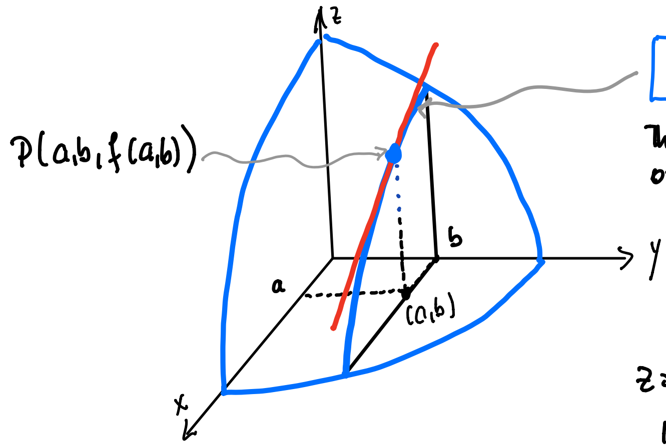
14.3. PARTIAL DERIVATIVES

Let $f(x,y)$ be a function of 2 variables. Suppose that ONLY x VARY while keeping y FIXED, say $y=b$ (b = constant).



$\boxed{z = f(x,b)}$ b = fixed

This is a function of only one variable, x.

We know how to differentiate a function of only one variable:

$z = f(x,a) = g(x)$

$g'(x)$ evaluated at $x=a$:

$g'(a)$ is the slope of the tangent line to the curve $z = g(x)$ in $\mathbb{R}^3$.

We denote $$g'(x)\Big|_{x=a} = g'(a) = f_x(x,b)\Big|_{x=a} = f_x(a,b)$$

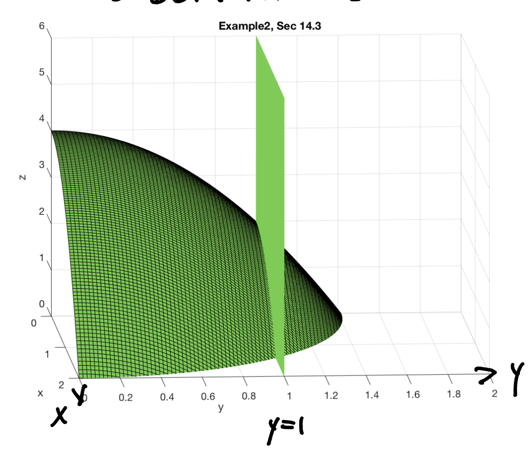
Notation: $f_x(a,b)$ or $\frac{\partial f}{\partial x}(a,b)$

Recall: $\frac{dg}{dx}$ is used for ORDINARY DERIVATIVES

$\frac{\partial f}{\partial x}$ is used for PARTIAL DERIVATIVES

THUS:

$$\boxed{f_x(a,b) = \lim_{h \to 0} \frac{f(a+h,b) - f(a,b)}{h}}$$



SIMILARLY:

The partial derivative of f with respect to y is obtained by KEEPING x FIXED $(x=a)$ and differentiating $G(y) = f(a,y)$ with respect to y:

$$\boxed{f_y(a,b) = \lim_{h\to 0} \frac{f(a,b+h) - f(a,b)}{h}}$$

Example: $f(x,y) = 4 - 2x^2 - y^2$. Find $f_x(1,1)$.

