ANNOUNCEMENTS

- FIRST MIDTERM IS ON Sep 24, 2020
  FROM 5 - 7 pm PST
- PRACTICE EXAM AND REVIEW on Sep 22, 2020
  IN CLASS
- PRACTICE MIDTERM WILL BE AVAILABLE ON
  bCOURSES on Fri, Sep 18 2020
- TOPICS COVERED: EVERYTHING WE COVERED SO FAR
  INCLUDING TODAY'S LECTURE WHICH WILL END
  WITH COVERING SECTION 12.5.
  (12.6 WILL NOT BE COVERED)
- HOMEWORK 3 WILL BE POSTED LATER TODAY AND IS DUE
  MON Sep 21. THE WILL BE A QUIZ ON THAT MONDAY
- NO HOMEWORK OR QUIZ WILL BE DUE ON MON, Sep 28 !

TRIPLE PRODUCT $\vec{a}\cdot(\vec{b}\times\vec{c})$

$$\vec{a}\cdot(\vec{b}\times\vec{c}) = (a_1\vec{i}+a_2\vec{j}+a_3\vec{k})\cdot\begin{vmatrix}\vec{i} & \vec{j} & \vec{k}\\ b_1 & b_2 & b_3\\ c_1 & c_2 & c_3\end{vmatrix} =$$

$$= (a_1\vec{i}+a_2\vec{j}+a_3\vec{k})\left[(b_2c_3-b_3c_2)\vec{i}+(b_3c_1-b_1c_3)\vec{j}+(\qquad)\vec{k}\right]$$

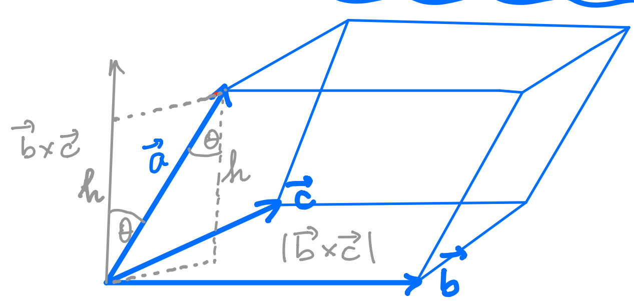
$$= a_1(b_2c_3-b_3c_2)+a_2(b_3c_1-b_1c_3)+(\qquad)a_3$$

(Like replacing $\vec{i},\vec{j},\vec{k}$ with a_1, a_2, a_3)

$$\Rightarrow \boxed{\vec{a}\cdot(\vec{b}\times\vec{c}) = \begin{vmatrix}a_1 & a_2 & a_3\\ b_1 & b_2 & b_3\\ c_1 & c_2 & c_3\end{vmatrix}}$$

SCALAR TRIPLE PRODUCT

GEOMETRIC INTERPRETATION



$V = |\vec{b}\times\vec{c}|\cdot h$

$h = \left|\text{comp}_{\vec{b}\times\vec{c}}\,\vec{a}\right| = \left|\dfrac{\vec{a}\cdot(\vec{b}\times\vec{c})}{|\vec{b}\times\vec{c}|}\right|$

THUS:

$V = \cancel{|\vec{b}\times\vec{c}|}\left|\dfrac{\vec{a}\cdot(\vec{b}\times\vec{c})}{\cancel{|\vec{b}\times\vec{c}|}}\right|$

$= \vec{a}\cdot(\vec{b}\times\vec{c})$

$|\vec{a}\cdot(\vec{b}\times\vec{c})|$ = VOLUME OF THE PARALLELEPIPED DETERMINED BY $\vec{a}$, $\vec{b}$, AND $\vec{c}$

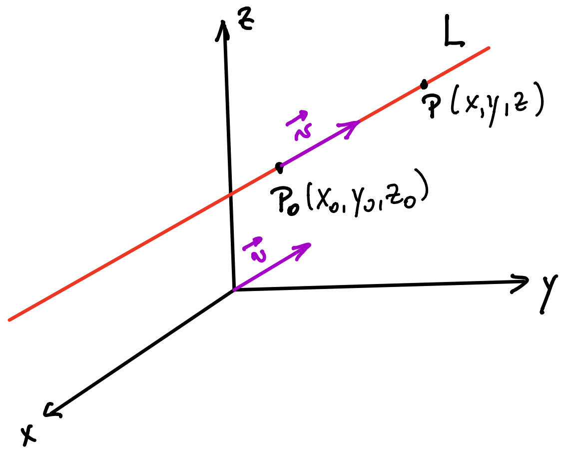
EXAMPLE: Use the scalar triple product to show that the vectors $\vec{a} = \langle 1, 4, -7 \rangle$, $\vec{b} = \langle 2, -1, 4 \rangle$, $\vec{c} = \langle 0, -9, 18 \rangle$ are COPLANAR.

