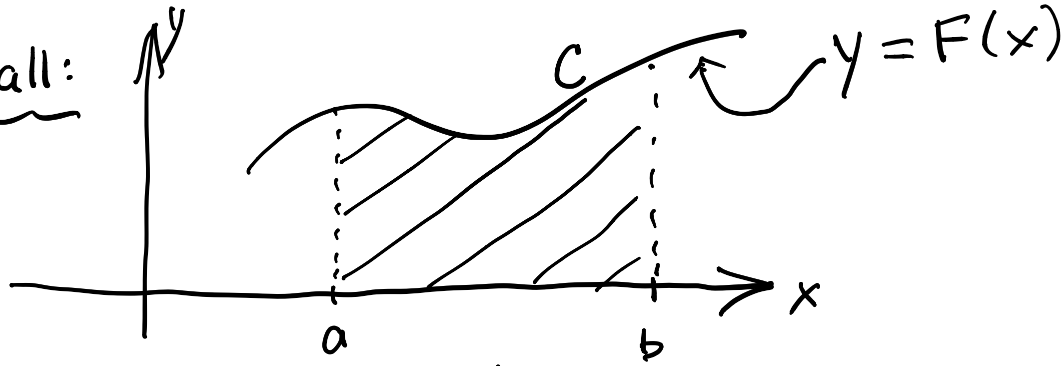
MOVING ON! STILL IN SECTION 10.2

AREA UNDER A PARAMETERIZED CURVE 😳

Recall:



$$\boxed{\text{AREA} = \int_a^b y\,dx}$$

But now:

$$\left.\begin{aligned} y &= g(t) \\ x &= f(t) \end{aligned}\right\} \text{curve } C$$

We know (change of variables) that

$$x = f(t) \Rightarrow dx = f'(t)\,dt$$

Thus, area in parametric form:

$$\boxed{\text{AREA} = \int_\alpha^\beta g(t)\,f'(t)\,dt}$$

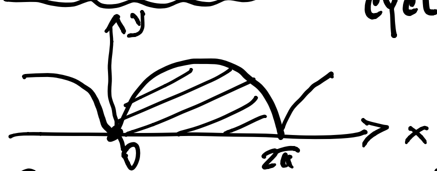
😎

where α and β are such that

$a = f(t=\alpha)$

$b = f(t=\beta)$

EXAMPLE: Find the area under one arch of the cycloid $\begin{cases} x = r(\theta - \sin\theta) \\ y = r(1-\cos\theta) \end{cases}$ for $0 \le \theta \le 2\pi$.



Solution: What is the parameter?

θ - changes $\Rightarrow$ θ is the parameter ; r is a constant

$$A = \int_0^{2\pi} \underbrace{[r(1-\cos\theta)]}_{g(\theta)} \underbrace{[r(1-\cos\theta)]d\theta}_{f'(\theta)d\theta} =$$

$$= r^2 \int_0^{2\pi} (1-\cos\theta)^2 d\theta = r^2 \int_0^{2\pi} (1 - 2\cos\theta + \cos^2\theta)\, d\theta$$

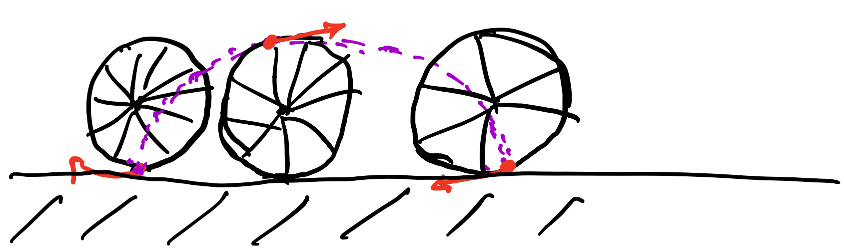
$$= r^2 \int_0^{2\pi} \left[1 - 2\cos\theta + \tfrac{1}{2}(1+\cos 2\theta)\right] d\theta$$

$$= r^2 \left[\theta - 2\sin\theta + \tfrac{1}{2}\left(\theta + \frac{\sin 2\theta}{2}\right)\right]_{\theta=0}^{\theta=2\pi}$$

$$= r^2 \left[\tfrac{3}{2}\theta - 2\sin\theta + \tfrac{1}{4}\sin 2\theta\right]_0^{2\pi}$$

