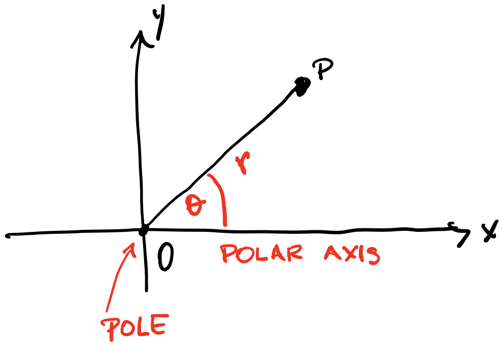
Section 10.3: $\boxed{\text{POLAR COORDINATES}}$ 🤗

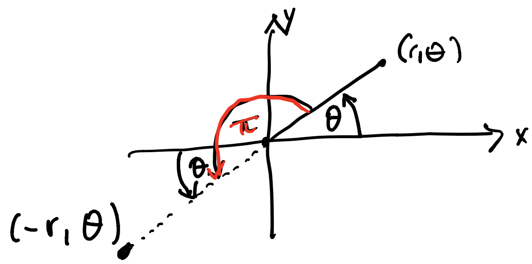
Polar coordinates are a different way to characterize $\mathbb{R}^2$.



- r is the distance from O to P
- θ is the angle between the line OP and polar axis

POLAR COORDINATE OF P: (r, θ)

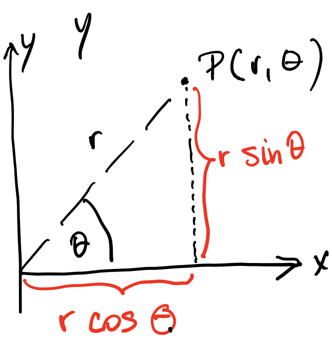
WE EXTEND THE MEANING OF (r,θ) TO THE CASE IN WHICH $r<0$:



THUS:

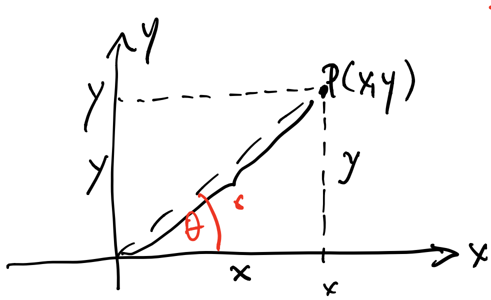
$$(-r,\theta) = (r, \theta + \pi)$$

TRANSFORMATION: POLAR-TO-CARTESIAN COORDINATES



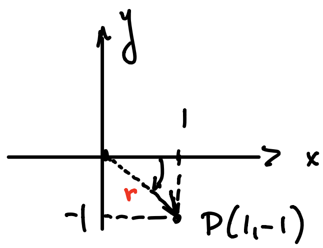
$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases}$$

TRANSFORMATION: CARTESIAN-TO-POLAR COORDINATES



$$\begin{cases} r = \sqrt{x^2 + y^2} \\ \tan\theta = \dfrac{y}{x} \end{cases}$$

EXAMPLE: Convert the point $P(1,-1)$ in Cartesian coordinates to polar coordinates.



$$r = \sqrt{1^2 + 1^2} = \sqrt{2}$$

$$\theta = -\frac{\pi}{4}$$

$$P\left(\sqrt{2}, -\frac{\pi}{4}\right),\ P\left(\sqrt{2}, \frac{7\pi}{4}\right),$$

$$P\left(-\sqrt{2}, \frac{3\pi}{4}\right) \ldots$$

GRAPHING IN POLAR COORDINATES

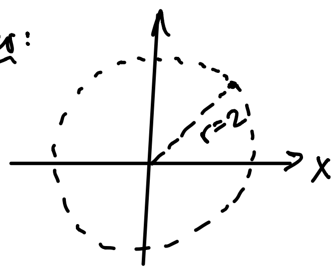
DISCLOSURE:

- NON - INTUITIVE
- IT'S A PAIN IN THE NECK
- BUT YOU SHOULD SEE IT ONCE IN YOUR LIFE
- AND TODAY IS THE DAY!

🥵

EXAMPLE: What is the curve $r=2$?