Solution: Calculate the volume of the parallelepiped determined by $\vec{a}$, $\vec{b}$, and $\vec{c}$:

DONE!

MOVING ON!

12.5. EQUATIONS OF LINES AND PLANES



A LINE THROUGH THE POINT $P_0\,(x_0, y_0, z_0)$ PARALLEL TO THE VECTOR $\vec{v} = \langle a, b, c \rangle$

Describe all the points $P(x,y,z)$ that belong to the line L :

Notice : if $P(x,y,z)$ is on the line L, then $\overrightarrow{P_0P}$ is PROPORTIONAL to $\vec{v}$:

$$\boxed{\overrightarrow{P_0P} = t\,\vec{v}}$$

THIS IS THE CONDITION FROM WHICH THE EQUATION OF THE LINE WILL FOLLOW

In components:

$$\overrightarrow{P_0P} = \langle x - x_0,\ y - y_0,\ z - z_0 \rangle$$

$$t\vec{v} = \langle ta,\ tb,\ tc \rangle$$

We use the following:

TWO VECTORS ARE EQUAL IF AND ONLY IF CORRESPONDING COMPONENTS ARE EQUAL

So, $\overrightarrow{P_0P} = t\vec{v}$ means: $\begin{cases} x - x_0 = ta \\ y - y_0 = tb \\ z - z_0 = tc \end{cases}$ where $t \in \mathbb{R}$

This defines PARAMETRIC EQUATIONS OF THE LINE L :

$$\boxed{x = x_0 + ta, \quad y = y_0 + tb, \quad z = z_0 + tc, \quad t \in \mathbb{R}} \qquad (\star)$$

By eliminating t we get SYMMETRIC EQUATIONS of L:

$$\boxed{\frac{x-x_0}{a} = \frac{y-y_0}{b} = \frac{z-z_0}{c}} \qquad (\star\star)$$

(Line through (x_0, y_0, z_0) parallel with $\langle a,b,c\rangle$)

VECTOR EQUATIONS OF L:

Denote by $\vec{r}$ the POSITION VECTOR of $P(x,y,z)$: $\vec{r} = \langle x, y, z\rangle$

Denote by $\vec{r_0}$ the POSITION VECTOR of $P_0(x_0,y_0,z_0)$: $\vec{r_0} = \langle x_0, y_0, z_0\rangle$

Then $(\star)$ can be written as: $\boxed{\vec{r} = \vec{r_0} + t\vec{v}}$ $\qquad (\star\star\star)$

<u>Example</u>: (a) Find a vector equation and parametric equations of the line that passes through the point $(5,1,3)$ and is parallel to $\vec{i}+4\vec{j}-2\vec{k}$.

(b) Also find two other points on the line.

<u>Solution</u>:

(a) Here $\vec{r_0} = \langle 5,1,3\rangle = 5\vec{i}+\vec{j}+3\vec{k}$

$\vec{v} = \langle 1,4,-2\rangle = \vec{i}+4\vec{j}-2\vec{k}$

Vector equation: $\vec{r} = \vec{r_0} + t\vec{v} = (5\vec{i}+\vec{j}+3\vec{k}) + t(\vec{i}+4\vec{j}-2\vec{k})$

$= (5+t)\vec{i} + (1+4t)\vec{j} + (3-2t)\vec{k}, \quad t \in \mathbb{R}$

Thus, a vector equation of the line is:

$$\boxed{\vec{r} = (5+t)\vec{i} + (1+4t)\vec{j} + (3-2t)\vec{k}, \quad t\in\mathbb{R}}$$

Parametric equations:

$$\boxed{\left.\begin{aligned} x &= 5+t \\ y &= 1+4t \\ z &= 3-2t \end{aligned}\right\} \quad t\in\mathbb{R}}$$

(b) Two other points: each choice of t defines one point.

