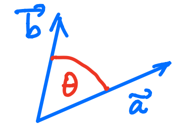
MATH 53 - Lecture 5

RECALL: For $\vec{a} \neq \vec{0}, \vec{b} \neq \vec{0}$ we have

$$\boxed{\cos\theta = \frac{\vec{a}}{|\vec{a}|} \cdot \frac{\vec{b}}{|\vec{b}|}}$$



Example: Let $\vec{a} = \vec{i}$, $\vec{b} = \vec{i} + \vec{j}$, what is the angle between $\vec{a}$ and $\vec{b}$?

$$\cos\theta = \frac{\vec{a}}{|\vec{a}|} \cdot \frac{\vec{b}}{|\vec{b}|} = \frac{\vec{i}}{1} \cdot \frac{\vec{i}+\vec{j}}{\sqrt{2}} = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}, \quad \theta = \frac{\pi}{4}$$

PERPENDICULAR OR ORTHOGONAL VECTORS

Definition:
- If $\vec{a} \neq \vec{0}, \vec{b} \neq \vec{0}$. We say that $\vec{a}$ is PERPENDICULAR to $\vec{b}$ if $\theta = \frac{\pi}{2}$. (Notation $\vec{a} \perp \vec{b}$)
- The zero-vector is perpendicular to all vectors.

In terms of the dot product: $\vec{a} \cdot \vec{b} = |\vec{a}||\vec{b}| \cos\frac{\pi}{2} = 0$

So, $\vec{a} \perp \vec{b} \Rightarrow \vec{a} \cdot \vec{b} = 0$.

The converse is also true: Suppose $\vec{a} \cdot \vec{b} = 0$. Then either $\vec{a} \neq \vec{0}, \vec{b} \neq \vec{0}$ but $\cos\theta = 0 \Rightarrow \theta = \frac{\pi}{2}$, or at least one of the vectors is the zero vector. $\Rightarrow \vec{a} \perp \vec{b}$.

THUS:

TWO VECTORS $\vec{a}$ AND $\vec{b}$ ARE PERPENDICULAR IF AND ONLY IF $\vec{a} \cdot \vec{b} = 0$

Perpendicular = Orthogonal

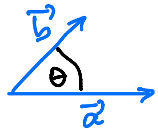
Example: Are $\vec{a} = 2\vec{i} + 2\vec{j} - \vec{k}$ and $\vec{b} = 5\vec{i} - 4\vec{j} + 2\vec{k}$ perpendicular?

Solution: $\vec{a} \cdot \vec{b} = \langle 2, 2, -1 \rangle \cdot \langle 5, -4, 2 \rangle = 10 + (-8) - 2 = 0$

Yes, $\vec{a} \perp \vec{b}$.

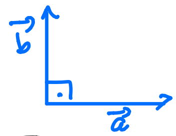
Question: What is the sign of $\vec{a}\cdot\vec{b}$ for the following 3 cases:

(a) $0 < \theta < \frac{\pi}{2}$



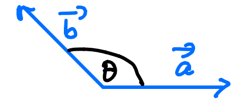
$$\boxed{\vec{a}\cdot\vec{b} = |\vec{a}||\vec{b}|\cos\theta > 0}$$

(b) $\theta = \frac{\pi}{2}$



$$\boxed{\vec{a}\cdot\vec{b} = 0}$$

(c) $\frac{\pi}{2} < \theta < \pi$

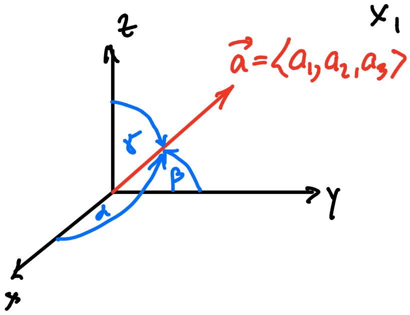


$$\boxed{\vec{a}\cdot\vec{b} < 0}$$

$\Rightarrow$ THE SIGN OF $(\vec{a}\cdot\vec{b})$ MEASURES THE EXTENT TO WHICH $\vec{a}$ AND $\vec{b}$ POINT IN THE SAME DIRECTION. ☺

