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LOGIC

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LECTURE 2a

Chapter 2

Introduction to Classical Logic Languages and Semantics

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Lecture 2

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Chapter 2

Introduction to Classical Logic Languages and Semantics

Part 5: Predicate Language

Predicate Language

We define a predicate language $\mathcal{L}$ following the pattern established by the definitions of symbolic and propositional language.

The predicate language **is much more complicated** in its structure.

Its alphabet $\mathcal{A}$ is **much richer**.

The definition of its set of formulas $\mathcal{F}$ is **more complicated**.

In order to define the set $\mathcal{F}$ define an **additional set T**, called a set of all **terms** of the predicate language $\mathcal{L}$.

We single out this set **T** of **terms** not only because we need it for the definition of formulas, but also because of its role in the development of other notions of **predicate logic**.

Predicate Language Definition

Definition

By a **predicate language** $\mathcal{L}$ we understand a triple

$$\mathcal{L} = (\mathcal{A}, \mathbf{T}, \mathcal{F})$$

where $\mathcal{A}$ is a predicate **alphabet**

$\mathbf{T}$ is the set of **terms**, and $\mathcal{F}$ is a set of **formulas**

Alphabet Components

Alphabet $\mathcal{A}$

The components of $\mathcal{A}$ are as follows

1. Propositional connectives

$$\neg, \; \cap, \; \cup, \; \Rightarrow, \; \Leftrightarrow$$

2. Quantifiers $\forall, \; \exists$

$\forall$ is the universal quantifier, and $\exists$ is the existential quantifier

3. Parenthesis (and)

Alphabet Components

4. Variables

We assume that we have, as we did in the propositional case a countably infinite set VAR of variables

The variables now have a different meaning than they had in the propositional case

We hence call them variables, or individual variables

We put

$$VAR = \{x_1, x_2, ...\}$$

5. Constants

The constants represent in "real life" concrete elements of sets. We assume that we have a countably. infinite set C of constants

$$\mathbf{C} = \{c_1, c_2, ...\}$$

Alphabet Components

6. Predicate symbols

The predicate symbols represent ”real life” relations

We denote them by P, Q, R, ..., with indices, if necessary

We use symbol **P** for the set of all predicate symbols

We assume that **P** is countably infinite and write

$$\mathbf{P} = \{P_1, P_2, P_3, ...\}$$

Alphabet Components

Logic notation

In "real life" we write symbolically $x < y$ to express that element x is smaller then element y according to the two argument relation $<$

In the **predicate language** $\mathcal{L}$ we represent the relation $<$ as a two argument predicate $P \in \mathbf{P}$

We write $P(x, y)$ as a **representation** of "real life" $x < y$.

The variables x, y in $P(x, y)$ are **individual variables** from the set VAR

Mathematical statements $n < 0,\ 1 < 2,\ 0 < m$ are **represented** in $\mathcal{L}$ by $P(x, c_1),\ P(c_2, c_3),\ P(c_1, y)$, respectively,

where c_1, c_2, c_3 are any **constants** and x, y any **variables**

Alphabet Components

7. Function symbols

The function symbols represent ”real life” functions

We denote function symbols by $f, g, h, ...$, with indices, if necessary

We use symbol **F** for the set of all function symbols

We assume that **F** is countably infinite and write

$$\mathbf{F} = \{f_1, f_2, f_3, ...\}$$

Set **T** of Terms

Definition

Terms are expressions built out of function symbols and variables.

They describe how we build compositions of functions.

We define the set **T** of all terms recursively as follows.

1. All variables are terms;

2. All constants are terms;

3. For any function symbol $f \in \mathbf{F}$ representing a function on n variables, and any terms $t_1, t_2, ..., t_n$, the expression $f(t_1, t_2, ..., t_n)$ is a term;

4. The set **T** of all terms of the predicate language $\mathcal{L}$ is the smallest set that fulfills the conditions **1. - 3.**

Example

Example

Here are some terms of $\mathcal{L}$

$$h(c_1),\ f(g(c,x)),\ g(f(f(c)),g(x,y)),$$

$$f_1(c,g(x,f(c))),\ g(g(x,y),g(x,h(c)))\ ....$$

Observe that to obtain the predicate language **representation** of for example $x+y$ we can first write it as $+(x,y)$ and then replace the addition symbol $+$ by any two argument function symbol $g \in \mathbf{F}$ and get the **term** $g(x,y)$.