Solution: Keep $y=1$ fixed: $f(x,1) = 4 - 2x^2 - 1 = 3 - 2x^2$

So: $f_x(1,1) = \frac{d}{dx}(3-2x^2)\Big|_{x=1} = -4x^2\Big|_{x=1} = -4$

PARTIAL DERIVATIVES AS FUNCTIONS

$$f_x(x,y) = \lim_{h\to 0} \frac{f(x+h,y) - f(x,y)}{h}$$

$$f_y(x,y) = \lim_{h\to 0} \frac{f(x,y+h) - f(x,y)}{h}$$

TO FIND $f_x(x,y)$ WE REGARD y AS A CONSTANT

TO FIND $f_y(x,y)$ WE REGARD x AS A CONSTANT

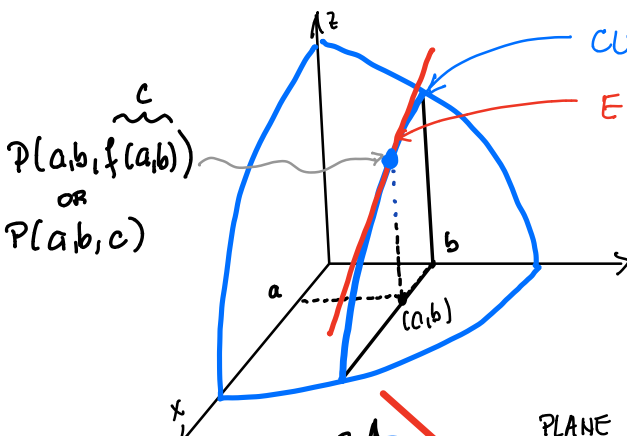
Example: $f(x,y) = x^3 + 3x^2y - 2xy^2 + xy - 3$. $f_x(x,y) = ?$ $f_y(x,y) = ?$

Solution: $f_x(x,y) = 3x^2 + 6xy - 2y^2 + y$

$f_y(x,y) = 3x^2 - 4xy + x$

INTERPRETATION OF PARTIAL DERIVATIVES

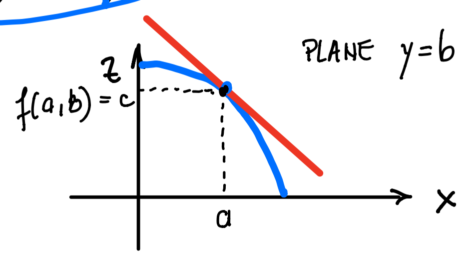
- Let $z = f(x,y)$
- Graph the surface that f represents in $\mathbb{R}^3$
- Consider a point $P(a,b,f(a,b)) = P(a,b,c)$ on the surface
- THE MEANING OF $f_x(a,b)$ is THE SLOPE OF THE TANGENT LINE TO THE CURVE C_1 OBTAINED BY FIXING $y=b$, AT THE POINT $P(a,b,f(a,b))$.



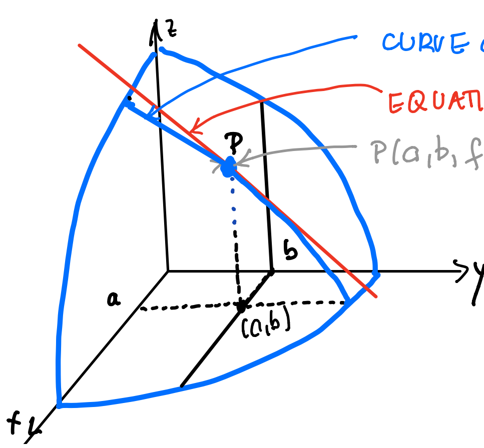
CURVE C_1 OBTAINED BY FIXING $y=b$

EQUATION OF THE TANGENT LINE

$$\boxed{z - c = f_x(a,b)(x-a)}$$



SIMILARLY:



CURVE C_2 OBTAINED BY FIXING $x=a$ ON $z=f(x,y)$

EQUATION OF THE TANGENT LINE AT $P(a,b,c)$:

$P(a,b,f(a,b)) = P(a,b,c)$

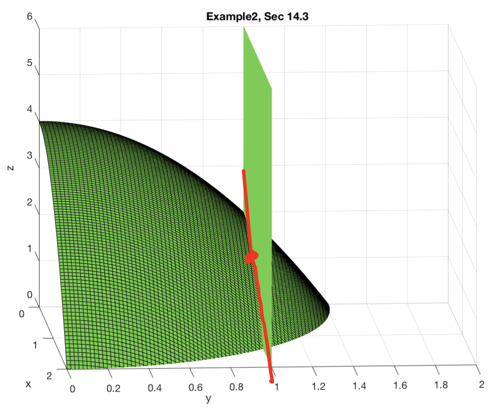
$$\boxed{z - c = f_y(a,b)(y-b)}$$

EXAMPLE: Find the slopes of the tangent lines $f_x(1,1)$ & $f_y(1,1)$ for $f(x,y) = 4 - x^2 - 2y^2$.

