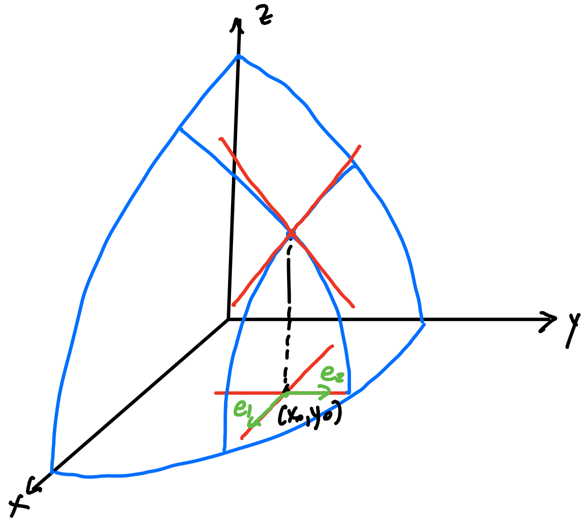
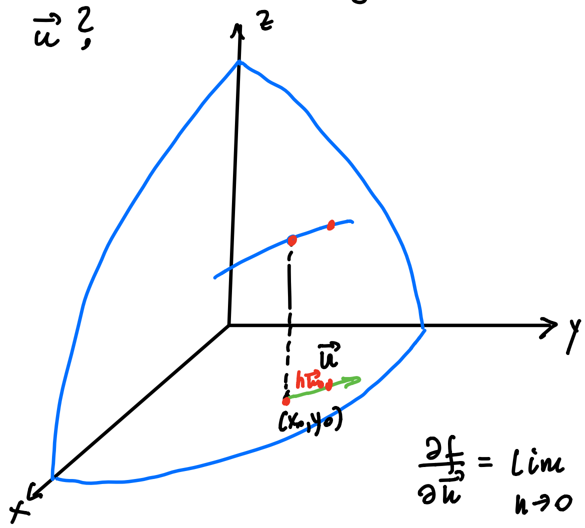
14.6. DIRECTIONAL DERIVATIVE AND THE GRADIENT

😎

Recall: $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ measure the change of f in the direction of the x axis and y axis, respectively. Namely, they measure the change of f in the direction of $\vec{e_1}$ and $\vec{e_2}$ vectors.



How to measure the change of f in the direction of an arbitrary vector $\vec{u}$?



$$\frac{\partial f}{\partial \vec{u}} = ?$$

$$\frac{\partial f}{\partial \vec{u}} = \lim_{h \to 0} \frac{f((x_0,y_0)+h\vec{u}_0) - f(x_0,y_0)}{h}$$

Let $\vec{u}_0 = \langle a,b \rangle$ be the UNIT VECTOR in the direction of $\vec{u}$.

Then

$$\boxed{\begin{array}{l} \dfrac{\partial f}{\partial \vec{u}} = \lim\limits_{h \to 0} \dfrac{f(x_0+ha, y_0+hb) - f(x_0,y_0)}{h} \\ \text{where } \langle a,b\rangle \text{ is the unit vector in the direction } \vec{u} \end{array}}$$

HOW TO CALCULATE $\frac{\partial f}{\partial \vec{u}}$?

- Look at $f(x_0+ha, y_0+hb)$ as a function of only h, since (x_0,y_0) is given, and $\langle a,b\rangle$ is given. The only thing that changes is h, measuring the distance along vector $\vec{u}$ from (x_0,y_0).

  So, introduce notation to denote that function of one variable:

$$g(h) := f(x_0+ha, y_0+hb)$$

- Then

$$g'(0) = \lim_{h\to 0} \frac{g(0+h)-g(0)}{h} = \lim_{h\to 0} \frac{f(x_0+ha, y_0+hb)-f(x_0,y_0)}{h} = \frac{\partial f}{\partial \vec{u}}$$

  So, $g'(0)$ is exactly $\frac{\partial f}{\partial \vec{u}}(x_0,y_0)$.