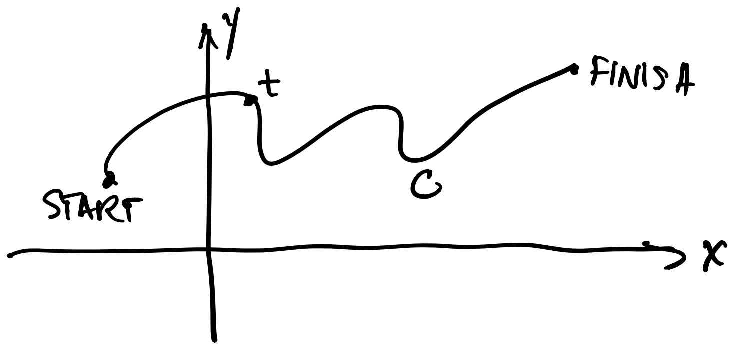
$$= r^2 \left[\tfrac{3}{2}\cdot 2\pi - 0 + 0 - 0\right] = 3\pi r^2$$

$\Rightarrow$ 3 times the area of the rolling circle



MOVING ON!

ARC LENGTH 🤔



$x = f(t)$

$y = g(t)$

LENGTH ?

Recall: If $y = F(x)$. The Length of a curve $y = F(x)$ from $x = a$ to $x = b$ is:

$$\boxed{L = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx}$$

From here we can express this in terms of t :

$$\frac{dy}{dt} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} \Rightarrow L = \int_\alpha^\beta \sqrt{1 + \left[\frac{\frac{dy}{dt}}{\frac{dx}{dt}}\right]^2}\; \frac{dx}{dt}\, dt$$

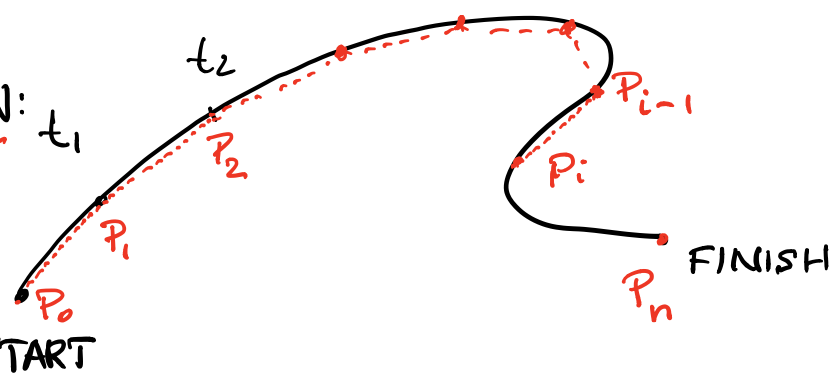
where $a = f(\alpha)$, $b = f(\beta)$.

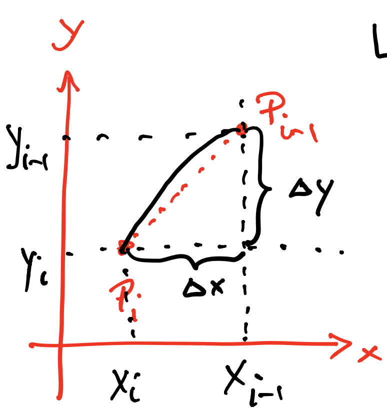
By manipulation:

$$L = \int_{\alpha}^{\beta} \sqrt{\left[1 + \frac{\left(\frac{dy}{dt}\right)^2}{\left(\frac{dx}{dt}\right)^2}\right]\left(\frac{dx}{dt}\right)^2}\, dt$$

$$\boxed{L = \int_{\alpha}^{\beta} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt}$$

GEOMETRIC INTERPRETATION:





$$L \approx \sum_{i=1}^{n} |P_{i-1}\ P_i|$$

$$= \sum_{i=1}^{n} |(x_{i-1}, y_{i-1})\ (x_i, y_i)|$$

$$= \sum \sqrt{(x_i - x_{i-1})^2 + (y_i - y_{i-1})^2}$$

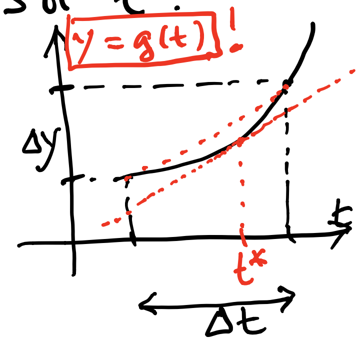
$$= \sum_{i=1}^{n} \sqrt{(\Delta x_i)^2 + (\Delta y_i)^2} \quad (*)$$

