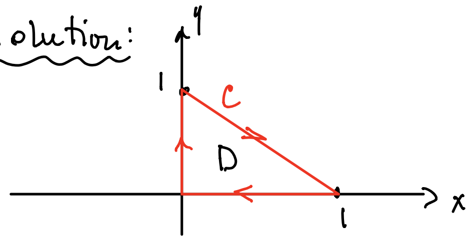
SOLVING PROBLEMS FROM Sections 16.4 – 16.9

GREEN'S THEOREM:

Problem 7 (Practice Final Exam B): Use Green's Thrm to evaluate the work done by the force $\vec{F} = x(x+y)\vec{i} + xy^2\vec{j}$ in moving a particle from the origin along the y-axis to $(0,1)$, then along the line segment to $(1,0)$, and then back to the origin along the x-axis.

Solution:



1. C is negatively oriented $\Rightarrow$ Green's theorem:

$$\int_C P\,dx + Q\,dy = -\iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy$$

2. $P(x,y) = x(x+y)$, $Q(x,y) = xy^2$

$$\int_C P\,dx + Q\,dy = -\iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy = \iint_D \left(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}\right) dx\,dy =$$

$$= \iint_D (x - y^2)\,dx\,dy = \int_0^1 \int_0^{1-x} (x - y^2)\,dy\,dx =$$

straight line from $(0,1)$ to $(1,0)$ is $y = 1-x$

$$= \int_0^1 \left[xy - \frac{y^3}{3}\right]_0^{1-x} dx$$

IT'S EASIER TO INTEGRATE WITH RESPECT TO $dx\,dy$ NOT $dy\,dx$ $\Rightarrow$

$$= \int_0^1 \int_0^{1-y} (x - y^2)\,dx\,dy = \int_0^1 \left[\frac{x^2}{2} - y^2 x\right]_{x=0}^{x=1-y} dy =$$

STRAIGHT LINE $x = 1-y$

$$= \int_0^1 \left(\frac{1}{2}(1-y)^2 - y^2(1-y)\right) dy = \int_0^1 \left[\frac{1}{2}(1-2y+y^2) - y^2 + y^3\right] dy$$

$$= \int_0^1 \left(\frac{1}{2} - y + \frac{1}{2}y^2 - y^2 + y^3\right) dy = \int_0^1 \left(\frac{1}{2} - y - \frac{1}{2}y^2 + y^3\right) dy$$

$$= \left[\frac{1}{2}y - \frac{y^2}{2} - \frac{y^3}{6} + \frac{y^4}{4}\right]_0^1 = \frac{1}{2} - \frac{1}{2} - \frac{1}{6} + \frac{1}{4} = \frac{1}{12}$$

$$\text{div}\,\vec{F} = \nabla \cdot \vec{F} = \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right\rangle \cdot \langle P, Q, R \rangle$$

SECTION 16.5: $\text{curl}\,\vec{F} = \nabla \times \vec{F}$

Problem 8 (Practice Final B): Is there a vector field $\vec{H}$ such that

$$\text{curl}\,\vec{H} = \langle x\sin y, \cos y, z - xy \rangle \,?$$

Explain.

Solution: By Theorem 11 on pg 1106, if there exists a vector field $\vec{H}$ such that $\text{curl}\,\vec{H} = \langle x\sin y, \cos y, z-xy \rangle$ then the divergence $\text{div}\,\text{curl}\,\vec{H}$ would be zero.

