ANNOUNCEMENTS

1. Last lecture on new material
2. Thursday Lecture = solving problems related to 16.7 - 16.9
3. RRR week = Tuesday Dec 8 = solving Practice Final
4. Last homework is due Mon, Dec 7. No quiz that week.
5. Final exam:

   Dec 17, 2020, 11:30 am - 2:00 pm PST

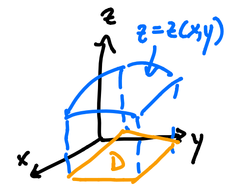
   It is going to be a 2 hour exam with extra 30 minutes for submission on Gradescope.

16.7. SURFACE INTEGRALS

(1) SURFACE INTEGRALS OF SCALAR FUNCTIONS f

(2) SURFACE INTEGRALS OF VECTOR FUNCTIONS $\vec{F}$

(1) SURFACE INTEGRALS OF SCALAR FUNCTIONS f



OVER PARAMETRIC SURFACES

$\vec{r}(u,v) = x(u,v)\vec{i} + y(u,v)\vec{j} + z(u,v)\vec{k}$

$$\boxed{\iint_S f(x,y,z)\,dS = \iint_D f(\underbrace{\vec{r}(u,v)})\,\underbrace{|\vec{r}_u \times \vec{r}_v|\,du\,dv}_{dS}}$$

D: region in parameter space $(u,v) \in D$.

Here $dS = |\vec{r}_u \times \vec{r}_v|\,du\,dv$.

OVER SURFACES DEFINED AS GRAPHS OF FUNCTIONS

$z = z(x,y)$

$$\boxed{\iint_S f(x,y,z)\,dS = \iint_D f(x,y,z(x,y))\sqrt{1+\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2}\,dx\,dy}$$

(☆)

Here $dS = \sqrt{1+\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2}$.

HINT: Notice that $z = z(x,y)$ can be regarded as a parametric surface with parametric equation $\vec{r}(x,y) = x\vec{i} + y\vec{j} + z(x,y)\vec{k}$ in terms of two parameters x and y: $\begin{cases} x = x \\ y = y \\ z = z(x,y) \end{cases}$.

Then: $|\vec{r}_x \times \vec{r}_y| = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 0 & \frac{\partial z}{\partial x} \\ 0 & 1 & \frac{\partial z}{\partial y} \end{vmatrix} = \boxed{-\frac{\partial f}{\partial x}\vec{i} - \frac{\partial f}{\partial y}\vec{j} + \vec{k}}$. So:

$\vec{r}_x = \vec{i} + \frac{\partial f}{\partial x}\vec{k}$

$\vec{r}_y = \vec{j} + \frac{\partial f}{\partial y}\vec{k}$

$$|\vec{r}_x \times \vec{r}_y| = \sqrt{1+\left(\frac{\partial f}{\partial x}\right)^2+\left(\frac{\partial f}{\partial y}\right)^2}$$

(2) SURFACE INTEGRALS OF VECTOR FUNCTIONS $\vec{F}$

Definition: If $\vec{F}$ is a continuous vector field defined on an oriented surface S with unit normal vector $\vec{n}$, then the SURFACE INTEGRAL OF $\vec{F}$ OVER S is:

$$\boxed{\iint_S \vec{F}\cdot d\vec{S} = \iint_S \underbrace{\vec{F}\cdot\vec{n}}_{\text{scalar function}}\, dS}$$

$d\vec{S} = \vec{n}\, dS$

where:

$$\vec{n} = \begin{cases} \boxed{\dfrac{\vec{r}_u \times \vec{r}_v}{|\vec{r}_u \times \vec{r}_v|}} & \text{for PARAMETRIC SURFACES} \\[2ex] \boxed{\dfrac{\langle -\frac{\partial f}{\partial x}, -\frac{\partial f}{\partial y}, 1\rangle}{\sqrt{\left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2 + 1}}} & \text{for } z = f(x,y) \end{cases}$$

$$dS = \begin{cases} \boxed{|\vec{r}_u \times \vec{r}_v|} & \text{for PARAMETRIC SURFACES} \\[1ex] \boxed{\sqrt{\left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2 + 1}} & \text{for } z = f(x,y) \end{cases}$$

