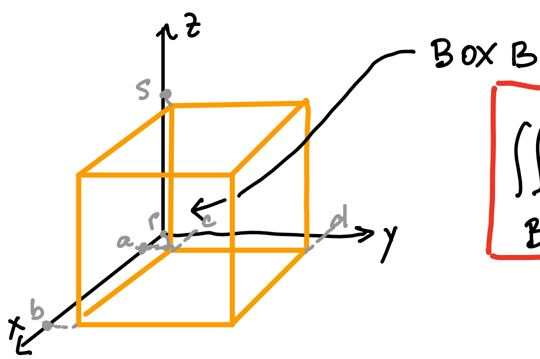
15.6. TRIPLE INTEGRALS

Double integrals: functions of two variables $z = f(x,y)$

Triple integrals: functions of three variables $w = f(x,y,z)$

$$\iiint_R f(x,y,z)\,dx\,dy\,dz = ?$$

① TRIPLE INTEGRALS OVER THE BOX $B = [a,b] \times [c,d] \times [r,s]$



$$\boxed{\iiint_B f(x,y,z)\,dV = \int_r^s \int_c^d \int_a^b f(x,y,z)\,dx\,dy\,dz}$$

FUBINI'S THEOREM $\Rightarrow \iiint_B f(x,y,z)\,dV = \int_r^s \int_a^b \int_c^d f(x,y,z)\,dy\,dx\,dz$

We have 6 possibilities to choose the order of integration!

EXAMPLE: $\iiint\limits_B xyz^2\,dV = ?$, $\quad B = [0,1]\times[-1,2]\times[0,3]$

$$\underset{x}{\uparrow}\qquad\underset{y}{\uparrow}\qquad\underset{z}{\uparrow}$$

Solution: We could choose any order of integration. We could also use the product integral rule.

APPROACH 1: $\iiint\limits_B xyz^2\,dV = \int_0^3\left[\int_{-1}^2\left(\int_0^1 x\,y\,z^2\,dx\right)dy\right]dz$

$$= \int_0^3\int_{-1}^2\left[\frac{x^2}{2}yz^2\right]_{x=0}^{1} dy\,dz = \int_0^3\int_{-1}^2 \frac{1}{2}yz^2\,dy\,dz =$$

$$= \frac{1}{2}\int_0^3 z^2\int_{-1}^2 y\,dy\,dz = \frac{1}{2}\int_0^3 z^2\left[\frac{y^2}{2}\right]_{y=-1}^{y=2} dz = \frac{1}{4}\int_0^3 z^2[4-1]\,dz =$$

$$= \frac{3}{4}\left[\frac{z^3}{3}\right]_{z=0}^{z=3} = \frac{27}{4}$$

APPROACH 2: PRODUCT INTEGRAL

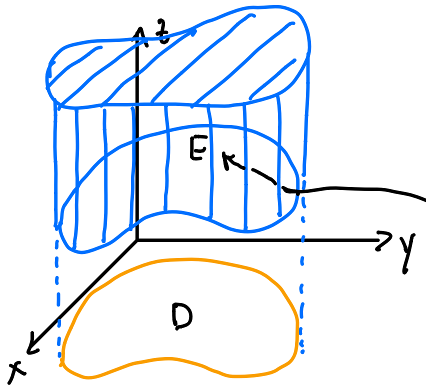
$$\iiint\limits_B xyz^2\,dV = \left(\int_0^3 z^2\,dz\right)\cdot\left(\int_{-1}^2 y\,dy\right)\cdot\left(\int_0^1 x\,dx\right) = \ldots = \frac{27}{4}$$

TRIPLE INTEGRALS OVER A GENERAL REGION

REGION OF TYPE I: Solid region lies between the graphs of two functions:

$z = u_1(x,y)$

$z = u_2(x,y)$ (u_1 and u_2 continuous)

