Initial Conditions

① Give the constants of motion : E, J, s
to obtain a motion bounded to a compact region of spacetime
(Here we need the effective potential)

② We will set this initial condition on the equatorial plane with
$\theta_0 = \frac{\pi}{2}$ $\quad \varphi_0 = 0$ $\quad r = r_0$

③ We give the spatial components of the spin tensor

$$S^{\theta\phi}\Big|_0 = \frac{J}{r_0^2}\cot\theta_0 = 0$$

$\longrightarrow$ The initial spin is perpendicular to the radial direction

$$S^{r\theta}\Big|_0 = -\frac{P_\theta|_0}{r_0}$$

$$S^{\varphi r}\Big|_0 = \frac{1}{r_0}\left(-J + \frac{P_\varphi}{\sin^2\theta_0}\right) = \frac{1}{r_0}\left(-J + P_\varphi|_0\right)$$

④ Define the parameter

$$\alpha = \arctan\left(\frac{S^{r\theta}|_0}{S^{\varphi r}|_0}\right)$$

This gives the initial direction of the spin.

$$\left.\begin{aligned} \alpha = 0 &: \text{ anti-parallel spin (w.r.t } z) \\ \alpha = \pi &: \text{ parallel spin (w.r.t. } z) \end{aligned}\right\}$$

Particle is always in the equatorial plane
$\rightarrow$ NO CHAOS

In order to obtain a chaotic behavior we consider

$$\frac{\pi}{2} \leqslant \alpha \leqslant \frac{3\pi}{2}$$

Then, we have

$$\tan\alpha = \frac{S^{r\theta}|_0}{S^{\varphi r}|_0}$$

and

$$S^{r\theta}\Big|_0 = -\frac{P_\theta|_0}{r_0} \longrightarrow P_\theta|_0 = -r_0 S^{r\theta}|_0$$

$$S^{\varphi r}\Big|_0 = \frac{1}{r_0}\left(-J + P_\varphi|_0\right) \longrightarrow P_\varphi|_0 = r_0 S^{\varphi r}|_0 + J$$

$$S^{tr}\Big|_0 = -\frac{1}{r_0 P_t|_0}\left(P_\theta^2|_0 + P_\varphi^2|_0 - J P_\varphi|_0\right) = -\frac{1}{r_0 P_t|_0}\left[P_\theta^2|_0 + P_\varphi|_0\left(P_\varphi|_0 - J\right)\right]$$

$$= -\frac{1}{r_0 P_t|_0}\left[r_0^2 (S^{r\theta}|_0)^2 + (r_0 S^{\varphi r}|_0 + J)(r_0 S^{\varphi r}|_0)\right]$$

$$= -\frac{1}{r_0 P_t|_0}\left[r_0^2\left[(S^{r\theta}|_0)^2 + (S^{\varphi r}|_0)^2\right] + r_0 J S^{\varphi r}|_0\right]$$

$$= -\frac{1}{P_t|_0}\left[r_0\left[(S^{r\theta}|_0)^2 + (S^{\varphi r}|_0)^2\right] + J S^{\varphi r}|_0\right]$$

$$S^{t\theta}\Big|_0 = \frac{1}{r_0 P_t|_0} P_r|_0 P_\theta|_0 = -\frac{P_r|_0}{r_0 P_t|_0} r_0 S^{r\theta}|_0 = -\frac{P_r|_0}{P_t|_0} S^{r\theta}|_0$$

$$S^{t\varphi}\Big|_0 = -\frac{1}{r_0 P_t|_0}\left(J P_r|_0 - P_r|_0 P_\varphi|_0\right) = -\frac{P_r|_0}{r_0 P_t|_0}\left(J - P_\varphi|_0\right) = \frac{P_r|_0}{r_0 P_t|_0} r_0 S^{\varphi r}|_0$$

$$= \frac{P_r|_0}{P_t|_0} S^{\varphi r}|_0$$

The spin magnitude is

$$S^2 = \frac{1}{2} S^{\mu\nu} S_{\mu\nu}$$

$$S^2 = S^{tr} S_{tr} + S^{t\theta} S_{t\theta} + S^{t\varphi} S_{t\varphi} + S^{r\theta} S_{r\theta} + S^{r\varphi} S_{r\varphi} + S^{\theta\varphi} S_{\theta\varphi}$$

$$S^2 = g_{tt} g_{rr} (S^{tr})^2 + g_{tt} r^2 (S^{t\theta})^2 + g_{tt} r^2 \sin^2\theta (S^{t\varphi})^2 + g_{rr} r^2 (S^{r\theta})^2$$
$$+ g_{rr} r^2 \sin^2\theta (S^{r\varphi})^2 + r^4 \sin^2\theta (S^{\theta\varphi})^2$$