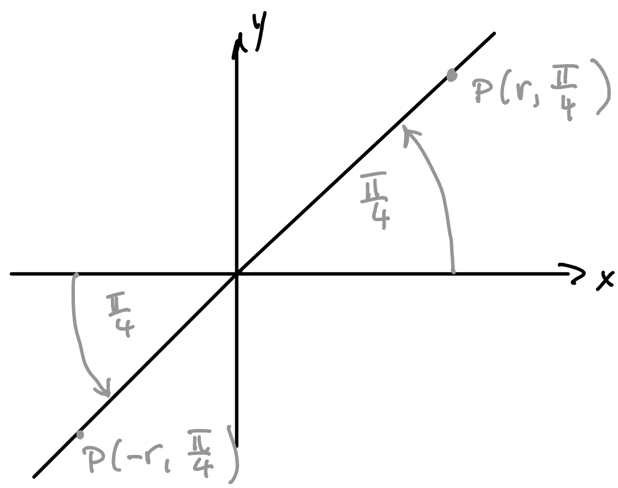
Answer:



CIRCLE OF RADIUS 2.

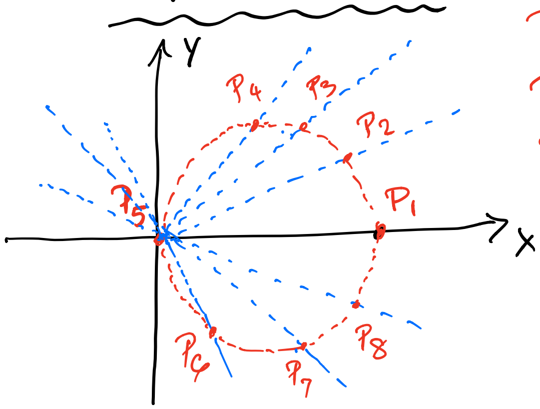
EXAMPLE: What does $\theta = \frac{\pi}{4}$ look like?

Answer:



EXAMPLE: (a) Sketch the curve $r = 2\cos\theta$.
(b) Find a Cartesian equation for this curve.

TRY SOME POINTS:



	θ	r	x	y
P_1	0	2	2	0
P_2	$\pi/6$	$\sqrt{3}$	$\sqrt{3}\cos \pi/6$	$\sqrt{3}\sin \pi/6$
P_3	$\pi/4$	$\sqrt{2}$	$\sqrt{2}\cos \pi/4$	$\sqrt{2}\sin \pi/4$
P_4	$\pi/3$	1	$\cos \pi/3$	$\sin \pi/3$
P_5	$\pi/2$	0	0	0
P_6	$2\pi/3$	-1	$-\cos 2\pi/3$	$-\sin \frac{2\pi}{3}$
P_7	$3\pi/4$	$-\sqrt{2}$	$-\sqrt{2}\cos 3\pi/4$	$-\sqrt{2}\sin 3\pi/4$
P_8	$5\pi/6$	$-\sqrt{3}$	$-\sqrt{3}\cos 5\pi/6$	$-\sqrt{3}\sin 5\pi/6$

(b) $r = 2\cos\theta$. Try to eliminate r, θ. Recall:

$$x = r\cos\theta.$$

Thus $\cos\theta = \frac{x}{r}$.

Eliminate $\cos\theta$ in $r = 2\cos\theta$ to obtain:

$$r = 2\frac{x}{r}.$$

Multiply by r to obtain:

$$r^2 = 2x$$

Recall: $r^2 = x^2 + y^2$, use it to eliminate r^2:

$$x^2 + y^2 = 2x.$$

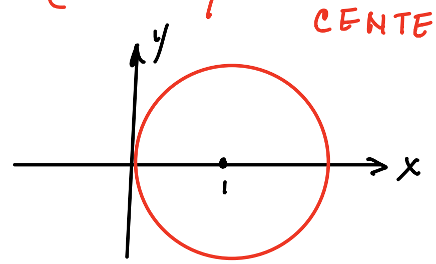
Rewrite so that it is easier to recognize this curve:

$x^2 - 2x + y^2 = 0$; complete the square:

$$\underbrace{x^2 - 2x + 1}_{(x-1)^2} \; \underbrace{-1}\; + y^2 = 0$$

$$(x-1)^2 + y^2 = 1$$

$(x-1)^2 + y^2 = 1$ CIRCLE OF RADIUS 1 CENTERED AT $(1,0)$

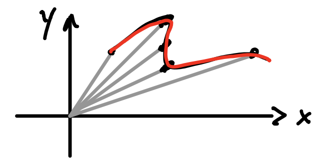


READ PAGES 662, 663

TANGENTS TO POLAR CURVES 😀

SUPPOSE WE ARE GIVEN A POLAR CURVE $r = r(\theta)$.

