6M^2}{r^5}\right)+\frac{3qQ}{4\pi m}\left(\frac{M}{r^5}\right)+O\left(\frac{1}{r^6}\right)$$

$$\left(-\frac{3M}{r^4}+\frac{6M^2+4Q^2}{r^5}\right)\dot{t}=-E\left(-\frac{3M}{r^4}+\frac{12M^2+4Q^2}{r^5}\right)+\frac{3qQM}{4\pi m r^5}+O\left(\frac{1}{r^6}\right)$$

$$\left(\frac{1}{r^2}-\frac{2M}{r^3}+\frac{Q^2}{r^4}\right)\dot{t}=\left(\frac{1}{r^2}-\frac{2M}{r^3}+\frac{Q^2}{r^4}\right)\cdot$$

$$\cdot\left[-\frac{E}{W}-\frac{qQ}{4\pi m W r}-\frac{hs}{m}\left(\frac{M}{r^3}+\frac{2M^2-Q^2}{r^4}+\frac{4M^3-3MQ^2}{r^5}\right)\right]$$

$$=-EW^{-1}\left(\frac{1}{r^2}-\frac{2M}{r^3}+\frac{Q^2}{r^4}\right)-\frac{qQ}{4\pi m}\frac{W^{-1}}{r}\left(\frac{1}{r^2}-\frac{2M}{r^3}+\frac{Q^2}{r^4}\right)$$

$$-\frac{hs}{m}\left(\frac{M}{r^3}\right)\left(\frac{1}{r^2}-\frac{2M}{r^3}+\frac{Q^2}{r^4}\right)+O\left(\frac{1}{r^6}\right)$$

$$=-EW^{-1}\left(\frac{1}{r^2}-\frac{2M}{r^3}+\frac{Q^2}{r^4}\right)-\frac{qQ}{4\pi m}W^{-1}\left(\frac{1}{r^3}-\frac{2M}{r^4}\right)$$

$$-\frac{hs}{m}\left(\frac{M}{r^5}\right)+O\left(\frac{1}{r^6}\right)$$

$$=-E\left(\frac{1}{r^2}-\frac{2M}{r^3}+\frac{Q^2}{r^4}\right)\left(1-\frac{2M}{r}+\frac{4M^2-Q^2}{r^2}-\frac{4MQ^2}{r^3}+\frac{Q^4}{r^4}\right)$$

$$-\frac{qQ}{4\pi m}\left(\frac{1}{r^3}-\frac{2M}{r^4}\right)\left(1-\frac{2M}{r}+\frac{4M^2-Q^2}{r^2}-\frac{4MQ^2}{r^3}+\frac{Q^4}{r^4}\right)$$

$$-\frac{hsM}{mr^5}+O\left(\frac{1}{r^6}\right)$$

$$=-E\left(\frac{1}{r^2}-\frac{2M}{r^3}+\frac{Q^2}{r^4}-\frac{2M}{r^3}+\frac{4M^2}{r^4}-\frac{2MQ^2}{r^5}+\frac{4M^2-Q^2}{r^4}-\frac{8M^3-2MQ^2}{r^5}-\frac{4MQ^2}{r^5}\right)$$

$$-\frac{qQ}{4\pi m}\left(\frac{1}{r^3}-\frac{2M}{r^4}-\frac{2M}{r^4}+\frac{4M^2}{r^5}+\frac{4M^2-Q^2}{r^5}\right)-\frac{hsM}{mr^5}+O\left(\frac{1}{r^6}\right)$$

$$\left(\frac{1}{r^2}-\frac{2M}{r^3}+\frac{Q^2}{r^4}\right)\dot{t}=-E\left(\frac{1}{r^2}-\frac{4M}{r^3}+\frac{8M^2}{r^4}-\frac{8M^3+4MQ^2}{r^5}\right)$$

$$-\frac{qQ}{4\pi m}\left(\frac{1}{r^3}-\frac{4M}{r^4}+\frac{8M^2-Q^2}{r^5}\right)-\frac{hsM}{mr^5}+O\left(\frac{1}{r^6}\right)$$

$$\frac{d\dot{r}}{d\tau} + \frac{2M}{r^2} - \frac{2Q^2 + h^2}{r^3} + \frac{2Mh^2}{r^4} - \frac{Q^2h^2}{r^5} + \frac{sh}{m}\left[-E\left(-\frac{3M}{r^4} + \frac{12M^2 + 4Q^2}{r^5} \right) + \frac{3qQM}{4\pi m r^5} \right]$$

$$- \frac{qQ}{4\pi m}\left[-E\left(\frac{1}{r^2} - \frac{4M}{r^3} + \frac{8M^2}{r^4} - \frac{8M^3 + 4MQ^2}{r^5} \right) \right.$$

$$\left. - \frac{qQ}{4\pi m}\left(\frac{1}{r^3} - \frac{4M}{r^4} + \frac{8M^2 - Q^2}{r^5} \right) - \frac{hsM}{mr^5} \right] + O\left(\frac{1}{r^6}\right) = 0$$

$$\rightarrow \alpha = 2M + \frac{EqQ}{4\pi m}$$

$$\rightarrow \beta = -2Q^2 - h^2 - \frac{4EqMQ}{4\pi m} + \frac{q^2Q^2}{16\pi^2 m^2}$$

$$\rightarrow \gamma = 2Mh^2 + 3\frac{EMh}{m}s + \frac{8EqQM^2}{4\pi m} - \frac{4q^2MQ^2}{16\pi^2 m^2}$$

$$\rightarrow \delta = -Q^2h^2 - E\frac{sh}{m}\left(12M^2 + 4Q^2\right) + \frac{3}{4}\frac{sh\,qQM}{4\pi m^2} - \frac{qQ}{4\pi m}E\left(8M^3 + 4MQ^2\right)$$

$$+ \frac{q^2Q^2}{16\pi^2 m^2}\left(8M^2 - Q^2\right) + \frac{qQ}{4\pi m^2}hsM$$