MATH 53 001
Suncica (Sunny) Canic - Evans 911

1. ALL COURSE INFORMATION is on bCourses.

2. COURSE LOGISTICS:

(A) Homework assignments will be posted on bCourses. They are due on Monday in section, uploaded in Gradescope.
(B) The two midterm exams will be “in class” administered during class time on Thursday, Sept 24, from 5-7pm PST and Tuesday, Nov 17, from 5-7pm
(C) Quizzes are given in section. On Mondays. On-line submission.

3. I cannot move the final. Please check that you do not have a conflict ahead of time. The same holds for the Midterm exams. There will be no makeup quizzes or exams.

4. Lectures will be posted on bCourses within 4 days of when they were given.

5. Grading: Midterms and Quizzes will be graded by GSIs.
There will be no curving.

All complaints about grading should be addressed to GSIs.

6. Grade breakdown:

Homework: 10%; Quizzes: 10%; Each Midterm: 25%; Final: 30%

7. Two worse quiz scores will be dropped. Homework will be graded by participation.
Two homeworks will be dropped, which means that you are allowed to miss two homeworks.
The final exam score can be used to replace the worse Midterm exam score.

TIPS TO DO WELL AND LOVE THIS CLASS 🥰

1. LISTEN TO ALL THE LECTURES. I discuss and emphasize things I think are important.

3. DO ALL HOMEWORK PROBLEMS. I encourage working in groups. However, the homework answers should be written individually.

   Doing lots of homework increases your ability to understand, recognize how to do a problem, and work quickly on exams.

4. SLEEP AT LEAST 6 HOURS EVERY NIGHT; 10 Hours would be optimal 😊

HOW SHOULD YOU PREPARE YOURSELF TO DO WELL IN CLASS? 🤓

1. Ask questions, attend discussions!

2. You should expect to spend at least 10 hours a week on this class, NOT INCLUDING class time.

3. Treat this class like a job!

WHAT IS THIS COURSE ABOUT? 😎

1. Instead of functions of one variable, we want to learn how to work with functions of two or more variables.

2. Why?

3. Many real-life problems involve functions of many variables.

   (A) The price of a car.

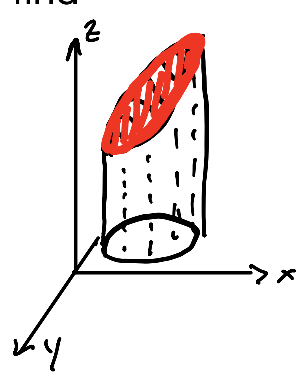
   (B) Your health.

4. Just like in 1D, we want to optimize those functions.

   (A) Find maxima or minima (differentiate; how?)
   (B) Add constraints

4. Just like in 1D we want to integrate to be able to find

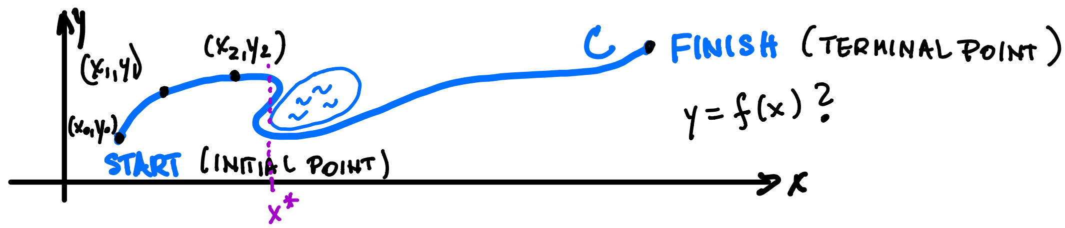
   (A) area (volume) under a function



HOW? 🤔

SECTION 10.1 CURVES DEFINED BY PARAMETRIC EQUATIONS

1. Suppose we went jogging in a park, and our path traced the following curve C:



Suppose we want to express this curve with an equation which will describe our position (x,y).

Can I write a formula for this curve in the form y = f(x)?

Since this curve C does not satisfy the Vertical Line Test, it is not possible to express C as y = f(x).

We can take another approach by expressing x and y as functions of another variable, such as, for example, time t:

$$\begin{cases} x = f(t) \\ y = g(t) \end{cases} \qquad t_0 \le t \le t_{END}$$

This defines our location at every time t.

