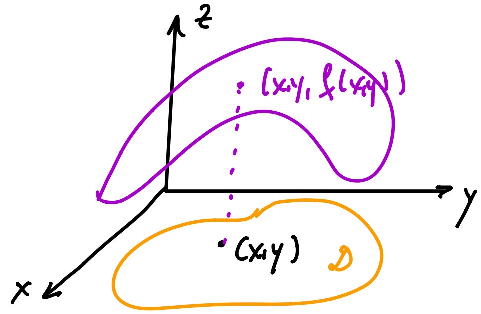
GRAPHS

VISUALIZING A FUNCTION OF TWO VARIABLES



Definition: If f is a function of TWO VARIABLES within domain $\mathcal{D}$, then the GRAPH of f is the set of all points $(x,y,z)\in\mathbb{R}^3$ such that $z=f(x,y)$ and $(x,y)\in\mathcal{D}$.

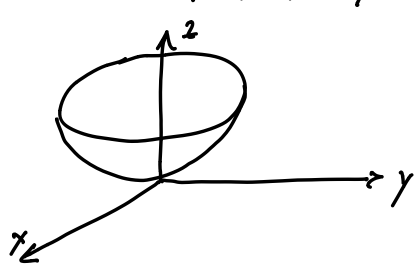
Thus, the graph of a function of two variables is a SURFACE in $\mathbb{R}^3$ with equation: $z=f(x,y)$.

EXAMPLE: Find domain, range, and sketch the graph of

$$f(x,y)=x^2+y^2.$$

Solution: $z=x^2+y^2$: $\mathcal{D}=\mathbb{R}^2$, $\mathcal{R}=\{z\in\mathbb{R} \mid z\geq 0\}=[0,\infty)$

Graph:
- horizontal traces at z^* are circles $x^2+y^2=z^*$
- vertical traces are parabolas; for example for $x=0 \Rightarrow z=y^2$, for $y=0\Rightarrow z=x^2$.



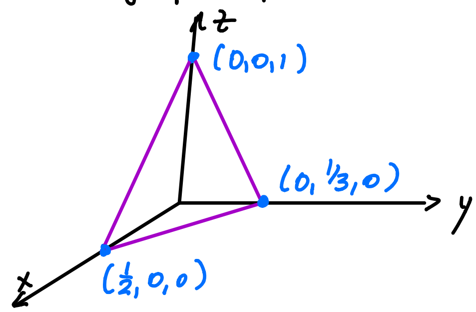
EXAMPLE: Domain, range, and graph of $g(x,y) = 1-2x-3y$.

Solution: $g(x,y)$ is a LINEAR FUNCTION

GRAPH OF A LINEAR FUNCTION IS A PLANE

$z = 1-2x-3y$ or $\boxed{2x+3y+z-1=0}$

To plot the graph, find the x, y, z intercepts :



$\mathcal{D} = \mathbb{R}$

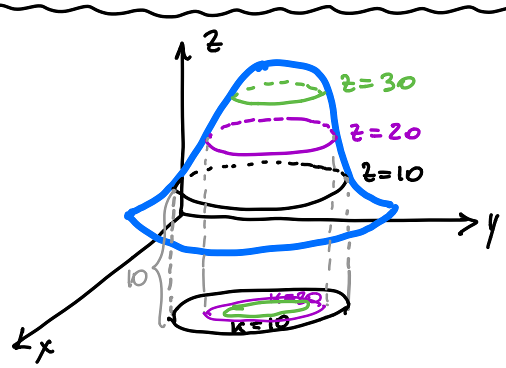
$\mathcal{R} = \mathbb{R}$.

LEVEL CURVES

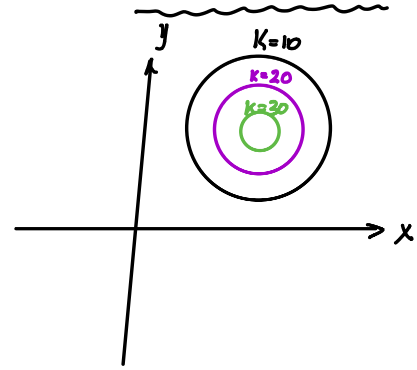
Definition: The LEVEL CURVES of a function f of two variables are the curves defined by equations $\boxed{f(x,y) = k}$, where k is a constant.

(curves at constant height k, but plotted in xy-plane!)

VISUALIZATION OF LEVEL CURVES:



LEVEL CURVES



EXAMPLE: Sketch the Level curves of $f(x,y) = 6-3x-2y$ for the values of $k=-6, 0, 6, 12$.