HOW CAN WE FIND A TANGENT?



STEP 1: We know how to find tangents to parametric curves

$$\left.\begin{array}{l} x = f(t) \\ y = g(t) \end{array}\right\}$$

We also know that in polar coordinates $\begin{cases} x = r\cos\theta \\ y = r\sin\theta. \end{cases}$

So, if we use $r = r(\theta)$, we get:

$$\begin{cases} x = r(\theta)\cos\theta & = f(\theta) \\ y = r(\theta)\sin\theta & = g(\theta) \end{cases}, \quad \theta = \text{PARAMETER}$$

STEP 2: FOR PARAMETRIC CURVES WITH θ DENOTING THE PARAMETER

$$\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}}.$$

WHAT ARE $\frac{dy}{d\theta}$ AND $\frac{dx}{d\theta}$?

$$\left.\begin{array}{l} \frac{dy}{d\theta} = r'(\theta)\sin\theta + r(\theta)\cos\theta \\ \frac{dx}{d\theta} = r'(\theta)\cos\theta - r(\theta)\sin\theta \end{array}\right\} \Rightarrow \boxed{\frac{dy}{dx} = \frac{r'(\theta)\sin\theta + r(\theta)\cos\theta}{r'(\theta)\cos\theta - r(\theta)\sin\theta}}$$

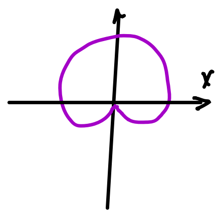
TANGENT:

$$y - y_0 = \frac{dy}{dx}(x_0, y_0)\,(x - x_0)$$

CARDIOID

EXAMPLE: $r = 1+\sin\theta$. Find the slope at $\theta = \pi/3$.

Answer: $\frac{dy}{dx}\left(\theta = \frac{\pi}{3}\right) = ?$



$$\frac{dy}{dx} = \frac{(1+\sin\theta)'\sin\theta + (1+\sin\theta)\cos\theta}{(1+\sin\theta)'\cos\theta - (1+\sin\theta)\sin\theta} = \frac{\overbrace{\cos\theta\sin\theta + \sin\theta\cos\theta}^{2\cos\theta\sin\theta} + \cos\theta}{\cos^2\theta - \sin^2\theta - \sin\theta}$$

$$\frac{dy}{dx}\left(\theta = \frac{\pi}{3}\right) = \frac{\frac{1}{2}(1+\sqrt{3})}{\frac{1}{4} - \frac{3}{4} - \frac{\sqrt{3}}{2}} = \frac{\frac{1}{2}(1+\sqrt{3})}{\frac{1}{2}\left(-\sqrt{3}+\frac{1}{2}-\frac{3}{2}\right)} = \frac{\cancel{1+\sqrt{3}}}{-\cancel{(1+\sqrt{3})}} = -1$$

RECALL:

$\cos\frac{\pi}{3} = \frac{1}{2}$

$\sin\frac{\pi}{3} = \frac{\sqrt{3}}{2}$

FINISH PART (b) IN EXAMPLE 9 ON PAGE 664 IN TEXTBOOK

SUMMARY

I POLAR COORDINATES

$x = r\cos\theta \qquad r = \sqrt{x^2+y^2}$

$y = r\sin\theta \qquad \tan\theta = y/x$

II PLOTTING IN POLAR COORDINATES

III TANGENT TO CURVE IN POLAR COORDINATES

$$\boxed{(y - y_0) = \frac{dy}{dx}(x_0)\,(x - x_0)}$$

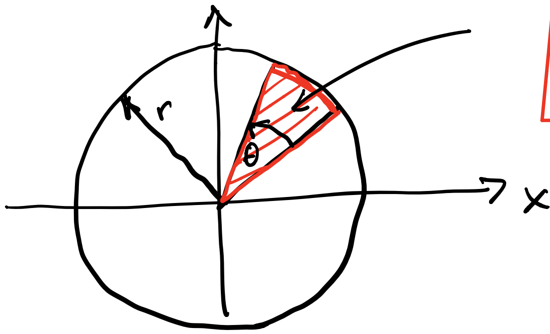
where:

$$\boxed{\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{\frac{dr}{d\theta}\sin\theta + r(\theta)\cos\theta}{\frac{dr}{d\theta}\cos\theta - r(\theta)\sin\theta}}$$

