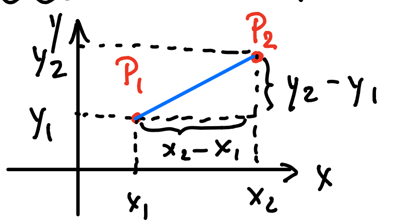
12.1. DISTANCES AND SPHERES

Recall: In $\mathbb{R}^2$, the distance between $P_1(x_1, x_2)$ and $P_2(x_2, y_2)$:

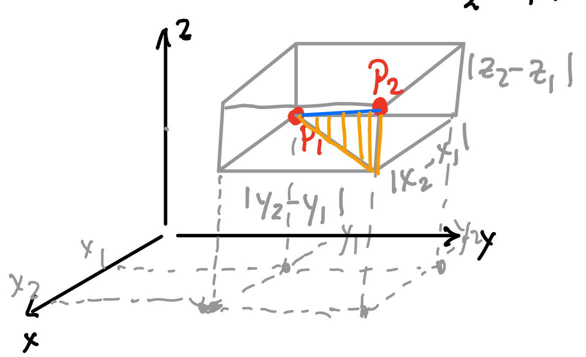


$$\boxed{|P_1P_2| = \sqrt{(x_2-x_1)^2 + (y_2-y_1)^2}}$$

In $\mathbb{R}^3$, similarly:

$$\boxed{|P_1P_2| = \sqrt{(x_2-x_1)^2 + (y_2-y_1)^2 + (z_2-z_1)^2}}$$

Rectangurar box with sides: $|x_2-x_1|$, $|y_2-y_1|$, $|z_2-z_1|$



EXAMPLE: Find an equation of a sphere with radius r and center $C(h,k,l)$.

Answer: By definition, a sphere is a set of all points $P(x,y,z)$ such that the distance $|PC| = r$.

Using the distance formula $\Rightarrow$ $\sqrt{(x-h)^2+(y-k)^2+(z-\ell)^2} = r$

Squaring both sides $\Rightarrow$ $\boxed{(x-h)^2+(y-k)^2+(z-\ell)^2 = r^2}$

Recommended: Do EXAMPLE 6 on page 796!!!

EXAMPLE: What region in $\mathbb{R}^3$ is represented by the following inequalities:

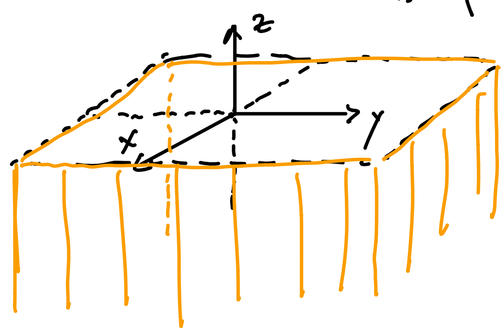
$$\begin{cases} 1 \le x^2+y^2+z^2 \le 9 \\ z \le 0 \end{cases}$$

Solution: 1. Recall $x^2+y^2+z^2=9$ is a sphere of radius 3
$x^2+y^2+z^2=1$ is a sphere of radius 1

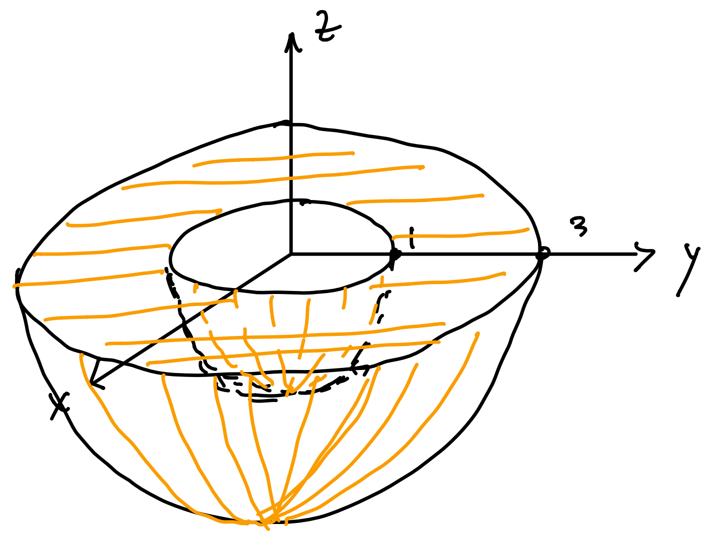
2. $x^2+y^2+z^2 \le 9$ is the sphere of radius 3 PLUS the INTERIOR of the sphere
$x^2+y^2+z^2 \ge 1$ is the sphere of radius 1 PLUS the EXTERIOR of the sphere

