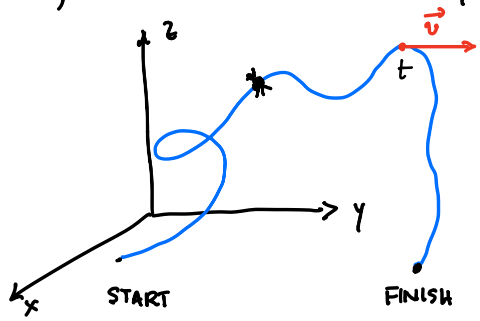
13.1 & 13.2 VECTOR FUNCTIONS

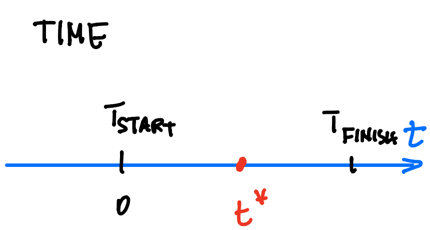
Example: Velocity as a function of time: a bug is flying in 3D, what is its velocity at time t ?

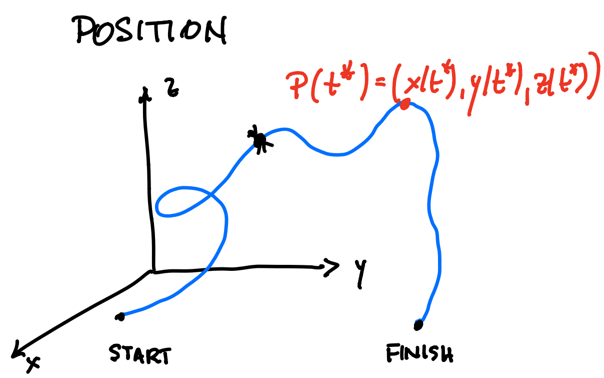


$\vec{v}$ = velocity at time t (has 3 components).

For example: $\vec{v}(t) = \langle \cos t, \sin t, t \rangle$

Example: Position in $\mathbb{R}^3$ as a function of time: a bug is flying in 3D, what is its position as a function of time ?



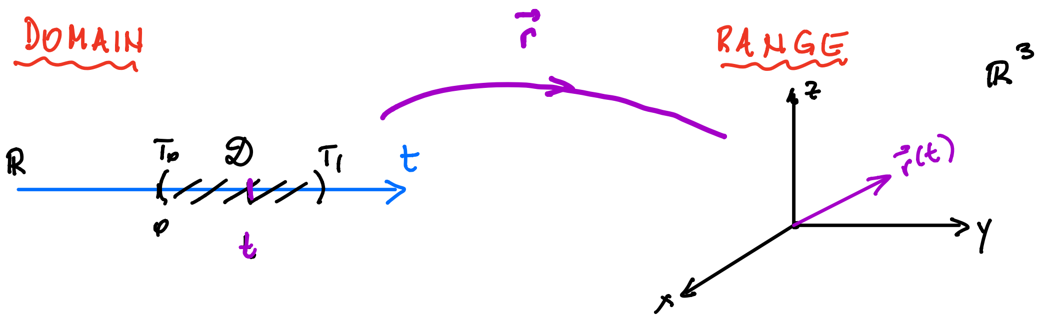


To every $t \in [T_{START}, T_{FINISH}]$ we assign a vector (POSITION VECTOR) of point $P(t)$, $\vec{r}(t) = \langle x(t), y(t), z(t) \rangle$. For example $\vec{r}(t) = \langle \sin(t), -\cos(t), \frac{1}{2}t^2 \rangle$. This defines a vector function:

$$t \longrightarrow \langle \sin(t), -\cos t, \frac{t^2}{2} \rangle$$

IN GENERAL: A vector function of one variable is a function whose domain is a set of real numbers, and whose RANGE is a set of VECTORS:

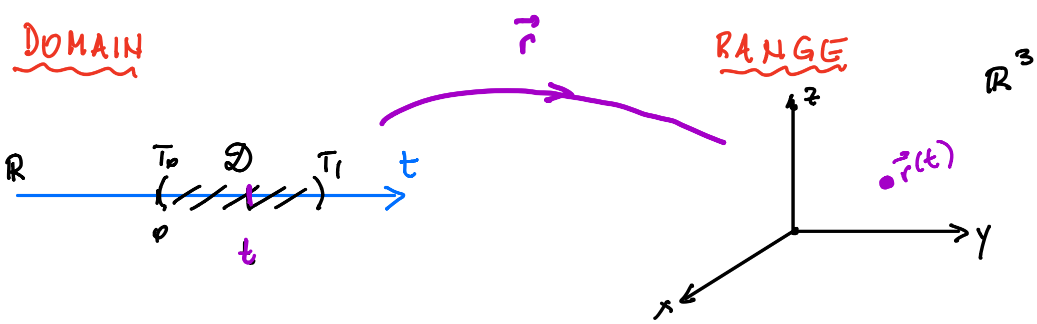
$$\boxed{\vec{r}(t) = \langle f(t), g(t), h(t) \rangle}$$



GEOMETRIC INTERPRETATION / DEPICTION

Instead of plotting vectors in $\mathbb{R}^3$, we can think of $\vec{r}(t) = \langle f(t), g(t), h(t) \rangle$ as a POSITION VECTOR of a point in $\mathbb{R}^3$ with coordinates

$$P(f(t), g(t), h(t))$$



Thus, when t runs througt the set of real numbers ($\mathcal{D}$omain) points $P(f(t), g(t), h(t))$ trace A CURVE IN $\mathbb{R}^3$.

EXAMPLE: Describe the curve defined by the vector function

$$\vec{r}(t) = t\,\vec{i} + (1+t)\,\vec{j} + 3t\,\vec{k}$$

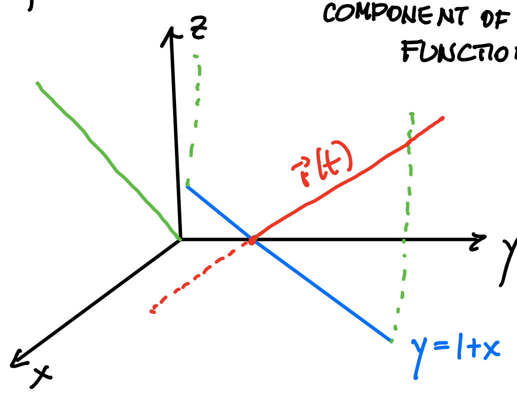
Answer: $x(t) = t, \quad y(t) = 1+t, \quad z(t) = 3t$ ← PARAMETRIC EQUATION

OR

$f(t) = t, \quad g(t) = 1+t, \quad h(t) = 3t$ ← THE CORRESPONDING COMPONENT OF VECTOR FUNCTION

$$\left.\begin{array}{l} x = t \\ y = 1+t \end{array}\right\} \Rightarrow \quad y = 1+x$$

$$z = 3t \quad \Rightarrow \quad z = 3x$$



The curve is a straight line in $\mathbb{R}^3$ with vector equation:

$$\vec{r}(t) = \vec{r_0} + t\vec{v} \qquad \text{where } \vec{r_0} = \langle 0, 1, 0\rangle, \ \vec{v} = \langle 1, 1, 3\rangle$$

passing through the point $P_0(0,1,0)$ and parallel to $\vec{v} = \langle 1,1,3\rangle$.

LIMITS AND CONTINUITY

- THE LIMIT OF A VECTOR FUNCTION $\vec{r}(t)$ IS DEFINED BY TAKING THE LIMITS OF ITS COMPONENT FUNCTIONS:

$$\boxed{\lim_{t\to a} \vec{r}(t) = \left\langle \lim_{t\to a} f(t), \ \lim_{t\to a} g(t), \ \lim_{t\to a} h(t) \right\rangle}$$

PROVIDED THAT THE LIMITS OF COMPONENT FUNCTIONS EXIST.

