BLACK HOLES

OBSERVATORIO
ASTRONÓMICO
NACIONAL

Classical Black Holes

05. Hypersurfaces

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Outline for Part 1

Hypersurfaces

- Σ: 3-dimensional submanifold that can be timelike, null or spacelike
- $\Phi(x)$ is a smooth function of the coordinates x^μ
- $\Phi(x) = \mathrm{constant}$ defines a family of hypersurfaces
- $\partial_\mu \Phi$ is normal to the hypersurfaces

Normal Vector to a Hypersurface

$$\mathbf{n} = N \left(\partial_\mu \Phi \right) \partial^\mu = N g^{\mu\nu} \partial_\mu \Phi \partial_\nu,$$

$N = N\left(x^\mu\right)$ is an arbitrary "normalization" function.

$$n^2 = n^\mu n_\mu = \begin{cases} 1 & \text{if } \Sigma \text{ is timelike} \\ -1 & \text{if } \Sigma \text{ is spacelike} \\ 0 & \text{if } \Sigma \text{ is null.} \end{cases}$$

Tangent Vector to a hypersurface

The vector t^μ is called a *Tangent Vector* to the hypersurface if

$$t^\mu n_\mu = 0$$

Null Hypersurfaces

- A hypersurface $\mathscr{N}$ is called *null hypersurface* if its normal vector satisfies

$$n^2 = n_\mu n^\mu = 0$$

- Since the normal vector is orthogonal to itself, **n** is also a tangent vector.
- $n^\mu = \frac{dx^\mu}{d\lambda}$
- $x^\mu(\lambda)$ are geodesics.

Null Hypersurfaces

- $x^\mu(\lambda)$ are geodesics.

$$n^\sigma \nabla_\sigma n^\mu|_{\mathcal{N}} \propto n^\mu$$

Choosing appropriately function $N(x^\mu)$,

$$n^\sigma \nabla_\sigma n^\mu|_{\mathcal{N}} = 0$$

Example 1: Schwarzschild in Ingoing Eddington-Finkelstein Coordinates

Coordinates: (v, r, θ, φ)

$$\begin{aligned} g_{vv} &= -\left(1 - \frac{2M}{r}\right) \\ g_{vr} = g_{rv} &= 1 \\ g_{\theta\theta} &= r^2 \\ g_{\varphi\varphi} &= r^2 \sin^2\theta \end{aligned}$$

Hypersurfaces: $\Phi = r - 2M$,

Example 1: Schwarzschild in Ingoing Eddington-Finkelstein Coordinates

Normal vector:

$$\mathbf{n} = N g^{\mu\sigma} \partial_\mu \Phi \partial_\sigma = N \partial_v$$

Magnitude of the normal vector:

$$n^2 = g_{\mu\nu} n^\mu n^\nu$$

$$n^2 = -\left(1 - \frac{2M}{r}\right) N^2$$

At the surface $\Phi = 0$ (i.e. $r = 2M$) we have $n^2 = 0$

Example 1: Schwarzschild in Ingoing Eddington-Finkelstein Coordinates

$$n^\sigma \nabla_\sigma n^\mu|_{\mathcal{N}} = \left[\partial_v \left(\ln |N| \right) - \frac{1}{4M} \right] N n^\mu$$

Choosing $N = e^{\frac{v}{4M}}$ we obtain

$$n^\sigma \nabla_\sigma n^\mu|_{\mathcal{N}} = 0$$

$$\mathbf{n} = e^{\frac{v}{4M}} \partial_v$$

Example 2: Kruskal Manifold

Coordinates: (V, U, θ, φ)

$$g_{UV} = -\frac{16M^3}{r} e^{-r/2M}$$
$$g_{\theta\theta} = r^2$$
$$g_{\varphi\varphi} = r^2 \sin^2\theta$$