3. $1 \le x^2+y^2+z^2 \le 9$ contains the points that lie BOTH in the interior of the sphere of radius 3, and the exterior of the sphere of radius 1.

4. $z \le 0$ describes the half space where $z \le 0$



5. Thus: $\left\{\begin{array}{l} 1 \le x^2 + y^2 + z^2 \le 9 \\ z \le 0 \end{array}\right\}$ contains all the points inside the sphere with $r = 3$, outside the sphere with $r = 1$, and below the xy plane:



MOVING ON !

12.2. VECTORS

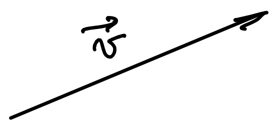
VECTOR is a quantity that has both:

- MAGNITUDE (LENGTH) and
- DIRECTION

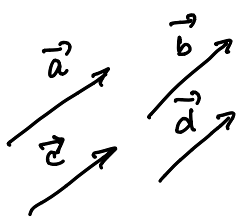
Examples: velocity, force, displacement, ...

REPRESENTATION OF VECTORS: ARROW, or A DIRECTED LINE SEGMENT

DENOTING VECTORS: bold letters, or an arrow above letter



EQUIVALENT VECTORS: VECTORS WITH THE SAME LENGTH AND DIRECTION ARE CALLED EQUIVALENT (OR EQUAL)



How many vectors?

($\vec{a}=\vec{b}=\vec{c}=\vec{d}$; ONE VECTOR REPRESENTED BY FOR LINE SEGMENTS)

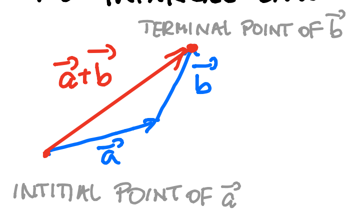
WE CAN DO LOTS WITH VECTORS

- ADDITION
- SUBTRACTION
- MULTIPLICATION BY A NUMBER

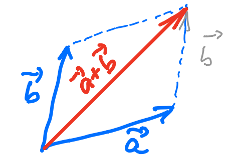
etc

VECTOR ADDITION:

THE TRIANGLE LAW



THE PARALLELOGRAM LAW



SUBTRACTION:

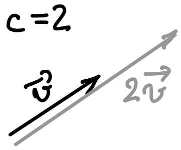
What is $\vec{a}-\vec{b}$?

$$\vec{a}-\vec{b} = \vec{a} + \underbrace{(-1)\vec{b}}_{?}$$

SCALAR MULTIPLICATION:

Multiplication of a vector by a real number, also called a scalar (to distinguish it from a vector)

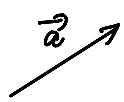
$c=2$



If c is a scalar and $\vec{v}$ a vector, then the SCALAR MULTIPLE $c\vec{v}$ is a vector such that

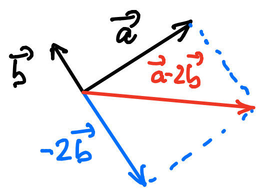
- LENGTH OF $c\vec{v}$ = $|c|$ times the length of $\vec{v}$
- DIRECTION OF $c\vec{v}$: $\begin{cases} \text{SAME as direction of } \vec{v} \text{ if } c>0 \\ \text{OPPOSITE DIRECTION of } \vec{v} \text{ if } c<0 \end{cases}$
- IF $c=0$ THEN $c\vec{v}=\vec{0}$ (THE ZERO VECTOR)

