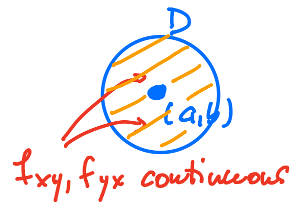
CLAIRAUT'S THEOREM

Suppose f is defined on a disk D that contains (a,b).
If f_{xy} and f_{yx} are both CONTINUOUS on D, then

$$\boxed{f_{xy}(a,b) = f_{yx}(a,b)}$$



EXAMPLE: (Problem 59, pg 925) $u = x^4y^3 - y^4$. Does Clairaut's Theorem hold on $\mathbb{R}^2$, and verify conclusions.

Solution: u is a polynomial $\Rightarrow$ continuous on $\mathbb{R}^2 \Rightarrow$ C. Theorem holds

- verify: $u_{xy} \overset{?}{=} u_{yx}$:

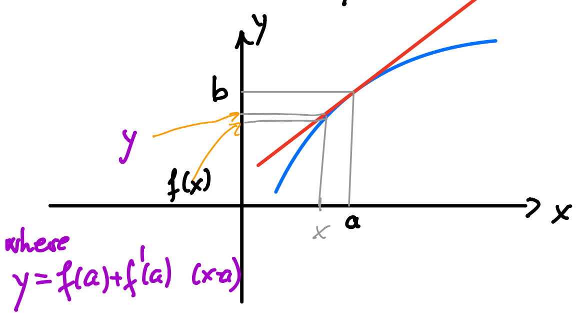
$$u_{xy} = \frac{\partial}{\partial y}(u_x) = \frac{\partial}{\partial y}(4x^3y^3) = 12x^3y^2$$

$$u_{yx} = \frac{\partial}{\partial x}(u_y) = \frac{\partial}{\partial x}(3x^4y^2 - 4y^3) = 12x^3y^2$$

$$= \checkmark$$

14.4. TANGENT PLANES AND LINEAR APPROXIMATIONS

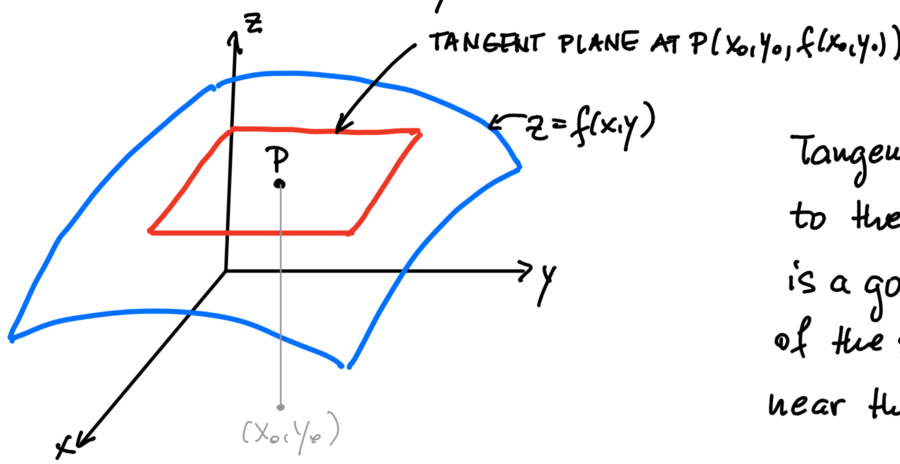
Recall. For $y=f(x)$:



If we zoom in closer and closer near the point (a,b), the tangent line to $y=f(x)$ at (a,b) becomes indistinguishable from the curve $y=f(x)$. Thus:

$$\boxed{f(x) \approx f(a)+f'(a)(x-a)}$$

Similarly: For $z=f(x,y)$:



Tangent plane at P to the surface $z=f(x,y)$ is a good approximation of the surface $z=f(x,y)$ near the point P.

TANGENT PLANES

Tangent plane through $P(x_0,y_0,z_0)$, where $z_0=f(x_0,y_0)$:

$$A(x-x_0)+B(y-y_0)+C(z-z_0)=0$$

What are A, B, and C?

