DSGE Problem Set WriteUp

OSE Bootcamp 2019 Week 3

Alan Shenkerman

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DSGE Models

Exercise 1

We substitute $K_{t+1} = Ae^{z_t}K_t^\alpha$ into each side of the Euler equation and find the correct value of A in terms of the model parameters. Simplifying the LHS:

$$\frac{1}{e^{z_t}K_t^\alpha - K_{t+1}} = \frac{1}{e^{z_t}K_t^\alpha - Ae^{z_t}K_t^\alpha} = \frac{1}{e^{z_t}K_t^\alpha(1-A)}$$

On the RHS, note $E(z_{t+1}) = \rho z_t$. We get.

$$\begin{aligned}\beta E_t\left[\frac{\alpha e^{z_{t+1}}K_{t+1}^{\alpha-1}}{e^{z_{t+1}}K_{t+1}^\alpha - K_{t+2}}\right] &= \beta E_t\left[\frac{\alpha e^{z_{t+1}}(Ae^{z_t}K_t^\alpha)^{\alpha-1}}{e^{z_{t+1}}(Ae^{z_t}K_t^\alpha)^\alpha - Ae^{z_{t+1}}(Ae^{z_t}K_t^\alpha)^\alpha}\right] \\ &= \frac{\alpha\beta}{Ae^{z_t}K_t^\alpha(1-A)}\end{aligned}$$

Combining the two, most terms cancel and we're left with

$$A = \alpha\beta,$$

so our steady-state policy function becomes:

$$K_{t+1} = \alpha\beta e^{z_t}K_t^\alpha.$$

Exercise 2

The characterizing equations are:

$$\begin{aligned} c_t &= (1-\tau)\left[w_t l_t + (r_t - \delta)k_t\right] + k_t + T_t - k_{t+1} \\ \frac{1}{c_t} &= \beta E\left\{\frac{1}{c_{t+1}}\left[(r_{t+1} - \sigma)(1-\tau) + 1\right]\right\} \\ \frac{a}{1-l_t} &= \frac{1}{c_t}w_t(1-\tau) \\ r_t &= \alpha e^{z_t}k_t^{\alpha-1}l_t^{1-\alpha} \\ w_t &= (1-\alpha)e^{z_t}k_t^\alpha l_t^{-\alpha} \\ T_t &= \tau\left[w_t l_t + (r_t - \delta)k_t\right] \\ z_t &= (1-\rho_z)\overline{z} + \rho_z z_{t-1} + \epsilon_t^z \end{aligned}$$

Note since here we have to consider leisure as well as consumption, the policy function is more complex and the simple form from Exercise 1 will not hold.

Exercise 3

The characterizing equations are:

$$\begin{aligned}
c_t &= (1-\tau)\left[w_t l_t + (r_t - \delta) k_t\right] + k_t + T_t - k_{t+1} \\
c_t^{-\gamma} &= \beta E_t \left\{ c_{t+1}^{-\gamma} \left[(r_{t+1} - \delta)(1-\tau) + 1\right]\right\} \\
\frac{a}{1-l_t} &= c_t^{-\gamma} w_t (1-\tau) \\
r_t &= \alpha e^{z_t} k_t^{\alpha-1} l_t^{1-\alpha} \\
w_t &= (1-\alpha) e^{z_t} k_t^{\alpha} l_t^{-\alpha} \\
T_t &= \tau\left[w_t l_t + (r_t - \delta) k_t\right] \\
z_t &= (1-\rho_z)\,\overline{z} + \rho_z z_{t-1} + \epsilon_t^z
\end{aligned}$$

Exercise 4

The characterizing equations are:

$$\begin{aligned}
c_t &= (1-\tau)\left[w_t l_t + (r_t - \delta) k_t\right] + k_t + T_t - k_{t+1} \\
c_t^{-\gamma} &= \beta E_t \left\{ c_{t+1}^{-\gamma} \left[(r_{t+1} - \delta)(1-\tau) + 1\right]\right\} \\
a\,(1-l_t)^{-\xi} &= c_t^{-\gamma} w_t (1-\tau) \\
r_t &= \alpha e^{z_t} k_t^{\eta-1} [\alpha k_t^{\eta} + (1-\alpha) l_t^{\eta}]^{\frac{1-\eta}{\eta}} \\
w_t &= (1-\alpha) e^{z_t} l_t^{\eta-1} [\alpha k_t^{\eta} + (1-\alpha) l_t^{\eta}]^{\frac{1-\eta}{\eta}} \\
T_t &= \tau[w_t l_t + (r_t - \delta) k_t] \\
z_t &= (1-\rho_z)\overline{z} + \rho_z z_{t-1} + \epsilon_t^z
\end{aligned}$$