10.4. AREAS AND LENGTHS

1. AREAS

(I) Recall: THE AREA OF A SECTOR OF A CIRCLE:



$$\boxed{A = \frac{1}{2} r^2 \theta}$$

θ = RADIAN MEASURE of the central angle

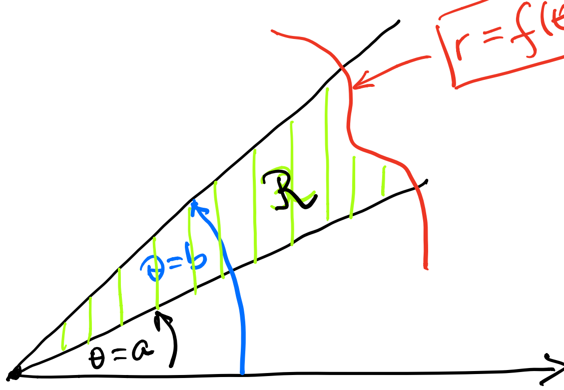
Area enclosed by the circle of radius r:

$A_{circ} = r^2 \pi$ corresponds to $\theta = 2\pi$ radians.

$$\Rightarrow \quad A = \frac{r^2\pi}{2\pi} \cdot \theta = \frac{1}{2} r^2 \pi$$

(θ measures the fraction of 2π)

(II) AREA OF A REGION WHOSE BOUNDARY IS GIVEN BY A POLAR EQUATION:

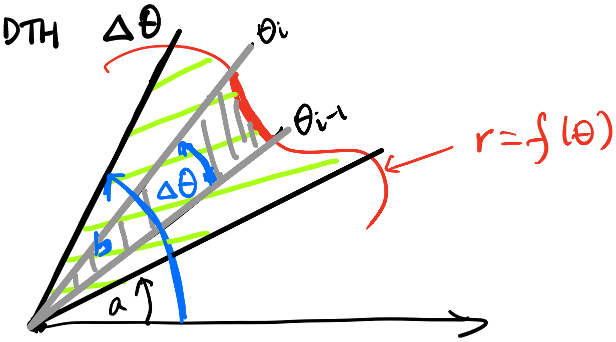


WHAT IS THE AREA OF REGION $\mathcal{R}$?

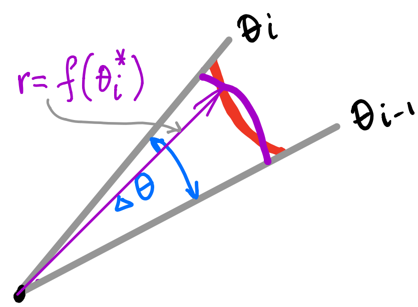
$\mathcal{R}$ IS BOUNDED BY: 1. POLAR CURVE $r = f(\theta)$ 2. RAYS $\begin{cases} \theta = a \\ \theta = b \end{cases}$ WHERE $0 < b - a \leq 2\pi$

APPROACH:

1. SUBDIVIDE REGION $\mathcal{R}$ INTO SMALL SLIVERS (SECTORS) OF WIDTH $\Delta\theta$



2. CALCULATE THE AREA OF EACH SLIVER:



APPROXIMATE THE AREA OF THE SLIVER BY THE AREA OF THE SECTOR OF A CIRCLE WITH CENTRAL ANGLE $\Delta\theta$ AND RADIUS $r=f(\theta_i^*)$ FOR SOME $\theta_i^* \in (\theta_{i-1}, \theta_i)$

$$\Delta A_i \approx \frac{1}{2}\left[f(\theta_i^*)\right]^2 \Delta\theta$$

3. SUM THEM ALL UP TO OBTAIN AN APROXIMATION OF THE TOTAL AREA:

$$A \approx \sum_{i=1}^{n} \frac{1}{2}\left[f(\theta_i^*)\right]^2 \Delta\theta$$

