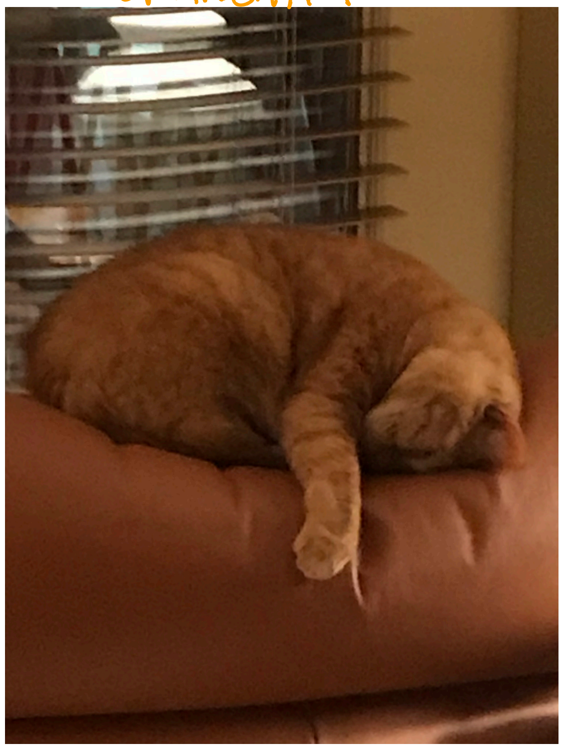
FINISH PROBLEMS FROM LECTURE 13

ORANGINA

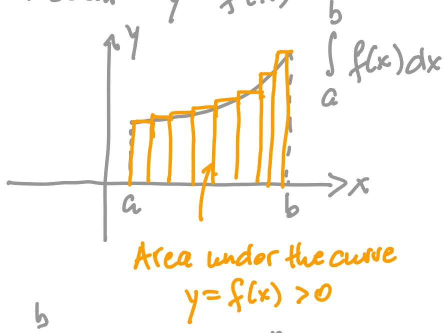


15. MULTIPLE INTEGRALS

15.1. DOUBLE INTEGRALS OVER RECTANGLES

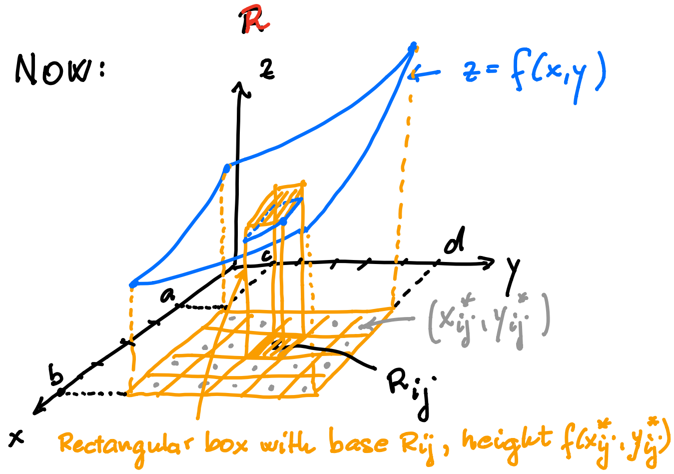
$R = [a,b] \times [c,d]$; what is $\iint_R f(x,y)\,dx\,dy$?

Recall: $y = f(x)$



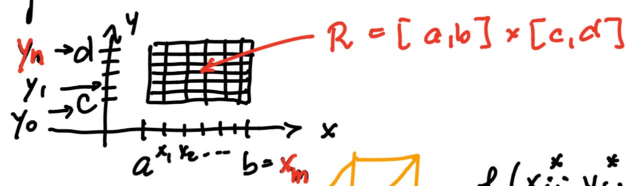
$$\int_a^b f(x)dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\, dx$$

Now:

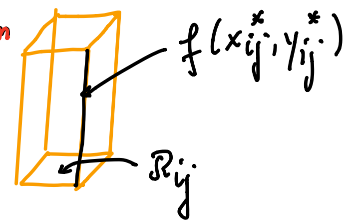


$$\iint_R f(x,y)\,dx\,dy = \text{VOLUME UNDER THE GRAPH OF } z=f(x,y),\text{ above the } xy \text{ plane}$$

$$\iint_R f(x,y)\,dx\,dy \approx \sum_{i=1}^{m}\sum_{j=1}^{n} V_{ij}$$ where m and n are the number of points in the subdivision of R:



where V_{ij} is the VOLUME of the column:



So: $$V_{ij} = \left(\text{Area of } R_{ij}\right)\cdot\left(f(x_{ij}^*, y_{ij}^*)\right)$$

Denote the small area of R_{ij} by ΔA. Then: $\boxed{V_{ij} = f(x_{ij}^*, y_{ij}^*)\,\Delta A}$

Thus:

$$\boxed{\iint_R f(x,y)\,\Delta A = \lim_{\substack{m\to\infty\\ n\to\infty}} \underbrace{\sum_{i=1}^{m}\sum_{j=1}^{n} f(x_{ij}^*, y_{ij}^*)\,\Delta A}}$$

DOUBLE RIEMANN SUM

If $f(x,y) \geq 0 \Rightarrow \iint_R f(x,y)\,dA$ = VOLUME OF THE SOLID ABOVE R AND BELOW $z = f(x,y)$

If the sample points x_{ij}^*, y_{ij}^* are chosen to be the CENTER of R_{ij}, in other words, if $\overline{x_i}$ is the MID POINT of $[x_{i-1}, x_i]$, and $\overline{y_j}$ is the MID POINT of $[y_{j-1}, y_j]$, the the approximation

$$\iint_R f(x,y)\,dx\,dy \approx \sum_{i=1}^{m}\sum_{j=1}^{n} f(\overline{x_i}, \overline{y_j})\,\Delta x\,\Delta y$$

is called THE MID POINT RULE FOR DOUBLE INTEGRALS.

HOW TO CALCULATE $\iint_R f(x,y)\,dx\,dy$

ITERATED INTEGRALS

- express double integral as an iterated integral which can be evaluated by calculating two single integrals

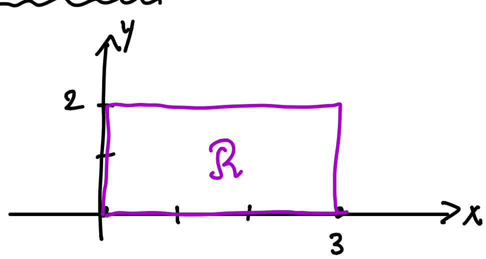
$$\iint_R f(x,y)\,dx\,dy = \int_c^d \int_a^b f(x,y)\,dx\,dy = \int_c^d \left[\int_a^b f(x,y)\,dx \right] dy$$

EXAMPLE: $\int_0^2 \int_0^3 (x^2 y)\,dx\,dy = ?$

Solution:

FREEZE y

$$\int_0^2 \left[\underbrace{\int_0^3 (x^2 y)\,dx} \right] dy =$$



FIRST INTEGRATE THE INTERIOR INTEGRAL BY FREEZING y (like partial differentiation)

$$= \int_0^2 \left[y \int_0^3 x^2\,dx \right] dy = \int_0^2 y \cdot \left[\frac{x^3}{3} \right]_0^3 dy = \int_0^2 y \left[\frac{3^3}{3} - \frac{0^3}{3} \right] dy =$$

$$= \int_0^2 y\,[9]\,dy = \underbrace{9 \int_0^2 y\,dy}_{\text{THIS IS NOW ONLY A FUNCTION OF } y} = 9 \left[\frac{y^2}{2} \right]_0^2 = 9 \left[\frac{2^2}{2} - \frac{0^2}{2} \right] = 18$$

The procedure where we integrate with respect to x by freezing y is called PARTIAL INTEGRATION.