Calculate $\text{div}\,\text{curl}\,\vec{F} = \text{div}\langle x\sin y, \cos y, z-xy \rangle =$

$$= \frac{\partial}{\partial x}(x\sin y) + \frac{\partial}{\partial y}(\cos y) + \frac{\partial}{\partial z}(z - xy) = \cancel{\sin y} - \cancel{\sin y} + 1 = 1 \neq 0$$

Conclusion: there is no $\vec{H}$ such that $\text{curl}\,\vec{H} = \langle x\sin y, \cos y, z-xy \rangle$

STOKES' THEOREM:

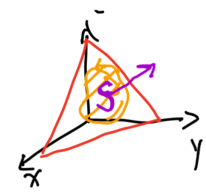
Problem 11 (Practice Final Exam B): Use Stokes' Theorem to show that the line integral

$$\int_C z\,dx - 2x\,dy + 3y\,dz,$$

where C is a simple, closed curve that lies in the plane $x+y+z=1$, depends only on the area of the region enclosed by C and not on the shape of the curve C or its location in the plane.

Solution:

Stokes': $\displaystyle\int_C \vec{F}\cdot d\vec{r} = \iint_S \operatorname{curl}\vec{F}\cdot\vec{n}\,dS$



Here $\vec{F} = \langle z, -2x, 3y\rangle$ and the surface S is the region in the plane $x+y+z=1$ bounded by C. So, $\vec{n} = \langle 1,1,1\rangle/\sqrt{3}$.

So, Stokes' implies:

$$I = \int_C z\,dx - 2x\,dy + 3y\,dz = \int_C \vec{F}\cdot d\vec{r} = \iint_S \operatorname{curl}\vec{F}\cdot\vec{n}\,dS = \iint_S \operatorname{curl}\vec{F}\cdot \overbrace{\frac{\langle 1,1,1\rangle}{\sqrt{3}}}^{\vec{n}}\,dS$$

Now: $$\operatorname{curl}\vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ z & -2x & 3y \end{vmatrix} = \langle 3, 1, -2\rangle.$$

Thus: $$\iint_S \operatorname{curl}\vec{F}\cdot\vec{n}\,dS = \iint_S \langle 3,1,-2\rangle \frac{\langle 1,1,1\rangle}{\sqrt{3}}\,dS =$$

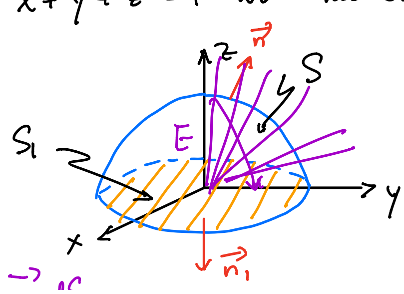
$$= \frac{1}{\sqrt{3}}\iint_S 2\,dS = \frac{2}{\sqrt{3}}\iint_S dS = \frac{2}{\sqrt{3}}A(S)$$

Thus the integral I depends only on the surface area $A(S)$ enclosed by C!

THE DIVERGENCE THEOREM:

Problem 12. (Practice Final Exam A): Use the Divergence Theorem to evaluate $\displaystyle\iint_S \vec{F}\cdot d\vec{S}$ where $\vec{F} = \langle z^2x, \frac{1}{3}y^3 + \tan z, x^2z + y^2\rangle$ and S is the top half of the sphere $x^2+y^2+z^2=1$ with the outward normal pointing away from the origin.

Solution: Must add surface S_1 to have a closed surface $S \cup S_1$ and apply the Divergence Theorem



outward unit normal (away from E) $\rightarrow \vec{n}$ as

The Divergence Theorem $\Rightarrow$ $$\iint_{S\cup S_1} \vec{F}\cdot d\vec{S} = \iiint_E \operatorname{div}\vec{F}\, dV$$

or: $$\iint_S \vec{F}\cdot d\vec{S} + \iint_{S_1} \vec{F}\cdot d\vec{S} = \iiint_E \operatorname{div}\vec{F}\, dV$$

Thus: $$\iint_S \vec{F}\cdot d\vec{S} = \iiint_E \operatorname{div}\vec{F}\, dV - \iint_{S_1} \vec{F}\cdot d\vec{S}.$$

Or: $$\iint_S \vec{F}\cdot\vec{n}\, dS = \iiint_E \operatorname{div}\vec{F}\, dV - \iint_{S_1} \vec{F}\cdot \underbrace{\vec{n}_1}_{-\vec{k}}\, dS \qquad (\star)$$

where $\vec{n}$ and $\vec{n}_1$ are OUTWARD normals to the boundary of E.