4. TAKE THE LIMIT AS THE NUMBER OF SLIVERS GOES TO ∞ (WHICH IS EQUIVALENT TO SAYING THAT $\Delta\theta$ BECOMES INFINITESIMALLY SMALL) TO OBTAIN:

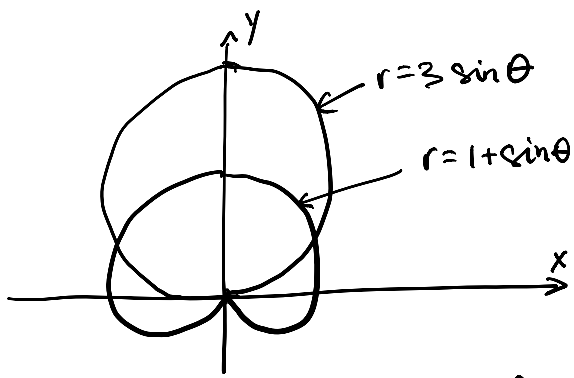
$$\boxed{A = \lim_{n\to\infty} \sum_{i=1}^{n} \frac{1}{2}\left[f(\theta_i^*)\right]^2 \Delta\theta = \frac{1}{2}\int_a^b \left[f(\theta)\right]^2 d\theta}$$

ANOTHER NOTATION :

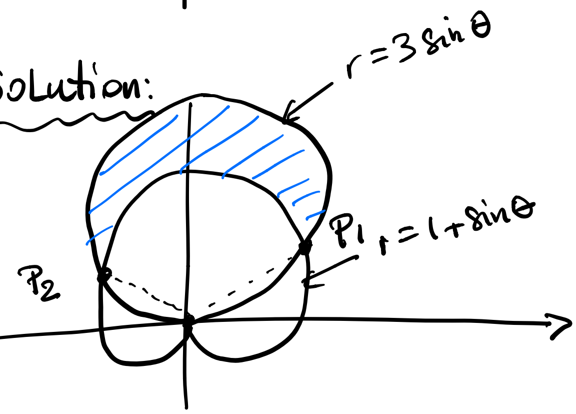
$$\boxed{A = \frac{1}{2}\int_a^b r^2\,d\theta}$$

WHERE $r = f(\theta)$.

EXAMPLE : Find the area of the region that lies inside the circle $r = 3\sin\theta$ and outside the cardioid $r = 1 + \sin\theta$.



Solution:



① INTERSECTION POINTS:

$r = 3\sin\theta$, $r = 1+\sin\theta \Rightarrow$

$3\sin\theta = 1 + \sin\theta \Rightarrow$

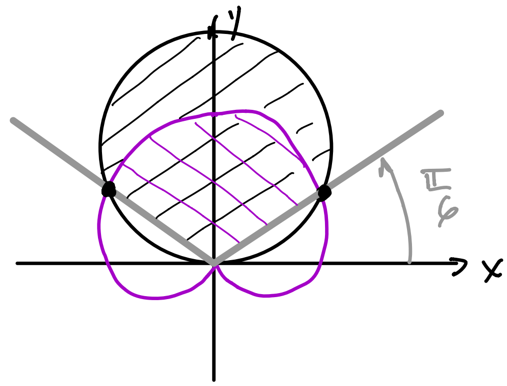
$2\sin\theta = 1 \Rightarrow \sin\theta = \frac{1}{2}$

$\Rightarrow \theta_1 = \frac{\pi}{6}$, $\theta_2 = \frac{5\pi}{6}$

② HOW TO FIND THE SHADED AREA?

WE CAN FIND THE AREA ENCLOSED BY THE CIRCLE BETWEEN $\theta_1 = \frac{\pi}{6}$ AND $\theta_2 = \frac{5\pi}{6}$, AND THEN SUBTRACT THE AREA ENCLOSED BY THE

CARDIOID BETWEEN $\theta_1 = \frac{\pi}{6}$ AND $\theta_2 = \frac{5\pi}{6}$

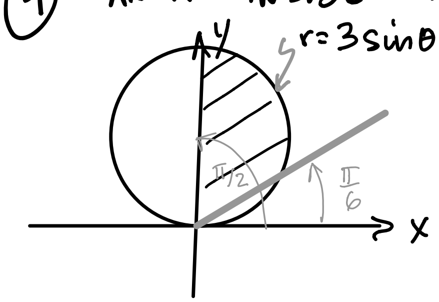


(3) CHECK FOR SYMMETRIES:

REGION IS SYMMETRIC AROUND $\theta = \frac{\pi}{2}$

$\Rightarrow$ CALCULATE INTEGRAL FROM $\theta_1 = \frac{\pi}{6}$ TO $\theta = \frac{\pi}{2}$ AND THEN MULTIPLY BY 2.