FUBINI'S THEOREM: If $R = [a,b] \times [c,d]$ and f is bounded on R, discontinuous possibly at a finite number of smooth curves and such that the iteratet integrals exist, then

$$\boxed{\iint_R f(x,y)\,dA = \int_a^b \int_c^d f(x,y)\,dy\,dx = \int_c^d \int_a^b f(x,y)\,dx\,dy}$$

(the order of integration does not matter) 😎

EXAMPLE: $\displaystyle\iint_R y \sin(xy)\,dA = ?$ $\qquad R = [1,2] \times [0,\pi]$

Solution:

APPROACH 1: Integrate first with respect to x:

$$\iint_R y\sin(xy)\,dA = \int_0^\pi \int_1^2 y \sin(x,y)\,dx\,dy = \int_0^\pi \left[\int_1^2 y \sin(xy)\,dx\right] dy$$

$$= \int_0^\pi y \left[\int_1^2 \sin(xy)\,dx\right] dy = \left|\begin{array}{l}\text{SUBSTITUTION:} \\ u = xy \\ du = y\,dx \\ \Rightarrow dx = \frac{1}{y}\,du\end{array}\right| = \int_0^\pi y \left[\int \sin(u)\frac{1}{y}\,du\right] dy$$

$$= \int_0^\pi \cancel{y}\cdot\frac{1}{\cancel{y}}\left[\int \sin(u)\,du\right] dy = \int_0^\pi \left[-\cos(u)\right] dy = \int_0^\pi \left[-\cos(xy)\right]_{x=1}^{x=2} dy =$$

$$= \int_0^\pi \left[-\cos(2y) + \cos(y)\right] dy = \left[-\sin(2y)\cdot\frac{1}{2}\right]_{y=0}^{\pi} + \left[\sin y\right]_{y=0}^{\pi}$$

$= -\frac{1}{2}\left[\sin 2\pi - \sin 0\right] + \left[\sin \pi - \sin 0\right] = 0$

APPROACH 2: Integrate first with respect to y:

$$\iint_R y\sin(xy)\,dA = \int_1^2 \Bigg[\underbrace{\int_0^{\pi} y\sin(xy)\,dy}_{\text{INTEGRATION BY PARTS TWICE!}}\Bigg]\,dx$$

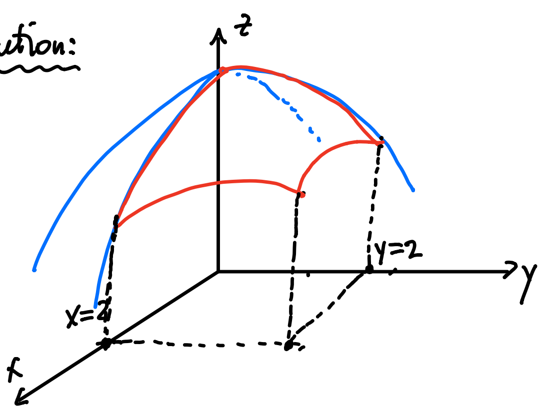
This approach is much more difficult!

$\Rightarrow$ $\boxed{\text{When evaluating double integrals, choose the order of integration that gives simpler integrals! USE FUBINI TO YOUR ADVANTAGE!}}$

Recall: INTEGRATION BY PARTS: $\displaystyle\int_a^b u\,dv = \big[uv\big]_a^b - \int_a^b v\,du$

EXAMPLE: Find the volume of the solid S that is bounded by the elliptic paraboloid $x^2+2y^2+z=16$, the planes $x=2$, and $y=2$, and the three coordinate planes.

Solution:



$z = 16-(x^2+2y^2) = f(x,y)$

$R = [0,2]\times[0,2]$

$V = \iint_R f(x,y)\,dA$

$$V = \int_0^2 \int_0^2 (16 - x^2 - 2y^2)\, dx\, dy = \int_0^2 \Big[\underbrace{\int_0^2 (16 - x^2 - 2y^2)\, dx}_{\text{KEEP } y \text{ FIXED}} \Big] dy =$$

$$= \int_0^2 \left[16x - \frac{x^3}{3} - 2y^2 x \right]_{x=0}^{2} dy =$$

$$= \int_0^2 \left[16 \cdot 2 - \frac{2^3}{3} - 2y^2 \cdot 2 - 0 \right] dy = \int_0^2 \Big[\underbrace{32 - \tfrac{8}{3}}_{\frac{88}{3}} - 4y^2 \Big] dy =$$

$$= \left[\frac{88}{3} y + 4 \frac{y^3}{3} \right]_{y=0}^{2} = \frac{88}{3} \cdot 2 + \frac{4}{3} 2^3 - 0 = 48$$

