St. Petersburg paradox

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What is the probability of running out of money in a game with infinite expected return?

Question formulation

We consider a repeatable game, where the player pays a fixed fee m to enter. The pot starts at \$1, and doubles each time a head (H) is thrown. The game stops when the first tail (T) appears, and the player receives the current pot value, i.e.

$$r(\mathrm{T}) = \$1, \quad r(\mathrm{HT}) = \$2, \quad r(\mathrm{HHT}) = \$4, \quad r(\mathrm{HHHT}) = \$8, \quad \ldots .$$

The player begins with a balance of s, and can only play the game if $s \geq m$ (i.e. if they have enough money to pay the entry fee). We want to find $p_m(s)$, which is the probability that the gambler can play the game indefinitely (or that they do not run out of money).

The paradox

This game is paradoxical because the expected return, $\langle r \rangle$, is infinite, so any rational gambler would play, no matter the cost of the entry fee m:

$$\begin{aligned} r_n &= 2^n && \text{Return after } n \text{ heads,} \\ p_n &= \left(\frac{1}{2}\right)^n \left(1 - \frac{1}{2}\right) && \text{Probability of } n \text{ heads followed by a tail,} \\ \langle r \rangle = \left(\sum_{n=0}^{\infty} p_n r_n\right) - m &= \left(\frac{1}{2}\sum_{n=0}^{\infty} 1\right) - m = \infty && \text{Expected return.} \end{aligned}$$

Solution: Markov chain

The repeated game can be formulated as a Markov chain. Let t denote the number of games played, and $P_i(t)$ be the probability that we have a balance of i after t games ($i \in \{1, 2, \ldots\}$, $t \in \{0, 1, 2, \ldots\}$). Therefore, $P_i(0) = \delta_{i,s}$. We now want to express $P_i(t+1)$ as a function of $P_i(t)$. As the states $i < m$ are absorbing (we can't play any more if our balance is less than the entry fee m), we write down an equation for the absorbing states and an equation for the remainder:

$$P_i(t+1) = \sum_{n\,:\,i-(2^n-m)\geq m} p_n P_{i-(2^n-m)}(t) + P_i(t) \qquad \text{for } 1 \leq i < m, \tag{1a}$$

$$P_i(t+1) = \sum_{n\,:\,i-(2^n-m)\geq m} p_n P_{i-(2^n-m)}(t) \qquad \text{for } m \leq i < \infty, \tag{1b}$$

where the limit on the sum accounts for no transitions out of the absorbing states.

This Markov chain can be forward-integrated, starting form $P_i(0)$, to find the distribution of balances after t games. The probability that we can continue playing ($i > m$) after this time is then

$$p_m(s) = \lim_{t\to\infty} \left[1 - \sum_{i=1}^{m-1} P_i(t)\right]. \tag{2}$$

To solve the Markov chain computationally we have to impose an artificial upper bound on the size of the state space, as well as on the number of iterations t.

Mathematica code

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(*Construct the transition matrix for cost m and state-space size nStates*)
m = 2;
nStates = 2^10;
weights = Table[(p^n (1 - p)) /. p -> 0.5, {n, 0, Floor[Log2[nStates]]}];
dim = {nStates, nStates};

mat =
Total[
   Map[
      SparseArray[
         Table[{i, m + i - 2^#} -> weights[[# + 1]], {i, 2^#,
           Min[nStates - m + 2^#, nStates]}], dim] &,
        Range[0, Floor[Log2[nStates]] ]
    ]
];
mat += SparseArray[Table[{i, i} -> 1., {i, 1, m - 1}], dim];

(*Build the initial state vector*)
s = 5;
P = Normal[SparseArray[s -> 1, nStates]];
(*Apply the matrix nIter times to the state vector*)
nIter = 1000;
Pfinal = Nest[Dot[mat, #] &, P, nIter];
(*Calculate the probability in the states i>m*)
1 - Sum[Pfinal[[k]], {k, 1, m - 1}]
```

Solution: Linear system

An alternative approach to this problem is to consider a linear system for the variables $p_m(s)$ themselves. I.e.

$$p_m(s) = \sum_{n=0}^{\infty} p_n p_m(s + 2^n - m), \tag{3}$$

with $p_m(s) = 0$ for $s < m$. The solution Eq. (3) is directly related to the stationary solution of Eqs. (1).

Computationally, as above, we have to put an upper bound on the state space