Lecture Notes on Electrodynamics

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Chapter 1

Maxwell Equations

1.1 Lorentz Force

The force acting on a particle with electric charge q moving with velocity $\mathbf{v}$ due to its interaction with electric and magnetic fields, $\mathbf{E}$ and $\mathbf{B}$, is given by the expression

$$\mathbf{F} = q\left(\mathbf{E} + \mathbf{v} \times \mathbf{B}\right). \tag{1.1}$$

1.1.1 Electric Charge

The electric charge is measured in the units called *Coulomb* and the fundamental charge corresponds to that of an electron. Its value is

$$e = 1.602177 \times 10^{-19}\ C. \tag{1.2}$$

The electric charge satisfies a superposition principle, so a set of n discrete point charges give a total electric charge given by

$$Q = \sum_{i=1}^{n} q_i. \tag{1.3}$$

Similarly, a continuous distribution of electric charge represented by a density function $\rho = \rho(t, \mathbf{r})$ over a volume V, gives a total charge

$$Q = \int_V \rho(t, \mathbf{r}) d^3 r \tag{1.4}$$

1.1.2 Dirac Delta Function

In one dimension, the Dirac delta function has the properties:

1. $\delta(x - a) = 0$ for $x \neq a$

2. $\int \delta(x-a)dx = 1$ if the region of integration includes $x = a$ and is zero otherwise.

3. $\int f(x)\delta(x-a)dx = f(a)$

4. $\int f(x)\delta'(x-a)dx = -f'(a)$, where prime denotes derivative with respect to the argument.

5. $\delta(f(x)) = \sum_i \frac{1}{\left|\frac{df}{dx}(x_i)\right|}\delta(x-x_i)$ where $f(x)$ is assumed to have only simple zeros, located at $x = x_i$.

6. $\delta(ax) = \frac{1}{|a|}\delta(x)$

7. $x\delta(x) = 0$

8. $\delta\left(x^2 - e^2\right) = \frac{1}{2|e|}\left[\delta(x+e) + \delta(x-e)\right]$

9. $\delta(\mathbf{r} - \mathbf{R}) = \delta(x-X)\delta(y-Y)\delta(z-Z)$

10. $\int f(x)\nabla\delta(\mathbf{r} - \mathbf{r}_0)d^3x = -\left.\nabla f\right|_{\mathbf{r}=\mathbf{r}_0}$

11. $\delta(\mathbf{r} - \mathbf{r}_0) = \delta(\mathbf{r}_0 - \mathbf{r})$

Example

For n point charges q_i moving along the trajectories $\mathbf{r}_i(t)$, the charge density may be written in terms of the Dirac delta function (see Appendix A) as

$$\rho(t, \mathbf{r}) = \sum_{i=1}^{n} q_i \delta\left(\mathbf{r} - \mathbf{r}_i(t)\right) \tag{1.5}$$

1.1.3 Electric Current

Electric charge in organized motion is called *electric current* and it is defined as

$$I = \frac{dQ}{dt} \tag{1.6}$$

and it is measured in units of *Coulomb per second* known as *Ampere*. The electric current can be written also as

$$I = \int \mathbf{j} \cdot \mathbf{dS} \tag{1.7}$$

where $\mathbf{j}$ is the electric current density (current per unit area) and $\mathbf{dS} = \hat{\mathbf{n}}dS$ represents a surface element vector across which charge is moving.

1.1.4 Electric Charge Conservation and the Equation of Continuity

Consider a region of space with volume V and some amount of electric charge. If this charge moves away from this region, it will produce an electric current given by

$$I = -\frac{dQ}{dt}, \tag{1.8}$$

where the minus sign indicates that the charge in that region is decreasing (charge conservation). Introducing a current density, this can be written as

$$\oint \mathbf{j} \cdot \mathbf{dS} = -\frac{dQ}{dt}, \tag{1.9}$$

where the integration is considered over all the closed surface surrounding the volume in which the charge is located. This integral corresponds to the flux of electric charge across the closed surface. Now, using the concept of charge density we have

$$\oint \mathbf{j} \cdot \mathbf{dS} = -\frac{d}{dt}\int_V \rho d^3r \tag{1.10}$$

$$\oint \mathbf{j} \cdot \mathbf{dS} = -\int_V \frac{\partial \rho}{\partial t} d^3r \tag{1.11}$$

Using Gauss' theorem in the integral in the left hand side we transform the surface integral into a volume one,

$$\int_V \boldsymbol{\nabla} \cdot \mathbf{j} d^3r = -\frac{d}{dt}\int_V \rho d^3r, \tag{1.12}$$

which gives the final result

$$\frac{\partial \rho}{\partial t} + \boldsymbol{\nabla} \cdot \mathbf{j} = 0, \tag{1.13}$$

known as the *continuity equation*.

Example

Given the density function for a set of n point charges in Eq. (1.5) and the particles velocities $\mathbf{v} = \dot{\mathbf{r}}_i(t)$, we have

$$\frac{\partial \rho}{\partial t} = \sum_{i=1}^{n} q_i \frac{\partial}{\partial t}\delta(\mathbf{r} - \mathbf{r}_i) \tag{1.14}$$

$$\frac{\partial \rho}{\partial t} = \sum_{i=1}^{n} q_i \frac{d\mathbf{r}_i}{dt} \cdot \boldsymbol{\nabla}_i \delta(\mathbf{r} - \mathbf{r}_i) \tag{1.15}$$

$$\frac{\partial \rho}{\partial t} = -\sum_{i=1}^{n} q_i \frac{d\mathbf{r}_i}{dt} \cdot \boldsymbol{\nabla} \delta(\mathbf{r} - \mathbf{r}_i) \tag{1.16}$$

$$\frac{\partial \rho}{\partial t} = -\sum_{i=1}^{n} q_i \mathbf{v}_i \cdot \boldsymbol{\nabla} \delta(\mathbf{r} - \mathbf{r}_i), \tag{1.17}$$

where the minus sign in the third line comes from the derivative properties of the delta function. Since $\boldsymbol{\nabla} \cdot \mathbf{v}_i = 0$, we write

$$\frac{\partial \rho}{\partial t} = -\boldsymbol{\nabla} \cdot \sum_{i=1}^{n} q_i \mathbf{v}_i \delta(\mathbf{r} - \mathbf{r}_i). \tag{1.18}$$

Comparison with the continuity equation (1.13) let us identify the corresponding density current as

$$\mathbf{j}(t, \mathbf{r}) = \sum_{i=1}^{n} q_i \mathbf{v}_i \delta(\mathbf{r} - \mathbf{r}_i). \tag{1.19}$$

1.1.5 Electric and Magnetic Fields

From Lorentz force (1.1) we identify the electric and magnetic fields,

$$\mathbf{E} = \mathbf{E}(t, \mathbf{r}) \tag{1.20}$$
$$\mathbf{B} = \mathbf{B}(t, \mathbf{r}). \tag{1.21}$$

The electric field is measured in units of *Newtons per Coulomb*, N/C while the magnetic field is measured in the units $\frac{N}{Cm/s} = \frac{N}{A\ m} = T$ known as *Tesla*. Electric fields are known to work up to distances $\sim 10^5 m$ (atmospheric electrostatic discharges) while magnetic fields have been observed in distances $\sim 10^{20} m$ (cosmic magnetic fields).

1.2 Maxwell Equations

The description and evolution of electric and magnetic fields in vacuum is given by Maxwell equations,

$$\boldsymbol{\nabla} \cdot \mathbf{E} = \frac{\rho}{\epsilon_0} \tag{1.22}$$
$$\boldsymbol{\nabla} \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \tag{1.23}$$
$$\boldsymbol{\nabla} \cdot \mathbf{B} = 0 \tag{1.24}$$
$$\boldsymbol{\nabla} \times \mathbf{B} = \mu_0 \mathbf{j} + \mu_0 \epsilon_0 \frac{\partial \mathbf{E}}{\partial t} \tag{1.25}$$

where

$$\epsilon_0 = 8.854 \times 10^{-12}\ F/m \tag{1.26}$$

is called the *permitivity of free space* and

$$\mu_0 = \frac{1}{\epsilon_0 c^2} \tag{1.27}$$

is the *permeability of free space.*

1.3 Maxwell Equations in Matter

We introduce the notion of "free" densities of charge and current,

$$\rho_f = \rho_f(t, \mathbf{r}) \tag{1.28}$$
$$\mathbf{j}_f = \mathbf{j}_f(t, \mathbf{r}), \tag{1.29}$$

which will produce electric and magnetic fields contributing to the total fields in a given region of space. The *polarization* of a dielectric is characterized by a vector

$$\mathbf{P} = \mathbf{P}(r, \mathbf{r}) \tag{1.30}$$

while the *magnetization* of a magnet is characterized by a vector

$$\mathbf{M} = \mathbf{M}(r, \mathbf{r}). \tag{1.31}$$

In order to incorporate these quantities into Maxwell equations we use the relations

$$\rho(t, \mathbf{r}) = \rho_f(t, \mathbf{r}) - \boldsymbol{\nabla} \cdot \mathbf{P}(t, \mathbf{r}) \tag{1.32}$$

and

$$\mathbf{j}(t, \mathbf{r}) = \mathbf{j}_f(t, \mathbf{r}) + \boldsymbol{\nabla} \times \mathbf{M}(t, \mathbf{r}) + \frac{\partial \mathbf{P}(t, \mathbf{r})}{\partial t}. \tag{1.33}$$

We also define the auxiliary macroscopic fields

$$\mathbf{D}(t, \mathbf{r}) = \epsilon_0 \mathbf{E}(t, \mathbf{r}) + \mathbf{P}(t, \mathbf{r}) \tag{1.34}$$

and

$$\mathbf{H}(t, \mathbf{r}) = \frac{\mathbf{B}(t, \mathbf{r})}{\mu_0} - \mathbf{M}(t, \mathbf{r}). \tag{1.35}$$

Hence, using these definitions, the first of Maxwell equations (1.22) writes

$$\boldsymbol{\nabla} \cdot \mathbf{E} = \frac{\rho}{\epsilon_0} \tag{1.36}$$
$$\boldsymbol{\nabla} \cdot \left(\frac{\mathbf{D} - \mathbf{P}}{\epsilon_0} \right) = \frac{\rho_f - \boldsymbol{\nabla} \cdot \mathbf{P}}{\epsilon_0} \tag{1.37}$$
$$\boldsymbol{\nabla} \cdot \mathbf{D} = \rho_f. \tag{1.38}$$

Similarly, equation (1.25) becomes

$$\boldsymbol{\nabla} \times \mathbf{B} = \mu_0 \mathbf{j} + \mu_0 \epsilon_0 \frac{\partial \mathbf{E}}{\partial t} \tag{1.39}$$
$$\mu_0 \boldsymbol{\nabla} \times (\mathbf{H} + \mathbf{M}) = \mu_0 \left(\mathbf{j}_f(t, \mathbf{r}) + \boldsymbol{\nabla} \times \mathbf{M}(t, \mathbf{r}) + \frac{\partial \mathbf{P}(t, \mathbf{r})}{\partial t} \right) + \mu_0 \epsilon_0 \frac{\partial \mathbf{E}}{\partial t}$$
$$\boldsymbol{\nabla} \times \mathbf{H} = \mathbf{j}_f + \frac{\partial \mathbf{D}}{\partial t} \tag{1.40}$$

Therefore, we conclude that Maxwell equations in matter are written as

$$\begin{aligned}
\boldsymbol{\nabla} \cdot \mathbf{D} &= \rho_f && (1.41) \\
\boldsymbol{\nabla} \times \mathbf{E} &= -\frac{\partial \mathbf{B}}{\partial t} && (1.42) \\
\boldsymbol{\nabla} \cdot \mathbf{B} &= 0 && (1.43) \\
\boldsymbol{\nabla} \times \mathbf{H} &= \mathbf{j}_f + \frac{\partial \mathbf{D}}{\partial t} && (1.44)
\end{aligned}$$

Chapter 2

Electrostatics

From Maxwell equations (1.22 - 1.25), every time-independent charge distribution $\rho(\mathbf{r})$ will produce an electric vector field independent of time, $\mathbf{E}(\mathbf{r})$, described by the equations

$$\boldsymbol{\nabla} \cdot \mathbf{E} = \rho \tag{2.1}$$

and

$$\boldsymbol{\nabla} \times \mathbf{E} = 0. \tag{2.2}$$

Given a charge distribution represented by the density $\tilde{\rho}(\mathbf{r})$, the electric field $\mathbf{E}(\mathbf{r})$ will exert on it a force given by

$$\mathbf{F} = \int \tilde{\rho}(\mathbf{r})\mathbf{E}(\mathbf{r}) d^3 r \tag{2.3}$$

and a torque given by

$$\mathbf{N} = \int \mathbf{r} \times [\tilde{\rho}(\mathbf{r})\mathbf{E}(\mathbf{r})]\, d^3 r. \tag{2.4}$$

2.1 Helmholtz Theorem

Helmholtz Theorem guarantees that equations (2.1) and (2.2) determines the field $\mathbf{E}(\mathbf{r})$ uniquely.

Statement of the Theorem

Any arbitrary vector field $\mathbf{E}(\mathbf{r})$ can always be decomposed into the sum of two vector fields, one with zero divergence and one with zero curl,

$$\mathbf{E} = \mathbf{E}_\perp + \mathbf{E}_\parallel, \tag{2.5}$$

where

$$\boldsymbol{\nabla} \cdot \mathbf{E}_\perp = 0 \tag{2.6}$$

$$\boldsymbol{\nabla} \times \mathbf{E}_\parallel = 0 \tag{2.7}$$

2.2 Scalar Potential

Equation (2.2) states that the electrostatic field is conservative, so that it can be derived from the gradient of some scalar function (because the curl of any well-behaved scalar function vanishes). This function is called the *scalar potential* and hence

$$\mathbf{E} = -\boldsymbol{\nabla}\Phi. \tag{2.8}$$

2.3 Poisson and Laplace Equations

Using the scalar potential defined in Eq. (2.8) in Maxwell equation for the electrostatic field, (2.1), gives one partial differential for $\Phi(\mathbf{r})$ called the *Poisson equation*,

$$\nabla^2\Phi = -\frac{\rho}{\epsilon_0}. \tag{2.9}$$

In a region of space with no electric charge, this equation reduces to the *Laplace equation*,

$$\nabla^2\Phi = 0. \tag{2.10}$$

2.4 Coulomb Field

A particular solution of Maxwell equations is given by Coulomb's field, which has been experimentally established in the 18th century by Priestley, Cavendish and Coulomb. It is defined by Coulomb's law, which gives the force acting on a charge q due to n point charges q_i located at positions $\mathbf{r}_i$,

$$\mathbf{F} = q\mathbf{E}, \tag{2.11}$$

where

$$\mathbf{F} = \frac{1}{4\pi\epsilon_0}\sum_{i=1}^{n} qq_i\frac{\mathbf{r}-\mathbf{r}_i}{|\mathbf{r}-\mathbf{r}_i|^3}. \tag{2.12}$$

Thus, Coulomb's electrostatic field in vacuum is

$$\mathbf{E} = \frac{1}{4\pi\epsilon_0}\sum_{i=1}^{n} q_i\frac{\mathbf{r}-\mathbf{r}_i}{|\mathbf{r}-\mathbf{r}_i|^3}, \tag{2.13}$$

where the proportionality constant has the value

$$k = \frac{1}{4\pi\epsilon_0} = 10^{-7}c^2. \tag{2.14}$$

For a general charge density $\rho(\mathbf{r}')$ the electrostatic Coulomb's field is

$$\mathbf{E} = \frac{1}{4\pi\epsilon_0}\int \rho(\mathbf{r}')\frac{\mathbf{r}-\mathbf{r}'}{|\mathbf{r}-\mathbf{r}'|^3}d^3r', \tag{2.15}$$

Since

$$\frac{\mathbf{r}-\mathbf{r}'}{|\mathbf{r}-\mathbf{r}'|^3} = -\boldsymbol{\nabla}\left(\frac{1}{|\mathbf{r}-\mathbf{r}'|}\right) \tag{2.16}$$

we can write Coulomb's field (2.15) as

$$\mathbf{E} = -\frac{1}{4\pi\epsilon_0}\boldsymbol{\nabla}\int\frac{\rho(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}d^3r', \tag{2.17}$$

and therefore, the scalar potential for the Coulomb's field is written

$$\Phi(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\int\frac{\rho(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}d^3r'. \tag{2.18}$$

In order to show that this potential is a solution of Poisson equation, consider each term in the integral, $\frac{1}{|\mathbf{r}-\mathbf{r}'|}$, which can be written as $\frac{1}{r}$ by choosing coordinates such that $\mathbf{r}'=0$. In cartesian coordinates we have $\mathbf{r} = x^1\hat{\mathbf{n}}_1 + x^2\hat{\mathbf{n}}_2 + x^3\hat{\mathbf{n}}_3$ and $r = \sqrt{(x^1)^2+(x^2)^2+(x^3)^2}$. Therefore, for $\mathbf{r}\neq 0$,

$$\frac{\partial r}{\partial x^j} = \frac{\partial}{\partial x^j}\sqrt{(x^1)^2+(x^2)^2+(x^3)^2} = \frac{x^j}{r} = n^j \tag{2.19}$$

and

$$\frac{\partial n^j}{\partial x^k} = \frac{\partial}{\partial x^k}\left(\frac{x^j}{r}\right) = \frac{\delta_{jk}}{r} - \frac{x^j}{r^2}\frac{\partial r}{\partial x^k} = \frac{1}{r}\left[\delta_{jk} - n^j n^k\right]. \tag{2.20}$$