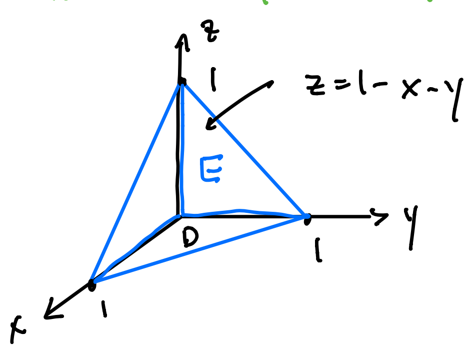


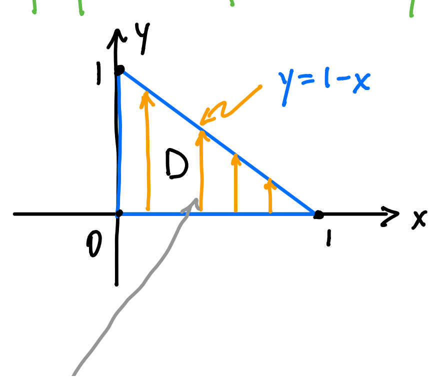
$E = \{ (x,y,z) \mid u_1(x,y) \le z \le u_2(x,y),\ (x,y) \in D \}$

$$\boxed{\iiint_E f(x,y,z)\,dV = \iint_D \left[\int_{u_1(x,y)}^{u_2(x,y)} f(x,y,z)\,dz \right] dA}$$

EXAMPLE: $\iiint_E z\,dV = \,?$, where E = solid tetrahedron bounded by the 4 planes: $x=0$, $y=0$, $z=0$, $x+y+z=1$.

Solution: It is helpful to draw two sketches: one for the solid E and the other for the region D (projection of E onto xy plane).





$$\iiint_E z\,dV = \iint_D \left[\int_0^{1-x-y} z\,dz\right] dA = \iint_D \left[\frac{z^2}{2}\right]_{z=0}^{z=1-x-y} dA =$$

$$= \frac{1}{2}\iint_D (1-x-y)^2\,dA = \frac{1}{2}\int_0^1\left[\int_0^{1-x}(1-x-y)^2\,dy\right]dx = \frac{1}{2}\int_0^1\left[\frac{(1-x-y)^3}{3}\cdot(-1)\right]_{y=0}^{1-x}dx$$

$$= -\frac{1}{6}\int_0^1\left[(1-x-(1-x))^3-(1-x)^3\right]dx = +\frac{1}{6}\int_0^1(1-x)^3\,dx = -\frac{1}{6}\left[\frac{(1-x)^4}{4}\right]_{x=0}^{1} =$$

$$= -\frac{1}{24}\left[0^4-1^4\right] = \frac{1}{24}$$

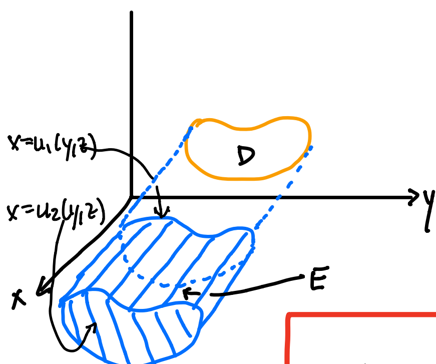
REGION OF TYPE II:

Solid region lies between the graphs of two functions:

$x = u_1(y,z)$

$x = u_2(y,z)$

(u_1 and u_2 continuous)

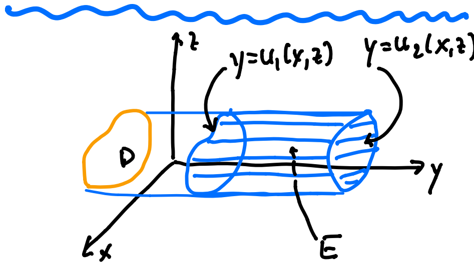


$$E = \{(x,y,z) \mid u_1(y,z) \le x \le u_1(y,z),\ (y,z)\in D\}$$

$$\boxed{\iiint_E f(x,y,z)\,dV = \iint_D \left[\int_{u_1(y,z)}^{u_2(y,z)} f(x,y,z)\,dx\right] dA}$$