So, what we did to describe this curve was to introduce a new variable t, called a **parameter,** and expressed the curve C by defining the two equations x = f(t) and y = g(t), called **parametric equations,** which define the curve C.

As t varies, the point (x,y) = (f(t),g(t)) varies and traces out the curve C, which we call a **parametric curve.**

Parameter t does not always have to represent time, but in many applications it does, and so we can interpret

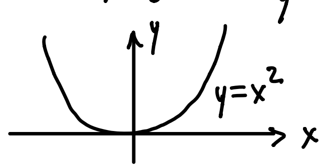
$$\boxed{\text{(x,y) = (f(t),g(t)) as the position of a particle at time t.}}$$

EXAMPLE. Sketch and identify the curve along which a bug crawls, given by $\begin{cases} x = t \\ y = t^2 \end{cases}$

Try to find explicit formula for $y=f(x)$:

From $x=t \Rightarrow y=t^2=x^2 \Rightarrow \boxed{y=x^2}$

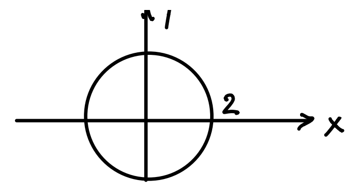
EXPLICIT FORMULA FOR CURVE C



we eliminated t

The bug crawls along a parabola.

EXAMPLE. $\left.\begin{array}{l} x = 2\cos t \\ y = 2\sin t \end{array}\right\}$ $0 \le t \le 2\pi$



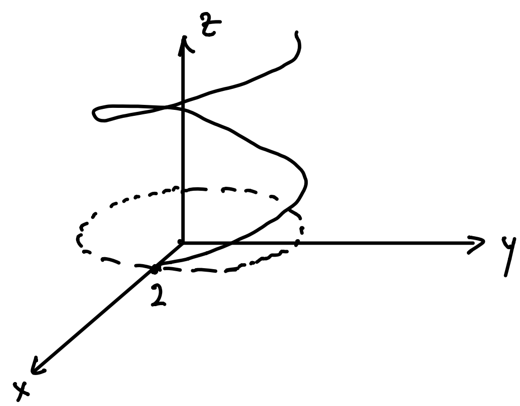
Two approaches:

① Plug values for t and trace the curve

② Try to eliminate t $\Rightarrow$ $x^2+y^2 = 4\cos^2 t + 4\sin^2 t = 4 \Rightarrow x^2+y^2=4$

The bug crawls along a circle of radius 2. $F(x,y) = x^2+y^2-4=0$ (IMPLICIT EQUATION)

EXAMPLE. A flying mosquito traces a curve $\left.\begin{array}{l} x = 2\cos t \\ y = 2\sin t \\ z = t \end{array}\right\}$

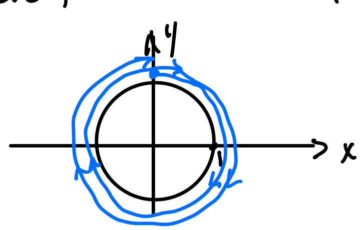


A mosquito traces out a helix in 3D.

EXAMPLE: $\left.\begin{array}{l} x = \sin 2t \\ y = \cos 2t \end{array}\right\}$ $0 \le t \le 2\pi$

Solution: $x^2+y^2 = \sin^2 2t + \cos^2 2t = 1 \Rightarrow x^2+y^2=1$ CIRCLE OF RADIUS 1.

However, notice that for $0 \le t \le 2\pi$ the circle is traversed twice:



EXAMPLE:

$$\left.\begin{aligned} x &= \sin t \\ y &= \cos t \end{aligned}\right\} \quad 0 \le t \le 2\pi$$

Solution: $x^2+y^2 = \sin^2 t + \cos^2 t = 1 \Rightarrow x^2+y^2=1$ CIRCLE OF RADIUS 1

The circle is traversed only once.

REMARKS:

① TWO DIFFERENT SETS OF PARAMETRIC EQUATIONS CAN DESCRIBE THE SAME CURVE, BUT THEY DO NOT DESCRIBE THE SAME PARAMETRIC CURVE!