- We know how to differentiate $g(h)$ (function of one variable):

$$\frac{dg}{dh} = \frac{d}{dh} f(x_0+ha, y_0+hb) \underset{\text{CHAIN RULE}}{=} \frac{\partial f}{\partial x}\frac{dx}{dh} + \frac{\partial f}{\partial y}\frac{dy}{dh}$$

$$= \frac{\partial f}{\partial x} a + \frac{\partial f}{\partial y} b$$

- THUS:

$$\boxed{\frac{\partial f}{\partial \vec{u}} = \frac{\partial f}{\partial x} a + \frac{\partial f}{\partial y} b, \text{ where } \langle a,b\rangle \text{ is the unit vector in the direction of } \vec{u}}$$

- Another notation: $\dfrac{\partial f}{\partial \vec{u}}$, $D_{\vec{u}} f$

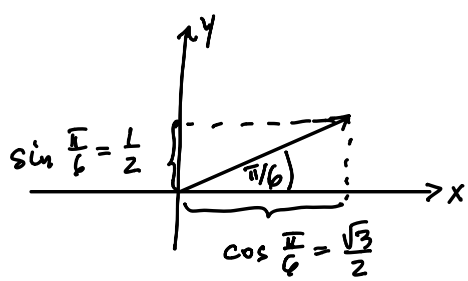
> If f is differentiable, then f has a directional derivative in the direction of ANY UNIT VECTOR $\vec{u} = \langle a, b\rangle$ and
> $$D_{\vec{u}} f = \frac{\partial f}{\partial x} a + \frac{\partial f}{\partial y} b$$

EXAMPLE: $f(x,y) = x^3 - 4xy^2 + y^2$, $\vec{u}$ = unit vector given by $\theta = \frac{\pi}{6}$.

Find $D_{\vec{u}} f(x,y)$. What is $D_{\vec{u}} f(1,0)$?

Solution: • First, f is a polynomial, so it is differentiable, and so $D_{\vec{u}} f$ exists.

- Unit vector $\vec{u} = \langle a, b\rangle$:



$\vec{u} = \langle \cos\frac{\pi}{6}, \sin\frac{\pi}{6}\rangle = \langle \frac{\sqrt{3}}{2}, \frac{1}{2}\rangle$

- $D_{\vec{u}} f(x,y) = \dfrac{\partial f}{\partial x} \cdot \dfrac{\sqrt{3}}{2} + \dfrac{\partial f}{\partial y} \cdot \dfrac{1}{2} = \left[3x^2 - 4y^2\right]\dfrac{\sqrt{3}}{2} + \left[-8xy + 2y\right]\cdot\dfrac{1}{2}$

$$= \frac{3\sqrt{3}}{2}x^2 - 4xy - 2\sqrt{3}y^2 + y$$

- $D_u f(1,0) = \dfrac{3\sqrt{3}}{2}$

NOTICE: We can write $D_{\vec{u}}f$ as the **DOT PRODUCT** between $\vec{u} = \langle a, b \rangle$ and $\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \rangle$:

$$D_{\vec{u}}f = \langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \rangle \cdot \langle a, b \rangle = \frac{\partial f}{\partial x} a + \frac{\partial f}{\partial y} b$$

INTRODUCE NOTATION: $\nabla f = \langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \rangle$ (vector function)

GRADIENT OF f or grad f

Definition: $\nabla f = \langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \rangle = \frac{\partial f}{\partial x}\vec{i} + \frac{\partial f}{\partial y}\vec{j}$ = GRADIENT OF f

Thus:

$$\boxed{D_{\vec{u}}f = \nabla f \cdot \vec{u}, \text{ where } \vec{u} = \text{unit vector}}$$

DIRECTIONAL DERIVATIVE OF f IN THE DIRECTION OF $\vec{u}$

Generalization to 3 dimensions: If $w = f(x,y,z)$ is a differentiable function $\vec{u} = \langle a, b, c \rangle$ is a UNIT VECTOR, then

- $\nabla f = \langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \rangle$
- $D_{\vec{u}}f = \nabla f \cdot \vec{u} = \frac{\partial f}{\partial x} a + \frac{\partial f}{\partial y} b + \frac{\partial f}{\partial z} c$

Example: Let $f(x,y,z) = x \sin yz$. Find ∇f and the directional derivative of f at $(1,3,0)$ in the direction of $\vec{v} = \vec{i} + 2\vec{j} - \vec{k}$.