Divide by $C \Rightarrow$ $\quad \frac{A}{C}(x-x_0)+\frac{B}{C}(y-y_0)+(z-z_0)=0$

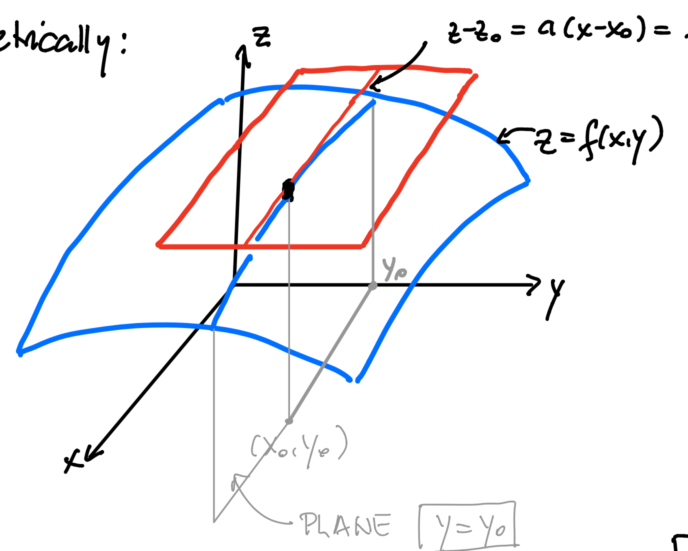
OR

(**) $z - z_0 = a(x-x_0) + b(y-y_0)$ where $\begin{cases} a = -\frac{A}{C} \\ b = -\frac{B}{C} \end{cases}$

Choose $y = y_0$ and let it be fixed. Then we get from (**)

$$z - z_0 = a(x - x_0)$$

geometrically:



If $z-z_0 = a(x-x_0)+b(y-y_0)$ represents the tangent plane, then $z-z_0 = a(x-x_0)$ is the TANGENT LINE to the curve at the intersection of $y=y_0$ and $z=f(x,y)$.

Thus:

$$\boxed{z-z_0 = a(x-x_0) = f_x(x_0,y_0)(x-x_0)}$$

THUS, $a = f_x(x_0, y_0)$.

Choose $x = x_0$: Similarly, we get $b = f_y(x_0, y_0)$.

Conclusion: The tangent plane to $z=f(x,y)$ at $P(x_0, y_0, z_0)$ is given by:

$$\boxed{z - z_0 = f_x(x_0,y_0)(x-x_0) + f_y(x_0,y_0)(y-y_0)}$$

Example: Tangent plane to the elliptic paraboloid $z = 2x^2 + y^2$ at $P(1,1,3)$?

Solution: $z - 3 = (4x)\Big|_{x=1}(x-1) + (2y)\Big|_{y=1}(y-1)$

$\Rightarrow \quad z-3 = 4(x-1) + 2(y-1)$

$\Rightarrow \quad \boxed{z = 4x+2y-3}$ or $\boxed{4x+2y-z-3=0}$

LINEAR APPROXIMATIONS

Tangent plane to $z=f(x,y)$ at $(a,b,f(a,b))$ is:

$$z - f(a,b) = f_x(a,b)(x-a) + f_y(a,b)(y-b)$$

OR

$$z = f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b)$$

Near $P_0(a,b,f(a,b))$, the tangent plane is a good approximation of the surface $z=f(x,y)$. Namely:

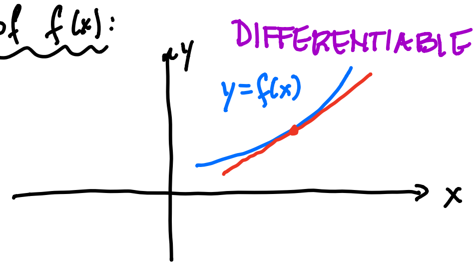
$$\boxed{f(x,y) \approx f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b)} = L(x,y)$$

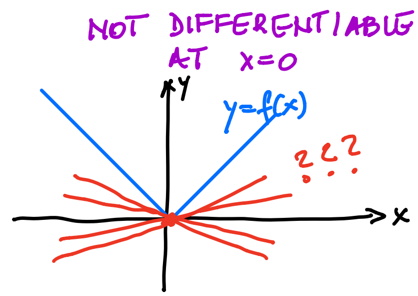
LINEARIZATION

THE TANGENT PLANE IS CALLED THE LINEAR APPROXIMATION OF $f(x,y)$ AT (a,b) (ALSO KNOWN AS THE TANGENT PLANE APPROXIMATION OF f)

LINEAR APPROXIMATION IS A GOOD APPROXIMATION OF $f(x,y)$ WHEN $f(x,y)$ IS DIFFERENTIABLE!

