BLACK HOLES

OBSERVATORIO
ASTRONÓMICO
NACIONAL

Classical Black Holes

05. Astrophysics of Black Holes

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Outline for Part 1

Astrophysical Evidence of Black Holes existence

Astrophysical observations have discovered evidence of the existence of at least three classes of black holes:

- **Stellar-mass black holes**. Masses from about $3M_\odot$ and up to $\sim 100M_\odot$.
- **Supermassive black holes**. At the center of galaxies with masses in the range $M \sim 10^5 - 10^{10} M_\odot$
- **Intermediate-mass black holes**. Objects with a masses in the range $M \sim 10^2 - 10^4 M_\odot$, filling the gap between the stellar-mass and the supermassive ones.

Origin of Black Holes

The principal mechanism for black hole production is the *complete gravitational collapse* of massive bodies, for example dying stars. When a star exhausts all its nuclear fuel, it shrinks to find a new

equilibrium configuration.The degenerate pressure of electrons can, eventually, stop the collapse to produce a **white dwarf**.

If the mass of the star beginning the collapse is greater than the Chandrasekhar’s Limit, the degenerate pressure of electrons can not stop the collapse.

Origin of Black Holes

The degenerate pressure of neutrons can stop the collapse to produce a **neutron star**.

The Oppenheimer-Volkof-Tolman Limit indicates the maximum mass for which the degenerate pressure of neutrons can stop the collapse. For very heavy stars, beyond the OVT limit, there is no known physical mechanism to stop the collapse !

The final product in this stage is thought to be a **Black Hole**.

Origin of Black Holes

Other mechanisms to produce Black holes include:

- The collapse of primordial inhomogeneities in the early Universe.
- The collapse of primordial inhomogeneities during phase transitions, such as inflation.
- Gravitational collapse of dark matter.
- Gravitational collapse of the first generation of stars (the so-called Population III stars).
- Gas-dynamical instabilities.
- Stellar-dynamical stabilities.

A First Model of Stellar Structure

XIX Century Model by Lane, Kelvin and Helmholtz.

Hydrostatic equilibrium

$$\frac{dP(r)}{dr} = -\frac{GM(r)\rho(r)}{r^2}$$

P: Pressure

$$P = \frac{\rho k T}{\mu m_p}$$

k: Boltzmann's Constant

μ: mean molecular weight

T: temperature

ρ: mass density

m_p: mass of the proton

A First Model of Stellar Structure

Kelvin-Helmholtz: Source of the heat is the gravitational contraction of the gas.
The total energy of the Sun will be released in just 10^7 years.

A Better Model of Stellar Structure

A. S. Eddington (1926):

1. The source of energy is thermonuclear.
2. Outward pressure of radiation should be included in the equations.

A Better Model of Stellar Structure

Hydrostatic Equilibrium Equation

$$\frac{d}{dr}\left[\frac{\rho k T}{\mu m_p} + \frac{1}{3} a T^4\right] = -\frac{GM(r)\rho(r)}{r^2}$$

Energy Transport Equation

$$\frac{dP_{rad}(r)}{dr} = -\frac{L(r)}{4\pi r^2 c}\frac{1}{\ell}$$

Energy Equation

$$\frac{dL(r)}{dr} = 4\pi r^2 \varepsilon \rho$$

ℓ: mean free path of the protons

L: luminosity

ε: energy generated per gram of material per unit time

Physical Aspects of the Stellar Death

Nuclear reactions $\rightarrow$ thermal pressure $\rightarrow$ support gravity

For massive stars $(M > 5M_{\odot})$,
$H \rightarrow He \rightarrow \ldots \rightarrow C \rightarrow \ldots \rightarrow Fe$

White Dwarf. Degenerate Gas of Electrons

Equation of State for the degenerate gas of electrons

$$P_{rel} = K\rho^{4/3}$$

Using $\frac{dP(r)}{dr} = -\frac{GM(r)\rho(r)}{r^2}$ we obtain

$$\frac{M^{4/3}}{r^5} \propto \frac{GM^2}{r^5}$$

Chandrasekhar's Limit (1931)

$$\frac{M^{4/3}}{r^5} \propto \frac{GM^2}{r^5}$$

This relation is satisfied by the unique mass

$$M_C = 0.197 \left[\left(\frac{hc}{G} \right)^3 \frac{1}{m_p^2} \right] \frac{1}{\mu_e^2}$$

μ_e: mean molecular wight of the electrons

$$M_C \approx 1.4 M_{\odot}$$

Mass of a completely degenerated star.