Solution:

① $\nabla f = \langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \rangle = \langle \sin yz, \overbrace{xz \cos yz}^{\text{chain rule}}, \overbrace{xy \cos yz}^{\text{chain rule}} \rangle$

② $D_{\vec{u}}f = \nabla f \cdot \vec{v}_0$, $\vec{v}_0$ = unit vector in the direction of $\vec{v}$:

$$\vec{v}_0 = \frac{\vec{v}}{|\vec{v}|} = \frac{\vec{i} + 2\vec{j} - \vec{k}}{\sqrt{1+4+1}} = \frac{1}{\sqrt{6}} \langle 1, 2, -1 \rangle$$

$$D_{\vec{u}}f = \nabla f \cdot \vec{v}_0 = \langle \sin yz, xz \cos yz, xy \cos yz \rangle \cdot \langle 1, 2, -1 \rangle \frac{1}{\sqrt{6}}$$

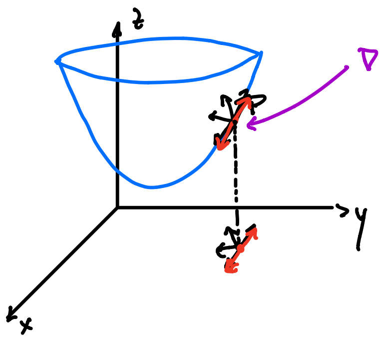
$$D_{\vec{u}} f = \frac{1}{\sqrt{6}} \left(\sin yz + 2xz\cos yz - xy\cos yz \right)$$

3. Evaluate $D_{\vec{u}} f$ at $(1,3,0)$:

$$D_{\vec{u}} f(1,3,0) = \frac{1}{\sqrt{6}} \left(\sin 0 + 2\cdot 1\cdot 0 \cos 0 - 1\cdot 3 \cos 0 \right) = -\frac{3}{\sqrt{6}}$$

$$= -\frac{\sqrt{6}}{2}$$

FINDING THE DIRECTION IN WHICH f CHANGES THE MOST



$\nabla f(P)$ = direction of fastest change of f

Indeed: To find the direction in which f changes the most, we calculate $D_{\vec{u}} f$ and ask ourselves what is $\vec{u}$ in which f changes the most?

$$D_{\vec{u}} f = \nabla f \cdot \vec{u} = |\nabla f|\,|\vec{u}| \cos\theta$$

FROM THE DEFINITION OF THE DOT PRODUCT

Notice the $D_{\vec{u}} f$ is maximal (Largest change of f) when $\cos\theta = 1$, or $\theta = 0$!

Thus $D_{\vec{u}} f$ is maximal when ∇f and $\vec{u}$ are PARALLEL!

Namely, $D_{\vec{u}} f$ is maximal when $\vec{u}$ is proportional to ∇f!

$\boxed{\text{CONCLUSION: The direction in which } f \text{ changes the most is the direction of } \nabla f!}$

What is the rate of change $D_{\vec{u}} f$ in that case?

Calculate $D_{\vec{u}} f$ when $\vec{u} = \underbrace{\dfrac{\nabla f}{|\nabla f|}}_{\text{unit vector}}$:

PROPERTY OF DOT PRODUCT: $\vec{a} \cdot \vec{a} = |\vec{a}|^2$

$$D_{\vec{u}} f = \nabla f \cdot \vec{u} = \nabla f \cdot \frac{\nabla f}{|\nabla f|} = \frac{\nabla f \cdot \nabla f}{|\nabla f|} = \frac{|\nabla f|^{\cancel{2}}}{\cancel{|\nabla f|}} = |\nabla f|$$

CONCLUSION: The maximal value of directional derivative $D_{\vec{u}} f$ is $|\nabla f|$.