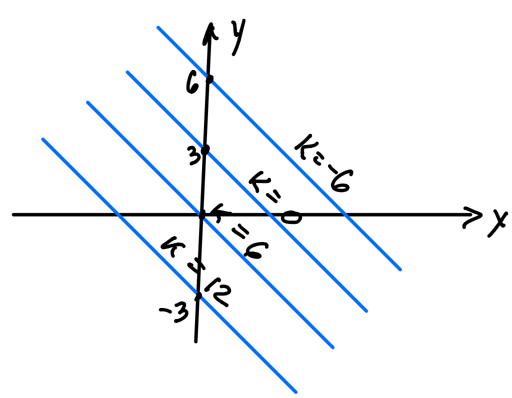
Solution: The Level curves are $6-3x-2y=k$ or $\boxed{3x+2y+(k-6)=0}$

So, for $k=-6$: $3x+2y-12=0 \Rightarrow y=-\frac{3}{2}x+6$

$k=0$: $3x+2y-6=0 \Rightarrow y=-\frac{3}{2}x+3$

$k=6$: $3x+2y=0 \Rightarrow y=-\frac{3}{2}x$

$k=12$: $3x+2y+6=0 \Rightarrow y=-\frac{3}{2}x-3$



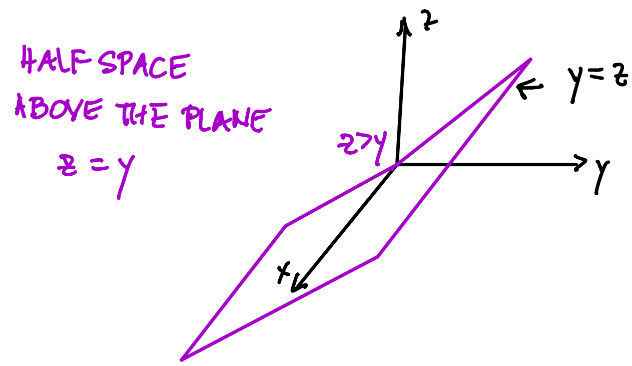
FUNCTIONS OF 3 VARIABLES

= a rule that assigns to each triple (x,y,z) in a domain $D \subset \mathbb{R}^3$ a unique real number, denoted by $f(x,y,z)$.

Example: Find the domain of $f(x,y,z) = \ln(z-y) + xy \sin z$. Sketch it.

Solution: $\mathcal{D} = \{ (x,y,z) \mid z - y > 0 \}$

Plot the set of points $(x,y,z) \in \mathbb{R}^3$ such that $z > y$:

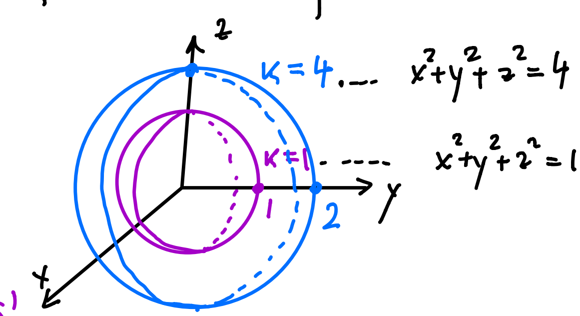


EXAMPLE: What are level surfaces of $f(x,y,z) = x^2+y^2+z^2$?

Answer: Level surface is defined by $\boxed{f(x,y,z) = k}$ (this is generalization to $\mathbb{R}^3$ of the level curve definition $f(x,y)=k$)

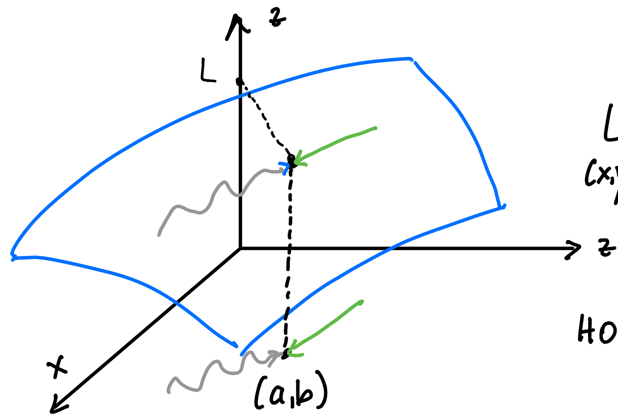
So, level surfaces in $\mathbb{R}^3$ are surfaces $\boxed{x^2+y^2+z^2 = k}$, $k \geq 0$.

$\Rightarrow$ family of concentric spheres!