Set $\mathcal{F}$ of Formulas

Formulas are build out of elements of the **alphabet** $\mathcal{A}$ and the set $\mathbf{T}$ of all **terms**.

We denote the formulas by $A, B, C, ...$, with indices, if necessary.

We build them, as before in recursive steps.

The first recursive step says:

all atomic formulas are formulas.

The atomic formulas are the simplest formulas, as the propositional variables were in the case of the propositional language.

We define the atomic formulas as follows.

Atomic Formulas

Definition

An **atomic formula** is any expression of the form

$$R(t_1, t_2, ..., t_n),$$

where R is any n-argument predicate $R \in \mathbf{P}$ and $t_1, t_2, ..., t_n$ are terms, i.e. $t_1, t_2, ..., t_n \in \mathbf{T}$.

Some atomic formulas of $\mathcal{L}$ are:

$$Q(c),\ Q(x),\ Q(g(x_1, x_2)),$$

$$R(c, d),\ R(x, f(c)),\ R(g(x, y), f(g(c, z))), ...$$

Set $\mathcal{F}$ of Formulas

Definition

The set $\mathcal{F}$ of formulas of predicate language $\mathcal{L}$ is the smallest set meeting the following conditions.

1. All **atomic formulas** are formulas;

2. If A, B are formulas, then $\neg A, (A \cap B), (A \cup B), (A \Rightarrow B), (A \Leftrightarrow B)$ are formulas;

3. If A is a formula, then $\forall x A$, $\exists x A$ are formulas for any variable $x \in VAR$.

Set $\mathcal{F}$ of Formulas

Example

Some formulas of $\mathcal{L}$ are:

$$R(c,d),\quad \exists y R(y, f(c)),\quad R(x,y),$$

$$(\forall x R(x, f(c)) \Rightarrow \neg R(x,y)),\quad (R(c,d) \cap \forall z R(z, f(c))),$$

$$\forall y R(y,\ g(c, g(x, f(c)))),\quad \forall y \neg \exists x R(x,y)$$

Set $\mathcal{F}$ of Formulas

Let's look now closer at the following formulas.

$$R(c_1, c_2), \quad R(x, y), \quad ((R(y, d) \Rightarrow R(a, z)),$$

$$\exists x R(x, y), \quad \forall y R(x, y), \quad \exists x \forall y R(x, y).$$

Observations

1. Some formulas are without quantifiers:

$$R(c_1, c_2), \quad R(x, y), \quad (R(y, d) \Rightarrow R(a, z)).$$

A formula without quantifiers is called an **open formula**

Variables x, y in $R(x, y)$ are called **free variables**.

The variable y in R(y, d) and z in R(a,z) are also **free**.

Set $\mathcal{F}$ of Formulas

Observations

2. Quantifiers **bind variables** within formulas.

The variable x is bounded by $\exists x$ in the formula $\exists x R(x, y)$, the variable y is free.

The variable y is bounded by $\forall y$ in the formula $\forall y R(x, y)$, the variable y is free.

3. The formula $\exists x \forall y R(x, y)$ **does not** contain any free variables, **neither does** the formula $R(c_1, c_2)$.

4. A formula **without** any free variables is called a **closed formula** or a **sentence**.

Mathematical Statements

We often use logic symbols, while writing mathematical statements in a more symbolic way.
For example, mathematicians to say "all natural numbers are greater then zero and some integers are equal 1" often write

$$x \geq 0, \quad \forall_{x \in N} \quad \text{and} \quad \exists_{y \in Z}, \quad y = 1$$

Some of them who are more "logic oriented" would write it as

$$\forall_{x \in N}\, x \geq 0 \;\cap\; \exists_{y \in Z}\, y = 1,$$

or even as

$$(\forall_{x \in N}\, x \geq 0 \;\cap\; \exists_{y \in Z}\, y = 1).$$

Observe that none of the above symbolic statement are formulas of the predicate language.
These are mathematical statements written with mathematical and logic symbols. They are written with different degree of "logical precision", the last being, from a logician point of view **the most precise**.

