Problem 1

a. Prove that if A is an n x n (real) symmetric matrix, then there exists an n x n (real) orthogonal matrix U and n x n (real) diagonal matrix D such that $A = U \cdot D \cdot U^T$.

Proof:

Proof: (1a)

- Assume that matrix A has non-degenerate eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_K$ and corresponding linearly independent eigenvectors $X_1, X_2, \ldots X_K$ which can be denoted as

$$\begin{bmatrix} x_{11} \\ x_{12} \\ \vdots \\ x_{1K} \end{bmatrix}, \begin{bmatrix} x_{21} \\ x_{22} \\ \vdots \\ x_{2K} \end{bmatrix}, \ldots \begin{bmatrix} x_{K1} \\ x_{K2} \\ \vdots \\ x_{KK} \end{bmatrix} \quad \ldots\ldots \text{(i)}$$

- We define U as a matrix composed of eigenvectors, i.e

$$U = [X_1 \ X_2 \ \ldots \ X_K]$$
$$= \begin{bmatrix} x_{11} & x_{21} & \cdots & x_{K1} \\ \vdots & \vdots & & \vdots \\ x_{1K} & x_{2K} & & x_{KK} \end{bmatrix} \quad \ldots\ldots \text{(ii)}$$

- We also define D as a diagonal matrix composed of all eigenvalues

$$D = \begin{bmatrix} \lambda_1 & & O \\ & \ddots & \\ O & & \lambda_K \end{bmatrix} \quad \ldots\ldots \text{(iii)}$$

- Then

$$AU = A[X_1 \ X_2 \ \ldots \ X_K]$$
$$= [AX_1 \ AX_2 \ \ldots \ AX_K]$$
$$= \begin{bmatrix} \lambda_1 x_{11} & \lambda_2 x_{21} & & \lambda_K x_{K1} \\ \lambda_1 x_{12} & \lambda_2 x_{22} & \cdots & \lambda_K x_{K2} \\ \vdots & \vdots & & \vdots \\ \lambda_1 x_{1K} & \lambda_2 x_{2K} & & \lambda_K x_{KK} \end{bmatrix}$$

$$\Rightarrow AU = \begin{bmatrix} x_{11} & x_{21} & & x_{K1} \\ x_{12} & x_{22} & \cdots & x_{K2} \\ \vdots & \vdots & & \vdots \\ x_{1K} & x_{2K} & & x_{KK} \end{bmatrix} \begin{bmatrix} \lambda_1 & & 0 \\ & \lambda_2 & \\ 0 & & \ddots \; \lambda_K \end{bmatrix}$$