Neutron Stars. Degenerate Gas of Neutrons

Chandrasekhar (1939)
If the degenerate star attains sufficiently high densities, the protons and electrons will combine to form neutrons.

Baade and Zwicky (1934)
"...supernovae represent the transitions from ordinary stars into neutron stars which in their final stages consist of extremely closely packed neutrons."

Neutron Stars. Degenerate Gas of Neutrons

Oppenheimer and Volkoff (1939)
Oppenheimer and Snyder (1939)

"...when all thermonuclear sources of energy are exhausted a sufficiently heavy star will collapse. This contraction will continue indefinitely till the radius of the star approaches asymptotically its gravitational radius. Light from the surface of the star will be progressively reddened and can escape over a progressively narrower range of angles till eventually the star tends to close itself off from any communication with a distant observer".

Black Holes

If the collapsing core is too massive to be supported by the degenerate pressure of neutrons, there is no known mechanism capable of finding a new equilibrium configuration, and the body should undergo a complete collapse.

In this case, the final product is a black hole.

Outline for Part 2

Spherically Symmetric Collapse

- Spherical Symmetry:

$$ds^2 = -e^{2A(t,r)}dt^2 + e^{2B(t,r)}dr^2 + R^2(t,r)d\Omega^2$$

- Suppose that the collapsing body can be described as a perfect fluid.
  Then, we consider that coordinates t and r are attached to every collapsing particle (co-moving coordinates).

Spherically Symmetric Collapse

- In the co-moving frame, the 4-velocity of the fluid is just

$$u^\mu = \left(e^{-A}, 0, 0, 0\right)$$

and thus $u^2 = -1$.

- The energy-momentum tensor in the co-moving frame is

$$T^\mu_\nu = diag(\rho, P, P, P)$$

Spherically Symmetric Collapse

The Einstein equations for this metric give

$$
\begin{aligned}
G^t_t = 8\pi T^t_t \quad &\Rightarrow \quad \frac{F'}{R^2 R'} = 8\pi\rho \\
G^r_r = 8\pi T^r_r \quad &\Rightarrow \quad \frac{\dot{F}}{R^2 \dot{R}} = -8\pi P \\
G^t_r = 0 \quad &\Rightarrow \quad \dot{R}' - \dot{R}A' - \dot{B}R' = 0
\end{aligned}
$$

$$F = R\left(1 - e^{-2B}R'^2 + e^{-2A}\dot{R}^2\right)$$

Misner-Sharp mass

Spherically Symmetric Collapse

The Misner-Sharp mass

$$F = R\left(1 - e^{-2B}R'^2 + e^{-2A}\dot{R}^2\right)$$

is defined by the relation

$$1 - \frac{F}{R} = g_{\mu\nu}\left(\partial^\mu R\right)\left(\partial^\nu R\right)$$

* Note that $n^\mu = \partial^\mu R$ is normal to the surfaces $R = \mathrm{constant}$.
Thus, when $1 - \frac{F}{R} = 0$, the corresponding surface is null.

Spherically Symmetric Collapse

From (t, t)-component of the Field equations we obtain the Misner-Sharp mass as

$$G^t_t = 8\pi T^t_t \Rightarrow \frac{F'}{R^2 R'} = 8\pi\rho$$

$$F(r) = \int_0^r F' d\tilde{r} = 8\pi \int_0^r \rho R^2 R' d\tilde{r} = 2M(r)$$

Spherically Symmetric Collapse

Finally, the conservation of the energy-momentum tensor gives the equation

$$\nabla_\mu T^\mu_\nu = 0 \Rightarrow A' = -\frac{P'}{\rho + P}$$

Dust Collapse

Dust Collapse

Dust is characterized by $P = 0$.

$$\begin{aligned}
G^t_t = 8\pi T^t_t \quad &\Rightarrow \quad \frac{F'}{R^2 R'} = 8\pi\rho \\
G^r_r = 8\pi T^r_r \quad &\Rightarrow \quad \frac{\dot{F}}{R^2 \dot{R}} = 0 \\
G^t_r = 0 \quad &\Rightarrow \quad \dot{R}' - \dot{R}A' - \dot{B}R' = 0 \\
\nabla_\mu T^\mu_\nu = 0 \quad &\Rightarrow \quad A' = 0
\end{aligned}$$

Then

$$\begin{aligned}
F &= F(r) \\
&= (t)
\end{aligned}$$

Dust Collapse

$F = F(r)$: No inflow or outflow of energy through spherical shells.
Thus, the exterior metric is Schwarzschild.
(* If $P \neq 0$ the exterior metric must be a non-vacuum Vaidya spacetime.)