Mathematical Statements

Our goal now is to "translate " mathematical and natural language statement into correct formulas of the predicate language $\mathcal{L}$.

Let's start with some **observations**.

O1 The quantifiers in $\forall_{x \in N}, \ \exists_{y \in Z}$ **are not** the one used in logic.

O2 The predicate language $\mathcal{L}$ **admits only** quantifiers $\forall x, \ \exists y$, for any variables $x, y \in VAR$.

O3 The quantifiers $\forall_{x \in N}, \ \exists_{y \in Z}$ are called **quantifiers with restricted domain**.

The **restriction** of the quantifier domain can, and often is given by more complicated statements.

Quantifiers with Restricted Domain

The quantifiers $\forall_{A(x)}$ and $\exists_{A(x)}$ are called quantifiers with **restricted domain** , or **restricted quantifiers** , where $A(x) \in \mathcal{F}$ is any formula with a free variable $x \in VAR$.

Definition

$\forall_{A(x)} B(x)$ stands for a formula $\forall x(A(x) \Rightarrow B(x)) \in \mathcal{F}$.

$\exists_{A(x)} B(x)$ stands for a formula $\exists x(A(x) \cap B(x)) \in \mathcal{F}$.

We write it as the following **transformations rules** for **restricted quantifiers**

$$\forall_{A(x)} B(x) \equiv \forall x(A(x) \Rightarrow B(x))$$

$$\exists_{A(x)} B(x) \equiv \exists x(A(x) \cap B(x))$$

Translations to Formulas of $\mathcal{L}$

Given a mathematical statement **S** written with logical symbols.

We obtain a formula $A \in \mathcal{F}$ that is a **translation** of **S** into $\mathcal{L}$ by conducting a following sequence of steps.

Step 1 We **identify** basic statements in **S**, i.e. mathematical statements that involve only relations. They are to be translated into atomic formulas.

We **identify** the relations in the basic statements and **choose** the predicate symbols as their names.

We **identify** all functions and constants (if any) in the basic statements and **choose** the function symbols and constant symbols as their names.

Step 2 We **write** the basic statements as **atomic formulas** of $\mathcal{L}$.

Translations to Formulas of $\mathcal{L}$

Remember that in the predicate language $\mathcal{L}$ we write a function symbol in front of the function arguments not between them as we write in mathematics.

The same applies to relation symbols.

For example we re-write a basic mathematical statement $x + 2 > y$ as $> (+(x, 2), y)$

Then we write it as an **atomic formula** $P(f(x, c), y)$

$P \in \mathbf{P}$ stands for two argument relation $>$,

$f \in \mathbf{F}$ stands for two argument function $+$,

and $c \in \mathbf{C}$ stands for the number 2.

Translations to Formulas of $\mathcal{L}$

Step 3 We **write** the statement **S** a formula with restricted quantifiers (if needed)

Step 4. We **apply** the **transformations rules** for **restricted quantifiers** to the formula from Step 3 and **obtain** a proper formula A of $\mathcal{L}$ as a result, i.e. as a **translation** of the given mathematical statement **S**

In case of a translation from mathematical statement written without logical symbols **we add** a following step.

Step 0 We **identify** propositional connectives and quantifiers and use them to re-write the statement in a form that is as close to the structure of a logical formula as possible

Translations Examples

Exercise

Given a mathematical statement **S** written with logical symbols

$$(\forall_{x \in N}\, x \geq 0 \;\cap\; \exists_{y \in Z}\, y = 1)$$

1. Translate it into a proper logical formula with restricted quantifiers i.e. into a formula of $\mathcal{L}$ that **uses** the restricted domain quantifiers.