THEOREM: Let f be differentiable. The maximum value of $D_{\vec{u}} f$ is $|\nabla f|$ and it occurs when $\vec{u}$ has the same direction as ∇f.

Example: Let $z = xe^y$.

(a) Find the rate of change of $z = xe^y$ at $P(2,0)$ in the direction from P to $Q(\frac{1}{2}, 2)$.

(b) In which direction does f have maximum rate of change?

(c) What is the maximum rate of change?

Solution:
(a)
- $\vec{PQ} = \langle \frac{1}{2} - 2, 2 - 0 \rangle = \langle -\frac{3}{2}, 2 \rangle$. Unit vector $\vec{u} = \dfrac{\vec{PQ}}{|\vec{PQ}|} = \dfrac{\langle -\frac{3}{2}, 2 \rangle}{\sqrt{\frac{9}{4} + 4}}$

  $\vec{u} = \dfrac{1}{\sqrt{\frac{25}{4}}} \langle -\frac{3}{2}, 2 \rangle = \frac{2}{5} \langle -\frac{3}{2}, 2 \rangle = \langle -\frac{3}{5}, \frac{4}{5} \rangle$

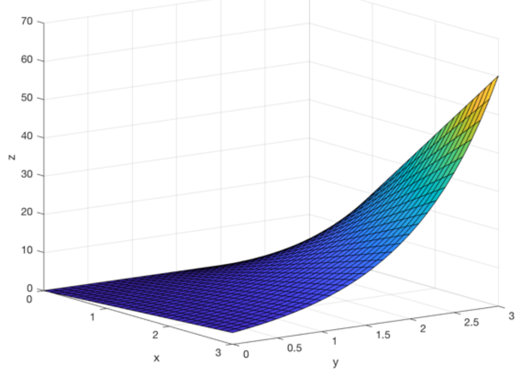
- $\nabla f = \langle e^y, xe^y \rangle$, $\nabla f(P) = \langle e^0, 2 \cdot e^0 \rangle = \langle 1, 2 \rangle$

- $D_{\vec{u}} f(P) = \nabla f(P) \cdot \vec{u} = \langle 1, 2 \rangle \cdot \langle -\frac{3}{5}, \frac{4}{5} \rangle = 1$

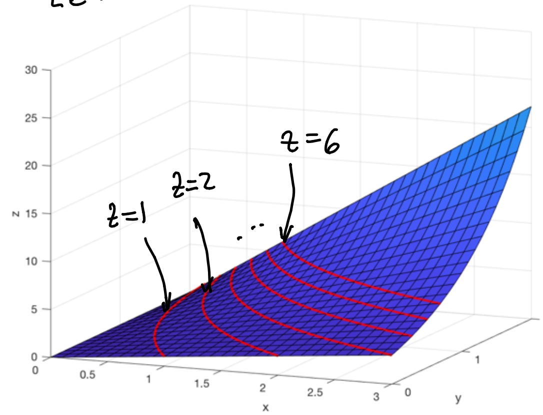
(b) f increases fastest in the direction of $\nabla f(P) = \langle 1, 2 \rangle$

(c) The maximum rate of change is $|\nabla f| = |\langle 1, 2 \rangle| = \sqrt{1+4} = \sqrt{5}$