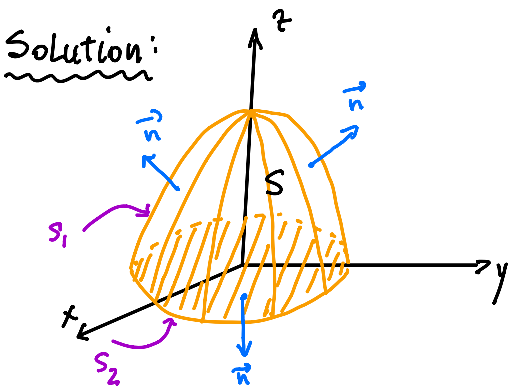
THUS:

$$d\vec{S} = \begin{cases} (\vec{r}_u \times \vec{r}_v)\, du\, dv & \text{for PARAMETRIC SURFACES} \\[1ex] \left(-\frac{\partial f}{\partial x}\vec{i} - \frac{\partial f}{\partial y}\vec{j} + \vec{k}\right) dx\, dy & \text{for } z = f(x,y) \end{cases}$$

$$\boxed{\iint_S \vec{F}\cdot d\vec{S} = \begin{cases} \displaystyle\iint_D \vec{F}\cdot(\vec{r}_u \times \vec{r}_v)\, \overbrace{dA}^{du\,dv} & D \leftarrow \text{region in parametric space } (u,v) \\[2ex] \displaystyle\iint_D \vec{F}\cdot\left\langle -\frac{\partial f}{\partial x}, -\frac{\partial f}{\partial y}, 1\right\rangle \underbrace{dA}_{dx\,dy} & D \leftarrow \text{region in the xy plane} \end{cases}} \quad (\star)$$

Example: Evaluate $\iint_S \vec{F}\cdot d\vec{S}$ where $\vec{F} = y\vec{i} + x\vec{j} + z\vec{k}$ and S is the boundary of the solid region E enclosed by the paraboloid $z = 1 - x^2 - y^2$ and the plane $z = 0$.

Solution:



1. First notice that S is a CLOSED surface, which consists of two parts:

S_1 = paraboloid, and

S_2 = plane.

Since $d\vec{S} = \vec{n}\,dS$, positive orientation corresponds to $\vec{n}$ = OUTWARD unit normal.

2. $$I = \iint_S \vec{F}\cdot d\vec{S} = \iint_{S_1} \vec{F}\cdot d\vec{S} + \iint_{S_2} \vec{F}\cdot d\vec{S} = I_1 + I_2$$

3. Calculate $I_1 = \iint_{S_1} \vec{F}\cdot d\vec{S}$, where S_1 is given by $z = 1 - x^2 - y^2 = f(x,y)$.

Thus, $\frac{\partial f}{\partial x} = -2x$, $\frac{\partial f}{\partial y} = -2y$.

From $(\star) \Rightarrow$ $$I_1 = \iint_D \underbrace{\langle y, x, z\rangle}_{\vec{F}} \cdot \underbrace{\langle 2x, 2y, 1\rangle}_{\langle -\frac{\partial f}{\partial x}, -\frac{\partial f}{\partial y}, 1\rangle} \,dx\,dy$$

