Fuerza y Torque

Tanto p^μ como $S^{\mu\nu}$ dependen del punto x_0 que yace sobre la línea de mundo. Si se utiliza la parametrización $x_0 = x_0(\lambda)$ se tiene

$$\frac{dC_\xi}{d\lambda} = 0$$

$$\frac{d}{d\lambda}\left[p^\mu(x_0,\Sigma)\,\xi_\mu + \frac{1}{2} S^{\mu\nu}(x_0,\Sigma)\,\nabla_{[\mu}\xi_{\nu]} \right] = 0$$

$$\frac{dp^\mu}{d\lambda}\xi_\mu + p^\mu \frac{d\xi_\mu}{d\lambda} + \frac{1}{2}\frac{dS^{\mu\nu}}{d\lambda}\nabla_{[\mu}\xi_{\nu]} + \frac{1}{2} S^{\mu\nu}\frac{d}{d\lambda}\left(\nabla_{[\mu}\xi_{\nu]}\right) = 0$$

Por otra parte,

$$\frac{d\xi_\mu}{d\lambda} = \frac{dx^\alpha}{d\lambda}\nabla_\alpha \xi_\mu = \dot{x}^\alpha \nabla_{[\alpha}\xi_{\mu]}$$

Además

$$\frac{d}{d\lambda}\left(\nabla_{[\mu}\xi_{\nu]}\right) = \dot{x}^\alpha \nabla_\alpha \nabla_\mu \xi_\nu = \dot{x}^\alpha R_{\mu\nu\alpha\beta}\,\xi^\beta$$

Reemplazando estos resultados tenemos

$$\frac{dp^\mu}{d\lambda}\xi_\mu + p^\mu \dot{x}^\alpha \nabla_{[\alpha}\xi_{\mu]} + \frac{1}{2}\frac{dS^{\mu\nu}}{d\lambda}\nabla_{[\mu}\xi_{\nu]} + \frac{1}{2} S^{\mu\nu}\dot{x}^\alpha R_{\mu\nu\alpha\beta}\,\xi^\beta = 0$$

$$\frac{dp^\mu}{d\lambda}\xi_\mu - p^\mu \dot{x}^\nu \nabla_{[\mu}\xi_{\nu]} + \frac{1}{2}\frac{dS^{\mu\nu}}{d\lambda}\nabla_{[\mu}\xi_{\nu]} + \frac{1}{2} S^{\alpha\beta}\dot{x}^\gamma R_{\alpha\beta\gamma}{}^{\mu}\,\xi_\mu = 0$$

$$\left(\frac{dp^\mu}{d\lambda} + \frac{1}{2} S^{\alpha\beta}\dot{x}^\gamma R_{\alpha\beta\gamma}{}^{\mu}\right)\xi_\mu + \frac{1}{2}\left(\frac{dS^{\mu\nu}}{d\lambda} - 2p^\mu\dot{x}^\nu\right)\nabla_{[\mu}\xi_{\nu]} = 0$$

Los términos en parentesis se identifican con la fuerza, F^μ, y torque $\tau^{\mu\nu}$, externos actuando sobre el cuerpo,

$$\begin{cases} F^\mu = \dfrac{dp^\mu}{d\lambda} + \dfrac{1}{2} S^{\alpha\beta}\dot{x}^\gamma R_{\alpha\beta\gamma}{}^{\mu} \\[2ex] \tau^{\mu\nu} = \dfrac{dS^{\mu\nu}}{d\lambda} - 2p^{[\mu}\dot{x}^{\nu]} \end{cases}$$

En ausencia de fuertas y torques externos , $F^{\mu}=0$ y $\tau^{\mu\nu}=0$, estas ecuaciones se reducen al sistema de Mathisson-Papapetrou-Dixon,

$$\boxed{\begin{aligned} \frac{dp^{\mu}}{d\lambda} &= -\frac{1}{2} S^{\alpha\beta} \dot{x}^{\gamma} R_{\alpha\beta\gamma}{}^{\mu} \\ \frac{dS^{\mu\nu}}{d\lambda} &= 2p^{[\mu} \dot{x}^{\nu]} = p^{\mu}\dot{x}^{\nu} - p^{\nu}\dot{x}^{\mu} \end{aligned}}$$

ECUACIONES MPD

Referencias

[1] I. Burak Ilhan. "Dynamics of Extended Objects in General Relativity".
arXiv: 0911.3645 [gr-qc]