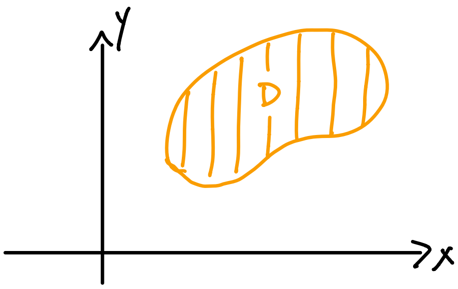
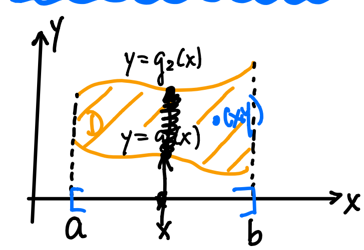
15.2. DOUBLE INTEGRALS OVER GENERAL REGIONS



$$\iint_D f(x,y)\,dA = ?$$

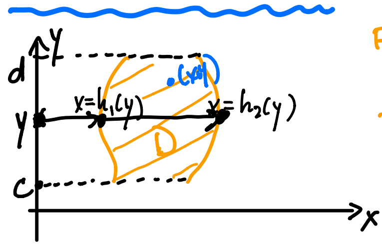
FORMULAS FOR CALCULATING $\boxed{\iint_D f(x,y)\,dA}$ OVER 2 TYPES OF REGIONS:

(I) D IS OF TYPE I if D lies between the graphs of two CONTINUOUS FUNCTIONS OF x:



$$D = \{(x,y) \mid a \le x \le b,\ g_1(x) \le y \le g_2(x)\}$$

(II) D IS OF TYPE II if D lies between the graps of two CONTINUOUS FUNCTIONS OF y:



$$D = \{(x,y) \mid c \le y \le d,\ h_1(y) \le x \le h_2(y)\}$$

INTEGRATION FORMULAS

TYPE I REGION

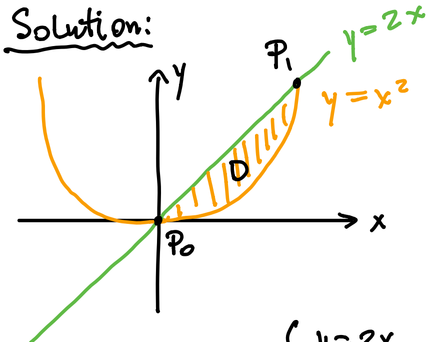
$$\boxed{\iint_D f(x,y)\,dA = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x,y)\,dy\,dx}$$

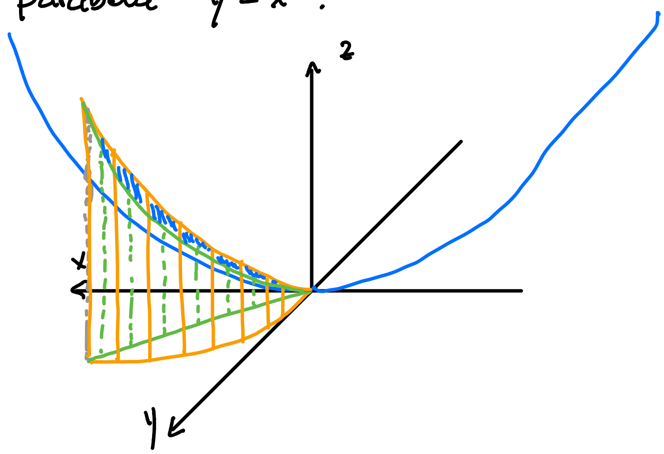
TYPE II REGION

$$\boxed{\iint_D f(x,y)\,dA = \int_c^d \int_{h_1(y)}^{h_2(y)} f(x,y)\,dx\,dy}$$

EXAMPLE: Find the volume of the solid that lies under the paraboloid $z = x^2+y^2$ and above the region D in the xy plane, bounded by $y=2x$ and the parabola $y=x^2$.