Since S is an invariant, we evaluate at the initial time,

$$S^2 = g_{tt}(r_0) g_{rr}(r_0) \frac{1}{P_t^2|_0} \left[r_0 \left[(S^{r\theta}|_0)^2 + (S^{\varphi r}|_0)^2 \right] + J S^{\varphi r}|_0 \right]^2$$
$$+ g_{tt}(r_0) r_0^2 \frac{P_r^2|_0}{P_t^2|_0} (S^{r\theta}|_0)^2 + g_{tt}(r_0) r_0^2 \frac{P_r^2|_0}{P_t^2|_0} (S^{\varphi r}|_0)^2$$
$$+ g_{rr}(r_0) r_0^2 (S^{r\theta}|_0)^2 + g_{rr}(r_0) r_0^2 (S^{\varphi r}|_0)^2$$

$$S^2 = \frac{g_{tt}(r_0) g_{rr}(r_0)}{P_t^2|_0} \left[r_0 \left[(S^{r\theta}|_0)^2 + (S^{\varphi r}|_0)^2 \right] + J S^{\varphi r}|_0 \right]^2$$
$$+ g_{tt}(r_0) r_0^2 \frac{P_r^2|_0}{P_t^2|_0} \left[(S^{r\theta}|_0)^2 + (S^{\varphi r}|_0)^2 \right]$$
$$+ g_{rr}(r_0) r_0^2 \left[(S^{r\theta}|_0)^2 + (S^{\varphi r}|_0)^2 \right]$$

Notice that

$$(S^{r\theta})_0^2 + (S^{\varphi r})_0^2 = (S^{\varphi r})_0^2 \left[\left(\frac{S^{r\theta}}{S^{\varphi r}} \right)\Big|_0 + 1 \right] = (S^{\varphi r}|_0)^2 \left[\tan^2\alpha + 1 \right]$$

$$(S^{r\theta})_0^2 + (S^{\varphi r})_0^2 = \frac{(S^{\varphi r}|_0)^2}{\cos^2\alpha}$$

$$s^2 = \frac{g_{tt}(r_0)\, g_{rr}(r_0)}{P_t^2|_0} \left[r_0 \frac{(S^{\varphi r}|_0)^2}{\cos^2\alpha} + J\, S^{\varphi r}|_0 \right]^2$$

$$+ g_{tt}(r_0)\, r_0^2 \frac{P_r^2|_0}{P_t^2|_0} \frac{(S^{\varphi r}|_0)^2}{\cos^2\alpha} + g_{rr}(r_0)\, r_0^2 \frac{(S^{\varphi r}|_0)^2}{\cos^2\alpha}$$

$$s^2 = \frac{g_{tt}(r_0)\, g_{rr}(r_0)}{P_t^2|_0} \left[r_0 \frac{(S^{\varphi r}|_0)^2}{\cos^2\alpha} + J\, S^{\varphi r}|_0 \right]^2$$

$$+ r_0^2 \frac{(S^{\varphi r}|_0)^2}{\cos^2\alpha} \left[g_{tt}(r_0) \frac{P_r^2|_0}{P_t^2|_0} + g_{rr}(r_0) \right]$$

$$s^2 = \frac{g_{tt}(r_0)\, g_{rr}(r_0)}{P_t^2|_0} \left[r_0^2 \frac{(S^{\varphi r}|_0)^4}{\cos^4\alpha} + J^2 (S^{\varphi r}|_0)^2 + 2 r_0 \frac{(S^{\varphi r}|_0)^3}{\cos^2\alpha} J \right]$$

$$+ r_0^2 \frac{(S^{\varphi r}|_0)^2}{\cos^2\alpha} \left[g_{tt}(r_0) \frac{P_r^2|_0}{P_t^2|_0} + g_{rr}(r_0) \right]$$

$$s^2 = \frac{g_{tt}(r_0)\, g_{rr}(r_0)}{P_t^2|_0} \left[r_0^2 \frac{(S^{\varphi r}|_0)^4}{\cos^4\alpha} + J^2 (S^{\varphi r}|_0)^2 + 2 r_0 \frac{(S^{\varphi r}|_0)^3}{\cos^2\alpha} J \right]$$

$$+ r_0^2 \frac{(S^{\varphi r}|_0)^2}{\cos^2\alpha} \frac{g_{tt}(r_0)\, g_{rr}(r_0)}{P_t^2|_0} \left[\frac{P_r^2|_0}{g_{rr}(r_0)} + \frac{P_t^2|_0}{g_{tt}(r_0)} \right]$$