WE CAN VISUALIZE FUNCTIONS OF 3 VARIABLES THROUGH LEVEL SURFACES!

14.2. LIMITS AND CONTINUITY



$$\lim_{(x,y)\to(a,b)} f(x,y) = L$$

HOW TO DEFINE THIS LIMIT?

DEFINITION: Let f be a function of two variables whose domain D includes points arbitrarily close to (a,b). Then we say that THE LIMIT OF f AS (x,y) APPROACHES (a,b) IS L and we write

$$\lim_{(x,y)\to(a,b)} f(x,y) = L$$

if for every number $\varepsilon > 0$ there is a corresponding number $\delta > 0$ such that if $(x,y) \in D$ and $\underbrace{d((x,y),(a,b))}_{\text{distance from } (x,y) \text{ to } (a,b) = \sqrt{(x-a)^2+(y-b)^2}} < \delta$ then $|f(x,y) - L| < \varepsilon$

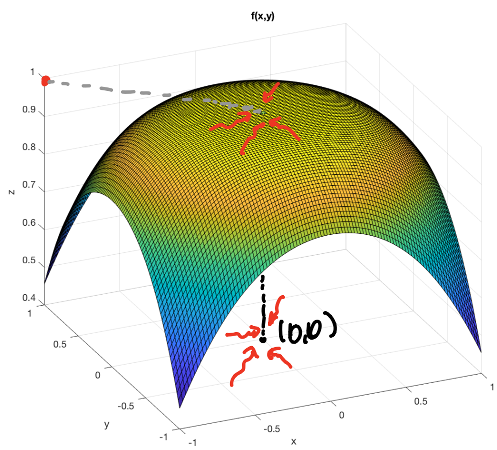
OTHER NOTATION: $\displaystyle\lim_{\substack{x\to a\\ y\to b}} f(x,y) = L$; $f(x,y) \to L$ as $(x,y) \to (a,b)$.

EXAMPLE: Does the limit as $(x,y) \to (0,0)$ exist for the following two functions:

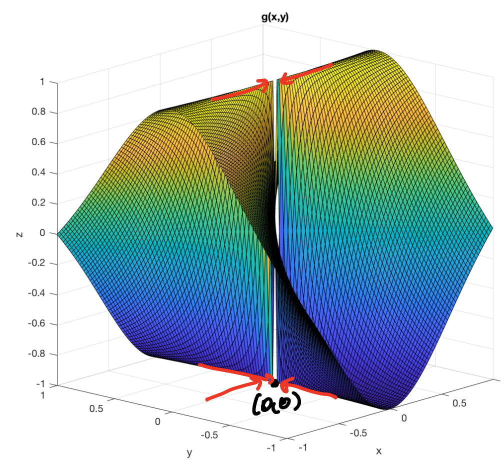
$$f(x,y) = \frac{\sin(x^2+y^2)}{x^2+y^2}, \quad g(x,y) = \frac{x^2-y^2}{x^2+y^2} ?$$

Solution: First notice that f and g are not defined at $(x,y)=(0,0)$.
But they are both defined everywhere else, in particular, for all the points close to $(0,0)$.

We can calculate the values of f and g for several points near $(0,0)$ to see how they behave, or plot the surfaces:



$f(x,y)$



$g(x,y)$

From the images, we can see that $\lim_{(x,y)\to(0,0)} g(x,y)$ does not exist,

while $\lim_{(x,y)\to(0,0)} f(x,y) = 1$.

Proof using the definition of the limit:

$\lim_{(x,y)\to(0,0)} g(x,y) = \lim_{(x,y)\to(0,0)} \frac{x^2-y^2}{x^2+y^2} \ldots$

- Take the path C_1 = x axis $(y=0)$. Then $\lim_{\substack{(x,y)\to(0,0)\\ \text{ALONG } C_1}} \frac{x^2-y^2}{x^2+y^2} = \lim_{x\to 0} \frac{x^2}{x^2} = 1$

- Take the path C_2 = y axis $(x=0)$. Then $\lim_{\substack{(x,y)\to(0,0)\\ \text{ALONG } C_2}} \frac{x^2-y^2}{x^2+y^2} = \lim \frac{-y^2}{y^2} = -1$

$\Rightarrow$ LIMIT DOES NOT EXIST!

IN GENERAL:

The are ∞ many paths along which (x,y) can approach (a,b).