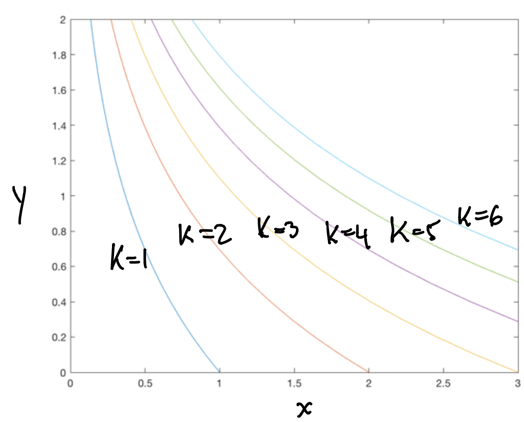
$z = xe^y$



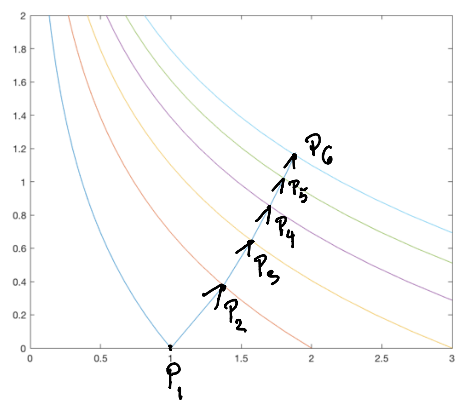
LEVEL CURVES FOR $z = xe^y$



LEVEL CURVES $xe^y = k$

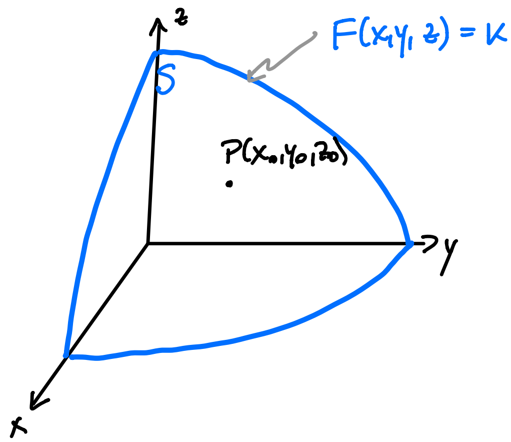


GRADIENT VECTORS ∇f AT POINTS $P_1, \ldots, P_6$



TANGENT PLANES TO LEVEL SURFACES

Suppose we are given $w = F(x,y,z)$ for which we need to find TANGENT PLANES to LEVEL SURFACES $\boxed{F(x,y,z) = k}$, where k = constant.



The tangent plane to S must be tangent to any curve passing through P that lies on the surface S.

Curves in 3D are described by vector functions:

$$\vec{r}(t) = \langle x(t), y(t), z(t) \rangle$$

where t is one parameter.

Curve C that lies on S must satisfy the equation $F(x,y,z) = k$:

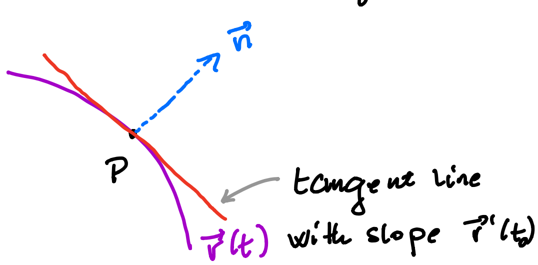
$$\boxed{F(\,x(t), y(t), z(t)\,) = k, \quad t = \text{parameter}}$$

(For example, if $\langle x(t), y(t), z(t) \rangle = \langle t, t^2+1, t^3 \rangle$ then $F(t, t^2+1, t^3) = k$.
So, if $F(x,y,z) = x^2+y^2+z^2$, then $F(t, t^2+1, t^3) = k$ reads $(t)^2 + (t^2+1)^2 + (t^3)^2 = k$)

Let t_0 correspond to the point P so that

$$\vec{r}(t_0) = \langle x_0, y_0, z_0 \rangle.$$

We said that the tangent line to $\vec{r}(t)$ has slope $\vec{r}\,'(t)$. Then



the NORMAL to the tangent plane must be perpendicular to $\vec{r}\,'(t_0)$.

HOW TO FIND THE NORMAL: Need to find a vector $\vec{n}$ such that

$$\vec{n} \cdot \vec{r}\,'(t) = 0 \quad \text{or} \quad \vec{n} \cdot \left\langle \frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt} \right\rangle \Big|_{t=t_0} = 0$$

for an arbitrary curve $\vec{r}(t)$ that lies on S and passes through P,

Thus, $\vec{r}(t)$ satisfies:

$$(\star) \quad F(x(t), y(t), z(t)) = k.$$

A way to find what $\vec{n}$ is, is to differentiate $(\star)$ with respect to t, to obtain (using CHAIN RULE):

$$\frac{\partial F}{\partial x}\frac{dx}{dt} + \frac{\partial F}{\partial y}\frac{dy}{dt} + \frac{\partial F}{\partial z}\frac{dz}{dt} = 0$$