These relations let us write the derivative

$$\frac{\partial}{\partial x^j}\left(\frac{1}{r}\right) = -\frac{1}{r^2}\frac{\partial r}{\partial x^j} = -\frac{n^j}{r^2} \tag{2.21}$$

and the second derivative

$$\frac{\partial^2}{\partial x^k x^j}\left(\frac{1}{r}\right) = \frac{\partial}{\partial x^k}\left(-\frac{n^j}{r^2}\right) = -\frac{1}{r^2}\frac{\partial n^j}{\partial x^k} + 2\frac{n^j}{r^3}\frac{\partial r}{\partial x^k} \tag{2.22}$$

$$\frac{\partial^2}{\partial x^k x^j}\left(\frac{1}{r}\right) = -\frac{1}{r^2}\frac{1}{r}\left[\delta_{jk} - n^j n^k\right] + 2\frac{n^j}{r^3}n^k \tag{2.23}$$

$$\frac{\partial^2}{\partial x^k x^j}\left(\frac{1}{r}\right) = \frac{1}{r^3}\left[3n^j n^k - \delta_{jk}\right]. \tag{2.24}$$

Thus, the Laplacian for $\mathbf{r}\neq 0$ is given by

$$\nabla^2\left(\frac{1}{r}\right) = \frac{\partial^2}{\partial x^k x^k}\left(\frac{1}{r}\right) = \frac{1}{r^3}\left[3n^k n^k - \delta_{kk}\right] = 0. \tag{2.25}$$

The behavior at $\mathbf{r}=0$ is studied by integrating the term $\nabla^2\left(\frac{1}{r}\right)$ over a tiny spherical volume V centered at the origin. In this case the divergence theorem gives

$$\int_V\left(\nabla^2\frac{1}{r}\right)d^3r = \int_V\boldsymbol{\nabla}\cdot\left(\boldsymbol{\nabla}\frac{1}{r}\right)d^3r = \oint_S\left(\boldsymbol{\nabla}\frac{1}{r}\right)\cdot d\mathbf{S} = -\oint_S\frac{\hat{\mathbf{r}}}{r^2}\cdot d\mathbf{S}. \tag{2.26}$$

Since the volume is a sphere, the surface element is $d\mathbf{S} = r^2 \sin^2\theta d\theta d\varphi \hat{\mathbf{r}}$. Thus,

$$\int_V \left(\nabla^2 \frac{1}{r}\right) d^3r = -\int_0^{2\pi} d\varphi \int_0^{\pi} \sin\theta d\theta = -4\pi. \tag{2.27}$$

These results can be summarized in the expression

$$\int_V \left(\nabla^2 \frac{1}{r}\right) d^3r = \begin{cases} 0 \text{ if } r \neq 0 \\ -4\pi \text{ if } r = 0 \end{cases} \tag{2.28}$$

which can be obtained by writing

$$\nabla^2 \frac{1}{r} = -4\pi\delta(\mathbf{r}). \tag{2.29}$$

In general, recovering the term $\mathbf{r}'$, we have that

$$\nabla^2 \left(\frac{1}{|\mathbf{r}-\mathbf{r}'|}\right) = -4\pi\delta(\mathbf{r}-\mathbf{r}'), \tag{2.30}$$

and therefore, replacing this result in the scalar potential (2.8), we recover the Poisson equation,

$$\nabla^2\Phi(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\int \rho(\mathbf{r}')\nabla^2\left(\frac{1}{|\mathbf{r}-\mathbf{r}'|}\right) d^3r' \tag{2.31}$$

$$\nabla^2\Phi(\mathbf{r}) = -\frac{1}{4\pi\epsilon_0}\int \rho(\mathbf{r}')4\pi\delta(\mathbf{r}-\mathbf{r}')d^3r' \tag{2.32}$$

$$\nabla^2\Phi(\mathbf{r}) = -\frac{\rho(\mathbf{r})}{\epsilon_0}. \tag{2.33}$$

2.5 Potential Energy

The product of a charge times a scalar potential, $U = q\Phi$ is interpreted as the potential energy of that charge in the presence of the corresponding electrostatic field. Similarly, the work done in moving the charge q from a point A to a point B in space against the electric field is given by

$$W = -\int_A^B \mathbf{F}\cdot d\mathbf{l} = -q\int_A^B \mathbf{E}\cdot d\mathbf{l}, \tag{2.34}$$

or in terms of the electrostatic potential

$$W = q\int_A^B \boldsymbol{\nabla}\Phi\cdot d\mathbf{l} = q\int_{\Phi_A}^{\Phi_B} d\Phi = q\left(s\Phi_A - \Phi_B\right). \tag{2.35}$$

2.6 Discontinuities in the Electric Field and Potential

Suppose a surface S with surface charge density $\sigma(\mathbf{r})$ and with normal unit vector $\hat{\mathbf{n}}$ going from side 1 to side 2 of S. Consider also that the electric field has values $\mathbf{E}_1$ and $\mathbf{E}_2$ on each side of the surface. Hence, considering a cylindrical surface crossing S, Gauss' law gives

$$\oint \mathbf{E} \cdot \hat{\mathbf{n}} dS = \frac{1}{\epsilon_0} \int_V \rho d^3x = \frac{1}{\epsilon_0} \int_S \sigma dS \tag{2.36}$$

$$\int_S \mathbf{E}_2 \cdot \hat{\mathbf{n}} dS - \int_S \mathbf{E}_1 \cdot \hat{\mathbf{n}} dS = \frac{1}{\epsilon_0} \int_S \sigma dS \tag{2.37}$$

from which

$$(\mathbf{E}_2 - \mathbf{E}_1) \cdot \hat{\mathbf{n}} = \frac{\sigma}{\epsilon_0}. \tag{2.38}$$

This relation doesn't specify completely the electric field, but tells that there is a discontinuity at the surface due to the charge density σ. On the other hand, the scalar potential can be written in general terms as in equation (2.18), but replacing the density ρ with σ, i.e.

$$\Phi(\mathbf{r}) = \frac{1}{4\pi\epsilon_0} \int_S \frac{\sigma(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|} dS'. \tag{2.39}$$

2.7 Dipole-Layer Distribution

A dipole-layer corresponds to configuration in which a surface S with charge density $\sigma(\mathbf{r})$ is accompanied of another surface S' with charge density $-\sigma(\mathbf{r})$. If S' approach infinitesimally close to S while $\sigma(\mathbf{r})$ becomes infinite, the product of this charge density and the local separation between surfaces, $d(\mathbf{r})$, corresponds to the *dipole-layer distribution strength*,

$$D(\mathbf{r}) = \lim_{d(\mathbf{r}) \to 0} \sigma(\mathbf{r}) d(\mathbf{r}). \tag{2.40}$$

This quantity has a direction defined as normal to the surface S and in going from the negative to the positive charge.
In order to obtain the scalar potential produce by this configuration we use equation (2.39) to write

$$\Phi(\mathbf{r}) = \frac{1}{4\pi\epsilon_0} \int_S \frac{\sigma(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|} dS' - \frac{1}{4\pi\epsilon_0} \int_{S'} \frac{\sigma(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}' + \hat{\mathbf{n}} d|} dS'. \tag{2.41}$$

Now, it is possible to expand the integrand in the second term for small d using the expression

$$\frac{1}{|\mathbf{x} - \boldsymbol{\alpha}|} = \frac{1}{x} + \boldsymbol{\alpha} \cdot \boldsymbol{\nabla} \left(\frac{1}{x} \right) + ... \tag{2.42}$$

when $|\boldsymbol{\alpha}| \ll |\mathbf{x}|$. This gives

$$\Phi(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\int_S \frac{\sigma(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}dS' - \frac{1}{4\pi\epsilon_0}\int_{S'} \sigma(\mathbf{r}')\left[\frac{1}{|\mathbf{r}-\mathbf{r}'|} + d\hat{\mathbf{n}}\cdot\boldsymbol{\nabla}\left(\frac{1}{|\mathbf{r}-\mathbf{r}'|}\right) + ...\right]dS' \tag{2.43}$$

$$\Phi(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\int_{S'} \sigma(\mathbf{r}')d\hat{\mathbf{n}}\cdot\boldsymbol{\nabla}\left(\frac{1}{|\mathbf{r}-\mathbf{r}'|}\right)dS' \tag{2.44}$$

and then, in the limit $d \to 0$, we obtain the dipole-layer distribution strength,

$$\Phi(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\int_S D(\mathbf{r}')\hat{\mathbf{n}}\cdot\boldsymbol{\nabla}\left(\frac{1}{|\mathbf{r}-\mathbf{r}'|}\right)dS'. \tag{2.45}$$

Note that

$$\hat{\mathbf{n}}\cdot\boldsymbol{\nabla}\left(\frac{1}{|\mathbf{r}-\mathbf{r}'|}\right)dS' = -\hat{\mathbf{n}}\cdot\left(\frac{\mathbf{r}-\mathbf{r}'}{|\mathbf{r}-\mathbf{r}'|^3}\right)dS' = -\frac{\cos\theta dS'}{|\mathbf{r}-\mathbf{r}'|^2} = -d\Omega, \tag{2.46}$$

where $d\Omega$ is the solid angle swept by dS' as seen from the position $\mathbf{r}$. Hence, the scalar potential is simply

$$\Phi(\mathbf{r}) = -\frac{1}{4\pi\epsilon_0}\int_S D(\mathbf{r}')d\Omega. \tag{2.47}$$

2.8 Energy Density and Capacitance

If a point charge q_i is brought from infinity to a point $\mathbf{r}_i$ in a region with an electric field described by the potnetial $\Phi(\mathbf{r}_i)$ (vanishing at infinity), the work done on the charge is

$$W_i = q_i\Phi(\mathbf{r}_i). \tag{2.48}$$

This also corresponds to the potential energy of the charged particle at point $\mathbf{r}$. If the potential is produced by an array of $(n-1)$ point charges q_j with $j = 1, 2, ..., n-1$ located at positions $\mathbf{r}_j$, we can write

$$\Phi(\mathbf{r}_i) = \frac{1}{4\pi\epsilon_0}\sum_{j=1}^{n-1}\frac{q_j}{|\mathbf{r}_i-\mathbf{r}_j|}. \tag{2.49}$$

Hence, we have

$$W_i = \frac{q_i}{4\pi\epsilon_0}\sum_{j=1}^{n-1}\frac{q_j}{|\mathbf{r}_i-\mathbf{r}_j|}. \tag{2.50}$$

If we consider that we add each charge in succession to build the whole system, we sum over i and j (with $i \neq j$) and divide by 2 to obtain the *total potential energy* of the system,

$$W = \frac{1}{8\pi\epsilon_0}\sum_{i=1}^{n}\sum_{j=1}^{n}\frac{q_iq_j}{|\mathbf{r}_i-\mathbf{r}_j|}. \tag{2.51}$$

This expression can be generalized to charge distributions as

$$W = \frac{1}{8\pi\epsilon_0}\int\int \frac{\rho(\mathbf{r})\rho(\mathbf{r}')}{|\mathbf{r}_i - \mathbf{r}_j|} d^3x d^3x' \tag{2.52}$$

or as

$$W = \frac{1}{2}\int \rho(\mathbf{r})\Phi(\mathbf{r}) d^3x. \tag{2.53}$$

Using Poisson equation we can write this equation as

$$W = -\frac{\epsilon_0}{2}\int \Phi(\mathbf{r})\nabla^2\Phi(\mathbf{r}) d^3x. \tag{2.54}$$

Since

$$\boldsymbol{\nabla}\cdot(\Phi\boldsymbol{\nabla}\Phi) = |\boldsymbol{\nabla}\Phi|^2 + \Phi\nabla^2\Phi, \tag{2.55}$$

we have

$$W = -\frac{\epsilon_0}{2}\int \left[\boldsymbol{\nabla}\cdot(\Phi\boldsymbol{\nabla}\Phi) - |\boldsymbol{\nabla}\Phi|^2\right] d^3x. \tag{2.56}$$

The first term is zero because the integration is over all space and the potential vanishes at infinity. Therefore we are left with

$$W = \frac{\epsilon_0}{2}\int |\boldsymbol{\nabla}\Phi|^2 d^3x = \frac{\epsilon_0}{2}\int |\mathbf{E}|^2 d^3x. \tag{2.57}$$

The integrand in the last expression is identified with the energy density of the electrostatic field,

$$\mathcal{W} = \frac{\epsilon_0}{2}|\mathbf{E}|^2. \tag{2.58}$$

Example Consider a conductor with a surface charge density σ. Gauss' law gives the field in the surroundings of the conductor as

$$|\mathbf{E}|^2 = \frac{\sigma^2}{\epsilon_0^2}. \tag{2.59}$$

and then

$$\mathcal{W} = \frac{\sigma^2}{2\epsilon_0}. \tag{2.60}$$

If an area element Δa of the conducting surface is displaced outwards in a small distance Δx, the electrostatic energy decreases in the amount

$$\Delta W = -\mathcal{W}\Delta a\Delta x = -\frac{\sigma^2}{2\epsilon_0}\Delta a\Delta x \tag{2.61}$$

2.8.1 Capacitance

Consider a system of n conductors, each with potential V_i and total carge Q_i in empty space. The electrostatic potential energy of this system can be expressed in terms of the potentials and some geometrical quantities called coefficients of capacity. Since the potential is proportional to the electric charge we write

$$V_i = \sum_{j=1}^{n} p_{ij} Q_j \tag{2.62}$$

where the coefficients p_{ij} depend on the geometry of the conductors. Inverting these n equations we get expressions in the form

$$Q_i = \sum_{j=1}^{n} C_{ij} V_j \tag{2.63}$$

where the coefficients C_{ii} are called *capacities* or *capacitances* and the terms C_{ij} with $i \neq j$ are called *coefficients of induction.*
Te capacitance of a conductor is defined as the total charge on the conductor when it is maintained at unit potential and all other conductors are held at zero potential.

The potential energy for the system of conductors is written as

$$W = \frac{1}{2}\sum_{i=1}^{n} Q_i V_i = \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n} C_{ij} V_i V_j \tag{2.64}$$

2.9 Green's Theorems

Now we will show two identities or theorems due to George Green (1824). The first one uses the divergence theorem, which writes

$$\int_V \boldsymbol{\nabla} \cdot \mathbf{A} d^3x = \oint_S \mathbf{A} \cdot \mathbf{dS} \tag{2.65}$$

for any well-behaved vector field $\mathbf{A}$. Considering two arbitrary scalar fields ϕ and ψ such that $\mathbf{A} = \phi \boldsymbol{\nabla} \psi$, we have the vector identity

$$\boldsymbol{\nabla} \cdot \mathbf{A} = \boldsymbol{\nabla} \cdot (\phi \boldsymbol{\nabla} \psi) = \phi \nabla^2 \psi + \boldsymbol{\nabla} \phi \cdot \boldsymbol{\nabla} \psi. \tag{2.66}$$

Similarly, writing the surface element as $\mathbf{dS} = \hat{\mathbf{n}} da$, we have

$$\mathbf{A} \cdot \mathbf{dS} = \phi \boldsymbol{\nabla} \psi \cdot \mathbf{dS} = \phi \boldsymbol{\nabla} \psi \cdot \hat{\mathbf{n}} da = \phi \frac{\partial \psi}{\partial n} da, \tag{2.67}$$

where $\frac{\partial}{\partial n}$ represents the normal derivative at the surface S. Replacing these relations in the divergence theorem gives the *Green's first identity*,

$$\int_V \left[\phi \nabla^2 \psi + \boldsymbol{\nabla} \phi \cdot \boldsymbol{\nabla} \psi\right] d^3x = \oint_S \phi \frac{\partial \psi}{\partial n} da. \tag{2.68}$$

Considering now the field $\mathbf{A} = \psi \boldsymbol{\nabla} \phi$ gives the relation

$$\int_V \left[\psi \nabla^2 \phi + \boldsymbol{\nabla}\psi \cdot \boldsymbol{\nabla}\phi\right] d^3x = \oint_S \psi \frac{\partial \phi}{\partial n} da. \tag{2.69}$$

Subtracting these two relations we obtain the *Green's second identity*,

$$\int_V \left[\phi \nabla^2 \psi - \psi \nabla^2 \phi\right] d^3x = \oint_S \left[\phi \frac{\partial \psi}{\partial n} - \psi \frac{\partial \phi}{\partial n}\right] da. \tag{2.70}$$

As a particular example of this theorem, consider the functions

$$\psi = \frac{1}{|\mathbf{r} - \mathbf{r}'|} \tag{2.71}$$

$$\phi = \Phi(\mathbf{r}'). \tag{2.72}$$

This gives

$$\int_V \left[\Phi(\mathbf{r}')\nabla'^2 \left(\frac{1}{|\mathbf{r} - \mathbf{r}'|}\right) - \frac{1}{|\mathbf{r} - \mathbf{r}'|}\nabla'^2 \Phi(\mathbf{r}')\right] d^3x' = \oint_S \left[\Phi \frac{\partial}{\partial n'} \left(\frac{1}{|\mathbf{r} - \mathbf{r}'|}\right) - \frac{1}{|\mathbf{r} - \mathbf{r}'|}\frac{\partial \Phi}{\partial n'}\right] da'. \tag{2.73}$$

Using Poisson equation and the Dirac delta function this becomes

$$\int_V \left[-4\pi\Phi(\mathbf{r}')\delta(\mathbf{r} - \mathbf{r}') + \frac{1}{|\mathbf{r} - \mathbf{r}'|}\frac{\rho(\mathbf{r}')}{\epsilon_0}\right] d^3x' = \oint_S \left[\Phi \frac{\partial}{\partial n'} \left(\frac{1}{|\mathbf{r} - \mathbf{r}'|}\right) - \frac{1}{|\mathbf{r} - \mathbf{r}'|}\frac{\partial \Phi}{\partial n'}\right] da'. \tag{2.74}$$

If the point $\mathbf{r}$ lies within the volume V the first term is integrated to give

$$-4\pi\Phi(\mathbf{r}) + \int_V \left[\frac{1}{|\mathbf{r} - \mathbf{r}'|}\frac{\rho(\mathbf{r}')}{\epsilon_0}\right] d^3x' = \oint_S \left[\Phi \frac{\partial}{\partial n'} \left(\frac{1}{|\mathbf{r} - \mathbf{r}'|}\right) - \frac{1}{|\mathbf{r} - \mathbf{r}'|}\frac{\partial \Phi}{\partial n'}\right] da' \tag{2.75}$$

$$\Phi(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\int_V \frac{\rho(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|} d^3x' + \frac{1}{4\pi}\oint_S \left[\frac{1}{|\mathbf{r} - \mathbf{r}'|}\frac{\partial \Phi}{\partial n'} - \Phi \frac{\partial}{\partial n'} \left(\frac{1}{|\mathbf{r} - \mathbf{r}'|}\right)\right] da'. \tag{2.76}$$

On the other hand, if the point $\mathbf{r}$ doesn't lie within the volume V we obtain

$$\frac{1}{\epsilon_0}\int_V \frac{\rho(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|} d^3x' = \oint_S \left[\Phi \frac{\partial}{\partial n'} \left(\frac{1}{|\mathbf{r} - \mathbf{r}'|}\right) - \frac{1}{|\mathbf{r} - \mathbf{r}'|}\frac{\partial \Phi}{\partial n'}\right] da' \tag{2.77}$$

...

