Measure Theory Problem Set

OSE Bootcamp 2019 Week 3

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Exercise 1.3

1. Not an algebra. The complement of an open set is closed, so for example, if $A = (-\infty, 3)$ then $A^c = [3, \infty) \notin G_1$.
2. We claim G_2 is an algebra, but not a σ-algebra. To see G_2 is an algebra, note that by definition G_2 is closed under finite union. Moreover, note that

$$(a, b]^c = (-\infty, a] \cup (b, \infty) \in G_2$$
$$(-\infty, b]^c = (b, \infty) \in G_2$$
$$(a, \infty)^c = [-\infty, a] \in G_2,$$

   and clearly finite unions of such intervals will have their complements in G_2 as well, because their complements are just finite unions of the exact same types of intervals.
   Note, however, that G_2 is not a sigma algebra because by definition it does not contain countable unions of such intervals.
3. This is a σ algebra. As in G_2, the set is closed under complements and by definition closed under countably infinite unions.

Exercise 1.7

Suppose A was a 'smaller' σ-algebra than $\{\emptyset, X\}$. Since by definition $\emptyset \in A$ then we must have $A = \{\emptyset\}$ but then $\emptyset^c = X \notin A$, contradiction. Thus $\{\emptyset, X\}$ is minimal.

Suppose B is 'larger' than $\mathcal{P}(X)$. Then $\exists$ a set S such that $S \in B$ and $S \notin \mathcal{P}(X)$ since a σ-algebra is a family of subsets of the space X, $S \subset X$, which automatically implies $S \subset \mathcal{P}(X)$ by definition, contradiction.

Exercise 1.10

Suppose a set $S \in \bigcap_\alpha S_\alpha$, we must have $\emptyset \in \bigcap_\alpha S_\alpha$ as the empty set is in each σ-algebra by definition. Moreover, we have that $S \in S_\alpha \forall \alpha$. Since each S_α is a σ-algebra, we see that $S^c \in S_\alpha \forall \alpha$ and thus S^c is in the intersection.

Next, take countably infinite unions of A_i such that $A_i \in \bigcap_\alpha S_\alpha$, then $A_i \in S_\alpha \forall \alpha$. Again by the fact that S_α are σ-algebras, we have that:

$$\bigcup_{i=1}^{\infty} A_i \in S_\alpha \forall \alpha \implies \bigcup_{i=1}^{\infty} A_i \in \bigcap_\alpha S_\alpha,$$

which completes the proof.

Exercise 1.22

First we show monotonicity. Take C such that $B = A \cup C$ where $A \cap C = \{\emptyset\}$. Then we have:

$$\mu(B) = \mu(A \cup C) = \mu(A) + \mu(C) \geq \mu(A),$$

which proves the result.

Next we show countable subadditivity. For $\bigcup_{i=1}^{\infty} A_i$, construct a sequence of sets B_i such that: $B_1 = A_1, B_2 = A_2 \backslash A_1, \ldots B_n = A_n \backslash A_{n-1} \backslash \ldots A_1$. By construction, these B_i are disjoint, $B_i \subset A_i$ and moreover, their union is equivalent to the union of the A_i. We have:

$$\mu(\bigcup_{i=1}^{\infty} A_i) = \sum_{i=1}^{\infty} \mu(B_i) \leq \sum_{i=1}^{\infty} \mu(A_i),$$

which completes the proof.

Exercise 1.23

Note $\lambda(\emptyset) = \mu(\emptyset \cap B) = \mu(\emptyset) = 0$. Next take $\{A_i\}_{i=1}^{\infty} \subset S$ such that $A_i \cap A_j = \emptyset$. We see that

$$\begin{aligned}
\lambda(\bigcup_{i=1}^{\infty} A_i) &= \mu(\bigcup_{i=1}^{\infty} A_i \cap B) \\
&= \mu(\bigcup_{i=1}^{\infty} (A_i \cap B)) \\
&= \sum_{i=1}^{\infty} \mu(A_i \cap B) \\
&= \sum_{i=1}^{\infty} \lambda(A_i),
\end{aligned}$$

which completes the proof.

Exercise 1.26

Note that $A_1 = \bigcap_{i=1}^{\infty} A_i \cup A_1 \backslash \bigcap_{i=1}^{\infty} A_i$. Taking the measure on both sides, we get:

$$\begin{aligned}
\mu(A_1) &= \mu(\bigcap_{i=1}^{\infty} A_i) + \mu(A_1 \backslash \bigcap_{i=1}^{\infty} A_i) \\
&= \mu(\bigcap_{i=1}^{\infty} A_i) + \mu(\bigcup_{n=1}^{\infty} A_1 \backslash A_n) \\
&= \lim_{n\to\infty} \mu(A_1 \backslash A_n) \\
&= \lim_{n\to\infty} (\mu(A_1) - \mu(A_n)),
\end{aligned}$$

which implies the result.