2. Translate your restricted quantifiers formula into a correct formula **without** restricted domain quantifiers, i.e. into a proper formula of $\mathcal{L}$

A **long** and **detailed solution** is given in the book Chapter 2, pages 43-45.

A short statement of the exercise and a short solution follows

Translations Examples

Exercise

Given a mathematical statement **S** written with logical symbols

$$(\forall_{x \in N}\; x \geq 0 \;\cap\; \exists_{y \in Z}\; y = 1)$$

Translate it into a proper formula of $\mathcal{L}$.

Short Solution

The basic statements in **S** are: $x \in N,\; x \geq 0,\; y \in Z, y = 1$

The corresponding atomic formulas of $\mathcal{L}$ are:
$N(x),\; G(x, c_1),\; Z(y),\; E(y, c_2)$, for
$n \in N,\; x \geq 0,\; y \in Z,\; y = 1$, respectively.

The statement **S** becomes restricted quantifiers formula

$$(\forall_{N(x)} G(x, c_1) \;\cap\; \exists_{Z(y)}\; E(y, c_2))$$

By the **transformation rules** we get $A \in \mathcal{F}$:

$$(\forall x(N(x) \Rightarrow G(x, c_1)) \cap \exists y(Z(y) \cap E(y, c_2)))$$

Translations Examples

Exercise

Here is a mathematical statement **S**:

”For all real numbers x the following holds: If $x < 0$, then there is a natural number n, such that $x + n < 0$.”

1. Re-write **S** as a symbolic mathematical statement SF that only uses mathematical and logical symbols.

2. Translate the symbolic statement SF into to a corresponding formula $A \in \mathcal{F}$ of the predicate language $\mathcal{L}$

Translations Examples

Solution

The statement **S** is:

"For all real numbers x the following holds: If $x < 0$, then there is a natural number n, such that $x + n < 0$."

S becomes a symbolic mathematical statement SF

$$\forall_{x \in R}(x < 0 \Rightarrow \exists_{n \in N}\ x + n < 0)$$

We write R(x) for $x \in R$, N(y) for $n \in N$,

We use the constant c for the number 0

We use $L \in \mathbf{P}$ to denote the relation $<$

We use $f \in \mathbf{F}$ to denote the function $+$

The statement $x < 0$ becomes an **atomic formula** L(x, c).

The statement $x + n < 0$ becomes L(f(x,y), c)

Translations Examples

Solution c.d.

The symbolic mathematical statement SF

$$\forall_{x \in R}(x < 0 \Rightarrow \exists_{n \in N}\ x + n < 0)$$

becomes a restricted quantifiers formula

$$\forall_{R(x)}(L(x, c) \Rightarrow \exists_{N(y)} L(f(x, y), c))$$

We apply now the **transformation rules** and get a corresponding formula $A \in \mathcal{F}$:

$$\forall x(N(x) \Rightarrow (L(x, c) \Rightarrow \exists y(N(y) \cap L(f(x, y), c)))$$

PROPOSITIONAL LANGUAGE TRANSLATION

ATTENTION

The **TRANSLATION** of the same mathematical statement **S**:

"For all real numbers x the following holds: If $x < 0$, then there is a natural number n, such that $x + n < 0$."

into a **PROPOSITIONAL LANGUAGE** is a formula

$$a$$

where a is any propositional variable denoting the statement "For all real numbers x the following holds: If $x < 0$, then there is a natural number n, such that $x + n < 0$."

Translations from Natural Language

Exercise

Translate a natural language statement

S: ”Any friend of Mary is a friend of John and Peter is not John’s friend. Hence Peter is not May’s friend”

into a formula $A \in \mathcal{F}$ of the predicate language $\mathcal{L}$.

Solution

1. We identify the basic relations and functions (if any) and **translate** them into atomic formulas

We have only one relation of ”being a friend”.