2.10 Dirichlet and Neumann Boundary Conditions

In order to specify completely a solution of Poisson or Laplace equation it is necessary to give some boundary conditions. When the value of the potential on a closed surface is given we are talking about a *Dirichlet boundary condition*. If the value of electric field (normal derivative of the potential) is given on the surface, it is known as a *Neumann boundary condition*.

In order to probe that the solution of the Poisson equations is unique given D- or N- boundary conditions, consider a volume V surrounded by a closed surface S at which boundary conditions are specified. Suppose that there exist two solutions of the Poisson equation, Φ_1 and Φ_2, satisfying the boundary conditions. Then the quantity

$$\Psi = \Phi_1 - \Phi_2 \tag{2.78}$$

will satisfy the relation

$$\nabla^2\Psi = \nabla^2\Phi_1 - \nabla\Phi_2 = -\frac{\rho}{\epsilon_0} + \frac{\rho}{\epsilon_0} = 0 \tag{2.79}$$

inside the volume V. It also satisfies

$$\Psi|_S = \Phi_1|_S - \Phi|_S = 0 \tag{2.80}$$

for Dirichlet boundary conditions or

$$\left.\frac{\partial\Psi}{\partial n}\right|_S = \left.\frac{\partial\Phi_1}{\partial n}\right|_S - \left.\frac{\partial\Phi_2}{\partial n}\right|_S = 0 \tag{2.81}$$

for Neumann boundary conditions. However, Green's first identity (2.68) give, using $\phi = \psi = \Psi$,

$$\int_V \left[\Psi\nabla^2\Psi + \boldsymbol{\nabla}\Psi \cdot \boldsymbol{\nabla}\Psi\right] d^3x = \oint_S \Psi\frac{\partial\Psi}{\partial n}da \tag{2.82}$$

and reduces for both boundary conditions to

$$\int_V \boldsymbol{\nabla}\Psi \cdot \boldsymbol{\nabla}\Psi d^3x = 0 \tag{2.83}$$

or

$$\int_V |\boldsymbol{\nabla}\Psi|^2 \, d^3x = 0 \tag{2.84}$$

from which we conclude that $\boldsymbol{\nabla}\Psi = 0$ or equivalently $\Psi =$ constant inside V. For Dirichlet boundary conditions we have $\Psi = 0$ on S, which implies that $\Phi_1 = \Phi_2$ inside V, i.e. the solution is unique.
On the other hand, for Neumann boundary conditions $\frac{\partial\Psi}{\partial n} = 0$ on S, and after integration it gives $\Phi_1 - \Phi_2 =$ constant inside V, i.e. the solution is unique apart from an unimportant additive constant.

2.11 Formal Solution of the Poisson Equation with Boundary Conditions. Green Functions

We have shown that

$$\nabla^2 \left(\frac{1}{|\mathbf{r} - \mathbf{r}'|} \right) = -4\pi\delta(\mathbf{r} - \mathbf{r}'). \tag{2.85}$$

This is an example of the *Green functions*, which satisfy the equation

$$\nabla^2 G(\mathbf{r}, \mathbf{r}') = -4\pi\delta(\mathbf{r} - \mathbf{r}'). \tag{2.86}$$

In general we can write G as

$$G(\mathbf{r}, \mathbf{r}') = \frac{1}{|\mathbf{r} - \mathbf{r}'|} + F(\mathbf{r}, \mathbf{r}') \tag{2.87}$$

with F a function satisfying

$$\nabla^2 G(\mathbf{r}, \mathbf{r}') = 0 \tag{2.88}$$

inside V. Green's theorem (2.70) let us write the general solution for the potential and the additional freedom given by function F can be used to eliminate one of the two surface integrals in the right hand side of the equation, giving an electrostatic potential satisfying either Dirichlet or Neumann boundary conditions. To show this, let us write equation (2.70) using $\phi = \Phi$ and $\psi = G(\mathbf{r}, \mathbf{r}')$,

$$\int_V \left[\Phi(\mathbf{r}')\nabla^2 G(\mathbf{r}, \mathbf{r}') - G(\mathbf{r}, \mathbf{r}')\nabla^2\Phi(\mathbf{r}')\right] d^3x' = \oint_S \left[\Phi \frac{\partial G(\mathbf{r}, \mathbf{r}')}{\partial n'} - G(\mathbf{r}, \mathbf{r}') \frac{\partial \Phi}{\partial n'}\right] da' \tag{2.89}$$

$$\int_V \left[-4\pi\Phi(\mathbf{r}')\delta(\mathbf{r} - \mathbf{r}') + G(\mathbf{r}, \mathbf{r}')\frac{\rho(\mathbf{r}')}{\epsilon_0}\right] d^3x' = \oint_S \left[\Phi \frac{\partial G(\mathbf{r}, \mathbf{r}')}{\partial n'} - G(\mathbf{r}, \mathbf{r}') \frac{\partial \Phi}{\partial n'}\right] da' \tag{2.90}$$

$$-4\pi\Phi(\mathbf{r}) + \int_V G(\mathbf{r}, \mathbf{r}')\frac{\rho(\mathbf{r}')}{\epsilon_0} d^3x' = \oint_S \left[\Phi \frac{\partial G(\mathbf{r}, \mathbf{r}')}{\partial n'} - G(\mathbf{r}, \mathbf{r}') \frac{\partial \Phi}{\partial n'}\right] da' \tag{2.91}$$

$$\Phi(\mathbf{r}) = \frac{1}{4\pi\epsilon_0} \int_V \rho(\mathbf{r}')G(\mathbf{r}, \mathbf{r}')d^3x' + \frac{1}{4\pi} \oint_S \left[G(\mathbf{r}, \mathbf{r}') \frac{\partial \Phi}{\partial n'} - \Phi \frac{\partial G(\mathbf{r}, \mathbf{r}')}{\partial n'}\right] da'. \tag{2.92}$$

For Dirichlet boundary conditions, we choose the function F so that

$$G_D(\mathbf{r}, \mathbf{r}') = 0 \tag{2.93}$$

for $\mathbf{x}'$ on S. Hence, the first term in the surface integral vanishes and the solution is

$$\Phi(\mathbf{r}) = \frac{1}{4\pi\epsilon_0} \int_V \rho(\mathbf{r}')G_D(\mathbf{r}, \mathbf{r}')d^3x' - \frac{1}{4\pi} \oint_S \Phi \frac{\partial G_D(\mathbf{r}, \mathbf{r}')}{\partial n'} da'. \tag{2.94}$$

On the other hand, for Neumann boundary conditions we must take another fact into account. Note that Gauss's theorem applied to the volume integral of equation (2.86) over the whole space gives

$$\int_V \nabla^2 G(\mathbf{r},\mathbf{r}')d^3x' = -4\pi \int \delta(\mathbf{r}-\mathbf{r}')d^3x' \tag{2.95}$$

$$\int_V \boldsymbol{\nabla}\cdot\boldsymbol{\nabla} G(\mathbf{r},\mathbf{r}')d^3x' = -4\pi \tag{2.96}$$

$$\oint_S \boldsymbol{\nabla} G(\mathbf{r},\mathbf{r}')\mathbf{dS}' = -4\pi \tag{2.97}$$

$$\oint_S \frac{\partial G(\mathbf{r},\mathbf{r}')}{\partial n'}da' = -4\pi. \tag{2.98}$$

Therefore, the condition that we will impose to obtain Neumann boundary conditions will be

$$\frac{\partial G_N(\mathbf{r},\mathbf{r}')}{\partial n'} = -\frac{4\pi}{S} \tag{2.99}$$

for $\mathbf{x}'$ on S. Hence the general solution becomes this time

$$\Phi(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\int_V \rho(\mathbf{r}')G(\mathbf{r},\mathbf{r}')d^3x' + \frac{1}{4\pi}\oint_S G(\mathbf{r},\mathbf{r}')\frac{\partial\Phi}{\partial n'}da' + \frac{1}{S}\oint_S \Phi da' \tag{2.100}$$

$$\Phi(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\int_V \rho(\mathbf{r}')G(\mathbf{r},\mathbf{r}')d^3x' + \frac{1}{4\pi}\oint_S G(\mathbf{r},\mathbf{r}')\frac{\partial\Phi}{\partial n'}da' + \langle\Phi\rangle_S, \tag{2.101}$$

where

$$\langle\Phi\rangle_S = \frac{1}{S}\oint_S \Phi da' \tag{2.102}$$

is the average value of the potential over the whole surface.
In order to give a physical meaning for the function $F(\mathbf{r},\mathbf{r}')$, note that it is a

solution of the Laplace equation inside V, so it represents the potential of a system of charges *external to the volume* V. This external distribution of charges is chosen to satisfy the homogeneous boundary conditions of zero potential or zero normal derivative on the surface S (D- or N- boundary conditions) when combined with the other term to give the total potential. This interpretation will be important in the method of images, which will be equivalent to find the appropiate function $F(\mathbf{r},\mathbf{r}')$ to satisfy the boundary conditions.

2.12 Variational Approach to Poisson Equation and Boundary Conditions

Consider the integral functional,

$$I[\psi] = \frac{1}{2}\int_V \boldsymbol{\nabla}\psi\cdot\boldsymbol{\nabla}\psi d^3x - \int_V g\psi d^3x, \tag{2.103}$$

where $\psi(\mathbf{r})$ is a well behaved function inside V and on the boundary surface S while $g(\mathbf{r})$ is a "source" function without singularities within V. Making the infinitesimal transformation $\psi \to \psi + \delta\psi$, we obtain

$$\delta I = \int_V \boldsymbol{\nabla}\psi \cdot \boldsymbol{\nabla}(\delta\psi) d^3x - \int_V g\delta\psi d^3x + \mathcal{O}(\delta\psi^2). \tag{2.104}$$

Using Green's first identity (2.68) with $\phi = \delta\psi$ and $\psi = \psi$ gives

$$\delta I = \int_V \left[-\nabla^2\psi\right] \delta\psi d^3x + \oint_S \delta\psi \frac{\partial\psi}{\delta n} da - \int_V g\delta\psi d^3x + \mathcal{O}(\delta\psi^2) \tag{2.105}$$

$$\delta I = \int_V \left[-\nabla^2\psi - g\right] \delta\psi d^3x + \oint_S \delta\psi \frac{\partial\psi}{\delta n} da + \mathcal{O}(\delta\psi^2). \tag{2.106}$$

If $\delta\psi = 0$ on the boundary surface, the second integral vanishes and we conclude that δI vanishes to first order in $delta\psi$ if

$$\nabla^2\psi = -g. \tag{2.107}$$

Hence, this variational derivation gives Poisson equation for electrostatics if we chose $\psi = \Phi$ and $g = \frac{\rho}{\epsilon_0}$. Note that Dirichlet's boundary conditions are given by the assumed condition $\delta\psi = \delta\Phi = 0$ on the boundary surface S.

In order to obtain the Poisson equation together with the Neumann boundary conditions we use the integral functional

$$I[\psi] = \frac{1}{2}\int_V \boldsymbol{\nabla}\psi \cdot \boldsymbol{\nabla}\psi d^3x - \int_V g\psi d^3x - \oint_s f\psi da, \tag{2.108}$$

and we suppose that the boundary conditions on ψ are given by

$$\left.\frac{\partial\psi}{\partial n}\right|_S = f(\mathbf{s}) \tag{2.109}$$

with s a point on the surface S. Considering the infinitesimal transformation $\psi \to \psi + \delta\psi$ and Green's first identity as before, we obtain

$$\delta I = \int_V \left[-\nabla^2\psi - g\right] \delta\psi d^3x + \oint_S \left[\frac{\partial\psi}{\delta n} - f(\mathbf{s})\right] \delta\psi da + \mathcal{O}(\delta\psi^2). \tag{2.110}$$

Thus, it is clear that requiring that $\delta I = 0$ independently of $\delta\psi$ implies that

$$\nabla^2\psi = -g \text{ within } V \tag{2.111}$$

and

$$\left.\frac{\partial\psi}{\partial n}\right|_S = f(\mathbf{s}) \text{ on } S. \tag{2.112}$$

Chapter 3

Boundary Value Problems in Electrostatics

3.1 Method of Images

The *Method of Images* works for problems in which there is one (or more) point charges in the presence of boundary surfaces. The method is based in the fact that, under some favorable conditions, it is possible to infer from the given geometry that a small number of charges, called *image charges*, with appropriate magnitudes and location (out of the region of interest) can simulate the required boundary conditions. Since the image charges are external to the volume of interest, their potentials are solution of Laplace's equation inside that volume.

3.1.1 Point Charge in the Presence of a Grounded Infinite Conductor Plane

As shown in the Figure, consider a point charge located at a position $\mathbf{r}' = \boldsymbol{\ell} = \ell \mathbf{e}_1$ in front of a vertical infinite conductor plane at zero potential. Hence, the Dirichlet boundary condition are $\Phi = 0$ at the points in the line $x = 0$ and $\Phi = 0$ at $\{x \to \infty, y \to \pm\infty, z \to \pm\infty\}$. From the geometry, it is easy to infer that this problem is equivalent to the problem of the original point charge together with an equal and opposite charge located at the mirror-image point $\mathbf{r}'_I = -\boldsymbol{\ell}$, behind the conductor surface. Therefore, the potential at any point $\mathbf{r}$ is given by the superposition

$$\Phi(\mathbf{r}) = \frac{1}{4\pi\epsilon_0} \frac{q}{|\mathbf{r} - \mathbf{r}'|} - \frac{1}{4\pi\epsilon_0} \frac{q}{|\mathbf{r} - \mathbf{r}'_I|}. \tag{3.1}$$

Here we identify the Green function

$$G(\mathbf{r}, \mathbf{r}') = \frac{1}{|\mathbf{r} - \mathbf{r}'|} - \frac{1}{|\mathbf{r} - \mathbf{r}'_I|} \tag{3.2}$$

in the region with $x \geq 0$. Note that this functions satisfies the imposed Dirichlet boundary conditions . It is also possible to identify that the method of images fixes the function $F(\mathbf{r}, \mathbf{r}')$ as

$$F(\mathbf{r}, \mathbf{r}') = -\frac{1}{|\mathbf{r} - \mathbf{r}'_I|}. \tag{3.3}$$

3.1.2 Finite Line of Charge

3.1.3 Point Charge in the Presence of a Grounded Conducting Sphere

Consider a point charge q located at the position $\mathbf{r}$ relative to the origin, around which is centered a grounded conducting sphere of radius R. We want to find the potential $\Phi(\mathbf{r})$ subject to the boundary conditions $\Phi(r = R) = 0$ and $\Phi(r \to \infty) = 0$. We will assume that it is needed only one image charge q_I lying on the ray going from the origin to the charge q. If q lies outside the sphere, the position $\mathbf{r}'_I$ of the image charge lies inside the sphere. Hence the potential due to both charges is given by

$$\Phi(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\frac{q}{|\mathbf{r} - \mathbf{r}'|} + \frac{1}{4\pi\epsilon_0}\frac{q_I}{|\mathbf{r} - \mathbf{r}'_I|}, \tag{3.4}$$

but we must chose q_I and $\mathbf{r}'_I$ to satisfy the boundary conditions. Define the unit vectors $\mathbf{n}$ along the direction $\mathbf{r}$ and $\mathbf{n}'$ along $\mathbf{r}'$. The we have

$$\Phi(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\frac{q}{|r\mathbf{n} - r'\mathbf{n}'|} + \frac{1}{4\pi\epsilon_0}\frac{q_I}{|r\mathbf{n} - r'_I\mathbf{n}'|}. \tag{3.5}$$

Factorizing r in the first term and r'_I in the second, we obtain

$$\Phi(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\frac{q}{r\left|\mathbf{n} - \frac{r'}{r}\mathbf{n}'\right|} + \frac{1}{4\pi\epsilon_0}\frac{q_I}{r'_I\left|\frac{r}{r'_I}\mathbf{n} - \mathbf{n}'\right|}, \tag{3.6}$$

and evaluating at the conductor sphere, $r = R$, yields

$$\Phi(r = R) = \frac{1}{4\pi\epsilon_0}\left[\frac{q}{R\left|\mathbf{n} - \frac{r'}{R}\mathbf{n}'\right|} + \frac{q_I}{r'_I\left|\frac{R}{r'_I}\mathbf{n} - \mathbf{n}'\right|}\right] = 0. \tag{3.7}$$