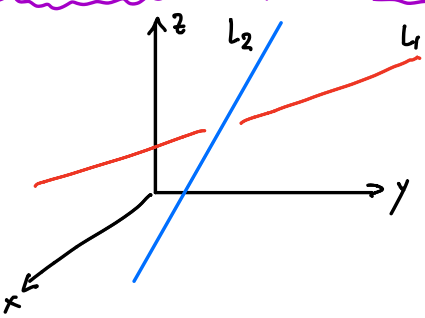
So, take, e.g., $t = 1 \Rightarrow P_1(x,y,z) = (6,5,1)$

$t = -1 \Rightarrow P_2(x,y,z) = (4,-3,5)$

REMARK: The vector equations, the parametric equations, and the symmetric equations are NOT unique!

If $\vec{v} = \langle a,b,c \rangle$ describes direction of L, the numbers a, b, and c are called **DIRECTION NUMBERS.**

EXAMPLE: Show that the lines L_1 and L_2 are SKEW LINES, i.e., they do not intersect and are not parallel:



L_1: $x = 1+t, \quad y = -2+3t, \quad z = 4-t$

L_2: $x = 2s, \quad y = 3+s, \quad z = -3+4s$

Solution: 1. Parallel? Compare direction vectors: $\begin{cases} \vec{v}_1 = \langle 1,3,-1 \rangle \\ \vec{v}_2 = \langle 2,1,4 \rangle \end{cases}$

They are not a scalar multiple of each other. $\Rightarrow$ NOT PARALLEL!

You can double check that the system:

$$\langle 1,3,-1 \rangle = \alpha \langle 2,1,4 \rangle, \quad \alpha = ?$$

does not have a solution.

2. Intersect?

$$\left.\begin{aligned} 1+t &= 2s \\ -2+3t &= 3+s \\ 4-t &= -3+4s \end{aligned}\right\}$$

$\Rightarrow t = 2s-1$

$\Rightarrow -2+3(2s-1) = 3+s$

$-2+6s-3 = 3+s$

$5s = 8$

$s = 8/5$

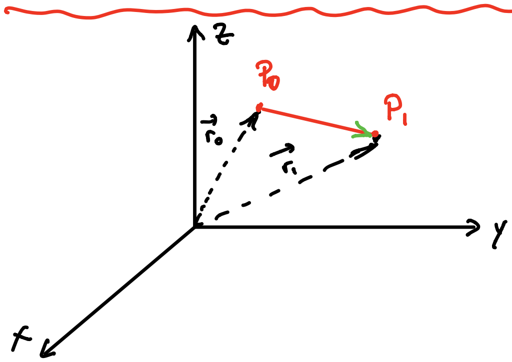
From $t = 2s-1 \Rightarrow t = \frac{16}{5} - 1 = \frac{11}{5}$

Third equation: $4 - \frac{11}{5} = -3 + \frac{32}{5}$?

$\Rightarrow \frac{9}{5} \neq \frac{17}{5} \Rightarrow$ NO INTERSECTION!

CONCLUSION: L_1 AND L_2 ARE SKEW LINES.

THE LINE SEGMENT FROM P_0 TO P_1



Let P_0 and P_1 be given. What is the vector equation for the line segment from P_0 to P_1?

1. P_0 and P_1 determine a line with direction vector $\vec{v} = \overrightarrow{P_0P_1}$:

$$\vec{v} = \overrightarrow{P_0P_1} = \vec{r}_1 - \vec{r}_0 = \langle x_1 - x_0, y_1 - y_0, z_1 - z_0 \rangle$$

2. All the points on the line through P_0, with direction vector $\vec{v}$ are:

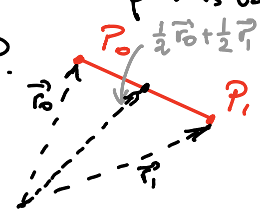
$$\vec{r} = \vec{r}_0 + t(\vec{r}_1 - \vec{r}_0), \quad t \in \mathbb{R}$$

OR

$$\boxed{\vec{r} = (1-t)\vec{r}_0 + t\vec{r}_1}, \quad t \in \mathbb{R}$$

3. We see that $\begin{cases} \text{for } t=0 \text{ we get } \vec{r} = \vec{r}_0, \text{ which corresponds to } P_0 \\ \text{for } t=1 \text{ we get } \vec{r} = \vec{r}_1, \text{ which corresponds to } P_1 \end{cases}$

Notice that for $t = \frac{1}{2}$, we get $\vec{r} = \frac{1}{2}\vec{r}_0 + \frac{1}{2}\vec{r}_1$.