Here D is the region in the xy plane under the surface $z = f(x,y)$:

$$D = \{ (x,y) \mid x^2 + y^2 \le 1 \}$$

$$I_1 = \iint_{x^2+y^2\le 1} (2xy + 2xy + z)\,dx\,dy = \left| \begin{array}{l} \text{Since } z = f(x,y) \text{ on } S_1, \\ \text{we replace } z \text{ with} \\ 1 - x^2 - y^2 \end{array} \right| =$$

$$= \iint_{x^2+y^2\le 1} (4xy + 1 - x^2 - y^2)\,dx\,dy = \left| \begin{array}{l} \text{POLAR COORDINATES} \\ x = r\cos\theta \\ y = r\sin\theta \\ dx\,dy = r\,dr\,d\theta \end{array} \right| =$$

$$= \int_0^{2\pi} \int_0^1 \left(1 - r^2 - 4r^2 \sin\theta \cos\theta\right) r\,dr\,d\theta =$$

$$= \int_0^{2\pi} \int_0^1 \left(r - r^3\right) dr\,d\theta + \int_0^{2\pi} \int_0^1 \left(2r^3 \sin 2\theta\right) dr\,d\theta =$$

$$= 2\pi \left[\frac{r^2}{2} - \frac{r^4}{4}\right]_0^1 + \int_0^{2\pi} 2\left[\frac{r^4}{4}\right]_0^1 \sin 2\theta\,d\theta =$$

$$= 2\pi \cdot \frac{1}{4} \qquad + \frac{1}{2} \underbrace{\int_0^{2\pi} \sin 2\theta\,d\theta}_{=0} = \frac{\pi}{2}$$

4. Calculate $I_2 = \iint_{S_2} \langle y, x, z\rangle \cdot d\vec{S}$. Surface S_2 is $z=0$. So

$z = f(x,y) = 0 \Rightarrow \frac{\partial f}{\partial x} = \frac{\partial f}{\partial y} = 0 \Rightarrow$ OUTER NORMAL $= -\vec{k}$.

$$I_2 = \iint_{x^2+y^2 \le 1} \langle x, y, \underset{\substack{\uparrow \\ z=0 \text{ on } S_2}}{0} \rangle \cdot \langle 0, 0, -1\rangle\,dx\,dy = \iint 0\,dx\,dy = 0$$

5. Answer: $I = I_1 + I_2 = \frac{\pi}{2} + 0 = \frac{\pi}{2}$

So, the total flux of vector field $\vec{F} = \langle y, x, z\rangle$ through S is $\frac{\pi}{2}$.

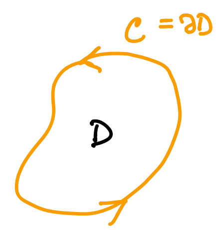
16.8 and 16.9. THE STOKES AND THE DIVERGENCE THEOREMS

RECALL GREEN'S THEOREM (in $\mathbb{R}^2$)

VERSION I:

$$\oint_C \vec{F}\cdot d\vec{r} = \iint_D \operatorname{curl} \vec{F}\cdot \vec{k}\, dA$$

$$\oint_C \vec{F}\cdot \vec{n}\, ds = \iint_D \operatorname{curl} \vec{F}\cdot \vec{k}\, dA$$



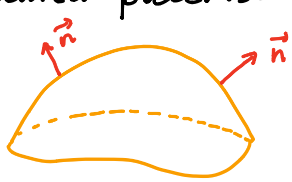
POSITIVE ORIENTATION
COUNTER CLOCKWISE

VERSION II:

$$\oint_C \vec{F}\cdot \vec{n}\, ds = \iint_D \operatorname{div} \vec{F}\, dA$$

16.8. THE STOKES' THEOREM

Let S be an oriented piecewise smooth surface (which means that you have unit normal):



and let C be a simple, closed, piecewise smooth boundary curve with POSITIVE ORIENTATION (which means that if you walk along the curve with your head pointing in the direction of $\vec{n}$, then the surface always must be on your left):

Let $\vec{F}$ be a vector field with continuous partial derivatives.