Calculate the right hand-side of $(\star)$:

$$\iiint_E \operatorname{div}\vec{F}\, dV = \iiint_E \left(\frac{\partial}{\partial x}(z^2x) + \frac{\partial}{\partial y}\left(\frac{1}{3}y^3+\tan z\right) + \frac{\partial}{\partial z}(x^2z+y^2)\right) dV$$

$$= \iiint_E (z^2+y^2+x^2)\, dV = \text{SPHERICAL COORDINATES} =$$

$$= \int_0^{2\pi}\int_0^{\pi/2}\int_0^1 \rho^2\cdot\rho^2\sin\phi\, d\rho\, d\phi\, d\theta$$

$$= 2\pi\left[\frac{\rho^5}{5}\right]_0^1 \int_0^{\pi/2} \sin\phi\, d\phi = \frac{2\pi}{5}\left[-\cos\phi\right]_{\phi=0}^{\phi=\pi/2} = \frac{2\pi}{5}$$

$$\iint_{S_1} \vec{F}\cdot\vec{n}_1\, dS \underset{\vec{n}_1=-\vec{k}}{=} \iint_{S_1} -(x^2z+y^2)\, dS = -\int_0^{2\pi}\int_0^1 (x^2\cdot 0 + r^2\sin^2\theta)\, r\,dr\,d\theta$$

S_1 -- $x^2+y^2\le 1$ and $z=0$

dS... dA in polar coordinates:

$dS = dA = r\,dr\,d\theta$

$$= -\int_0^{2\pi}\int_0^1 r^3\sin^2\theta\, dr\, d\theta = -\left[\frac{r^4}{4}\right]_0^1 \int_0^{2\pi}\frac{1}{2}(1-\cos 2\theta)\, d\theta = -\frac{1}{4}\cdot\frac{1}{2}\left[\theta - \frac{1}{2}\sin 2\theta\right]_0^{2\pi} =$$

$$= -\frac{1}{8}[2\pi - 0] = -\frac{\pi}{4}$$

Conclusion: $(\star) \Rightarrow \iint_S \vec{F}\cdot d\vec{S} = \frac{2\pi}{5} - \left(-\frac{\pi}{4}\right) = \frac{2\pi}{5} + \frac{\pi}{4} = \frac{13\pi}{20}$ //

SECTION 16.5:

Problem 8 (Final Exam Practice Test A): Prove:

$$\operatorname{div}(f\vec{F}) = f \operatorname{div}\vec{F} + \vec{F}\cdot\nabla f$$

Proof: Let $\vec{F} = \langle P, Q, R\rangle$. Then:

$$\operatorname{div}(f\vec{F}) = \operatorname{div}\langle fP, fQ, fR\rangle = \frac{\partial(fP)}{\partial x} + \frac{\partial(fQ)}{\partial y} + \frac{\partial(fR)}{\partial z} =$$

$$\overset{\text{PRODUCT RULE}}{=} \frac{\partial f}{\partial x}P + f\frac{\partial P}{\partial x} + \frac{\partial f}{\partial y}Q + f\frac{\partial Q}{\partial y} + \frac{\partial f}{\partial z}R + f\frac{\partial R}{\partial z} =$$

$$= \frac{\partial f}{\partial x}P + \frac{\partial f}{\partial y}Q + \frac{\partial f}{\partial z}R + f\left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}\right) =$$

$$= \nabla f\cdot\vec{F} + f\operatorname{div}\vec{F} = \vec{F}\cdot\nabla f + f\operatorname{div}\vec{F}$$

Q.E.D. //

(Quod Erat Demonstrandum)

INDEPENDENCE OF PATH INTEGRATION:

Problem 6 (Practice Final A): Show that the following line integral is independent of path and evaluate the integral:

$$\int_C \sin y\, dx + (x\cos y - y^2)\, dy = \int_C \nabla f\cdot d\vec{r}$$

where C is any path from $(4,0)$ to $(5,\pi)$.