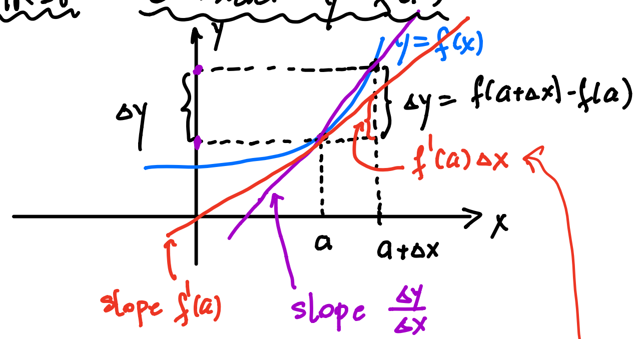
Examples of $f(x)$:





DIFFERENTIABILITY OF $f(x,y)$

FIRST: Consider $y = f(x)$



$\Delta y = f(a+\Delta x) - f(a)$ = INCREMENT OF y

$y = f(x)$ is differentiable at a if Δy can be expressed in the form

$$\Delta y = f'(a)\Delta x + \varepsilon \Delta x$$

where $\varepsilon \to 0$ as $\Delta x \to 0$.

Differentiability:

$$\lim_{h \to 0} \frac{f(a+h) - f(a)}{h} = \text{exists} = f'(a)$$

OR

$$\lim_{\Delta x \to 0} \frac{f(a+\Delta x) - f(a)}{\Delta x} = f'(a)$$

$$\frac{f(a+\Delta x) - f(a)}{\Delta x} \to f'(a) \text{ as } \Delta x \to 0$$

SO

$$f(a+\Delta x) - f(a) \approx f'(a)\Delta x \quad \text{FOR SMALL } \Delta x$$

AND

$$\boxed{\Delta y = f'(a)\Delta x + \varepsilon \Delta x}$$

where $\varepsilon \to 0$ as $\Delta x \to 0$

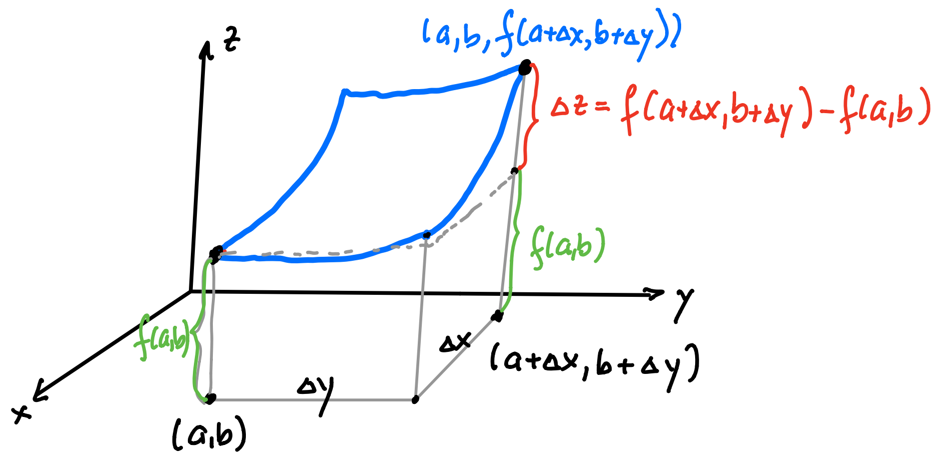
Tangent Line through $(a, f(a))$: $\quad y - f(a) = f'(a)(x-a)$

For $x = a + \Delta x \Rightarrow$

$$\underbrace{y - f(a)}_{\text{CHANGE IN HEIGHT OF THE TANGENT LINE}} = f'(a)(a + \Delta x - a) = f'(a)\Delta x$$

The increment Δy can be expressed as the tangent line increment $f'(a)\Delta x$ plus something small that goes to zero as Δx goes to zero. (Differentiability of $y = f(x)$)

Now, $z = f(x,y)$



Definition: $z = f(x,y)$ is DIFFERENTIABLE at (a,b) if Δz can be expressed as

$$\Delta z = f_x(a,b)\,\Delta x + f_y(a,b)\,\Delta y + \varepsilon_1\,\Delta x + \varepsilon_2\,\Delta y$$

where $\varepsilon_1 \to 0,\ \varepsilon_2 \to 0$ as $(\Delta x,\Delta y) \to (0,0)$.

Theorem: If partial derivatives f_x and f_y exist near (a,b) and are continuous at (a,b), then f is differentiable.

Example: Show that $f(x,y) = x e^{xy}$ is differentiable at $(1,0)$ and find its Linearization there. Then use it to approximate $f(1.1,-0.1)$.