$$\Rightarrow AU = UD$$

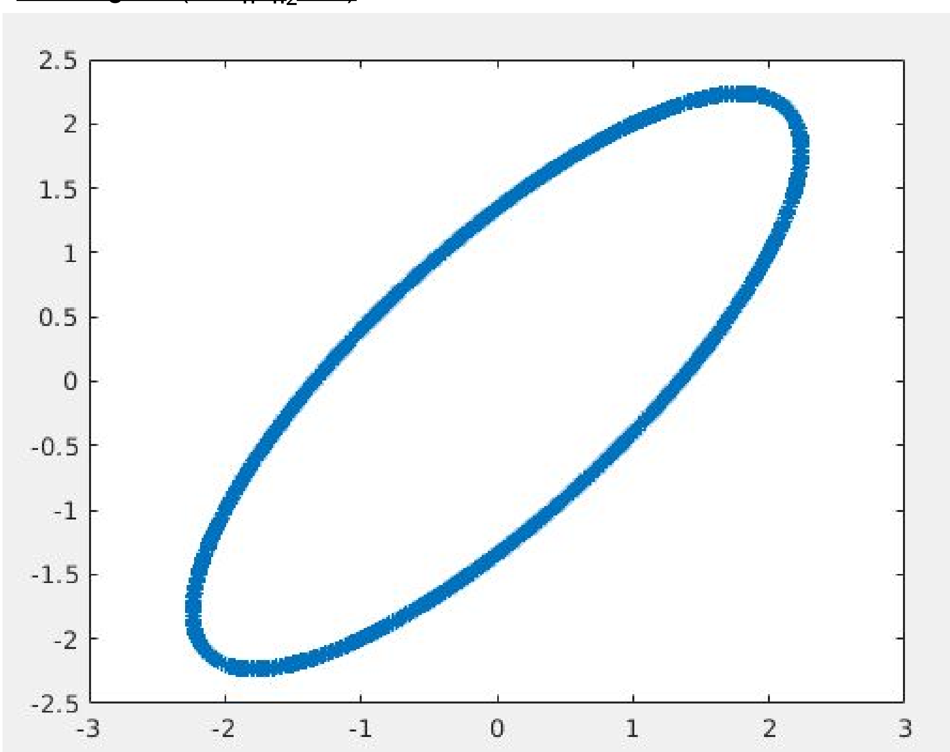
$$\Rightarrow A = UDU^{-1}$$

Since U is an orthogonal matrix $U^{-1} = U^T$

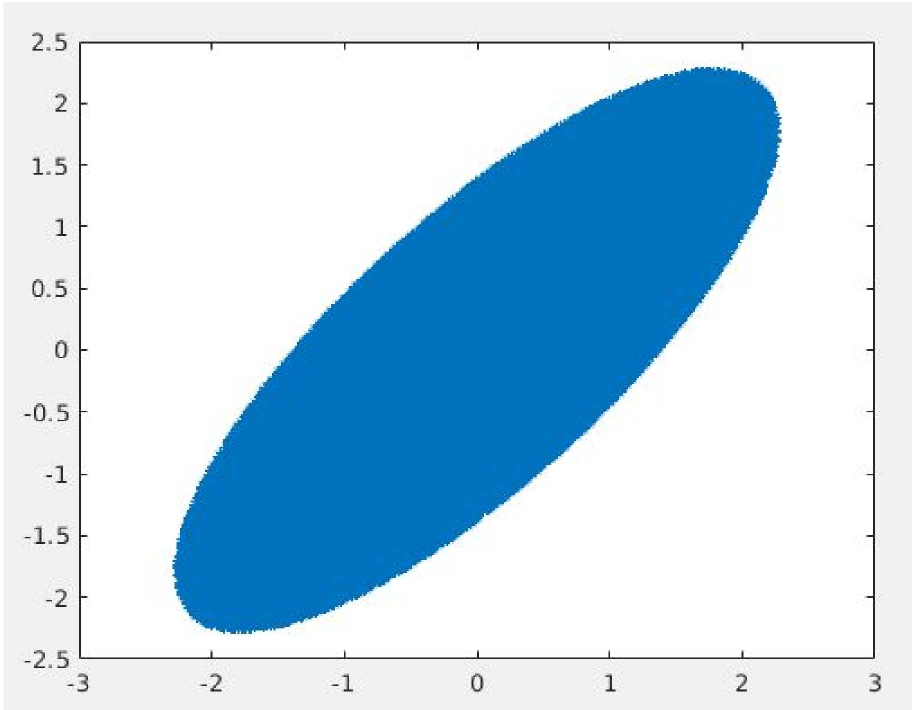
Hence, $\boxed{A = UDU^T}$

b. For this question, I assumed that x is a 2x1 matrix and plotted the region for Ax

- <u>First region: (For $||x||_2 = 1$)</u>



- <u>Second region: (For $||x||_2$ <= 1)</u>



- Here, for the first part, I took **x** as a 2x1 vector with elements [sin(theta); cos(theta)] and plotted the values of Ax for 10^5 different values of theta (ranging from 0 to 10). The reason for choosing the above vector is that its norm is 1 for any value of theta.

- For the second part, I chose 10^5 different random values (ranging from -1 to 1) for a 2x1 vector **x** and plotted the values of Ax for which the 2-norm of x is less than or equal to 1.

- The code for these outputs is in file: **<u>Prob_1b_Plot_Regions.m</u>**

Problem 3

a. Let R_1 and R_2 be two rotation matrices on the plane. Prove or disprove the following:
$R_1 R_2 = R_2 R_1$.

Proof:

- Matrices commute if they preserve each others' eigenspaces: there is a set of eigenvectors that, taken together, describe all the eigenspaces of both matrices, in possibly varying partitions.
- This makes intuitive sense: this constraint means that a vector in one matrix's eigenspace won't leave that eigenspace when the other is applied, and so the original matrix's transformation still works fine on it
- In two dimensions, no matter what, the eigenvectors of a rotation matrix are [i,1] and [−i,1]. So since all such matrices have the same eigenvectors, they will commute.
- An example of proof is shown below:

Consider two rotation matrices R_1 and R_2 as follows:

$$R_1 = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \qquad R_2 = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$$

$$R_1 R_2 = \begin{bmatrix} \cos^2\theta + \sin^2\theta & \sin\theta\cdot\cos\theta - \sin\theta\cdot\cos\theta \\ \sin\theta\cdot\cos\theta - \cos\theta\cdot\sin\theta & \sin^2\theta + \cos^2\theta \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Similarly,

$$R_2 R_1 = \begin{bmatrix} \cos^2\theta + \sin^2\theta & -\sin\theta\cdot\cos\theta + \sin\theta\cdot\cos\theta \\ -\sin\theta\cos\theta + \cos\theta\cdot\sin\theta & \sin^2\theta + \cos^2\theta \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Here, $R_1 R_2 = R_2 R_1$

Since, R_1, R_2 represent general form for 2D rotation matrices, we can say that $\boxed{R_1 R_2 = R_2 R_1}$ on a plane.

b. Let R_1 and R_2 be two rotation matrices in 3D space. Prove or disprove the following: $R_1 R_2 = R_2 R_1$.

- We saw that rotation matrices in 2D space satisfy the commutative property but in three dimensions, there's always one real eigenvalue for a real matrix such as a rotation matrix, so that eigenvalue has a real eigenvector associated with it: the axis of rotation. But this eigenvector doesn't share values with the rest of the eigenvectors for the rotation matrix (because the other two are necessarily complex)! So the axis is an eigenspace of dimension 1, so rotations with different axes can't possibly share eigenvectors, so they cannot commute

→ Rotations in 3D are generally represented by the following matrices:

$$R_x(\theta) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix}, \quad R_y(\theta) = \begin{bmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{bmatrix}, \quad R_z(\theta) = \begin{bmatrix} \sin\theta & -\sin\theta & 0 \\ \cos\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Considering the 1st two matrices:

$$R_x(\theta) \cdot R_y(\theta) = \begin{bmatrix} \cos\theta+0+0 & 0\cdot & \sin\theta \\ \sin^2\theta & \cos\theta & -\sin\theta\cos\theta \\ -\sin\theta\cos\theta & \sin\theta & \cos^2\theta \end{bmatrix} = \begin{bmatrix} \cos\theta & 0 & \sin\theta \\ \sin^2\theta & \cos\theta & -\sin\theta\cos\theta \\ -\sin\theta\cos\theta & \sin\theta & \cos^2\theta \end{bmatrix}$$

Similarly,

$$R_y(\theta) R_x(\theta) = \begin{bmatrix} \cos\theta & \sin^2\theta & \sin\theta\cos\theta \\ 0 & \cos\theta & -\sin\theta \\ -\sin\theta & \sin\theta\cos\theta & \cos^2\theta \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin^2\theta & \sin\theta\cos\theta \\ 0 & \cos\theta & -\sin\theta \\ -\sin\theta & \sin\theta\cos\theta & \cos^2\theta \end{bmatrix}$$

Here, $\boxed{R_x(\theta) R_y(\theta) \neq R_y(\theta) R_x(\theta)}$

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Problem 4

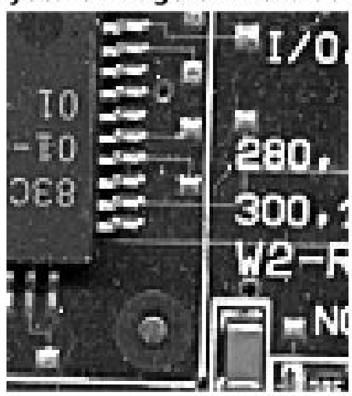
Let R be a 3D rotation matrix. Claim: 1 is an eigenvalue of R. Is the claim true? If so, what is the physical meaning of the corresponding eigenvector?

- Yes, 1 is an eigenvalue of R. This claim is true.

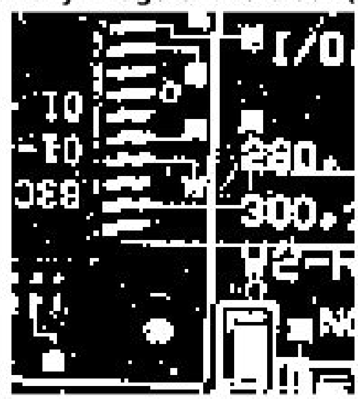
- Explanation:
    - In 3D space, any two Cartesian coordinate systems with a common origin are related by a rotation about some fixed axis. This also means that the product of two rotation matrices is again a rotation matrix and that for a non-identity rotation matrix one eigenvalue is 1 and the other two are both complex, or both equal to −1. The eigenvector corresponding to this eigenvalue is the **axis of rotation connecting the two systems**.