We **translate** it into an **atomic formula** F(x, y),

where F(x, y) stands for ”x is a friend of y”

Translations from Natural Language

S: ”Any friend of Mary is a friend of John and Peter is not John’s friend. Hence Peter is not Mary’s friend”

We use **constants** m, j, p for Mary, John, and Peter, respectively

We hence have the following **atomic formulas**:

F(x, m), F(x, j), F(p, j), where

F(x, m) stands for ”x is a friend of Mary”,

F(x, j) stands for ”x is a friend of John”, and

F(p, j) stands for ”Peter is a friend of John”

Translations from Natural Language

2. Statement "Any friend of Mary is a friend of John"

translates into a restricted quantifier formula $\forall_{F(x,m)} F(x,j)$

"Peter is not John's friend" **translates** into $\neg F(p,j)$, and

"Peter is not May's friend" **translates** into $\neg F(p,m)$

3. Restricted quantifiers formula for **S** is

$$((\forall_{F(x,m)} F(x,j) \cap \neg F(p,j)) \Rightarrow \neg F(p,m))$$

and the formula $A \in \mathcal{F}$ of $\mathcal{L}$ is

$$((\forall x(F(x,m) \Rightarrow F(x,j)) \cap \neg F(p,j)) \Rightarrow \neg F(p,m))$$

Rules of Translations

Rules of translation from natural language to the predicate language $\mathcal{L}$

1. Identify the basic relations and functions (if any) and **translate** them into **atomic formulas**

2. Identify propositional connectives and use symbols $\neg, \cup, \cap, \Rightarrow, \Leftrightarrow$ for them

3. Identify quantifiers: restricted $\forall_{A(x)}, \exists_{A(x)}$, and non-restricted $\forall x, \exists x$

4. Use the symbols from **1.** - **3.** and restricted quantifiers **transformation rules** to write $A \in \mathcal{F}$ of the predicate language $\mathcal{L}$

Translation Example

Exercise

Given a natural language statement

S: "For any bird one can find some birds that white"

Show that the **translation** of **S** into a formula of the predicate language $\mathcal{L}$ is $\forall x(B(x) \Rightarrow \exists x(B(x) \cap W(x)))$

Solution

We follow the rules of translation to **verify** the correctness of the translation

1. Atomic formulas: B(x), W(x).

B(x) stands for " x is a bird" and W(x) stands for " x is white"

2. There is no propositional connectives in **S**

Translation Example

3. Restricted quantifiers:

$\forall_{B(x)}$ for "any bird " and

$\exists_{B(x)}$ for "one can find some birds".

Restricted quantifiers formula for **S** is

$$\forall_{B(x)} \exists_{B(x)} \, W(x)$$

4. By the **transformation rules** we get a required formula of the predicate language $\mathcal{L}$:

$$\forall x (B(x) \Rightarrow \exists x (B(x) \cap W(x)))$$

Translation Example

Exercise

Translate into $\mathcal{L}$ a natural language statement
S: " Some patients like all doctors."

Solution

1. Atomic formulas: P(x), D(x), L(x, y).

P(x) stands for " x is a patient",

D(x) stands for " x is a doctor", and

L(x,y) stands for " x likes y"

2. There is no propositional connectives in **S**

Translation Example

3. Restricted quantifiers:

$\exists_{P(x)}$ for "some patients " and $\forall_{D(x)}$ for "all doctors"

Observe that we **can't** write L(x, D(y)) for "x likes doctor y"

D(y) is a predicate, not a term, and hence L(x, D(y)) **is not a formula**

We have to express the statement " x likes all doctors y" in terms of restricted quantifiers and the predicate L(x,y) only

Translation Example

Observe that the statement " x likes all doctors y" means also " all doctors y are liked by x"

We can **re- write** it as "for all doctors y, x likes y" what translates to a formula $\forall_{D(y)} L(x, y)$

Hence the statement **S** translates to

$$\exists_{P(x)} \forall_{D(x)} L(x, y)$$

4. By the transformation rules we get the following **translation** of **S** into $\mathcal{L}$

$$\exists x (P(x) \cap \forall y (D(y) \Rightarrow L(x, y)))$$

Chapter 2

Introduction to Classical Logic Languages and Semantics

Part 6: Predicate Tautologies- Laws for Quantifiers

Predicate Tautologies

The notion of predicate tautology is much more **complicated** then that of the propositional