Solution:





Intersections: $\begin{cases} y=2x \\ y=x^2 \end{cases}$

$\Rightarrow \quad 2x = x^2 \Rightarrow 2x - x^2 = 0$

$x(2-x)=0$

$x=0$	$x=2$
$y=0$	$y=4$
$P_0(0,0)$	$P_1(2,4)$

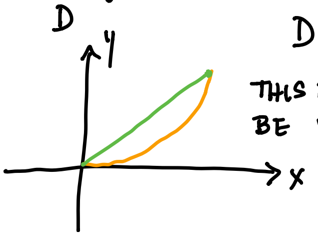
Since $f(x,y) \geq 0$ above D, the volume under the surface, above D, is defined by:

$$V = \iint_D f(x,y)\,dA$$

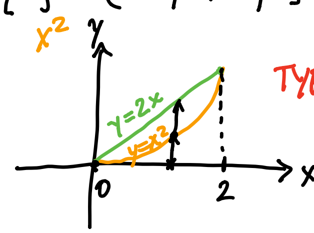
Calculate the double integral:

$$V = \iint_D f(x,y)\,dA = \iint_D (x^2+y^2)\,dA = \int_0^2 \left[\int_{x^2}^{2x} (x^2+y^2)\,dy \right] dx$$

TYPE I



THIS REGION CAN BE VIEWED AS BOTH TYPE I AND TYPE II REGION !

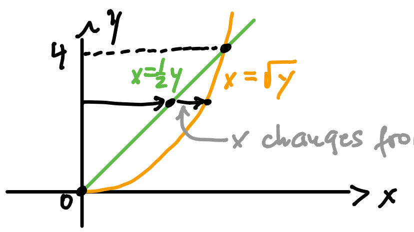


TYPE I

$$= \int_0^2 \left[x^2 y + \frac{y^3}{3} \right]_{y=x^2}^{y=2x} dx = \int_0^2 \left[x^2(2x-x^2) + \frac{1}{3}\left((2x)^3 - (x^2)^3\right) \right] dx$$

$$= \int_0^2 \left[2x^3 - x^4 + \frac{1}{3}8x^3 - \frac{1}{3}x^6\right] dx = \frac{14}{3}\left[\frac{x^4}{4}\right]_0^2 - \left[\frac{x^5}{5}\right]_0^2 - \frac{1}{3}\left[\frac{x^7}{7}\right]_0^2 = \frac{216}{35}$$

APPROACH 2 (TYPE II INTEGRAL)



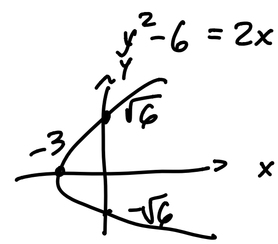
$$V = \int_0^4 \left(\int_{\frac{y}{2}}^{\sqrt{y}} (x^2+y^2)\, dx \right) dy$$

$$= \int_0^4 \left(\frac{x^3}{3} + y^2 x \right)\Bigg|_{x=\frac{y}{2}}^{x=\sqrt{y}} dy$$

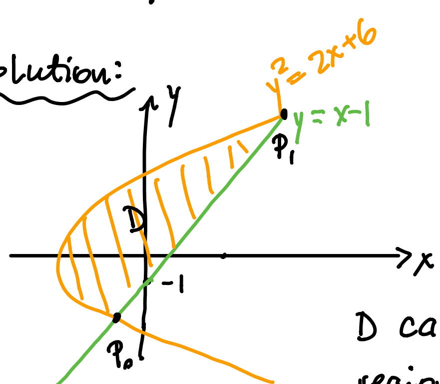
$$= \int_0^4 \left(\frac{1}{3} y^{3/2} - \frac{1}{3}\left(\frac{y}{2}\right)^3 + \underbrace{y^2\sqrt{y}}_{y^{5/2}} - y^2 \cdot \frac{y}{2} \right) dy$$

$$= \left(\frac{2}{15} y^{\frac{5}{2}} + \frac{2}{7} y^{\frac{7}{2}} - \frac{13}{96} y^4 \right)\Bigg|_{y=0}^{4} = \frac{216}{35}$$

EXAMPLE: $\iint_D xy \, dA = ?$ where D is the region bounded by the line $y = x-1$ and the parabola $y^2 = 2x+6$.