- A VECTOR FUNCTION $\vec{r}(t)$ is CONTINUOUS AT $t=a$ IF

$$\boxed{\lim_{t\to a} \vec{r}(t) = \vec{r}(a)}$$

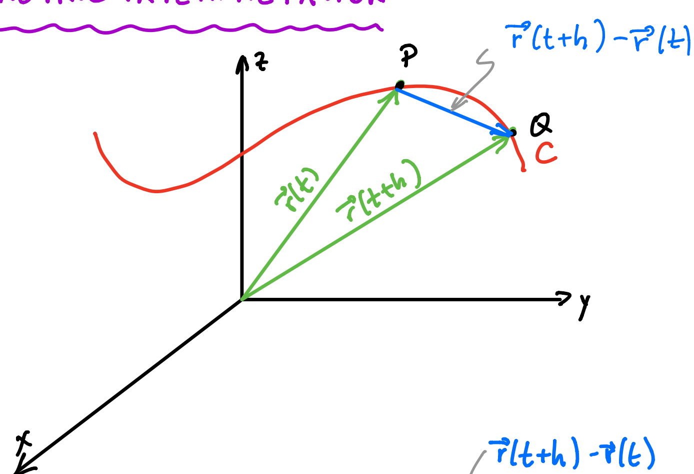
THUS, $\vec{r}$ is continuous at $t=a$ IF AND ONLY IF f, g, and h are continuous at $t=a$.

13.2 DERIVATIVES AND INTEGRALS

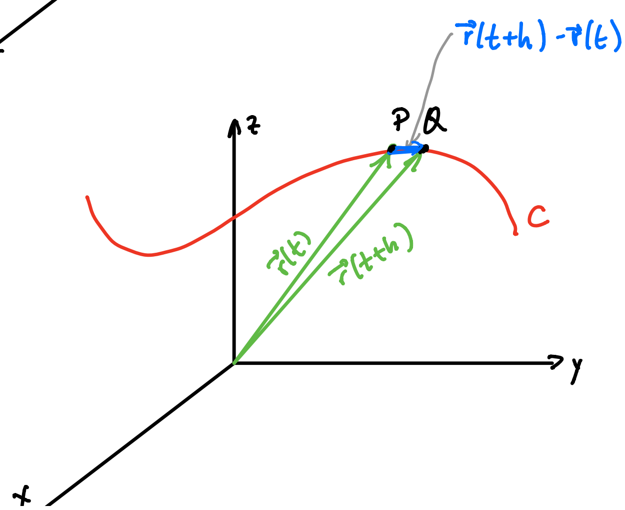
DERIVATIVE:

$$\boxed{\frac{d\vec{r}}{dt} = \vec{r}\,'(t) = \lim_{h\to 0} \frac{\vec{r}(t+h) - \vec{r}(t)}{h}}$$

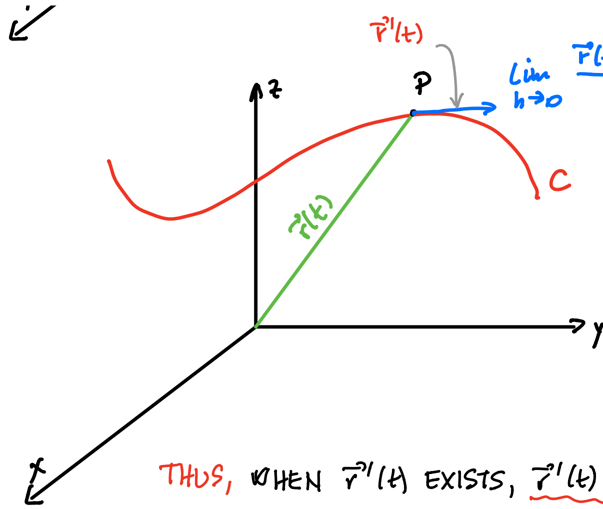
GEOMETRIC INTERPRETATION:



$\vec{r}(t+h)-\vec{r}(t)$ = DIRECTION VECTOR OF THE LINE THROUGH P AND Q



$\vec{r}(t+h)-\vec{r}(t)$ = DIRECTION VECTOR OF THE LINE THROUGH P AND Q FOR SMALL h



$$\lim_{h\to 0} \frac{\vec{r}(t+h)-\vec{r}(t)}{h} = \vec{r}\,'(t)$$

IN THE LIMIT WE OBTAIN THE DIRECTION VECTOR OF THE TANGENT LINE TO CURVE $\vec{r}(t)$.