We **define** it formally in later chapters

Predicate tautologies are also called valid formulas, or laws of quantifiers to **distinguish them** from the propositional case

We **provide** here a motivation, examples and an intuitive definitions

We also list and discuss the most used and useful tautologies and equational laws of quantifiers

Interpretation

The formulas of the predicate language $\mathcal{L}$ have a **meaning** only when an **interpretation** is given for its symbols

We **define** the **interpretation** I in a set $U \neq \emptyset$

by **interpreting** predicate and functional symbols of $\mathcal{L}$ as concrete relations and functions defined in the set U

We **interpret** constants symbols as elements of the set U

The set U is called the **universe** of the **interpretation** I.

These two items specify a **model structure** for $\mathcal{L}$

We write it as a pair $\mathbf{M} = (U, I)$

Model Structure

Given a formula A of $\mathcal{L}$, and the **model structure** $\mathbf{M} = (U, I)$

We **denote** by A_I a statement written with logical symbols determined by the formula A and the **interpretation** I in the **universe** U

When A **is a sentence**, it means it is a formula without free variables, A_I **represents** a proposition that is **true** or **false**

When A **is not a sentence**, it contains free variables and may be **satisfied** (i.e. true) for some values in the universe U and **not satisfied** (i.e. false) for the others

Lets look at few simple examples

Examples

Example

Let A be a formula $\exists x P(x, c)$

Consider a **model structure** $\mathbf{M}_1 = (N, I_1)$

The **universe** of the **interpretation** I_1 is the set N of natural numbers

We **define** the **interpretation** I_1 as follows

We **interpret** the two argument predicate P as a relation $=$ and the constant c as number 5, i.e we put

$$P_{I_1} := \quad \text{and} \quad c_{I_1} : 5$$

Examples

The formula A: $\exists x P(x,c)$ under the interpretation I_1 becomes a mathematical statement $\exists x\; x=5$ defined in the set N of natural numbers

We write it for short

$$A_{I_1} : \; \exists_{x \in N}\; x = 5$$

A_{I_1} is obviously a **true** mathematical statement

In this case we say: the formula A: $\exists x P(x,c)$ is **true** under the interpretation I_1 in the universe of $\mathbf{M}_1$, or for short:

A is **true** int the model structure $\mathbf{M}_1$.

We write it **symbolically** as

$$\mathbf{M}_1 \models \exists x P(x,c)$$

and say: $\mathbf{M}_1$ is a **model** for the formula A

Examples

Example

Consider now a **model structure** $\mathbf{M}_2 = (N, I_2)$ and the formula A: $\exists x P(x, c)$.

We interpret now the predicate P as relation $<$ in the set N of natural numbers and the constant c as number 0

We write it as

$$P_{I_2} : < \quad \text{and} \quad c_{I_2} : \ 0$$

Examples

The formula A: $\exists x P(x, c)$ under the interpretation I_2 becomes a mathematical statement $\exists x\ x < 0$ defined in the set N of natural numbers

We write it for short

$$A_{I_2}: \quad \exists_{x \in N}\ x < 0$$

A_{I_2} is obviously a **false** mathematical statement.

We say: the formula A: $\exists x P(x, c)$ is **false** under the interpretation I_2 in $\mathbf{M}_2$, or we say for short: A is **false** in $\mathbf{M}_2$

We write it **symbolically** as

$$\mathbf{M}_2 \not\models \exists x P(x, c)$$

and say that $\mathbf{M}_2$ is a **counter-model** for the formula A

Examples

Example

Consider now a **model structure**

$\mathbf{M}_3 = (Z, I_3)$ and the formula A: $\exists x P(x, c)$

We **define** an interpretation I_3 in the set of all integers Z exactly as the interpretation I_1 was defined, i.e. we put

$$P_{I_3} : < \quad \text{and} \quad c_{I_3} : 0$$

Examples

In this case we get

$$A_{I_3} : \; \exists_{x \in Z} \; x < 0$$

Obviously A_{I_3} is a **true** mathematical statement

The formula A is **true** under the interpretation I_3 in $\mathbf{M}_3$

(A is **satisfied, true** in $\mathbf{M}_3$)