Solution:

$f_x = -2x \qquad f_x(1,1) = -2$

$f_y = -4y \qquad f_y(1,1) = -4$



PARTIAL DERIVATIVES CAN ALSO BE INTERPRETED AS RATES OF CHANGE

EXAMPLE 3, pg. 916: We define the body mass index of a person as

$$B(m,h) = \frac{m}{h^2}$$

where m = weight (in kg) and h = height (in meters).

Calculate the partial derivatives of B for a young man with $m = 64$ kg and $h = 1.68$ m.

Solution: $\frac{\partial B}{\partial m} = \frac{1}{h^2} \qquad \frac{\partial B}{\partial h} = -2\frac{m}{h^3}$

$\frac{\partial B}{\partial m}(64, 1.68) = \frac{1}{(1.68)^2} \approx 0.35 \Rightarrow$ The rate of change at which his Body Mass Index increases with respect to his weight. So, if his weight increases by one kilogram, his BMI will increase by about 0.35.

$\frac{\partial B}{\partial h}(64, 1.68) = -\frac{64}{(1.68)^3} \approx -27$

COMPOSITE FUNCTIONS:

Example: $f(x,y) = \sin\left(\frac{x}{1+y}\right)$, $f_x = ?$, $f_y = ?$

Solution:
$$f_x(x,y) = \cos\left(\frac{x}{1+y}\right) \cdot \frac{\partial}{\partial x}\left(\frac{x}{1+y}\right)$$
$$= \cos\left(\frac{x}{1+y}\right) \cdot \frac{1}{1+y}$$
$$f_y(x,y) = \cos\left(\frac{x}{1+y}\right) \cdot \frac{\partial}{\partial y}\left(\frac{x}{1+y}\right)$$
$$= \cos\left(\frac{x}{1+y}\right) \cdot \left(-\frac{x}{(1+y)^2}\right)$$

IMPLICITLY DEFINED FUNCTIONS

$$\boxed{F(x,y,z)=0}$$

Insted of $z=f(x,y)$, we have z defined as a function of x and y through an equation $F(x,y,z)=0$.

Example: Find $\frac{\partial z}{\partial x}$, $\frac{\partial z}{\partial y}$ if z is defined implicity as:

$$x^3+y^3+z^3+6xyz=1$$

Solution: $\frac{\partial z}{\partial x} = ?$

STEP 1: Implicit differentiation with respect to x (y is assumed constant):

Differentiate with respect to x the following:

$$x^3+y^3+z^3(x,y)+6xy\,z(x,y)=1 \quad \Big/ \frac{\partial}{\partial x} \text{ both sides}$$
$$3x^2+0+3z^2\frac{\partial z}{\partial x}+6yz+6xy\frac{\partial z}{\partial x}=0$$

OR:
$$3x^2+6yz+(3z^2+6xy)\frac{\partial z}{\partial x}=0 \qquad (*)$$

STEP 2: Solve for $\frac{\partial z}{\partial x}$:

From $(*)$ we have: $\frac{\partial z}{\partial x} = -\frac{3x^2+6yz}{3z^2+6xy} = -\frac{\cancel{3}(x^2+2yz)}{\cancel{3}(z^2+2xy)}$

$$= -\frac{x^2+2yz}{z^2+2xy}$$

Similarly we calculate $\frac{\partial z}{\partial y}$:

STEP 1 $\quad x^3+y^3+z^3(x,y)+6xyz(x,y)=1 \quad \Big/ \frac{\partial}{\partial y}$

$$0+3y^2+3z^2\cdot\frac{\partial z}{\partial y}+6xz+6xy\frac{\partial z}{\partial y}=0$$

$$3y^2+6xz+(3z^2+6xy)\frac{\partial z}{\partial y}=0$$

STEP 2: $\quad \frac{\partial z}{\partial y} = -\frac{3y^2+6xz}{3z^2+6xy} = -\frac{y^2+2xz}{z^2+2xy}$

HIGHER DERIVATIVES

$$f_{xx} = (f_x)_x = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial x^2} = \frac{\partial^2 z}{\partial x^2}$$

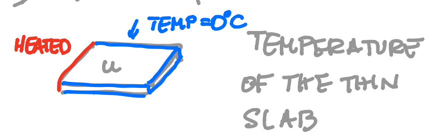
$$f_{xy} = (f_x)_y = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial y \partial x} = \frac{\partial^2 z}{\partial y\, \partial x}$$

$$f_{yx} = (f_y)_x = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 z}{\partial x\, \partial y}$$

$$f_{yy} = (f_y)_y = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial y^2} = \frac{\partial^2 z}{\partial y^2}$$

EXAMPLE: Check whether $u = x^3 + 3xy^2$ is a solution to the Laplace's equation: $\boxed{u_{xx} + u_{yy} = 0}$

(This is a PARTIAL DIFFERENTIAL EQUATION. It desribes e.g. heat transfer, in which case u = TEMPERATURE; fluid flow, in which case u = FLUID VELOCITY, e.t.c.)



Solution: $u = x^3 + 3xy^2$

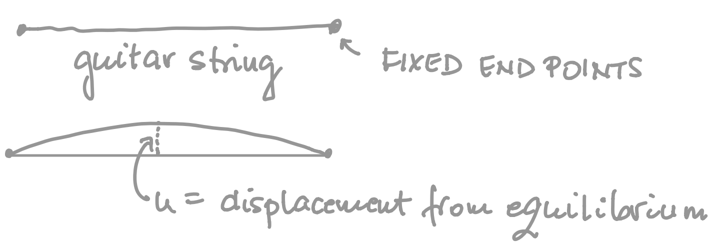
$$u_x = 3x^2 + 3y^2 \Rightarrow u_{xx} = 6x + 0$$
$$u_y = 0 + 6xy \Rightarrow u_{yy} = 6x$$

$u_{xx} + u_{yy} = 12x \not\equiv 0$

This is not equal to 0 for ALL x!

EXAMPLE: (PROBLEM 79, pg 925) Show that $u(x,y) = f(x+at) + g(x-at)$ solves THE WAVE EQUATION $u_{tt} = a^2 u_{xx}$.

(This PDE (partial differential equation) describes wave propagation. E.g.



)

Solution: $u(x,t) = f(\underbrace{x+at}_{\xi}) + g(\underbrace{x-at}_{\eta})$

$$u_t = f' \cdot \frac{\partial}{\partial t}(x+at) + g' \cdot \frac{\partial}{\partial t}(x-at)$$

$$= f' \cdot a \quad + \quad g' \cdot (-a)$$

$$u_{tt} = \frac{\partial}{\partial t}(u_t) = \frac{\partial}{\partial t}\left(f'(x+at) \cdot a - g'(x-at) \cdot a \right)$$

$$= f'' \cdot \frac{\partial}{\partial t}(x+at) \cdot a - g''(x-at) \frac{\partial}{\partial t}(x-at) \cdot a$$

$$= f'' \cdot a \cdot a - g'' \cdot (-a) \cdot a$$

$$= f'' a^2 + g'' a^2$$

$$u_x = f' \cdot \frac{\partial}{\partial x}(x+at) + g' \cdot \frac{\partial}{\partial x}(x-at)$$