Solution: Show that there exists a function f such that

$$\nabla f = \langle \sin y, x\cos y - y^2\rangle = \langle P, Q\rangle = \vec{F}$$

(1) $\frac{\partial Q}{\partial x} = \frac{\partial P}{\partial y}$? $\quad \frac{\partial Q}{\partial x} = \cos y = \frac{\partial P}{\partial y}$ ✓ $\Rightarrow$ yes, $\vec{F}$ is conservative, i.e., there exist an f such that $\vec{F} = \nabla f$.

(2) Find f: $\frac{\partial f}{\partial x} = \sin y \Rightarrow f(x,y) = x\sin y + g(y)$

$$\frac{\partial f}{\partial y} = \cancel{x\cos y} + g'(y) = Q = \cancel{x\cos y} - y^2$$

$$\Rightarrow g'(y) = -y^2 \Rightarrow g(y) = -\frac{y^3}{3} + C$$

$$\Rightarrow f(x,y) = x\sin y - \frac{y^3}{3} + C$$

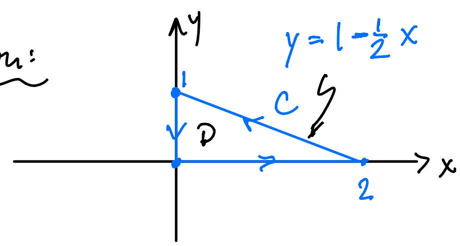
(3) $\int_C \vec{F}\cdot d\vec{r} = f(B) - f(A) = f((5,\pi)) - f((4,0)) =$

$$= 5\sin\pi - \frac{\pi^3}{3} - 4\sin 0 + \frac{0^3}{3} = -\frac{\pi^3}{3}$$

GREEN'S THEOREM:

Problem 7 (Practice Final A): Use Green's Theorem to evaluate the line integral $\int_C (100x^5+y^2)\,dx + (x^2 - 100y^5)\,dy$ where C is a triangle with vertices $(0,0)$, $(2,0)$ and $(0,1)$ oriented counterclockwise.

Solution:



Green's Theorem:

$$\int_C P\,dx + Q\,dy = \iint_C \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy$$

$$I = \int_C (100x^5+y^2)\,dx + (x^2-100y^5)\,dy = \iint_D (2x-2y)\,dx\,dy =$$

$$= \int_0^2 \int_0^{1-\frac{x}{2}} (2x-2y)\,dy\,dx = \int_0^2 \left[2xy - y^2\right]_{y=0}^{1-\frac{x}{2}} dx = \int_0^2 \left(2x\left(1-\frac{x}{2}\right) - \left(1-\frac{x}{2}\right)^2\right) dx$$

$$= \int_0^2 \left(2x - x^2 - 1 + x - \frac{x^2}{4}\right) dx = \int_0^2 \left(3x - \frac{5}{4}x^2 - 1\right) dx = \left[\frac{3}{2}x^2 - \frac{5}{12}x^3 - x\right]_0^2 = \frac{2}{3}$$

TANGENT PLANE TO PARAMETRIC SURFACE

Problem 9 (Practice Final A): Find an equation of the tangent plane to the parametric surface $x = 2u^3+5$, $y = v^2+10$, $z = u+v$ at $(7, 11, 2)$.