(4) AREA INSIDE THE CIRCLE:



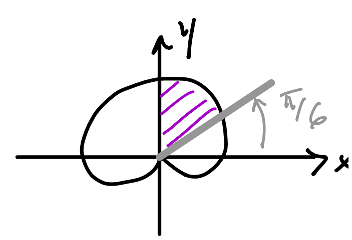
$$A^*_{\text{CIRCLE}} = 2\, A^*_{\text{HALF CIRCLE}}$$

$$A^*_{\text{HALF CIRCLE}} = \frac{1}{2}\int_{\frac{\pi}{6}}^{\frac{\pi}{2}} r^2\, d\theta =$$

$$= \frac{1}{2}\int_{\frac{\pi}{6}}^{\pi/2} (3\sin\theta)^2\, d\theta$$

(5) AREA INSIDE THE CARDIOID:

$$A^*_{\text{CARDIO}} = 2\, A^*_{\text{HALF CARDIO}}$$



$$A^*_{\text{HALFCARDIO}} = \frac{1}{2}\int_{\pi/6}^{\pi/2} r^2\,d\theta = \frac{1}{2}\int_{\pi/6}^{\pi/2}(1+\sin\theta)^2 d\theta =$$

$$= \frac{1}{2}\int_{\pi/6}^{\pi/2}(1+2\sin\theta+\sin^2\theta)\,d\theta$$

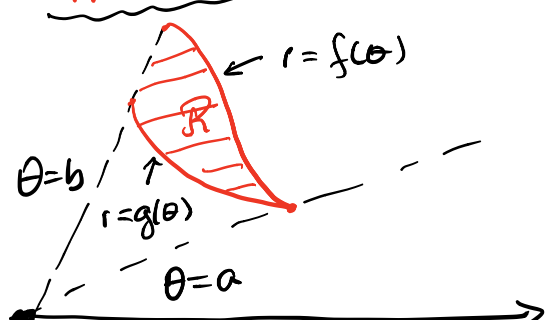
⑥ $$A = A^*_{\text{CIRCE}} - A^*_{\text{CARDIO}} = 2\left[\frac{1}{2}\int_{\pi/6}^{\pi/2} 9\sin^2\theta\,d\theta - \frac{1}{2}\int_{\pi/6}^{\pi/2}(1+2\sin\theta+\sin^2\theta)\,d\theta\right]$$

$$= \int_{\pi/6}^{\pi/2}(8\sin^2\theta - 1 - 2\sin\theta)\,d\theta = \int_{\pi/6}^{\pi/2}(3-4\cos 2\theta - 2\sin\theta)\,d\theta =$$

$\sin^2\theta = \frac{1}{2}(1-\cos 2\theta)$

$$= \left[3\theta - 2\sin 2\theta + 2\cos\theta\right]_{\pi/6}^{\pi/2} = \pi$$

IN GENERAL



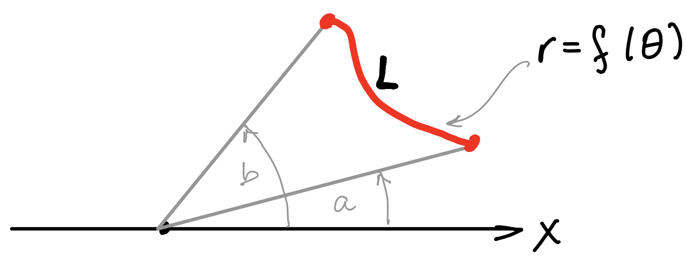
$$\boxed{\begin{aligned} A &= \int_a^b \frac{1}{2}[f(\theta)]^2 d\theta - \int_a^b \frac{1}{2}[g(\theta)]^2 d\theta \\ &= \frac{1}{2}\int_a^b \left([f(\theta)]^2 - [g(\theta)]^2\right) d\theta \end{aligned}}$$

