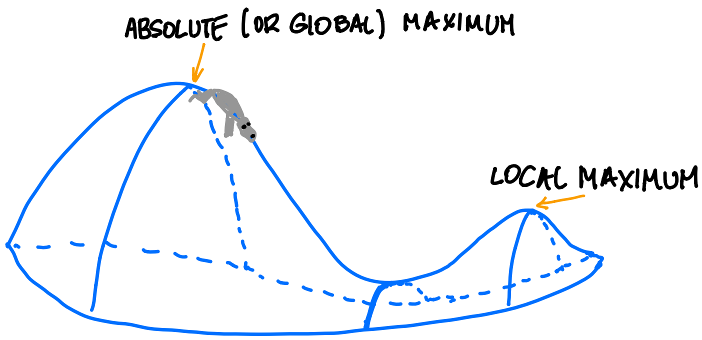
14.7. MAXIMUM AND MINIMUM VALUES



DEFINITION: f = function of two variables has:

- A LOCAL MAXIMUM at (a,b) if

$$f(a,b) \geq f(x,y) \quad \text{for all } (x,y) \text{ in some disk centered at } (a,b)$$

- A LOCAL MINIMUM at (a,b) if

$$f(a,b) \leq f(x,y) \quad \text{for all } (x,y) \text{ in some disk centered at } (a,b)$$

- A GLOBAL MAXIMUM at (a,b) if

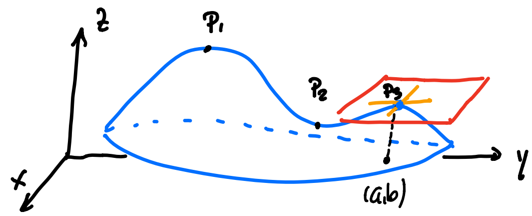
$$f(a,b) \geq f(x,y) \quad \text{for all } (x,y) \text{ in the domain } \mathcal{D} \text{ of } f$$

- A GLOBAL MINIMUM at (a,b) if

$$f(a,b) \leq f(x,y) \quad \text{for all } (x,y) \text{ in the domain } \mathcal{D} \text{ of } f$$

THEOREM: If f has a LOCAL MAX or MIN at (a,b), and f_x and f_y exist at (a,b), then

$$\boxed{f_x(a,b) = f_y(a,b) = 0}$$



CRITICAL POINT (OR STATIONARY POINT)

Point (a,b) is called a CRITICAL POINT of f if $\nabla f = \langle f_x, f_y \rangle = \langle 0, 0 \rangle$ or one of these derivatives does not exist.

NOTE: Not all critical points are extrema. For example, P_2 above is a saddle point (minimum in one direction, maximum in other)

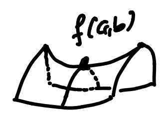
SECOND DERIVATIVE TEST. Suppose (a,b) is a critical point.

Let $\boxed{D(a,b) = f_{xx}(a,b)\, f_{yy}(a,b) - [f_{xy}(a,b)]^2} = \begin{vmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{vmatrix}$

(a) If $D > 0$ and $f_{xx}(a,b) > 0$ THEN $f(a,b)$ IS A LOCAL MINIMUM

(b) If $D > 0$ and $f_{xx}(a,b) < 0$ THEN $f(a,b)$ IS A LOCAL MAXIMUM

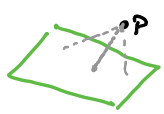
(c) If $D < 0$ THEN $f(a,b)$ IS NOT A LOCAL MINIMUM OR MAXIMUM



(SADDLE POINT)

NOTE: If $D = 0$ the test gives no information.

EXAMPLE: Find the shortest distance from the point $P(1,0,-2)$ to the plane $x + 2y + z = 4$.



Solution: The distance from any point (x,y,z) to $P(1,0,-2)$ is:

$$d = \sqrt{(x-1)^2 + y^2 + (z+2)^2}$$

Since we are interested in points (x,y,z) that lie on the plane $x+2y+z=4$, we know that for such points: $\boxed{z = 4 - x - 2y}$.