In order to satisfy this boundary condition, we choose

$$\frac{q_I}{r'_I} = -\frac{q}{R} \tag{3.8}$$

and

$$\frac{R}{r'_I} = \frac{r'}{R}. \tag{3.9}$$

These relations are combined to obtain the image charge

$$q_I = -\frac{R}{r'}q \tag{3.10}$$

and its position

$$r'_I = \frac{R^2}{r'}. \tag{3.11}$$

If the charge q is brought closer to the sphere (i.e. $r' \to R$) the image charge grows in magnitude, $q_I \to -q$, and it is moves to the center of the sphere, $r'_I \to R$.
The surface charge density in the conducting sphere is calculated using the image charge and the normal derivative of the potential at the surface (remember that $\mathbf{E}\cdot\mathbf{n} = -\boldsymbol{\nabla}\Phi\cdot\mathbf{n} = \frac{\sigma}{\epsilon_0}$),

$$\sigma = -\epsilon_0 \left.\boldsymbol{\nabla}\Phi\cdot\mathbf{n}\right|_{r=R} = -\frac{1}{4\pi}\left.\boldsymbol{\nabla}\left[\frac{q}{|r\mathbf{n} - r'\mathbf{n}'|} + \frac{q_I}{|r\mathbf{n} - r'_I\mathbf{n}'|}\right]\cdot\mathbf{n}\right|_{r=R} \tag{3.12}$$

$$\sigma = \frac{1}{4\pi}\left.\left[\frac{q(r\mathbf{n} - r'\mathbf{n}')\cdot\mathbf{n}}{|r\mathbf{n} - r'\mathbf{n}'|^3} + \frac{q_I(r\mathbf{n} - r'_I\mathbf{n}')\cdot\mathbf{n}}{|r\mathbf{n} - r'_I\mathbf{n}'|^3}\right]\right|_{r=R} \tag{3.13}$$

$$\sigma = \frac{1}{4\pi}\left.\left[\frac{q(r - r'\mathbf{n}'\cdot\mathbf{n})}{|r\mathbf{n} - r'\mathbf{n}'|^3} + \frac{q_I(r - r'_I\mathbf{n}'\cdot\mathbf{n})}{|r\mathbf{n} - r'_I\mathbf{n}'|^3}\right]\right|_{r=R} \tag{3.14}$$

$$\sigma = \frac{1}{4\pi}\left.\left[\frac{q(r - r'\mathbf{n}'\cdot\mathbf{n})}{(r^2 + r'^2 - 2rr'\mathbf{n}\cdot\mathbf{n}')^{3/2}} + \frac{q_I(r - r'_I\mathbf{n}'\cdot\mathbf{n})}{(r^2 + r'^2_I - 2rr'_I\mathbf{n}\cdot\mathbf{n}')^{3/2}}\right]\right|_{r=R} \tag{3.15}$$

$$\sigma = \frac{1}{4\pi}\left[\frac{q(R - r'\mathbf{n}'\cdot\mathbf{n})}{(R^2 + r'^2 - 2Rr'\mathbf{n}\cdot\mathbf{n}')^{3/2}} + \frac{q_I(R - r'_I\mathbf{n}'\cdot\mathbf{n})}{(R^2 + r'^2_I - 2Rr'_I\mathbf{n}\cdot\mathbf{n}')^{3/2}}\right]. \tag{3.16}$$

Replacing the values of q_I and r'_I we have

$$\sigma = \frac{qR}{4\pi}\left[\frac{(1 - \frac{r'}{R}\mathbf{n}'\cdot\mathbf{n})}{(R^2 + r'^2 - 2Rr'\mathbf{n}\cdot\mathbf{n}')^{3/2}} - \frac{\frac{R}{r'}(1 - \frac{R}{r'}\mathbf{n}'\cdot\mathbf{n})}{\left(R^2 + \left(\frac{R^2}{r'}\right)^2 - 2R\left(\frac{R^2}{r'}\right)\mathbf{n}\cdot\mathbf{n}'\right)^{3/2}}\right] \tag{3.17}$$

$$\sigma = \frac{qR}{4\pi}\left[\frac{(1 - \frac{r'}{R}\mathbf{n}'\cdot\mathbf{n})}{r'^3\left(\left(\frac{R}{r'}\right)^2 + 1 - 2\frac{R}{r'}\mathbf{n}\cdot\mathbf{n}'\right)^{3/2}} - \frac{\frac{R}{r'}(1 - \frac{R}{r'}\mathbf{n}'\cdot\mathbf{n})}{R^3\left(1 + \left(\frac{R}{r'}\right)^2 - 2\left(\frac{R}{r'}\right)\mathbf{n}\cdot\mathbf{n}'\right)^{3/2}}\right] \tag{3.18}$$

$$\sigma = \frac{qR}{4\pi}\frac{1}{\left(1 + \left(\frac{R}{r'}\right)^2 - 2\frac{R}{r'}\mathbf{n}\cdot\mathbf{n}'\right)^{3/2}}\left[\frac{1}{r'^3} - \frac{1}{Rr'^2}\mathbf{n}'\cdot\mathbf{n} - \frac{1}{R^2r'} + \frac{1}{Rr'^2}\mathbf{n}'\cdot\mathbf{n}\right] \tag{3.19}$$

$$\sigma = \frac{qR}{4\pi} \frac{1}{\left(1 + \left(\frac{R}{r'}\right)^2 - 2\frac{R}{r'}\mathbf{n}\cdot\mathbf{n}'\right)^{3/2}} \left[\frac{1}{r'^3} - \frac{1}{R^2 r'}\right] \tag{3.20}$$

$$\sigma = \frac{qR}{4\pi} \frac{R}{r'} \frac{1}{\left(1 + \left(\frac{R}{r'}\right)^2 - 2\frac{R}{r'}\mathbf{n}\cdot\mathbf{n}'\right)^{3/2}} \left[\frac{1}{r'^2 R} - \frac{1}{R^3}\right] \tag{3.21}$$

$$\sigma = -\frac{q}{4\pi R^2} \frac{R}{r'} \frac{\left(1 - \frac{R^2}{r'^2}\right)}{\left(1 + \left(\frac{R}{r'}\right)^2 - 2\frac{R}{r'}\mathbf{n}\cdot\mathbf{n}'\right)^{3/2}}. \tag{3.22}$$

Defining $\mathbf{n}\cdot\mathbf{n}' = \cos\gamma$ we write

$$\sigma = -\frac{q}{4\pi R^2} \frac{R}{r'} \frac{\left(1 - \frac{R^2}{r'^2}\right)}{\left(1 + \left(\frac{R}{r'}\right)^2 - 2\frac{R}{r'}\cos\gamma\right)^{3/2}}. \tag{3.23}$$

The force acting on q is calculated using the image charge q_I and the distance between them:

$$r_{qq_I} = r' - r'_I = r'\left(1 - \frac{R^2}{r'^2}\right). \tag{3.24}$$

Hence, this force is simply

$$\mathbf{F} = \frac{1}{4\pi\epsilon_0} \frac{qq_I}{r_{qq_I}^2}\mathbf{n} \tag{3.25}$$

$$\mathbf{F} = \frac{1}{4\pi\epsilon_0} \frac{q\left(-\frac{R}{r'}q\right)}{r'^2\left(1 - \frac{R^2}{r'^2}\right)^2}\mathbf{n} \tag{3.26}$$

$$\mathbf{F} = -\frac{1}{4\pi\epsilon_0} \frac{q^2}{R^2} \left(\frac{R}{r'}\right)^3 \left(1 - \frac{R^2}{r'^2}\right)^{-2} \mathbf{n}. \tag{3.27}$$

If the point charge q lies inside the sphere, the the same results apply.

3.1.4 Point Charge in the Presence of a Charged, Insulated, Conducting Sphere

In order to obtain the potential due to a point charge q near a charged, insulated, conducting sphere we begin with the result of the previous section in which the grounded conducting sphere acquires a total charge of q_I distributed on its surface. The the ground is disconnected and we add to the sphere the charge $(Q - q_I)$ which will bring the total charge of the sphere up to Q. However, we may think as if this added charge will simply distribute uniformly over the surface of the sphere (because the external point charge q is already balanced by the image q_I). Therefore, the total potential is obtained by adding the potential found in the previous section, (3.4), and the potential of a point charge $(Q - q_I)$ at the origin. This gives

$$\Phi(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\frac{q}{|\mathbf{r}-\mathbf{r}'|} + \frac{1}{4\pi\epsilon_0}\frac{q_I}{|\mathbf{r}-\mathbf{r}'_I|} + \frac{1}{4\pi\epsilon_0}\frac{Q-q_I}{|\mathbf{r}|} \tag{3.28}$$

$$\Phi(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\left[\frac{q}{|\mathbf{r}-\mathbf{r}'|} + \frac{q_I}{|\mathbf{r}-\mathbf{r}'_I|} + \frac{Q-q_I}{|\mathbf{r}|}\right] \tag{3.29}$$

$$\Phi(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\left[\frac{q}{|\mathbf{r}-\mathbf{r}'|} - \frac{Rq}{r'\left|\mathbf{r}-\frac{R^2}{r'^2}\mathbf{r}'\right|} + \frac{Q+\frac{R}{r'}q}{|\mathbf{r}|}\right]. \tag{3.30}$$

Similarly, the force acting on the charge q is obtained by superposition of equation (3.27) with the Coulomb force produced by $(Q - q_I)$,

$$\mathbf{F} = -\frac{1}{4\pi\epsilon_0}\frac{q^2}{R^2}\left(\frac{R}{r'}\right)^3\left(1-\frac{R^2}{r'^2}\right)^{-2}\mathbf{n} + \frac{1}{4\pi\epsilon_0}\frac{q(Q-q_I)}{r'^2}\mathbf{n} \tag{3.31}$$

$$\mathbf{F} = -\frac{1}{4\pi\epsilon_0}\frac{q^2}{R^2}\left(\frac{R}{r'}\right)^3\left(1-\frac{R^2}{r'^2}\right)^{-2}\mathbf{n} + \frac{1}{4\pi\epsilon_0}\frac{q(Q+\frac{R}{r'}q)}{r'^2}\mathbf{n} \tag{3.32}$$

$$\mathbf{F} = \frac{1}{4\pi\epsilon_0}\frac{q}{r'^2}\left[Q - q\left(\frac{R}{r'}\right)\left(1-\frac{R^2}{r'^2}\right)^{-2} + \frac{R}{r'}q\right]\mathbf{n} \tag{3.33}$$

$$\mathbf{F} = \frac{1}{4\pi\epsilon_0}\frac{q}{r'^2}\left[Q - q\left(\frac{R}{r'}\right)\left(\left(1-\frac{R^2}{r'^2}\right)^{-2} - 1\right)\right]\mathbf{n} \tag{3.34}$$

$$\mathbf{F} = \frac{1}{4\pi\epsilon_0}\frac{q}{r'^2}\left[Q - q\left(\frac{R}{r'}\right)\frac{1-\left(1-\frac{R^2}{r'^2}\right)^2}{\left(1-\frac{R^2}{r'^2}\right)^2}\right]\mathbf{n} \tag{3.35}$$

$$\mathbf{F} = \frac{1}{4\pi\epsilon_0}\frac{q}{r'^2}\left[Q - q\left(\frac{R}{r'}\right)\frac{r'^4-\left(r'^2-R^2\right)^2}{\left(r'^2-R^2\right)^2}\right]\mathbf{n} \tag{3.36}$$

$$\mathbf{F} = \frac{1}{4\pi\epsilon_0}\frac{q}{r'^2}\left[Q - q\left(\frac{R}{r'}\right)\frac{2r'^2R^2-R^4}{\left(r'^2-R^2\right)^2}\right]\mathbf{n} \tag{3.37}$$

$$\mathbf{F} = \frac{1}{4\pi\epsilon_0}\frac{q}{r'^2}\left[Q - q\left(\frac{R^3}{r'}\right)\frac{2r'^2-R^2}{\left(r'^2-R^2\right)^2}\right]\mathbf{n}. \tag{3.38}$$

Note that in the linimt $r' \gg R$ the force reduces to the Coulomb's law,

$$\lim_{r' \gg R}\mathbf{F} = \frac{1}{4\pi\epsilon_0}\frac{qQ}{r'^2}\mathbf{n} \tag{3.39}$$

3.1.5 Point Charge in the Presence of a Conducting Sphere at Fixed Potential

In this case we consider a conducting sphere at a fixed potential V. The problem is treated exactly as that of a charged, insulated conducting sphere but, instead of the charge $(Q-q_I)$ at its center, we use the value $V(4\pi\epsilon_0 R)$. Hence, equation (3.28) becomes

$$\Phi(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\frac{q}{|\mathbf{r}-\mathbf{r}'|} + \frac{1}{4\pi\epsilon_0}\frac{q_I}{|\mathbf{r}-\mathbf{r}'_I|} + \frac{VR}{|\mathbf{r}|}. \qquad (3.40)$$

The force acting on the charge q is obtained by adding to equation (3.27) a Coulomb force produced by V ,

$$\mathbf{F} = -\frac{1}{4\pi\epsilon_0}\frac{q^2}{R^2}\left(\frac{R}{r'}\right)^3\left(1-\frac{R^2}{r'^2}\right)^{-2}\mathbf{n} + \frac{qVR}{r'^2}\mathbf{n} \qquad (3.41)$$

$$\mathbf{F} = \frac{q}{r'^2}\left[VR - \frac{1}{4\pi\epsilon_0}\frac{qR}{r'}\left(1-\frac{R^2}{r'^2}\right)^{-2}\mathbf{n}\right]\mathbf{n} \qquad (3.42)$$

$$\mathbf{F} = \frac{q}{r'^2}\left[VR - \frac{1}{4\pi\epsilon_0}\frac{qRr'^3}{\left(r'^2-\frac{R^2}{r'^2}\right)^2}\mathbf{n}\right]\mathbf{n}. \qquad (3.43)$$

3.1.6 Conducting Sphere ina Uniform Electric Field

Consider now a conducting sphere of radius R in the presence of an uniform electric field $\mathbf{E} = E_0\mathbf{e}_z$. Since the uniform electric field may be thought as produced by an appropriate distribution of electric charge at infinity, we may use the method of images, using some point charges, to solve this problem.

Consider two point charges Q and $-Q$ at positions $z=-\ell$ and $z=\ell$, respectively. The electric field produced in a region near the origin is approximately constant with the value $\mathbf{E}_0 \approx \frac{2Q}{4\pi\epsilon_0\ell^2}\mathbf{e}_z$ (if the dimensions of the region are small compared with the distance ℓ). This approximated value becomes exact if we take the limits $\ell\to\infty$ and $Q\to\infty$ maintaining the value of $\frac{Q}{\ell^2}$ constant.

Considering some of our previous results, the system of the conducting sphere in the presence of the uniform electric field may be replaced, by the method of images, by the charges $\pm Q$ located at $z=\mp\ell$ together with the corresponding image charges $\mp\frac{R}{\ell}Q$ at $z=\mp\frac{R^2}{\ell}$ (see equations (3.10) and (3.11)) and taking the appropriate limit. Therefore the resulting potential of the system is obtained from

$$\begin{aligned}\Phi(\mathbf{r}) &= \frac{1}{4\pi\epsilon_0}\frac{Q}{\sqrt{r^2+\ell^2+2\ell r\cos\theta}} - \frac{1}{4\pi\epsilon_0}\frac{Q}{\sqrt{r^2+\ell^2-2\ell r\cos\theta}} \\ &-\frac{1}{4\pi\epsilon_0}\frac{QR}{\ell\sqrt{r^2+\left(\frac{R^2}{\ell}\right)^2+2\frac{R^2r}{\ell}\cos\theta}} \\ &+\frac{1}{4\pi\epsilon_0}\frac{QR}{\ell\sqrt{r^2+\left(\frac{R^2}{\ell}\right)^2-2\frac{R^2r}{\ell}\cos\theta}}. \end{aligned} \tag{3.44}$$

Using the expansions for $\ell \gg 1$

$$\begin{aligned}\frac{Q}{\sqrt{r^2+\ell^2\pm 2\ell r\cos\theta}} &= \frac{Q}{\ell}\left(1+\left(\frac{r}{\ell}\right)^2 \pm 2\frac{r}{\ell}\cos\theta\right)^{-1/2} \\ &= \frac{Q}{\ell}\left(1-\frac{1}{2}\left(\frac{r}{\ell}\right)^2 \mp \frac{r}{\ell}\cos\theta + \mathcal{O}\left(\left(\frac{r}{\ell}\right)^3\right)\right) \\ &= \frac{Q}{\ell} \mp \frac{Qr}{\ell^2}\cos\theta + \mathcal{O}\left(\left(\frac{r}{\ell}\right)^3\right) \end{aligned} \tag{3.45}$$

and

$$\begin{aligned}\frac{QR}{\ell\sqrt{r^2+\left(\frac{R^2}{\ell}\right)^2 \pm 2\frac{R^2r}{\ell}\cos\theta}} &= \frac{QR}{\ell r\sqrt{1+\left(\frac{R^2}{r}\frac{1}{\ell}\right)^2 \pm 2\frac{R^2}{r}\frac{1}{\ell}\cos\theta}} \\ &= \frac{QR}{\ell r}\left(1-\frac{1}{2}\left(\frac{R^2}{r}\frac{1}{\ell}\right)^2 \mp \frac{R^2}{r}\frac{1}{\ell}\cos\theta + \mathcal{O}\left(\left(\frac{1}{\ell}\right)^3\right)\right) \\ &= \frac{QR}{\ell r} \mp \frac{Q}{\ell^2}\frac{R^3}{r^2}\cos\theta + \mathcal{O}\left(\left(\frac{1}{\ell}\right)^3\right). \end{aligned} \tag{3.46}$$

Taking the limits $\ell \to \infty$ and $Q \to \infty$, maintaining the value of $\frac{Q}{\ell}$ constant, gives the non-vanishing terms in the electrostatic potential

$$\Phi(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\left[-2\frac{Qr}{\ell^2}\cos\theta + 2\frac{Q}{\ell^2}\frac{R^3}{r^2}\cos\theta\right] + ... \tag{3.47}$$

Note that the considered limit implies the constant value of the field $E_0 = \frac{2Q}{4\pi\epsilon_0\ell^2}$ and therefore we can write

$$\Phi(\mathbf{r}) = -E_0\left(r+\frac{R^3}{r^2}\right)\cos\theta. \tag{3.48}$$

Here we can identify the first term as the potential contribution of the external uniform field,

$$\Phi_0 = -E_0 r\cos\theta = -E_0 z, \tag{3.49}$$

while the second term is the potential contribution due to the induced surface charge density on the sphere, i.e. due to the image charges,

$$\Phi_I = E_0 \frac{R^3}{r^2} \cos\theta. \tag{3.50}$$

The corresponding induced surface charge density on the sphere is given by

$$\sigma = -\epsilon_0 \left.\frac{\partial \Phi}{\partial r}\right|_{r=R} = \epsilon_0 E_0 \left.\frac{\partial}{\partial r}\left(r + \frac{R^3}{r^2}\right)\right|_{r=R} \cos\theta = \epsilon_0 E_0 \left.\left(1 + 2\frac{R^3}{r^3}\right)\right|_{r=R} \cos\theta \tag{3.51}$$

$$\sigma = 3\epsilon_0 E_0 \cos\theta. \tag{3.52}$$

3.2 Green Function for the Sphere

As seen from the previous examples, the Green function describing a sphere of radius R and satisfying Dirichlet boundary conditions is

$$G(\mathbf{r}, \mathbf{r}') = \frac{1}{|\mathbf{r} - \mathbf{r}'|} - \frac{R}{r'\left|\mathbf{r} - \frac{R^2}{r'^2}\mathbf{r}'\right|}. \tag{3.53}$$

Here $\mathbf{r}'$ corresponds to the location of the point source and $\mathbf{r}$ corresponds to the point at which the potential will be evaluated. Using spherical coordinates we write

$$|\mathbf{r} - \mathbf{r}'| = \sqrt{r^2 + r'^2 - 2rr'\cos\gamma} \tag{3.54}$$

and

$$\left|\mathbf{r} - \frac{R^2}{r'^2}\mathbf{r}'\right| = \sqrt{r^2 + \frac{R^4}{r'^2} - 2\frac{R^2}{r'} r\cos\gamma}, \tag{3.55}$$

where γ is the angle between $\mathbf{r}$ and $\mathbf{r}'$. Then, Green' function becomes

$$G(\mathbf{r}, \mathbf{r}') = \frac{1}{\sqrt{r^2 + r'^2 - 2rr'\cos\gamma}} - \frac{R}{r'\sqrt{r^2 + \frac{R^4}{r'^2} - 2\frac{R^2}{r'} r\cos\gamma}} \tag{3.56}$$

$$G(\mathbf{r}, \mathbf{r}') = \frac{1}{\sqrt{r^2 + r'^2 - 2rr'\cos\gamma}} - \frac{1}{\sqrt{\frac{r^2 r'^2}{R^2} + R^2 - 2rr'2\cos\gamma}}. \tag{3.57}$$

Note the symmetry of this relation in the variables $\mathbf{r}$ and $\mathbf{r}'$ and that $G = 0$ if either $\mathbf{r} = R$ or $\mathbf{r}' = R$.