Hypersurfaces: $\Phi = U$,

Example 2: Kruskal Manifold

Normal vector:

$$\mathbf{n} = N g^{\mu\sigma} \partial_\mu \Phi \partial_\sigma = -N \frac{r}{16M^3} e^{r/2M} \partial_V$$

Magnitude of the normal vector:

$$n^2 = g_{\mu\nu} n^\mu n^\nu$$

$$n^2 = g_{VV} n^V n^V = 0$$

Any surface $\Phi = U = \text{constant}$ has $n^2 = 0$

Example 2: Kruskal Manifold

$$n^{\sigma}\nabla_{\sigma}n^{\mu}|_{\mathcal{N}} = n^{\sigma}\partial_{\sigma}\left(\ln|N|\right)n^{\mu}|_{\mathcal{N}}$$

Choosing $N = \mathrm{constant} = \frac{8M^2}{e}$ we obtain

$$n^{\sigma}\nabla_{\sigma}n^{\mu}|_{\mathcal{N}} = 0$$

$$\mathbf{n} = -\frac{\partial}{\partial V}$$

Killing Horizons

A null hypersurface $\mathcal{N}$ is a *Killing Horizon* if there exist a Killing vector ξ that is normal to $\mathcal{N}$.

$$\xi = f\mathbf{n}$$

f is an arbitrary non-null function.

Surface Gravity

$$n^\sigma \nabla_\sigma n^\mu |_{\mathcal{N}} = 0$$

$$\xi^\sigma \nabla_\sigma \xi^\mu |_{\mathcal{N}} = \kappa \xi^\mu$$

The proportionality constant κ is called *surface gravity*.

Surface Gravity

$$\kappa = \xi^\sigma \partial_\sigma \ln f$$

Theorem

κ^2 is constant at orbits of the Killing vector ξ.

Example 3: Kruskal Manifold

Null hypersurfaces: $\mathscr{N} = \{U = 0\} \cup \{V = 0\}$

Killing Vector: $\xi = \frac{\partial}{\partial t}$

This can be written at te null hypersurfaces as

$$\xi = \frac{1}{4M}\left[V\frac{\partial}{\partial V} - U\frac{\partial}{\partial U}\right].$$

Example 3: Kruskal Manifold

This vector is normal to the hypersurface $\mathscr{N}$, so it is a Killing horizon.

$$\xi = \frac{V}{4M}\frac{\partial}{\partial V} \text{ en } \{U = 0\}$$

$$\xi = -\frac{U}{4M}\frac{\partial}{\partial U} \text{ en } \{V = 0\},$$

Example 3: Kruskal Manifold

$$\xi = f\mathbf{n},$$

$$\mathbf{n} = \begin{cases} \frac{\partial}{\partial V} \text{ en } \{U = 0\} \\ \frac{\partial}{\partial U} \text{ en } \{V = 0\} \end{cases}$$

$$f = \begin{cases} \frac{V}{4M} \text{ en } \{U = 0\} \\ -\frac{U}{4M} \text{ en } \{V = 0\} \end{cases}$$

Example 3: Kruskal Manifold

Surface Gravity

$$\kappa = \xi^{\sigma}\partial_{\sigma}\ln f,$$

$$\kappa = \begin{cases} \frac{1}{4M} \text{ en } \{U=0\} \\ -\frac{1}{4M} \text{ en } \{V=0\} \end{cases}$$

Example 3: Kruskal Manifold

Surface Gravity

$$\kappa = \xi^{\sigma} \partial_{\sigma} \ln f,$$

$$\kappa = \begin{cases} \frac{1}{4M} \text{ en } \{U = 0\} \\ -\frac{1}{4M} \text{ en } \{V = 0\} \end{cases}$$

$$\kappa^2 = \left(\frac{1}{4M} \right)^2$$

Asymptotically Simple Spacetime

A manifold $(\mathscr{M}, \mathbf{g})$ is called *asymptotically simple* if there is a manifold $(\tilde{\mathscr{M}}, \tilde{\mathbf{g}})$ with boundary $\partial\tilde{\mathscr{M}}$ and a continuous embedding

