BLACK HOLES

OBSERVATORIO
ASTRONÓMICO
NACIONAL

Classical Black Holes

03. Schwarzschild Solution. Properties

Edward Larrañaga

Outline for Part 1

Physical Conditions to obtain Schwarzschild's Metric

- Spherical Symmetry
- Asymptotically flat spacetime
- Static Metric
- Empty Spacetime

Spherically Symmetric Solution in Empty Space

$$ds^2 = -\left(1 - \frac{K}{r}\right) dt^2 + \left(1 - \frac{K}{r}\right)^{-1} dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2$$

K: Integration constant to be determined

Newtonian Approximation of the metric

$$g_{\mu\nu}(x) = \eta_{\mu} + h_{\mu\nu}(x)$$

where

- $\eta_{\mu\nu}$ is the Minkowskian metric representing a flat spacetime.
- $h_{\mu\nu}(x)$ is a perturbative term that contains the information of a weak gravitational field.

$$\left|h_{\mu\nu}\right| \sim h \ll 1$$

Newtonian Approximation of the metric

The inverse of the metric, up to linear terms in the perturbation is

$$g^{\mu\nu}(x) = \eta^{\mu\nu} - h^{\mu\nu}(x) + \mathcal{O}(h^2)$$

where

$$h^{\mu\nu} = \eta^{\mu\alpha}\eta^{\nu\beta}h_{\alpha\beta}$$

Newtonian Approximation of the Connections

Inserting these expressions for the metric and its inverse, the connections are linearised as

$$\Gamma^{\alpha}_{\mu\nu} = \frac{1}{2}\eta^{\alpha\sigma}\left[\partial_{\mu}h_{\sigma\nu} + \partial_{\nu}h_{\mu\sigma} - \partial_{\sigma}h_{\mu\nu}\right] + \mathcal{O}\left(h^{2}\right)$$

Newtonian Approximation of the Curvature Tensors

Riemann and Ricci tensors are approximated, up to linear terms in the perturbation, as

$$R_{\alpha\mu\beta\nu} = \frac{1}{2}\left[\partial_\mu \partial_\beta h_{\alpha\nu} + \partial_\alpha \partial_\nu h_{\mu\beta} - \partial_\mu \partial_\nu h_{\alpha\beta} - \partial_\alpha \partial_\beta h_{\mu\nu}\right] + \mathcal{O}\left(h^2\right)$$

$$R_{\mu\nu} = \frac{1}{2}\left[\partial_\sigma \partial_\mu h^\sigma_\nu + \partial_\sigma \partial_\nu h^\sigma_\mu - \partial_\mu \partial_\nu h - \Box h_{\mu\nu}\right] + \mathcal{O}\left(h^2\right)$$

The Energy-Momentum Tensor

In order to have a consistent picture of a nearly flat spacetime, we must consider that the energy-momentum contribution of the source is small and that the characteristic velocities of the matter distribution are small compared with the speed of light. Consider the general form of the energy-momentum tensor for a perfect fluid,

$$T^{\mu\nu} = \frac{\varepsilon + P}{c^2} u^\mu u^\nu + P g^{\mu\nu},$$

where ε and P are the energy density and pressure of the distribution and u^μ is the velocity 4-vector associated with the matter distribution.

The Energy-Momentum Tensor

The assumptions described above let us neglect the pressure contribution of a matter distribution such an ideal gas. Therefore, we will consider the energy-momentum tensor of an incoherent matter (dust) source,

$$T^{\mu\nu} = \frac{\varepsilon}{c^2} u^{\mu} u^{\nu}.$$

The Energy-Momentum Tensor

Considering that $\frac{v}{c} \ll 1$ we have that

$$\gamma = \left(1 - \frac{v^2}{c^2}\right)^{-1/2} = 1 + \mathcal{O}\left(\frac{v^2}{c^2}\right)$$