$\vec r = (2u^3+5)\vec i + (v^2+10)\vec j + (u+v)\vec k$

Solution:

$$\vec n = \vec r_u \times \vec r_v = \begin{vmatrix} \vec i & \vec j & \vec k \\ 6u^2 & 0 & 1 \\ 0 & 2v & 1 \end{vmatrix} = \langle -2v, -6u^2, 12uv \rangle$$

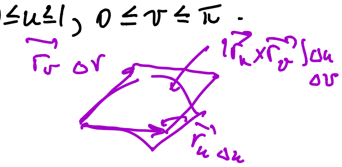
Parameter values for $(7, 11, 2)$:
$$\begin{cases} 7 = 2u^3+5 \Rightarrow u = 1, \ldots \\ 11 = v^2+10 \Rightarrow v = \pm 1 \\ 2 = u+v \Rightarrow \boxed{u=1,\ v=1} \end{cases}$$

So: $\vec n(P) = \langle -2, -6, 12 \rangle$

Tangent plane:
$$-2(x-7) - 6(y-11) + 12(z-2) = 0$$
$$\boxed{2x + 6y - 12z = 5\cancel{9}}\ 6$$

SURFACE AREA OF A PARAMETRIC SURFACE:

Problem 9 (Practice Final B): Find the surface area of the surface given by $\vec r(u,v) = \cos v\,\vec i + \sin v\,\vec j + u\,\vec k$, $0 \le u \le 1$, $0 \le v \le \pi$.



Solution:

$$A(S) = \iint_S dS = \iint_D \underbrace{|\vec r_u \times \vec r_v|\, dA}_{dS}$$

$$\left.\begin{aligned} \vec r_u &= \langle 0, 0, 1 \rangle \\ \vec r_v &= \langle -\sin v, \cos v, 0 \rangle \end{aligned}\right\} \quad \vec r_u \times \vec r_v = \begin{vmatrix} \vec i & \vec j & \vec k \\ 0 & 0 & 1 \\ -\sin v & \cos v & 0 \end{vmatrix} = \langle -\cos v, -\sin v, 0 \rangle$$

$$|\vec r_u \times \vec r_v| = \sqrt{\sin^2 v + \cos^2 v} = 1$$

$$A(S) = \int_0^1 \int_0^\pi du\, dv = \pi$$

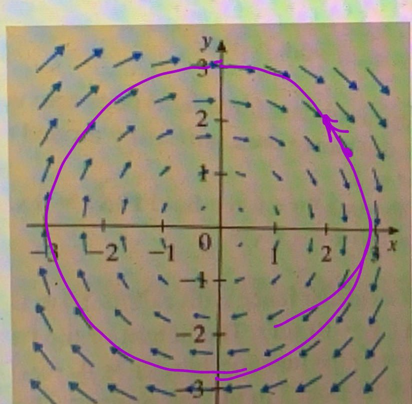
Problem 26 pg 1146: Prove $V(E) = \frac{1}{3}\iint_S \vec{F}\cdot d\vec{S}$ where $\vec{F} = x\vec{i} + y\vec{j} + z\vec{k}$ where S and E satisfy the conditions of the Divergence Theorem.

Proof: $\frac{1}{3}\iint_S \vec{F}\cdot d\vec{S} =$ by Div-Theorem $= \frac{1}{3}\iiint_E \operatorname{div}\vec{F}\, dV =$

$$= \frac{1}{3}\iiint_E \left(\frac{\partial x}{\partial x} + \frac{\partial y}{\partial y} + \frac{\partial z}{\partial z}\right) dV = \frac{1}{3}\iiint_E 3\, dV = \iiint_E dV = V(E) \quad //$$

Problem 10 (Practice Final B):

10. (5 points total) Let **F** be a vector field shown in Figure 1 below.
(a) If C_1 is the vertical line segment from $(0,-3)$ to $(0,3)$, determine whether $\int_{C_1} \mathbf{F}\cdot d\mathbf{r}$ is positive, negative, or zero.
(b) If C_2 is the counterclockwise oriented circe with radius 3 and center the origin, determine whether $\int_{C_2} \mathbf{F}\cdot d\mathbf{r}$ is positive, negative, or zero.