$$f : \mathscr{M} \longrightarrow \tilde{\mathscr{M}},$$

such that

1. $f(\mathscr{M}) = \tilde{\mathscr{M}} - \partial\tilde{\mathscr{M}}$
2. There is a smooth function ω in $\tilde{\mathscr{M}}$ satisfying
   - 2.1 $\omega > 0$ in $f(\mathscr{M})$
   - 2.2 $\omega = 0$ in $\partial\tilde{\mathscr{M}}$
   - 2.3 $d\omega \neq 0$ in $\partial\tilde{\mathscr{M}}$
   - 2.4 $\tilde{\mathbf{g}} = \omega^2\mathbf{g}$
3. Every null geodesic in $\mathscr{M}$ has two end points in $\partial\tilde{\mathscr{M}}$

$\tilde{\mathscr{M}}$ is called the conformal compactification of $\mathscr{M}$.

Weak Asymptotically Simple Spacetime

A manifold $(\mathscr{M}, \mathbf{g})$ is called *weak asymptotically simple* if there is an open set $U \subset \mathscr{M}$ isometric to an open neighbourhood of $\partial\tilde{\mathscr{M}}$ where $\tilde{\mathscr{M}}$ is the conformal compactification of some asymptotically simple manifold.

Asymptotically Empty Spacetime

A manifold is called *asymptotically empty* if $R_{\mu\nu} = 0$ in an open neighbourhood of $\partial\tilde{\mathscr{M}}$ where $\tilde{\mathscr{M}}$ is the conformal compactification of some asymptotically simple manifold.

Asymptotically Flat Spacetime

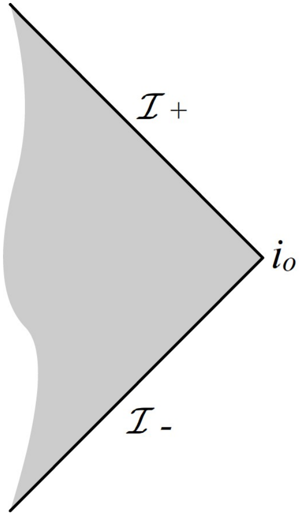
A manifold is called *asymptotically flat* if it is weak asymptotically simple and asymptotically empty.

Example 4: Minkowski's Manifold

Taking $\mathscr{M}$ as Minkowski's manifold and $\tilde{\mathscr{M}}$ its conformal compactification, it is clear that conditions 1., 2. and 3. are satisfied, i.e. Minkowski is an asymptotically simple spacetime.

It is also weak asymptotically simple, asymptotically empty and asymptotically flat.

Asymptotically Flat Manifold Diagram



Asymptotically Flat Manifold Diagram

Asymptotically flat manifolds admit vectors that are asymptotically equivalent to the Killing vectors of Minkowski spacetime near i_0.

Therefore, these Killing vectors allow the definition of total mass, momentum and angular momentum on spacelike hypersurfaces.

Example 5: Kruskal Manifold

Taking $\mathscr{M}$ as Kruskal's manifold and $\tilde{\mathscr{M}}$ its conformal compactification, condition 3 above is NOT satisfied because there are some null geodesics that end up at the singularity and not in $\partial\tilde{\mathscr{M}}$.
Therefore, Kruskal is not asymptotically simple.

However, this manifold is weak asymptotically simple and asymptotically flat. Thus we conclude that it is also asymptotically flat as is shown in its Carter-Penrose diagram.

Causal Curve

A *causal curve* $\mathscr{C}(\lambda)$ is any non space-like smooth curve (i.e. it is time-like or null everywhere).

Causal Past

The *causal past* $\mathscr{J}^-(p)$ of point p is the set of all the events that causally preceded point p, i.e. is the set of all points q for which there is at least one future directed causal curve from q to p.