Solution:

$$\left.\begin{aligned} f_x(x,y) &= e^{xy} + x\cdot e^{xy}\cdot y \\ f_x(1,0) &= 1 \end{aligned}\right\}$$

f_x exists and is continuous everywhere, including near $(1,0)$

$$f_y(x,y) = e^{xy}\cdot x$$
$$f_y(1,0) = 1$$

} f_y exists and is continuous everywhere, including near $(1,0)$

$\Rightarrow$ f is differentiable at $(1,0)$.

Linearization at $(1,0)$:

$$L(x,y) = f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b)$$
$$= f(1,0) + f_x(1,0)(x-1) + f_y(1,0)(y-0)$$
$$= 1 + 1(x-1) + 1\cdot y = 1 + x - 1 + y$$

$\boxed{L(x,y) = x+y}$ Linearization at $(1,0)$.

So $f(x,y) \approx L(x,y)$ near $(1,0)$:

$$x e^{xy} \approx x+y \quad \text{near } (1,0)$$

To approximate $f(1.1, -0.1)$ we calculate:

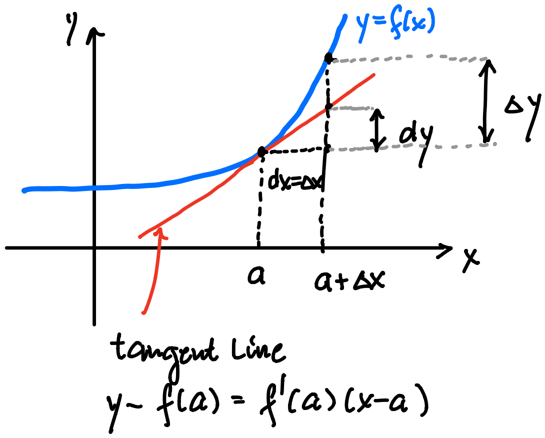
$$f(1.1, -0.1) \approx L(1.1, -0.1) = 1.1 - 0.1 = 1$$

(The actual value is $f(1.1, -0.1) = 1.1\, e^{(1.1)\cdot(-0.1)} \approx 0.98542$.)

DIFFERENTIALS

Recall: For $y = f(x)$: We define the differential dx as an independent variable, which can be given any value (usually small). Then:

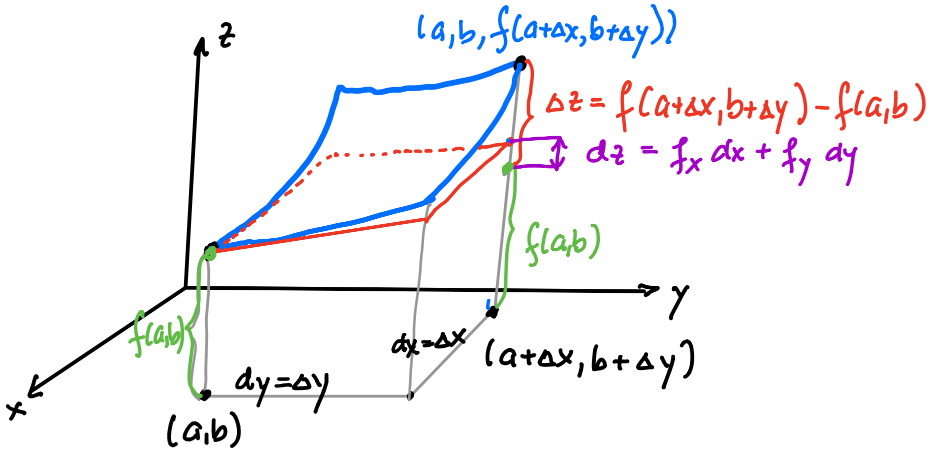
$\boxed{dy = f'(x)\,dx}$ DEFINES THE DIFFERENTIAL dy



$$\leftarrow \begin{cases} \Delta y = f(a+\Delta x) - f(a) \\ dy = f'(a)\,dx \end{cases}$$

SIMILARLY: For $z=f(x,y)$, differentials dx and dy are independent variables, and THE DIFFERENTIAL dz, ALSO CALLED THE TOTAL DIFFERENTIAL is defined by

$$\boxed{dz = f_x(x,y)\,dx + f_y(x,y)\,dy = \frac{\partial z}{\partial x}\,dx + \frac{\partial z}{\partial y}\,dx}$$



EXAMPLE: (a) If $z=f(x,y) = x^2+3xy-y^2$, find dz;

(b) If x changes from 2 to 2.05 and y from 3 to 2.96, compare dz and Δz.