So, the distance formula becomes:

$$d = \sqrt{(x-1)^2 + y^2 + (\underbrace{4-x-2y}_{z} + 2)^2} = \sqrt{(x-1)^2 + y^2 + (6-x-2y)^2}.$$

This is a function of two variables for which we would like to find its min:

$$f(x,y) = \sqrt{(x-1)^2 + y^2 + (6-x-2y)^2} \qquad \text{min ?}$$

First we need to find critical points by solving $f_x = 0$ and $f_y = 0$. Differentiating f would give the square root in the denominator and would look complicated. We can simplify our life by finding the point of the minimum of d^2 instead (this point would be the same as the point of the minimum of d).

So, consider $\tilde{f}(x,y) = d^2 = (x-1)^2 + y^2 + (6-x-2y)^2 \longrightarrow$ min ?

• CRITICAL POINTS OF $\tilde{f}(x,y)$: $\frac{\partial \tilde{f}}{\partial x}(x,y) = 0$ and $\frac{\partial \tilde{f}}{\partial y}(x,y) = 0$:

$$\left.\begin{aligned} \frac{\partial f}{\partial x} &= 2(x-1) + 0 + 2(6-x-2y)(-1) = 0 \\ \frac{\partial f}{\partial y} &= 0 + 2y + 2(6-x-2y)(-2) = 0 \end{aligned}\right\}$$

$$\Rightarrow \quad \left.\begin{aligned} 4x + 4y - 14 &= 0 \\ 4x + 10y - 24 &= 0 \end{aligned}\right\} \Rightarrow \text{SOLUTION} \quad \begin{aligned} x &= 11/6 \\ y &= 5/3 \end{aligned}$$

ONE CRITICAL POINT $P_c = \left(\frac{11}{6}, \frac{5}{3}\right)$.

• Is this critical point a MINIMUM? Second derivative test:

$$\tilde{f}_{xx} = 4, \quad \tilde{f}_{yy} = 10, \quad \tilde{f}_{xy} = 4$$

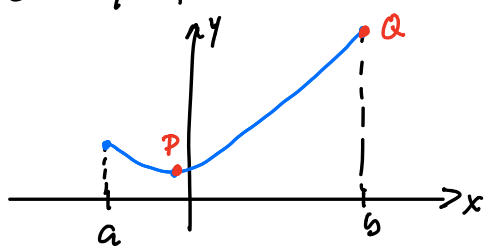
$$\left.\begin{aligned} D &= \tilde{f}_{xx}\tilde{f}_{yy} - (\tilde{f}_{xy})^2 = 4 \cdot 10 - 4^2 = 40 - 16 > 0 \\ \tilde{f}_{xx} &= 4 > 0 \end{aligned}\right\} \Rightarrow \text{LOCAL MINIMUM !}$$

The shortest distance: $d(x,y) = d\left(\frac{11}{6}, \frac{5}{3}\right) = \sqrt{\left(\frac{11}{6}-1\right)^2 + \left(\frac{5}{3}\right)^2 + \left(6 - \frac{11}{6} - 2\cdot\frac{5}{3}\right)^2} = \frac{5\sqrt{6}}{6}$

The shortest distance from $(1,0,-2)$ to $x+2y+z=4$ is $\frac{5\sqrt{6}}{6}$

ABSOLUTE MAXIMUM AND MINIMUM VALUES

Recall: $y = f(x)$ on $[a,b]$ = CLOSED AND BOUNDED INTERVAL



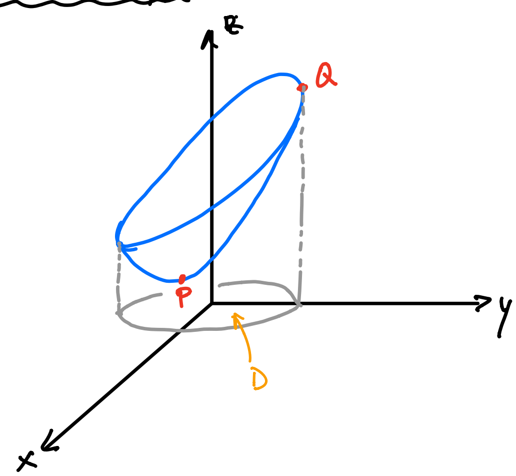
P = GLOBAL MINIMUM

Q = GLOBAL MAXIMUM

f attains its global max and min on $[a,b]$!

