ANNOUNCEMENTS

1. MIDTERM 2: Nov 17, in class 5-7 pm
2. MIDTERM REVIEW: Nov 12, in class
3. PRACTICE MIDTERM WILL BE POSTED ON Nov 10 on bCourses
4. Thursday Lecture will be pre-recorded (serving on NSF review panel from 7 am till 5 pm Th, Fri)

TOPICS ON MIDTERM 2

1. 13.1 and 13.2
2. 14.1 through 14.8
3. 15.1 through 15.3 and 15.5 through 15.9
4. 16.1 through 16.5 ???

16. VECTOR CALCULUS

1. FUNCTIONS OF ONE VARIABLE

→ SCALAR FUNCTIONS
$z = f(x)$, $\boxed{x \in \mathbb{R}}$

→ VECTOR FUNCTIONS
$\vec{r} = \langle r_1(t), r_2(t) \rangle$, $\boxed{t \in \mathbb{R}}$
(e.g. $\vec{r} = \langle x(t), y(t) \rangle$)

2. FUNCTIONS OF MORE THAN ONE VARIABLE:

→ SCALAR FUNCTIONS:
$z = f(x, y)$

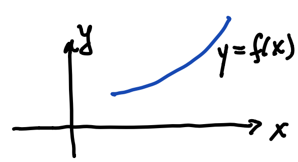
→ VECTOR FUNCTIONS:
$\vec{F} = \langle P(x,y), Q(x,y) \rangle$

THIS IS WHAT WE PLAN TO STUDY NOW

GEOMETRIC REPRESENTATION:

① ONE VARIABLE :

→ $y = f(x)$

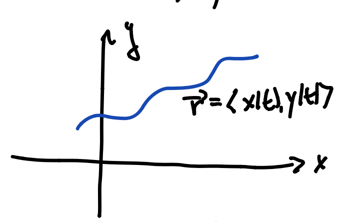


CURVE IN $\mathbb{R}^2$

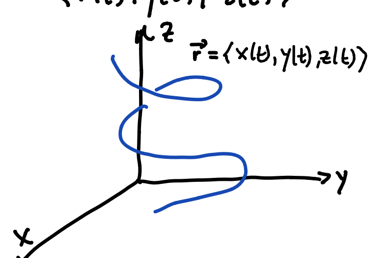
→ $\vec{r} = \langle r_1(t), r_2(t) \rangle$
$= \langle x(t), y(t) \rangle$

OR

$\vec{r} = \langle r_1(t), r_2(t), r_3(t) \rangle$
$= \langle x(t), y(t), z(t) \rangle$



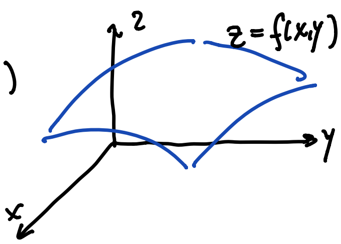
CURVE IN $\mathbb{R}^2$
(ORIENTED CURVE)



CURVE IN $\mathbb{R}^3$

② MORE THAN ONE VARIABLE:

- $z = f(x,y)$



SURFACE IN $\mathbb{R}^3$

- $\vec{F} = \langle P(x,y), Q(x,y) \rangle$ HOW TO REPRESENT THIS FUNCTION?

WE WOULD NEED 4-dimensional space: 2 dimensions to represent (x,y) plus two dimensions to represent $\langle P, Q \rangle$. Cannot do that.

Instead, for each (x,y) in $\mathbb{R}^2$ we draw a vector $\langle P(x,y), Q(x,y) \rangle$!

For example: Let $\vec{F}(x,y) = \langle -y, x \rangle$. Describe $\vec{F}$ by sketching some vectors.