DIRECTION ANGLES AND DIRECTION COSINES

DIRECTION ANGLES = ANGLES THAT $\vec{a}$ MAKES WITH THE POSITIVE x, y AND z AXES



$$\left.\begin{aligned} \cos\alpha &= \frac{\vec{a}\cdot\vec{i}}{|\vec{a}|} = \frac{a_1}{|\vec{a}|} \\ \cos\beta &= \frac{\vec{a}\cdot\vec{j}}{|\vec{a}|} = \frac{a_2}{|\vec{a}|} \\ \cos\gamma &= \frac{\vec{a}\cdot\vec{k}}{|\vec{a}|} = \frac{a_3}{|\vec{a}|} \end{aligned}\right\} \text{DIRECTION COSINES} \quad (\star)$$

QUESTION: What is $\cos^2\alpha + \cos^2\beta + \cos^2\gamma$?

Answer:

$$\cos^2\alpha + \cos^2\beta + \cos^2\gamma = \frac{a_1^2}{|\vec{a}|^2} + \frac{a_2^2}{|\vec{a}|^2} + \frac{a_3^2}{|\vec{a}|^2} = \frac{|\vec{a}|^2}{|\vec{a}|^2} = 1$$

$$\Rightarrow \boxed{\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1}$$

DIRECTION ANGLES

CONSEQUENCE:

FROM (★)

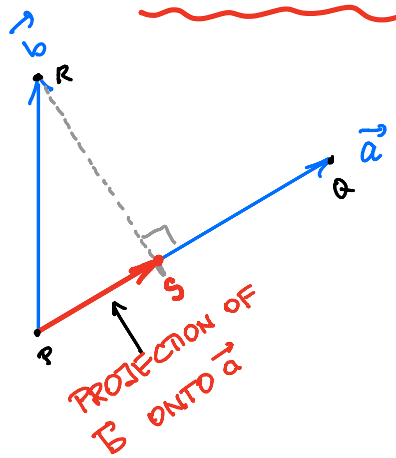
$$\vec{a} = \langle a_1, a_2, a_3 \rangle = \langle |\vec{a}|\cos\alpha, |\vec{a}|\cos\beta, |\vec{a}|\cos\gamma \rangle =$$

$$= |\vec{a}| \langle \cos\alpha, \cos\beta, \cos\gamma \rangle$$

$$\Rightarrow \boxed{\langle \cos\alpha, \cos\beta, \cos\gamma \rangle = \frac{\vec{a}}{|\vec{a}|}}$$

UNIT VECTOR IN THE DIRECTION OF $\vec{a}$

PROJECTIONS



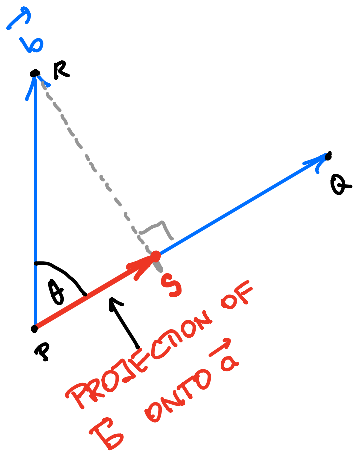
VECTOR $\vec{PS}$ IS CALLED THE VECTOR PROJECTION OF $\vec{b}$ ONTO $\vec{a}$ AND DENOTED BY:

$$\boxed{\vec{PS} = \text{proj}_{\vec{a}} \vec{b}}$$

THE LENGTH OF $\vec{PS}$, $|\vec{PS}|$, IS CALLED THE SCALAR PROJECTION OF $\vec{b}$ ONTO $\vec{a}$, ALSO CALLED THE COMPONENT OF $\vec{b}$ ALONG $\vec{a}$:

$$\boxed{|\vec{PS}| = \text{comp}_{\vec{a}} \vec{b}}$$

How to calculate $\text{proj}_{\vec{a}} \vec{b}$ and $\text{comp}_{\vec{a}} \vec{b}$ from the components of $\vec{a}$ and $\vec{b}$?