Exercise 2.10

Note that $B = (B \cap E) \cup (B \cap E^c)$ and by subadditivity,

$$\mu^*(B) \leq \mu^*(B \cap E) + \mu^*(B \cap E^c),$$

so equality follows in the theorem condition.

Exercise 2.14

Since $\sigma(O)$ is a sigma algebra, for any open set $S \in \sigma(O)$, $S^c \in \sigma(O)$, note S^c is closed. Moreover, since $\sigma(O)$ is closed under countable union, it contains unions of all closed/open sets in $\mathbb{R}$. Now consider $\sigma(A)$, which is nearly A but with the addition of countable unions of these intervals. Any countable intersection of intervals is an intersection of intervals is an intersection of open and closed sets in $\mathbb{R}$. Moreover, any open/closed set in $\mathbb{R}$ is an intersection of countably many intervals, this implies that there is a bijection between $\sigma(O)$ and $\sigma(A)$, so they are equivalent. Caratheodory implies the result.

Exercise 3.1
Note the Lebesgue measure of a point is 0, as a point is a trivial interval. If we have a countable subset of $\mathbb{R}$, this is just a countable union of disjoint points P_i, thus by definition of a measure:

$$\mu(\bigcup_{i=1}^{\infty} P) = \sum_{i=1}^{\infty} \mu(P_i) = 0.$$

Exercise 3.7
We can replace w/ less than or equal to as a single point has measure zero in $\mathbb{R}$. Note that if $\{x \in X : f(x) < a\} \in M$ then $\{x \in X : f(x) < a\}^c \in M$ as M is a sigma-algebra, thus we get the greater than or equal to case and from previous logic we see that we also get the greater than case.

Exercise 3.10
For $f + g$, $f * g$, $|f|$, take: $F(x, y) = x + y$; $F(x, y) = x * y$; $F(x, y) = \sqrt{x^2} = |x|$ note all of these are continuous, so we see that by (4), each of these functions are measureable.

Since we have the sup and inf condition, the functions:

$$F(x, y) = \frac{1}{2}(x + y + |x - y|) = max(x, y)$$

and

$$F(x, y) = \frac{1}{2}(x + y - |x - y|) = min(x, y)$$

are well defined and clearly continuous, and so max and min are measurable. These statements imply condition 1.

Exercise 3.17
Suppose $f \leq M$, $f \geq M$ for some $M \in \mathbb{R}$. For any x, since we have $f \in [-M, M]$, for every ϵ there exists N such that

$$\frac{1}{2^n} < \epsilon; \quad n \geq N.$$

We can construct *finitely* many E_i (same E_i as in the constructive proof), that cover all of f, in that the union of the E_i is the entire real line, as we can partition the range into finitely many intervals, none of which contains infinity. Choose an N that fits the bound for ϵ, and then by the construction of s_n we see that

$$|f(x) - s_N(x)| < \epsilon$$

for all x, which implies the uniform convergence of $s_n \to f$.

Exercise 4.13
If $|f| < M$ then we see that $f^+ < M, f^- > -M$, so both are finite on E. Thus from the definition of the Lebesgue integral we find that we are taking the *sup* over simple measurable bounded functions for both f^+ and f^-, which are necessarily bounded and thus $\int_E f^+ d\mu < \infty$ and $\int_E f^- d\mu < \infty$, which by definition implies that f is integrable.

Exercise 4.14
Suppose otherwise, then there exists some $E_\alpha \subset E$ such that $\mu(E_\alpha) > 0$ and where $f(x) = \infty$. Consider the Lebesgue integral on E_α. For any $a \in \mathbb{R}$, take an $x \in E_\alpha$ such that $f(x) > a$. Then by proposition 4.5 we have that:

$$a\mu(E_\alpha) \leq \int_E f d\mu$$

. Since $\mu(E_\alpha)$ is greater than 0, and for any a there exists $x \in E_\alpha$ such that $f(x) > a$, we can take the *sup* of both sides, to find that the Lebesgue integral is infinite. This is a contradiction, hence the result.

Exercise 4.15 Note that since $f \leq g$, it must be true that:

$$\{s : 0 \leq s \leq f^+\} \subset \{s : 0 \leq s \leq g^+\},$$

a similar argument holds true for f^- and g^-. Then the result follows from Proposition 4.7.

Exercise 4.16
Observe that

$$\int_A f^+ d\mu = sup\{\int_A s d\mu, 0 \leq s \leq f\} \leq sup\{\int_E s d\mu, 0 \leq s \leq f\} < \infty,$$

where the inequality comes from the fact that $A \subset E$. Similarly f^- is finite, so we are done.