$$= f' \cdot 1 \quad + g' \cdot 1$$

$$= f' + g'$$

$$u_{xx} = \frac{\partial}{\partial x}\left(f'(x+at) + g'(x-at) \right) = f'' \cdot \frac{\partial}{\partial x}(x+at) + g'' \cdot \frac{\partial}{\partial x}(x-at)$$

$$= f'' + g''$$

$$u_{tt} - a^2 u_{xx} = f'' a^2 + g'' a^2 - a^2(f'' + g'') = 0$$

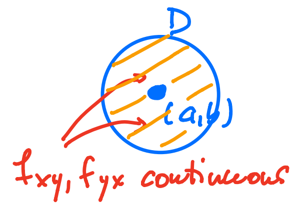
$\Rightarrow$ Indeed, $u(x,y) = f(x+at) + g(x-at)$ solves

$$u_{tt} = a^2 u_{xx}$$

CLAIRAUT'S THEOREM

Suppose f is defined on a disk D that contains (a,b).
If f_{xy} and f_{yx} are both CONTINUOUS on D, then

$$\boxed{f_{xy}(a,b) = f_{yx}(a,b)}$$



EXAMPLE: (Problem 59, pg 925) $u = x^4y^3 - y^4$. Does Clairaut's Theorem hold on $\mathbb{R}^2$, and verify conclusions.

Solution: u is a polynomial $\Rightarrow$ continuous on $\mathbb{R}^2 \Rightarrow$ C. Theorem holds

• verify: $u_{xy} \stackrel{?}{=} u_{yx}$:

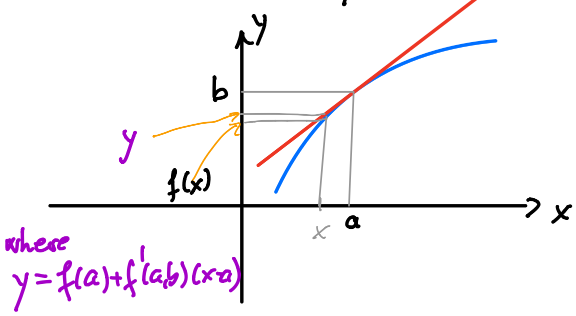
$$u_{xy} = \frac{\partial}{\partial y}(u_x) = \frac{\partial}{\partial y}(4x^3y^3) = 12x^3y^2$$

$$u_{yx} = \frac{\partial}{\partial x}(u_y) = \frac{\partial}{\partial x}(3x^4y^2 - 4y^3) = 12x^3y^2$$

$\Big\} = \checkmark$

14.4. TANGENT PLANES AND LINEAR APPROXIMATIONS

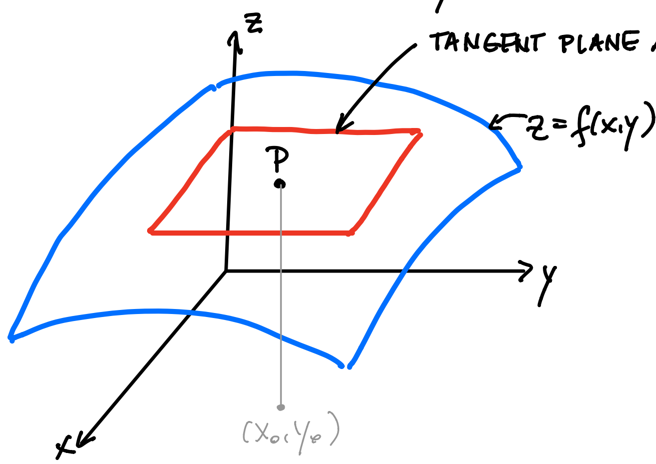
Recall. For $y=f(x)$:



If we zoom in closer and closer near the point (a,b), the tangent line to $y=f(x)$ at (a,b) becomes indistinguishable from the curve $y=f(x)$. Thus:

$$\boxed{f(x) \approx f(a)+f'(a,b)(x-a)}$$

Similarly: For $z=f(x,y)$:



Tangent plane at P to the surface $z=f(x,y)$ is a good approximation of the surface $z=f(x,y)$ near the point P.