Solution:



D can be viewed as both region of TYPE I and region of TYPE II

Points of intersection:

$$\begin{cases} y^2 = 2x+6 \\ y = x-1 \end{cases} \Rightarrow$$

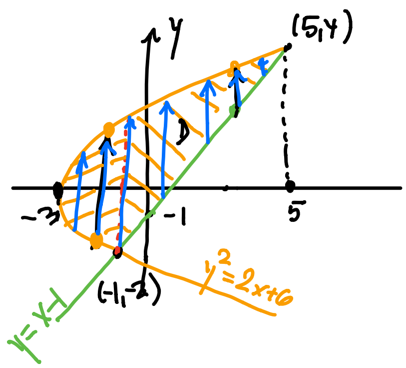
$$(x-1)^2 = 2x+6$$

$$x^2-2x+1 = 2x+6$$

$$x^2-4x-5=0 \Rightarrow \begin{array}{l} x=-1 \\ x=5 \end{array}$$

$P_0(-1,-2)$, $P_1(5,4)$

Approach 1 (REGION OF TYPE I):



For x between -3 and -1, y changes from

$y=-\sqrt{2x+6}$ to $y=\sqrt{2x+6}$

For x between -1 and 5, y changes from

$y=x-1$ to $y=\sqrt{2x+6}$

$$I=\iint_D xy\,dA \underset{\text{TYPE I}}{=} I_1+I_2=\underbrace{\int_{-3}^{-1}\left(\int_{-\sqrt{2x+6}}^{\sqrt{2x+6}} xy\,dy\right)dx}_{I_1}+\underbrace{\int_{-1}^{5}\left(\int_{x-1}^{\sqrt{2x+6}} xy\,dy\right)dx}_{I_2}$$

$$=\int_{-3}^{-1}\left[x\frac{y^2}{2}\right]_{y=-\sqrt{2x+6}}^{\sqrt{2x+6}}dx+\int_{-1}^{5}\left[x\frac{y^2}{2}\right]_{y=x-1}^{\sqrt{2x+6}}dx=$$

$$=\int_{-3}^{-1}\left[\cancel{\frac{1}{2}x(2x+6)}-\cancel{\frac{1}{2}x(2x+6)}\right]dx+\frac{1}{2}\int_{-1}^{5}\left[x(2x+6)-x(x-1)\right]dx$$

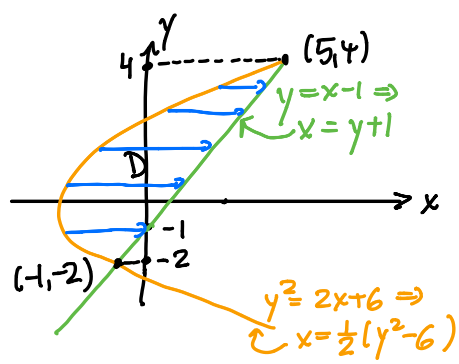
$$=\frac{1}{2}\int_{-1}^{5}\left(2x^2+6x-x^2+x\right)dx=\ldots=36$$

Approach 2 (REGION OF TYPE II):

For y between -2 and 4,

x changes from

$x=\frac{1}{2}(y^2-6)$ to $x=y+1$



$$I = \iint_D xy\, dA \underset{\uparrow}{=} \int_{-2}^{4} \left(\int_{\frac{1}{2}(y^2-6)}^{y+1} xy \, dx \right) dy =$$

