Mathisson - Papapetrou - Dixon (MPD) Equations

$$\frac{DP^\alpha}{D\tau} = \frac{dP^\alpha}{d\tau} + \Gamma^\alpha_{\lambda\eta} u^\lambda P^\eta = -\frac{1}{2} R^\alpha{}_{\nu\rho\sigma} S^{\rho\sigma} u^\nu - q F^\alpha{}_\mu u^\mu$$

$$\frac{DS^{\mu\nu}}{D\tau} = P^\mu u^\nu - P^\nu u^\mu = 0$$

$$P^\alpha = m u^\alpha$$

Spherically Symmetric Static Spacetime

$$ds^2 = -W(r)\, dt^2 + \frac{dr^2}{W(r)} + r^2 d\Omega^2$$

Tulczyjew Spin Condition

$$(35)\begin{cases} P_\nu S^{\mu\nu} = 0 & \leftarrow \text{Tulczyjew condition: conservation of the dynamical mass of the particle.} \\ s^2 = s^\mu s_\mu = \frac{1}{2} S^{\mu\nu} S_{\mu\nu} & \leftarrow \text{Pirani condition: The magnitude of } S^{\mu\nu} \text{ is conserved} \end{cases}$$

Equatorial Motion

We assume $\theta = \frac{\pi}{2}$ and $\dot{\theta} = 0 \longrightarrow P_2 = 0$

Spin Tensor

The defınion of the spin vector s^μ is [An 2018]

$$s^\mu = -\frac{1}{2m} \epsilon^\mu{}_{\nu\rho\sigma} u^\nu S^{\rho\sigma} \qquad \epsilon_{0123} = 1$$

From this expression we get

$$S^{\mu\nu} = m \, \epsilon^{\mu\nu}{}_{\sigma\rho} u^\sigma s^\rho = \epsilon^{\mu\nu}{}_{\sigma\rho} P^\sigma s^\rho$$

* We will restrict the possibilities of s^μ to one component only

$$s^\mu = (s^0, s^1, s^2, s^3) = (0, 0, -s, 0)$$

Since we will assume that the particle is moving in the equatorial plane, this corresponds to the following possibilities

$s > 0$: spin along the z axis

$s < 0$: spin along the $-z$ axis

Since

$$S^{\mu\nu} = m\,\epsilon^{\mu\nu}{}_{\sigma\rho}\, u^{\sigma} s^{\rho} = \epsilon^{\mu\nu}{}_{\sigma\rho}\, P^{\sigma} s^{\rho}$$

we have

$$S^{02} = \epsilon^{02}{}_{\sigma\rho} P^{\sigma} s^{\rho} = \epsilon^{02}{}_{13} P^{1} \cancel{s^{3}} + \epsilon^{02}{}_{31} P^{3} \cancel{s^{1}}$$

$$S^{02} = 0 = -S^{20}$$

$$S^{12} = \epsilon^{12}{}_{\sigma\rho} P^{\sigma} s^{\rho} = \epsilon^{12}{}_{30} P^{3} \cancel{s^{0}} + \epsilon^{12}{}_{03} P^{0} \cancel{s^{3}}$$

$$S^{12} = 0 = -S^{21}$$

$$S^{23} = \epsilon^{23}{}_{\sigma\rho} P^{\sigma} s^{\rho} = \epsilon^{23}{}_{01} P^{0} \cancel{s^{1}} + \epsilon^{23}{}_{10} P^{1} \cancel{s^{0}}$$

$$S^{23} = 0 = -S^{32}$$

$$S^{01} = \epsilon^{01}{}_{\sigma\rho} P^{\sigma} s^{\rho} = \epsilon^{01}{}_{23} P^{2} \cancel{s^{3}} + \epsilon^{01}{}_{32} P^{3} s^{2}$$

$$S^{01} = \epsilon^{01}{}_{32} P^{3} s^{2} = -S^{10}$$

$$S^{03} = \epsilon^{03}{}_{\sigma\rho} P^{\sigma} s^{\rho} = \epsilon^{03}{}_{12} P^{1} s^{2} + \epsilon^{03}{}_{21} P^{2} \cancel{s^{1}}$$

$$S^{03} = \epsilon^{03}{}_{12} P^{1} s^{2} = -S^{30}$$

$$S^{13} = \epsilon^{13}{}_{\sigma\rho} P^{\sigma} s^{\rho} = \epsilon^{13}{}_{02} P^{0} s^{2} + \epsilon^{13}{}_{20} P^{2} \cancel{s^{0}}$$

$$S^{13} = \epsilon^{13}{}_{02} P^{0} s^{2} = -S^{31}$$

Summary:

$$S^{02} = -S^{20} = 0 \qquad S^{01} = -S^{10} = -\epsilon^{01}{}_{32} P^{3} s$$
$$S^{12} = -S^{21} = 0 \qquad S^{03} = -S^{30} = -\epsilon^{03}{}_{12} P^{1} s$$
$$S^{23} = -S^{32} = 0 \qquad S^{13} = -S^{31} = -\epsilon^{13}{}_{02} P^{0} s$$