① FROM THE TRIANGLE PSR $\Rightarrow$

$$\cos\theta = \frac{|\vec{PS}|}{|\vec{b}|}$$

THUS: $|\vec{PS}| = |\vec{b}|\cos\theta$

EXPRESS $\cos\theta$ IN TERMS OF THE DOT PRODUCT:

$$|\vec{PS}| = |\vec{b}|\,\frac{\vec{a}\cdot\vec{b}}{|\vec{a}||\vec{b}|} \Rightarrow |\vec{PS}| = \frac{\vec{a}\cdot\vec{b}}{|\vec{a}|}$$

$$\Rightarrow \boxed{\operatorname{comp}_{\vec{a}}\vec{b} = \frac{\vec{a}\cdot\vec{b}}{|\vec{a}|}}$$

② SINCE $\vec{PS}$ IS IN THE DIRECTION OF $\vec{a}$, AND WE KNOW THE LENGTH OF $\vec{PS}$, WE JUST NEED TO MULTIPY $|\vec{PS}|$ BY THE UNIT VECTOR IN THE DIRECTION OF $\vec{a}$:

$$\boxed{\operatorname{proj}_{\vec{a}}\vec{b} = \frac{\vec{a}\cdot\vec{b}}{|\vec{a}|}\,\frac{\vec{a}}{|\vec{a}|} = \underbrace{\frac{\vec{a}\cdot\vec{b}}{|\vec{a}|^2}}_{\text{scalar}}\;\underbrace{\vec{a}}_{\text{vector}}}$$

EXAMPLE: (Prob 43, pg 813) Find $\operatorname{proj}_{\vec{a}}\vec{b}$ and $\operatorname{comp}_{\vec{a}}\vec{b}$ if $\vec{a} = 3\vec{i} - 3\vec{j} + \vec{k}$, $\vec{b} = 2\vec{i} + 4\vec{j} - \vec{k}$.

Solution:

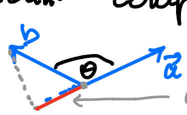
$$\operatorname{comp}_{\vec{a}}\vec{b} = \frac{\vec{a}\cdot\vec{b}}{|\vec{a}|} = \frac{\langle 3,-3,1\rangle\cdot\langle 2,4,-1\rangle}{\sqrt{3^2+3^2+1}} = \frac{6-12-1}{\sqrt{19}} = -\frac{7}{\sqrt{19}}$$

$$\operatorname{proj}_{\vec{a}}\vec{b} = -\frac{7}{19}\langle 3,-3,1\rangle = \left\langle -\frac{21}{19}, \frac{21}{19}, -\frac{1}{19}\right\rangle$$ //

QUESTION: What is the meaning of $\operatorname{comp}_{\vec{a}}\vec{b} < 0$???

Answer: Recall: $\operatorname{comp}_{\vec{a}}\vec{b} = \dfrac{\vec{a}\cdot\vec{b}}{|\vec{a}|}$ and $\vec{a}\cdot\vec{b} < 0$ means $\theta > \frac{\pi}{2}$.

Thus:

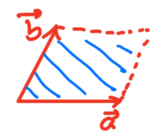


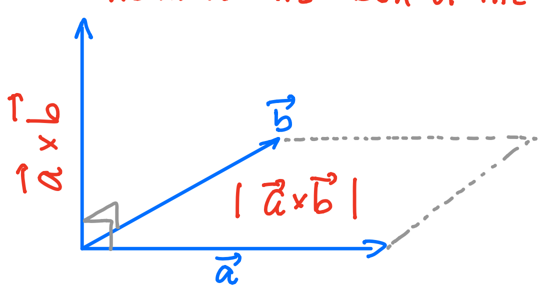
$\operatorname{comp}_{\vec{a}}\vec{b} < 0$ for $\frac{\pi}{2} < \theta < \pi$.