We write it symbolically as

$$\mathbf{M}_3 \models \; \exists x P(x, c)$$

$\mathbf{M}_3$ is yet another **model** for the formula A

Examples

When a formula **is not a closed** (not a sentence) the situation gets more complicated

Given a model structure $\mathbf{M} = (U, I)$,

a formula can be **satisfied** (i.e. true) for some values in the universe U and **not satisfied** (i.e. false) for the others

Examples

Example

Given the formulas:

1. $A_1 : R(x,y)$, **2**. $A_2 : \forall y R(x,y)$, **3**. $A_3 : \exists x \forall y R(x,y)$

Let $\mathbf{M} = (N, I)$ be a **model structure**

where the predicate R is **interpreted** as a relation $\leq$, i.e.

$$R_I : \leq$$

1. Consider the formula $A_1 : R(x,y)$. Obviously,

$$A_{1I} : \ x \leq y$$

and $A_1 : \ R(x,y)$ is **satisfied** in $\mathbf{M} = (N, I)$ by all $n, m \in N$ such that $n \leq m$

Examples

2. Consider the formula $A_2 : \forall y R(x, y)$

By definition of $\mathbf{M} = (N, I)$ we have that

$$R_I : \leq$$

and hence

$$A_{2I} : \forall_{y \in N}\, x \leq y$$

and the formula $A_2 : \forall y R(x, y)$ is **satisfied** in the model structure $\mathbf{M} = (N, I)$ by the natural number 0 only

Examples

3. Consider the formula $A_3 : \exists x \forall y R(x, y)$

3. $A_{3_I} : \exists_{x \in N} \forall_{y \in N}\ x \leq y$ asserts that there is a smallest natural number what is a **true** statement, i.e. **M** is a **model** for A_3

Observe that changing the universe of $\mathbf{M} = (N, I)$ to the set of all Integers Z, we get a different a model structure $\mathbf{M}_1 = (Z, I)$.
In this case $A_{3_I} :\ \exists_{x \in Z} \forall_{y \in Z}\ x \leq y$
asserts that there is a smallest integer and A_3 is a **false** sentence in $\mathbf{M}_1$, i.e. $\mathbf{M}_1$ is a **counter-model** for A_3

Predicate Tautology Definition

We want the predicate language **tautologies** to have the same property as the propositional, namely to be **always true**

In this case, we intuitively agree that it means that we want the **predicate tautologies** to be formulas that are **true** under any interpretation in any possible universe

A rigorous definition of the **predicate tautology** is provided in a later chapter on **Predicate Logic**

Predicate Tautology Definition

We construct the rigorous definition in the following steps.

1. We first define **formally** the notion of interpretation I of symbols of $\mathcal{L}$ in a set $U \neq \emptyset$, i.e. in the **model structure** $\mathbf{M} = (U, I)$ for the predicate language $\mathcal{L}$.

2. Then we define formally a notion " a formula A of $\mathcal{L}$ a is **true** in $\mathbf{M} = (U, I)$"

We write it symbolically

$$\mathbf{M} \models A$$

and call the model structure $\mathbf{M} = (U, I)$ a **model** for A

3. We define a notion "A is a predicate tautology" as follows.

Predicate Tautology Definition

Defintion For any formula A of predicate language $\mathcal{L}$,
A is a **predicate tautology (valid formula)** if and only if

$$\mathbf{M} \models A$$

for all model structures $\mathbf{M} = (U, I)$ for $\mathcal{L}$

4. Directly from the above definition we get the following definition of a notion " A is not a predicate tautology"

Defintion

For any formula A of predicate language $\mathcal{L}$,

A **is not a predicate tautology** if and only if **there is** a model structure $\mathbf{M} = (U, I)$ for $\mathcal{L}$, such that

$$\mathbf{M} \not\models A$$

We call such model structure **M** a **counter-model** for A

Predicate Tautology Definition

The definition of a notion " A is not a predicate tautology" says: to prove that A **is not** a predicate tautology one has to show a **counter- model**