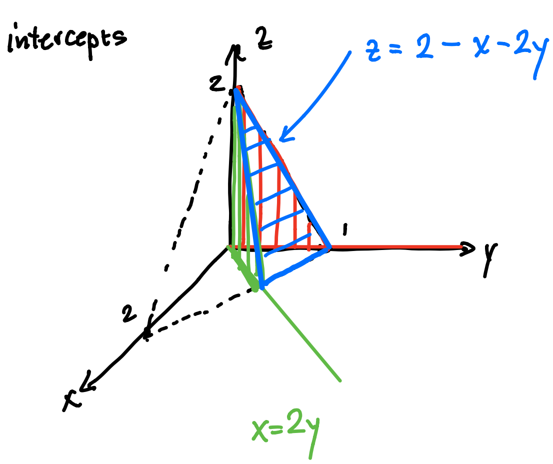
TYPE 2

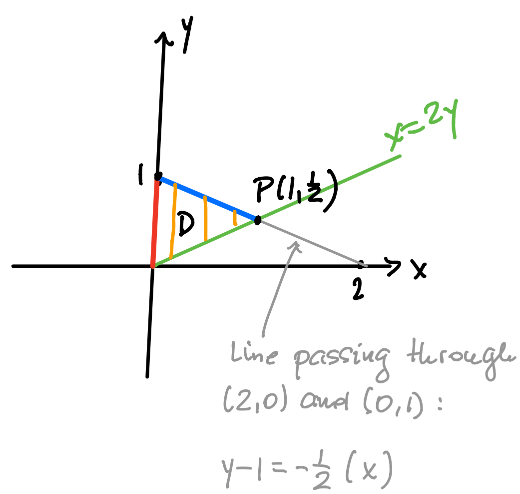
$$= \int_{-2}^{4} \left(\frac{x^2}{2} y \right)_{x=\frac{1}{2}(y^2-6)}^{x=y+1} dy = \frac{1}{2} \int_{-2}^{4} \left[(y+1)^2 y - \left(\frac{1}{2}(y^2-6) \right)^2 y \right] dy$$

$$= \frac{1}{2} \int_{-2}^{4} y \left[y^2 + 2y + 1 - \frac{1}{4} y^4 + \frac{12}{4} y^2 - \frac{36}{4} \right] dy = \dots = 36$$

EXAMPLE: Find the volume of the tetrahedron bounded by the planes $x+2y+z=2$, $x=2y$, $x=0$, $z=0$.

Solution: Draw two diagrams: one of the solid, and one of the plane region D above which the solid lies.





Intersection point P:

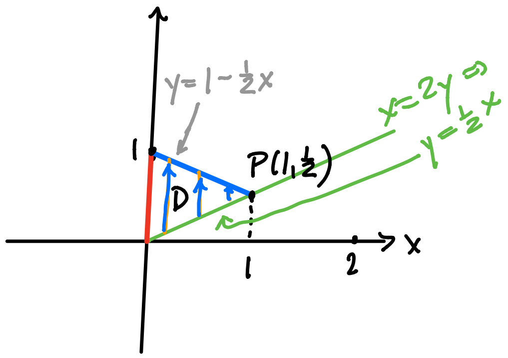
$$\left.\begin{aligned} x &= 2y \\ x &= 2-2y \end{aligned}\right\} \Rightarrow \begin{aligned} 2y &= 2-2y \\ y &= 1-y \\ y &= \tfrac{1}{2} \end{aligned}$$

$x = 1$

$P(1, \tfrac{1}{2}) \Leftarrow$

$$V = \iint_D (2-x-2y)\,dA$$ = TYPE I OR TYPE II ? = simpler to consider TYPE I

$$\underset{\text{TYPE I}}{\overset{=}{\uparrow}} \int_0^1 \left[\int_{\frac{x}{2}}^{1-\frac{x}{2}} (2-x-2y)\,dy \right] dx = \int_0^1 \left[2y - xy - \cancel{2}\frac{y^2}{\cancel{2}} \right]_{y=\frac{x}{2}}^{y=1-\frac{x}{2}} dx$$



$$= \int_0^1 \left[2\left(1-\frac{x}{2}-\frac{x}{2}\right) - x\left(1-\frac{x}{2}-\frac{x}{2}\right) - \left(\left(1-\frac{x}{2}\right)^2 - \left(\frac{x}{2}\right)^2\right) \right]$$

$$= \int_0^1 2\left(1-x-x(1-x) - \left(1-x+\cancel{\frac{x^2}{4}}-\cancel{\frac{x^2}{4}}\right)\right] dx$$

$$= \ldots\ldots = \frac{1}{3}$$

EXAMPLE: $\displaystyle\int_0^1 \int_x^1 \sin(y^2)\,dy\,dx = ?$

Solution: $\displaystyle\int_x^1 \sin(y^2)\,dy = ?$

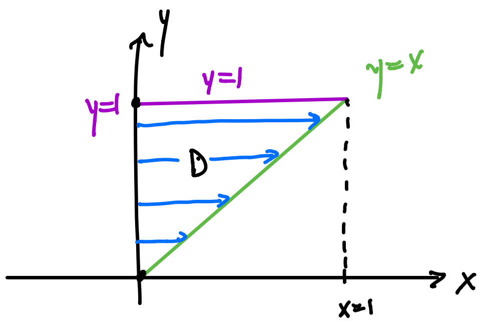
Cannot integrate since $\sin(y^2)$ is not an elementary function.

Can use numerical (computer) simulations.

What if we change the order of integration ?

$$\int_0^1 \int_x^1 \sin(y^2)\,dy\,dx = \iint_{D\,?} \sin(y^2)\,dx\,dy$$

What is D ? For $\displaystyle\int_x^1 \sin(y^2)\,dy$ we see that y goes from $y=x$ to $y=1$.