In order to calculate the electrostatic potential we will use the general relation

$$\Phi(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\int_V \rho(\mathbf{r}')G(\mathbf{r}, \mathbf{r}')d^3x' - \frac{1}{4\pi}\oint_S \Phi \frac{\partial G(\mathbf{r}, \mathbf{r}')}{\partial n'} da', \tag{3.58}$$

so we need the derivative

$$\frac{\partial G(\mathbf{r},\mathbf{r}')}{\partial n'} = \boldsymbol{\nabla} G(\mathbf{r},\mathbf{r}')\cdot\mathbf{n}'. \tag{3.59}$$

Due to the symmetry of G in the variables $\mathbf{r}$ and $\mathbf{r}'$, this derivative gives the same result as the surface charge density in equation (3.23),

$$\left.\frac{\partial G(\mathbf{r},\mathbf{r}')}{\partial n'}\right|_{r'=R} = -\frac{1}{R^2}\frac{R}{r}\frac{\left(1-\frac{R^2}{r^2}\right)}{\left(1+\left(\frac{R}{r}\right)^2-2\frac{R}{r}\cos\gamma\right)^{3/2}} \tag{3.60}$$

$$\left.\frac{\partial G(\mathbf{r},\mathbf{r}')}{\partial n'}\right|_{r'=R} = -\frac{1}{R^2}\frac{R}{r}\frac{\left(1-\frac{R^2}{r^2}\right)}{\frac{1}{r^3}\left(r^2+R^2-2Rr\cos\gamma\right)^{3/2}} \tag{3.61}$$

$$\left.\frac{\partial G(\mathbf{r},\mathbf{r}')}{\partial n'}\right|_{r'=R} = -\frac{r^2}{R}\frac{\left(1-\frac{R^2}{r^2}\right)}{\left(r^2+R^2-2Rr\cos\gamma\right)^{3/2}} \tag{3.62}$$

$$\left.\frac{\partial G(\mathbf{r},\mathbf{r}')}{\partial n'}\right|_{r'=R} = -\frac{\left(r^2-R^2\right)}{R\left(r^2+R^2-2Rr\cos\gamma\right)^{3/2}}. \tag{3.63}$$

Replacing this result in the expression for the potential, we conclude that the solution of the Laplace equation *outside* a sphere (where the density vanishes, $\rho = 0$), with the potential specified on its surface (Dirichlet boundary conditions) is simply

$$\Phi(\mathbf{r}) = \frac{1}{4\pi}\int \Phi(R,\theta'\varphi')\frac{\left(r^2-R^2\right)}{R\left(r^2+R^2-2Rr\cos\gamma\right)^{3/2}}R^2 d\Omega', \tag{3.64}$$

$$\Phi(\mathbf{r}) = \frac{1}{4\pi}\int \Phi(R,\theta'\varphi')\frac{R\left(r^2-R^2\right)}{\left(r^2+R^2-2Rr\cos\gamma\right)^{3/2}}d\Omega', \tag{3.65}$$

where $d\Omega'$ is the element of solid angle at the point (R,θ',φ'). The angle γ is related with these angles through the relation

$$\cos\gamma = \cos\theta\cos\theta' + \sin\theta\sin\theta'\cos(\varphi-\varphi'). \tag{3.66}$$

3.2.1 Conducting Sphere with Hemispheres at Different Potentials

Consider a conducting sphere of radius R, made up of two hemispherical shells separated by a small insulated ring. Consider that the insulating ring lies in the $z = 0$ plane and that the upper hemisphere is kept at the potential $+V_0$ while the lower hemisphere is kept at $-V_0$. Using equation (3.65), the potential in the exterior of the conducting sphere is given by

$$\begin{aligned}\Phi(r,\theta,\varphi) &= \frac{1}{4\pi}\int_0^{2\pi}\int_0^1 V_0 \frac{R\left(r^2-R^2\right)}{\left(r^2+R^2-2Rr\cos\gamma\right)^{3/2}} d\varphi' d(\cos\theta') \\ &+\frac{1}{4\pi}\int_0^{2\pi}\int_{-1}^0 (-V_0)\frac{R\left(r^2-R^2\right)}{\left(r^2+R^2-2Rr\cos\gamma\right)^{3/2}} d\varphi' d(\cos\theta')\end{aligned}$$

$$\begin{aligned}\Phi(r,\theta,\varphi) &= \tfrac{V_0}{4\pi} \int_0^{2\pi}\left[\int_0^1 \frac{R\left(r^2-R^2\right)}{\left(r^2+R^2-2Rr\cos\gamma\right)^{3/2}} d(\cos\theta') \right.\\ &\left. - \int_{-1}^0 \frac{R\left(r^2-R^2\right)}{\left(r^2+R^2-2Rr\cos\gamma\right)^{3/2}} d(\cos\theta')\right] d\varphi'.\end{aligned}$$

Making the change of variables in the second integral,

$$\begin{aligned}\theta' &\to \pi-\theta' \\ \varphi' &\to \varphi'+\pi\end{aligned}$$

we have

$$\begin{aligned}\cos\theta' &\to \cos(\pi-\theta') = \cos\pi\cos\theta' + \sin\pi\sin\theta' = -\cos\theta' \\ \sin\theta' &\to \sin(\pi-\theta') = \sin\pi\cos\theta' - \cos\pi\sin\theta' = \sin\theta' \\ \cos(\varphi-\varphi') &\to \cos(\varphi-\varphi'+\pi) = \cos(\varphi-\varphi')\cos(\pi) - \sin(\varphi-\varphi')\sin\pi = -\cos(\varphi-\varphi')\end{aligned}$$

and

$$\begin{aligned}\cos\gamma &\to \cos\theta\cos(\pi-\theta') + \sin\theta\sin(\pi-\theta')\cos(\varphi-\varphi'-\pi)\cos\gamma \\ &= -\cos\theta\cos\theta' - \sin\theta\sin\theta'\cos(\varphi-\varphi') \\ &= -\cos\gamma\end{aligned}$$

Therefore, under these coordinate transformations the potential becomes

$$\begin{aligned}\Phi(r,\theta,\varphi) &= \tfrac{V_0}{4\pi} \int_0^{2\pi}\left[\int_0^1 \frac{R\left(r^2-R^2\right)}{\left(r^2+R^2-2Rr\cos\gamma\right)^{3/2}} d(\cos\theta') \right.\\ &\left. - \int_1^0 \frac{R\left(r^2-R^2\right)}{\left(r^2+R^2+2Rr\cos\gamma\right)^{3/2}} d(-\cos\theta')\right] d\varphi'\end{aligned}$$

$$\begin{aligned}\Phi(r,\theta,\varphi) &= \tfrac{V_0 R\left(r^2-R^2\right)}{4\pi} \int_0^{2\pi}\left[\int_0^1 \left(r^2+R^2-2Rr\cos\gamma\right)^{-3/2} d(\cos\theta') \right.\\ &\left. - \int_0^1 \left(r^2+R^2+2Rr\cos\gamma\right)^{-3/2} d(\cos\theta')\right] d\varphi'\end{aligned}$$

$$\begin{aligned}\Phi(r,\theta,\varphi) &= \tfrac{V_0 R\left(r^2-R^2\right)}{4\pi} \quad \int_0^{2\pi}\int_0^1 \Big[\left(r^2+R^2-2Rr\cos\gamma\right)^{-3/2} \\ &\qquad - \left(r^2+R^2+2Rr\cos\gamma\right)^{-3/2}\Big]\, d(\cos\theta')d\varphi' \qquad (3.67)\end{aligned}$$

The complex dependence of the angle γ on θ' and φ' makes the integral impossible in closed form. However some special cases can studied. For example, the electrostatic potential along the z axis ($\theta = 0$ and $r = z$) gives the simple relation $\cos\gamma = \cos\theta'$ and the integral becomes

$$\begin{aligned}\Phi(z) &= \tfrac{V_0 R\left(z^2-R^2\right)}{4\pi} \quad \int_0^{2\pi}\int_0^1 \Big[\left(z^2+R^2-2Rz\cos\theta'\right)^{-3/2} \\ &\qquad - \left(z^2+R^2+2Rz\cos\theta'\right)^{-3/2}\Big]\, d(\cos\theta')d\varphi'\end{aligned}$$

$$\begin{aligned}\Phi(z) &= \tfrac{V_0 R\left(z^2-R^2\right)}{2} \quad \int_0^1 \Big[\left(z^2+R^2-2Rz\cos\theta'\right)^{-3/2} \\ &\qquad - \left(z^2+R^2+2Rz\cos\theta'\right)^{-3/2}\Big]\, d(\cos\theta').\end{aligned}$$

Note that

$$\begin{aligned}\int_0^1 \left(z^2+R^2-2Rz\cos\theta'\right)^{-3/2} d(\cos\theta') &= \int_{z^2+R^2}^{z^2+R^2-2Rz} \alpha^{-3/2}\frac{d\alpha}{-2Rz} \\ &= -\frac{1}{2Rz}\,(-2\alpha^{-1/2})\Big|_{z^2+R^2}^{z^2+R^2-2Rz} \\ &= \frac{1}{Rz}\left[(z^2+R^2-2Rz)^{-1/2}-(z^2+R^2)^{-1/2}\right] \\ &= \frac{1}{Rz}\left[((z-R)^2)^{-1/2}-(z^2+R^2)^{-1/2}\right] \\ &= \frac{1}{Rz}\left[(z-R)^{-1}-(z^2+R^2)^{-1/2}\right]\end{aligned}$$

and

$$\begin{aligned}\int_0^1 \left(z^2+R^2+2Rz\cos\theta'\right)^{-3/2} d(\cos\theta') &= \int_{z^2+R^2}^{z^2+R^2+2Rz} \alpha^{-3/2}\frac{d\alpha}{2Rz} \\ &= \frac{1}{2Rz}\,(-2\alpha^{-1/2})\Big|_{z^2+R^2}^{z^2+R^2+2Rz} \\ &= \frac{1}{Rz}\left[(z^2+R^2)^{-1/2}-(z^2+R^2+2Rz)^{-1/2}\right] \\ &= \frac{1}{Rz}\left[(z^2+R^2)^{-1/2}-((z+R)^2)^{-1/2}\right] \\ &= \frac{1}{Rz}\left[(z^2+R^2)^{-1/2}-(z+R)^{-1}\right].\end{aligned}$$

Thus, the potential gives

$$\Phi(z) = \frac{V_0 R\left(z^2 - R^2\right)}{2} \frac{1}{Rz} \left[(z-R)^{-1} - (z^2+R^2)^{-1/2} - (z^2+R^2)^{-1/2} + (z+R)^{-1}\right] \tag{3.68}$$

$$\Phi(z) = \frac{V_0\left(z^2 - R^2\right)}{2z} \left[\frac{1}{z-R} + \frac{1}{z+R} - \frac{2}{\sqrt{z^2+R^2}}\right] \tag{3.69}$$

$$\Phi(z) = \frac{V_0\left(z^2 - R^2\right)}{2z} \left[\frac{2z}{z^2-R^2} - \frac{2}{\sqrt{z^2+R^2}}\right] \tag{3.70}$$

$$\Phi(z) = V_0 \left[1 - \frac{z^2-R^2}{z\sqrt{z^2+R^2}}\right]. \tag{3.71}$$

Note that evaluating this equation at $z = R$ gives $\Phi(z = R) = V_0$ as needed and taking the limit $z \gg R$ gives the asymptotic value

$$\Phi = V_0 \left[1 - \frac{z^2\left(1 - \left(\frac{R}{z}\right)^2\right)}{z^2\sqrt{1 + \left(\frac{R}{z}\right)^2}}\right] \tag{3.72}$$

$$\Phi = V_0 \left[1 - \left(1 - \left(\frac{R}{z}\right)^2\right)\left(1 + \left(\frac{R}{z}\right)^2\right)^{-1/2}\right] \tag{3.73}$$

$$\Phi \approx V_0 \left[1 - \left(1 - \left(\frac{R}{z}\right)^2\right)\left(1 - \frac{1}{2}\left(\frac{R}{z}\right)^2 + ...\right)\right] \tag{3.74}$$

$$\Phi \approx V_0 \left[1 - \left(1 - \left(\frac{R}{z}\right)^2 - \frac{1}{2}\left(\frac{R}{z}\right)^2 + ...\right)\right] \tag{3.75}$$

$$\Phi \approx V_0 \frac{3}{2}\left(\frac{R}{z}\right)^2. \tag{3.76}$$

Integration

Equation (3.67) can be expanded in power series to be integrated. Factorization gives

$$\begin{aligned}\Phi(r,\theta,\varphi) &= \tfrac{V_0 R\left(r^2-R^2\right)}{4\pi(r^2+R^2)^{3/2}} \int_0^{2\pi}\int_0^1 \Big[(1-2\alpha\cos\gamma)^{-3/2} \\ &\quad - (1+2\alpha\cos\gamma)^{-3/2}\Big]\, d(\cos\theta')d\varphi' \end{aligned} \tag{3.77}$$

where $\alpha = \frac{Rr}{r^2+R^2}$. The expansion of the radical terms are

$$\begin{aligned}(1-2\alpha\cos\gamma)^{-3/2} &= 1 + \frac{3}{2}2\alpha\cos\gamma - \frac{1}{2!}\frac{3}{2}\frac{5}{2}(2\alpha\cos\gamma)^2 + \frac{1}{3!}\frac{3}{2}\frac{5}{2}\frac{7}{2}(2\alpha\cos\gamma)^3 - ... \\ &= 1 + 3\alpha\cos\gamma - \frac{15}{2}\alpha^2\cos^2\gamma + \frac{35}{2}\alpha^3\cos^3\gamma - ... \end{aligned} \tag{3.78}$$

and

$$
\begin{aligned}
(1+2\alpha\cos\gamma)^{-3/2} &= 1-\frac{3}{2}2\alpha\cos\gamma+\frac{1}{2!}\frac{3}{2}\frac{5}{2}(2\alpha\cos\gamma)^2-\frac{1}{3!}\frac{3}{2}\frac{5}{2}\frac{7}{2}(2\alpha\cos\gamma)^3+... \\
&= 1-3\alpha\cos\gamma+\frac{15}{2}\alpha^2\cos^2\gamma-\frac{35}{2}\alpha^3\cos^3\gamma+... \qquad (3.79)
\end{aligned}
$$

Then, the integrand in the potential expands only in odd powers terms,

$$
\left[(1-2\alpha\cos\gamma)^{-3/2}-(1+2\alpha\cos\gamma)^{-3/2}\right]=6\alpha\cos\gamma+35\alpha^3\cos^3\gamma+.... \quad (3.80)
$$

Integration of these odd powers in the solid angle can be done easily,

$$
\begin{aligned}
\int_0^{2\pi}\int_0^1\cos\gamma d(\cos\theta')d\varphi' &= \pi\cos\theta \qquad (3.81)\\
\int_0^{2\pi}\int_0^1\cos^3\gamma d(\cos\theta')d\varphi' &= \frac{\pi}{4}\cos\theta\left(3-\cos^2\theta\right) \qquad (3.82)\\
\cdots &= \cdots
\end{aligned}
$$

These results let us write the general potential for the sphere as the expansion

$$
\Phi(r,\theta,\varphi)=\frac{V_0R\left(r^2-R^2\right)}{4\pi(r^2+R^2)^{3/2}}\left[6\alpha\pi\cos\theta+35\alpha^3\frac{\pi}{4}\cos\theta\left(3-\cos^2\theta\right)+...\right] \quad (3.83)
$$

$$
\Phi(r,\theta,\varphi)=\frac{3V_0R\left(r^2-R^2\right)}{2(r^2+R^2)^{3/2}}\alpha\cos\theta\left[1+\frac{35}{24}\alpha^2\left(3-\cos^2\theta\right)+...\right] \quad (3.84)
$$

$$
\Phi(r,\theta,\varphi)=\frac{3V_0R^2}{2r^2}\frac{r^3\left(r^2-R^2\right)}{(r^2+R^2)^{5/2}}\cos\theta\left[1+\frac{35}{24}\frac{R^2r^2}{(r^2-R^2)^2}\left(3-\cos^2\theta\right)+...\right] \quad (3.85)
$$

This expression is rewritten using the expansion parameter $\frac{R^2}{x^2}$ rather than α. To do so, note that

$$
\begin{aligned}
\frac{r^3\left(r^2-R^2\right)}{(r^2+R^2)^{5/2}} &= \frac{r^5\left(1-\left(\frac{R}{r}\right)^2\right)}{r^5\left(1+\left(\frac{R}{r}\right)^2\right)^{5/2}}=\left(1-\left(\frac{R}{r}\right)^2\right)\left(1+\left(\frac{R}{r}\right)^2\right)^{-5/2}\\
&= \left(1-\left(\frac{R}{r}\right)^2\right)\left(1-\frac{5}{2}\left(\frac{R}{r}\right)^2+...\right)\\
&= 1-\left(\frac{R}{r}\right)^2-\frac{5}{2}\left(\frac{R}{r}\right)^2+...\\
&= 1-\frac{7}{2}\left(\frac{R}{r}\right)^2+... \qquad (3.86)
\end{aligned}
$$

and

$$\begin{aligned}
\frac{R^2 r^2}{(r^2-R^2)^2} &= \frac{R^2 r^2}{r^4\left(1-\left(\frac{R}{r}\right)^2\right)^2} = \frac{R^2}{r^2}\left(1-\left(\frac{R}{r}\right)^2\right)^{-2} \\
&= \frac{R^2}{r^2}\left(1+2\left(\frac{R}{r}\right)^2+...\right) \\
&= \frac{R^2}{r^2}+... \qquad (3.87)
\end{aligned}$$

Hence, the potential approximates to

$$\begin{aligned}
\Phi(r,\theta,\varphi) &= \frac{3V_0R^2}{2r^2}\left(1-\frac{7}{2}\frac{R^2}{r^2}+...\right)\cos\theta\left[1+\frac{35}{24}\frac{R^2}{r^2}\left(3-\cos^2\theta\right)+...\right] \\
&= \frac{3V_0R^2}{2r^2}\left[\left(1-\frac{7}{2}\frac{R^2}{r^2}+...\right)\cos\theta+\frac{35}{24}\frac{R^2}{r^2}\left(3-\cos^2\theta\right)\cos\theta+...\right] \\
&= \frac{3V_0R^2}{2r^2}\left[\cos\theta-\frac{7}{12}\frac{R^2}{r^2}\left(\frac{5}{2}\cos^3\theta-\frac{3}{2}\cos\theta\right)+...\right] \qquad (3.88)
\end{aligned}$$

Note that the terms including $\cos\theta$ correspond to the well-known Legendre polynomials $P_1(\cos\theta)$ and $P_3(\cos\theta)$.