(Corinaldesi and Papapetrou)

$$\epsilon^{01}{}_{32} = g_{3\alpha} g_{2\beta}\, \epsilon^{01\alpha\beta} = g_{33} g_{22}\, \epsilon^{0132} = -g_{22} g_{33}$$
$$\epsilon^{03}{}_{12} = g_{1\alpha} g_{2\beta}\, \epsilon^{03\alpha\beta} = g_{11} g_{22}\, \epsilon^{0312} = g_{11} g_{22}$$

$$\epsilon^{13}{}_{02} = g_{0\alpha} g_{2\beta}\, \epsilon^{13\alpha\beta} = g_{00} g_{22}\, \epsilon^{1302} = -g_{00} g_{22}$$

$$
\begin{aligned}
S^{02} &= -S^{20} = 0 & \qquad S^{01} &= -S^{10} = -\epsilon^{01}{}_{32}\, P^3 s = g_{22} g_{33} P^3 s \\
S^{12} &= -S^{21} = 0 & \qquad S^{03} &= -S^{30} = -\epsilon^{03}{}_{12}\, P^1 s = -g_{11} g_{22} P^1 s \\
S^{23} &= -S^{32} = 0 & \qquad S^{13} &= -S^{31} = -\epsilon^{13}{}_{02}\, P^0 s = g_{00} g_{22} P^0 s
\end{aligned}
\tag{36}
$$

We'll use the Tulczyjew spin-supplementary condition,

$$P_\mu S^{\mu\nu} = 0$$

to obtain:

$$P_0 S^{0\alpha} + P_1 S^{1\alpha} + \underbrace{\cancel{P_2}}_{\hookrightarrow\, =0} S^{2\alpha} + P_3 S^{3\alpha} = 0$$

(Motion is restricted to equatorial plane!)

$$P_0 S^{0\alpha} + P_1 S^{1\alpha} + P_3 S^{3\alpha} = 0$$

Then, we have

For $\alpha = 1$

$$P_0 S^{01} + P_1 \cancel{S^{11}} + P_3 S^{31} = 0$$

$$S^{01} = -\frac{P_3}{P_0} S^{31} = \frac{P_3}{P_0} S^{13}$$

For $\alpha = 3$

$$P_0 S^{03} + P_1 S^{13} + P_3 \cancel{S^{33}} = 0$$

$$S^{03} = -\frac{P_1}{P_0} S^{13} = \frac{P_1}{P_0} S^{31}$$

$$
\begin{aligned}
P_0 &= g_{0\alpha} P^\alpha = g_{00} P^0 = g_{00}\, m \dot{t} \\
P_1 &= g_{1\alpha} P^\alpha = g_{11} P^1 = g_{11}\, m \dot{r} \\
P_3 &= g_{3\alpha} P^\alpha = g_{33} P^3 = g_{33}\, m \dot{\phi}
\end{aligned}
$$

$$
\begin{cases}
S^{01} = \dfrac{g_{33}}{g_{00}} \dfrac{\dot{\phi}}{\dot{t}} S^{13} \\[2ex]
S^{03} = -\dfrac{g_{11}}{g_{00}} \dfrac{\dot{r}}{\dot{t}} S^{13}
\end{cases}
\tag{37}
$$

Now, consider the definition of the spin angular momentum

$$s^2 = \frac{1}{2} S^{\mu\nu} S_{\mu\nu}$$

$$2s^2 = g_{\mu\alpha} g_{\nu\beta} S^{\mu\nu} S^{\alpha\beta}$$

Remember that the non-vanishing components of the spin tensor are S^{01}, S^{03} and S^{13}

$$2s^2 = g_{0\alpha} g_{\nu\beta} S^{0\nu} S^{\alpha\beta} + g_{1\alpha} g_{\nu\beta} S^{1\nu} S^{\alpha\beta} + g_{3\alpha} g_{\nu\beta} S^{3\nu} S^{\alpha\beta}$$

$$\begin{aligned} 2s^2 = {} & g_{0\alpha} g_{1\beta} S^{01} S^{\alpha\beta} + g_{0\alpha} g_{3\beta} S^{03} S^{\alpha\beta} \\ & + g_{1\alpha} g_{0\beta} S^{10} S^{\alpha\beta} + g_{1\alpha} g_{3\beta} S^{13} S^{\alpha\beta} \\ & + g_{3\alpha} g_{0\beta} S^{30} S^{\alpha\beta} + g_{3\alpha} g_{1\beta} S^{31} S^{\alpha\beta} \end{aligned}$$

$$\begin{aligned} 2s^2 = {} & g_{0\alpha} g_{11} S^{01} S^{\alpha 1} + g_{00} g_{3\beta} S^{03} S^{0\beta} \\ & + g_{11} g_{0\beta} S^{10} S^{1\beta} + g_{11} g_{3\beta} S^{13} S^{1\beta} \\ & + g_{3\alpha} g_{00} S^{30} S^{\alpha 0} + g_{3\alpha} g_{11} S^{31} S^{\alpha 1} \end{aligned}$$