Under this aproximation, the components of the energy-momentum tensor can be approximated as

$$T^{00} = \frac{\varepsilon}{c^2} u^0 u^0 = \varepsilon + \mathcal{O}\left(\frac{v^2}{c^2}\right) \tag{1}$$

$$T^{0j} = \frac{\varepsilon}{c^2} u^0 u^j = \varepsilon \frac{v^j}{c} + \mathcal{O}\left(\frac{v^2}{c^2}\right) \tag{2}$$

$$T^{ij} = \frac{\varepsilon}{c^2} u^i u^j = 0 + \mathcal{O}\left(\frac{v^2}{c^2}\right) \tag{3}$$

The Energy-Momentum Tensor

The trace of the energy momentum tensor simplifies to

$$T = g_{\mu\nu}T^{\mu\nu} = -\varepsilon + \mathcal{O}\left(\frac{v^2}{c^2}\right)$$

The Linearized Field Equations

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = \frac{8\pi G}{c^4} T_{\mu\nu}$$

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$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = \frac{8\pi G}{c^4}T_{\mu\nu}$$

$$R = -\frac{8\pi G}{c^4}T$$

$$R_{\mu\nu} = \frac{8\pi G}{c^4}\left[T_{\mu\nu} - \frac{1}{2}g_{\mu\nu}T\right]$$

The Linearized Field Equations

Component $\mu = \nu = 0$ of the field equation

$$R^{00} = \frac{8\pi G}{c^4} \left[T^{00} - \frac{1}{2} g^{00} T \right]$$

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$$\frac{1}{2}\left[2\partial_\sigma\partial^0 h^{0\sigma} - \partial^0\partial^0 h - \Box h^{00}\right] + \mathscr{O}(h^2) = \frac{8\pi G}{c^4}\left[\varepsilon + \frac{1}{2}\eta^{00}\varepsilon\right] + \mathscr{O}\left(\frac{v^2}{c^2}\right)$$

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$$2\partial_\sigma \partial^0 h^{0\sigma} - \partial^0 \partial^0 h - \Box h^{00} = \frac{8\pi G}{c^4} \varepsilon$$

The Linearized Field Equations

Considering that the gravitational field is quasi-static (i.e. it does not change appreciably in time), we can neglect time derivatives of the metric tensor. Hence,

$$-\nabla^2 h^{00} = \frac{8\pi G}{c^4}\varepsilon$$

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Introducing the mass density $\rho = \frac{\varepsilon}{c^2}$ gives

$$-\frac{c^2}{2}\nabla^2 h^{00} = 4\pi G\rho$$

The Linearized Field Equations

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Comparison with the Poisson equation,

$$\nabla^2\phi = 4\pi G\rho$$

gives the identification

$$h^{00} = -\frac{2\phi}{c^2}$$

or

$$h_{00} = -\frac{2\phi}{c^2}$$

Back to the Spherically Symmetric Solution

$$g_{00} = \eta_{00} + h_{00} = -\left(1 - \frac{K}{r}\right)$$

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Back to the Spherically Symmetric Solution

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Considering the Newtonian gravitational potential associated with a spherical mass M gives the identification of the integration constant,

$$-\frac{2}{c^2}\left(-\frac{GM}{r}\right) = \frac{K}{r}$$

$$K = \frac{2GM}{c^2}$$

Schwarzschild's Solution

$$ds^2 = -\left(1 - \frac{2GM}{c^2 r}\right) dt^2 + \left(1 - \frac{2GM}{c^2 r}\right)^{-1} dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2$$

M: Mass associated to the central object
(Total energy content in spacetime)

Schwarzschild's Solution

$$ds^2 = -\left(1 - \frac{2M}{r}\right) dt^2 + \left(1 - \frac{2M}{r}\right)^{-1} dr^2 + r^2 d\Omega^2$$

$$d\Omega^2 = d\theta^2 + \sin^2\theta d\phi^2$$

Geometrical units: $G = c = 1$

Physical Properties of Schwarzschild's Solution

The limit $r \longrightarrow \infty$ gives Minkowski's metric,

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Asymptotically flat spacetime

Interpretation of Coordinates in Schwarzschild's Solution

1. If we consider a surface with $t = constant$ and $r = R = constant$, i.e. such that $dt = dr = 0$, the remaining terms in the line element are

$$ds^2 = R^2 d\Omega^2 = R^2 \left(d\theta^2 + \sin^2\theta d\phi^2 \right),$$

which corresponds to the measure of length on the surface of a sphere of radius R.