If C_1 and C_2 are two different paths, and if

$$f(x,y) \to L_1 \quad \text{as} \quad (x,y) \to (a,b) \text{ along path } C_1$$
$$f(x,y) \to L_2 \quad \text{as} \quad (x,y) \to (a,b) \text{ along path } C_2$$

where $L_1 \neq L_2$, then $\lim\limits_{(x,y)\to(a,b)} f(x,y)$ DOES NOT EXIST.

EXAMPLE: Does the limit $\lim\limits_{(x,y)\to(0,0)} \dfrac{xy^2}{x^2+y^4}$ exist?

Solution: Hint: • check the lines $y = mx$ and show $f(x, mx) \to 0$ as $x \to 0$

• check the parabola $x = y^2$ and show $f(y^2, y) \to \frac{1}{2}$ as $y \to 0$

$\Rightarrow$ NO LIMIT!

CALCULATING THE LIMIT:

EXAMPLE: Prove that $\lim_{(x,y)\to(0,0)} \frac{3x^2y}{x^2+y^2} = 0$.

Solution: Given any $\varepsilon>0$, find $\delta>0$, such that whenever $0<\sqrt{x^2+y^2}<\delta$ then $\left|\frac{3x^2y}{x^2+y^2}-0\right|<\varepsilon$.

Try to estimate (express) $\left|\frac{3x^2y}{x^2+y^2}\right|$ in term of δ:

$$\left|\frac{3x^2y}{x^2+y^2}\right| = \frac{3x^2|y|}{x^2+y^2} \le \frac{3(x^2+y^2)|y|}{x^2+y^2} = 3|y| = 3\sqrt{y^2}$$

$$\le 3\sqrt{y^2+x^2} = 3\underbrace{\sqrt{x^2+y^2}}_{<\delta} < 3\delta$$

So, whenever $\sqrt{x^2+y^2}<\delta$, we have that $\left|\frac{3x^2y}{x^2+y^2}\right|<3\delta$

But we want $\left|\frac{3x^2y}{x^2+y^2}\right| < 3\delta < \varepsilon$.

From here we can see that for any $\varepsilon>0$ there is a $\delta=\frac{\varepsilon}{3}>0$ given in terms of ε, such that whenever $0<\sqrt{x^2+y^2}<\delta$ we have $\left|\frac{3x^2y}{x^2+y^2}\right|<\varepsilon$.

This completes the proof. (Q.E.D.)

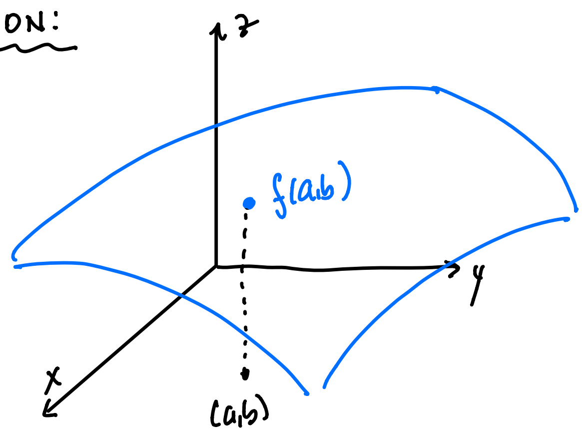
CONTINUITY

DEFINITION:

(I) f is CONTINUOUS AT (a,b) if $\lim_{(x,y)\to(a,b)} f(x,y) = f(a,b)$

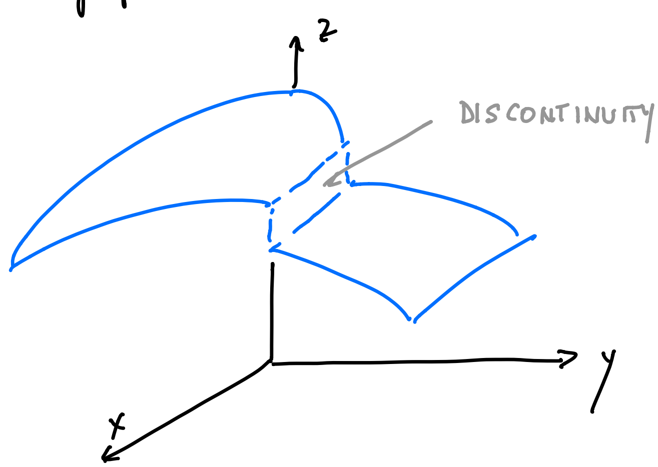
(II) f is CONTINUOS ON D if f is continuous at every point on D.

INTUITION:



For continuos functions: • if (x,y) changes by a small amount, then also $f(x,y)$ changes by a small amount.

- the surface graph has no holes or breaks.