Solution: (a) $dz = \frac{\partial z}{\partial x}dx + \frac{\partial z}{\partial y}dy = (2x+3y)dx + (3x-2y)dy$

(b) $dx = \Delta x = 2.05 - 2 = 0.05$

$dy = \Delta y = 2.96 - 3 = -0.04$

$(a,b) = (2,3)$

$dz = \frac{\partial z}{\partial x}(2,3)\,dx + \frac{\partial z}{\partial y}(2,3)\,dy = (2\cdot 4 + 3\cdot 3)\cdot 0.05 + (3\cdot 2 - 2\cdot 3)(-0.04)$

$= 0.65$ ← EASIER TO COMPUTE

$dz \approx \Delta z$

$\Delta z = f(2.05, 2.96) - f(2,3) = \cdots = 0.6449$

$f = x^2 + 3xy - y^2$

DIFFERENTIALS FOR FUNCTIONS OF MORE THAN TWO VARIABLES:

- $w = f(x,y,z) \approx \underbrace{f(a,b,c) + f_x(a,b,c)(x-a) + f_y(a,b,c)(y-b) + f_z(a,b,c)(z-c)}_{\text{LINEAR APPROXIMATION}}$

- Δw = INCREMENT IN w = $\underbrace{f(x+\Delta x, y+\Delta y, z+\Delta z) - f(x,y,z)}_{\Delta w \text{ INCREMENT}}$

- dw = DIFFERENTIAL = $\underbrace{\frac{\partial w}{\partial x}dx + \frac{\partial w}{\partial y}dy + \frac{\partial w}{\partial z}dz}_{dw \,=\, \text{TOTAL DIFFERENTIAL}}$

14.5. THE CHAIN RULE

CASE I

$$\boxed{z=f(x,y)\,,\ x=g(t),\ y=h(t)}$$

$$z=f(g(t),h(t))$$

$$\boxed{\frac{dz}{dt}=\frac{\partial f}{\partial x}\frac{dx}{dt}+\frac{\partial f}{\partial y}\frac{dy}{dt}}$$

OR

$$\frac{dz}{dt}=\frac{\partial f}{\partial x}\frac{dg}{dt}+\frac{\partial f}{\partial y}\frac{dh}{dt}$$

CASE II

$$\boxed{z=f(x,y),\ x=g(s,t),\ y=h(s,t)}$$

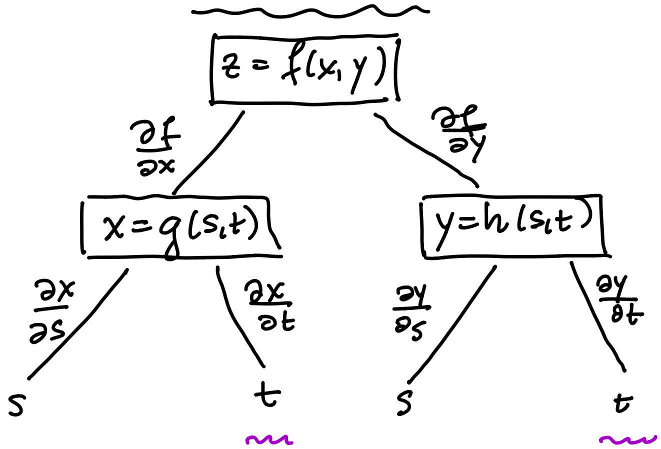
$$z=f(g(s,t),h(s,t))$$

$$\boxed{\begin{aligned}\frac{\partial z}{\partial s}&=\frac{\partial f}{\partial x}\frac{\partial x}{\partial s}+\frac{\partial f}{\partial y}\frac{\partial y}{\partial s}\\ \frac{\partial z}{\partial t}&=\frac{\partial f}{\partial x}\frac{\partial x}{\partial t}+\frac{\partial f}{\partial y}\frac{\partial y}{\partial t}\end{aligned}}$$

OR

$$\begin{cases}\dfrac{\partial z}{\partial s}=\dfrac{\partial f}{\partial x}\dfrac{\partial g}{\partial s}+\dfrac{\partial f}{\partial y}\dfrac{\partial h}{\partial s}\\[2ex] \dfrac{\partial z}{\partial t}=\dfrac{\partial f}{\partial x}\dfrac{\partial g}{\partial t}+\dfrac{\partial f}{\partial y}\dfrac{\partial h}{\partial t}\end{cases}$$

DIAGRAM



EXAMPLE: $z = e^x \sin y$, $x = st^2$, $y = s^2 t$

Find $\frac{\partial z}{\partial s}$ and $\frac{\partial z}{\partial t}$.