Thus, region D is determined by x between 0 and 1, and y changing from $y=x$ to $y=1$

$$I = \int_0^1 \int_x^1 \sin(y^2)\,dy\,dx = \int_0^1 \left(\int_0^y \sin(y^2)\,dx \right) dy =$$

$$= \int_0^1 \sin(y^2) \left[\int_0^y dx \right] dy = \int_0^1 y \sin(y^2)\,dy =$$

substitution
$u = y^2$
$du = 2y\,dy$
$y\,dy = \frac{1}{2}\,du$

$$= \int (\sin u)\,\frac{1}{2}\,du = -\frac{1}{2}\cos u = -\frac{1}{2}\Big[\cos(y^2)\Big]_{y=0}^{1} =$$

$$= \frac{1}{2}\left[\cos 1 - \cos 0\right] = \frac{1}{2}\left[\cos 1 - 1\right]$$

PROPERTIES OF DOUBLE INTEGRALS

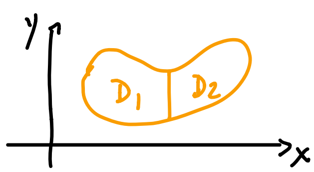
1. $$\iint_D [f(x,y) + g(x,y)]\, dA = \iint_D f(x,y)\, dA + \iint_D g(x,y)\, dA$$

2. $$\iint_D c\, f(x,y)\, dA = c \iint_D f(x,y)\, dA, \qquad c = \text{constant}$$

3. If $f(x,y) \geq g(x,y)$ for all (x,y) in D, then

$$\iint_D f(x,y)\, dA \geq \iint_D g(x,y)\, dA$$

4. If $D = D_1 \cup D_2$, wher D_1 and D_2 DO NOT OVERLAP except on their boundaries, then



$$\iint_D f(x,y)\, dA = \iint_{D_1} f(x,y)\, dA + \iint_{D_2} f(x,y)\, dA$$

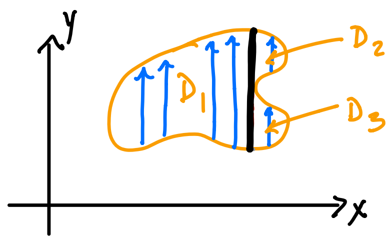
5. $$\iint_D 1\, dA = \text{AREA OF } D = A(D)$$

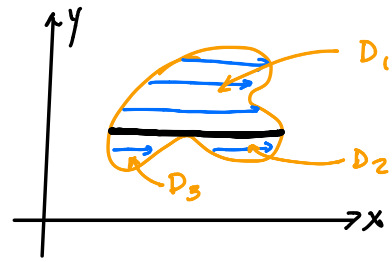
6. If $m \leq f(x,y) \leq M$ for all (x,y) in D, then

$$m\, A(D) \leq \iint_D f(x,y)\, dA \leq M\, A(D).$$

REMARKS

(I) Property 4 can be used to evaluate double integrals over regions D that are neither TYPE I nor TYPE II, but can be expressed as a UNION OF REGIONS OF TYPE I OR TYPE II

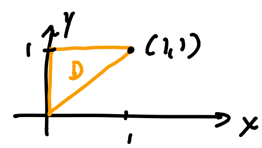




(II) Property 6 can be used estimate integrals.

For example:

$$\int_0^1 \int_x^1 \sin(y^2)\,dy\,dx$$



We know: ① $0 \le \sin(y^2) \le 1$ everywhere on D

② $A(D) = \frac{1}{2}$

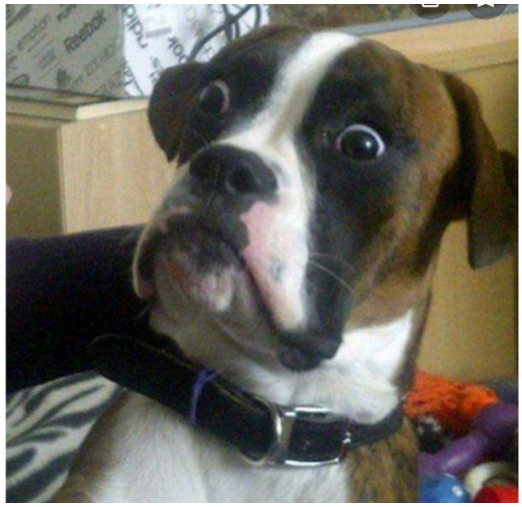
Therefore:

$$0 \le \int_0^1 \int_x^1 \sin(y^2)\,dy\,dx \le \frac{1}{2}$$

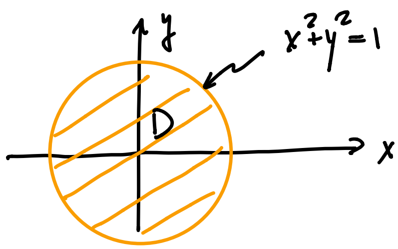
😎

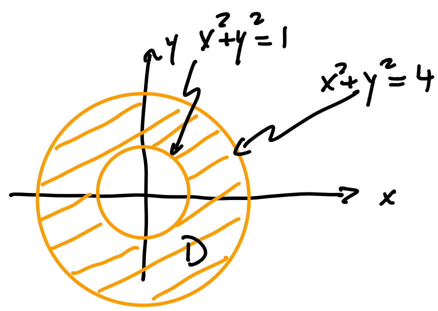
MOVING ON!

15.3. DOUBLE INTEGRALS IN POLAR COORDINATES



Used to evaluate integrals over regions easily representable in polar coordinates, such as:



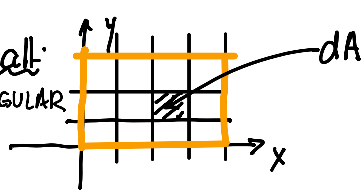


Recall:
$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases}$$

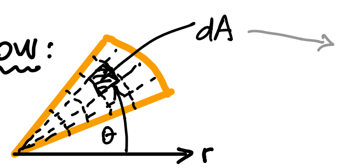
These regions are a special case of a POLAR RECTANGLE:

(★) $$R = \{ (r,\theta) \mid a \le r \le b \,,\ \alpha \le \theta \le \beta \}$$

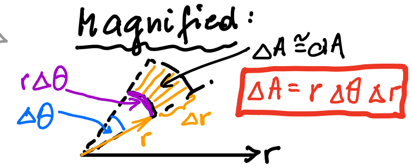
Recall:
RECTANGULAR



Now:
POLAR

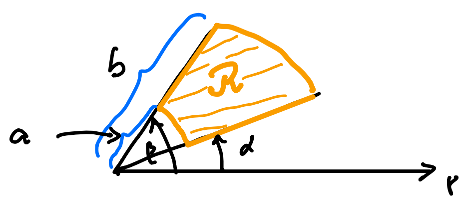


Magnified:



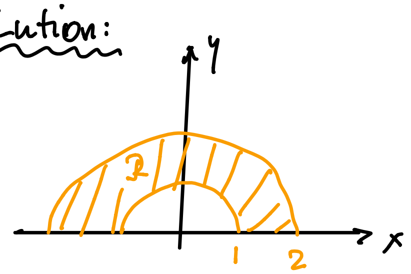
DEFINITION: If f is continuous on a polar rectangle R given by (A) above, where $\alpha \le \beta - \alpha \le 2\pi$, then

$$\boxed{\iint_R f(x,y)\,dA = \int_\alpha^\beta \int_a^b f(r\cos\theta, r\sin\theta)\, r\,dr\,d\theta}$$



EXAMPLE: $\iint_R (3x+4y^2)\,dA = ?$ where R is the region in the upper half plane bounded by the circles $x^2+y^2=1$ and $x^2+y^2=4$.

Solution:



$R = \{(r,\theta) \mid 1 \le r \le 2,\ 0 \le \theta \le \pi\}$

$$\iint_R (3x+4y^2)\,dA = \int_1^2 \int_0^\pi \left(3r\cos\theta + 4r^2\sin^2\theta\right) r\,dr\,d\theta$$

$$= 3\int_1^2\int_0^\pi \underbrace{r^2\cos\theta}_{\text{PRODUCT FUNCTION}}\,dr\,d\theta + 4\int_1^2\int_0^\pi \underbrace{r^3\sin^2\theta}_{\text{PRODUCT}}\,dr\,d\theta$$