EXAMPLE: What is $\vec{a}-2\vec{b}$ if



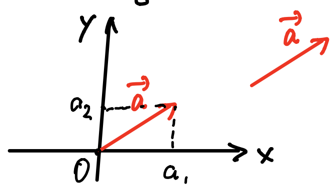
:

Solution:



VECTOR COMPONENTS

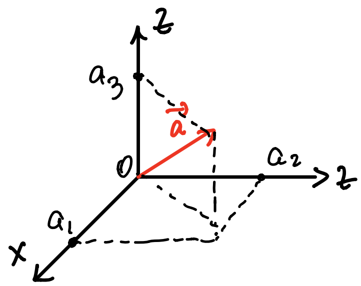
- describing vectors in a coordinate system



1. Place the initial point of $\vec{a}$ at the origin O
2. Then the terminal point of $\vec{a}$ defines a point with coordinates

   (a_1, a_2) in $\mathbb{R}^2$

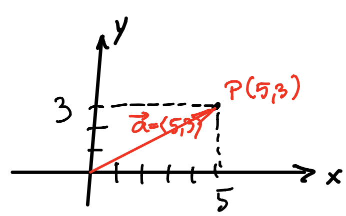
   (a_1, a_2, a_3) in $\mathbb{R}^3$



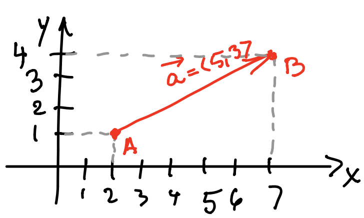
3. These coordinates define COMPONENTS of $\vec{a}$, and we write:

$$\boxed{\begin{array}{l} \vec{a} = \langle a_1, a_2 \rangle \text{ in } \mathbb{R}^2 \\ \vec{a} = \langle a_1, a_2, a_3 \rangle \text{ in } \mathbb{R}^3 \end{array}}$$

EXAMPLE: Sketch $\vec{a} = \langle 5, 3 \rangle$



ANY OTHER REPRESENTATION OF $\vec{a} = \langle 5, 3 \rangle$?



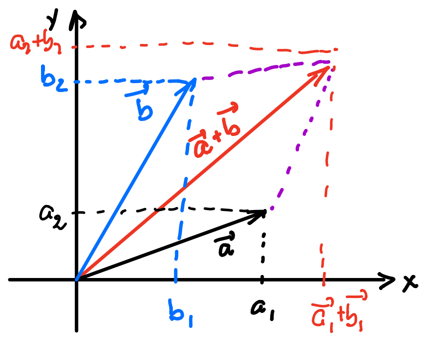
GIVEN THE POINTS $A(x_1, y_1, z_1)$ and $B(x_2, y_2, z_2)$ THE VECTOR $\vec{a}$ WITH REPRESENTATION $\vec{AB}$ IS:

$$\boxed{\vec{AB} = \langle x_2 - x_1, y_2 - y_1, z_2 - z_1 \rangle}$$

LENGTH: In $\mathbb{R}^2$: $\vec{a} = \langle a_1, a_2 \rangle$, $|\vec{a}| = \sqrt{a_1^2 + a_2^2}$

In $\mathbb{R}^3$: $\vec{a} = \langle a_1, a_2, a_3 \rangle$, $|\vec{a}| = \sqrt{a_1^2 + a_2^2 + a_3^2}$

ADDING VECTORS ALGEBRAICALLY



If $\vec{a}=\langle a_1,a_2\rangle$, $\vec{b}=\langle b_1,b_2\rangle$ then:

- $\vec{a}+\vec{b}=\langle a_1+b_1\ ,\ a_2+b_2\rangle$
- $\vec{a}-\vec{b}=\langle a_1-b_1\ ,\ a_2-b_2\rangle$

If c is a scalar, then:

- $c\vec{a}=\langle ca_1\ ,\ ca_2\rangle$

NOTATION:

V_2 - set of all two-dimensional vectors

V_3 - set of all three-dimensional vectors

V - set of all n-dimensional vectors $\vec{v}=\langle v_1, v_2, \ldots, v_n\rangle$