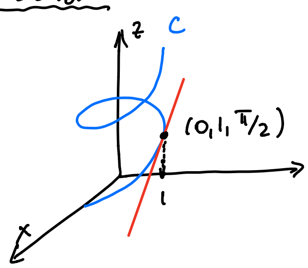
THUS, WHEN $\vec{r}\,'(t)$ EXISTS, $\vec{r}\,'(t)$ IS THE TANGENT VECTOR TO $\vec{r}(t)$.

THEOREM: If $\vec{r}(t) = \langle f(t), g(t), h(t)\rangle = f(t)\,\vec{i} + g(t)\,\vec{j} + h(t)\,\vec{k}$ and f, g, h are differentiable functions of t, then

$$\boxed{\vec{r}\,'(t) = \langle f'(t), g'(t), h'(t)\rangle = f'(t)\,\vec{i} + g'(t)\,\vec{j} + h'(t)\,\vec{k}}$$

EXAMPLE: Helix $\vec{r}(t) = \langle \cos t, \sin t, t\rangle$, find parametric equations of the tangent line at $(0, 1, \pi/2)$.

Solution:



Vector equation for tangent:

$$\vec{r} = \vec{r}_0 + t\vec{v} \qquad (\star)$$

where $\vec{r}_0 = \langle 0, 1, \frac{\pi}{2}\rangle$ and

$\vec{v} = \langle -1, 0, 1\rangle$.

From $(\star)$ we get:

$$\left.\begin{aligned} x &= 0 + t(-1) = -t \\ y &= 1 + t\cdot 0 = 1 \\ z &= \pi/2 + t\cdot 1 = \pi/2 + t \end{aligned}\right\} \text{PARAMETRIC EQUATIONS}$$

Slope of the tangent line:

$\vec{r}\,'(t) = \langle -\sin t, \cos t, 1\rangle = \vec{v}$

DIRECTION VECTOR

Direction vector at $(0, 1, \frac{\pi}{2})$?

What is the value of t corresponding to $(0, 1, \frac{\pi}{2})$?

$\vec{r}(t) = \langle \cos t, \sin t, t\rangle$

For which t do we get the direction vector $\vec{r}_0 = \langle 0, 1, \pi/2\rangle$?

Answer: $t = \frac{\pi}{2}$

DIRECTION VECTOR AT $t = \frac{\pi}{2}$:

$$\vec{v} = \langle -\sin(\tfrac{\pi}{2}), \cos\tfrac{\pi}{2}, 1\rangle$$
$$= \langle -1, 0, 1\rangle$$

Parametric equations of the tangent line at $(0, 1, \frac{\pi}{2})$ are:

$$\left.\begin{aligned} x &= -t \\ y &= 1 \\ z &= \tfrac{\pi}{2} + t \end{aligned}\right\} \quad t \in \mathbb{R}$$

SECOND-ORDER DERIVATIVE

$$\boxed{\frac{d^2\vec{r}}{dt^2}(t) = \vec{r}\,''(t) = \langle f''(t), g''(t), h''(t) \rangle}$$

DIFFERENTIATION RULES

Suppose $\vec{u}$ and $\vec{v}$ are differentiable functions, c is a scalar, and f is a real valued function. Then:

1. $\frac{d}{dt}\left[\vec{u}(t) + \vec{v}(t)\right] = \frac{d\vec{u}(t)}{dt} + \frac{d\vec{v}(t)}{dt} = \vec{u}\,'(t) + \vec{v}\,'(t)$

2. $\frac{d}{dt}\left[c\,\vec{u}(t)\right] = c\,\frac{d\vec{u}}{dt}$

3. $\frac{d}{dt}\left[f(t)\,\vec{u}(t)\right] = f'(t)\,\vec{u}(t) + f(t)\,\vec{u}\,'(t)$ PRODUCT RULE

4. $\frac{d}{dt}\left[\vec{u}(t) \cdot \vec{v}(t)\right] = \vec{u}\,'(t) \cdot \vec{v}(t) + \vec{u}(t) \cdot \vec{v}\,'(t)$ PRODUCT RULE

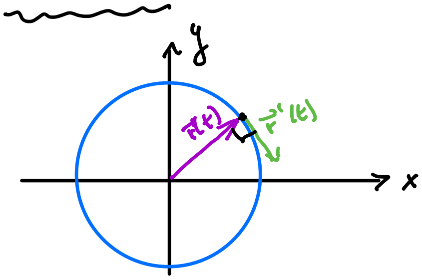
   DOT PRODUCT

5. $\frac{d}{dt}\left[\vec{u}(t) \times \vec{v}(t)\right] = \vec{u}\,'(t) \times \vec{v}(t) + \vec{u}(t) \times \vec{v}\,'(t)$ PRODUCT RULE