Solution:

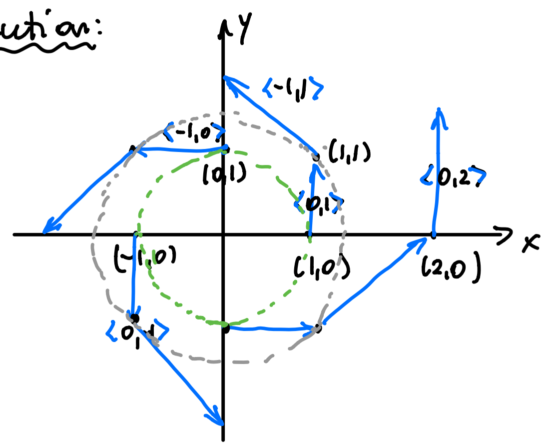
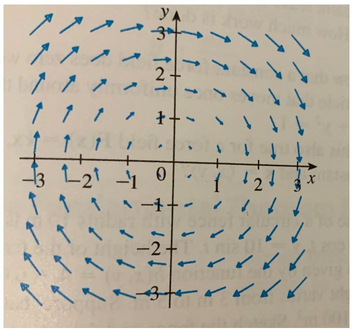


TABLE:

(x,y)	$\vec{F}(x,y) = \langle -y, x \rangle$
$(1,0)$	$\langle 0,1 \rangle$
$(-1,0)$	$\langle 0,-1 \rangle$
$(2,2)$	$\langle -2,2 \rangle$
$(0,1)$	$\langle -1,0 \rangle$
$(0,-1)$	$\langle 1,0 \rangle$
$\vdots$	$\vdots$

It appears that this vector function takes a point (x,y) and moves it in the direction tangent to the circle passing through that point (rotation). More precisely, it look like $\langle P, Q \rangle$ is perpendicular to $\langle x, y \rangle$ position vector of (x,y). Indeed, the dot product:

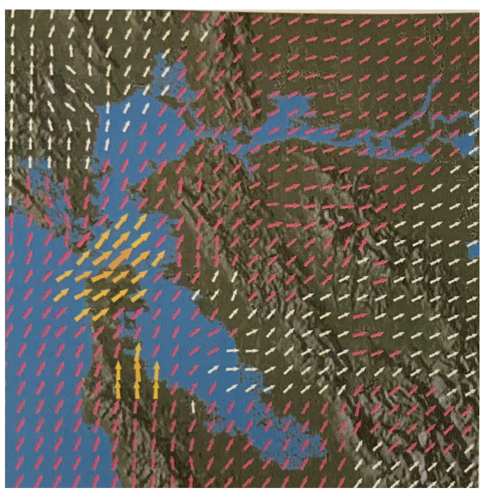
$$\underbrace{\langle P, Q\rangle} \cdot \langle x, y\rangle = \underbrace{\langle -y, x\rangle} \cdot \langle x, y\rangle = -yx + xy = 0 \checkmark$$



Thus, $\vec{F}(x,y) = \langle -y, x\rangle = -y\vec{i} + x\vec{j}$ is tangent to any circle with center at the origin!

EXAMPLE: Velocity field describing WIND VELOCITY at every Location (x,y) (measured 10 meters above the surface) on the map of the Bay Area:

$\vec{F}_{WIND}(x,y)$:



Don't have a formula for function $\vec{F}_{WIND}$

😂

DEFINITION: (VECTOR FIELD) Let D be a set $\mathbb{R}^2$. A VECTOR FIELD on $\mathbb{R}^2$ is a function $\vec{F}$ that assigns to each point (x,y) in D a two-dimensional vector $\vec{F}(x,y) = P(x,y)\vec{i} + Q(x,y)\vec{j}$.

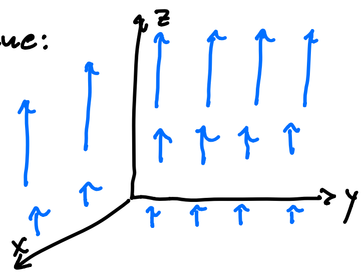
COMPONENT FUNCTIONS OF $\vec{F}$

VISUALIZATION: Draw arrows representing the vector $\vec{F}$ starting at several different points (x,y).

SIMILARLY, IN $\mathbb{R}^3$: $\vec{F}(x,y,z) = P(x,y,z)\,\vec{i} + Q(x,y,z)\,\vec{j} + R(x,y,z)\,\vec{k}$

EXAMPLE: Sketch $\vec{F}(x,y,z) = z\,\vec{k}$ in $\mathbb{R}^3$.

Solution: All the vectors are vertical and with magnitude equal to the distance from the xy-plane:



GRADIENT FIELDS

Let $f(x,y,z)$ be a SCALAR FUNCTION.