We can write this a the dot product:

$$\left\langle \frac{\partial F}{\partial x}, \frac{\partial F}{\partial y}, \frac{\partial F}{\partial z} \right\rangle \cdot \underbrace{\left\langle \frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt} \right\rangle}_{\vec{r}\,'(t)} = 0$$

and recognize $\longrightarrow \vec{r}\,'(t)$

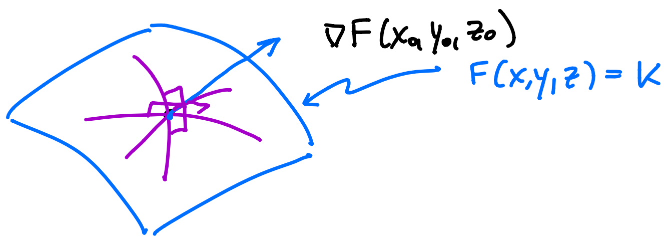
Thus, we have found a vector which is perpendicular to $\vec{r}\,'(t)$:

$$\boxed{\vec{n} = \left\langle \frac{\partial F}{\partial x}, \frac{\partial F}{\partial y}, \frac{\partial F}{\partial z} \right\rangle = \nabla F}$$

In particular, at $P(x_0, y_0, z_0)$ we have

$$\boxed{\nabla F(x_0, y_0, z_0) \cdot \vec{r}\,'(t_0) = 0}$$

CONCLUSION: THE GRADIENT VECTOR $\nabla F(x_0, y_0, z_0)$ IS PERPENDICULAR TO ANY CURVE C ON THE SURFACE S PASSING THROUGH $P(x_0, y_0, z_0)$.



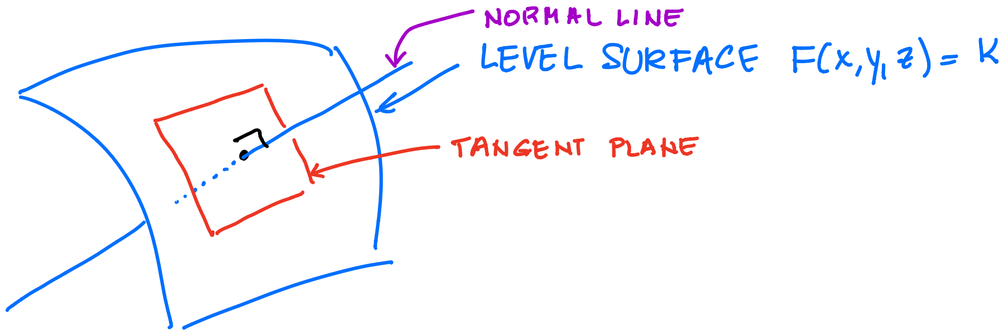
$\Rightarrow$ TANGENT PLANE TO THE LEVEL SURFACE $F(x,y,z)=k$ AT $P(x_0, y_0, z_0)$ IS :

$$\boxed{\frac{\partial F}{\partial x}(x_0,y_0,z_0)\,(x-x_0)+\frac{\partial F}{\partial y}(x_0,y_0,z_0)\,(y-y_0)+\frac{\partial F}{\partial z}(x_0,y_0,z_0)\,(z-z_0)=0}$$

THE NORMAL LINE TO THE LEVEL SURFACE S AT $P(x_0,y_0,z_0)$:

$$\boxed{\frac{x-x_0}{\frac{\partial F}{\partial x}(x_0,y_0,z_0)}=\frac{y-y_0}{\frac{\partial F}{\partial y}(x_0,y_0,z_0)}=\frac{z-z_0}{\frac{\partial F}{\partial z}(x_0,y_0,z_0)}}$$