EXTREME VALUE THEOREM

For $z = f(x,y)$:



D = CLOSED AND BOUNDED SET IN $\mathbb{R}^2$

CLOSED = THE SET CONTAINS ALL ITS BOUNDARY POINTS

BOUNDED = IT IS CONTAINED WITHIN SOME DISK

EXTREME VALUE THEOREM: If f is CONTINUOUS on a closed, bounded set D in $\mathbb{R}^2$ then f attains an ABSOLUTE MAXIMUM VALUE $f(x_M, y_M)$ and an ABSOLUTE MINIMUM VALUE $f(x_m, y_m)$ at some points (x_M, y_M) and (x_m, y_m) in D.

HOW TO FIND THE ABSOLUTE EXTREMA

(1) Find the values of f at the CRITICAL POINTS of f in D.

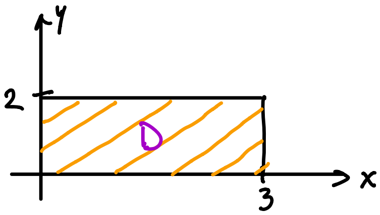
(2) Find the extreme values of f at the BOUNDARY of D.

(3) The Largest of the values from steps (1) and (2) is the MAX

The smallest of the values from steps (1) and (2) is the MIN

Example: Find absolute extrema of $f(x,y) = x^2 - 2xy + 2y$ on the rectangle $D = \{(x,y) \mid 0 \le x \le 3,\ 0 \le y \le 2\}$.

Solution:

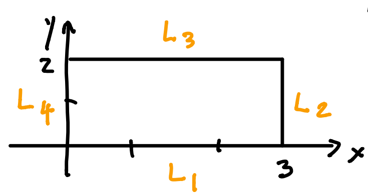


- f is a polynomial so it is continuous on D. By the Extreme Value Theorem it attains its absolute extrema on D.

- CRITICAL POINTS: $\left.\begin{array}{l} f_x = 2x - 2y = 0 \\ f_y = -2x + 2 = 0 \end{array}\right\} \Rightarrow \begin{array}{l} x = 1 \\ y = 1 \end{array}$

$P(1,1)$ is the only critical point, and $\boxed{f(1,1) = 1}$

- f on the BOUNDARY of D: The boundary of D consists of 4 line segments:



On L_1: $y = 0,\ 0 \le x \le 3$. So $f(x,0) = x^2$, $0 \le x \le 3$.

This is an increasing function on $[0,3]$.

Its minimum is at $x=0$: $\boxed{f(0,0)=0}$

Its maximum is at $x=3$: $\boxed{f(3,0)=9}$

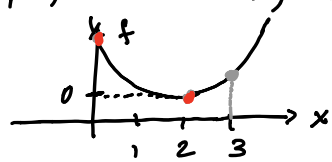
On L_2: $x=3$, $0\le y\le 2$. So $f(3,y)=9-4y$, $0\le y\le 2$.

This is a decreasing function so $\boxed{f_{min}(3,2)=1}$

$\boxed{f_{max}(3,0)=9}$

On L_3: $y=2$, $0\le x\le 3$. So $f(x,2)=x^2-4x+4$, $0\le x\le 3$.

Or, $f(x,y)=(x-2)^2$



$\boxed{f_{min}(2,2)=0}$

$\boxed{f_{max}(0,2)=4}$

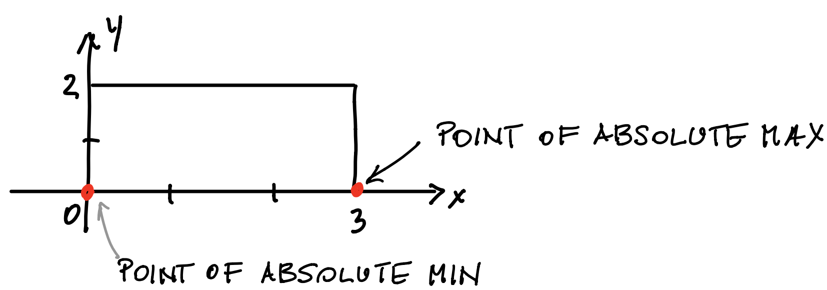
On L_4: $x=0$, $0\le y\le 2$: $f(0,y)=2y \Rightarrow \boxed{f_{min}(0,0)=0}$