GRADIENT OF f:

$$\boxed{\nabla f(x,y,z) = \frac{\partial f}{\partial x}(x,y,z)\,\vec{i} + \frac{\partial f}{\partial y}(x,y,z)\,\vec{j} + \frac{\partial f}{\partial z}(x,y,z)\,\vec{k}}$$

This is a GRADIENT VECTOR FIELD
(defined for all (x,y,z) in some D)

EXAMPLE: Let $f = x^2y - y^3$. Find its gradient vector field in $\mathbb{R}^2$.

Solution:

$$\nabla f(x,y) = \langle 2xy,\ x^2 - 3y^2 \rangle = 2xy\,\vec{i} + (x^2 - 3y^2)\,\vec{j}$$

This is a vector function in $\mathbb{R}^2$.

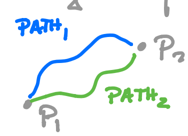
EXAMPLE: Let $\vec{F}(x,y) = 2xy\,\vec{i} + (x^2 - 3y^2)\,\vec{j}$. Can $\vec{F}$ be written as a gradient of a scalar function? Namely, is there a scalar function f such that

$$\vec{F} = \nabla f \quad ?$$

Answer: Yes, $f(x,y) = x^2y - y^3$.

Definition: A vector field $\vec{F}$ is called CONSERVATIVE if it is a gradient of some scalar function, that is, if $\vec{F} = \nabla f$. In this case, f is called a POTENTIAL function of $\vec{F}$.

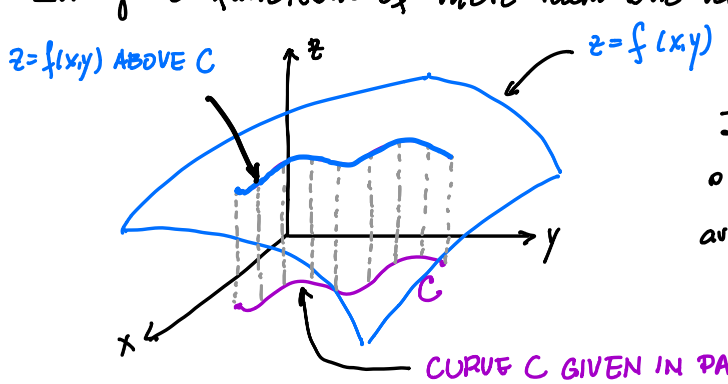
(We will see later that integrating conservative vector fields along two paths connecting the same two points gives the same result.)



(In physics, the work done by a force on an object in motion is done by taking a line integral of the force vector field along the path of motion. A force is called conservative if the work it does on an object moving from any point A to another point B is always the same, no matter what path is taken. One example is GRAVITY $= \vec{F}$. The function f is potential energy (gravity potential).) 😎

16.2. LINE INTEGRALS

Integrate functions of more than one variables over a CURVE!



GEOMETRIC INTERPRETATION:

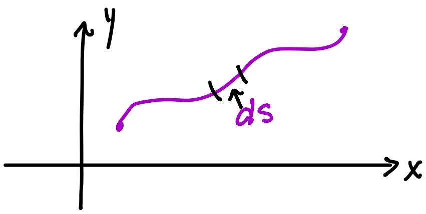
If $f(x,y) \geq 0$, geometric interpretation of the integral $\int_C f\,ds$ is the area of the "curtain" under f, above C.

CURVE C GIVEN IN PARAMETRIC FORM $\begin{cases} x = x(t) \\ y = y(t) \end{cases}$, $t \in [a,b]$

$$\boxed{I = \int_C f(x,y)\, ds = \,?}$$

LINE INTEGRAL WITH RESPECT TO ARC LENGTH ds

Here ds is "infinitesimally" small piece of the ARC LENGTH of C:



Recall:

$$\text{Arc Length } L = \int_a^b \underbrace{\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt}_{ds}$$

DEFINITION: If C is a smooth curve given by $\begin{cases} x = x(t) \\ y = y(t) \end{cases}$ $t \in [a,b]$,

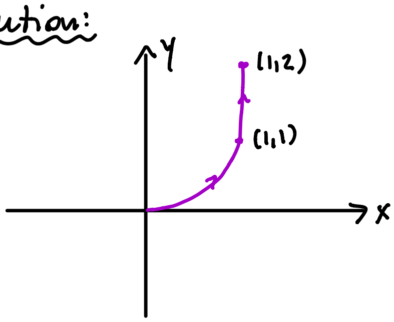
then the LINE INTEGRAL OF f ALONG C is

$$\boxed{\int_C f(x,y)\, ds = \int_C f(x(t), y(t)) \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt}$$

LINE INTEGRAL OF f OVER C WITH RESPECT TO ARC LENGTH

EXAMPLE: Evaluate $\int_C 2x\, ds$, where C consists of the arc C_1 of the parabola $y = x^2$ from $(0,0)$ to $(1,1)$, followed by the vertical line segment C_2 from $(1,1)$ to $(1,2)$.