REGION OF TYPE III:

Solid region lies between the graphs of two functions:



$y = u_1(x,z)$

$y = u_2(x,z)$

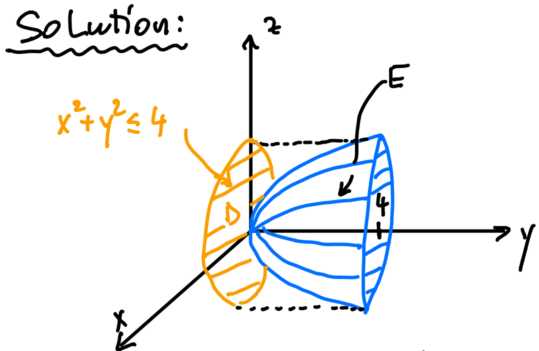
(u_1 and u_2 continuous)

$$E = \{(x,y,z) \mid u_1(x,z) \le y \le u_1(x,z),\ (x,z)\in D\}$$

$$\boxed{\iiint_E f(x,y,z)\,dV = \iint_D \left[\int_{u_1(x,z)}^{u_2(x,z)} f(x,y,z)\,dy\right] dA}$$

EXAMPLE: $\iiint_E \sqrt{x^2+z^2}\,dV = ?$ where E = region bounded by the paraboloid $y = x^2+z^2$ and the plane $y=4$.

Solution:



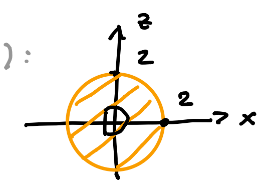
$E = \{(x,y,z) \mid x^2+z^2 \le y \le 4\}$

$D = \{(x,z) \mid x^2+z^2 \le 4\}$

$$\iiint_E \sqrt{x^2+z^2}\,dV = \iint_D \left[\int_{x^2+z^2}^{4} \underbrace{\sqrt{x^2+z^2}}_{\text{independent of } y}\,dy\right] dA = \iint_D \sqrt{x^2+z^2}\int_{x^2+z^2}^{4} dy\;dA =$$

$$= \iint_D \underbrace{\sqrt{x^2+z^2}}\;\underbrace{\left[4-(x^2+z^2)\right]}\;\underbrace{dA} =$$

Polar coordinates in (x,z):
$x = r\cos\theta$
$z = r\sin\theta$
$dx\,dz = r\,dr\,d\theta$



$$= \int_0^{2\pi}\int_0^{2} r\left[4-r^2\right] r\,dr\,d\theta =$$

PRODUCT INTEGRAL
FIXED BOUNDS OF INTEGRATION

$$= \left(\int_0^{2\pi} d\theta\right)\cdot\left(\int_0^{2} r^2\left[4-r^2\right]dr\right)$$

$$= 2\pi\cdot\left[4\frac{r^3}{3}-\frac{r^5}{5}\right]_{r=0}^{2} = 2\pi\left[4\cdot\frac{8}{3}-\frac{32}{5}\right] = \underline{\underline{\frac{128\pi}{15}}}$$

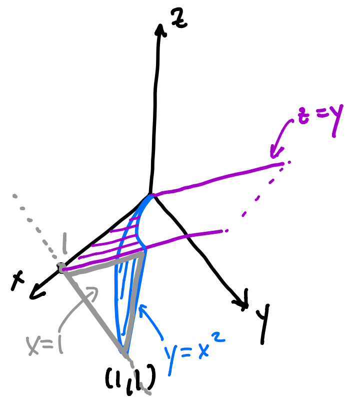
EXAMPLE: Express the iterated integral

$$I = \int_0^1 \int_0^{x^2} \int_0^y f(x,y,z)\, dz\, dy\, dx$$

first as a triple integral over E (write what is E) and then rewrite as an iterated integral in a different order, integrating first with respect to x, then z, then y.