ARC LENGTH FOR $r = f(\theta)$

$$x = r\cos\theta = f(\theta)\cos\theta$$

$$y = r\sin\theta = f(\theta)\sin\theta$$

$$\boxed{L = \int_a^b \sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2}\, d\theta}$$



SUMMARY

I. AREA ENCLOSED BY POLAR CURVE:

$$A = \frac{1}{2}\int_a^b r^2 \, d\theta$$

II. AREA ENCLOSED BETWEEN TWO POLAR CURVES $r_1 = f(\theta)$, $r_2 = g(\theta)$

$$A = \frac{1}{2}\int_a^b \left([f(\theta)]^2 - [g(\theta)]^2 \right) d\theta$$

III. ARC LENGTH: $L = \int_a^b \sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2} \, d\theta$

FOR $r = f(\theta)$

WHERE

$$\begin{cases} x = f(\theta)\cos\theta \\ y = f(\theta)\sin\theta \end{cases}$$

😎

DONE WITH CHAPTER 10

MOVE ON!

PREVIEW INTO

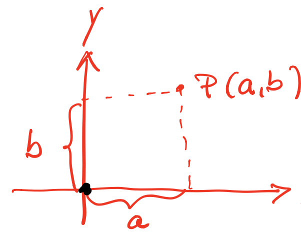
CHAPTER 12

12.1. 3D COORDINATE SYSTEM

RECALL: Points in a plane:

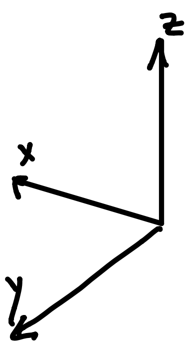
We need 2 numbers to specify P

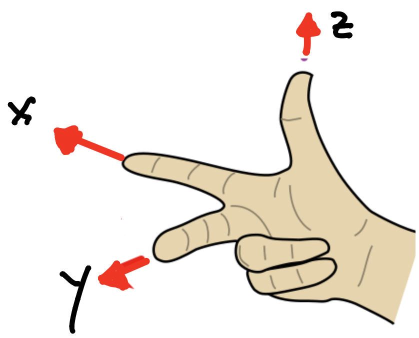
(a,b) = ORDERED PAIR which describes coordinates in $\mathbb{R}^2$



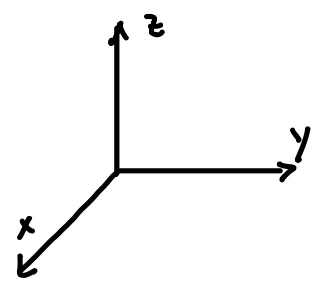
NOW: Points in space: need a coordinate system

- First choose a fixed point, THE ORIGIN, denoted by O.
- Draw 3 lines through O that are perpendicular to each other, called COORDINATE AXIS
- Label them x, y, z usually following the RIGHT-HAND RULE