$\boxed{f_{max}(0,2)=4}$

- Compare min and max values:

ABSOLUTE MAX: $\boxed{f_{max}=f(3,0)=9}$

ABSOLUTE MIN: $\boxed{f_{min}=f(0,0)=0}$

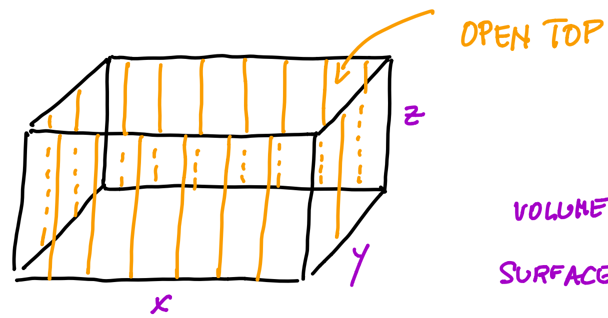


14.8. LAGRANGE MULTIPLIERS

- CONSTAINED OPTIMIZATION

  FINDING A MAX OR MIN SUBJECT TO A CONSTRAINT

Example: Maximaze the volume of a rectangular box, subject to a constraint that the surface area is $12\ m^2$.



VOLUME: $V = xyz$

SURFACE AREA:

$A = 2yz + 2xz + xy$

$$\begin{cases} f(x,y,z) = xyz \longrightarrow \max \\ 2xz + 2yz + xy = 12 \quad (\text{constraint}) \end{cases}$$

In general:

$f(x,y,z) \to \max$

SUBJECT TO A CONSTRAINT:

$g(x,y,z) = k$

CONSTRAINED OPTIMIZATION PROBLEM

Solution: MAIN IDEAS EXPLAINED ON A 2D PROBLEM

$f(x,y) \to \max$ (or min)

SUBJECT TO A CONSTRAINT:

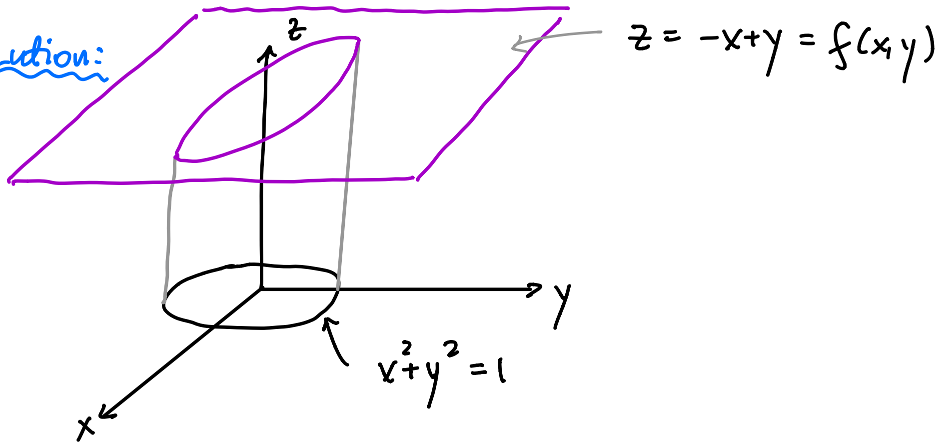
$g(x,y) = k$

EXAMPLE: Find the maximum and minimum of

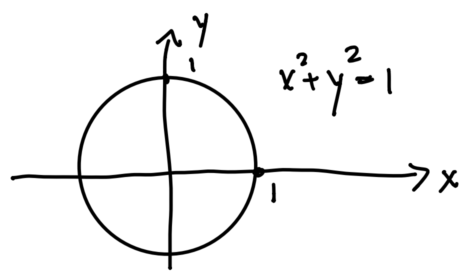
$$z = f(x,y) = -x+y$$

subject to the constraint: $x^2+y^2=1$.