PROPERTIES OF VECTORS

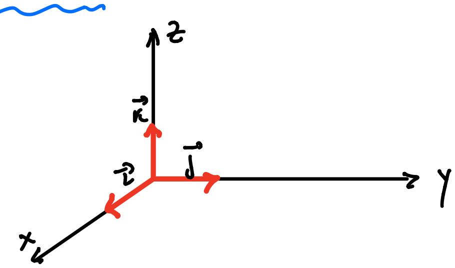
If $\vec{a},\vec{b},\vec{c}$ are vectors in V_n, and c and d are salars. Then:

1. $\vec{a}+\vec{b}=\vec{b}+\vec{a}$
2. $\vec{a}+(\vec{b}+\vec{c})=(\vec{a}+\vec{b})+\vec{c}$
3. $\vec{a}+\vec{0}=\vec{a}$
4. $\vec{a}+(-\vec{a})=\vec{0}$
5. $c(\vec{a}+\vec{b})=c\vec{a}+c\vec{b}$
6. $(c+d)\vec{a}=c\vec{a}+d\vec{a}$
7. $(cd)\vec{a}=c(d\vec{a})$

Easily verified geometrically.

SPECIAL VECTORS: THE STANDARD BASIS VECTORS

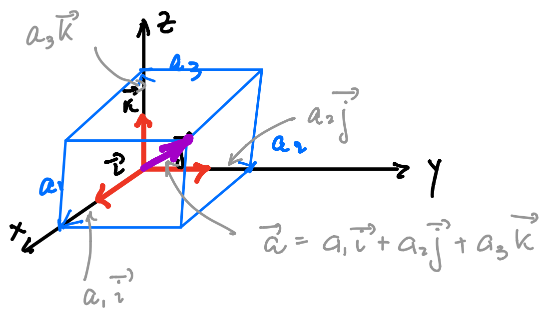
$\vec{i}=\langle 1,0,0\rangle$
$\vec{j}=\langle 0,1,0\rangle$
$\vec{k}=\langle 0,0,1\rangle$



ANY VECTOR CAN BE REPREZENTED IN TERMS OF $\vec{i}, \vec{j}, \vec{k}$

EXAMPLE: $\vec{a} = \langle a_1, a_2, a_3 \rangle$ can be represented as:

$$\vec{a} = a_1 \vec{i} + a_2 \vec{j} + a_3 \vec{k}$$



EXAMPLE: $\vec{a} = \vec{i} + 3\vec{j} - 5\vec{k}$, $\vec{b} = 2\vec{j} + \vec{k}$. Express $\vec{a} + 2\vec{b}$ in terms of $\vec{i}, \vec{j}, \vec{k}$.

Solution:

$$\vec{a} + 2\vec{b} = \vec{i} + 3\vec{j} - 5\vec{k} + 2(2\vec{j} + \vec{k}) = \vec{i} + 3\vec{j} - 5\vec{k} + 4\vec{j} + 2\vec{k} =$$

$$= \vec{i} + (3+4)\vec{j} + (-5+2)\vec{k} = \vec{i} + 7\vec{j} - 3\vec{k} \;//$$

UNIT VECTOR

= a vector of Length 1

If $\vec{a} \neq \vec{0}$, then the unit vector that has the SAME DIRECTION as $\vec{a}$ is:

$$\boxed{\vec{u} = \frac{\vec{a}}{|\vec{a}|} = \underbrace{\frac{1}{|\vec{a}|}}_{\text{scalar}} \vec{a}}$$

EXAMPLE: Find the unit vector in the direction of $\vec{a} = 2\vec{i} - \vec{j} - 2\vec{k}$.