If the boundary of the cloud of dust is located at the co-moving coordinate $r = r_b$, we have $F(r_b) = 2M$ with M the Schwarzschild's mass of the exterior metric.

Dust Collapse

$A = A(t)$: One can re-define the time coordinate such that

$$e^{A(t)}dt \Rightarrow dt$$

and therefore $g_{tt} = -1$

Dust Collapse

$$\dot{R}' - \dot{R}A' - \dot{B}R' = 0$$

$$\dot{R}' - \dot{B}R' = 0$$

$$\frac{1}{R'}\frac{d(R')}{dt} = \frac{dB}{dt}$$

$$\frac{d(\log R')}{dt} = \frac{dB}{dt}$$

$$R' = e^{B+h(r)}$$

Dust Collapse

$$R' = e^{B+h(r)}$$

Introducing $f(r) = e^{2h(r)} - 1$ we have

$$e^{2B} = \frac{R'^2}{1+f}$$

Dust Collapse

$$ds^2 = -dt^2 + \frac{R'^2(t,r)}{1+f(r)}dr^2 + R^2(t,r)d\Omega^2$$

Lemaitre-Tolman-Bondi Metric

Dust Collapse

Kretschmann Scalar:

$$K = 12\frac{F'^2}{R^4 R'^2} - 32\frac{FF'}{R^5 R'} + 48\frac{F^2}{R^6}$$

Divergence at $R = 0$

Dust Collapse

One can re-scale to set $R(t,r)$ to equal the co-moving radius r at the time $t=0$, i.e. we impose $R(0,r)=r$ and introduce a scale factor a such that

$$R(t,r) = ra(t,r)$$

$a(0,r)=1$
$a(t_s,r)=0$ (at the time of the formation of the singularity, $t=t_s$)

The collapse also impose the condition $\dot{a}<0$

Dust Collapse

One of the field equations,

$$\frac{F'}{R^2 R'} = 8\pi\rho$$

implies that a regular value of the density at $t = 0$ is obtained if the Misner-Sharp mass has the form

$$F(r) = r^3 m(r)$$

where $m(r)$ is a (sufficiently) regular function in the region $[\,0, r_b\,]$.

Dust Collapse

We have $F' = 3r^2m + r^3m'$ and therefore

$$\frac{F'}{R^2R'} = 8\pi\rho$$

$$\frac{3r^2m + r^3m'}{(ra)^2(a + ra')} = 8\pi\rho$$

$$\frac{3m + rm'}{a^2(a + ra')} = 8\pi\rho$$

Dust Collapse

It is usual to consider $m(r)$ as a polynomial expansion around $r = 0$,

$$m(r) = \sum_{k=0}^{\infty} m_k r^k$$

where m_k are constants.

* In order to prevent the existence of cusps of ρ at $R = 0$, we require that $m_1 = 0$.

Dust Collapse

From the definition of the Misner-Sharp mass we have

$$\dot{R}^2 = \frac{F}{R} + f$$

Using the proposed forms for F and R we obtain

$$r^2\dot{a}^2 = \frac{mr^3}{ra} + f$$

$$\dot{a} = -\sqrt{\frac{m}{a} + \frac{f}{r^2}}$$

Dust Collapse

Evaluating at the time $t = 0$ we have $a(0, r) = 1$ and thus

$$\dot{a}(0, r) = -\sqrt{m + \frac{f}{r^2}}$$

Note that the function f defines the initial velocity of the particles in the cloud.

It is usual to write this function as an expansion around $r = 0$ as

$$f(r) = r^2 b(r)$$

with

$$b(r) = \sum_{k=0}^{\infty} b_k r^k$$

Homogeneous Dust Collapse

Homogeneous Dust Collapse

Simplest Model of Gravitational Collapse. Oppenheimer-Snyder (1939)

Homogeneous: $\rho = \rho(t)$

This implies that
$m' = 0 \longrightarrow m = m_0$

Homogeneous Dust Collapse

Simplest Model of Gravitational Collapse. Oppenheimer-Snyder (1939)