Solution:



In the xy plane:

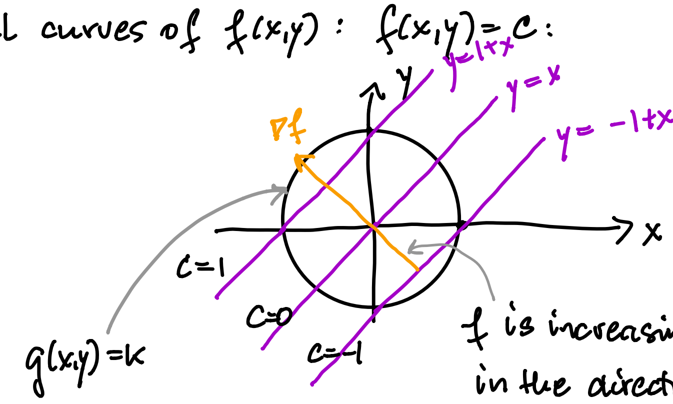


Plot the level curves of $f(x,y)$: $f(x,y)=c$:

$-x+y=0$

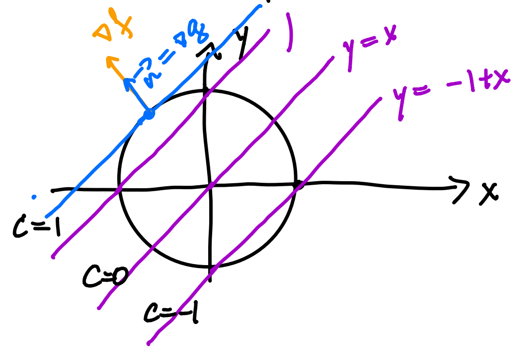
$-x+y=1$

$-x+y=-1$



f is increasing in the direction transverse to level curves. Fastest increase is in the ∇f direction.

At the extrema, the Level curve of f is tangent to $g(x,y)=k$.



So, THE NORMAL TO $g(x,y)=k$ IS PARALLEL TO ∇f!

What is the normal to $g(x,y)=k$:

$\vec{n} = \nabla g$ FOR $g(x,y)=k$.

(Last Lecture)

THUS, $\boxed{\nabla f = \lambda \nabla g}$ AT THE EXTREMA!

CONCLUSION: TO FIND THE POINTS OF CONSTRAINED MIN OF MAX:

(1) SOLVE $\begin{cases} \nabla f = \lambda \nabla g \\ g(x,y) = k \end{cases}$ FOR (x,y) AND λ

LAGRANGE MULTIPLIER

(2) EVALUATE f AT THE POINTS FROM (1). The Largest is the MAX, the smallest is the MIN.

In 3D: **Problem:** $\begin{cases} f(x,y,z) \longrightarrow \text{extremum} \\ g(x,y,z) = k \end{cases}$

Solution: (1) Solve for (x,y,z) and λ the following system:

$$\begin{cases} \nabla f(x,y,z) = \lambda \nabla g(x,y,z) \\ g(x,y,z) = k \end{cases}$$

(2) Evaluate f at the points found in (1)

The Largest value of f = MAX

The smallest value of f = MIN

EXAMPLE:
$$\begin{cases} f(x,y) = -x+y \to \max \\ x^2+y^2 = 1 \end{cases}$$

Solution:
- $$\left.\begin{matrix} \nabla f = \lambda \nabla g \\ g(x,y) = k \end{matrix}\right\} \Rightarrow \begin{cases} \langle -1, 1\rangle = \lambda \langle 2x, 2y\rangle \\ x^2+y^2 = 1 \end{cases}$$

$$\Rightarrow \begin{cases} 2\lambda x = -1 \\ 2\lambda y = 1 \\ x^2+y^2 = 1 \end{cases} \Rightarrow \quad \begin{aligned} &x = -\frac{1}{2\lambda} \\ &y = \frac{1}{2\lambda} \\ &x^2+y^2 = \frac{1}{4\lambda^2} + \frac{1}{4\lambda^2} = \frac{1}{2\lambda^2} = 1 \Rightarrow \boxed{\lambda = \pm\frac{1}{\sqrt{2}}} \end{aligned}$$

Two Lagrange multipliers

For $\lambda = \frac{1}{\sqrt{2}} \Rightarrow P_1\left(-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$

For $\lambda = -\frac{1}{\sqrt{2}} \Rightarrow P_2\left(\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}\right)$

- **EXTREMA:**
$$f\left(-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right) = \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2} = \sqrt{2} \Rightarrow \text{MAX}$$
$$f\left(\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}\right) = -\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2} = -\sqrt{2} \Rightarrow \text{MIN}$$

At home, solve: EXAMPLE 1, EXAMPLE 2, and EXAMPLE 4, pg 973-975

EXAMPLE 3, pg 974:

Global extrema of $f(x,y) = x^2 + 2y^2$ on the disk $x^2+y^2 \le 1$.