12.4. THE CROSS PRODUCT

$$\vec{a} \times \vec{b} = ?$$

$$\vec{a} \times \vec{b} = \begin{cases} \text{VECTOR PERPENDICULAR TO } \vec{a} \text{ AND } \vec{b} \text{ (right-hand rule)} \\ \text{ITS LENGTH IS THE AREA OF THE PARALLELOGRAM} \end{cases}$$





If $\vec{a} = \langle a_1, a_2, a_3 \rangle$ AND $\vec{b} = \langle b_1, b_2, b_3 \rangle$ WHAT IS $\vec{a} \times \vec{b}$ IN COMPONENTS?

DETERMINANTS

DETERMINANT OF ORDER 2:
$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

(Multiply across the diagonals and subtract.)

DETERMINANT OF ORDER 3:

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = a_1 \begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} - a_2 \begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix} + a_3 \begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix}$$

EXAMPLE:

$$\begin{vmatrix} 1 & 2 & -1 \\ 1 & 0 & 1 \\ 3 & -1 & 1 \end{vmatrix} = 1 \cdot \begin{vmatrix} 0 & 1 \\ -1 & 1 \end{vmatrix} - 2 \begin{vmatrix} 1 & 1 \\ 3 & 1 \end{vmatrix} - 1 \begin{vmatrix} 0 & 0 \\ 3 & -1 \end{vmatrix}$$

$$= (0 \cdot 1 - 1 \cdot (-1)) - 2(1-3) - 1(0-3)$$

$$= 1 + 4 + 3 = 8$$ 😎

BACK TO THE CROSS PRODUCT

Let $\vec{a} = a_1 \vec{i} + a_2 \vec{j} + a_3 \vec{k}$ and $\vec{b} = b_1 \vec{i} + b_2 \vec{j} + b_3 \vec{k}$.

Then:

$$\vec{a} \times \vec{b} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} \vec{i} - \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} \vec{j} + \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} \vec{k}$$

DEFINITION: The cross product $\vec{a} \times \vec{b}$ is a vector defined by:

$$\vec{a} \times \vec{b} = (a_2 b_3 - a_3 b_2)\vec{i} - (a_1 b_3 - a_3 b_1)\vec{j} + (a_1 b_2 - a_2 b_1)\vec{k}$$

$$= \langle a_2 b_3 - a_3 b_2,\ a_3 b_1 - a_1 b_3,\ a_1 b_2 - a_2 b_1 \rangle \qquad (**)$$

EXAMPLE: If $\vec{a} = \langle 1, 0, 1 \rangle$, $\vec{b} = \langle 1, 2, 3 \rangle$, calculate $\vec{a} \times \vec{b}$.

Solution:

$$\vec{a} \times \vec{b} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 0 & 1 \\ 1 & 2 & 3 \end{vmatrix} = -2\vec{i} - 2\vec{j} + 2\vec{k}$$

PROBLEM: Show that $\vec{a} \times \vec{b}$ is orthogonal to $\vec{a}$ and $\vec{b}$.

Solution: $\vec{a} \times \vec{b} \perp \vec{a}$ if and only if $(\vec{a} \times \vec{b}) \cdot \vec{a} = 0$

Calculate $(\vec{a} \times \vec{b}) \cdot \vec{a}$:

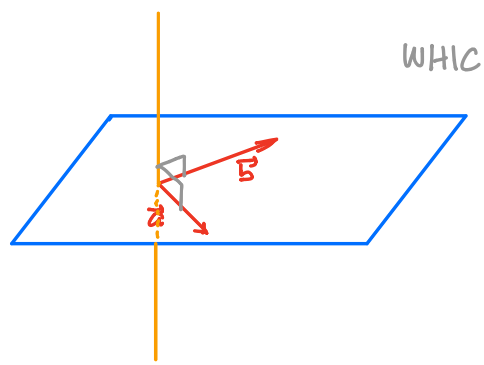
$$(\vec{a} \times \vec{b}) \cdot \vec{a} = \langle a_2 b_3 - a_3 b_2,\ a_3 b_1 - b_3 a_1,\ a_1 b_2 - b_1 a_2 \rangle \cdot \langle a_1, a_2, a_3 \rangle =$$