(a) $\int_{C_1} \vec{F}\cdot d\vec{r} = \int_{C_1} \underbrace{\vec{F}\cdot\vec{t}}_{=0}\, ds = 0$

(b) $\int_{C_2} \vec{F}\cdot d\vec{r} = \int_{C_2} \underbrace{\vec{F}\cdot\vec{t}}_{<0}\, ds < 0$

Figure 1: Vector Field in Problem 10.

LINE INTEGRAL:

Problem 5 (Practice Final A): Evaluate $\int_C y^3\,ds$ where C is the parametric curve $x=t^3$, $y=2t$, with $0\le t\le 2$.

Solution:

$$\int_C y^3\,ds = \int_C y^3\sqrt{[x'(t)]^2+[y'(t)]^2}\,dt = \int_0^2 (2t)^3\sqrt{[3t^2]^2+[2]^2}\,dt$$

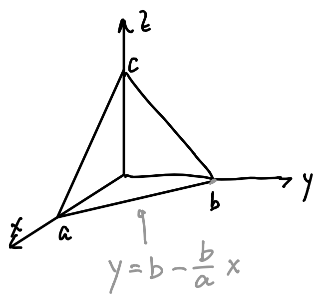
$$= 8\int_0^2 t^3\sqrt{9t^4+4}\,dt = \left| \begin{array}{l} \text{Substitution:}\\ 9t^4+4=u \\ 36t^3\,dt=du \\ t^3\,dt = \frac{1}{36}du\end{array}\right| = \frac{8}{36}\int u^{1/2}\,du =$$

$$= \frac{2}{9}\,\frac{2}{3}\,u^{3/2} = \frac{4}{27}\left[(9t^4+4)\sqrt{9t^4+4}\right]_0^2 =$$

$$= \frac{4}{27}\left[(9\cdot16+4)\sqrt{9\cdot16+4} - 4\sqrt{4}\right] =$$

$$= \frac{4}{27}\left[148\sqrt{148}-8\right]$$

Problem 12 (Practice Final B): Using the Divergence Theorem calculate the flux of $\vec{F} = x\vec{i}+y\vec{j}+z\vec{k}$ across the surface S of the solid enclosed by the coordinate planes and the plane $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1$, where $a,b,c>0$.



$$\text{Flux of } \vec{F} = \iint_S \vec{F}\cdot d\vec{S} = \iiint_E \operatorname{div}\vec{F}\,dV = 3\iiint_E dV = 3V(E)$$

$$= \cancel{3}\frac{abc}{\cancel{6}\,2} = \frac{abc}{2}$$

OR: $V(E) = \int_0^a \int_0^{b-\frac{b}{a}x} \int_0^{c-\frac{c}{a}x-\frac{c}{b}y} dz\,dy\,dx =$

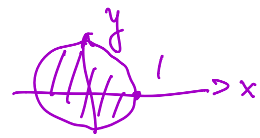
$$= \int_0^a \int_0^{b-b/a\,x} \left(c - \frac{c}{a}x - \frac{c}{b}y\right) dy\,dx = \int_0^a \left[cy - \frac{c}{a}xy - \frac{c}{b}\frac{y^2}{2}\right]_0^{b-b/a\,x} dx =$$

$$= \int_0^a \left[c\left(b-\frac{b}{a}x\right) - \frac{c}{a}x\left(b-\frac{b}{a}x\right) - \frac{c}{2b}\left(b-\frac{b}{a}x\right)^2\right]dx = \int_0^a cb - \frac{cb}{a}x - \frac{cb}{a}x + \frac{cb}{a^2}x^2 - \frac{c}{2b}\left(b^2 - \frac{2b^2}{a}x + \frac{b^2}{a^2}x^2\right)$$

$$= \int_0^a \left(\frac{cb}{2} - \frac{cb}{a}x + \frac{cb}{2a^2}x^2\right)dx = \frac{cb}{2}\int_0^a \left(1 - \frac{2}{a}x + \frac{1}{a^2}x^2\right)dx = \frac{cb}{2}\left[a - \frac{2}{\cancel{a}}\frac{a^{\cancel{2}}}{2} + \frac{1}{\cancel{a^2}}\frac{a^{\cancel{3}}}{3}\right] = \frac{cb}{2}\left[\cancel{a}-\cancel{a}+\frac{a}{3}\right] = \frac{abc}{6}$$