PROPERTIES OF CONTINUOUS FUNCTIONS

SUMS, PRODUCTS, DIFFERENCES, AND QUOTIENTS OF CONTINUOUS FUNCTIONS ARE CONTINUOUS FUNCTIONS ON THEIR DOMAINS

EXAMPLES OF CONTINUOUS FUNCTIONS

1. POLYNOMIALS (sums of the terms of the form $c\,x^m y^n$)

EXAMPLE: $f(x,y) = 3x^3y + 2yx + 3x + y - 1$

ALL POLYNOMIALS ARE CONTINUOUS FUNCTIONS ON $\mathbb{R}^2$

2. RATIONAL FUNCTIONS ON THEIR DOMAINS (QUOTIENTS OF POLYNOMIALS)

EXAMPLE: $g(x,y) = \dfrac{3xy + x}{x^2+y^2}$ is continuous for all $(x,y) \neq (0,0)$

EXAMPLE: Let $g(x,y) = \begin{cases} \dfrac{3x^2y}{x^2+y^2} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0) \end{cases}$

Is this function continuous on $\mathbb{R}^2$?

Answer:

1. $\dfrac{3x^2y}{x^2+y^2}$ is a rational function and is therefore continuous everywhere except at $(x,y) = (0,0)$.
2. At $(0,0)$ the function g is defined to be 0.
3. If $\lim\limits_{(x,y)\to(0,0)} g(x,y) = g(0,0) = 0$ then g is continuos.
4. We proved earlier that $\lim\limits_{(x,y)\to(0,0)} \dfrac{3x^2y}{x^2+y^2} = 0 = g(0,0)$ ✓✓ CONTINUOS EVERYWHERE

EXAMPLE: Is $g(x,y) = \begin{cases} \dfrac{x^2-y^2}{x^2+y^2} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0) \end{cases}$ continuous?

Answer: We showed earlier that $\lim\limits_{(x,y)\to(0,0)} \dfrac{x^2-y^2}{x^2+y^2}$ does not exist,

so it is not true that $\lim\limits_{(x,y)\to(0,0)} \dfrac{x^2-y^2}{x^2+y^2}$ exists and is equal to 0.

Thus g is NOT CONTINUOUS.

CONTINUITY OF COMPOSITE FUNCTIONS $h = g(f(x,y))$ or $h = g \circ f$

If $\left\{ \begin{array}{l} (1)\ f \text{ is continuous} \\ (2)\ g \text{ is continuous} \\ \text{and defined on the range of } f \end{array} \right\}$, then $h = g \circ f$ is continuous on the domain $\mathcal{D}$ of f.

Notice: Here g is a function of one variable, and f is a function of (x,y).

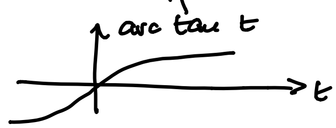
EXAMPLE: Let $f(x,y) = \dfrac{y}{x}$ and $g(t) = \arctan t$.

Where is $h = g(f(x,y))$ continuous?

Answer: $\bullet\,(x,y) \xrightarrow{f} \bullet\, f(x,y) \xrightarrow{g} g(\underbrace{f(x,y)}_{t})$

① $f(x,y) = \dfrac{y}{x}$ is continuous for all (x,y) such that $x \neq 0$

② $g(t) = \arctan t$



is continuous everywhere

$\Rightarrow$ $h = g(f(x,y))$ is continuous at all (x,y) except at points $(0,y)$

LIMITS   FUNCTIONS OF 3 OR MORE VARIABLES

Let $y = f(\vec{x})$, $\vec{x} = \langle x_1, x_2, \dots, x_n \rangle \in \mathbb{R}^n$.

Definition: $\lim\limits_{\vec{x} \to \vec{a}} f(\vec{x}) = L$ means that

FOR EVERY $\varepsilon > 0$ THERE IS A CORRESPONDING NUMBER $\delta > 0$ SUCH THAT

if $0 < |\vec{x} - \vec{a}| < \delta$ AND $\vec{x} \in D$, THEN $|f(\vec{x}) - L| < \varepsilon$

In $\mathbb{R}^3$: $\vec{x} = \langle x, y, z \rangle$, $\vec{a} = \langle a, b, c \rangle$, $|\vec{x} - \vec{a}| = \sqrt{(x-a)^2 + (y-b)^2 + (z-c)^2}$

Definition: f is CONTINUOUS AT $\vec{a}$ if $\lim\limits_{\vec{x} \to \vec{a}} f(\vec{x}) = f(\vec{a})$

NEXT TIME: PARTIAL DERIVATIVES

😎