6. $\frac{d}{dt}\left[\vec{u}(f(t))\right] = \vec{u}\,'(f(t))\, f'(t)$ CHAIN RULE

EXAMPLE: Show that if $|\vec{r}(t)| = c$ (constant), then $\vec{r}\,'(t)$ is ORTHOGONAL to $\vec{r}(t)$.

Solution:



Want to show that if $|\vec{r}(t)| = c$, then $\vec{r}\,'(t) \cdot \vec{r}(t) = 0$.

What is $|\vec{r}(t)|$?

$$|\vec{r}(t)| = \sqrt{\vec{r}(t) \cdot \vec{r}(t)} = c$$

Thus:

$$|\vec{r}(t)|^2 = \vec{r}(t) \cdot \vec{r}(t) = c^2$$

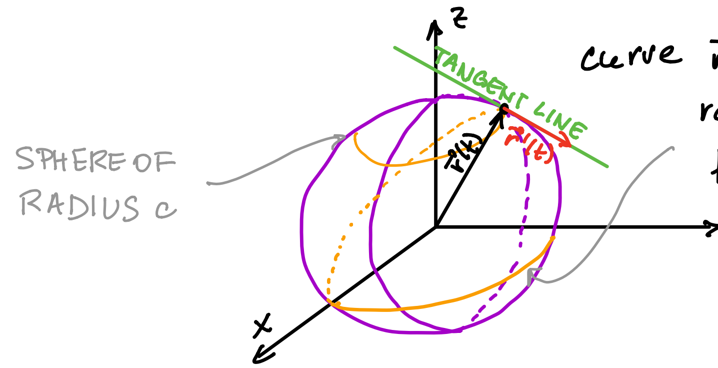
How to see what $\vec{r}\,'(t)$ satisfies from knowing that $\vec{r}(t) \cdot \vec{r}(t) = c^2$?

Differentiate $\quad \vec{r}(t) \cdot \vec{r}(t) = c^2 \quad \Big/ \ \frac{d}{dt}$ (both sides)

$$\vec{r}\,'(t) \cdot \vec{r}(t) + \vec{r}(t) \cdot \vec{r}\,'(t) = 0$$

Thus: $\quad 2\,\vec{r}\,'(t) \cdot \vec{r}(t) = 0$, and so $\vec{r}\,'(t) \cdot \vec{r}(t) = 0$!

GEOMETRIC INTERPRETATION IN $\mathbb{R}^3$:



curve $\vec{r}(t)$ on the sphere of radius c has the property that $|\vec{r}(t)| = c$

INTEGRALS

$$\int_a^b \vec{r}(t)\,dt = \left[\int_a^b f(t)\,dt\right]\vec{i} + \left[\int_a^b g(t)\,dt\right]\vec{j} + \left[\int_a^b h(t)\,dt\right]\vec{k}$$

EXAMPLE: If $\vec{r}(t) = 2\cos t\,\vec{i} + \sin t\,\vec{j} + 2t\,\vec{k}$, calculate $\int \vec{r}(t)\,dt$.

Solution:

$$\int \vec{r}(t)\,dt = \left[\int 2\cos t\,dt\right]\vec{i} + \left[\int \sin t\,dt\right]\vec{j} + \left[\int 2t\,dt\right]\vec{k}$$

$$= \left[2\sin t + C_1\right]\vec{i} - \left[\cos t + C_2\right]\vec{j} + \left[t^2 + C_3\right]\vec{k}$$

$$= 2\sin t\,\vec{i} - \cos t\,\vec{j} + t^2\vec{k} + \vec{C}$$

VECTOR CONSTANT

☺

The DEFINITE INTEGRAL:

$$\int_0^{\pi/2} \vec{r}(t)\,dt = \left[2\sin t\right]_0^{\pi/2}\vec{i} - \left[\cos t\right]_0^{\pi/2}\vec{j} + \left[t^2\right]_0^{\pi/2}\vec{k}$$

$$= 2\vec{i} + \vec{j} + \frac{\pi^2}{4}\vec{k}$$

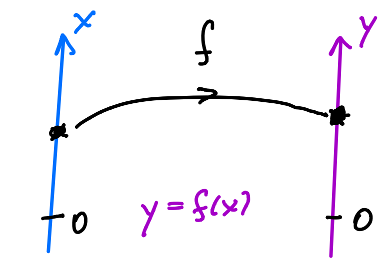
MOVING ON!