② WE DISTINGUISH BETWEEN
- A CURVE (A SET OF POINTS)
- A PARAMETRIC CURVE (A CURVE IN WHICH THE POINTS ARE TRACED IN A PARTICULAR WAY)

MOVING ON

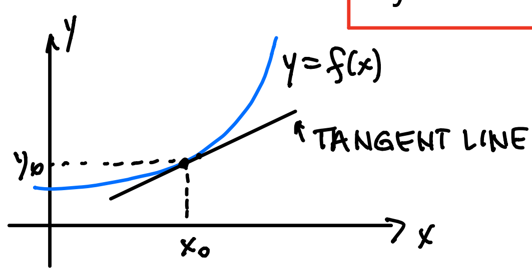
SECTION 10.2

CALCULUS WITH PARAMETRIC CURVES

Well, now that we have parametric curves, we want tangents and derivatives!

Recall, for a function y = f(x), the slope of the tangent line is given by

$$\boxed{dy/dx = f'(x).}$$



$$\boxed{y - y_0 = f'(x_0)(x - x_0)}$$

$f'(x_0)$ = SLOPE OF THE TANGENT LINE AT x_0

This is the geometric interpretation.

What to do when x = f(t) and y = g(t)? How to calculate dy/dx?

EXAMPLE: $\begin{cases} x = t \\ y = t^2 \end{cases}$

😳

So, in general, suppose that f and g are differentiable functions, and that we want to find the tangent line to the parametric curve

$$x = f(t), \quad y = g(t)$$

where y is also a differentiable function of x; y = F(x).
Then we can use the chain rule:

$$y = F(x) = F(\overbrace{f(t)}^{x}) \quad \leftarrow \quad y \text{ is a function of } t!$$

Apply Chain Rule to calculate $\frac{dy}{dt}$:

$$\frac{dy}{dt} = F'(x)\frac{dx}{dt} = \frac{dy}{dx}\,\frac{dx}{dt}$$

If $\frac{dx}{dt} \neq 0$ we can solve for $\frac{dy}{dx}$:

$$\boxed{\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}}\;, \quad \frac{dx}{dt} \neq 0$$

(You can remember this formula by "canceling" the dt's) 😎

EXAMPLE: Find dy/dx at t = 2 for $\begin{cases} x = f(t) = t \\ y = g(t) = t^2 \end{cases}$

Solution:

1. "THE OLD WAY" (eliminate t, and express $y = F(x)$)
2. "THE NEW WAY" (use formula above)

(1) THE OLD WAY: Eliminate t: $\left.\begin{matrix} x = t \\ y = t^2 \end{matrix}\right\} \Rightarrow \boxed{y = x^2}$

Differentiate: $\frac{dy}{dx} = 2x$

Evaluate at $t=2$: What is x at $t=2$?

x at $t=2$ equals 2 (from $x=t$)

Thus, $\frac{dy}{dx}$ at $t=2$ equals $\frac{dy}{dx}$ at $x=2$.

$$\frac{dy}{dx}(x=2) = 2\cdot 2 = 4$$

③ THE NEW WAY:

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{2t}{1} = 2t$$

$$\frac{dy}{dx}(t=2) = 2 \cdot 2 = 4$$ 🤗

QUESTION: Why not always do it the old way?

ANSWER: Because we are not going to always have such nice formulas.

EXAMPLE: Find $\frac{dy}{dx}$ at $t=4$ if $\begin{cases} x = t^2 \\ y = t^3 - 3t \end{cases}$.

Solution:

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{3t^2 - 3}{2t}$$

$$\frac{dy}{dx}(t=4) = \frac{3 \cdot 4^2 - 3}{2 \cdot 4} = \frac{45}{8}$$

EXAMPLE. For the equations from the previous example:

(a) Show that the curve C described by these equations has two tangents at the point (3,0) and find their equations;
(b) Find the points on C where the tangent is horizontal or vertical;
(c) Determine where the curve is concave upward or downward;
(d) Sketch the curve.

Solution: Parametric curve $\begin{cases} x = t^2 \\ y = t^3 - 3t \end{cases}$

FIRST: $(x=3, y=0)$ corresponds to which t ?

From the equations $\Rightarrow$ $t^2 = 3$, $t^3 - 3t = 0$.

From $t^2 = 3 \Rightarrow t = \pm\sqrt{3}$;

From $t^3 - 3t = 0 \Rightarrow t(t^2-3) = 0 \Rightarrow t=0$, $t = \pm\sqrt{3}$.