Solution:



This is a piecewise smooth curve, consisting of two pieces that are smooth. We can break the integral into two parts, each along a smooth curve C_1 and C_2.

$$\int_C f(x,y)\, ds = \int_{C_1} f(x,y)\, ds + \int_{C_2} f(x,y)\, ds$$

① Calculate $\int_{C_1} f(x,y)\,ds$, where C_1 is $y=x^2$. Parametric form: take x as parameter, then $C_1 = \begin{cases} x=x \\ y=x^2 \end{cases}, \; x\in[0,1]$

$$\int_{C_1} 2x\,ds = \int_{C_1} 2x\sqrt{\left(\frac{dx}{dx}\right)^2+\left(\frac{dy}{dx}\right)^2}\,dx =$$

$$= \int_0^1 2x\sqrt{1+(2x)^2}\,dx = \int_0^1 2x\sqrt{4x^2+1}\,dx = \left| \begin{matrix} 4x^2+1=u \\ 8x\,dx=du \end{matrix} \right|$$

$$= \frac{1}{4}\int \sqrt{u}\,du = \frac{1}{4}\cdot\frac{2}{3}u^{3/2} = \frac{1}{6}\left[\left(\sqrt{4x^2+1}\right)^3\right]_{x=0}^{1} = \frac{1}{6}\left[5\sqrt{5}-1\right]$$

② Calculate $\int_{C_2} 2x\,ds$, where C_2 is the segment from $(1,1)$ to $(1,2)$.

We can choose y as a parameter, then $C_2 = \begin{cases} x=1 \\ y=y \end{cases}, \; y\in[1,2]$

$$\int_{C_2} 2x\,ds = \int_{C_2} 2\cdot 1\sqrt{0^2+\left(\frac{\partial y}{\partial y}\right)^2}\,dy = \int_1^2 2\sqrt{1}\,dy = 2$$

③ $$\int_C 2x\,ds = \int_{C_1} 2x\,ds + \int_{C_2} 2x\,ds = \frac{1}{6}\left[5\sqrt{5}-1\right]+2$$

- MASS OF WIRE OF SHAPE C AND LINEAR DENSITY $\rho(x,y)$:

$$\boxed{m = \int_C \rho(x,y)\,ds}$$

- CENTER OF MASS:

$$\boxed{\bar{x} = \frac{1}{m}\int_C x\,\rho(x,y)\,ds, \quad \bar{y} = \frac{1}{m}\int_C y\,\rho(x,y)\,ds}$$

THE LINE INTEGRALS OF f ALONG C WITH RESPECT TO x and y

(compare c.f. partial derivatives with respect to x and y, versus $\frac{\partial f}{\partial \vec{v}}$ $\vec{v}$ vector)

$$\int_C f(x,y)\,dx = ? \qquad \int_C f(x,y)\,dy = ?$$

😳

where C is given by parametric equations $\begin{cases} x = x(t) \\ y = y(t) \end{cases}$, $t \in [a,b]$.

Recall: If $x = x(t)$ then $dx = x'(t)dt$. Similarly: for $y = y(t)$, $dy = y'(t)dt$.

Thus:

$$\int_C f(x,y)\,dx = \int_C f(x(t), y(t))\, x'(t)\,dt$$

LINE INTEGRAL OVER C WITH RESPECT TO x

$$\int_C f(x,y)\,dy = \int_C f(x(t), y(t))\, y'(t)\,dt$$

LINE INTEGRAL OVER C WITH RESPECT TO y

Recall:

$$\boxed{\int_C f(x,y)\,ds = \int_C f(x(t), y(t))\sqrt{[x'(t)]^2 + [y'(t)]^2}\,dt}$$

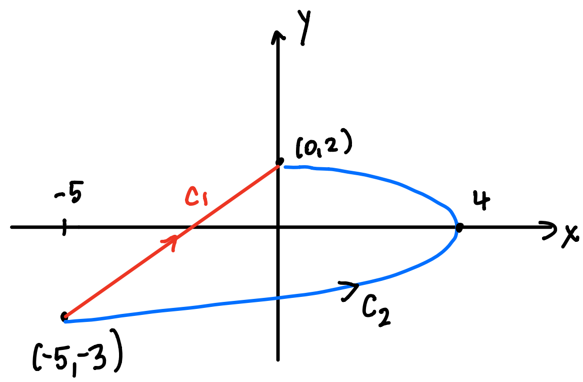
LINE INTEGRAL OVER C WITH RESPECT TO ARC LENGTH.