$$\begin{aligned} 2s^2 = {} & g_{00} g_{11} S^{01} S^{01} + g_{00} g_{33} S^{03} S^{03} + g_{11} g_{00} S^{10} S^{10} \\ & + g_{11} g_{33} S^{13} S^{13} + g_{33} g_{00} S^{30} S^{30} + g_{33} g_{11} S^{31} S^{31} \end{aligned}$$

$$2s^2 = 2 g_{00} g_{11} S^{01} S^{01} + 2 g_{00} g_{33} S^{03} S^{03} + 2 g_{11} g_{33} S^{13} S^{13}$$

$$s^2 = g_{00} g_{11} S^{01} S^{01} + g_{00} g_{33} S^{03} S^{03} + g_{11} g_{33} S^{13} S^{13}$$

$$s^2 = g_{00} g_{11} (S^{01})^2 + g_{00} g_{33} (S^{03})^2 + g_{11} g_{33} (S^{13})^2$$

Using the Tulczyjew relation, we obtain

$$s^2 = g_{00}\, g_{11}\left(-\frac{P_3}{P_0} S^{31}\right)^2 + g_{00}\, g_{33}\left(\frac{P_1}{P_0} S^{31}\right)^2 + g_{11}\, g_{33}\left(-S^{31}\right)^2$$

$$s^2 = \frac{(S^{31})^2}{P_0^2}\left[g_{00}\, g_{11}\, P_3^2 + g_{00}\, g_{33}\, P_1^2 + g_{11}\, g_{33}\, P_0^2\right]$$

$$s^2 P_0^2 = (S^{31})^2\left[g_{11}\, g_{33}\, P_0^2 + g_{00}\, g_{33}\, P_1^2 + g_{00}\, g_{11}\, P_3^2\right]$$

$$g_{\mu\nu} = \begin{pmatrix} g_{00} & 0 & 0 & 0 \\ 0 & g_{11} & 0 & 0 \\ 0 & 0 & g_{22} & 0 \\ 0 & 0 & 0 & g_{33} \end{pmatrix}$$

$$g = \det g_{\mu\nu} = g_{00}\, g_{11}\, g_{22}\, g_{33}$$

Inverse:

$$g^{\mu\nu} = \frac{1}{g}\begin{pmatrix} g_{11}\, g_{22}\, g_{33} & 0 & 0 & 0 \\ 0 & g/g_{11} & 0 & 0 \\ 0 & 0 & g/g_{22} & 0 \\ 0 & 0 & 0 & g_{00}\, g_{11}\, g_{22} \end{pmatrix}$$

$$\frac{g^{00}\, g}{g_{22}} = g_{11}\, g_{33}$$

$$g^{11}\frac{g}{g_{22}} = g_{00}\, g_{33}$$

$$\frac{g^{33}\, g}{g_{22}} = g_{00}\, g_{11}$$

Replacing these relations:

$$s^2 P_0^2 = (S^{31})^2 \left[g^{00} P_0^2 + g^{11} P_1^2 + g^{33} P_3^2 \right] \frac{g}{g_{22}}$$

Since $p^2 = g^{\mu\nu} P_\mu P_\nu = -m^2$; we have

$$s^2 P_0^2 = -m^2 (S^{31})^2 \frac{g}{g_{22}}$$

$$(S^{31})^2 = \frac{s^2 P_0^2}{m^2} \frac{g_{22}}{(-g)}$$

$$S^{31} = \sqrt{\frac{g_{22}}{-g}} \frac{s P_0}{m}$$

$$S^{31} = \sqrt{\frac{g_{22}}{-g}} g_{00} \dot{t} s$$

SPHERICALLY SYMMETRIC STATIC SPACETIME

$$ds^2 = -W(r)\,dt^2 + \frac{dr^2}{W(r)} + r^2 d\Omega^2$$

$$S^{01} = \frac{g_{33}}{g_{00}} \frac{\dot\phi}{\dot t} S^{13}$$

$$S^{03} = -\frac{g_{11}}{g_{00}} \frac{\dot r}{\dot t} S^{13}$$

$$S^{31} = \sqrt{\frac{g_{22}}{-g}}\, g_{00}\, \dot t\, s$$

$$g = g_{00} g_{11} g_{22} g_{33} = -r^4 \sin^2\theta$$

$$S^{31} = \sqrt{\frac{r^2}{r^4 \sin^2\theta}}\,(-W)\,\dot t\, s$$

$$S^{31} = -\frac{W \dot t s}{r \sin\theta}$$

$$S^{13} = \frac{W \dot t s}{r \sin\theta}$$

Using $\theta = \frac{\pi}{2}$: $\quad S^{13} = \frac{W \dot t s}{r} \qquad (47)$

$$S^{01} = \frac{r^2 \sin^2\theta}{-W} \frac{\dot\phi}{\dot t} \frac{W \dot t s}{r}$$

$$S^{01} = -r \sin^2\theta\, \dot\phi\, s \longrightarrow S^{01} = -r \dot\phi s \qquad (48)$$

$$S^{03} = -\frac{1/W}{-W} \frac{\dot r}{\dot t} \frac{W \dot t s}{r}$$

$$S^{03} = \frac{\dot r s}{W r} \qquad (49)$$