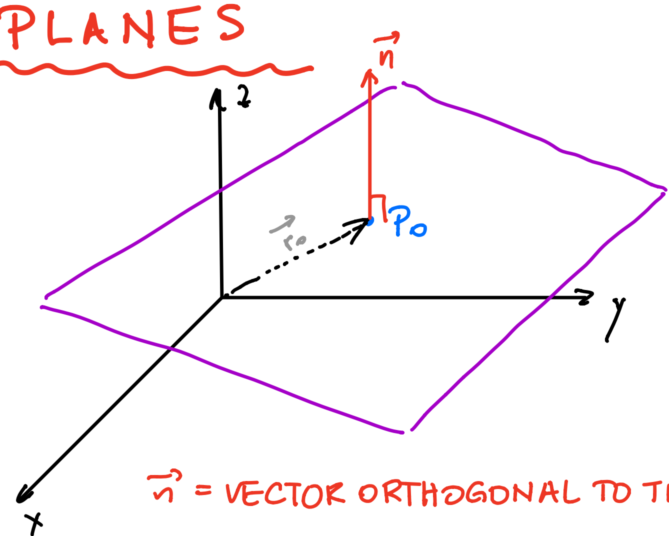


So, for t between $t=0$ and $t=1$ we get all the points inside the segment from P_0 to P_1. So, THE LINE SEGMENT FROM P_0 TO P_1 is given by

$$\boxed{\vec{r} = (1-t)\vec{r}_0 + t\vec{r}_1, \quad 0 \le t \le 1}$$

where $\vec{r}_0$ and $\vec{r}_1$ are position vectors of P_0 and P_1.

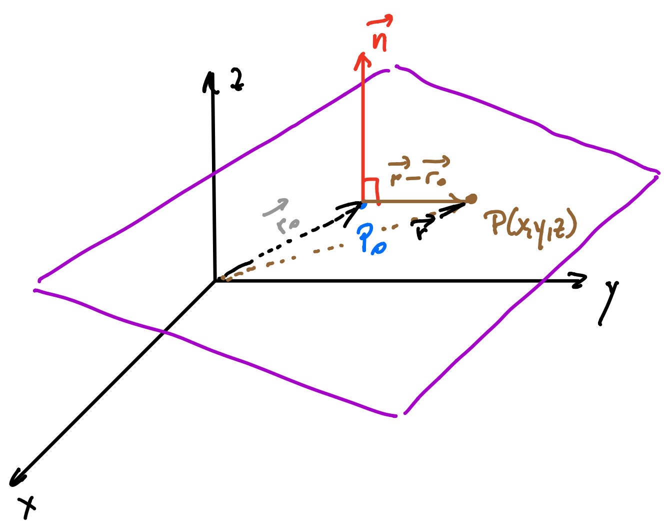
PLANES



$\vec{n}$ = VECTOR ORTHOGONAL TO THE PLANE CALLED A NORMAL

P_0 = POINT IN THE PLANE

Want to find the equations that would describe all the points in the plane orthogonal to $\vec{n}$, passing through P_0.



If $P(x,y,z)$ = arbitrary point in the plane, then $\overrightarrow{P_0P}$ must be orthogonal to $\vec{n}$:

$$\overrightarrow{P_0P} \cdot \vec{n} = 0$$

or

$$\boxed{(\vec{r}-\vec{r}_0)\cdot\vec{n} = 0}$$ VECTOR EQUATION

In components: if $\vec{n} = \langle a, b, c \rangle$, $\vec{r_0} = \langle x_0, y_0, z_0 \rangle$ and $\vec{r} = \langle x, y, z \rangle$, then:

$$\vec{n} \cdot (\vec{r} - \vec{r_0}) = 0$$

implies

$$\boxed{a(x - x_0) + b(y - y_0) + c(z - z_0) = 0}$$

SCALAR EQUATION OF THE PLANE THROUGH $P_0(x_0, y_0, z_0)$ WITH NORMAL $\vec{n} = \langle a, b, c \rangle$