$$s^2 = \frac{g_{tt}(r_0)\, g_{rr}(r_0)}{P_t^2|_0} \left[r_0^2 \frac{(S^{\varphi r}|_0)^4}{\cos^4\alpha} + J^2 (S^{\varphi r}|_0)^2 + 2 r_0 \frac{(S^{\varphi r}|_0)^3}{\cos^2\alpha} J \right]$$

$$+ r_0^2 \frac{(S^{\varphi r}|_0)^2}{\cos^2\alpha} \frac{g_{tt}(r_0)\, g_{rr}(r_0)}{P_t^2|_0} \left[g^{rr}(r_0)\, P_r^2|_0 + g^{tt}(r_0)\, P_t^2|_0 \right]$$

$$s^2 = \frac{g_{tt}(r_0)\, g_{rr}(r_0)}{P_t^2|_0} \left[r_0^2 \frac{(S^{\varphi r}|_0)^4}{\cos^4\alpha} + J^2 (S^{\varphi r}|_0)^2 + 2 r_0 \frac{(S^{\varphi r}|_0)^3}{\cos^2\alpha} J \right]$$

$$+ r_0^2 \frac{(S^{\varphi r}|_0)^2}{\cos^2\alpha} \frac{g_{tt}(r_0)\, g_{rr}(r_0)}{P_t^2|_0} \left[-\mu^2 - g^{\theta\theta}(r_0)\, P_\theta^2|_0 - g^{\varphi\varphi}(r_0)\, P_\varphi^2|_0 \right]$$

$$s^2 = \frac{g_{tt}(r_0)\, g_{rr}(r_0)}{P_t^2|_0}\left[r_0^2 \frac{(S^{\varphi r}|_0)^4}{\cos^4\alpha} + J^2 (S^{\varphi r}|_0)^2 + 2 r_0 \frac{(S^{\varphi r}|_0)^3}{\cos^2\alpha} J \right]$$

$$+ r_0^2 \frac{(S^{\varphi r}|_0)^2}{\cos^2\alpha} \frac{g_{tt}(r_0)\, g_{rr}(r_0)}{P_t^2|_0}\left[-\mu^2 - g^{\theta\theta}(r_0)\, r_0^2 (S^{r\theta}|_0)^2 - g^{\varphi\varphi}(r_0)(r_0 S^{\varphi r}|_0 + J)^2 \right]$$

$$\frac{s^2 P_t^2|_0}{g_{tt}(r_0)\, g_{rr}(r_0)} = \left[r_0^2 \frac{(S^{\varphi r}|_0)^4}{\cos^4\alpha} + J^2 (S^{\varphi r}|_0)^2 + 2 r_0 \frac{(S^{\varphi r}|_0)^3}{\cos^2\alpha} J \right]$$

$$+ r_0^2 \frac{(S^{\varphi r}|_0)^2}{\cos^2\alpha}\left[-\mu^2 - \frac{1}{r_0^2} r_0^2 (S^{r\theta}|_0)^2 - \frac{1}{r_0^2}(r_0 S^{\varphi r}|_0 + J)^2 \right]$$

$$\frac{s^2 P_t^2|_0}{g_{tt}(r_0)\, g_{rr}(r_0)} = r_0^2 \frac{(S^{\varphi r}|_0)^4}{\cos^4\alpha} + J^2 (S^{\varphi r}|_0)^2 + 2 r_0 \frac{(S^{\varphi r}|_0)^3}{\cos^2\alpha} J$$

$$+ \frac{(S^{\varphi r}|_0)^2}{\cos^2\alpha}\left[- r_0^2\mu^2 - r_0^2 (S^{r\theta}|_0)^2 - (r_0 S^{\varphi r}|_0 + J)^2 \right]$$

$$\frac{s^2 P_t^2|_0}{g_{tt}(r_0)\, g_{rr}(r_0)} = r_0^2 \frac{(S^{\varphi r}|_0)^4}{\cos^4\alpha} + J^2 (S^{\varphi r}|_0)^2 + 2 r_0 \frac{(S^{\varphi r}|_0)^3}{\cos^2\alpha} J$$

$$+ \frac{(S^{\varphi r}|_0)^2}{\cos^2\alpha}\left[- r_0^2\mu^2 - r_0^2 (S^{r\theta}|_0)^2 - r_0^2 (S^{\varphi r}|_0)^2 - J^2 - 2 r_0 J S^{\varphi r}|_0 \right]$$