TANGENT PLANES

Tangent plane through $P(x_0,y_0,z_0)$, where $z_0=f(x_0,y_0)$:

$$A(x-x_0)+B(y-y_0)+C(z-z_0)=0$$

What are A, B, and C?

Divide by C $\Rightarrow$ $\frac{A}{C}(x-x_0)+\frac{B}{C}(y-y_0)+(z-z_0)=0$

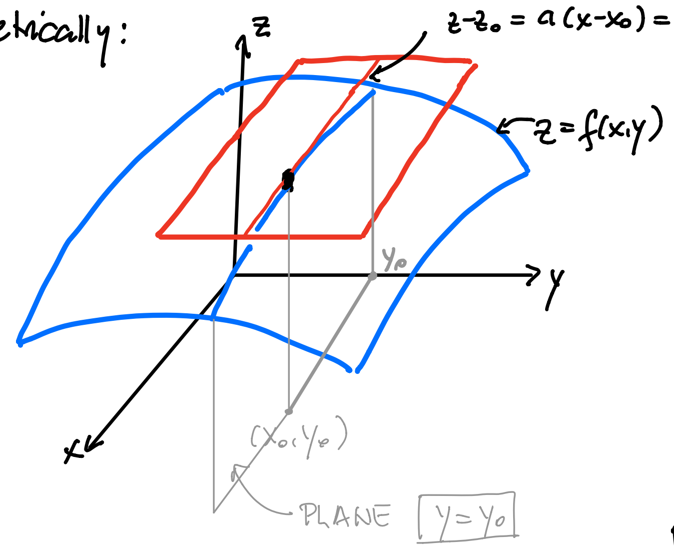
OR

$$(**) \quad z - z_0 = a(x - x_0) + b(y - y_0) \quad \text{where} \quad \begin{cases} a = -\frac{A}{C} \\ b = -\frac{B}{C} \end{cases}$$

Choose $y = y_0$ and let it be fixed. Then we get from $(**)$

$$z - z_0 = a(x - x_0)$$

Geometrically:



If $z - z_0 = a(x-x_0) + b(y-y_0)$ represents the tangent plane, then $z - z_0 = a(x - x_0)$ is the TANGENT LINE to the curve at the intersection of $y = y_0$ and $z = f(x,y)$.

Thus:

$$\boxed{z - z_0 = a(x-x_0) = f_x(x_0, y_0)(x - x_0)}$$

THUS, $\quad a = f_x(x_0, y_0)$.

Choose $x = x_0$: Similarly, we get $b = f_y(x_0, y_0)$.

Conclusion: The tangent plane to $z = f(x,y)$ at $P(x_0, y_0, z_0)$ is given by:

$$\boxed{z - z_0 = f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)}$$

Example: Tangent plane to the elliptic paraboloid $z = 2x^2 + y^2$ at $P(1,1,3)$?

Solution: $\quad z - 3 = (4x)\Big|_{x=1}(x-1) + (2y)\Big|_{y=1}(y-1)$

$$\Rightarrow \quad z-3 = 4(x-1) + 2(y-1)$$

$$\Rightarrow \quad \boxed{z = 4x+2y-3} \quad \text{or} \quad \boxed{4x+2y-z-3=0}$$

LINEAR APPROXIMATIONS

Tangent plane to $z=f(x,y)$ at $(a,b,f(a,b))$ is:

$$z - f(a,b) = f_x(a,b)(x-a) + f_y(a,b)(y-b)$$

OR

$$z = f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b)$$

Near $P_0(a,b,f(a,b))$, the tangent plane is a good approximation of the surface $z=f(x,y)$. Namely:

$$\boxed{f(x,y) \approx f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b)}$$

THE TANGENT PLANE IS CALLED THE LINEAR APPROXIMATION OF $f(x,y)$ AT (a,b) (ALSO KNOWN AS THE TANGENT PLANE APPROXIMATION OF f)