3.3 Orthogonal Functions and Expansions

Consider a variable ξ in an interval (a,b) and a set of (complex) functions $U_i(\xi)$ defined and square integrable in it. These functions are said to be *orthonormalized* if they satisfy

$$\int_a^b U_i^*(\xi)U_j(\xi)d\xi = \delta_{ij}. \qquad (3.89)$$

A set of orthonormal functions is said to be *complete* if any square integrable function $f/\xi)$ in the interval (a,b) can be written as the series representation

$$f(\xi) = \sum_{i=1}^{\infty} a_i U_i(\xi). \qquad (3.90)$$

The coefficients in this expansion can be easily obtained using the orthonormality condition as

$$a_j = \int_a^b U_j^*(\xi)f(\xi)d\xi. \qquad (3.91)$$

Using this expression for the coefficients, the series expansion can be written as

$$f(\xi) = \sum_{i=1}^{\infty}\int_a^b U_i^*(\xi')f(\xi')d\xi' U_i(\xi) = \int_a^b\left[\sum_{i=1}^{\infty}U_i^*(\xi')U_i(\xi)\right]f(\xi')d\xi', \qquad (3.92)$$

from which we recognize the relation

$$\sum_{i=1}^{\infty} U_i^*(\xi')U_i(\xi) = \delta(\xi' - \xi), \tag{3.93}$$

known as the *completeness* or *closure relation*.

If the space has more than one dimension the decomposition can be easily generalized. For example, suppose that ξ ranges over the interval (a, b) and η ranges over (c, d) and that there are two orthonormal complete set of functions $U_i(\xi)$ and $V_j(\eta)$. Then, any well behaved function $f(\xi, \eta)$ will be expanded as

$$f(\xi, \eta) = \sum_i \sum_j a_{ij} U_i(\xi) V_j(\eta) \tag{3.94}$$

where the coefficients are given by

$$a_{ij} = \int_a^b \int_c^d U_i^*(\xi) V_j^*(\eta) f(\xi, \eta) d\eta d\xi. \tag{3.95}$$

3.3.1 Fourier Series

An example of orthogonal functions in the interval $\left(-\frac{a}{2}, \frac{a}{2}\right)$ are

$$\left\{ \sqrt{\frac{2}{a}} \sin\left(\frac{2\pi nx}{a}\right) , \sqrt{\frac{2}{a}} \cos\left(\frac{2\pi nx}{a}\right) \right\}, \tag{3.96}$$

where $n = 1, 2, 3, ...$ The series representation of a function $f(x)$ is

$$f(x) = \frac{1}{2}A_0 + \sum_{n=1}^{\infty} \left[A_n \cos\left(\frac{2\pi nx}{a}\right) + B_n \sin\left(\frac{2\pi nx}{a}\right) \right] \tag{3.97}$$

and the coefficients can be obtained using the equations

$$A_n = \frac{2}{a} \int_{-\frac{a}{2}}^{\frac{a}{2}} f(x) \cos\left(\frac{2\pi nx}{a}\right) dx \tag{3.98}$$

$$B_n = \frac{2}{a} \int_{-\frac{a}{2}}^{\frac{a}{2}} f(x) \sin\left(\frac{2\pi nx}{a}\right) dx. \tag{3.99}$$

An alternative expansion can be accomplished by considering the orthonormal set of complex functions on the interval $\left(-\frac{a}{2}, \frac{a}{2}\right)$,

$$\left\{ U_n(x) = \frac{1}{\sqrt{a}} e^{\frac{2\pi i n x}{a}} \right\} \tag{3.100}$$

where $n = 0, \pm 1, \pm 2, ...$ A function $f(x)$ is written now as

$$f(x) = \frac{1}{\sqrt{a}} \sum_{n=-\infty}^{\infty} A_n e^{\frac{2\pi i n x}{a}} \tag{3.101}$$

and the coefficients are

$$A_n = \frac{1}{\sqrt{a}} \int_{-\frac{a}{2}}^{\frac{a}{2}} e^{\frac{-2\pi i n x}{a}} f(x')dx'. \tag{3.102}$$

3.3.2 Fourier Integral

When the interval becomes infinite in the previous example, i.e. $a \to \infty$, the sum in the expansion is transformed by an integral given by the transformations

$$\begin{cases} \frac{2\pi n}{a} \to k \\ \sum_n \to \int_{-\infty}^{\infty} dn = \frac{a}{2\pi} \int_{-\infty}^{\infty} dk \\ A_n \to \sqrt{\frac{2\pi}{a}A(k)} \end{cases} . \tag{3.103}$$

Hence, the expansion of $f(x)$ becomes the *Fourier Integral*

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} A(k)e^{ikx}dk \tag{3.104}$$

where

$$A(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ikx} f(x)dx. \tag{3.105}$$

Finally, it is interesting to note that, in this case, the orthogonality condition is

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} e^{i(k-k')x}dx = \delta(k-k') \tag{3.106}$$

while the completeness relation is

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} e^{ik(x-x')}dk = \delta(x-x'). \tag{3.107}$$

3.4 Laplace Equation in Rectangular Coordinates

Consider Laplace equation $\nabla^2\Phi = 0$ in rectangular coordinates,

$$\frac{\partial^2\Phi}{\partial x^2} + \frac{\partial^2\Phi}{\partial y^2} + \frac{\partial^2\Phi}{\partial z^2} = 0. \tag{3.108}$$

In order to solve this equation we will use the separation of variables

$$\Phi(x, y, z) = X(x)Y(y)Z(z) \tag{3.109}$$

which gives

$$\frac{1}{X}\frac{d^2X}{dx^2} + \frac{1}{Y}\frac{d^2Y}{dy^2} + \frac{1}{Z}\frac{d^2Z}{dz^2} = 0. \tag{3.110}$$

Introducing the separation constants α, β and γ we obtain

$$\begin{cases} \frac{1}{X}\frac{d^2X}{dx^2} = -\alpha^2 \\ \frac{1}{Y}\frac{d^2Y}{dy^2} = -\beta^2 \\ \frac{1}{Z}\frac{d^2Z}{dz^2} = \gamma^2 \end{cases} \tag{3.111}$$

subject to the condition

$$\alpha^2 + \beta^2 = \gamma^2. \tag{3.112}$$

The solution to the three separated equations are

$$\begin{cases} X(x) = e^{\pm i\alpha x} \\ Y(y) = e^{\pm i\beta y} \\ Z(z) = e^{\pm \gamma z} = e^{\pm\sqrt{\alpha^2+\beta^2}z} \end{cases} \tag{3.113}$$

and thus, the general electrostatic potential solution of Laplace equation is

$$\Phi(x, y, z) = e^{\pm i\alpha x} e^{\pm i\beta y} e^{\pm\sqrt{\alpha^2+\beta^2}z}. \tag{3.114}$$

3.4.1 Rectangular Box Boundary Conditions

Boundary conditions determine the constants α and β. For example, consider a rectangular box with dimensions $0 \le x \le a$, $0 \le y \le b$ and $0 \le z \le c$. Suppose that all surfaces are kept at zero potential, except for the surface $z = c$ which is at the potential $V(x, y)$.

Boundary condition $\Phi(0, y, z) = 0$ impose $X(x) = \sin(\alpha x)$.
Boundary condition $\Phi(x, 0, z) = 0$ impose $Y(y) = \sin(\beta y)$.
Boundary condition $\Phi(x, y, 0) = 0$ impose $Z(z) = \sinh(\sqrt{\alpha^2 + \beta^2}z)$.
Boundary condition $\Phi(a, y, z) = 0$ impose $\alpha a = n\pi$.
Boundary condition $\Phi(x, b, z) = 0$ impose $\beta b = m\pi$.

Last two of these conditions lead us to write the constants as

$$\begin{cases} \alpha_n = \frac{n\pi}{a} \\ \beta_m = \frac{m\pi}{b} \\ \gamma_{nm} = \pi\sqrt{\frac{n^2}{a^2} + \frac{m^2}{b^2}} \end{cases} \tag{3.115}$$

and therefore, we can write a partial electrostatic potential satisfying these 5 boundary conditions as

$$\Phi_{nm} = \sin(\alpha_n x)\sin(\beta_m y)\sinh(\gamma_{nm} z). \tag{3.116}$$

The complete electrostatic potential will be written as a superposition of these partial potentials as

$$\Phi(x, y, z) = \sum_{n,m=1}^{\infty} A_{nm}\Phi_{nm} = \sum_{n,m=1}^{\infty} A_{nm}\sin(\alpha_n x)\sin(\beta_m y)\sinh(\gamma_{nm} z) \tag{3.117}$$

where the coefficients A_{nm} will be chosen to satisfy the final boundary condition. Thus, the condition $\Phi(x,y,c) = V(x,y)$ impose

$$\Phi(x,y,c) = \sum_{n,m=1}^{\infty} A_{nm} \sin(\alpha_n x)\sin(\beta_m y)\sinh(\gamma_{nm} c) = V(x,y). \tag{3.118}$$

This equation can be interpreted as a double Fourier series of function $V(x,y)$ and therefore the coefficients A_{nm} are given by

$$A_{nm}\sinh(\gamma_{nm}c) = \frac{2}{a}\frac{2}{b}\int_0^a\int_0^b V(x,y)\sin(\alpha_n x)\sin(\beta_m y)dydx \tag{3.119}$$

$$A_{nm} = \frac{4}{ab\sinh(\gamma_{nm}c)}\int_0^a\int_0^b V(x,y)\sin(\alpha_n x)\sin(\beta_m y)dydx. \tag{3.120}$$

As a final note, if the box has potentials V_i at each of its six sides, the solution for the potential inside can be obtained by linear superposition of six solutions equivalent to that described above.

3.4.2 2-Dimensional Potential Problem

Consider a physical system in which the electrostatic potential is independent of coordinate z and therefore, described by Laplace equation in two dimensions. Separation of variables in the form

$$\Phi(x,y) = X(x)Y(y) \tag{3.121}$$

will give a Laplace equation in the form

$$\frac{1}{X}\frac{d^2X}{dx^2} + \frac{1}{Y}\frac{d^2Y}{dy^2} = 0. \tag{3.122}$$

Introducing the separation constant α gives

$$\begin{cases} \frac{1}{X}\frac{d^2X}{dx^2} = -\alpha \\ \frac{1}{Y}\frac{d^2Y}{dy^2} = \alpha \end{cases} \tag{3.123}$$

and therefore a general solution

$$\Phi(x,y,z) = e^{\pm i\alpha x}e^{\pm \alpha y}. \tag{3.124}$$

Constant α is determined by boundary conditions. For example, consider the system in Figure XX with boundary conditions $\Phi(0,y) = \Phi(a,y) = 0$, $\Phi(x,0) = V_0$ and $\Phi(x, y \to \infty) = 0$. Imposing three of these condition in the general solution (3.124) gives

Boundary condition $\Phi(0,y) = 0$ impose $X(x) = \sin(\alpha x)$.
Boundary condition $\Phi(a,y) = 0$ impose $\alpha a = n\pi$.

Boundary condition $\Phi(x, y \to \infty) = 0$ impose $Y(y) = e^{-\alpha y}$.

Then, the linear combination of solutions satisfying these three boundary conditions gives the potential

$$\Phi(x,y) = \sum_{n=1}^{\infty} A_n e^{-\frac{n\pi y}{a}} \sin\left(\frac{n\pi x}{a}\right), \tag{3.125}$$

where the coefficients A_n will be determined by the fourth boundary condition. In fact, evaluating at $y = 0$ we obtain

$$\Phi(x,0) = \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi x}{a}\right) = V_0, \tag{3.126}$$

which corresponds to a Fourier series. Therefore, the coefficients will be given by

$$A_n = \frac{2}{a}\int_0^a \Phi(x,0)\sin\left(\frac{n\pi x}{a}\right) dx = \frac{2}{a}\int_0^a V_0 \sin\left(\frac{n\pi x}{a}\right) dx. \tag{3.127}$$

Integration gives the coefficients

$$A_n = \frac{4V_0}{\pi n}\begin{cases} 1 \text{ for } n \text{ odd} \\ 0 \text{ for } n \text{ even}\end{cases} \tag{3.128}$$

and therefore, the electrostatic problem is written as

$$\Phi(x,y) = \frac{4V_0}{\pi}\sum_{n \text{ odd}} \frac{1}{n} e^{-\frac{n\pi y}{a}} \sin\left(\frac{n\pi x}{a}\right). \tag{3.129}$$

3.4.3 2-Dimensional Corners

Many physical systems have conducting surfaces coming together in such a way that they can be described as the intersection of two planes. For example, Figure XX shows two conducting planes intersecting at an angle β. Suppose that they are held at a constant potential V_0. From the geometry of the system, it is convenient to use polar coordinates (ρ, ϕ) for its description. Hence, the Laplace equation becomes

$$\nabla^2\Phi = \frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial\Phi}{\partial\rho}\right) + \frac{1}{\rho^2}\frac{\partial^2\Phi}{\partial\phi^2} = 0. \tag{3.130}$$

In order to solve this equation, we propose the separation

$$\Phi(\rho,\phi) = R(\rho)\Psi(\phi), \tag{3.131}$$

which leads to the equation

$$\frac{\rho}{R}\frac{d}{d\rho}\left(\rho\frac{dR}{d\rho}\right) + \frac{1}{\Psi}\frac{d^2\Psi}{d\phi^2} = 0, \tag{3.132}$$

and introducing the separation constant γ^2 we obtain the differential equations

$$\frac{\rho}{R}\frac{d}{d\rho}\left(\rho\frac{dR}{d\rho}\right) = \gamma^2 \tag{3.133}$$

$$\frac{1}{\Psi}\frac{d^2\Psi}{d\phi^2} = -\gamma^2. \tag{3.134}$$

The corresponding solutions are, for $\gamma \neq 0$

$$R(\rho) = a\rho^{\gamma} + b\rho^{-\gamma} \tag{3.135}$$

$$\Psi(\phi) = A\cos(\gamma\phi) + B\sin(\gamma\phi). \tag{3.136}$$

and, for $\gamma = 0$

$$R(\rho) = a_0 + b_0 \ln\rho \tag{3.137}$$

$$\Psi(\phi) = A_0 + B_0\phi. \tag{3.138}$$

The potential for a particular example will be obtained by linear superposition of these solutions. Hence the general solution for the potential is written as

$$\Phi(\rho,\phi) = [a_0 + b_0\ln\rho]\,[A_0 + B_0\phi] + \sum_n \left[a_n\rho^{\gamma_n} + b_n\rho^{-\gamma_n}\right][A_n\cos(\gamma_n\phi) + B_n\sin(\gamma_n\phi)]\,. \tag{3.139}$$

Examples

1. Consider the region between two cylindrical surfaces at $\rho = R_1$ and $\rho = R_2$. In this region, the electrostatic potential will be a function only of ϕ. If there are no restrictions on the coordinate ϕ, the solution with $\gamma = 0$ needs $B_0 = 0$ to ensure that the potential is single-valued. Similarly, to ensure that the potential is single valued, the solution with $\gamma \neq 0$ needs that the argument in the sin and cos functions behaves properly and therefore $\gamma_n = n$ with n a positive or negative integer. Then, the potential becomes

$$\Phi(\rho,\phi) = [a_0 + b_0\ln\rho]\,[A_0] + \sum_{n=-\infty}^{\infty} \left[a_n\rho^{n} + b_n\rho^{-n}\right][A_n\cos(n\phi) + B_n\sin(n\phi)]\,. \tag{3.140}$$

   Redefining some constants and using trigonometric identities we can write

$$\Phi(\rho,\phi) = a_0 + b_0\ln\rho + \sum_{n=0}^{\infty} a_n\rho^n\sin(n\phi + \alpha_n) + \sum_{n=0}^{\infty} b_n\rho^{-n}\sin(n\phi+\beta_n). \tag{3.141}$$

   Note that if the origin $\rho = 0$ is included in the solution, it is needed that $b_n = 0$. Finally, it is interesting to note that the logarithmic term describes a line of charge on the axis with a charge density per unit length $\lambda = -2\pi\epsilon_0 b_0$.