We interpret (θ, ϕ) *as the usual spherical angular coordinates.*

Interpretation of Coordinates in Schwarzschild's Solution

2. Consider a static observer at the coordinates $(r, \theta, \phi) = (R, \theta_0, \phi_0)$, i.e. with $dr = d\theta = d\varphi = 0$.
The line element becomes

$$ds^2 = -\left(1 - \frac{2M}{R}\right) dt^2$$

or in terms of the proper time τ_R of the static observer at radius R,

$$d\tau_R^2 = \left(1 - \frac{2M}{R}\right) dt^2.$$

Interpretation of Coordinates in Schwarzschild's Solution

- Now consider a static observer located at the asymptotic region, $R \longrightarrow \infty$. *The coordinate time t corresponds to the proper time of the static asymptotic observer,*

$$d\tau_{\infty}^{2} = dt^{2}.$$

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- For $R \longrightarrow 2M$, we have $d\tau_R \longrightarrow 0$.

Interpretation of Coordinates in Schwarzschild's Solution

3. To consider radial proper distance, let $dt = d\theta = d\phi = 0$ in the line element. The *proper radial length* is given by the differential quantity dl,

$$ds^2 = dl^2 = \left(1 - \frac{2M}{r}\right)^{-1} dr^2$$

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A ruler located radially in this spacetime measures dl and not dr.

Gravitational Redshift

The wavelength of an electromagnetic wave is proportional to its period, $\tau = \frac{\lambda}{c}$.

The relation between the wavelength λ as seen by an static observer at radius R and the wavelength λ_∞ as seen by an asymptotic static observer is

$$\lambda = \sqrt{1 - \frac{2M}{R}}\lambda_\infty$$

Gravitational Redshift

The redshift factor z is defined as

$$z = \frac{\lambda_\infty - \lambda}{\lambda}$$

from which

$$1 + z = \left(1 - \frac{2M}{R}\right)^{-\frac{1}{2}}$$

Physical Properties of Schwarzschild's Solution

Singularities

- $r = 0$
- $r = 2M$

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$$K = R_{\alpha\beta\mu\nu}R^{\alpha\beta\mu\nu} = \frac{48M^2}{r^6}$$

Kretchmann Scalar

Physical Properties of Schwarzschild's Solution

Singularities

- $r = 2M$: Schwarzschild's radius

Physical Properties of Schwarzschild's Solution

Singularities

- $r = 2M$: Schwarzschild's radius
  Coordinate Singularity

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The infinite value of the line element at this value can be removed with a change of coordinates (e.g. Eddington-Finkelstein coordinates)

Causal Structure of Schwarzschild's Metric

Ingoing radial light rays move according to the conditions

$$ds^2 = 0$$

$$d\Omega = 0$$

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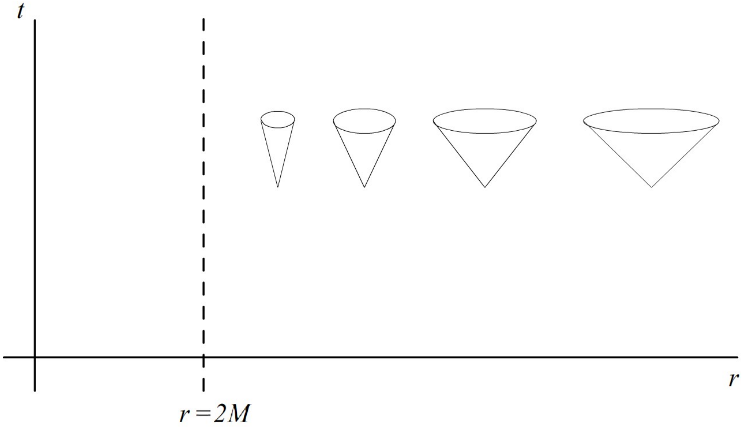

$$\frac{dt}{dr} = \pm \frac{1}{\left(1 - \frac{2M}{r}\right)}$$

Killing Vector Fields

$$\xi = \frac{\partial}{\partial t}$$

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- Temporal Inversion Symmetry