14.1. FUNCTIONS OF SEVERAL VARIABLES

So far, we have studied:

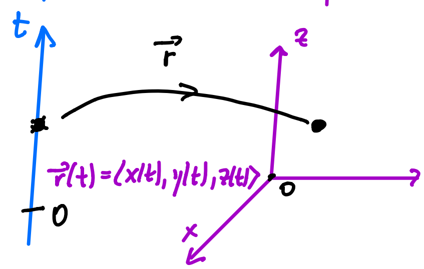
(1.) $f : \mathbb{R} \longrightarrow \mathbb{R}$

Real-valued functions of ONE variable



(2.) $\vec{r} : \mathbb{R} \longrightarrow \mathbb{R}^n$, $n = 2, 3, \ldots$

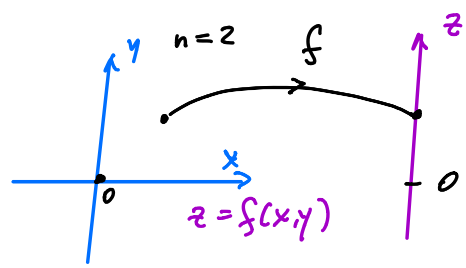
Vector-valued functions of ONE variable

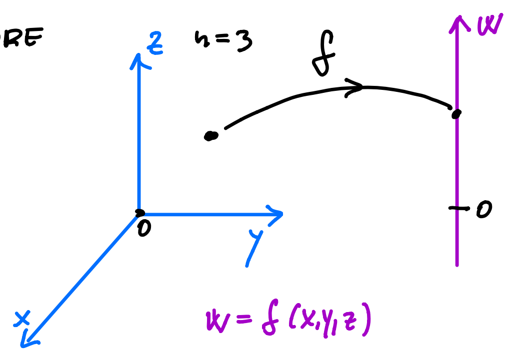


We will learn about:

(3.) $f : \mathbb{R}^n \longrightarrow \mathbb{R}$, $n = 2, 3, \ldots$

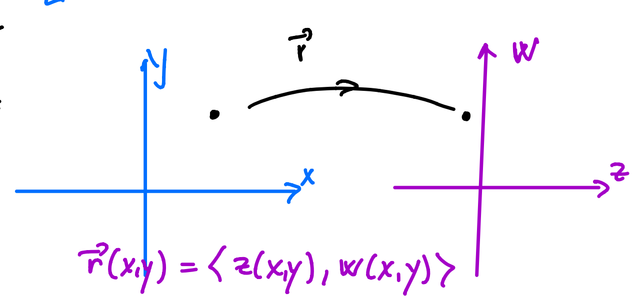
Real-valued functions of MORE THAN ONE VARIABLE





(4) $\vec{r} : \mathbb{R}^n \longrightarrow \mathbb{R}^m$, $n, m = 2, 3, \ldots$

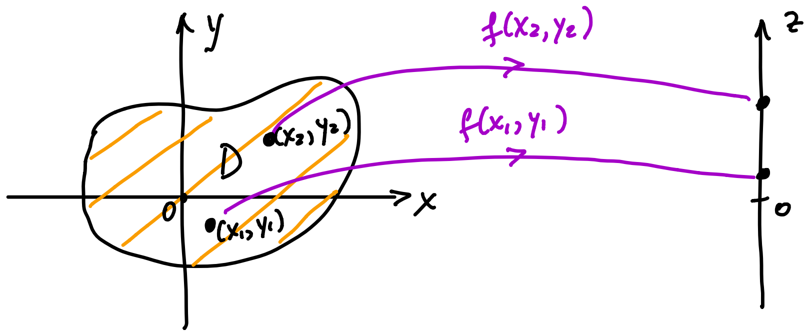
Vector-valued functions of MORE THAN ONE VARIABLE



DEFINITION: A FUNCTION OF TWO VARIABLES is a rule that assigns to each ordered pair of real numbers (x,y) in a set D, a UNIQUE real number, denoted by $f(x,y)$.