Solution: $E = \{ (x,y,z) \mid 0 \le x \le 1,\ 0 \le y \le x^2,\ 0 \le z \le y \}$

Sketch of E:



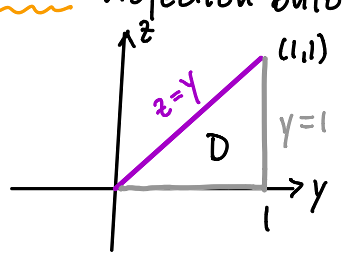
$$I = \iiint_E f(x,y,z)\, dV$$

$$= \int_0^1 \int_0^{x^2} \int_0^y f(x,y,z) \quad dz\, dy\, dx$$

$$= \int_?^? \int_?^? \left[\int_?^? f(x,y,z)\, dx \right] dz\, dy$$

$$= \iint_{D?} \left[\int_?^? f(x,y,z)\, dx \right] dz\,dy$$

STEP 1 Projection onto the yz plane:



STEP 2: For an arbitrary point (y,z) in D, as we move in the POSITIVE x direction, when do we first enter the solid E? (which x?)

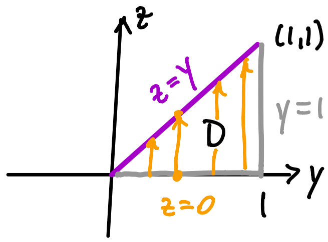
Answer: For $x = \sqrt{y}$

STEP 3: As we continue moving through E in the positive x direction, where do we exit E?

Answer: For $x = 1$.

STEP 4: So now we have: $I = \iint_D \left[\int_{\sqrt{y}}^{1} f(x,y,z)\, dx \right] dz\, dy$

STEP 5: Now focus on D and $dz\, dy$:



THE OUTER INTEGRAL with respect to y:

The bounds must be numbers (not functions).

What is the strip (interval in y) that contains D:

Answer: $0 \le y \le 1$

THE INTEGRAL WITH RESPECT TO z: For an arbitrary point $y \in (0,1)$, as we move in the POSITIVE z direction, where do we enter D, and where do we exit D (for which z)?

Answer: $z_{ENTER} = 0$, $z_{EXIT} = y$

STEP 6: Now we have the bounds over D, and the final integral:

$$I = \iint_D \left[\int_{\sqrt{y}}^{1} f(x,y,z)\, dx \right] dz\, dy = \int_0^1 \int_0^y \int_{\sqrt{y}}^{1} f(x,y,z)\, dx\, dz\, dy$$

APPLICATIONS

1. VOLUME OF SOLID E :

$$\boxed{V(E) = \iiint_E dV}$$

2. MASS OF SOLID E : Let $\rho(x,y,z)$ denote solid DENSITY.

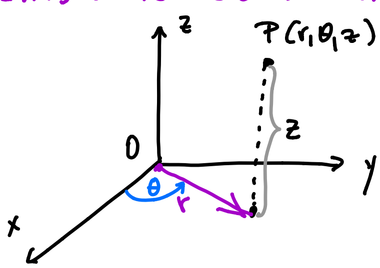
Then

$$\boxed{m(E) = \iiint_E \rho(x,y,z)\, dV}$$

TODO: Read the rest of APPLICATIONS OF TRIPLE INTEGRALS (pg 1035-1037)

15.7. TRIPLE INTEGRALS IN CYLINDRICAL COORDINATES

CYLINDRICAL COORDINATES ARE POLAR COORDINATES IN 3D!



Point P in 3D is represented by an ordered triple (r, θ, z) where (r, θ) are polar coordinates of the PROJECTION of P onto the xy plane, and z is the directed distance from the xy plane to P.