WHAT ARE Δx AND Δy IN TERMS OF t

- FIRST: WHAT IS Δy IN TERMS OF t:

Recall: $\boxed{y = g(t)}$

By the Mean Value Theorem, there exists a number t^* such that $\dfrac{\Delta y}{\Delta t} = g'(t^*)$

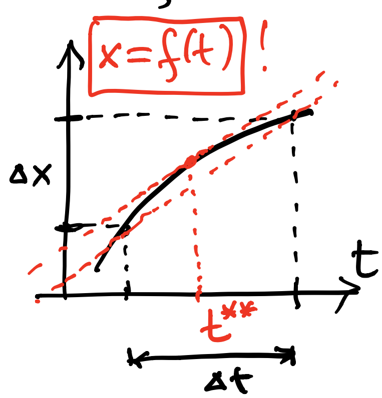


So, Δy in terms of t is: $\boxed{\Delta y = g'(t^*)\Delta t}$

- SIMILARLY: FOR x given by $x = f(t)$

We get:

$$\boxed{\Delta x = f'(t^{**})\Delta t}$$



GOING BACK TO $(*)$:

$$L \approx \sum_{i=1}^{n} \sqrt{(\Delta x_i)^2 + (\Delta y_i)^2} =$$

$$= \sum_{i=1}^{n} \sqrt{\left[f'(t_i^{**})\right]^2 + \left[g'(t_i^{*})\right]^2}\, \Delta t$$

THE ACTUAL LENGTH

$$\boxed{L = \lim_{n\to\infty} \sum_{i=1}^{n} \sqrt{[f'(t_i^{**})]^2 + [g'(t_i^*)]^2}\,\Delta t = \int_\alpha^\beta \sqrt{[f'(t)]^2 + [g'(t)]^2}\,dt}$$

😎

THEOREM: If a curve C is described by parametric equations $\begin{cases} x = f(t) \\ y = g(t) \end{cases}$ for $\alpha \le t \le \beta$, where f' and g' are continuous on $[\alpha, \beta]$ and C is traversed exactly once as t increases from α to β, then THE LENGTH OF C IS

$$\boxed{L = \int_\alpha^\beta \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt}$$

EXAMPLE: Length of C ... $\begin{cases} x = \cos t \\ y = \sin t \end{cases}$, $0 \le t \le 2\pi$

SOLUTION:

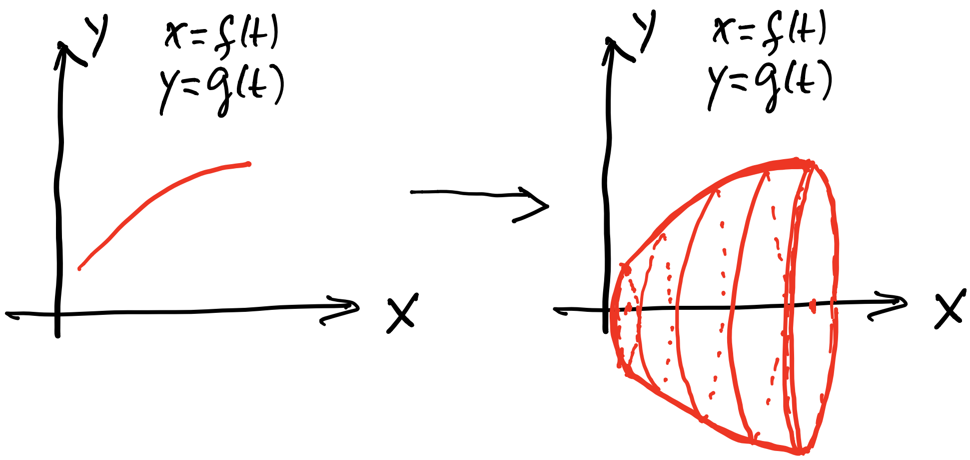
$$L = \int_0^{2\pi} \sqrt{(-\sin t)^2 + (\cos t)^2}\,dt = \int_0^{2\pi} 1\,dt = 2\pi$$

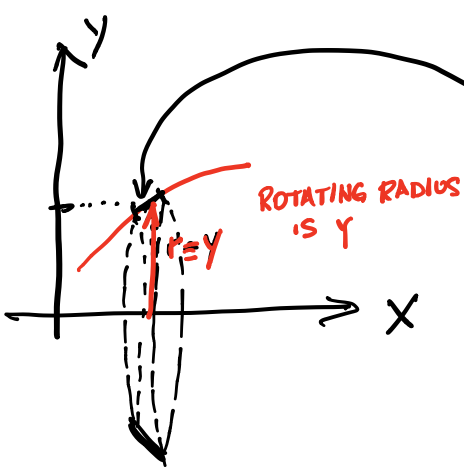
MAKE SURE THAT THIS PARAMETERIZATION TRAVERSES C ONCE

SUGGESTION: Solve EXAPLE 5 in your textbook.