EXAMPLE: $\int_C y^2 dx + x\,dy = ?$ where:

(a) $C = C_1$ is the Line segment from $(-5,-3)$ to $(0,2)$

(b) $C = C_2$ is the arc of the parabola $x = 4 - y^2$ from $(-5,-3)$ to $(0,2)$.

Solution:



(a) Line segment from $(-5,-3)$ to $(0,2)$:

Recall: Line segment from (x_0, y_0) to (x_1, y_1): $\vec{r}(t) = (1-t)\vec{r_0} + t\vec{r_1}$, $t \in [0,1]$

$$\vec{r} = (1-t)\langle -5,-3\rangle + t\langle 0,2\rangle = \langle -5(1-t), -3(1-t)\rangle + \langle 0, 2t\rangle = \langle 5t-5, 5t-3\rangle$$

$\Rightarrow$ Line segment parametric equations $= \begin{cases} x = 5t-5 \\ y = 5t-3 \end{cases}$ $t \in [0,1]$.

$$\int_{C_1} y^2\,dx + x\,dy = \int_0^1 \underbrace{(5t-3)}_{y}{}^2 \underbrace{5}_{x'(t)}\,dt + \int_0^1 \underbrace{(5t-5)}_{x} \underbrace{5}_{y'(t)}\,dt = \ldots = -\frac{5}{6}$$

(b) C_2 = parabola $x = 4-y^2$. Parametric equations: $\begin{cases} x = 4-y^2 \\ y = y \end{cases}$, $y \in [-3,2]$

$$\int_{C_2} y^2\,dx + x\,dy = \int_{-3}^{2} y^2 \cdot \underbrace{(-2y)}_{x'(y)}\,dy + \int_{-3}^{2} \underbrace{(4-y^2)}_{x}\,dy = \ldots = 40\frac{5}{6}$$

NOTICE: In Line integrals, the answer depends not only on the end points of the curve, but also on the path C!

CURVE ORIENTATION

EXAMPLE: $\int_C y^2 dx + x\,dy$ over the curve $C=-C_1$, which is the line segment FROM $(0,2)$ TO $(-5,-3)$.

Solution: Now $\vec{r_0} = \langle 0,2\rangle$, $\vec{r_1} = \langle -5,-3\rangle$, so that

$$\vec{r} = (1-t)\vec{r_0} + t\vec{r_1} = (1-t)\langle 0,2\rangle + t\langle -5,-3\rangle = \langle -5t, 2-5t\rangle, \quad t\in[0,1]$$

So, we calculate:

$$\int_{-C_1} y^2 dx + x\,dy = \int_0^1 (2-5t)^2(-5)\,dt + \int_0^1 (-5t)(-5)\,dt = \ldots = \frac{5}{6}$$

Thus:
$$\int_{-C_1} y^2 dx + x\,dy = -\int_{C_1} y^2 dx + x\,dy \;!$$

THE FOLLOWING STATEMENTS ARE TRUE:

1. A given parameterization determines an ORIENTATION of a curve C, with the positive direction corresponding to INCREASING values of parameter t.