Thus, $t = \pm\sqrt{3}$ satisfy both equations, and so there are two values of t that correspond to $(3,0)$!

TANGENTS:

$$y - y_0 = \frac{dy}{dx}(x_0, y_0)(x - x_0)$$

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{3t^2-3}{2t}$$

$$\frac{dy}{dx}(t=-\sqrt{3}) = \frac{3\cdot3-3}{-2\sqrt{3}} = \frac{6}{-2\sqrt{3}}\cdot\frac{\sqrt{3}}{\sqrt{3}} = -\sqrt{3}$$

$$\frac{dy}{dx}(t=\sqrt{3}) = \sqrt{3}$$

TANGENT SLOPE AT $t = -\sqrt{3} = -\sqrt{3}$

TANGENT SLOPE AT $t = \sqrt{3} = \sqrt{3}$

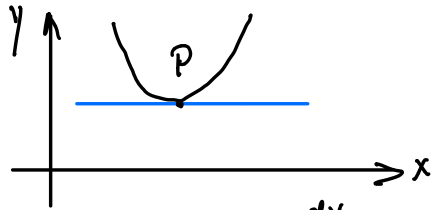
TWO TANGENT LINES

TANGENT 1: $\boxed{y - 0 = -\sqrt{3}\,(x-3)}$

TANGEN 2: $\boxed{y - 0 = \sqrt{3}\,(x-3)}$

(b) HORIZONTAL TANGENT AT P:

$$\boxed{\frac{dy}{dx}(P) = 0}$$



Since $\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}$ for parametric curves, we see that

$$\boxed{\frac{dy}{dt}(P) = 0 \quad \text{and} \quad \frac{dx}{dt}(P) \neq 0 \quad \Rightarrow \quad \frac{dy}{dx}(P) = 0}$$

$$\frac{dy}{dt} = 3t^2 - 3 = 3(t^2-1) = 0 \quad \begin{array}{l} \nearrow t = 1 \\ \searrow t = -1 \end{array}$$

$\frac{dx}{dt} = 2t \neq 0$ at both $t = 1$ and $t = -1$

What are the corresponding x and y coordinates:

AT $t = 1$, $x = 1^2 = 1$, $y = 1^3 - 3 \cdot 1 = 1 - 3 = -2$

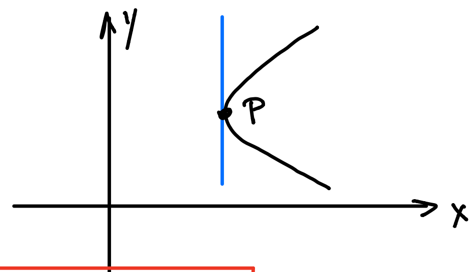
$x = t^2$ $\quad$ $y = t^3 - 3t$

AT $t = -1$, $x = (-1)^2 = 1$, $y = (-1)^3 - 3 \cdot (-1) = 2$

$P_1 (1, -2)$ $t = 1$, $P_2 = (1, 2)$ $t = -1$.

HORIZONTAL TANGENTS

VERTICAL TANGENT AT P:



$$\boxed{\frac{dx}{dy}(P) = 0}$$

$$\boxed{\frac{dx}{dy} = \frac{\frac{dx}{dt}}{\frac{dy}{dt}}} \Rightarrow \boxed{\frac{dx}{dt}(P) = 0 \quad \text{AND} \quad \frac{dy}{dt}(P) \neq 0 \Rightarrow \frac{dx}{dy}(P) = 0}$$

CALCULATE:

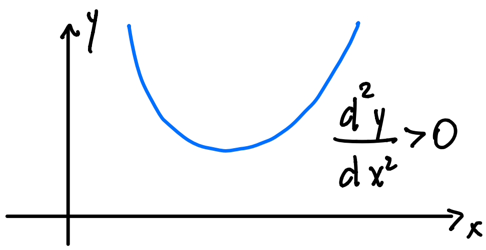
$$\left.\begin{aligned} \frac{dx}{dt} &= 2t \underset{t=0}{=} 0 \Rightarrow t = 0 \\ \frac{dy}{dt} &= 3t^2 - 3 \underset{t=0}{=} -3 \neq 0 \end{aligned}\right\} \Rightarrow \text{VERTICAL TANGENT AT } t = 0$$