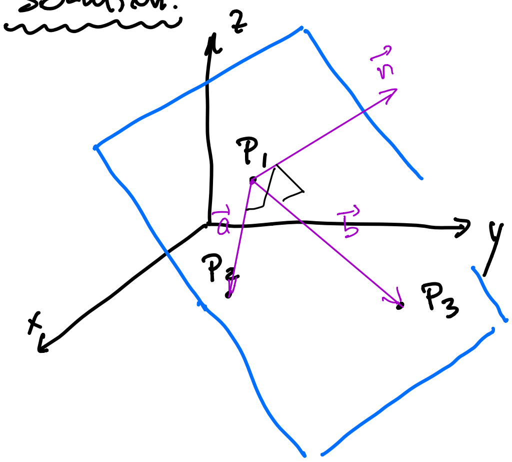
We can simplify it to obtain:

$$\boxed{ax + by + cz + d = 0}, \quad d = -(ax_0 + by_0 + cz_0)$$

LINEAR EQUATION OF THE PLANE

EXAMPLE: Find an equation of the plane that passes through $P_0(1,3,2)$, $P_1(3,-1,6)$ and $P_2(5,2,0)$.

Solution:



We need normal $\vec{n}$.

$$\vec{n} = \vec{a} \times \vec{b} = \overrightarrow{P_0P_1} \times \overrightarrow{P_0P_2}$$

$$\vec{a} = \overrightarrow{P_0P_1} = \langle 2, -4, 4 \rangle$$

$$\vec{b} = \overrightarrow{P_0P_2} = \langle 4, -1, -2 \rangle$$

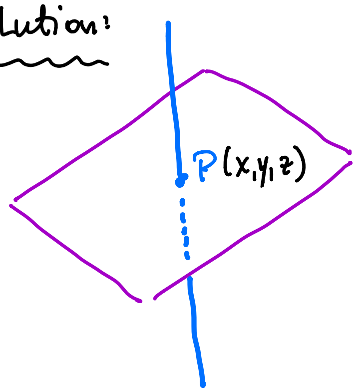
$$\vec{a} \times \vec{b} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2 & -4 & 4 \\ 4 & -1 & -2 \end{vmatrix} = \langle 12, 20, 14 \rangle = \vec{n}$$

Thus, the plane through $P_0(1,3,2)$ with normal $\vec{n} = \langle 12, 20, 14 \rangle$ is:

$$\boxed{12(x-1) + 20(y-3) + 14(z-2) = 0} \quad \text{OR} \quad \boxed{6x + 10y + 7z = 50}$$

EXAMPLE: Find the point at which the line $\begin{cases} x = 2+3t \\ y = -4t \\ z = 5+t \end{cases}$ intersects the plane $4x+5y-2z=18$.

Solution:



The point of intersection bellongs to both the line and the plane. Thus, it must satisfy both equations:

$$\begin{aligned} x &= 2+3t \\ y &= -4t \\ z &= 5+t \end{aligned} \qquad \text{and} \qquad 4x+5y-2z=18$$

We plug the coordinates $\begin{aligned} x &= 2+3t \\ y &= -4t \\ z &= 5+t \end{aligned}$ into the plane equation $4x+5y-2z=18$, to find the value of t which determines the point of intersection, i.e., satisfies both setc of equations.

We get
$$4(2+3t)+5(-4t)-2(5+t)=18$$
$$8+12t-20t-10-2t = 18$$
$$-10t = 18+2$$
$$-10t = 20 \Rightarrow \boxed{t=-2}$$

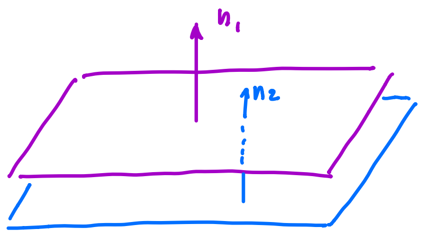
This determines the coordinates of the point P:

$$\begin{aligned} x &= 2+3\cdot(-2) = -4 \\ y &= -4\cdot(-2) = 8 \\ z &= 5+(-2) = +3 \end{aligned}$$