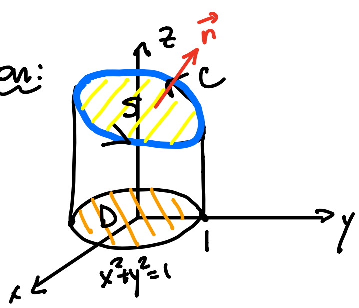
Then

$$\boxed{\oint_C \vec{F}\cdot d\vec{r} = \iint_S \text{curl}\,\vec{F}\cdot d\vec{S}}$$

STOKES THEOREM

Example: Evaluate $\oint_C \vec{F}\cdot d\vec{r}$, where $\vec{F} = \langle -y^2, x, z^2\rangle$ and C is the curve of intersection of the plane $y+z=2$ and the cylinder $x^2+y^2=1$. Orient C to be counterclockwise when viewed from above.

Solution:



We could choose any surface with boundary C

The simplest choice for S is $y+z=2$.

Stokes' Theorem (easier):

$$\oint_C \vec{F}\cdot d\vec{r} = \iint_S \text{curl}\,\vec{F}\cdot d\vec{S}$$

$$\text{curl}\,\vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ -y^2 & x & z^2 \end{vmatrix} = \langle 0, 0, 1+2y\rangle$$

$z = f(x,y) = 2-y$

$$I = \iint_D \langle 0, 0, 1+2y\rangle \cdot \langle -\tfrac{\partial f}{\partial x}, -\tfrac{\partial f}{\partial y}, 1\rangle \, dx\,dy$$

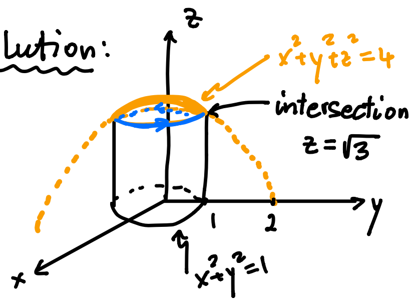
$$= \iint_{x^2+y^2\le 1} \langle 0, 0, 1+2y\rangle \cdot \langle 0, 1, 1\rangle \, dx\,dy$$

$$= \int_0^{2\pi}\int_0^1 (1+2\overset{r\cos\theta}{y})\, r\,dr\,d\theta = \int_0^{2\pi}\int_0^1 (1+2r\cos\theta)\, r\,dr\,d\theta =$$

$$= 2\pi\left[\frac{r^2}{2}\right]_0^1 + 2\left[\frac{r^3}{3}\right]_0^1 \underbrace{\int_0^{2\pi}\cos\theta\, d\theta}_{=0} = \underline{\underline{\pi}}$$

Example: Use Stokes' Theorem to compute $\iint_S \operatorname{curl} \vec{F} \cdot d\vec{S}$ where S is the part of the sphere $x^2+y^2+z^2=4$ that lies inside the cylinder $x^2+y^2=1$ and above the xy plane, where $\vec{F} = xz\,\vec{i} + yz\,\vec{j} + xy\,\vec{k}$.

Solution:



$$\iint_S \operatorname{curl} \vec{F} \cdot d\vec{S} = \oint_C \vec{F} \cdot d\vec{r}$$

curve C is defined by $x^2+y^2=1$ at $z=\sqrt{3}$,

or, in parametric form:

$$\begin{cases} x = \cos t \\ y = \sin t \\ z = \sqrt{3} \end{cases} \qquad \underset{\substack{\uparrow \\ \text{parameter}}}{0 \le t \le 2\pi}$$

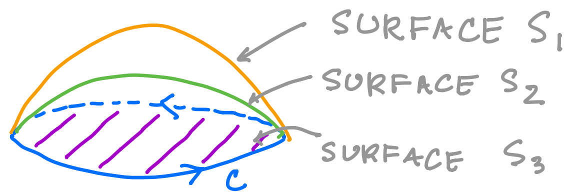
$$\oint_C \vec{F} \cdot d\vec{r} = \int_0^{2\pi} \langle \underbrace{\sqrt{3}\cos t}_{zx}, \underbrace{\sqrt{3}\sin t}_{zy}, \underbrace{\sin t \cos t}_{xy} \rangle \cdot \langle \underbrace{-\sin t}_{x'(t)}, \underbrace{\cos t}_{y'(t)}, \underbrace{0}_{z'(t)} \rangle dt$$

$$= \int_0^{2\pi} \left(-\sqrt{3}\sin t \cos t + \sqrt{3} \sin t \cos t\right) dt = 0$$

REMARK:

The integral of curl $\vec{F}$ over an oriented surface S_1 depends ONLY ON THE BOUNDARY CURVE C!