THIS CORRESPONDS TO: $x = 0^2 = 0$

$y = 3 \cdot 0^2 - 3 \cdot 0 = 0$

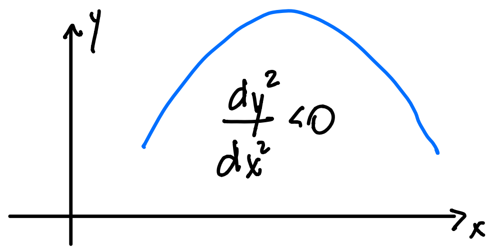
$P(0,0)$ $t=0$ VERTICAL TANGENT

(c) CONCAVE UP OR DOWN:



CONCAVE UP

$$\boxed{\frac{d^2y}{dx^2} > 0}$$



CONCAVE DOWN

$$\boxed{\frac{d^2y}{dx^2} < 0}$$

NEW

CALCULATING 2^{nd}-ORDER DERIVATIVES (PARAMETRIC CASE)

Hint:

- $$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right)$$
- Use: $$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}$$
- "Replace" y by $\frac{dy}{dt}$:

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d\left(\frac{dy}{dx}\right)}{dx} = \frac{\frac{d}{dt}\left(\frac{dy}{dx}\right)}{\frac{dx}{dt}}$$

THUS:

$$\boxed{\frac{d^2y}{dx^2} = \frac{\frac{d}{dt}\left(\frac{dy}{dx}\right)}{\frac{dx}{dt}}}$$

BACK TO OUR PROBLEM:

$$\begin{cases} x = t^2 \\ y = t^3 - 3t \end{cases}$$

CONCAVE UP OR DOWN?

CALCULATE $\frac{d^2y}{dx^2}$:

$$\frac{d^2y}{dx^2} = \frac{\frac{d}{dt}\left(\frac{dy}{dx}\right)}{\frac{dx}{dt}} = \frac{\frac{d}{dt}\left(\frac{3t^2-3}{2t}\right)}{2t} = \frac{\frac{d}{dt}\left(\frac{3}{2}t - \frac{3}{2}\frac{1}{t}\right)}{2t}$$

$$= \frac{\frac{3}{2} - \frac{3}{2}\left(-\frac{1}{t^2}\right)}{2t} = \frac{\frac{3}{2} + \frac{3}{2}\frac{1}{t^2}}{2t} = \frac{\frac{3}{2}\left(1 + \frac{1}{t^2}\right)}{2t}$$

$$= \frac{3}{2}\,\frac{\left(\frac{t^2+1}{t^2}\right)}{2t} = \frac{3}{2}\,\frac{\frac{t^2+1}{t^2}}{\frac{2t}{1}} =$$

$$= \frac{3}{2}\,\frac{(t^2+1)\cdot 1}{2t\cdot t^2} = \frac{3}{2}\,\frac{t^2+1}{2t^3} = \frac{3}{4}\,\frac{t^2+1}{t^3}$$

THUS: $\frac{d^2y}{dx^2} > 0$ FOR $t > 0$ .... CONCAVE UP

$\frac{d^2y}{dx^2} < 0$ FOR $t < 0$ .... CONCAVE DOWN

(d) SKETCH: SO FAR WE KNOW:

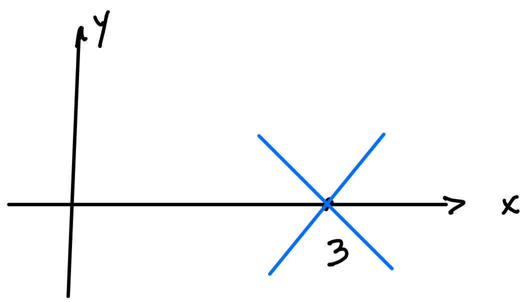
(a) AT $x=3, y=0$ THERE ARE TWO TANGENTS WITH SLOPES $\pm\sqrt{3}$