$=$ multiply through and show $= 0$

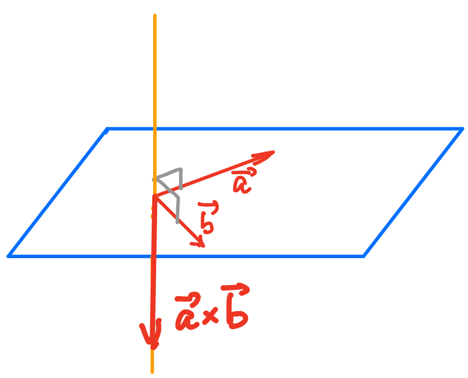
$(\vec{a} \times \vec{b}) \cdot \vec{b} = \ldots\ldots \quad = 0$

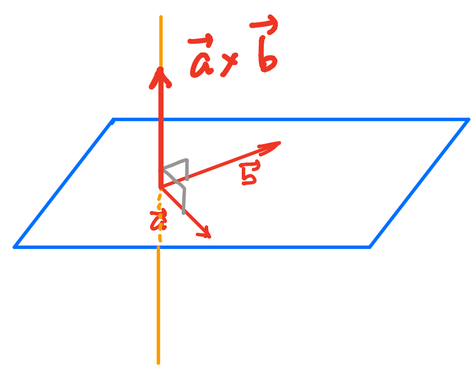
THUS: $\vec{a} \times \vec{b}$ IS PERPENDICULAR TO THE PLANE THROUGH $\vec{a}$ AND $\vec{b}$

WHICH DIRECTION???



From the definition (**) one can show that the direction of $\vec{a} \times \vec{b}$ is determined by THE RIGHT-HAND RULE



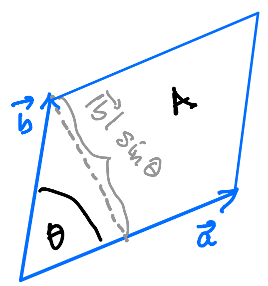


LENGTH $|\vec{a} \times \vec{b}|$ AND GEOMETRIC INTERPRETATION:

THEOREM: If θ is the angle between $\vec{a}$ and $\vec{b}$ $(0 \le \theta \le \pi)$, then:

$$|\vec{a} \times \vec{b}| = |\vec{a}|\,|\vec{b}| \sin \theta$$

GEOMERIC INTERPRETATION:



AREA OF THE PARALLELOGRAM:

$$A = \text{base} \cdot \text{altitude}$$
$$= |\vec{a}| \; |\vec{b}| \sin\theta$$
$$= |\vec{a} \times \vec{b}|$$

EXAMPLE: Are the following vectors parallel: $\begin{cases} \vec{a} = 3\vec{i} - \vec{j} + \vec{k} \\ \vec{b} = -\vec{i} + 3\vec{j} - 3\vec{k} \end{cases}$?

Solution: If $\vec{a} \parallel \vec{b}$ ($\vec{a}$ parallel to $\vec{b}$) then $\theta = 0$. This $\Rightarrow$

$$|\vec{a} \times \vec{b}| = |\vec{a}|\,|\vec{b}|\sin 0 = 0$$

The only vector with zero length is the zero vector.

Thus $\vec{a} \times \vec{b} = \vec{0}$.

$|\vec{a}| \neq 0$ $\quad$ $|\vec{b}| \neq 0$

The converse is also true: If $\vec{a} \times \vec{b} = \vec{0} \Rightarrow \ldots \Rightarrow \vec{a} \parallel \vec{b}$

Thus, if we can show $\vec{a} \times \vec{b} = \vec{0}$, then $\vec{a} \parallel \vec{b}$.

Calculate:

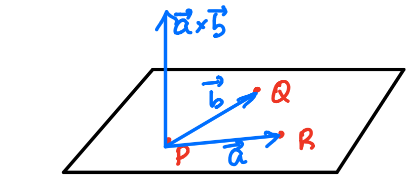
$$\vec{a} \times \vec{b} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 3 & -1 & 1 \\ -1 & 3 & -3 \end{vmatrix} = \langle 0, 0, 0 \rangle \; \checkmark$$

$$\Rightarrow \vec{a} \parallel \vec{b}$$

In general: $\boxed{\text{TWO NON-ZERO VECTORS } \vec{a} \text{ AND } \vec{b} \text{ ARE PARALLEL IF AND ONLY IF } \vec{a} \times \vec{b} = \vec{0}}$

PROBLEM: Find a vector perpendicular to the plane that passes through the points $P(1,4,6)$, $Q(-2,5,-1)$, $R(1,-1,1)$.