It also implies that

$a' = 0 \longrightarrow \frac{f}{r^2} = \mathrm{constant}$
$f(r) = r^2 b(r) = r^2 \mathrm{constant}$
which is accomplished by choosing $b = b_0$
Then

$$\dot{a} = -\sqrt{\frac{m_0}{a} + b_0}$$

Homogeneous Dust Collapse

The Lemaitre-Tolman-Bondi line element becomes

$$ds^2 = -dt^2 + \frac{R'^2(t,r)}{1+f(r)}dr^2 + R^2(t,r)d\Omega^2$$

$$ds^2 = -dt^2 + \frac{a^2(t)}{1+r^2b_0}dr^2 + r^2a^2(t)d\Omega^2$$

$$ds^2 = -dt^2 + a^2(t)\left[\frac{dr^2}{1+b_0r^2} + r^2d\Omega^2\right]$$

Homogeneous Dust Collapse

$$ds^2 = -dt^2 + a^2(t)\left[\frac{dr^2}{1+b_0r^2} + r^2 d\Omega^2\right]$$

For $b_0 = 0$ this line element becomes

$$ds^2 = -dt^2 + a^2(t)\left[dr^2 + r^2 d\Omega^2\right]$$

and describes the counterpart of a flat spacetime, in which the collapse is marginally bound (i.e. the falling particles have vanishing velocity at infinity).

Homogeneous Dust Collapse

For $b_0 = 0$ the scale factor is

$$\dot{a} = -\sqrt{\frac{m_0}{a} + b_0}$$

$$\frac{da}{dt} = -\sqrt{\frac{m_0}{a}}$$

$$\sqrt{a}da = -\sqrt{m_0}dt$$

$$\frac{2}{3}a^{3/2} - \frac{2}{3}a^{3/2}(0) = -\sqrt{m_0}t$$

Homogeneous Dust Collapse

The value of the initial condition was supposed as $a(0) = 1$. Thus

$$a(t) = \left[1 - \frac{3\sqrt{m_0}}{2}t\right]^{2/3}$$

Therefore, the singularity $R = 0$ occurs at the time t_s for which $a(t_s) = 0$. This is

$$t_s = \frac{2}{3\sqrt{m_0}}$$

Homogeneous Dust Collapse

The curve $t_H(r)$ describing the time at which the shell r crosses the horizon can be obtained from the condition

$$1 - \frac{F}{R} = 0$$

$$1 - \frac{m_0 r^3}{r a(t_H)} = 0$$

$$a(t_H) = m_0 r^2$$

Homogeneous Dust Collapse

$$a(t_H) = m_0 r^2$$

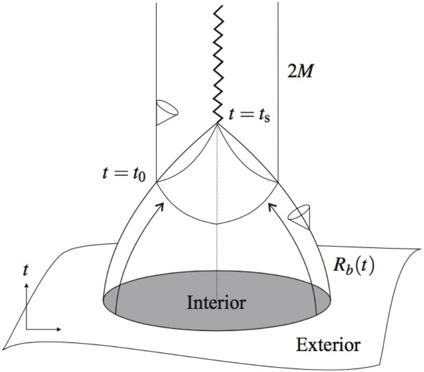
For $b_0 = 0$ we have

$$\left[1 - \frac{3\sqrt{m_0}}{2} t_H\right]^{2/3} = m_0 r^2$$

$$t_H(r) = \frac{2}{3\sqrt{m_0}} - \frac{2}{3} m_0 r^3$$

Gravitational Collapse of a Homogeneous Cloud of Dust

Eddington-Finkelstein Diagram



Gravitational Collapse of a Homogeneous Cloud of Dust

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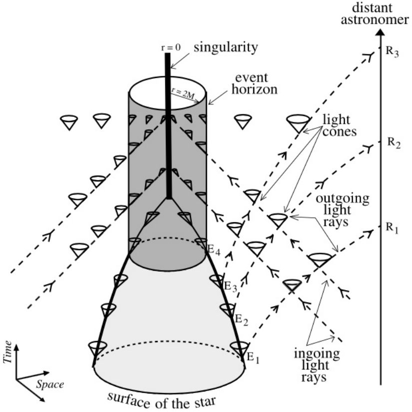

Inhomogeneous Dust Collapse

Inhomogeneous Dust Collapse

ρ depends on both t and r, and thus

$$\rho = \rho(t, r)$$
$$m = m(r)$$
$$b = b(r)$$
$$a = a(t, r)$$

Inhomogeneous Dust Collapse

The simplest form for the function $m(r)$ is

$$m(r) = m_0 + m_2 r^2$$

This function defines the density interms of the two parameter m_0 and m_2.

Inhomogeneous Dust Collapse

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This function defines the density interms of the two parameter m_0 and m_2.
The condition that ρ must be a decreasing function of r implies that $m_2 < 0$.