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- Azimuthal Symmetry

Schwarzschild's Embedding Diagram

Equatorial Spatial Slice: $dt = 0$ and $\theta = \frac{\pi}{2}$

Schwarzschild's Embedding Diagram

Equatorial Spatial Slice: $dt = 0$ and $\theta = \frac{\pi}{2}$

$$ds^2 = \left(1 - \frac{2M}{r}\right)^{-1} dr^2 + r^2 d\phi^2$$

Schwarzschild's Embedding Diagram

3-dimensional space in Cylindrical coordinates,

$$ds^2 = dz^2 + dr^2 + r^2 d\phi^2$$

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$$ds^2 = dz^2 + dr^2 + r^2 d\phi^2$$

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$$ds^2 = \left[1 + \left(\frac{dz}{dr}\right)^2\right] dr^2 + r^2 d\phi^2$$

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$$z = \sqrt{8M}\sqrt{r - 2M}$$

Regge-Wheeler Radial Coordinate

$$(t, r, \theta, \phi) \longrightarrow (t, r^*, \theta, \phi)$$

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While the original coordinate r takes values in the range $2M < r < \infty$, the new coordinate r^* takes values in $-\infty < r^* < \infty$.

Ingoing Eddington-Finkelstein Coordinates

$$(t, r, \theta, \phi) \longrightarrow (v, r, \theta, \phi)$$

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In these coordinates there is no singularity at $r = 2M$.

Causal Structure in Ingoing Eddington-Finkelstein Coordinates

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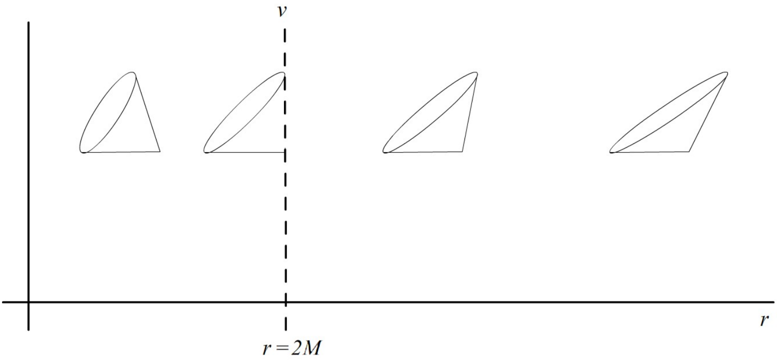

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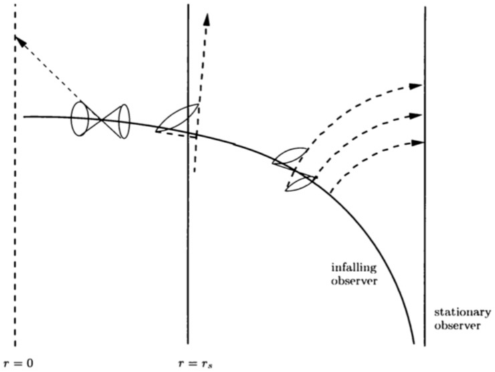

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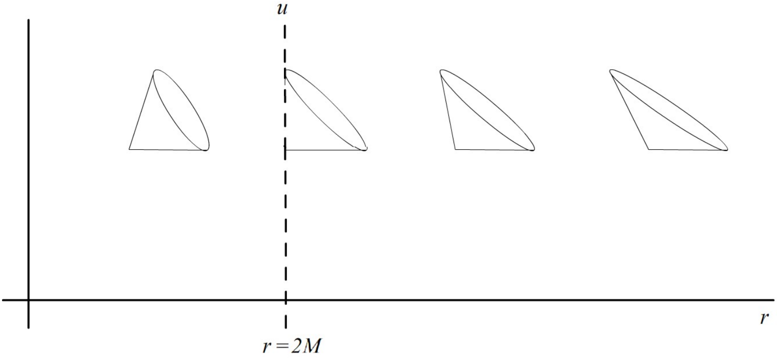

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Next Lecture

04. Penrose Diagrams, Hypersurfaces and Horizons