Solution:



$\vec{PR} = \vec{a} = \langle 0,-5,-5\rangle = -5\underbrace{\langle 0,1,1\rangle}_{\vec{c}}$

$\vec{PQ} = \vec{b} = \langle -3,1,-7\rangle$

$\vec{a}\times\vec{b}$ is perpendicular to both $\vec{a}$ & $\vec{b}$

$\vec{c}\times\vec{b}$ is also perpendicurar to $\vec{a}$ & $\vec{b}$ (easier to calculate)

$$\vec{c}\times\vec{b} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 0 & 1 & 1 \\ -3 & 1 & -7 \end{vmatrix} = \langle -8, -3, 3\rangle$$

(Book: $\langle -40,-15,15\rangle$)

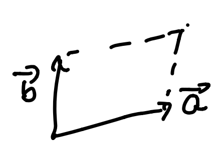
Notice: ∞ many solutions (all differ in scalar multiple)

PROBLEM: What is the area of the triangle from the previous example (triangle PQR)?

$$|\vec{a}\times\vec{b}| = |\langle 40,-15,15\rangle| = |+5\langle -8,-3,3\rangle| =$$

$$= 5\,|\langle -8,-3,3\rangle| = 5\sqrt{64+9+9} = 5\sqrt{82}$$

= AREA OF THE PARALLELOGRAM

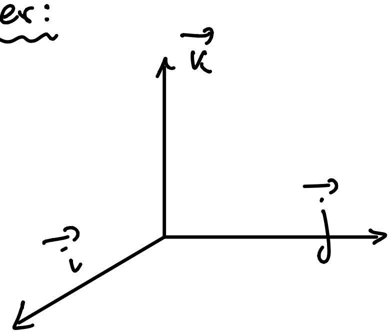


$$A_{\triangle PQR} = \frac{1}{2} A_{\text{PARALLELOGRAM}} = \frac{5}{2}\sqrt{82}$$

PROBLEM: $\vec{i} \times \vec{j} = ?$ $\qquad \vec{j} \times \vec{k} = ?$ $\qquad \vec{k} \times \vec{i} = ?$

$\vec{j} \times \vec{i} = ?$ $\qquad \vec{k} \times \vec{j} = ?$ $\qquad \vec{i} \times \vec{k} = ?$

Answer:



$$\vec{i} \times \vec{j} = \langle 1,0,0 \rangle \times \langle 0,1,0 \rangle = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{vmatrix} = \langle 0,0,1 \rangle = \vec{k}$$

$$\vec{j} \times \vec{i} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{vmatrix} = \langle 0,0,-1 \rangle = -\vec{k}$$

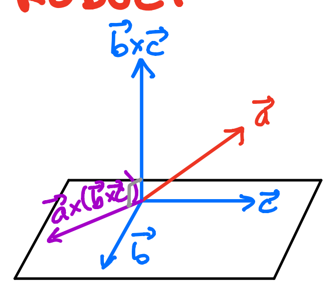
$\Rightarrow$ THE CROSS PRODUCT IS

NOT COMUTATIVE !

$$\boxed{\begin{aligned} \vec{i} \times \vec{j} &= \vec{k} \\ \vec{j} \times \vec{i} &= -\vec{k} \\ \vec{j} \times \vec{k} &= \vec{i} \\ \vec{k} \times \vec{j} &= -\vec{i} \\ \vec{k} \times \vec{i} &= \vec{j} \\ \vec{i} \times \vec{k} &= -\vec{j} \end{aligned}}$$