This means if we had another surface S_2, different from S_1, with the same boundary curve C, the integral of curl $\vec{F}$ over S_2 would be the SAME!



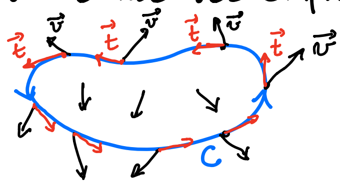
$$\iint_{S_1} \operatorname{curl} \vec{F} \cdot d\vec{S} = \iint_{S_2} \operatorname{curl} \vec{F} \cdot d\vec{S}$$

$$= \oint_C \vec{F} \cdot d\vec{r}$$

This is useful when it is difficult to integrate over one surface but it is easy to integrate over the other!

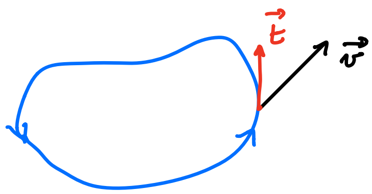
REMARK: MEANING OF THE CURL IN TERMS OF VELOCITY $\vec{v}$

Let $\vec{v}$ be the vector field describing velocity of some fluid.



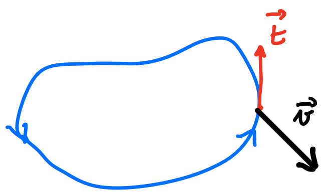
The integral $\boxed{\oint_C \vec{v}\cdot d\vec{r}}$ is called CIRCULATION

The integral $\oint_C \vec{v}\cdot d\vec{r} = \oint_C \vec{v}\cdot\vec{t}\,ds$ measures the TENDENCY OF THE FLUID TO MOVE AROUND C.



$\leftarrow \vec{v}\cdot\vec{t} > 0$

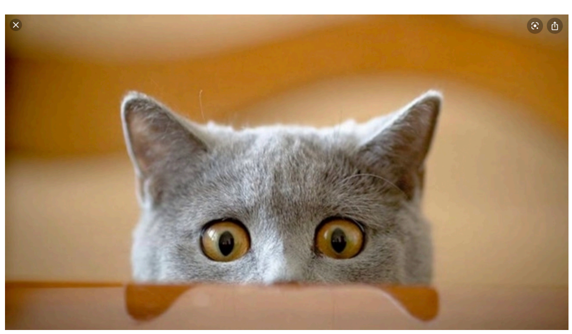
$\oint_C \vec{v}\cdot\vec{t}\,ds > 0 \Rightarrow$ POSITIVE CIRCULATION



$\leftarrow \vec{v}\cdot\vec{t} < 0$

$\oint_C \vec{v}\cdot\vec{t}\,ds < 0 \Rightarrow$ NEGATIVE CIRCULATION

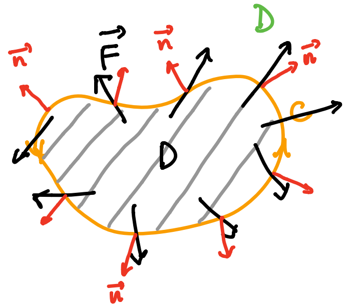
MOVING ON!