2. In the case of the intersection of two conducting planes defining an angle β, there is a restriction $0 \leq \phi \leq \beta$ and there will be boundary conditions $\Phi(\rho, 0) = \Phi(\rho, \beta) = V_0$ for $\rho \geq 0$. Since the origin is included, we impose that $b_n = 0$ in equation (3.139). Hence, the first boundary condition is

$$\Phi(\rho, 0) = [a_0 + b_0 \ln \rho]\,[A_0] + \sum_n [a_n \rho^{\gamma_n}]\,[A_n] = V_0 \tag{3.142}$$

from which it is needed that $b_0 = 0$, $A_n = 0$ and $a_0 A_0 = V_0$. This gives a potential of the form

$$\Phi(\rho, \phi) = V_0 + B_0 \phi + \sum_n a_n \rho^{\gamma_n} \sin(\gamma_n \phi). \tag{3.143}$$

Imposing the second boundary condition gives

$$\Phi(\rho, \beta) = V_0 + B_0 \beta + \sum_n a_n \rho^{\gamma_n} \sin(\gamma_n \beta) = V_0 \tag{3.144}$$

from which we need $B_0 = 0$ and $\gamma_n = \frac{n\pi}{\beta}$ with $n = 1, 2, ...$ Therefore, the solution is

$$\Phi(\rho, \phi) = V_0 + \sum_{n=1}^{\infty} a_n \rho^{\frac{n\pi}{\beta}} \sin\left(\frac{n\pi}{\beta}\phi\right). \tag{3.145}$$

The coefficients a_n will be determined by the potential far from the origin. However, if we are interested in the behavior near the corner, i.e. for small ρ, the first term in the expansion will be the most important . Hence we approximate

$$\Phi(\rho, \phi) \approx V_0 + a_1 \rho^{\frac{\pi}{\beta}} \sin\left(\frac{\pi}{\beta}\phi\right) \text{ near } \rho = 0. \tag{3.146}$$

The electric field components are given by the derivatives

$$\begin{cases} E_\rho(\rho, \phi) = -\frac{\partial \Phi}{\partial \rho} \approx -\frac{\pi a_1}{\beta} \rho^{\frac{\pi}{\beta} - 1} \sin\left(\frac{\pi}{\beta}\phi\right) \\ E_\phi(\rho, \phi) = -\frac{1}{\rho}\frac{\partial \Phi}{\partial \phi} \approx -\frac{\pi a_1}{\beta} \rho^{\frac{\pi}{\beta} - 1} \cos\left(\frac{\pi}{\beta}\phi\right) \end{cases}. \tag{3.147}$$

Finally, the surface charge densities at $\phi = 0$ and $\phi = \beta$ are

$$\sigma(\rho) = \epsilon_0 E_\phi(\rho, 0) = \epsilon_0 E_\phi(\rho, \beta) \approx -\frac{\pi a_1}{\beta} \rho^{\frac{\pi}{\beta} - 1}. \tag{3.148}$$

Chapter 4

Boundary Value Problems in Electrostatics II

4.1 Laplace Equation in Spherical Coordinates

The Laplace equation in spherical coordinates (r, θ, ϕ) is written

$$\nabla^2 \Phi = \frac{1}{r} \frac{\partial^2}{\partial r^2} (r\Phi) + \frac{1}{r^2 \sin\theta} \frac{\partial}{\partial \theta} \left(\sin\theta \frac{\partial \Phi}{\partial \theta} \right) + \frac{1}{r^2 \sin^2\theta} \frac{\partial^2 \Phi}{\partial \phi^2} = 0. \tag{4.1}$$

Considering the separation in the potential function

$$\Phi(r, \theta, \phi) = \frac{U(r)}{r} P(\theta) \Psi(\phi), \tag{4.2}$$

Laplace equation becomes

$$P\Psi \frac{d^2 U}{dr^2} + \frac{U\Psi}{r^2 \sin\theta} \frac{d}{d\theta} \left(\sin\theta \frac{dP}{d\theta} \right) + \frac{UP}{r^2 \sin^2\theta} \frac{d^2 \Psi}{d\phi^2} = 0 \tag{4.3}$$

$$r^2 \sin^2\theta \left[\frac{1}{U} \frac{d^2 U}{dr^2} + \frac{1}{Pr^2 \sin\theta} \frac{d}{d\theta} \left(\sin\theta \frac{dP}{d\theta} \right) \right] + \frac{1}{\Psi} \frac{d^2 \Psi}{d\phi^2} = 0. \tag{4.4}$$

Naming the separation constant for the ϕ therm as $-m^2$ gives the equation

$$\frac{1}{\Psi} \frac{d^2 \Psi}{d\phi^2} = -m^2 \tag{4.5}$$

and therefore

$$\Psi = e^{\pm i m \phi}. \tag{4.6}$$

If the range of the azimuthal variable is $0 \le \phi < 2\pi$, the constant m must be an integer in order to have a single valued function Ψ. On the other hand we have

$$r^2 \sin^2\theta \left[\frac{1}{U} \frac{d^2 U}{dr^2} + \frac{1}{Pr^2 \sin\theta} \frac{d}{d\theta} \left(\sin\theta \frac{dP}{d\theta} \right) \right] = m^2 \tag{4.7}$$

$$r^2 \frac{1}{U}\frac{d^2U}{dr^2} + \frac{1}{P\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{dP}{d\theta}\right) = \frac{m^2}{\sin^2\theta} \tag{4.8}$$

$$r^2 \frac{1}{U}\frac{d^2U}{dr^2} = -\frac{1}{P\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{dP}{d\theta}\right) + \frac{m^2}{\sin^2\theta} \tag{4.9}$$

Introducing the constant $l(l+1)$ to separate the functions U and P gives the equations

$$r^2 \frac{1}{U}\frac{d^2U}{dr^2} = l(l+1) \tag{4.10}$$

$$\frac{1}{P\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{dP}{d\theta}\right) - \frac{m^2}{\sin^2\theta} = -l(l+1) \tag{4.11}$$

or

$$\frac{d^2U}{dr^2} - \frac{l(l+1)}{r^2}U = 0 \tag{4.12}$$

$$\frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{dP}{d\theta}\right) + \left[l(l+1) - \frac{m^2}{\sin^2\theta}\right]P = 0. \tag{4.13}$$

The radial equation is integrated to obtain

$$U = Ar^{l+1} + Br^{-l}. \tag{4.14}$$

The equation for $P(\theta)$ is usually rewritten in terms of the variable $x = \cos\theta$. We have that $\frac{d}{dtheta} = \frac{dx}{d\theta}\frac{d}{dx} = -\sin\theta\frac{d}{dx}$ and then

$$\frac{\sin\theta}{\sin\theta}\frac{d}{dx}\left(\sin^2\theta\frac{dP}{dx}\right) + \left[l(l+1) - \frac{m^2}{\sin^2\theta}\right]P = 0. \tag{4.15}$$

$$\frac{d}{dx}\left[(1-x^2)\frac{dP}{dx}\right] + \left[l(l+1) - \frac{m^2}{1-x^2}\right]P = 0. \tag{4.16}$$

This equation is called the *generalized Legendre equation* and now we will present its solutions.

4.2 Legendre Equation with Azimuthal Symmetry

4.2.1 $m = 0$ and the Legendre Polynomials

In order to obtain the solution of equation (4.16) we will consider first the particular case $m = 0$,

$$\frac{d}{dx}\left[(1-x^2)\frac{dP}{dx}\right] + l(l+1)P = 0, \tag{4.17}$$

called the ordinary Legendre differential equation. In order to give a physical reasonable electrostatic potential, the solution of this equation must be a single valued, finite and continuous function in the range $-1 \leq x \leq 1$ (because $x = \cos\theta$). To obtain this solution, consider a power series representation,

$$P(x) = x^\alpha \sum_{j=0}^{\infty} a_j x^j, \tag{4.18}$$

with α an unknown parameter to be determined. Direct substitution gives

$$\frac{d}{dx}\left[(1-x^2)\sum_{j=0}^{\infty} a_j(\alpha+j)x^{\alpha+j-1}\right] + l(l+1)\sum_{j=0}^{\infty} a_j x^{\alpha+j} = 0 \tag{4.19}$$

$$\frac{d}{dx}\left[\sum_{j=0}^{\infty} a_j(\alpha+j)x^{\alpha+j-1} - \sum_{j=0}^{\infty} a_j(\alpha+j)x^{\alpha+j+1}\right] + l(l+1)\sum_{j=0}^{\infty} a_j x^{\alpha+j} = 0 \tag{4.20}$$

$$\sum_{j=0}^{\infty} a_j(\alpha+j)(\alpha+j-1)x^{\alpha+j-2} - \sum_{j=0}^{\infty} a_j(\alpha+j)(\alpha+j+1)x^{\alpha+j} + l(l+1)\sum_{j=0}^{\infty} a_j x^{\alpha+j} = 0 \tag{4.21}$$

$$\sum_{j=0}^{\infty} a_j\left\{(\alpha+j)(\alpha+j-1)x^{\alpha+j-2} - \left[(\alpha+j)(\alpha+j+1) - l(l+1)\right]x^{\alpha+j}\right\} = 0 \tag{4.22}$$

$$x^\alpha \sum_{j=0}^{\infty} a_j\left\{(\alpha+j)(\alpha+j-1)x^{j-2} - \left[(\alpha+j)(\alpha+j+1) - l(l+1)\right]x^{j}\right\} = 0. \tag{4.23}$$

Writting explicitly the first terms in this equation gives

$$\begin{aligned}
x^\alpha a_0\left\{\alpha(\alpha-1)x^{-2} - [\alpha(\alpha+1) - l(l+1)]\right\} & \\
+x^\alpha a_1\left\{(\alpha+1)\alpha x^{-1} - [(\alpha+1)(\alpha+2) - l(l+1)]\,x\right\} & \\
+x^\alpha \sum_{j=2}^{\infty} a_j\left\{(\alpha+j)(\alpha+j-1)x^{j-2} - [(\alpha+j)(\alpha+j+1) - l(l+1)]\,x^j\right\} &= 0.
\end{aligned}$$

Since this equation states that the coefficient of each power of x must vanish separately, we begin considering the first term. There, the coefficient with the factor x^{-2} must vanish, and therefore

$$\alpha(\alpha-1) = 0 \text{ if } a_0 \neq 0. \tag{4.24}$$

Similarly, from the second term in the expansion, the coefficient with the factor x^{-1} must vanish, and therefore

$$(\alpha+1)\alpha = 0 \text{ if } a_1 \neq 0. \tag{4.25}$$

The vanishing of these terms leave us with

$$-x^\alpha a_0 \left[\alpha(\alpha+1) - l(l+1)\right] - x^\alpha a_1 \left[(\alpha+1)(\alpha+2) - l(l+1)\right] x$$
$$+x^\alpha \sum_{j=2}^{\infty} a_j \left\{(\alpha+j)(\alpha+j-1)x^{j-2} - \left[(\alpha+j)(\alpha+j+1) - l(l+1)\right] x^j\right\} \quad = \quad 0.$$

Following the same idea, the coefficient of x^0 must vanish,

$$-a_0 \left[\alpha(\alpha+1) - l(l+1)\right] + a_2(\alpha+2)(\alpha+2-1) = 0 \tag{4.26}$$

or

$$a_2 = \left[\frac{\alpha(\alpha+1) - l(l+1)}{(\alpha+2)(\alpha+2-1)}\right] a_0. \tag{4.27}$$

Similarly, the coefficient of x^1 gives

$$-a_1 \left[(\alpha+1)(\alpha+2) - l(l+1)\right] + a_3(\alpha+3)(\alpha+3-1) = 0 \tag{4.28}$$

or

$$a_3 = \left[\frac{(\alpha+1)(\alpha+2) - l(l+1)}{(\alpha+3)(\alpha+3-1)}\right] a_1. \tag{4.29}$$

In general, the condition for the vanishing of the coefficient of x^j gives the condition

$$a_{j+2} = \left[\frac{(\alpha+j)(\alpha+j+1) - l(l+1)}{(\alpha+j+1)(\alpha+j+2)}\right] a_j. \tag{4.30}$$

Conditions (4.24) and (4.25) are equivalent, and therefore it is sufficient to choose either a_0 or a_1 different from zero, but not both. We will choose $a_1 = 0$ together with $a_0 \neq 0$ and thus $\alpha = 0$ or $\alpha = 1$.

The series with $\alpha = 0$ gives

$$P(x) = x^0 \sum_{j=0}^{\infty} a_j x^j = \sum_{j=0}^{\infty} a_j x^j \tag{4.31}$$

with

$$a_{j+2} = \left[\frac{j(j+1) - l(l+1)}{(j+1)(j+2)}\right] a_j, \tag{4.32}$$

due to equation (4.30). Note that only even powers of x exist in this series.

On the other hand, the series with $\alpha = 1$ gives

$$P(x) = x^1 \sum_{j=0}^{\infty} a_j x^j = \sum_{j=0}^{\infty} a_j x^{j+1} \tag{4.33}$$

with

$$a_{j+2} = \left[\frac{(j+1)(j+2) - l(l+1)}{(j+2)(j+3)}\right] a_j \tag{4.34}$$

and then, only odd powers of x exist in this case.

Both of these series converge for $x^2 < 1$, regardless of the value of l but they diverge ar $x = \pm 1$, unless it terminates. From recursion (4.30), where α and j are zero or positive integers, it is possible to see that the series terminate only if l is zero or a positive integer. In fact, if l is even (odd), only the series with $\alpha = 0$ ($\alpha = 1$) terminates. Since the convergence of the series depends on l, we define the *Legendre Polynomials* or order l as the corresponding converging function $P_l(x)$. Some of the polynomials are given by:

For $l = 0$, the $\alpha = 0$ terminates and gives

$$P_0(x) = \sum_{j=0}^{\infty} a_j x^j = a_0 + a_2 x^2 + a_4 x^4 + ... \tag{4.35}$$

However, note that using (4.32) the second coefficient is

$$a_2 = \left[\frac{0(0+1)}{(0+1)(0+2)} \right] a_0 = 0, \tag{4.36}$$

and therefore, only the first term in the series survives. Choosing $a_0 = 1$ this polynomial is

$$P_0(x) = 1. \tag{4.37}$$

For $l = 1$, the $\alpha = 1$ terminates,

$$P_1(x) = \sum_{j=0}^{\infty} a_j x^{j+1} = a_0 x + a_2 x^3 + a_4 x^5 + ... \tag{4.38}$$

However, note that using (4.34) the second coefficient is

$$a_2 = \left[\frac{(0+1)(0+2) - 1(1+1)}{(0+2)(0+3)} \right] a_0 = 0, \tag{4.39}$$

and therefore, only the first term in the series survives. Choosing $a_0 = 1$ to ensure a normalization such that the polynomial has the value $+1$ at $x = +1$ gives

$$P_1(x) = x. \tag{4.40}$$

For $l = 2$, the $\alpha = 0$ terminates and gives

$$P_2(x) = \sum_{j=0}^{\infty} a_j x^j = a_0 + a_2 x^2 + a_4 x^4 + ... \tag{4.41}$$

This time, the second coefficient is

$$a_2 = \left[\frac{0(0+1) - 2(2+1)}{(0+1)(0+2)} \right] a_0 = -3a_0, \tag{4.42}$$

while the third coefficient is

$$a_4 = \left[\frac{2(2+1) - 2(2+1)}{(2+1)(2+2)} \right] a_2 = 0, \tag{4.43}$$

and therefore the polynomial have just two terms,

$$P_2(x) = a_0 - 3a_0 x^2 = a_0(1 - 3x^2). \tag{4.44}$$

In order to ensure the normalization such that the polynomial has the value $+1$ at $x = +1$, we choose $a_0 = -\frac{1}{2}$, giving

$$P_2(x) = \frac{1}{2}(3x^2 - 1). \tag{4.45}$$

Following this procedure we can find all the polynomials,

$$\begin{aligned} P_3(x) &= \frac{1}{2}(5x^3 - 3x) && (4.46) \\ P_4(x) &= \frac{1}{8}(35x^4 - 30x^2 + 3) && (4.47) \\ \dots & \end{aligned}$$

From this power series it is possible to show that a compact representation of Legendre polynomials is given by Rodrigues' formula,

$$P_l(x) = \frac{1}{2^l l!} \frac{d^l}{dx^l} (x^2 - 1)^l. \tag{4.48}$$

Orthogonality of the Legendre Polynomials

Legendre polynomials form a set of complete orthonormal set of functions on the interval $-1 \leq x \leq 1$. Orthogonality is probed as follow: consider equation (4.17)

$$\frac{d}{dx}\left[(1 - x^2)\frac{dP_l}{dx}\right] + l(l+1)P_l(x) = 0, \tag{4.49}$$

multiply by $P_{l'}(x)$ and integrate,

$$\int_{-1}^{1} P_{l'}(x) \left\{ \frac{d}{dx}\left[(1 - x^2)\frac{dP_l}{dx}\right] + l(l+1)P_l(x) \right\} dx = 0. \tag{4.50}$$

The first term integrates by parts to give

$$\int_{-1}^{1} \left\{ \frac{dP_{l'}}{dx}\frac{dP_l}{dx} + l(l+1)P_{l'}P_l \right\} dx = 0. \tag{4.51}$$