Problem 2 (Test B):

$$\begin{cases} f(x,y) = 2x + y \to \text{extrema} \\ x^2+y^2 \le 1 \quad \text{constraint} \end{cases}$$



① Critical points inside D : none

② On the ∂D: $x^2+y^2=1$:

$$\nabla f = \lambda \nabla g$$

$$\begin{cases} \langle 2, 1\rangle = \lambda \langle 2x, 2y\rangle \\ x^2+y^2=1 \end{cases} \Rightarrow \begin{aligned} 2x\lambda &= 2 \\ 2y\lambda &= 1 \end{aligned}$$

$$\Downarrow$$

$$\frac{x}{y} = 2 \Rightarrow x = 2y$$

$$(2y)^2 + y^2 = 1 \Leftarrow$$

$$5y^2 = 1 \Rightarrow y = \pm\frac{\sqrt5}{5}, \quad x = \pm 2\frac{\sqrt5}{5}$$

$P_1 = \left(2\frac{\sqrt5}{5}, \frac{\sqrt5}{5}\right)$, $P_2 = \left(-2\frac{\sqrt5}{5}, -\frac{\sqrt5}{5}\right)$

③ $f\left(2\frac{\sqrt5}{5}, \frac{\sqrt5}{5}\right) = 4\frac{\sqrt5}{5} + \frac{\sqrt5}{5} = \sqrt5$ $\quad f\left(-2\frac{\sqrt5}{5}, -\frac{\sqrt5}{5}\right) = -\sqrt5$

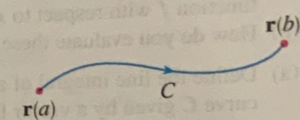
$\Rightarrow f_{MAX}\left(2\frac{\sqrt5}{5}, \frac{\sqrt5}{5}\right) = \sqrt5$ $\quad f_{MIN}\left(-2\frac{\sqrt5}{5}, -\frac{\sqrt5}{5}\right) = -\sqrt5$

Fundamental Theorem of Calculus

$$\int_a^b F'(x)\,dx = F(b) - F(a)$$

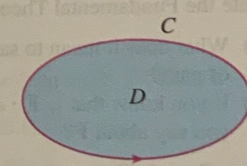
Fundamental Theorem for Line Integrals

$$\int_C \nabla f \cdot d\mathbf{r} = f(\mathbf{r}(b)) - f(\mathbf{r}(a))$$



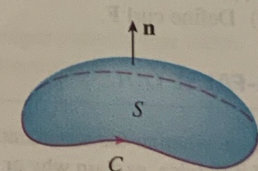
Green's Theorem

$$\iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA = \int_C P\,dx + Q\,dy$$



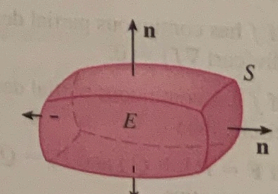
Stokes' Theorem

$$\iint_S \operatorname{curl} \mathbf{F} \cdot d\mathbf{S} = \int_C \mathbf{F} \cdot d\mathbf{r}$$



Divergence Theorem

$$\iiint_E \operatorname{div} \mathbf{F}\,dV = \iint_S \mathbf{F} \cdot d\mathbf{S}$$




GOOD LUCK ON YOUR FINALS
AND HAPPY HOLIDAYS!