Solution:

$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial s} = (e^x \sin y)\cdot(t^2) + (e^x \cos y)\cdot(2st)$$

$$= \left(e^{st^2}\sin(s^2t)\right)t^2 + \left(e^{st^2}\cos(s^2t)\right)(2st)$$

$$= t^2 e^{st^2}\sin(s^2t) + 2st\,e^{st^2}\cos(s^2t)$$

$$\frac{\partial z}{\partial t} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial t} = (e^x \sin y)\cdot(2st) + (e^x \cos y)(s^2)$$

$$= 2st\,e^{st^2}\sin(s^2t) + s^2 e^{st^2}\cos(s^2t)$$

VARIABLES: s, t = INDEPENDENT VARIABLES

x, y = INTERMEDIATE VARIABLES

z = DEPENDENT VARIABLE

GENERAL VERSION - CHAIN RULE

- If $u = f(x_1, x_2, \ldots, x_n)$ and $x_j = x_j(t_1, t_2, \ldots, t_m)$
- Then u is a function of $t_1, t_2, \ldots, t_m$ and

$$\boxed{\frac{\partial u}{\partial t_i} = \frac{\partial f}{\partial x_1}\frac{\partial x_1}{\partial t_i} + \frac{\partial f}{\partial x_2}\frac{\partial x_2}{\partial t_i} + \cdots + \frac{\partial f}{\partial x_n}\frac{\partial x_n}{\partial t_i}}$$

for each $i = 1, 2, \ldots, m$

Second-order derivative example:

$z = f(x,y)$, $x = r^2 + s^2$, $y = 2rs$.

Find $\dfrac{\partial z}{\partial r}$ and $\dfrac{\partial^2 z}{\partial r^2}$, where f has continuous 2^{nd}-order partial derivatives.

Solution:

$$\frac{\partial z}{\partial r} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial r} = \frac{\partial f}{\partial x}(2r) + \frac{\partial f}{\partial y}(2s)$$

$$\frac{\partial^2 z}{\partial r^2} = \frac{\partial}{\partial r}\left(\frac{\partial z}{\partial r}\right) = \frac{\partial}{\partial r}\left[\left(\frac{\partial f}{\partial x}(x,y)\right)(2r) + \left(\frac{\partial f}{\partial y}(x,y)\right)(2s)\right] = \text{PRODUCT RULE} =$$

$$= \underbrace{\frac{\partial}{\partial r}\left(\frac{\partial f}{\partial x}(x,y)\right)}_{\text{CHANE RULE}}\cdot(2r) + \left(\frac{\partial f}{\partial x}(x,y)\right)\cdot\frac{\partial}{\partial r}(2r) + \underbrace{\frac{\partial}{\partial r}\left(\frac{\partial f}{\partial y}\right)}_{\text{CHAIN RULE}}\cdot(2s) + \frac{\partial f}{\partial y}\cdot\frac{\partial}{\partial r}(2s) =$$

$$= \left[\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right)\frac{\partial x}{\partial r} + \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right)\frac{\partial y}{\partial r}\right]\cdot(2r) + 2\frac{\partial f}{\partial x} + \left[\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right)\frac{\partial x}{\partial r} + \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right)\frac{\partial y}{\partial r}\right]\cdot(2s) + 0$$

$$= \left[\frac{\partial^2 f}{\partial x^2}\cdot(2r) + \frac{\partial^2 f}{\partial y \partial x}\cdot(2s)\right]\cdot(2r) + 2\frac{\partial f}{\partial x} + \left[\frac{\partial^2 f}{\partial x \partial y}\cdot(2r) + \frac{\partial^2 f}{\partial y^2}\cdot(2s)\right]\cdot(2s)$$

$$= 4r^2\frac{\partial^2 f}{\partial x^2} + \underbrace{4rs\frac{\partial^2 f}{\partial y \partial x} + 4rs\frac{\partial^2 f}{\partial x \partial y}}_{\text{CONTINUOUS } 2^{nd}\text{-ORDER DERIVATIVES}} + 4s^2\frac{\partial^2 f}{\partial y^2} + 2\frac{\partial f}{\partial x}$$

$$= 4r^2\frac{\partial^2 f}{\partial x^2} + 8rs\frac{\partial^2 f}{\partial y \partial x} + 4s^2\frac{\partial^2 f}{\partial y^2} + 2\frac{\partial f}{\partial x}$$