The set D is THE DOMAIN OF f, and its RANGE is the set of values that f takes on, denoted by

$$R = \{ f(x,y) \mid (x,y) \in D \}.$$



We often write $\boxed{z = f(x,y)}$.

(x,y) ARE CALLED THE INDEPENDENT VARIABLES

z IS CALLED THE DEPENDENT VARIABLE

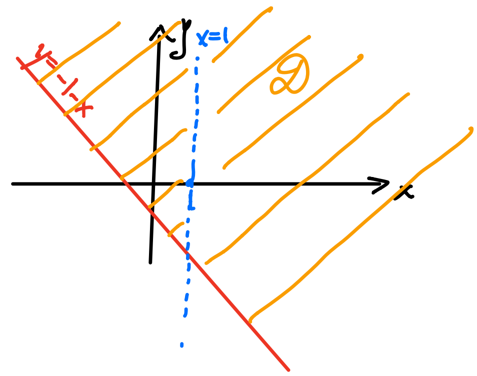
EXAMPLE: Evaluate $f(3,2)$ and sketch the domain if

$$f(x,y) = \frac{\sqrt{x+y+1}}{x-1}.$$

Solution: $f(3,2) = \dfrac{\sqrt{3+2+1}}{3-1} = \dfrac{\sqrt{6}}{2}$ //

Domain: $\left\{ \begin{array}{ll} ① & x+y+1 \geq 0 \Rightarrow y \geq -1-x \\ ② & x \neq 1 \end{array} \right\} \Rightarrow \mathcal{D} = \{ (x,y) \mid x+y+1 \geq 0 \text{ and } x \neq 1 \}$

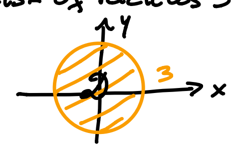
Sketch of the domain $\mathcal{D}$:



EXAMPLE: Domain and range of $g(x,y)=\sqrt{9-x^2-y^2}$?

Solution: Domain: $9-x^2-y^2 \geq 0 \Rightarrow x^2+y^2 \leq 9$ (disk of radius 3)

$$\mathcal{D}=\{(x,y)\in\mathbb{R}^2 \mid x^2+y^2\leq 9\}$$



Range: The definition of the range is:

$$\mathcal{R}=\{f(x,y) \mid (x,y)\in\mathcal{D}\} = \{z \mid z=\sqrt{9-x^2-y^2},\ (x,y)\in\mathcal{D}\}$$

What exactly is this set? Specify!

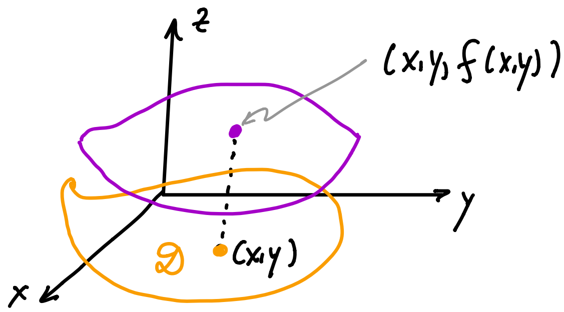
1. Since z is the positive square root $\Rightarrow z\geq 0$
2. Is there a restriction on the max value of z? Well, since x^2+y^2 is always greater than or equal to zero, we know that $-(x^2+y^2)\leq 0$. Thus, $9-(x^2+y^2)\leq 9$, and so

$$\sqrt{9-(x^2+y^2)} \leq \sqrt{9}=3$$

Thus, the range of z is: $\mathcal{R}=\{z\in\mathbb{R} \mid 0\leq z\leq 3\}$ //

GRAPHS

VISUALIZING A FUNCTION OF TWO VARIABLES



Definition: If f is a function of TWO VARIABLES with domain $\mathcal{D}$, then the GRAPH of f is the set of all points $(x,y,z) \in \mathbb{R}^3$ such that $z = f(x,y)$ and $(x,y) \in \mathcal{D}$.