Problem 6

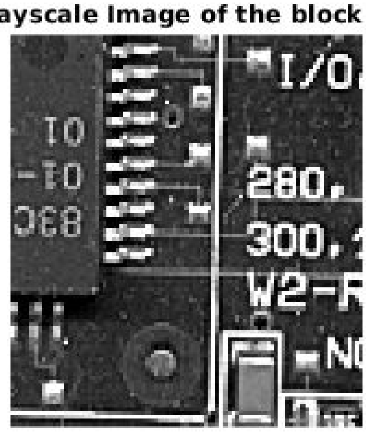
Grayscale Image of the block (i)



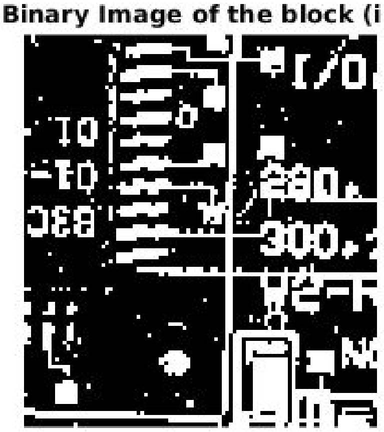
Binary Image of the block (i)



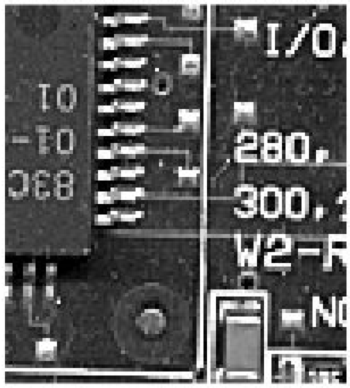
Grayscale Image of the block (ii)



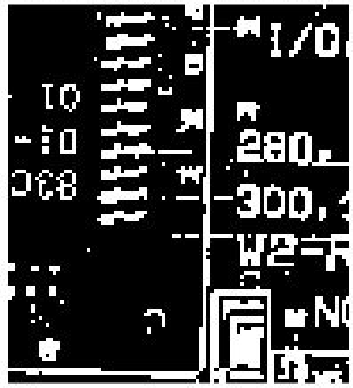
Binary Image of the block (ii)



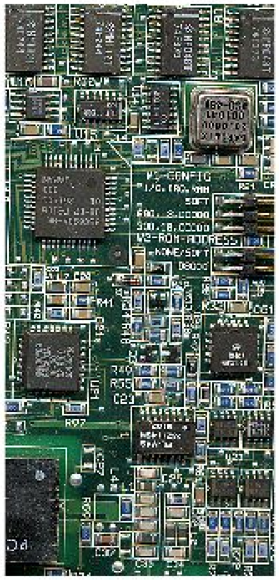
Grayscale Image of the block (iii)



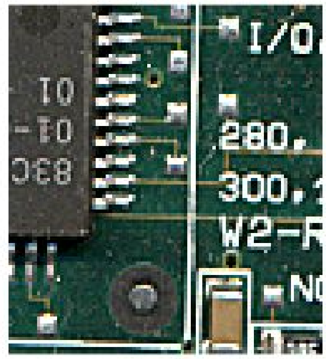
Binary Image of the block (iii)



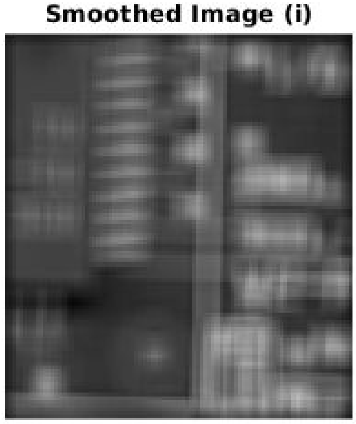
Original Image



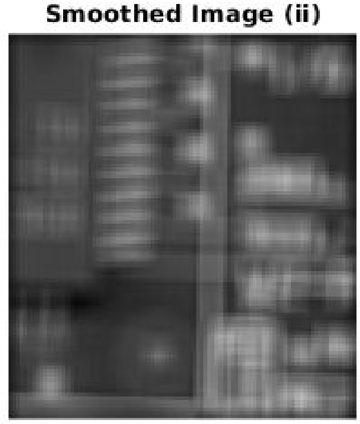
Crystal



Smoothed Image (i)



Smoothed Image (ii)



Runtimes:

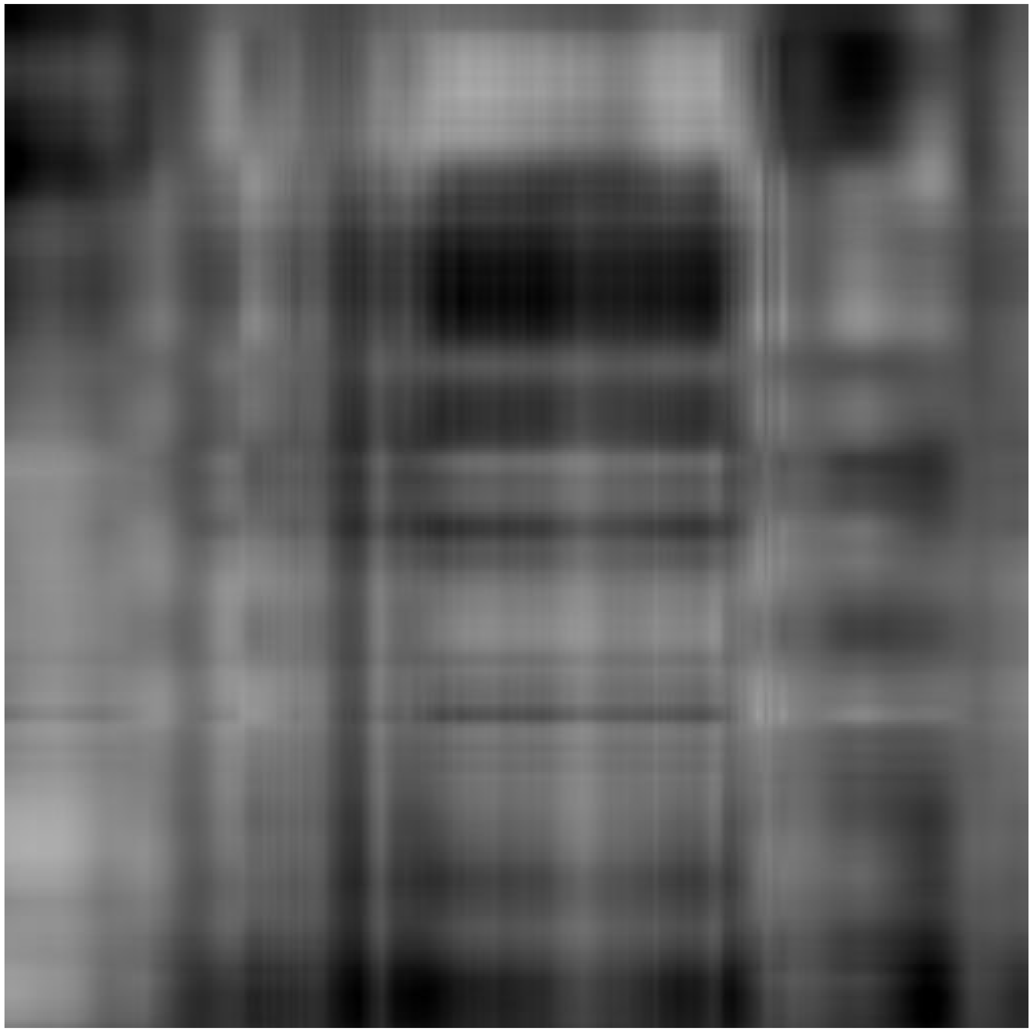
Method Name	Runtime (seconds)
RGB to Gray (Using Loops)	0.003634
RGB to Gray (Using MATLAB matrix operations)	0.000603
RGB to Gray (Using rgb2gray)	0.001796
Gray to BW (Using Loops)	0.001171
Gray to BW (Using MATLAB matrix operations)	0.000331
Gray to BW (Using Loops)	0.003770
Smoothed Image (Using Loops)	0.011381
Smoothed Image (Using conv2)	0.000740

- **Extra Credit:**

  (b) Original and the compressed images

  - Original Image

- Image with Top Singular Values: 3



- Image with Top Singular Values: 10

- Image with Top Singular Values: 20

- Image with Top Singular Values: 40

Relative Errors and Compression Ratios:

Image with selected Top Singular Values (TSV)	Relative Error (MSE w.r.t Original Gray Image)	Compression Ratio (size(C)/size(original))
TSV = 3	0.0085	0.46
TSV = 10	0.0026	0.59
TSV = 20	0.0010	0.67
TSV = 40	0.0004	0.75

Note: The MSEs were calculated using ***immse()*** function in MATLAB

a. **We denote by σ_i (B) the ith singular value of B (sorted in descending order). Prove that if A_1 and A_2 are m × n matrices, then for all i and j in N: $\sigma_{i+j-1}(A_1 + A_2) \leq \sigma_i(A_1) + \sigma_j(A_2)$.**

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