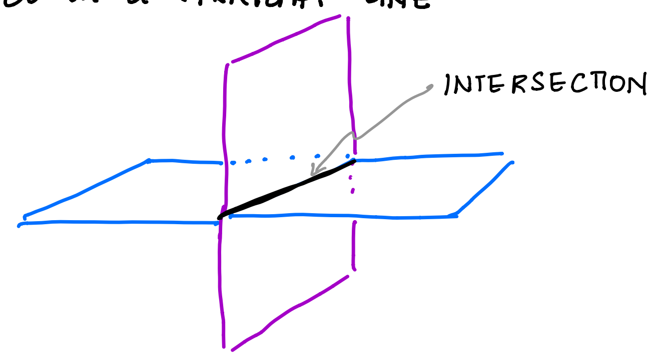
Thus, the point of intersection is: $P(4, 8, 3)$

PARALLEL PLANES: if their normal vectors are parallel

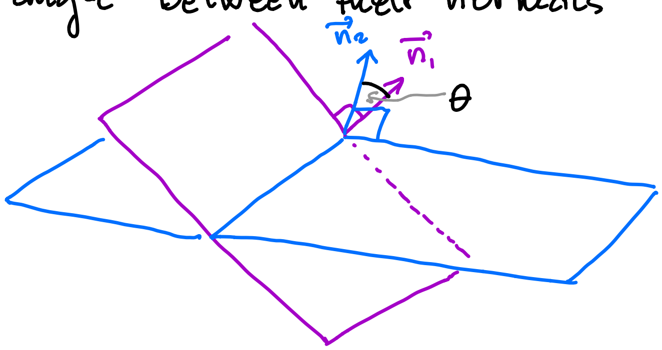
$\vec{n}_1 \parallel \vec{n}_2$



INTERSECTION OF PLANES: if two planes are not parallel, they intersect in a STRAIGHT LINE



ANGLE BETWEEN TWO PLANES is determined by the acute angle between their normals



EXAMPLE: Find the angle between the planes $x+y+z=1$ and $x-2y+3z=1$. Then find the symmetric equations for the line of intersection of the two planes.

Solution: • $\vec{n}_1 = \langle 1,1,1\rangle$, $\quad \vec{n}_2 = \langle 1,-2,3\rangle$

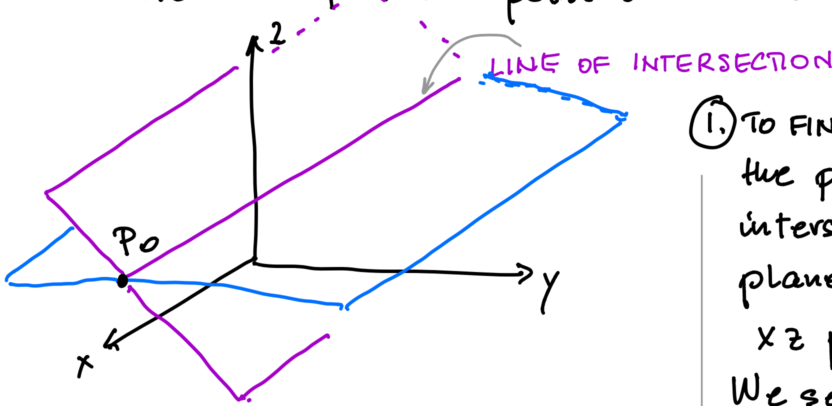
$$\cos\theta = \frac{\vec{n}_1}{|\vec{n}_1|}\cdot\frac{\vec{n}_2}{|\vec{n}_2|} = \frac{\langle 1,1,1\rangle}{\sqrt{3}}\cdot\frac{\langle 1,-2,3\rangle}{\sqrt{1+4+9}}$$

$$= \frac{1}{\sqrt{3\cdot 14}}\langle 1,1,1\rangle\cdot\langle 1,-2,3\rangle = \frac{1}{\sqrt{42}}(1-2+3) = \frac{2}{\sqrt{42}}$$

$$\theta = \cos^{-1}\left(\frac{2}{\sqrt{42}}\right) \approx 72^\circ$$

• Line of intersection:

Need to find one point on the line and the direction vector.