PROPERTIES OF THE CROSS PRODUCT

1. $\vec{a} \times \vec{b} = -\vec{b} \times \vec{a}$
2. $(c\vec{a}) \times \vec{b} = c(\vec{a} \times \vec{b}) = \vec{a} \times (c\vec{b})$
3. $\vec{a} \times (\vec{b} + \vec{c}) = \vec{a} \times \vec{b} + \vec{a} \times \vec{c}$
4. $(\vec{a} + \vec{b}) \times \vec{c} = \vec{a} \times \vec{c} + \vec{b} \times \vec{c}$
5. $\vec{a} \cdot (\vec{b} \times \vec{c}) = (\vec{a} \times \vec{b}) \cdot \vec{c}$
6. $\vec{a} \times (\vec{b} \times \vec{c}) = (\vec{a} \cdot \vec{c})\vec{b} - (\vec{a} \cdot \vec{b})\vec{c}$



THE TRIPLE PRODUCT $\vec a\cdot(\vec b\times\vec c)$

$$\vec a\cdot(\vec b\times\vec c) = (a_1\vec i + a_2\vec j + a_3\vec k)\cdot \begin{vmatrix} \vec i & \vec j & \vec k \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3\end{vmatrix} =$$

$$= (a_1\vec i+a_2\vec j+a_3\vec k)\left[(b_2c_3-b_3c_2)\vec i + (b_3c_1-b_1c_3)\vec j + (\qquad)\vec k\right]$$

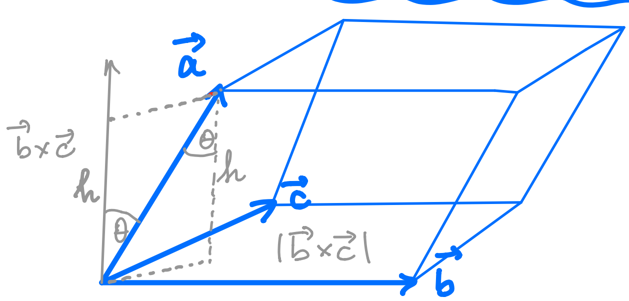
$$= a_1(b_2c_3-b_3c_2)+a_2(b_3c_1-b_1c_3)+(\qquad)a_3$$

(Like replacing $\vec i,\vec j,\vec k$ with a_1, a_2, a_3)

$$\Rightarrow \boxed{\vec a\cdot(\vec b\times\vec c) = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3\end{vmatrix}}$$

SCALAR TRIPLE PRODUCT

GEOMETRIC INTERPRETATION



$$V = |\vec b\times\vec c|\cdot h$$

$$h = \left|\operatorname{comp}_{\vec b\times\vec c}\vec a\right| = \frac{|\vec a\cdot(\vec b\times\vec c)|}{|\vec b\times\vec c|}$$

THUS:

$$V = \cancel{|\vec b\times\vec c|}\,\frac{|\vec a\cdot(\vec b\times\vec c)|}{\cancel{|\vec b\times\vec c|}}$$

$$= \vec a\cdot(\vec b\times\vec c)$$

$|\vec a\cdot(\vec b\times\vec c)|$ = VOLUME OF THE PARALLELEPIPED DETERMINED BY $\vec a$, $\vec b$, AND $\vec c$

EXAMPLE: Use the scalar triple product to show that the vectors $\vec{a} = \langle 1, 4, -7 \rangle$, $\vec{b} = \langle 2, -1, 4 \rangle$, $\vec{c} = \langle 0, -9, 18 \rangle$ are COPLANAR.

Solution: Calculate the volume of the parallelepiped determined by $\vec{a}$, $\vec{b}$, and $\vec{c}$:

$$V = |\vec{a} \cdot (\vec{b} \times \vec{c})| = ?$$

$$|\vec{a} \cdot (\vec{b} \times \vec{c})| = \left| \begin{matrix} 1 & 4 & -7 \\ 2 & -1 & 4 \\ 0 & -9 & 18 \end{matrix} \right| = (-18+36)+4(36)-7(-18)=0$$

Since the volume $V=0 \Rightarrow \vec{a}, \vec{b}, \vec{c}$ must be coplanar.

DONE!