16.9. THE DIVERGENCE THEOREM

Recall: GREEN'S THEOREM (2nd VECTOR FORM) (in $\mathbb{R}^2$):

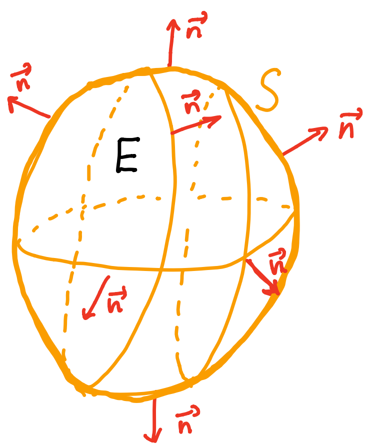
$$\oint_C \vec{F}\cdot\vec{n}\,ds = \iint_D \operatorname{div}\vec{F}(x,y)\,dA$$



POSITIVELY ORIENTED $\partial D = C$

If $\vec{F} = \rho\vec{v}$, $\vec{v}$ = fluid velocity, ρ = fluid density, then the divergence of mass from the fluid region D can be measured by the MASS FLUX $\rho\vec{v}\cdot\vec{n}$ through the boundary $\partial D = C$.

GENERALIZATION TO $\mathbb{R}^3$



E = SIMPLE SOLID REGION
(simultaneously of type I, II, and III)

S = BOUNDARY SURFACE WITH POSITIVE (OUTWARD) ORIENTATION

$\vec{F}$ has continuous partial derivatives

Then,

$$\boxed{\iint_S \vec{F}\cdot d\vec{S} = \iiint_E \operatorname{div}\vec{F}\,dV}$$

THE DIVERGENCE THEOREM

THE FLUX OF $\vec{F}$ ACROSS S IS EQUAL TO THE DIVERGENCE OF $\vec{F}$ INTEGRATED OVER THE ENTIRE SOLID BOUNDED BY S.

Example: Find the flux of $\vec{F} = z\vec{i} + y\vec{j} + x\vec{k}$ across the unit sphere $x^2 + y^2 + z^2 = 1$.

Solution: $\iint_S \vec{F}\cdot d\vec{S} = \text{FLUX} = \iiint_E \text{div}\,\vec{F}\,dV$

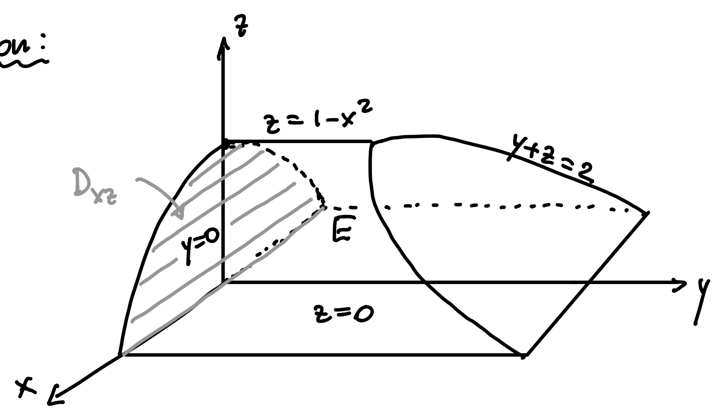
DIV THEOREM

$$\text{div}\,\vec{F} = \frac{\partial}{\partial x} z + \frac{\partial}{\partial y} y + \frac{\partial}{\partial z} x = 1$$

Thus: FLUX $\iint_S \vec{F}\cdot d\vec{S} = \iiint_V 1\,dV =$ VOLUME OF UNIT SPHERE $= \dfrac{4\pi}{3}$

Example: Evaluate $\iint_S \vec{F}\cdot d\vec{S}$, where $\vec{F} = \langle xy,\ y^2 + e^{xz^2},\ \sin(xy)\rangle$ and S is the surface boundary of the region E bounded by the parabolic cylinder $z = 1 - x^2$, and the planes $z=0$, $y=0$, and $y + z = 2$.