Exercise 5

Characterizing Equations, $l_t = 1$:

$$\begin{aligned}
c_t &= (1-\tau)\left[w_t + (r_t - \delta) k_t\right] + k_t + T_t - k_{t+1} \\
c_t^{-\gamma} &= \beta E_t \left\{ c_{t+1}^{-\gamma} \left[(r_{t+1} - \delta)(1-\tau) + 1\right]\right\} \\
r_t &= \alpha e^{z_t} k_t^{\alpha-1} e^{z_t(1-\alpha)} \\
w_t &= (1-\alpha) e^{z_t(1-\alpha} k_t^{\alpha} \\
T_t &= \tau\left[w_t + (r_t - \delta) k_t\right] \\
z_t &= (1-\rho_z)\,\overline{z} + \rho_z z_{t-1} + \epsilon_t^z
\end{aligned}$$

It follows that the steady state versions are:

$$\begin{aligned}
\bar{c} &= (1-\tau)[\bar{w} + (\bar{r} - \delta)\bar{k}] + \bar{k} + \bar{T} - \bar{k} \\
\bar{T} &= \tau[\bar{w} + (\bar{r} - \delta)\bar{k}] \\
\bar{c}^{-\gamma} &= \beta E_t \left[\bar{c}^{-\gamma}[(\bar{r} - \delta)(1-\tau) + 1]\right] \\
\bar{r} &= \alpha \bar{k}^{\alpha-1} (e^{\bar{z}})^{1-\alpha} \\
\bar{w} &= (1-\alpha) \bar{k}^{\alpha} (e^{\bar{z}})^{1-\alpha} \\
\bar{z} &= (1-\rho_z)\bar{z} + \rho_z \bar{z}
\end{aligned}$$

We solve these analytically for $\bar{K}$ as a function of the parameters of the model to see that:

$$\bar{K} = e^{\bar{z}} \sqrt[1-\alpha]{\frac{\alpha}{\delta + \frac{1-\beta}{\beta(1-\tau)}}}.$$

We see in our Jupyter Notebook that this coincides with very high accuracy to the numerical steady state solution.

Exercise 6

Characterizing Equations:

$$
\begin{aligned}
c_t &= (1-\tau)[w_t l_t + (r_t - \delta)k_t] + k_t + T_t - k_{t+1} \\
c_t^{-\gamma} &= \beta E_t \left[c_{t+1}^{-\gamma}[(r_{t+1} - \delta)(1-\tau) + 1] \right] \\
\frac{a}{(1-l_t)^{\xi}} &= c_t^{-\gamma} w_t (1-\tau) \\
r_t &= \alpha \left(\frac{l_t e^{z_t}}{k_t} \right)^{1-\alpha} \\
w_t &= (1-\alpha) e^{z_t} \left(\frac{k_t}{l_t e^{z_t}} \right)^{\alpha} \\
T_t &= \tau [w_t l_t + (r_t - \delta) k_t] \\
z_t &= (1-\rho_z)\bar{z} + \rho_z z_{t-1} + \epsilon_t^z; \quad \epsilon_t^z \sim i.i.d.(0, \sigma_z^2)
\end{aligned}
$$

Steady State:

$$
\begin{aligned}
\bar{c} &= (1-\tau)[\bar{w}\bar{l} + (\bar{r} - \delta)\bar{k}] + \bar{k} + \bar{T} - \bar{k} \\
\bar{c}^{-\gamma} &= \beta E_t \left[\bar{c}^{-\gamma}[(\bar{r} - \delta)(1-\tau) + 1] \right] \\
\frac{a}{(1-\bar{l})^{\xi}} &= \bar{c}\bar{w}(1-\tau) \\
\bar{r} &= \alpha \left(\frac{\bar{l} e^{\bar{z}}}{\bar{k}} \right)^{1-\alpha} \\
\bar{w} &= (1-\alpha) e^{\bar{z}} \left(\frac{\bar{k}}{\bar{l} e^{\bar{z}}} \right)^{\alpha} \\
\bar{T} &= \tau [\bar{w}\bar{l} + (\bar{r} - \delta)\bar{k}] \\
\bar{z} &= (1-\rho_z)\bar{z} + \rho_z \bar{z}
\end{aligned}
$$

Linearization Methods

Exercise 3

We simplify:

$$
\begin{aligned}
&E_t\{FX_{t+1} + GX_t + HX_{t-1} + LZ_{t+1} + MZ_t\} \\
&E_t\{F(PX_t + Z_{t+1}) + G(PX_{t-1} + QZ_t) + HX_{t-1} + L(NZ_t + \epsilon_{t+1}) + MZ_t\} \\
&E_t\{F(P(PX_{t-1} + QZ_{t+1}) + NZ_t + \epsilon_t) + GPX_{t-1} + GQZ_t + HX_{t-1} + LNZ_t + MZ_t\} \\
&FP^2X_{t-1} + FPQZ_t + FNQZ_t + GPX_{t-1} + GQZ_t + HX_{t-1} + LNZ_t + MZ_t \\
&[(FP+G)P + H]X_{t-1} + [Q(FP+G) + M + N(FQ+L)]Z_t,
\end{aligned}
$$

as desired.

Perturbation Methods

Exercise 1

After taking derivatives and simplifying the terms, we end up with the following closed form:

$$
x_{uuu} = \frac{F_{xxx}x_u^3 + 3[F_{xx}x_u x_{uu} + F_{xu}x_{uu} + F_{xxu}x_u^2 + F_{xuu}x_u] + F_{uuu}}{F_x}
$$