Solution:

First note that $x^2+y^2 \le 1$ is a CLOSED and BOUNDED subset of $\mathbb{R}^2$.

By the Extreme Value Theorem $\Rightarrow$ global extrema exist (f is continuous).

To find the extrema:

(1) Find CRITICAL POINTS in $x^2+y^2<1$ (interior of $x^2+y^2\le 1$).

(2) Find extreme values on the boundary $x^2+y^2=1$

(3) Compare values of f in (1) and (2).

STEP 1 CRITICAL POINTS in $x^2+y^2<1$: $\nabla f=0 \Rightarrow \left.\begin{array}{l} f_x=2x=0 \\ f_y=4y=0 \end{array}\right\} \Rightarrow P_1(0,0)$

Evaluate f at $P_1(0,0)$: $\boxed{f(0,0)=0}$

STEP 2: Constrained optimization: $\begin{cases} f(x,y)=x^2+2y^2 \longrightarrow \text{min, max} \\ \underbrace{x^2+y^2}_{g(x,y)}=1 \end{cases}$

Use method of Lagrange Multipliers to solve it:

$$\left\{\begin{array}{l} \nabla f=\lambda \nabla g \\ g(x,y)=k \end{array}\right\} \Rightarrow \begin{cases} \langle 2x,4y\rangle=\lambda\langle 2x,2y\rangle \\ x^2+y^2=1 \end{cases}$$

$$\Rightarrow \begin{cases} 2x=\lambda\cdot 2x & \Rightarrow & x-\lambda x=0 & \Rightarrow & x(1-\lambda)=0 & (1) \\ 4y=\lambda\cdot 2y & \Rightarrow & 2y-\lambda y=0 & \Rightarrow & y(2-\lambda)=0 & (2) \\ x^2+y^2=1 & & & & x^2+y^2=1 & (3) \end{cases}$$

From (1): Either $x=0$ or $\lambda=1$ (or both)

If $x=0$, (3) implies $y^2=1 \Rightarrow y=\pm 1 \Rightarrow P_2(0,1), P_3(0,-1)$ and $\lambda=2$.

Evaluate f at P_2 and P_3:

$\boxed{f(0,1)=2,\ f(0,-1)=2}$ $\lambda=2$

If $\lambda=1$, (2) implies: $y=0$, and (3) $\Rightarrow$ $\underline{\underline{x=\pm 1}}$

$\Rightarrow P_4(1,0)$, $P_5(-1,0)$.

Evaluate f at P_4 and P_5:

$$\boxed{\underline{\underline{f(1,0)=1}}, \quad \underline{\underline{f(-1,0)=1}}}$$

$\lambda=1$

Compare values in orange boxes $\Rightarrow$

MAXIMUM along $x^2+y^2=1$ is equal $f_{MAX}=2$ at $P_2(0,1)$, $P_3(0,-1)$

MINIMUM along $x^2+y^2=1$ is equal $f_{MIN}=1$ at $P_4(1,0)$, $P_5(-1,0)$

STEP 3 Compare values in purple boxes:

GLOBAL MAXIMUM: $f_{MAX}=2$ at $P_2(0,1)$ and $P_3(0,-1)$

GLOBAL MINIMUM: $f_{min}=0$ at $P_1(0,0)$

TWO CONSTRAINTS

$$\begin{cases} f(x,y,z) \to \text{extremum} \\ g(x,y,z)=K \\ h(x,y,z)=C \end{cases}$$

Two constraints $\Rightarrow$ two Lagrange multipliers λ, μ

To find extrema, solve for (x,y,z) and λ and μ:

3 equations $\longrightarrow$ $\nabla f(x,y,z) = \lambda \nabla g(x,y,z) + \mu \nabla h(x,y,z)$

1 equation $\longrightarrow$ $g(x,y,z) = K$

1 equation $\longrightarrow$ $h(x,y,z) = C$