Solution: $\vec{u} = \frac{1}{|\vec{a}|} \vec{a}$, $|\vec{a}| = \sqrt{2^2 + (-1)^2 + (-2)^2} = \sqrt{\cancel{5}9}$

$$\vec{u} = \frac{1}{\underset{3}{\cancel{\sqrt{5}}}}(2\vec{i} - \vec{j} - 2\vec{k}) = \frac{2}{\underset{3}{\cancel{\sqrt{5}}}}\vec{i} - \frac{1}{\underset{3}{\cancel{\sqrt{5}}}}\vec{j} - \frac{2}{\underset{3}{\cancel{\sqrt{5}}}}\vec{k} \;//$$

MULTIPLICATION OF VECTORS

THE DOT PRODUCT (Section 12.3)

THE CROSS PRODUCT (Section 12.4)

THE DOT PRODUCT

Definition

If $\vec{a}=\langle a_1, a_2, a_3\rangle$ and $\vec{b}=\langle b_1, b_2, b_3\rangle$, then THE DOT PRODUCT OF $\vec{a}$ AND $\vec{b}$ is a number

$$\vec{a}\cdot\vec{b} = a_1b_1 + a_2b_2 + a_3b_3$$

Since the result is NOT A VECTOR, but a scalar, the dot product is sometimes called the SCALAR PRODUCT or INNER PRODUCT.

Example: $\vec{a}=\vec{i}+\vec{j}$, $\vec{b}=\vec{j}+3\vec{k}$. $\vec{a}\cdot\vec{b}=?$

Solution: $\vec{a}\cdot\vec{b} = 1\cdot 0 + 1\cdot 1 + 0\cdot 3 = 1$

Notice: $\vec{a}=\vec{i}+\vec{j}$ in $\mathbb{R}^3$ is $\vec{a}=\vec{i}+\vec{j}+0\vec{k}$

Example: $\vec{a}=\vec{i}+\vec{j}$. What is $\vec{a}\cdot\vec{a}$?

Solution: $\vec{a}\cdot\vec{a} = 1\cdot 1 + 1\cdot 1 = 2$

In general: $\vec{a}=\langle a_1, a_2, a_3\rangle$, $\vec{a}\cdot\vec{a} = (a_1)^2 + (a_2)^2 + (a_3)^2 = |\vec{a}|^2$

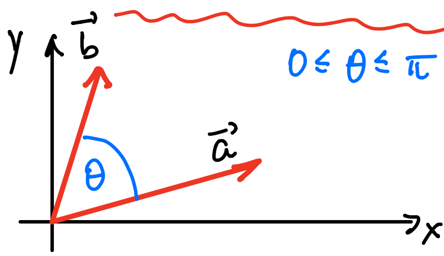
$$\boxed{\vec{a}\cdot\vec{a} = |\vec{a}|^2}$$

PROPERTIES OF THE DOT PRODUCT

1. $\vec{a}\cdot\vec{a} = |\vec{a}|^2$
2. $\vec{a}\cdot\vec{b} = \vec{b}\cdot\vec{a}$
3. $\vec{a}\cdot(\vec{b}+\vec{c}) = \vec{a}\cdot\vec{b}+\vec{a}\cdot\vec{c}$
4. $(c\vec{a})\cdot\vec{b} = c(\vec{a}\cdot\vec{b}) = \vec{a}\cdot(c\vec{b})$
5. $\vec{0}\cdot\vec{a} = \vec{0}$

For fun, prove these properties for vectors in $\mathbb{R}^2$!

GEOMETRIC INTERPRETATION



$0 \le \theta \le \pi$

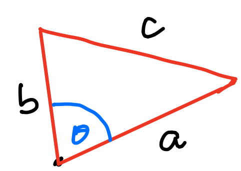
THEOREM: If θ is the angle between vectors $\vec{a}$ and $\vec{b}$, then

$$\boxed{\vec{a}\cdot\vec{b} = |\vec{a}|\,|\vec{b}|\cos\theta}$$

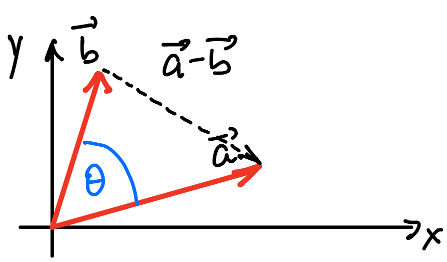
Proof: THE LAW OF COSINES:

$$\boxed{c^2 = a^2+b^2-2ab\cos\theta}$$

SIDE OPPOSITE TO θ



- EXPRESS THE LAW OF COSINES IN TERMS OF VECTORS:



$a^2 = |\vec{a}|^2, \quad b^2 = |\vec{b}|^2$

$c^2 = ?$

$c^2 = |\vec{a}-\vec{b}|^2$

So: $\boxed{|\vec{a}-\vec{b}|^2 = |\vec{a}|^2+|\vec{b}|^2-2|\vec{a}|\,|\vec{b}|\cos\theta}$ (★)

- Using the properties of the dot product we calculate:

$$|\vec{a}-\vec{b}|^2 \underset{\text{PROPERTY 1}}{=} (\vec{a}-\vec{b})\cdot(\vec{a}-\vec{b}) \underset{\text{PROPERTY 3}}{=} (\vec{a}-\vec{b})\cdot\vec{a} - (\vec{a}-\vec{b})\cdot\vec{b} =$$

$$= \vec{a}\cdot\vec{a} - \vec{b}\cdot\vec{a} - \vec{a}\cdot\vec{b} + \vec{b}\cdot\vec{b} \underset{\text{PROPERTY 2}}{=} \vec{a}\cdot\vec{a} - \vec{a}\cdot\vec{b} - \vec{a}\cdot\vec{b} + \vec{b}\cdot\vec{b}$$

$$\underset{\text{PROPERTY 1}}{=} |\vec{a}|^2 - 2\,\vec{a}\cdot\vec{b} + |\vec{b}|^2$$

- GOING BACK TO EQUATION ($\star$):

$$|\vec{a}|^2 - 2\vec{a}\cdot\vec{b} + |\vec{b}|^2 = |\vec{a}|^2 + |\vec{b}|^2 - 2|\vec{a}||\vec{b}|\cos\theta$$

THUS:

$$-2\,\vec{a}\cdot\vec{b} = -2|\vec{a}||\vec{b}|\cos\theta \qquad / \div(-2)$$

$$\Rightarrow \boxed{\vec{a}\cdot\vec{b} = |\vec{a}||\vec{b}|\cos\theta} \quad //$$

CONSEQUENCE: For $\vec{a} \neq \vec{0}$, $\vec{b} \neq \vec{0}$ we have

$$\boxed{\cos\theta = \frac{\vec{a}}{|\vec{a}|}\cdot\frac{\vec{b}}{|\vec{b}|}}$$

EXAMPLE: Let $\vec{a} = \vec{i}$, $\vec{b} = \vec{i} + \vec{j}$. What is the angle between $\vec{a}$ and $\vec{b}$?

Solution: $\cos\theta = \frac{\vec{a}}{|\vec{a}|}\cdot\frac{\vec{b}}{|\vec{b}|} = \frac{\vec{i}}{1}\cdot\frac{\vec{i}+\vec{j}}{\sqrt{2}} = \frac{1}{\sqrt{2}}\langle 1,0\rangle\cdot\langle 1,1\rangle$

$= \frac{1}{\sqrt{2}}(1+0) = \frac{\sqrt{2}}{2} \Rightarrow \theta = \frac{\pi}{4}$ //

PERPENDICULAR OR ORTHOGONAL VECTORS

Definition: If $\vec{a} \neq \vec{0}$, $\vec{b} \neq \vec{0}$, $\vec{a}$ is PERPENDICULAR TO $\vec{b}$ if $\theta = \frac{\pi}{2}$.

In terms of the dot product:

$$\vec{a} \cdot \vec{b} = |\vec{a}||\vec{b}| \cos \frac{\pi}{2} = 0$$

The $\vec{0}$ -vector is perpendicular to all vectors.

THUS:

$$\boxed{\text{TWO VECTORS } \vec{a} \text{ AND } \vec{b} \text{ ARE ORTHOGONAL IF AND ONLY IF } \vec{a} \cdot \vec{b} = 0}$$

EXAMPLE: Are $\vec{a} = 2\vec{i} + 2\vec{j} - \vec{k}$ and $\vec{b} = 5\vec{i} - 4\vec{j} + 2\vec{k}$ perpendicular?

Solution: $\vec{a} \cdot \vec{b} = \langle 2, 2, -1 \rangle \cdot \langle 5, -4, 2 \rangle = 10 - 8 - 2 = 0$ //

YES!