MOVING ON 🤢: A SLIGHT GENERALIZATION OF THE ARC-LENGTH FORMULA TO

CALCULATE SURFACE AREA OBTAINED BY ROTATING A CURVE PARAMETERIZED BY t FOR $\alpha \le t \le \beta$ AROUND THE x-AXIS





$$ds = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

$$dS = 2\pi y\, ds$$

$$\boxed{S = \int_\alpha^\beta 2\pi y \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt}$$

SUMMARY

I. AREA UNDER A CURVE
$$\begin{cases} x = f(t) \\ y = g(t) \end{cases}$$

$$\boxed{A = \int_{\alpha}^{\beta} g(t)\, f'(t)\, dt}$$

II. ARC LENGTH:

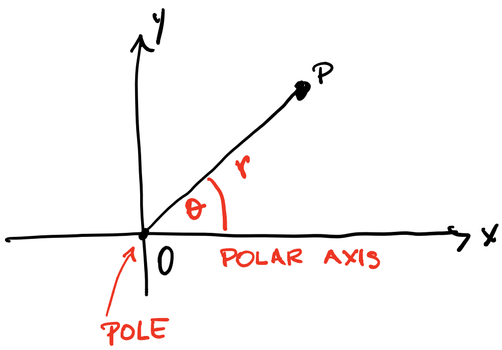
$$\boxed{L = \int_{\alpha}^{\beta} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt}$$

III. SURFACE AREA OBTAINED BY ROTATING C AROUND X-AXIS:

$$\boxed{S = \int_{\alpha}^{\beta} 2\pi y \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt}$$

Section 10.3: POLAR COORDINATES 🤗

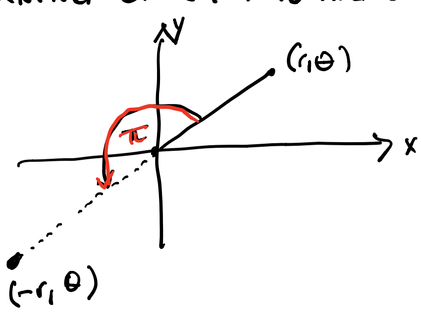
Polar coordinates are a different way to characterize $\mathbb{R}^2$.



- r is the distance from O to P
- θ is the angle between the line OP and polar axis

POLAR COORDINATE OF P: (r, θ)

WE EXTEND THE MEANING OF (r,θ) TO THE CASE IN WHICH $r<0$:

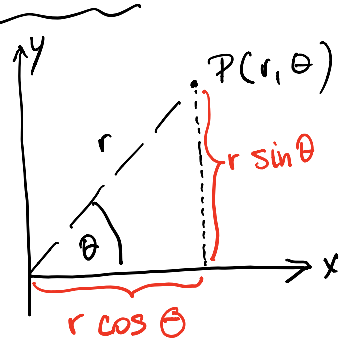


THUS:

$$(-r,\theta) = (r, \theta+\pi)$$

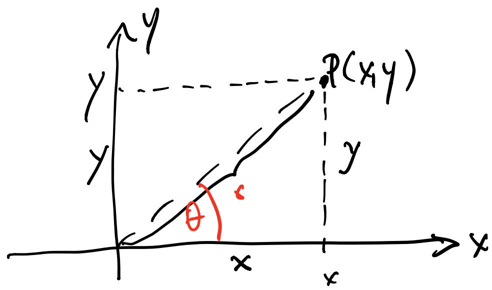
EXAMPLE: $(5,0)$, $(5,2\pi)$, $(5,4\pi)$, $(-5,-\pi)$ ALL THE SAME POINT!

QUESTION: How do we go from polar coordinates to (x,y)?