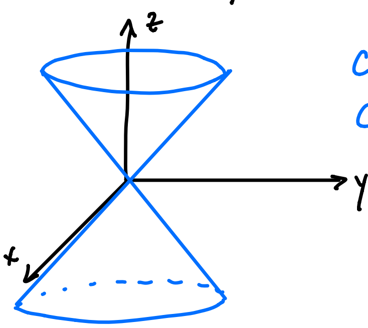
$$\boxed{\begin{aligned} x &= r\cos\theta \\ y &= r\sin\theta \\ z &= z \end{aligned}} \qquad \boxed{\begin{aligned} r^2 &= x^2+y^2 \\ \tan\theta &= \frac{y}{x} \\ z &= z \end{aligned}}$$

EXAMPLE: Describe the surface whose equation in cylindrical coordinates is $z = r$.

Solution: $z = r \Rightarrow z = \pm\sqrt{x^2+y^2} \Rightarrow \boxed{z^2 = x^2+y^2}$

Plot it:

$$\left.\begin{aligned} &\text{Set } x=0 \Rightarrow z^2 = y^2 \Rightarrow z = \pm y \\ &\text{Set } y=0 \Rightarrow z^2 = x^2 \Rightarrow z = \pm x \\ &\text{Set } z=0 \Rightarrow x^2+y^2=0 \Rightarrow x=y=0 \end{aligned}\right\} \Rightarrow \text{CONE}$$



CIRCULAR CONE

TRIPLE INTEGRALS: For regions:

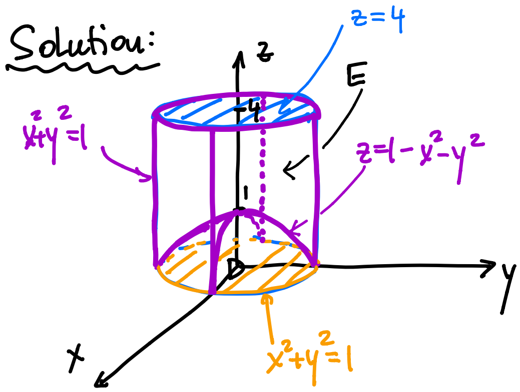
$$E = \{(x,y,z) \mid (x,y) \in D,\ u_1(x,y) \le z \le u_2(x,y)\}$$

where $D = \{(r,\theta) \mid \alpha \le \theta \le \beta,\ h_1(\theta) \le r \le h_2(\theta)\}$

$$\boxed{\iiint_E f(x,y,z)\,dV = \iint_D \left[\int_{u_1(x,y)}^{u_2(x,y)} f(x,y,z)\,dz\right] dA = \int_\alpha^\beta \int_{h_1(\theta)}^{h_2(\theta)} \int_{u_1(r\cos\theta,\, r\sin\theta)}^{u_2(r\cos\theta,\, r\sin\theta)} f(r\cos\theta, r\sin\theta, z)\,dz\, r\,dr\,d\theta}$$

EXAMPLE: A solid E lies within the cylinder $x^2+y^2=1$, below the plane $z=4$ and above the paraboloid $z=1-x^2-y^2$. The density at any point is proportional to its distance from the axis of the cylinder. Find the mass of E.

Solution:



DENSITY:

Distance of $P(x,y,z)$ to the z axis:

$$d=\sqrt{x^2+y^2}$$

So, density is given by:

$$\rho(x,y,z)=K\sqrt{x^2+y^2}, \quad K=\text{constant}.$$

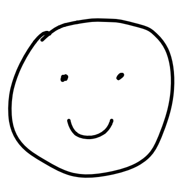
$$m(E)=\iiint_E K\sqrt{x^2+y^2}\,dV = K\iint_D\left[\int_{1-(x^2+y^2)}^{4}\underbrace{\sqrt{x^2+y^2}}\,dz\right]dA =$$

independent of z!