2. CURVE ORIENTATION MATTERS IN INTEGRALS WITH RESPECT TO X AND Y!

$$\boxed{\int_{-C} f(x,y)\,dx = -\int_C f(x,y)\,dx} \qquad \boxed{\int_{-C} f(x,y)\,dy = -\int_C f(x,y)\,dy}$$

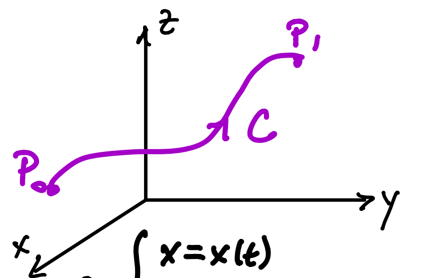
3. CURVE ORIENTATION DOES NOT MATTER IN INTEGRALS WITH RESPECT TO ARC LENGTH!

$$\boxed{\int_{-C} f(x,y)\,ds = \int_C f(x,y)\,ds}$$

LINE INTEGRALS IN SPACE

$$\int_C f(x,y,z)\,ds = ?$$

where



$$C \begin{cases} x = x(t) \\ y = y(t) & t \in [a,b] \\ z = z(t) \end{cases}$$

① The LINE INTEGRAL OF f WITH RESPECT TO ARC LENGTH:

$$\boxed{\int_C f(x,y,z)\,ds = \int_a^b f(x(t),y(t),z(t))\sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2+\left(\frac{dz}{dt}\right)^2}\,dt}$$

IN MORE COMPACT NOTATION:

Curve C in vector form: $\vec{r}(t) = x(t)\,\vec{i} + y(t)\,\vec{j} + z(t)\,\vec{k}$

Then: $\vec{r}\,'(t) = x'(t)\,\vec{i} + y'(t)\,\vec{j} + z'(t)\,\vec{k}$ $\quad t \in [a,b]$

and $|\vec{r}\,'(t)| = \sqrt{(x'(t))^2+(y'(t))^2+(z'(t))^2}$

Thus:

$$\boxed{\int_C f(x,y,z)\,ds = \int_a^b f(\vec{r}(t))\,|\vec{r}\,'(t)|\,dt}$$

SPECIAL CASE WHEN $f(x,y,z)=1 \Rightarrow$

$$\boxed{\int_C ds = \int_a^b |\vec{r}\,'(t)|\,dt = L}$$

CURVE ARC LENGTH

② The LINE INTEGRAL OF f WITH RESPECT TO x, y, and z:

$$\boxed{\begin{aligned}&\int_C P(x,y,z)\,dx + Q(x,y,z)\,dy + R(x,y,z)\,dz \\ &= \int_a^b P(x(t),y(t),z(t))\,x'(t)\,dt + Q(x(t),y(t),z(t))\,y'(t)\,dt + R(x(t),y(t),z(t))\,z'(t)\,dt\end{aligned}}$$

EXAMPLE: Evaluate $\int_C y\,dx + z\,dy + x\,dz$ where C consists of the Line segment C_1 from $(2,0,0)$ to $(3,4,5)$, followed by the vertical segment C_2 from $(3,4,5)$ to $(3,4,0)$.

Solution:

① C_1 ... Line segment from $(2,0,0)$ to $(3,4,5)$: $\vec{r}(t) = (1-t)\vec{r}_0 + t\vec{r}_1$

$\vec{r}(t) = (1-t)\langle 2,0,0\rangle + t\langle 3,4,5\rangle = \langle 2(1-t)+3t,\ 4t,\ 5t\rangle$

$$\Rightarrow \begin{cases} x(t) = 2+t \\ y(t) = 4t \\ z(t) = 5t \end{cases} \qquad t\in[0,1]$$

② C_2 --- Line segment from $(3,4,5)$ to $(3,4,0)$:

$\vec{r}(t) = (1-t)\langle 3,4,5\rangle + t\langle 3,4,0\rangle = \langle 3,\ 4,\ 5-5t\rangle, \quad t\in[0,1]$

③ $$\int_C y\,dx + z\,dy + x\,dz = \int_{C_1} (\qquad) + \int_{C_2} (\qquad) = I_1 + I_2$$

$$I_1 = \int_0^1 \left\{ 4t \cdot \underbrace{x'(t)}_{=1} dt + 5t \cdot \underbrace{y'(t)}_{=4} dt + (2+t) \underbrace{z'(t)}_{=5} dt \right\}$$

$$= \int_0^1 (4t + 20t + 5t + 10)\,dt = \ldots = 24.5$$

$$I_2 = \int_0^1 \left\{ 4 \cdot \underbrace{x'(t)}_{=0} dt + (5-5t) \cdot \underbrace{y'(t)}_{=0} dt + 3\,\underbrace{z'(t)}_{=0} dt \right\}$$

$$= \int_0^1 (-15)\,dt = -15$$

Thus: $\quad I = I_1 + I_2 = \underline{\underline{9.5}}$