Solution:

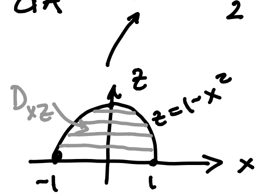


Several surfaces $\Rightarrow$ easier to integrate over the solid E.

$$\text{div}\,\vec{F} = y + 2y + 0 = 3y$$

$$\iint_S \vec{F}\cdot d\vec{S} = \iiint_E \text{div}\,\vec{F}\,dV = \iiint_E 3y\,dV = \iint_{D_{xz}} \int_0^{2-z} 3y\,dy\,dA$$

$$= \iint_{D_{xz}} 3\left[\frac{y^2}{2}\right]_0^{2-z} dA = \frac{3}{2}\iint_{D_{xz}} (2-z)^2\, dA = \frac{3}{2}\int_{-1}^{1}\int_{0}^{1-x^2} (2-z)^2\, dz\, dx$$



$$= \frac{3}{2}\int_{-1}^{1}\left[-\frac{(2-z)^3}{3}\right]_0^{1-x^2} dx =$$

$$= \frac{1}{2}\int_{-1}^{1}\left[\,2^3 - (2-(1-x^2))^3\,\right] dx = \frac{1}{2}\int_{-1}^{1}\left[8 - (1+x^2)^3\right] dx =$$

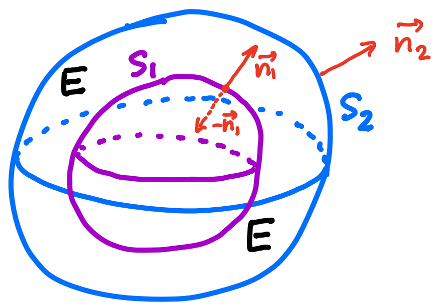
= binomial expansion $(1+b)^3 = 1-3b+3b^2-b^3$ =

$$= \frac{1}{2}\quad 8\,[x]_{-1}^{1} - \frac{1}{2}\int_{-1}^{1}(1-3x^2+3x^4-x^6)\, dx =$$

$$= \quad 8 \quad - \frac{1}{2}\left[x - 3\frac{x^3}{3} + 3\frac{x^5}{5} - \frac{x^7}{7}\right]_{-1}^{1} = \frac{184}{35}$$

DIVERGENCE THEOREM CAN BE EXTENDED TO REGIONS THAT ARE FINITE UNIONS OF SIMPLE REGIONS. ONE EXAMPLE IS:

DIVERGENCE THEOREM FOR A REGION BOUNDED BY AN INTERIOR AND AN EXTERIOR SURFACE



S_1 and S_2 are two positively oriented closed surfaces with (outward) normals $\vec{n}_1$ and $\vec{n}_2$.

E is the solid region between S_1 and S_2.

THE OUTWARD NORMAL TO THE BOUNDARY OF E IS GIVEN BY:

$$\vec{n} = \begin{cases} \vec{n}_2 & \text{on } S_2 \quad (\text{exterior surface}) \\ -\vec{n}_1 & \text{on } S_1 \quad (\text{interior surface}) \end{cases}$$

THE DIVERGENCE THEOREM:

OUTWARD UNIT NORMAL TO ∂E !!!

$$\iiint_E \operatorname{div} \vec{F}\, dV = \iint_{\partial E} \vec{F} \cdot d\vec{S} = \iint_{S_1 \cup S_2} \vec{F} \cdot \vec{n}\, dS =$$

$$= \iint_{S_1} \vec{F} \cdot (-\vec{n}_1)\, dS + \iint_{S_2} \vec{F} \cdot \vec{n}_2\, dS$$

$$= \iint_{S_2} \vec{F} \cdot d\vec{S} - \iint_{S_1} \vec{F} \cdot d\vec{S}$$

THUS:

$$\boxed{\iiint_E \operatorname{div} \vec{F}\, dV = \iint_{S_2\,(\text{exterior})} \vec{F} \cdot d\vec{S} - \iint_{S_1\,(\text{interior})} \vec{F} \cdot d\vec{S}}$$