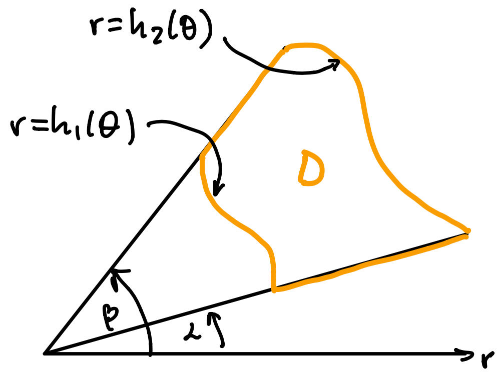
$$= 3\left(\int_1^2 r^2\,dr\right)\left(\int_0^\pi \cos\theta\,d\theta\right) + 4\left(\int_1^2 r^3\,dr\right)\left(\int_0^\pi \sin^2\theta\,d\theta\right)$$

PRODUCT FUNCTION: $f(r,\theta) = f_1(r) f_2(\theta)$
AND BOUNDS OF INTEGRATION ARE CONSTANTS
$\Rightarrow$ SIMPLIFY

$$= 3\left[\frac{r^3}{3}\right]_{r=1}^{2} \underbrace{\left[\sin\theta\right]_{\theta=0}^{\pi}}_{=0} + 4\underbrace{\left[\frac{r^4}{4}\right]_{r=1}^{2}} \int_0^{\pi} \frac{1}{2}(1-\cos 2\theta)\,d\theta$$

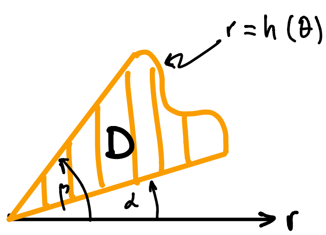
$$= \quad 0 \quad + \quad 15 \quad \left[\frac{1}{2}\left(\theta - \frac{\sin 2\theta}{2}\right)\right]_{\theta=0}^{\pi} = 15\frac{\pi}{2}$$

MORE GENERAL POLAR REGIONS



$$\boxed{\iint_D f(x,y)\,dA = \int_{\alpha}^{\beta}\int_{h_1(\theta)}^{h_2(\theta)} f(r\cos\theta, r\sin\theta)\, r\,dr\,d\theta}$$

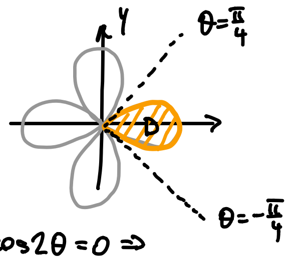
EXAMPLE: What is the area of the region bounded by $\theta=\alpha$, $\theta=\beta$ and $r=0$ and $r=h(\theta)$?



$$A(D) = \iint_D 1\,dA = \iint_D r\,dr\,d\theta$$

$$A(D) = \int_{\alpha}^{\beta}\int_0^{h(\theta)} r\,dr\,d\theta = \int_{\alpha}^{\beta}\left[\frac{r^2}{2}\right]_{r=0}^{h(\theta)} d\theta = \frac{1}{2}\int_{\alpha}^{\beta}\left[h(\theta)\right]^2 d\theta$$

EXAMPLE: Use a double integral to find the area enclosed by one Loop of the four leaved rose $r = \cos 2\theta$.



$\cos 2\theta = 0 \Rightarrow$

$2\theta = -\frac{\pi}{2}, \frac{\pi}{2}$

$\Rightarrow \theta = -\frac{\pi}{4}, \frac{\pi}{4}$

$$D = \{ (r,\theta) \mid 0 \le r \le \cos 2\theta, -\frac{\pi}{4} \le \theta \le \frac{\pi}{4} \}$$

$$A(D) = \iint_D 1 \, dA = \int_{-\frac{\pi}{4}}^{\pi/4} \int_0^{\cos 2\theta} r \, dr \, d\theta$$

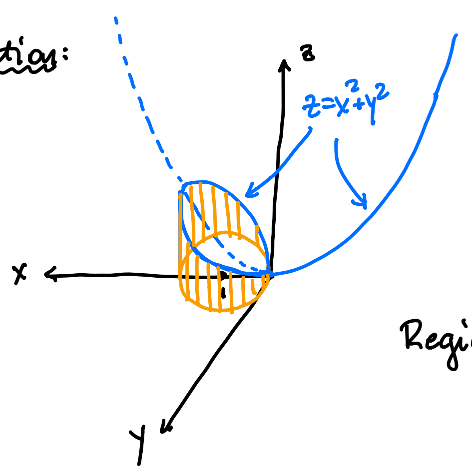
$$= \int_{-\frac{\pi}{4}}^{\pi/4} \left[\frac{r^2}{2} \right]_{r=0}^{\cos 2\theta} d\theta = \frac{1}{2} \int_{-\pi/4}^{\pi/4} \cos^2 2\theta \, d\theta$$