IMPLICIT DIFFERENTIATION (revisited)

If $z = f(x,y)$ is given implicitly by $F(x,y,z) = 0$, to find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ we:

(1) Differentiate both sides of $F(x,y,z) = 0$

(2) Express the derivative we are calculating.

$$F(x,y,z) = 0 \quad \Big/ \ \frac{\partial}{\partial x}$$

$$\frac{\partial F}{\partial x}\underbrace{\frac{\partial x}{\partial x}}_{=1} + \frac{\partial F}{\partial y}\underbrace{\frac{\partial y}{\partial x}}_{=0} + \frac{\partial F}{\partial z}\frac{\partial z}{\partial x} = 0$$

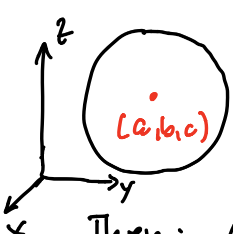
$$\frac{\partial F}{\partial x} + \frac{\partial F}{\partial z}\frac{\partial z}{\partial x} = 0 \Rightarrow \boxed{\frac{\partial z}{\partial x} = -\frac{\frac{\partial F}{\partial x}}{\frac{\partial F}{\partial z}}} \quad (*)$$

Similarly: $$\boxed{\frac{\partial z}{\partial y} = -\frac{\frac{\partial F}{\partial y}}{\frac{\partial F}{\partial z}}} \quad (**)$$

These formulas hold under the assumptions of the Implicit Function Theorem:

Assumptions: (1) F is defined within the sphere containing (a,b,c)

(2) $F(a,b,c) = 0$



(3) $\frac{\partial F}{\partial x}, \frac{\partial F}{\partial y}, \frac{\partial F}{\partial z}$ are continuous inside the sphere

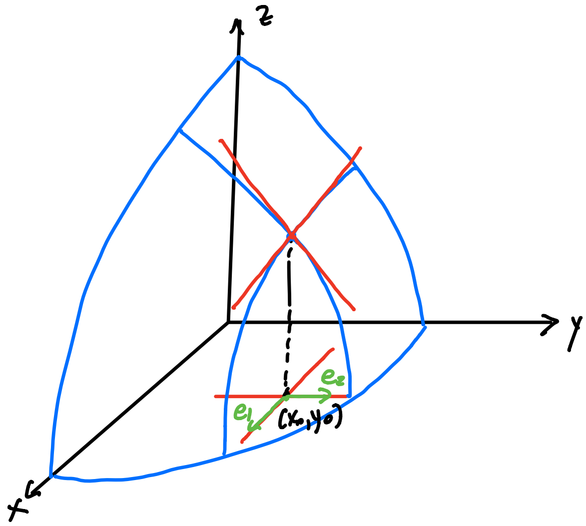
(4) $\frac{\partial F}{\partial z}(a,b,c) \neq 0$

Then: (a) $F(x,y,z) = 0$ defines z as a function of x and y

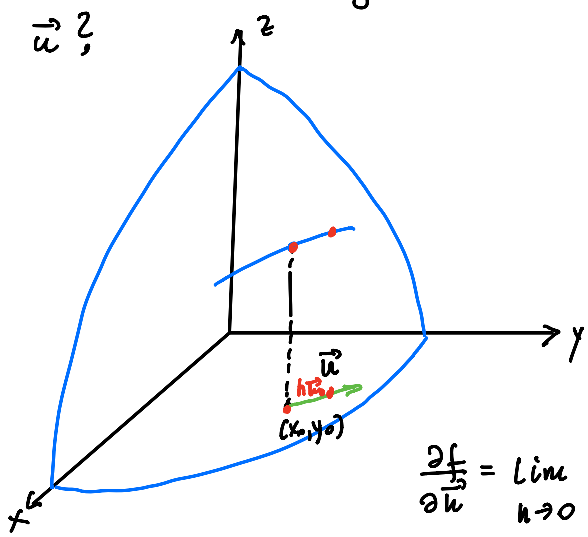
(b) Formulas $(*)$ and $(**)$ define its partial derivatives.

14.6. DIRECTIONAL DERIVATIVE

Recall: $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ measure the change of f in the direction of the x axis and y axis, respectively. Namely, they measure the change of f in the direction of $\vec{e_1}$ and $\vec{e_2}$ vectors.