SPECIAL CASE : $\boxed{f(x,y) = g(x) \cdot h(y)}$

$$\iint_{R=[a,b]\times[c,d]} f(x,y)\, dA = \iint_R g(x)\, h(y)\, dx\, dy = \left(\int_a^b g(x)\, dx \right) \cdot \left(\int_c^d h(y)\, dy \right)$$

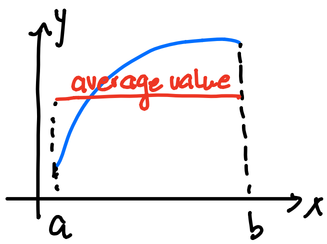
EXAMPLE: $R = [0, \frac{\pi}{2}] \times [0, \frac{\pi}{2}]$, $\iint_R \sin x \cos y \, dA = ?$

Solution:

$$\iint_R \sin x \cos y \, dA = \left(\int_0^{\pi/2} \sin x \, dx \right) \left(\int_0^{\pi/2} \cos y \, dy \right) = \dots\dots = 1$$

AVERAGE VALUE

Recall: Average of $y=f(x)$ over $[a,b]$ is

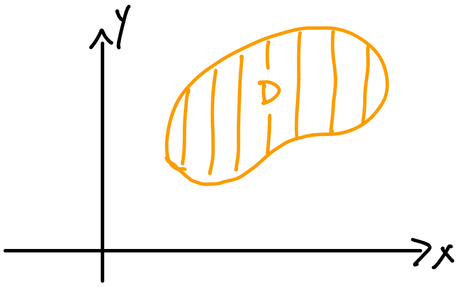


$$\boxed{f_{ave} = \frac{1}{b-a}\int_a^b f(x)\,dx}$$

Similarly: AVERAGE OF $z=f(x,y)$ OVER $R=[a,b]\times[c,d]$ is:

$$\boxed{f_{AVE} = \frac{1}{\text{Area of } R}\iint_R f(x,y)\,dA}$$

15.2. DOUBLE INTEGRALS OVER GENERAL REGIONS

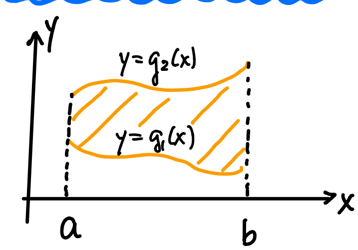


$$\iint_D f(x,y)\,dA = ?$$

FORMULAS FOR CALCULATING $\boxed{\iint_D f(x,y)\,dA}$ **OVER 2 TYPES OF REGIONS:**

(I) D IS OF TYPE I

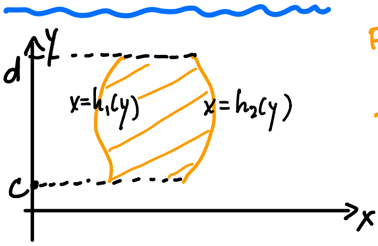
if D lies between the graphs of two CONTINUOUS FUNCTIONS OF x:



$$D = \{(x,y) \mid a \le x \le b,\ g_1(x) \le y \le g_2(x)\}$$

(II) D IS OF TYPE II

if D lies between the graps of two CONTINUOUS FUNCTIONS OF y:



$$D = \{(x,y) \mid c \le y \le d,\ h_1(y) \le x \le h_2(y)\}$$

INTEGRATION FORMULAS

TYPE I REGION

$$\boxed{\iint_D f(x,y)\,dA = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x,y)\,dy\,dx}$$

TYPE II REGION

$$\boxed{\iint_D f(x,y)\,dA = \int_c^d \int_{h_1(y)}^{h_2(y)} f(x,y)\,dx\,dy}$$

EXAMPLE: Find the volume of the solid that lies under the paraboloid $z = x^2 + y^2$ and above the region D in the xy plane, bounded by $y = 2x$ and the parabola $y = x^2$.