Interchanging l and l' in this relation gives

$$\int_{-1}^{1} \left\{ \frac{dP_l}{dx}\frac{dP_{l'}}{dx} + l'(l'+1)P_l P_{l'} \right\} dx = 0, \tag{4.52}$$

and substracting these two relations gives the result

$$\int_{-1}^{1} \left[l(l+1) - l'(l'+1)\right] P_l P_{l'} \, dx = 0, \tag{4.53}$$

or

$$\int_{-1}^{1} P_l P_{l'} \, dx = 0 \text{ for } l \neq l'. \tag{4.54}$$

On the other hand, for $l = l'$ the integral may not be zero. In fact, integration in this case is done by using Rodriguez' formula,

$$N_l = \int_{-1}^{1} P_l P_l dx = \int_{-1}^{1} \left[\frac{1}{2^l l!} \frac{d^l}{dx^l}(x^2-1)^l \frac{1}{2^l l!} \frac{d^l}{dx^l}(x^2-1)^l\right] dx \tag{4.55}$$

$$N_l = \frac{1}{2^{2l}(l!)^2} \int_{-1}^{1} \left[\frac{d^l}{dx^l}(x^2-1)^l \frac{d^l}{dx^l}(x^2-1)^l\right] dx. \tag{4.56}$$

Integration by parts l times in the r.h.s gives

$$N_l = \frac{(-1)^l}{2^{2l}(l!)^2} \int_{-1}^{1} \left[(x^2-1)^l \frac{d^{2l}}{dx^{2l}}(x^2-1)^l\right] dx. \tag{4.57}$$

Now, the derivatives in the integrand gives for $l = 1$

$$\frac{d^2}{dx^2}(x^2-1) \quad = \quad \frac{d}{dx}\left[2x\right] = 2 = 2! \ , \tag{4.58}$$

for $l = 2$,

$$\begin{aligned} \frac{d^4}{dx^4}(x^2-1)^2 \quad &= \quad \frac{d^3}{dx^3}\left[4x(x^2-1)\right] = \frac{d^2}{dx^2}\left[4(x^2-1) + 8x^2\right] \\ &= \quad \frac{d}{dx}\left[8x + 16x\right] = 24 = 4! \ , \end{aligned} \tag{4.59}$$

for $l = 3$,

$$\begin{aligned} \frac{d^6}{dx^6}(x^2-1)^3 \quad &= \quad \frac{d^5}{dx^5}\left[6x(x^2-1)^2\right] = \frac{d^4}{dx^4}\left[6(x^2-1)^2 + 24x^2(x^2-1)\right] \\ &= \quad \frac{d^3}{dx^3}\left[24x(x^2-1) + 48x(x^2-1) + 48x^3\right] \\ &= \quad \frac{d^2}{dx^2}\left[24(x^2-1) + 48x^2 + 48(x^2-1) + 96x^2 + 144x^2\right] \\ &= \quad \frac{d}{dx}\left[48x + 96x + 96x + 192x + 288x\right] \\ &= \quad 48 + 96 + 96 + 192 + 288 = 720 = 6! \end{aligned} \tag{4.60}$$

or in general,

$$\frac{d^{2l}}{dx^{2l}}(x^2-1)^l = (2l)!\ . \tag{4.61}$$

Thus, the integral becomes

$$N_l = \frac{(-1)^l(2l)!}{2^{2l}(l!)^2}\int_{-1}^{1}(x^2-1)^l dx \tag{4.62}$$

$$N_l = \frac{(2l)!}{2^{2l}(l!)^2}\int_{-1}^{1}(1-x^2)^l dx. \tag{4.63}$$

The integrand in this expression is

$$\begin{aligned}(1-x^2)^l &= (1-x^2)(1-x^2)^{l-1} = (1-x^2)^{l-1} - x^2(1-x^2)^{l-1} \\ &= (1-x^2)^{l-1} + \frac{x}{2l}\frac{d}{dx}(1-x^2)^l.\end{aligned} \tag{4.64}$$

Replacing in the integral,

$$N_l = \frac{(2l)!}{2^{2l}(l!)^2}\int_{-1}^{1}\left[(1-x^2)^{l-1} + \frac{x}{2l}\frac{d}{dx}(1-x^2)^l\right]dx \tag{4.65}$$

$$N_l = \frac{(2l)!}{2^{2l}(l!)^2}\int_{-1}^{1}(1-x^2)^{l-1}dx + \frac{(2l)!}{2^{2l}(l!)^2}\int_{-1}^{1}\left[\frac{x}{2l}\frac{d}{dx}(1-x^2)^l\right]dx \tag{4.66}$$

$$N_l = \frac{(2l)!}{2^{2l}(l!)^2}\frac{2^{2(l-1)}((l-1)!)^2}{(2(l-1))!}N_{l-1} + \frac{(2l)!}{2^{2l}(l!)^2}\int_{-1}^{1}\frac{x}{2l}d\left[(1-x^2)^l\right] \tag{4.67}$$

$$N_l = \frac{(2l)!}{l^2}\frac{2^{-2}}{(2l-2)!}N_{l-1} + \frac{(2l-1)!}{2^{2l}(l!)^2}\int_{-1}^{1}xd\left[(1-x^2)^l\right] \tag{4.68}$$

$$N_l = \frac{(2l)(2l-1)}{l^2}\frac{2^{-2}}{(2l-2)!}N_{l-1} + \frac{(2l-1)!}{2^{2l}(l!)^2}\int_{-1}^{1}xd\left[(1-x^2)^l\right] \tag{4.69}$$

$$N_l = \frac{(2l)(2l-1)(2l-2)!}{l^2}\frac{2^{-2}}{(2l-2)!}N_{l-1} + \frac{(2l-1)!}{2^{2l}(l!)^2}\int_{-1}^{1}xd\left[(1-x^2)^l\right] \tag{4.70}$$

$$N_l = \frac{(2l-1)}{2l}N_{l-1} + \frac{(2l-1)!}{2^{2l}(l!)^2}\int_{-1}^{1}xd\left[(1-x^2)^l\right]. \tag{4.71}$$

The integral in the last term can be done by parts,

$$\begin{aligned}\int_{-1}^{1}xd\left[(1-x^2)^l\right] &= \left[x(1-x^2)^l\right]_{-1}^{1} - \int_{-1}^{1}(1-x^2)^l dx \\ &= -\frac{2^{2l}(l!)^2}{(2l)!}N_l,\end{aligned} \tag{4.72}$$

and therefore

$$N_l = \frac{(2l-1)}{2l}N_{l-1} - \frac{(2l-1)!}{2^{2l}(l!)^2}\frac{2^{2l}(l!)^2}{(2l)!}N_l \tag{4.73}$$

$$N_l = \frac{(2l-1)}{2l} N_{l-1} - \frac{1}{2l} N_l \tag{4.74}$$

from which

$$(2l+1)N_l = (2l-1)N_{l-1}. \tag{4.75}$$

Note that for $l = 0$ the polynomial is $P_0(x) = 1$ and therefore $N_0 = 2$. Thus, the recursion relation above gives

$$N_1 = \frac{1}{2(1)+1} N_0 \tag{4.76}$$

$$N_2 = \frac{3}{2(2)+1} N_1 = \frac{3}{2(2)+1} \frac{1}{2(1)+1} N_0 = \frac{1}{2(2)+1} N_0 \tag{4.77}$$

$$N_3 = \frac{5}{2(3)+1} N_2 = \frac{5}{2(3)+1} \frac{1}{2(2)+1} N_0 = \frac{1}{2(3)+1} N_0 \tag{4.78}$$

$$N_4 = \frac{7}{2(4)+1} N_3 = \frac{7}{2(4)+1} \frac{1}{2(3)+1} N_0 = \frac{1}{2(4)+1} N_0 \tag{4.79}$$

$$\cdots \tag{4.80}$$

$$N_l = \frac{1}{2l+1} N_0 = \frac{2}{2l+1}. \tag{4.81}$$

This result, together with equation (4.54) may be summarized in one relation,

$$\int_{-1}^{1} P_l P_{l'} dx = \frac{2}{2l+1} \delta_{l'l}, \tag{4.82}$$

representing the orthonormality of Legendre polynomials.

Expansion of a Function in Legendre Polynomials

Given any function $f(x)$ defined in the interval $-1 \leq x \leq 1$ can be expanded in terms of the Legendre polynomials in the form

$$f(x) = \sum_{l=0}^{\infty} A_l P_l(x) \tag{4.83}$$

where the coefficients are given by

$$A_l = \frac{2l+1}{2} \int_{-1}^{1} f(x) P_l(x) dx. \tag{4.84}$$

Example
Consider the function

$$f(x) = \begin{cases} +1 \text{ for } x > 0 \\ -1 \text{ for } x < 0 \end{cases} . \tag{4.85}$$

The expansion of this function in Legendre polynomials will have coefficients

$$A_l = \frac{2l+1}{2}\left[\int_0^1 P_l(x)dx - \int_{-1}^0 P_l(x)dx\right]. \tag{4.86}$$

If l is even, the polynomial $P_l(x)$ is even and therefore the integrals cancel out. If l is odd, the polynomial is odd and the integrals sum to give

$$A_l = (2l+1)\int_0^1 P_l(x)dx \text{ for } l \text{ odd.} \tag{4.87}$$

Using Rodrigues' formula, this integral is evaluated to give

$$A_l = \left(-\frac{1}{2}\right)^{(l-1)/2} \frac{(2l+1)(l-2)!!}{2\left(\frac{l+1}{2}\right)!}. \tag{4.88}$$

Hence the first coefficients in this series gives

$$f(x) = \frac{3}{2}P_1(x) - \frac{7}{8}P_3(x) + \frac{11}{16}P_5(x) - ... \tag{4.89}$$

Some Recurrence Formulas

$$\frac{dP_{l+1}}{dx} - \frac{dP_{l-1}}{dx} - (2l+1)P_l = 0 \tag{4.90}$$

$$(l+1)P_{l+1} - (2l+1)xP_l + lP_{l-1} = 0 \tag{4.91}$$

$$\frac{dP_{l+1}}{dx} - x\frac{dP_l}{dx} - (l+1)P_l = 0 \tag{4.92}$$

$$(x^2-1)\frac{dP_l}{dx} - lxP_l + lP_{l-1} = 0 \tag{4.93}$$

4.2.2 Azimuthal Problems with Boundary Conditions

The general solution of Laplace equation in spherical coordinates with azimuthal symmetry ($m = 0$) is given by

$$\Phi(r,\theta) = \sum_{l=0}^{\infty}\left[A_l r^l + B_l r^{-(l+1)}\right] P_l(\cos\theta), \tag{4.94}$$

where the coefficients A_l and B_l will be determined from boundary conditions.
Example. Sphere with hemispheres at different potentials.
Consider that we want to find the electrostatic potential in the interior of a sphere of radius R and that the boundary condition states that $\Phi(R,\theta) = V(\theta)$. Note that if there are no charges at the origin ($r = 0$), the potential must be finite and thus $B_l = 0$ for all l. Hence

$$\Phi(r,\theta) = \sum_{l=0}^{\infty} A_l r^l P_l(\cos\theta), \tag{4.95}$$

and evaluating at the surface of the sphere, the boundary condition gives

$$\Phi(R,\theta) = \sum_{l=0}^{\infty} A_l R^l P_l(\cos\theta), = V(\theta). \tag{4.96}$$

This equation can be seen as an expansion of the function $V(\theta)$ in Legendre series and consequently, the coefficients A_l are given by

$$A_l = \frac{2l+1}{2R^l}\int_0^{\pi} V(\theta)P_l(\cos\theta)\sin\theta d\theta. \tag{4.97}$$

Consider now that $V(\theta)$ is given by the function

$$V(\theta) = \begin{cases} +V_0 \text{ for } 0 \le \theta < \frac{\pi}{2} \\ -V_0 \text{ for } \frac{\pi}{2} < \theta \le \pi \end{cases}, \tag{4.98}$$

as in a previous example. Then

$$A_l = \frac{2l+1}{2R^l}\left[\int_0^{\frac{\pi}{2}} V_0 P_l(\cos\theta)\sin\theta d\theta - \int_{\frac{\pi}{2}}^{\pi} V_0 P_l(\cos\theta)\sin\theta d\theta\right] \tag{4.99}$$

or under a change of variable,

$$A_l = \frac{2l+1}{2R^l}\left[\int_0^1 V_0 P_l(x)dx - \int_{-1}^0 V_0 P_l(x)dx\right]. \tag{4.100}$$

This equation is equal to (4.86) and therefore using the result (4.88) for the integral we obtain

$$A_l = \left(-\frac{1}{2}\right)^{(l-1)/2} \frac{(2l+1)(l-2)!!}{2R^l\left(\frac{l+1}{2}\right)!} V_0 \text{ with } l \text{ odd}. \tag{4.101}$$

Hence, the electrostatic potential is given by the series

$$\Phi(r,\theta) = V_0\left[\frac{3}{2}\frac{r}{R}P_1(\cos\theta) - \frac{7}{8}\left(\frac{r}{R}\right)^3 P_3(\cos\theta) + \frac{11}{6}\left(\frac{r}{R}\right)^5 P_5(\cos\theta)...\right] \tag{4.102}$$

Finally, it is important to note that the potential outside the sphere is obtained from this equation by replacing the factor $\left(\frac{r}{R}\right)^l$ by $\left(\frac{R}{r}\right)^{l+1}$.

Example. Expansion of the potential due to a point charge. In this example we will consider the function

$$f(\mathbf{r},\mathbf{r}') = \frac{1}{|\mathbf{r}-\mathbf{r}'|}. \tag{4.103}$$

In order to exploit the azimuthal symmetry described above, we will rotate the axes so that $\mathbf{r}'$ lies along the z axis. Hence, the angle θ will correspond to the angle between $\mathbf{r}$ and $\mathbf{r}'$ and thus we will name it γ. Due to the azimuthal symmetry, we expand in terms of Legendre polynomials as

$$f(\mathbf{r}) = \frac{1}{|\mathbf{r}-\mathbf{r}'|} = \sum_{l=0}^{\infty} \left[A_l r^l + B_l r^{-(l+1)} \right] P_l(\cos\gamma). \tag{4.104}$$

If the point $\mathbf{r}$ in on the z axis, we have $\gamma = 0$ and then, this expansion will reduce to

$$\frac{1}{|\mathbf{r}-\mathbf{r}'|} = \sum_{l=0}^{\infty} \left[A_l r^l + B_l r^{-(l+1)} \right], \tag{4.105}$$

and

$$\frac{1}{|\mathbf{r}-\mathbf{r}'|} = \frac{1}{\left(r^2 + r'^2 - 2rr'\cos\gamma\right)^{1/2}} = \frac{1}{\left(r^2 + r'^2 - 2rr'\right)^{1/2}} = \frac{1}{|r-r'|}. \tag{4.106}$$

Then we have the expansion as

$$\frac{1}{|r-r'|} = \sum_{l=0}^{\infty} \left[A_l r^l + B_l r^{-(l+1)} \right], \tag{4.107}$$

where the coefficients A_l and B_l may be functions of r'.

If $r < r'$, this expansion needs to be finite at $r = 0$ and therefore we $B_l = 0$. The resulting expression is

$$\frac{1}{|r-r'|} = \frac{1}{(r'-r)} = \sum_{l=0}^{\infty} A_l r^l \tag{4.108}$$

$$\frac{1}{r'\left(1-\frac{r}{r'}\right)} = \sum_{l=0}^{\infty} A_l r^l. \tag{4.109}$$

Using the convention $r = r_<$ and $r' = r_>$, this expression becomes

$$\frac{1}{r_>\left(1-\frac{r_<}{r_>}\right)} = \sum_{l=0}^{\infty} A_l r_<^l. \tag{4.110}$$

On the other hand, for points with $r > r'$, this expansion needs to be finite when $r \to \infty$ and therefore we $A_l = 0$. The resulting equation is now

$$\frac{1}{|r-r'|} = \frac{1}{(r-r')} = \sum_{l=0}^{\infty} B_l r^{-(l+1)} \tag{4.111}$$

$$\frac{1}{r\left(1-\frac{r'}{r}\right)} = \sum_{l=0}^{\infty} B_l r^{-(l+1)}. \tag{4.112}$$

Using the convention $r' = r_<$ and $r = r_>$, this expression becomes

$$\frac{1}{r_>\left(1 - \frac{r_<}{r_>}\right)} = \sum_{l=0}^{\infty} \frac{B_l}{r_>^{(l+1)}}. \tag{4.113}$$

Results (4.110) and (4.113) can be summarized into one equation

$$\frac{1}{r_>\left(1 - \frac{r_<}{r_>}\right)} = \sum_{l=0}^{\infty} C_l \frac{r_<^l}{r_>^{(l+1)}}, \tag{4.114}$$

provided that

$$C_l = \begin{cases} A_l r_>^{(l+1)} & \text{when } r < r' \\ \frac{B_l}{r_<^l} & \text{when } r > r' \end{cases}. \tag{4.115}$$

The expansion of both sides of equation (4.114) gives

$$\frac{1}{r_>\left(1 - \frac{r_<}{r_>}\right)} = \frac{1}{r_>}\left(1 - \frac{r_<}{r_>}\right)^{-1} = \frac{1}{r_>}\left(1 + \frac{r_<}{r_>} + \frac{r_<^2}{r_>^2} + ...\right) \tag{4.116}$$

and

$$\sum_{l=0}^{\infty} C_l \frac{r_<^l}{r_>^{(l+1)}} = \frac{1}{r_>}\sum_{l=0}^{\infty} C_l \frac{r_<^l}{r_>^{(l)}} = \frac{1}{r_>}\left(C_0 + C_1\frac{r_<}{r_>} + C_2\frac{r_<^2}{r_>^2} + ...\right) \tag{4.117}$$

Comparison of both expansions let us identify the coefficients $C_l = 1$ and therefore we obtain

$$\frac{1}{|r - r'|} = \sum_{l=0}^{\infty} \frac{r_<^l}{r_>^{(l+1)}}. \tag{4.118}$$

For points $\mathbf{r}$ off the z axis it is only necessary to multiply each term by the corresponding Legendre polynomial $P_l(\cos\theta)$, i.e.

$$\frac{1}{|r - r'|} = \sum_{l=0}^{\infty} \frac{r_<^l}{r_>^{(l+1)}} P_l(\cos\theta). \tag{4.119}$$