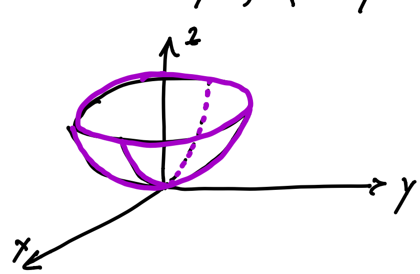
Thus, the graph of a function of two variables is a SURFACE in $\mathbb{R}^3$ with equation: $z = f(x,y)$.

EXAMPLE: Find domain, range, and sketch the graph of

$$f(x,y) = x^2 + y^2.$$

Solution: $z = x^2 + y^2$: $\mathcal{D} = \mathbb{R}^2$, $\mathcal{R} = \{ z \in \mathbb{R} \mid z \geq 0 \} = [0, \infty)$

Graph:
- horizontal traces at z^* are circles $x^2 + y^2 = z^*$
- vertical traces are parabolas; for example for $x = 0 \Rightarrow z = y^2$, for $y = 0 \Rightarrow z = x^2$.



PARABOLOID

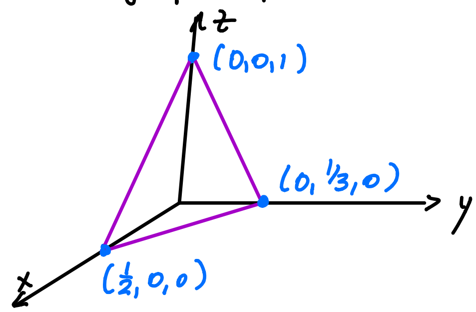
EXAMPLE: Domain, range, and graph of $g(x,y) = 1-2x-3y$.

Solution: $g(x,y)$ is a LINEAR FUNCTION

GRAPH OF A LINEAR FUNCTION IS A PLANE

$z = 1-2x-3y$ or $\boxed{2x+3y+z-1=0}$

To plot the graph, find the x, y, z intercepts:



$\mathcal{D} = \mathbb{R}$

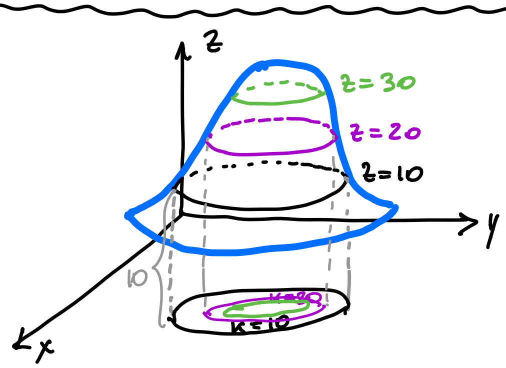
$\mathcal{R} = \mathbb{R}$.

LEVEL CURVES

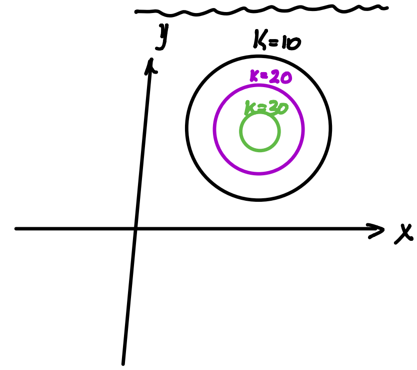
Definition: The LEVEL CURVES of a function f of two variables are the curves defined by equations $\boxed{f(x,y)=k}$, where k is a constant.

(curves at constant height k, but plotted in xy-plane!)

VISUALIZATION OF LEVEL CURVES:



LEVEL CURVES



EXAMPLE: Sketch the Level curves of $f(x,y) = 6-3x-2y$ for the values of $k = -6, 0, 6, 12$.

Solution: The Level curves are $6-3x-2y = k$ or $\boxed{3x+2y+(k-6)=0}$

So, for $k=-6$: $3x+2y-12=0 \Rightarrow y = -\frac{3}{2}x+6$

$k=0$: $3x+2y-6=0 \Rightarrow y = -\frac{3}{2}x+3$

$k=6$: $3x+2y=0 \Rightarrow y = -\frac{3}{2}x$

$k=12$: $3x+2y+6=0 \Rightarrow y = -\frac{3}{2}x-3$