$$\frac{s^2 P_t^2|_0}{g_{tt}(r_0)\, g_{rr}(r_0)} = r_0^2 \frac{(S^{\varphi r}|_0)^4}{\cos^4\alpha} + J^2 (S^{\varphi r}|_0)^2 + 2 r_0 \frac{(S^{\varphi r}|_0)^3}{\cos^2\alpha} J$$

$$+ \frac{(S^{\varphi r}|_0)^2}{\cos^2\alpha}\left[- r_0^2\mu^2 - r_0^2 \left[(S^{r\theta}|_0)^2 + (S^{\varphi r}|_0)^2 \right] - J^2 - 2 r_0 J S^{\varphi r}|_0 \right]$$

$$\frac{s^2 P_t^2|_0}{g_{tt}(r_0)\, g_{rr}(r_0)} = r_0^2 \frac{(S^{\varphi r}|_0)^4}{\cos^4\alpha} + J^2 (S^{\varphi r}|_0)^2 + 2 r_0 \frac{(S^{\varphi r}|_0)^3}{\cos^2\alpha} J$$

$$+ \frac{(S^{\varphi r}|_0)^2}{\cos^2\alpha}\left[- r_0^2\mu^2 - r_0^2 \frac{(S^{\varphi r}|_0)^2}{\cos^2\alpha} - J^2 - 2 r_0 J S^{\varphi r}|_0 \right]$$

$$\frac{s^2 P_t^2 b}{g_{tt}(r_0)\, g_{rr}(r_0)} = r_0^2 \frac{(S^{\varphi r}|_0)^4}{\cos^4\alpha} + J^2 (S^{\varphi r}|_0)^2 + 2 r_0 \frac{(S^{\varphi r}|_0)^3}{\cos^2\alpha} J$$

$$+ \frac{(S^{\varphi r}|_0)^2}{\cos^2\alpha}\left[- r_0^2 \mu^2 - r_0^2 \frac{(S^{\varphi r}|_0)^2}{\cos^2\alpha} - J^2 - 2 r_0 J S^{\varphi r}|_0 \right]$$

$$\frac{s^2 P_t^2 b}{g_{tt}(r_0)\, g_{rr}(r_0)} = J^2 (S^{\varphi r}|_0)^2 + \frac{(S^{\varphi r}|_0)^2}{\cos^2\alpha}\left[- r_0^2 \mu^2 - J^2 \right]$$

$$\frac{s^2 P_t^2 b}{g_{tt}(r_0)\, g_{rr}(r_0)} = \frac{(S^{\varphi r}|_0)^2}{\cos^2\alpha}\left(J^2 \cos^2\alpha - r_0^2 \mu^2 - J^2 \right)$$

$$\frac{s^2 P_t^2 b}{g_{tt}(r_0)\, g_{rr}(r_0)} = \frac{(S^{\varphi r}|_0)^2}{\cos^2\alpha}\left(- J^2 \sin^2\alpha - r_0^2 \mu^2 \right)$$

$$(S^{\varphi r}|_0)^2 = \frac{s^2 P_t^2 b \cos^2\alpha}{- g_{tt}(r_0)\, g_{rr}(r_0) \left(J^2 \sin^2\alpha + r_0^2 \mu^2 \right)}$$

$$S^{\varphi r}|_0 = \pm \sqrt{\frac{s^2 P_t^2 b \cos^2\alpha}{- g_{tt}(r_0)\, g_{rr}(r_0) \left(J^2 \sin^2\alpha + r_0^2 \mu^2 \right)}}$$

$$\boxed{S^{\varphi r}|_0 = \pm \frac{s P_t b \cos\alpha}{\sqrt{- g_{tt}(r_0)\, g_{rr}(r_0) \left(J^2 \sin^2\alpha + r_0^2 \mu^2 \right)}}}$$

$$S^{r\theta}|_0 = S^{\varphi r}|_0 \tan\alpha$$

$$\boxed{S^{r\theta}|_0 = = \pm \frac{s P_t b \sin\alpha}{\sqrt{- g_{tt}(r_0)\, g_{rr}(r_0) \left(J^2 \sin^2\alpha + r_0^2 \mu^2 \right)}}}$$