How to measure the change of f in the direction of an arbitrary vector $\vec{u}$?



$$\frac{\partial f}{\partial \vec{u}} = ?$$

$$\frac{\partial f}{\partial \vec{u}} = \lim_{h \to 0} \frac{f((x_0,y_0)+h\vec{u_0}) - f(x_0,y_0)}{h}$$

Let $\vec{u_0} = \langle a,b \rangle$ be the UNIT VECTOR in the direction of $\vec{u}$.

Then
$$\boxed{\begin{array}{l} \dfrac{\partial f}{\partial \vec{u}} = \lim\limits_{h\to 0} \dfrac{f(x_0+ha,\, y_0+hb) - f(x_0,y_0)}{h} \\ \text{where } \langle a,b\rangle \text{ is the unit vector in the direction } \vec{u} \end{array}}$$

HOW TO CALCULATE $\frac{\partial f}{\partial \vec{u}}$?

- Look at $f(x_0+ha,\, y_0+hb)$ as a function of only h, since (x_0,y_0) is given, and $\langle a,b\rangle$ is given. The only thing that changes is h, measuring the distance along vector $\vec{u}$ from (x_0,y_0).

  So, introduce notation to denote that function:
$$g(h) := f(x_0+ha,\, y_0+hb)$$

- Then
$$g'(0) = \lim_{h\to 0} \frac{g(0+h)-g(0)}{h} = \lim_{h\to 0} \frac{f(x_0+ha,\, y_0+hb)-f(x_0,y_0)}{h} = \frac{\partial f}{\partial \vec{u}}$$

  So, $g'(0)$ is exactly $\frac{\partial f}{\partial \vec{u}}(x_0,y_0)$.

- We know how to differentiate $g(h)$ (function of one variable):
$$\frac{dg}{dh} = \frac{d}{dh} f(x_0+ha,\, y_0+hb) \underset{\text{CHAIN RULE}}{=} \frac{\partial f}{\partial x}\frac{dx}{dh} + \frac{\partial f}{\partial y}\frac{dy}{dh}$$
$$= \frac{\partial f}{\partial x} a + \frac{\partial f}{\partial y} b$$

- THUS:
$$\boxed{\frac{\partial f}{\partial \vec{u}} = \frac{\partial f}{\partial x} a + \frac{\partial f}{\partial y} b, \text{ where } \langle a,b\rangle \text{ is the unit vector in the direction of } \vec{u}}$$

- Another notation: $\dfrac{\partial f}{\partial \vec{u}}$, $D_{\vec{u}} f$

If f is differentiable, then f has a directional derivative in the direction of ANY UNIT VECTOR $\vec{u} = \langle a, b \rangle$ and

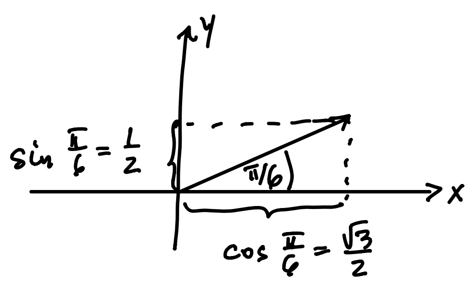
$$D_{\vec{u}} f = \frac{\partial f}{\partial x} a + \frac{\partial f}{\partial y} b$$

EXAMPLE: $f(x,y) = x^3 - 4xy^2 + y^2$, $\vec{u}$ = unit vector given by $\theta = \frac{\pi}{6}$.

Find $D_{\vec{u}} f(x,y)$. What is $D_{\vec{u}} f(1,0)$?

Solution: • First, f is a polynomial, so it is differentiable, and so $D_{\vec{u}} f$ exists.

- Unit vector $\vec{u} = \langle a, b \rangle$:



$\vec{u} = \langle \cos \frac{\pi}{6}, \sin \frac{\pi}{6} \rangle = \langle \frac{\sqrt{3}}{2}, \frac{1}{2} \rangle$

- $D_{\vec{u}} f(x,y) = \dfrac{\partial f}{\partial x} \cdot \dfrac{\sqrt{3}}{2} + \dfrac{\partial f}{\partial y} \cdot \dfrac{1}{2} = \left[3x^2 - 4y^2\right] \dfrac{\sqrt{3}}{2} + \left[-8xy + 2y\right] \cdot \dfrac{1}{2}$

$$= \frac{3\sqrt{3}}{2} x^2 - 4xy - 2\sqrt{3} y^2 + y$$

- $D_u f(1,0) = \dfrac{3\sqrt{3}}{2}$