$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases}$$

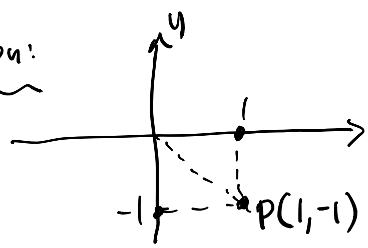
QUESTION: How do we go from Cartesian to polar?



$$r = \sqrt{x^2 + y^2}$$
$$\tan\theta = \frac{y}{x}$$

EXAMPLE: Convert the point $P(1,-1)$ in Cartesian coordinates to polar coordinates.

Solution:



$$r = \sqrt{1^2 + (-1)^2} = \sqrt{2}$$
$$\tan\theta = \frac{-1}{1} = -1$$

Since P lies in the 4^{th} quadrant we can choose $\theta = -\frac{\pi}{4}$

So, one answer is P in polar coord. $\left(\sqrt{2}, -\frac{\pi}{4}\right)$.

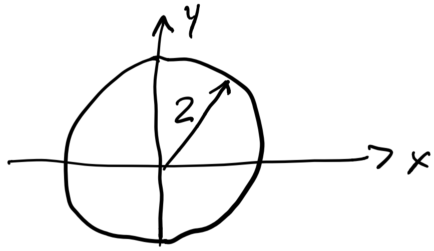
Other possibilities: $\left(\sqrt{2}, \frac{7\pi}{4}\right)$, $\left(-\sqrt{2}, \frac{3\pi}{4}\right)$.

GRAPHING IN POLAR COORDINATES

DISCLOSURE:
- NON-INTUITIVE
- IT'S A PAIN IN THE NECK
- BUT YOU SHOULD SEE IT ONCE IN YOUR LIFE
- AND TODAY IS THE DAY!

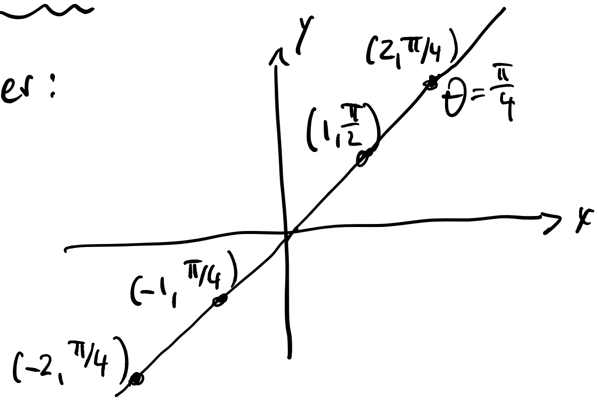
EXAMPLE: What is the curve $r = 2$?

Answer:



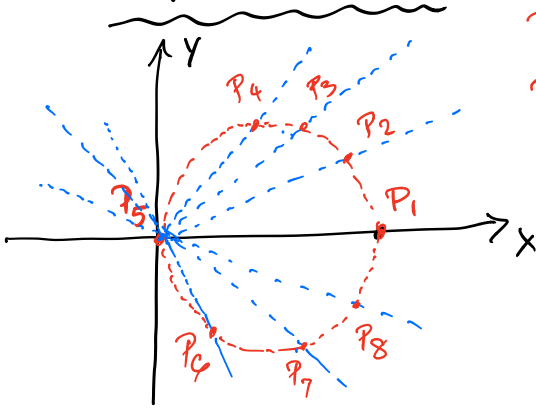
EXAMPLE: What does $\theta = \frac{\pi}{4}$ look like?

Answer:



EXAMPLE: (a) Sketch the curve $r = 2\cos\theta$.
(b) Find a Cartesian equation for this curve.

TRY SOME POINTS:



	θ	r	x	y
P_1	0	2	2	0
P_2	$\pi/6$	$\sqrt{3}$	$\sqrt{3}\cos \pi/6$	$\sqrt{3}\sin \pi/6$
P_3	$\pi/4$	$\sqrt{2}$	$\sqrt{2}\cos \pi/4$	$\sqrt{2}\sin \pi/4$
P_4	$\pi/3$	1	$\cos \pi/3$	$\sin \pi/3$
P_5	$\pi/2$	0	0	0
P_6	$2\pi/3$	-1	$-\cos 2\pi/3$	$-\sin \frac{2\pi}{3}$
P_7	$3\pi/4$	$-\sqrt{2}$	$-\sqrt{2}\cos 3\pi/4$	$-\sqrt{2}\sin 3\pi/4$
P_8	$5\pi/6$	$-\sqrt{3}$	$-\sqrt{3}\cos 5\pi/6$	$-\sqrt{3}\sin 5\pi/6$

(b) From $x = r\cos\theta \Rightarrow \cos\theta = \frac{x}{r}$

so $r = 2\cos\theta$ becomes $r = 2\cdot\frac{x}{r}$.