Using these results in the energy expression gives

$$E = -P_t - \frac{1}{2rP_t}\left(P_\theta^2 + \frac{P_\varphi^2}{\sin^2\theta} - JP_\varphi\right)\partial_r g_{tt} - \tilde{q}\Phi$$

$$E = -P_t|_0 - \frac{1}{2r_0P_t|_0}\left(r_0^2 (S^{r\theta}|_0)^2 + (r_0 S^{\varphi r}|_0 + J) r_0 S^{\varphi r}|_0 \right)\partial_r g_{tt} - \tilde{q}\Phi$$

$$E = -P_t|_0 - \frac{1}{2r_0P_t|_0}\left[r_0^2 \left[(S^{r\theta}|_0)^2 + (S^{\varphi r}|_0)^2\right] + r_0 J S^{\varphi r}|_0 \right]\partial_r g_{tt} - \tilde{q}\Phi$$

$$E = -P_t|_0 - \frac{1}{2r_0P_t|_0}\left[r_0^2 \frac{(S^{\varphi r}|_0)^2}{\cos^2\alpha} + r_0 J S^{\varphi r}|_0 \right]\partial_r g_{tt} - \tilde{q}\Phi$$

$$E = -P_t|_0 - \frac{1}{2r_0P_t|_0}\left[\frac{r_0^2}{\cos^2\alpha} \frac{s^2 P_t|_0^2 \cos^2\alpha}{-g_{tt}(r_0)\, g_{rr}(r_0)\left(J^2\sin^2\alpha + r_0^2\mu^2\right)} \right.$$

$$\left. + r_0 J \frac{s P_t|_0 \cos\alpha}{\sqrt{-g_{tt}(r_0)\, g_{rr}(r_0)\left(J^2\sin^2\alpha + r_0^2\mu^2\right)}} \right]\partial_r g_{tt} - \tilde{q}\Phi$$

$$E = -P_t|_0 - \frac{1}{2r_0P_t|_0}\left[- \frac{r_0^2 s^2 P_t|_0^2}{g_{tt}(r_0)\, g_{rr}(r_0)\left(J^2\sin^2\alpha + r_0^2\mu^2\right)} \right.$$

$$\left. + \frac{r_0 J s P_t|_0 \cos\alpha}{\sqrt{-g_{tt}(r_0)\, g_{rr}(r_0)\left(J^2\sin^2\alpha + r_0^2\mu^2\right)}} \right]\partial_r g_{tt} - \tilde{q}\Phi$$

$$E = -P_t|_0 + \frac{r_0 s^2 \partial_r g_{tt}}{2 g_{tt}(r_0)\, g_{rr}(r_0)\left(J^2\sin^2\alpha + r_0^2\mu^2\right)} P_t|_0$$

$$- \frac{J s \cos\alpha\, \partial_r g_{tt}}{2\sqrt{-g_{tt}(r_0)\, g_{rr}(r_0)\left(J^2\sin^2\alpha + r_0^2\mu^2\right)}} - \tilde{q}\Phi$$

$$P_t|_0 \left[\frac{r_0 s^2 \partial_r g_{tt}}{2 g_{tt}(r_0)\, g_{rr}(r_0) \left(J^2 \sin^2\alpha + r_0^2 \mu^2\right)} - 1 \right] =$$

$$E + \tilde{q}\Phi + \frac{J s \cos\alpha\, \partial_r g_{tt}}{2\sqrt{-g_{tt}(r_0)\, g_{rr}(r_0) \left(J^2 \sin^2\alpha + r_0^2 \mu^2\right)}}$$

$$P_t|_0 = \left[E + \tilde{q}\Phi + \frac{J s \cos\alpha\, \partial_r g_{tt}}{2\sqrt{-g_{tt}(r_0)\, g_{rr}(r_0) \left(J^2 \sin^2\alpha + r_0^2 \mu^2\right)}} \right] \cdot \left[\frac{r_0 s^2 \partial_r g_{tt}}{2 g_{tt}(r_0)\, g_{rr}(r_0) \left(J^2 \sin^2\alpha + r_0^2 \mu^2\right)} - 1 \right]^{-1}$$

We give:

$$\begin{cases} r_0 \\ \theta_0 = \frac{\pi}{2} \\ \varphi_0 = 0 \\ E \\ J \\ s \end{cases} \longrightarrow$$

$S^{\varphi r}|_0$

$S^{r\theta}|_0$

$S^{\theta\varphi}|_0 = 0$

$P_t|_0$

$P_\theta|_0$

$P_\varphi|_0$

$$P_r|_0 = \left[-\frac{1}{g^{rr}} \left(\mu^2 + g^{tt}(P_t|_0)^2 + \frac{(P_\theta|_0)^2 + (P_\varphi|_0)^2}{r_0^2} \right) \right]^{1/2}$$

$S^{tr}|_0$

$S^{t\theta}|_0$

$S^{t\varphi}|_0$