The causal past $\mathscr{J}^-(U)$ of a set of points $U \subset \mathscr{M}$ is the set of all points that causally preceded at least one point of U.

Causal Future

The *causal future* $\mathscr{J}^+(p)$ of a point p is the set of all points q for which there is at least one future directed causal curve from p to q.

The causal future $\mathscr{J}^+(U)$ of a set of points $U \subset \mathscr{M}$ is the set of all points that causally follow at least one point of U.

Topological Closure

$\overline{\mathscr{J}}^{-}(U)$ and $\overline{\mathscr{J}}^{+}(U)$ are the topological closure of the causal past and causal future of U, respectively.

$j^{-}(U)$ and $j^{+}(U)$ are the boundary of the topological closures,

$$
\begin{aligned}
j^{-}(U) &= \overline{\mathscr{J}}^{-}(U) - \mathscr{J}^{-}(U) \\
j^{+}(U) &= \overline{\mathscr{J}}^{+}(U) - \mathscr{J}^{+}(U) .
\end{aligned}
$$

Future Event Horizon

The *future event horizon* $\mathscr{H}^+$ is defined as the boundary of the closure of the causal past of the null future infinite $\mathscr{I}^+$.

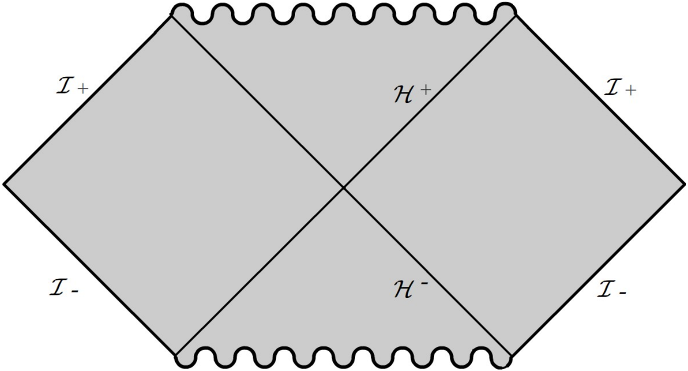
$$\mathscr{H}^+ = \dot{J}^- \left(\mathscr{I}^+ \right).$$

Past Event Horizon

Similarly, the *past event horizon* $\mathscr{H}^-$ is defined as the boundary of the closure of the causal future of the null past infinite $\mathscr{I}^-$.

$$\mathscr{H}^- = \mathrm{j}^+\left(\mathscr{I}^-\right).$$

Example 6: Kruskal Manifold



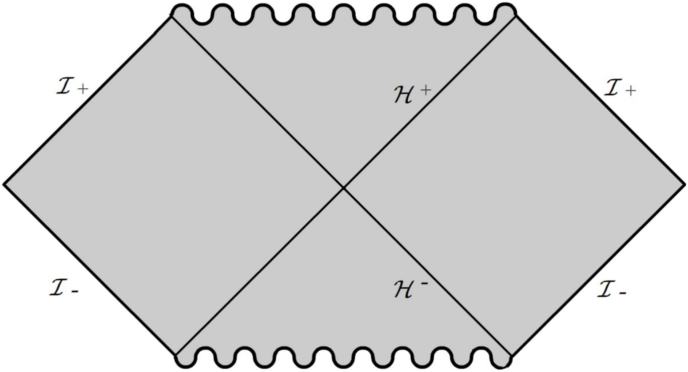
Black Hole

A *Black Hole* in an asymptotically flat spacetime $\mathscr{M}$ is defined as the set of events that do not belong to the causal past of the future null infinity $\mathscr{J}^-(\mathscr{I}^+)$, namely

$$\mathscr{B} = \mathscr{M} - \mathscr{J}^-(\mathscr{I}^+)$$

The *event horizon* is the boundary of $\mathscr{B}$.

Example 6: Kruskal Manifold



Next Lecture

06. Rotating Black Holes