Therefore: $r^2 = 2x$, or $\boxed{x^2+y^2 = 2x}$

SIMPLIFY TO RECOGNIZE THE CURVE:

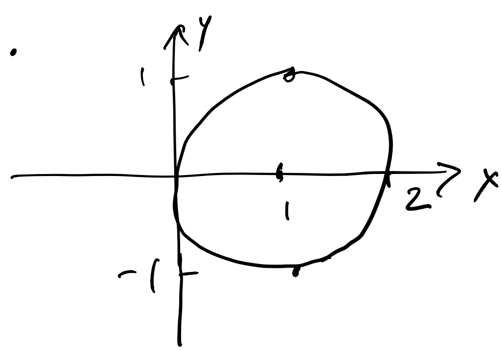
$x^2-2x+y^2=0 \Rightarrow \boxed{(x-1)^2+y^2 = 1}$

COMPLETE THE SQUARE

$x^2-2x+1-1+y^2=0$

$(x-1)^2+y^2-1=0 \Rightarrow (x-1)^2+y^2=1$

$\Rightarrow$ CIRCLE WITH THE CENTER AT $(1,0)$ OF RADIUS 1.



READ PAGES 662, 663

TANGENTS TO POLAR CURVES 😀

SUPPOSE WE ARE GIVEN A POLAR CURVE $r = r(\theta)$.

HOW CAN WE FIND A TANGENT?

STEP 1 We have: $x = r\cos\theta = r(\theta)\cos\theta = f(\theta)$
$y = r\sin\theta = r(\theta)\sin\theta = g(\theta)$

So, we are viewing θ as a parameter. 🤓

STEP 2. For parametric curves we know: $\dfrac{dy}{dx} = \dfrac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}}$

What are $\dfrac{dy}{d\theta}$ and $\dfrac{dx}{d\theta}$ in terms of r?

$$\frac{dy}{d\theta} = r'(\theta)\sin\theta + r(\theta)\cos\theta$$

$$\frac{dx}{d\theta} = r'(\theta)\cos\theta - r(\theta)\sin\theta$$

So:

$$\boxed{\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{\frac{dr}{d\theta}\sin\theta + r(\theta)\cos\theta}{\frac{dr}{d\theta}\cos\theta - r(\theta)\sin\theta}}$$

EXAMPLE: $r = 1 + \sin\theta$. Find the slope at $\theta = \pi/3$.

$$\frac{dy}{dx} = \frac{\cos\theta\sin\theta + (1+\sin\theta)\cos\theta}{\cos^2\theta - (1+\sin\theta)\sin\theta} = \frac{\cos\theta\,(1+2\sin\theta)}{\cos^2\theta - \sin^2\theta - \sin\theta}$$

$$\frac{dy}{dx}\left(\text{at } \theta = \frac{\pi}{3}\right) = \frac{\frac{1}{2}(1+\sqrt{3})}{\frac{1}{4} - \frac{3}{4} - \frac{\sqrt{3}}{2}} = \frac{\cancel{\frac{1}{2}}(1+\sqrt{3})}{\cancel{\frac{1}{2}}\left(-\sqrt{3} + \frac{1}{2} - \frac{3}{2}\right)} = \frac{\cancel{1+\sqrt{3}}}{-(\cancel{1+\sqrt{3}})} = -1$$

$\cos\frac{\pi}{3} = \frac{1}{2}$

$\sin\frac{\pi}{3} = \frac{\sqrt{3}}{2}$

FINISH (b) in EXAMPLE 9 ON PAGE 664 in textbook.

SUMMARY

I. POLAR COORDINATES

$x = r\cos\theta$ $\qquad$ $r = \sqrt{x^2 + y^2}$

$y = r\sin\theta$ $\qquad$ $\tan\theta = y/x$

II. PLOTTING IN POLAR COORDINATES

III. TANGENT TO CURVE IN POLAR COORDINATES

$$\boxed{(y - y_0) = \frac{dy}{dx}(x_0)\,(x - x_0)}$$

where:

$$\boxed{\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{\frac{dr}{d\theta}\sin\theta + r(\theta)\cos\theta}{\frac{dr}{d\theta}\cos\theta - r(\theta)\sin\theta}}$$