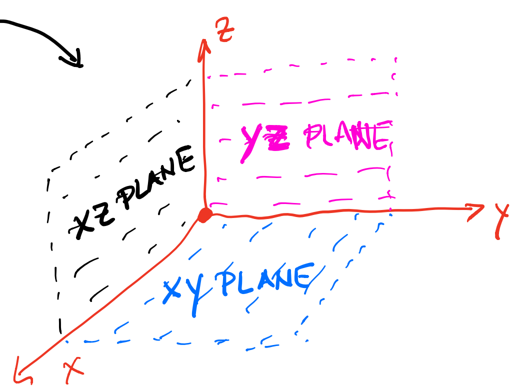


DIFFERENT VIEWING ANGLE

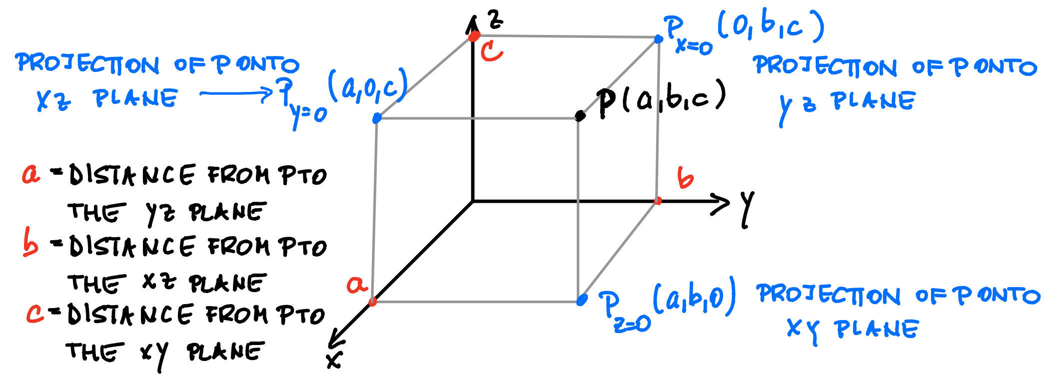


- COORDINATE PLANES

- SPACE IS DIVIDED INTO 8 PARTS CALLED OCTANTS
- FIRST OCTANT HAS x,y,z POSITIVE
- OCTANTS I, II, III, IV HAVE z POSITIVE



- COORDINATES OF A POINT :



a = DISTANCE FROM P TO THE YZ PLANE
b = DISTANCE FROM P TO THE XZ PLANE
c = DISTANCE FROM P TO THE XY PLANE

- THE CARTESIAN PRODUCT $\boxed{\mathbb{R}\times\mathbb{R}\times\mathbb{R} = \{(x,y,z) \mid x,y,z \in \mathbb{R}\}}$

IS THE SET OF ALL ORDERED TRIPLES OF REAL NUMBERS AND IS DENOTED BY $\mathbb{R}^3$.

SURFACES

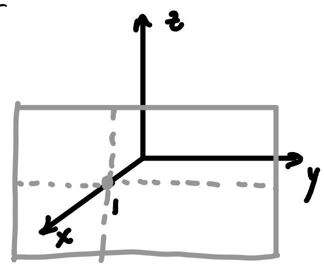
Recall: In $\mathbb{R}^2$: What is the graf of an equation involving x and y?

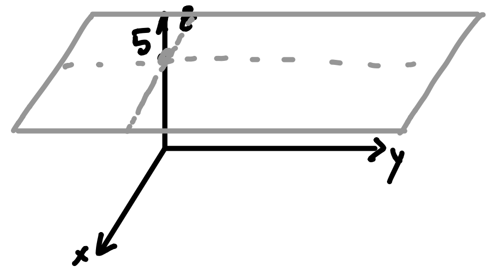
A CURVE

Now: In $\mathbb{R}^3$: What is the graf of an equation involving x, y, and z?

A SURFACE

EXAMPLE: What surfaces in $\mathbb{R}^3$ are represented by the following equations: (a) $x=1$ (b) $z=5$?





WHAT ARE THE EQUATIONS OF COORDINATE PLANES?

$z=0$ (xy)
$y=0$ (xz)
$x=0$ (yz)

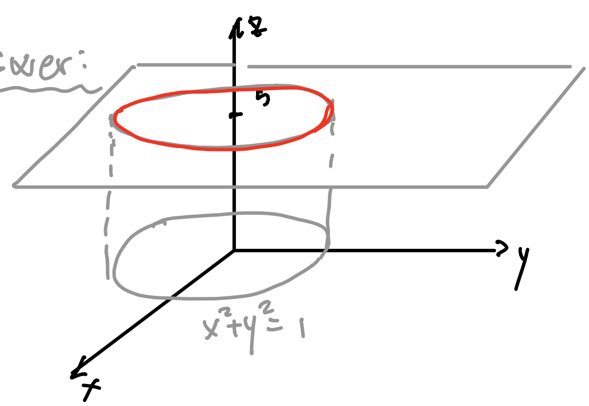
NOTE: When an equation is given, we must understand from the context of the problem whether we are working in $\mathbb{R}^2$ or $\mathbb{R}^3$.

EXAMPLE: (a) Which (x,y,z) satisfy: $\begin{cases} x^2+y^2=1 \\ z=5 \end{cases}$?

(b) What does $x^2+y^2=1$ represent as a surface?

Answer:

(a)



(b) $x^2+y^2=1$ is a cylinder of radius 1.

EXAMPLE: Sketch the surface in $\mathbb{R}^3$: $y=x$.