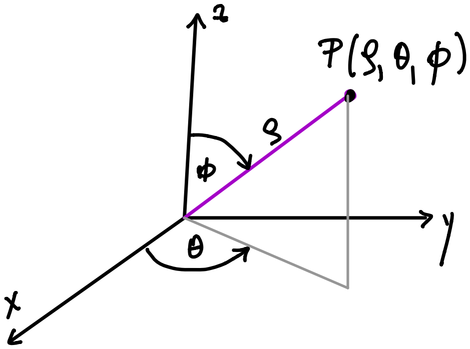
$$= K\iint_D \sqrt{x^2+y^2}\,[4-(1-(x^2+y^2))]\,dA = K\iint_D \sqrt{x^2+y^2}\,[3+(x^2+y^2)]\,dA$$

$$= K\int_0^{2\pi}\int_0^1 r\,[3+r^2]\,r\,dr\,d\theta = \left|\begin{array}{c}\text{PRODUCT INTEGRAL}\\ \text{WITH FIXED BOUNDS}\end{array}\right| = 2\pi K\int_0^1 r^2[3+r^2]\,dr$$

$$= 2\pi K\left[3\frac{r^3}{3}+\frac{r^5}{5}\right]_{r=0}^{1} = \frac{12\pi K}{5}$$



15.8. TRIPLE INTEGRALS IN SPHERICAL COORDINATES



Spherical coordinates
$P(\rho, \theta, \phi)$

ρ = distance from P to the origin

θ = same as the angle in cylindrical coordinates

ϕ = angle between positive z axis and $\overrightarrow{OP}$

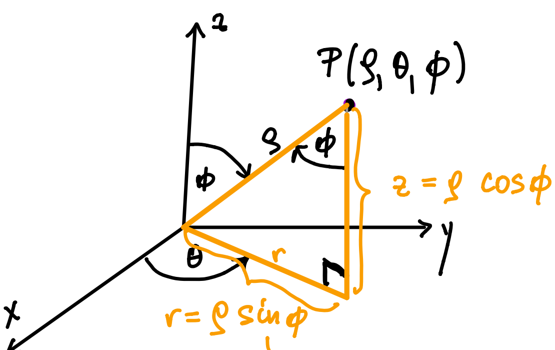
Note that $\rho \geq 0$, $\phi \in [0, \pi]$

Useful for integration over spherical regions.

RECTANGULAR VERSUS SPHERICAL COORDINATES:

$$\boxed{\begin{aligned} x &= \rho \cos\theta \sin\phi \\ y &= \rho \sin\theta \sin\phi \\ z &= \rho \cos\phi \end{aligned}}$$

$$\boxed{\rho^2 = x^2 + y^2 + z^2}$$



Polar: $x = r\cos\theta = \rho \sin\phi \cos\theta$

$y = r \sin\theta = \rho \sin\phi \sin\theta$

EXAMPLE: The point $(0, 2\sqrt{3}, -2)$ is given in rectangular coordinates. Find spherical coordinates for this point.

Solution: $\rho = ?$, $\phi = ?$, $\theta = ?$

From $\rho^2 = x^2 + y^2 + z^2 \Rightarrow \rho^2 = 0 + 12 + 4 = 16 \Rightarrow \rho = 4$ //

From $z = \rho \cos\phi \Rightarrow -2 = 4\cos\phi \Rightarrow \cos\phi = -\frac{1}{2} \Rightarrow \phi = \frac{2\pi}{3}$ //

From $x = \rho \sin\phi \cos\theta \Rightarrow 0 = 4 \sin\frac{2\pi}{3} \cos\theta \Rightarrow \cos\theta = 0 \Rightarrow \theta = \frac{\pi}{2}$ //

Note that $\theta \neq \frac{3\pi}{2}$ since $y > 0$!

Therefore, the spherical coordinates are $\begin{cases} \rho = 4 \\ \phi = \frac{2\pi}{3} \\ \theta = \frac{\pi}{2} \end{cases}$!