(b) HORIZONTAL TANGENTS ARE AT $P_1(1,-2)$, $P_2(1,2)$

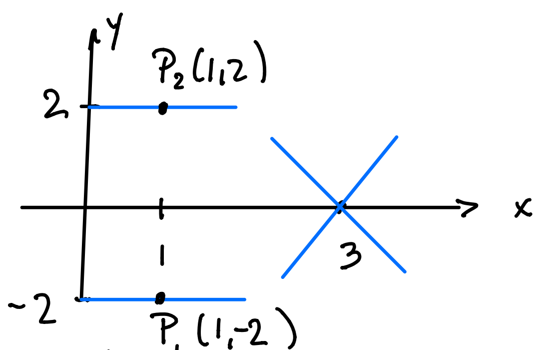
(c) CURVE C IS CONCAVE UP FOR $t>0$ AND CONCAVE DOWN FOR $t<0$

DRAW:

(a)

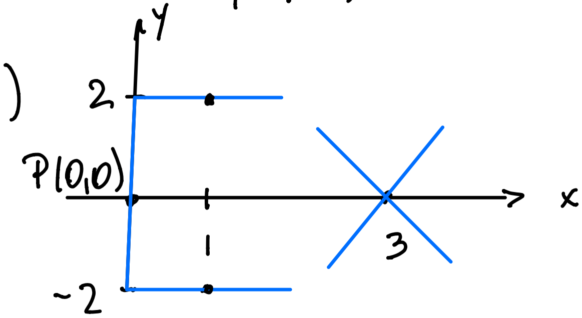


(b)



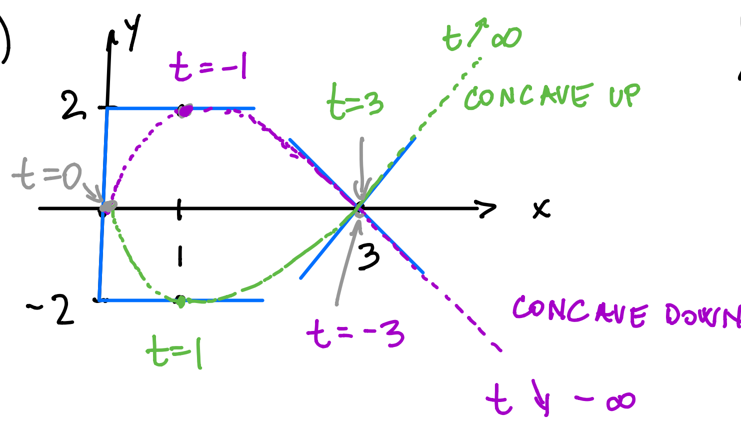
HORIZONTAL TANGENS

(b)



VERTICAL TANGENTS

(c)

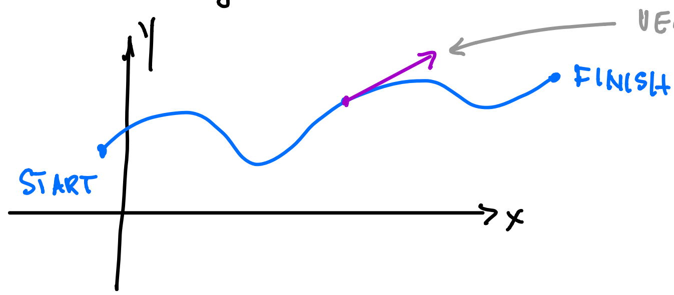


$$\begin{cases} x = t^2 \\ y = t^3 - 3t \end{cases}$$

YOU CAN PLUG IN A FEW VALUES FOR t TO DOUBLE CHECK

🤗🤗🤗

REMARK: Meaning of $\frac{dx}{dt}$, $\frac{dy}{dt}$:



VECTOR WITH COMPONENTS $\left\langle \frac{dx}{dt}, \frac{dy}{dt} \right\rangle$

= VELOCITY

SUMMARY

FOR $\begin{cases} x = f(t) \\ y = g(t) \end{cases}$

(I) DERIVATIVE :

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}$$

(II) TANGENT LINE THROUGH (x_0, y_0) :

$$y - y_0 = \left.\frac{dx}{dt}\right|_{(x_0, y_0)} (x - x_0)$$

(III) 2^{nd} - ORDER DERIVATIVE :

$$\frac{d^2y}{dx^2} = \frac{\frac{d}{dt}\left(\frac{dy}{dx}\right)}{\frac{dx}{dt}}$$

If $\frac{d^2y}{dx^2} > 0 \Rightarrow$ CONCAVE UP

If $\frac{dy^2}{dx^2} < 0 \Rightarrow$ CONCAVE DOWN