$$= \frac{1}{2} \cdot \frac{1}{2} \int_{-\pi/4}^{\pi/4} [1 + \cos 4\theta] \, d\theta =$$

$$= \frac{1}{4} \left[\theta + \frac{\sin 4\theta}{4} \right]_{\theta=-\pi/4}^{\pi/4} = \frac{\pi}{8}$$

EXAMPLE: Find the volume of the solid UNDER $z = x^2 + y^2$, ABOVE the xy plane, and inside the cylinder $x^2 + y^2 = 2x$.

Solution:

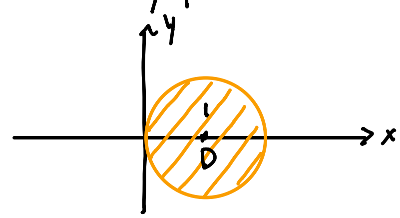


Cylinder $x^2 + y^2 = 2x \Rightarrow$

$x^2 - 2x + 1 + y^2 = 1$

$(x-1)^2 + y^2 = 1$

Region D in the xy plane:



VOLUME $\boxed{V = \iint_D f(x,y) \, dA}$

What determines D in polar coordinates: $(x-1)^2+y^2=1$

$$\left.\begin{array}{l} x = r\cos\theta \\ y = r\cos\theta \end{array}\right\} \Rightarrow \quad (x-1)^2+y^2=1 \quad \text{OR} \quad x^2+y^2 = 2x \quad \Rightarrow$$

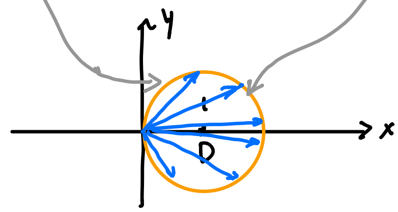
$$(r\cos\theta)^2 + (r\sin\theta)^2 = 2r\cos\theta$$

$$r^2\underbrace{(\cos^2\theta+\sin^2\theta)}_{=1} = 2r\cos\theta$$

$$r^2 = 2r\cos\theta$$

$$r=0 \qquad \boxed{r = 2\cos\theta}$$

$$D = \left\{ (r,\theta) \mid 0 \le r \le 2\cos\theta,\ -\tfrac{\pi}{2} \le \theta \le \tfrac{\pi}{2} \right\}$$



Volume:

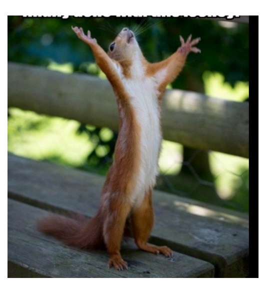
$$V = \iint_D f(x,y)\,dA = \iint_D (x^2+y^2)\,dA = \text{POLAR COORDINATES} =$$

$$= \int_{-\frac{\pi}{2}}^{\pi/2}\int_0^{2\cos\theta} \big(\underbrace{r^2\cos^2\theta + r^2\sin^2\theta}_{=r^2}\big)\, r\,dr\,d\theta = \int_{-\frac{\pi}{2}}^{\pi/2}\int_0^{2\cos\theta} r^3\,dr\,d\theta =$$

$$= \int_{-\frac{\pi}{2}}^{\pi/2} \left[\frac{r^4}{4}\right]_{r=0}^{2\cos\theta} d\theta = \frac{1}{4}\int_{-\frac{\pi}{2}}^{\pi/2} \left[16\cos^4\theta\right] d\theta = 4\int_{-\frac{\pi}{2}}^{\pi/2} \left[\cos^2\theta\right]^2 d\theta$$

$$= 4\int_{-\frac{\pi}{2}}^{\pi/2} \frac{1}{2}\left[1+\cos 2\theta\right]^2 d\theta = 4\cdot\frac{1}{4}\int_{-\frac{\pi}{2}}^{\pi/2}\left[1+2\cos 2\theta + \cos^2 2\theta\right] d\theta$$

$$= \ldots\ldots = \frac{3\pi}{2}$$



DONE!