SYMMETRIC EQUATIONS OF THE NORMAL LINE



EXAMPLE: If S is given in the form $z=f(x,y)$, then we can write

$$F(x,y,z)=f(x,y)-z=0$$

So, we can think of $z=f(x,y)$ as the 0-Level surface of $F(x,y,z)=f(x,y)-z$, and we can use the ideas developed above, to find the tangent plane to the surface $f(x,y)-z=0$ at (x_0,y_0,z_0):

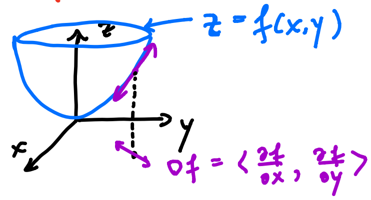
$$F(x,y,z)=f(x,y)-z=0 \Rightarrow \nabla F(x,y,z)=\left\langle \frac{\partial F}{\partial x}, \frac{\partial F}{\partial y}, \frac{\partial F}{\partial z}\right\rangle=\left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, -1\right\rangle=\vec{n}$$

Thus, tangent plane at (x_0,y_0,z_0):

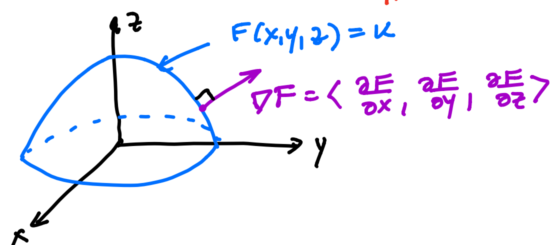
$$\boxed{\frac{\partial f}{\partial x}(x_0,y_0,z_0)(x-x_0)+\frac{\partial f}{\partial y}(x_0,y_0,z_0)(y-y_0)-(z-z_0)=0}$$

SIGNIFICANCE OF THE GRADIENT VECTOR

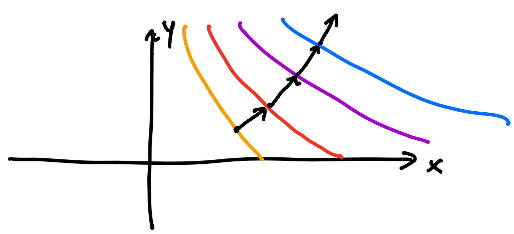
(I) For $z=f(x,y)$: $\nabla f(x_0,y_0)$ POINTS IN THE DIRECTION OF FASTEST CHANGE OF f



(II) For Level surfaces $F(x,y,z)=k$ IN 3D : $\nabla F=\langle \frac{\partial F}{\partial x}, \frac{\partial F}{\partial y}, \frac{\partial F}{\partial z}\rangle$ is PERPENDICULAR TO THE SURFACE $F(x,y,z)=k$.



(III) For Level curves $f(x,y)=k$ in 2D: $\nabla f=\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\rangle$ is PERPENDICULAR TO THE LEVEL CURVES



Example: Find the equation of the tangent plane to the ellipsoid $\frac{x^2}{4} + y^2 + \frac{z^2}{9} = 3$ at the point $P(-2,1,-3)$. What is the normal Line at this point?

Solution: We can regard $\frac{x^2}{4} + y^2 + \frac{z^2}{9} = 3$ as the Level surface of $F(x,y,z) = \frac{x^2}{4} + y^2 + \frac{z^2}{9}$ for $K=3$.

Tangent plane: $\left.\frac{\partial F}{\partial x}\right|_P (x-x_0) + \left.\frac{\partial F}{\partial y}\right|_P (y-y_0) + \left.\frac{\partial F}{\partial z}\right|_P (z-z_0) = 0$

$$\Rightarrow \left.\left(\frac{2x}{4}\right)\right|_P (x+2) + \left.(2y)\right|_P (y-1) + \left.\left(\frac{2z}{9}\right)\right|_P (z+3) = 0$$

$$\Rightarrow \boxed{-(x+2) + 2(y-1) - \left(\frac{2}{3}\right)(z+3) = 0}$$

TANGENT PLANE AT P

$\Rightarrow$ NORMAL LINE AT P:

$$\boxed{\frac{x+2}{-1} = \frac{y-1}{2} = \frac{z+3}{-2/3}}$$