① TO FIND ONE POINT we can find the point at which the line intersects one of the coordinate planes. For example, the xz plane $y=0$.
We set $y=0$ in both equations:

$$x + z = 1$$
$$x + 3z = 1$$

and solve this system:

$$2z = 0 \Rightarrow z = 0$$
$$x = 1 \Rightarrow P_0(1,0,0)$$

② TO FIND THE DIRECTION VECTOR $\vec{v}$ notice that $\vec{v}$ must be orthogonal to both normals $\vec{n}_1$ and $\vec{n}_2$.

Thus $\vec{v} = \vec{n}_1 \times \vec{n}_2$!

$$\vec{v} = \vec{n}_1 \times \vec{n}_2 = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 1 & 1 \\ 1 & -2 & 3 \end{vmatrix} = \ldots = \langle 5,-2,-3\rangle$$

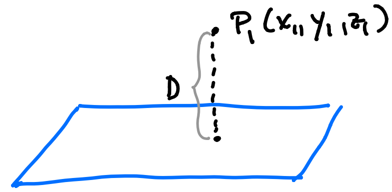
CONCLUSION: The symmetric equations of the line of intersection:

$$\frac{x-1}{5} = \frac{y}{-2} = \frac{z}{-3}$$

(Notice: ∞ many different forms of symmetric equations)

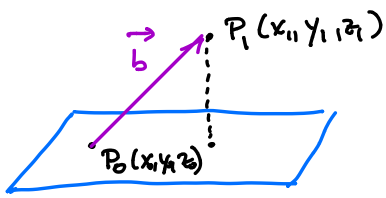
EXAMPLE: Find the formula for the distance of point $P_1(x_1, y_1, z_1)$ from the plane $ax + by + cz = d$.

Solution:

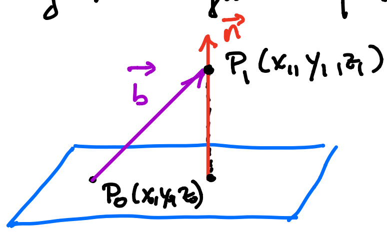


$D = ?$

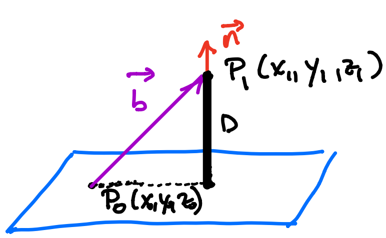
1. Take an arbitrary point $P_0(x_0, y_0, z_0)$ in the plane.
2. Denote by $\vec{b}$ the vector $\overrightarrow{P_0P_1}$, $\quad \vec{b} = \overrightarrow{P_0P_1}$



3. Consider the normal $\vec{n}$ to the plane, and imagine it passing through the point P_1



4. Then D is the LENGTH OF THE PROJECTION OF $\vec{b}$ ONTO $\vec{n}$!



$$\boxed{D = \left| \text{comp}_{\vec{n}} \vec{b} \right|}$$

5. We know how to calculate $\text{comp}_{\vec{n}} \vec{b}$:

$$\text{comp}_{\vec{n}} \vec{b} = \vec{b} \cdot \frac{\vec{n}}{|\vec{n}|} = \frac{\vec{b} \cdot \vec{n}}{|\vec{n}|}$$

6. Thus: $$D = \left| \frac{\vec{b}\cdot\vec{n}}{|\vec{n}|} \right|$$

In components: $\vec{n} = \langle a, b, c \rangle$

$\vec{b} = \langle x_1 - x_0,\ y_1 - y_0,\ z_1 - z_0 \rangle$

Thus:

$$D = \frac{|a(x_1-x_0)+b(y_1-y_0)+c(z_1-z_0)|}{\sqrt{a^2+b^2+c^2}}$$

$$= \frac{|ax_1+by_1+cz_1 - \overbrace{(ax_0+bx_0+cx_2)}^{=d}|}{\sqrt{a^2+b^2+c^2}}$$

$$= \frac{|ax_1+by_1+cy_1+d|}{\sqrt{a^2+b^2+c^2}}$$

7. In conclusion, the distance of $P_1(x_1, y_1, z_1)$ from the plane $ax+by+cz+d=0$ is given by

$$\boxed{D = \frac{|ax_1+by_1+cz_1+d|}{\sqrt{a^2+b^2+c^2}}}$$

Suggestion (STRONGLY RECOMMEND): Solve examples 9 and 10 on page 830!

WE ARE DONE! 😅🤪😊