Spin Tensor

The definion of the spin vector s^μ is [An 2018]

$$s^\mu = -\frac{1}{2m} \epsilon^\mu{}_{\nu\rho\sigma} u^\nu S^{\rho\sigma} \qquad \epsilon_{0123} = 1$$

From this expression we get

$$S^{\mu\nu} = m\, \epsilon^{\mu\nu}{}_{\sigma\rho} u^\sigma s^\rho$$

* We will restrict the possibilities of s^μ to one component only

$$s^\mu = (s^0, s^1, s^2, s^3) = (0, 0, -s, 0)$$

Since we will assume that the particle is moving in the equatorial plane, this corresponds to the possibilities

$s > 0$: particle's spin parallel to BH's spin

$s < 0$: particle's spin antiparallel to BH's spin

In the tetrad frame

$$S^{(\alpha)(\beta)} = m\, \epsilon^{(\alpha)(\beta)}{}_{(\gamma)(\delta)} u^{(\gamma)} s^{(\delta)}$$

$$s^{(\delta)} = (0, 0, -s, 0)$$

Then;

$$S^{(0)(2)} = m\, \epsilon^{(0)(2)}{}_{(\gamma)(\delta)} u^{(\gamma)} s^{(\delta)}$$

$$S^{(0)(2)} = m\, \epsilon^{(0)(2)}{}_{(1)(3)} u^{(1)} \underline{s^{(3)}} + m\, \epsilon^{(0)(2)}{}_{(3)(1)} u^{(3)} \underline{s^{(1)}}$$

$$S^{(0)(2)} = 0$$

$$S^{(1)(2)} = m\, \epsilon^{(1)(2)}{}_{(\gamma)(\delta)} u^{(\gamma)} s^{(\delta)}$$

$$S^{(1)(2)} = m\, \epsilon^{(1)(2)}{}_{(0)(3)} u^{(0)} \underline{s^{(3)}} + m\, \epsilon^{(1)(2)}{}_{(3)(0)} u^{(3)} \underline{s^{(0)}}$$

$$S^{(1)(2)} = 0$$

$$S^{(2)(3)} = m\,\epsilon^{(2)(3)}{}_{(\gamma)(\delta)}\, u^{(\gamma)}\, s^{(\delta)}$$

$$S^{(2)(3)} = m\,\epsilon^{(2)(3)}{}_{(0)(1)}\, u^{(0)}\, \underline{s^{(1)}} + m\,\epsilon^{(2)(3)}{}_{(1)(0)}\, u^{(1)}\, \underline{s^{(0)}}$$

$$S^{(2)(3)} = 0$$

The non-vanishing components are $S^{(0)(1)}$, $S^{(0)(3)}$ and $S^{(1)(3)}$

$$S^{(0)(1)} = m\,\epsilon^{(0)(1)}{}_{(\delta)(\gamma)}\, u^{(\delta)}\, s^{(\gamma)}$$

$$S^{(0)(1)} = m\,\epsilon^{(0)(1)}{}_{(2)(3)}\, u^{(2)}\, \underline{s^{(3)}} + m\,\epsilon^{(0)(1)}{}_{(3)(2)}\, u^{(3)}\, s^{(2)}$$

$$S^{(0)(1)} = -m\,\epsilon^{(0)(1)}{}_{(3)(2)}\, u^{(3)}\, s$$

$$S^{(0)(1)} = -\epsilon^{(0)(1)}{}_{(3)(2)}\, P^{(3)}\, s$$

$$S^{(0)(1)} = -\eta^{(0)(\alpha)}\,\eta^{(1)(\beta)}\,\epsilon_{(\alpha)(\beta)(3)(2)}\, s\, P^{(3)}$$

$$S^{(0)(1)} = \epsilon_{(0)(1)(3)(2)}\, s\, P^{(3)}$$

$$S^{(0)(1)} = -s\, P^{(3)}$$

$$S^{(0)(3)} = m\,\epsilon^{(0)(3)}{}_{(\delta)(\gamma)}\, u^{(\delta)}\, s^{(\gamma)}$$

$$S^{(0)(3)} = m\,\epsilon^{(0)(3)}{}_{(2)(1)}\, u^{(2)}\, \underline{s^{(1)}} + m\,\epsilon^{(0)(3)}{}_{(1)(2)}\, u^{(1)}\, s^{(2)}$$

$$S^{(0)(3)} = -m\,\epsilon^{(0)(3)}{}_{(1)(2)}\, u^{(1)}\, s$$

$$S^{(0)(3)} = -\epsilon^{(0)(3)}{}_{(1)(2)}\, P^{(1)}\, s$$

$$S^{(0)(3)} = -\eta^{(0)(\alpha)}\,\eta^{(3)(\beta)}\,\epsilon_{(\alpha)(\beta)(1)(2)}\, s\, P^{(1)}$$

$$S^{(0)(3)} = \epsilon_{(0)(3)(1)(2)}\, s\, P^{(1)}$$

$$S^{(0)(3)} = s\, P^{(1)}$$

$$S^{(1)(3)} = m\,\epsilon^{(1)(3)}{}_{(\delta)(\gamma)}\,u^{(\delta)}\,s^{(\gamma)}$$

$$S^{(1)(3)} = m\,\epsilon^{(1)(3)}{}_{(2)(0)}\,u^{(2)}\,\underline{s^{(0)}} + m\,\epsilon^{(1)(3)}{}_{(0)(2)}\,u^{(0)}\,s^{(2)}$$

$$S^{(1)(3)} = -m\,\epsilon^{(1)(3)}{}_{(0)(2)}\,u^{(0)}\,s$$

$$S^{(1)(3)} = -\epsilon^{(1)(3)}{}_{(0)(2)}\,P^{(0)}\,s$$

$$S^{(1)(3)} = -\eta^{(1)(\alpha)}\,\eta^{(3)(\beta)}\,\epsilon_{(\alpha)(\beta)(0)(2)}\,s\,P^{(0)}$$

$$S^{(1)(3)} = -\epsilon_{(1)(3)(0)(2)}\,s\,P^{(0)}$$

$$S^{(1)(3)} = s\,P^{(0)}$$

RESULTS:

$$S^{(0)(1)} = -s\,P^{(3)}$$

$$S^{(0)(3)} = s\,P^{(1)}$$

$$S^{(1)(3)} = s\,P^{(0)}$$