* $m_1 = 0$ in order to have a non divergent density.

Inhomogeneous Dust Collapse

The equation

$$\dot{R}^2 = \frac{F}{R} + f$$

gives this time

$$ra(\dot{t}, r) = -\sqrt{\frac{mr^2}{ra} + br^2}$$

Inhomogeneous Dust Collapse

The equation

$$\dot{R}^2 = \frac{F}{R} + f$$

gives this time

$$r\dot{a}(t, r) = -\sqrt{\frac{mr^2}{ra} + br^2}$$

$$\dot{a}(t, r) = -\sqrt{\frac{m}{a} + b}$$

Inhomogeneous Dust Collapse

In the marginally bound case, $b = 0$, it gives

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$$\dot{a}(t, r) = -\sqrt{\frac{m(r)}{a}}$$

Inhomogeneous Dust Collapse

In the marginally bound case, $b = 0$, it gives

$$a(\dot{t}, r) = -\sqrt{\frac{m(r)}{a}}$$

$$\sqrt{a(t, r)}da = -\sqrt{m(r)}dt$$

Inhomogeneous Dust Collapse

In the marginally bound case, $b = 0$, it gives

$$\dot{a}(t, r) = -\sqrt{\frac{m(r)}{a}}$$

$$\sqrt{a(t, r)}da = -\sqrt{m(r)}dt$$

$$\frac{2}{3}a^{3/2}(t, r) - \frac{2}{3}a^{3/2}(0, r) = -\sqrt{m(r)}t$$

Inhomogeneous Dust Collapse

The value of the initial condition was supposed as $a(0) = 1$. Thus

$$a(t,r) = \left[1 - \frac{3\sqrt{m(r)}}{2} t\right]^{2/3}$$

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$$a(t, r) = \left[1 - \frac{3\sqrt{m(r)}}{2} t \right]^{2/3}$$

This equation shows how each shell of the distribution (each r) collapses with a different scale factor and with a different velocity.

Inhomogeneous Dust Collapse

The time to reach the singularity and the horizon are, respectively,

Inhomogeneous Dust Collapse

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$$t_s(r) = \frac{2}{3\sqrt{m(r)}}$$

Inhomogeneous Dust Collapse

The time to reach the singularity and the horizon are, respectively,

$$t_s(r) = \frac{2}{3\sqrt{m(r)}}$$

$$t_H(r) = \frac{2}{3\sqrt{m(r)}} - \frac{2}{3}m(r)r^3$$

Inhomogeneous Dust Collapse

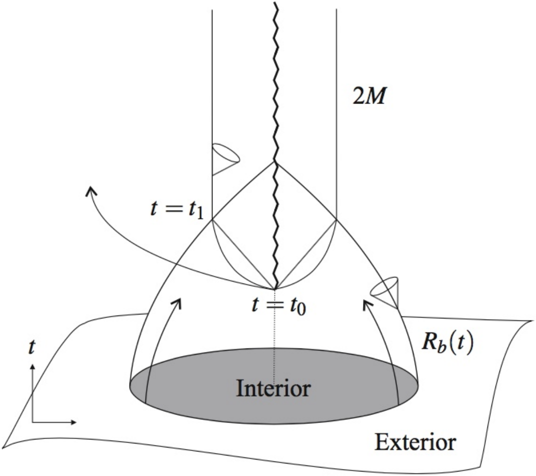
Using $m(r) = m_0 + m_2 r^2$ gives the functions

$$t_s(r) = \frac{2}{3\sqrt{m_0 + m_2 r^2}}$$

$$t_H(r) = \frac{2}{3\sqrt{m_0 + m_2 r^2}} - \frac{2}{3} r^3 \left(m_0 + m_2 r^2\right)$$

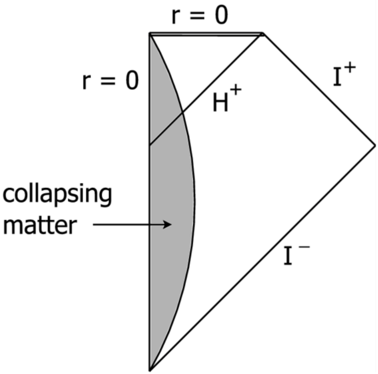
Gravitational Collapse of a Inhomogeneous Cloud of Dust

Eddington-Finkelstein Diagram



Carter-Penrose Diagram of a Gravitational Collapse

Carter-Penrose Diagram of a Gravitational Collapse



Next Lecture

06. Black Holes Astrophysics