It means one has to **define** a non-empty set U and define an interpretation I, such that we can prove that A_I is **false**

Predicate Tautology Definition

We use terms **predicate tautology** or **valid formula** instead of just saying a tautology in order to distinguish tautologies belonging to two very different languages

For the same reason we usually reserve the symbol $\models$ for **propositional** case

Sometimes we use symbols $\models_p$ or $\models_f$ to **denote** predicate tautologies

p stands for **predicate** and f stands **first order**.

The predicate tautologies are also called laws of quantifiers

We will use both names

Predicate Tautologies Examples

Here are some **examples** of predicate tautologies and **counter models** for formulas that are not tautologies.

Example

For any formula $A(x)$ with a free variable x:

$$\models_p \ (\forall x\, A(x) \Rightarrow \exists x\, A(x))$$

Observe that the formula

$$(\forall x\, A(x) \Rightarrow \exists x\, A(x))$$

represents an infinite number of formulas.

It is a **tautology** for **any** formula $A(x)$ of $\mathcal{L}$ with a free variable x

Predicate Tautologie Examples

The **inverse** implication to $(\forall x\, A(x) \Rightarrow \exists x\, A(x))$ **is not** a predicate tautology, i.e.

$$\not\models_p \ (\exists x\, A(x) \Rightarrow \forall x\, A(x))$$

To **prove it** we have to provide an example of a concrete formula $A(x)$ and construct a **counter-model** $\mathbf{M} = (U, I)$ for the formula $F : (\exists x\, A(x) \Rightarrow \forall x\, A(x))$

Let $A(x)$ be an **atomic** formula $P(x, c)$

We define $\mathbf{M} = (N, I)$ for N set of natural numbers and $P_I : <, \ c_I : \ 3$

The formula F becomes an obviously **false** mathematical statement

$$F_I : (\exists_{n \in N} n < 3 \Rightarrow \forall_{n \in N} n < 3)$$

Restricted Quantifiers Laws

We have to be very careful when we deal with quantifiers with restricted domain.

For example, the most basic **predicate tautology**

$(\forall x\, A(x) \Rightarrow \exists x\, A(x))$ **fails** when written with the restricted domain quantifiers.

Example

We show that $\not\models_p (\forall_{B(x)} A(x) \Rightarrow \exists_{B(x)} A(x))$.

To prove this we have to show that corresponding formula of $\mathcal{L}$ obtained by the restricted quantifiers transformations rules **is not** a predicate tautology, i.e. to prove:

$$\not\models_p (\forall x(B(x) \Rightarrow A(x)) \Rightarrow \exists x(B(x) \cap A(x))).$$

Restricted Quantifiers Laws

We construct a **counter-model M** for the formula

F : $(\forall x(B(x) \Rightarrow A(x)) \Rightarrow \exists x(B(x) \cap A(x)))$ as follows

We take $\mathbf{M} = (N, I)$, where N is the set of natural numbers, we take as $B(x), A(x)$ atomic formulas $Q(x, c), P(x, c),$ and the interpretation I is defined as $Q_I : <,\ \ P_I : >,\ \ c_I :\ 0$

The formula F becomes a mathematical statement

$$F_I : \ (\forall_{n \in N} (x < 0 \Rightarrow n > 0) \Rightarrow \exists_{n \in N}(n < 0 \cap n > 0))$$

F_I is a **false** because the statement $n < 0$ is **false** for all natural numbers and the implication $false \Rightarrow B$ is **true** for any logical value of B

Hence $\forall_{n \in N} (n < 0 \Rightarrow n > 0)$ is a **true** statement and $\exists_{n \in N}(n < 0 \cap n > 0)$ is obviously **false**

Restricted Quantifiers Laws

Restricted Quantifiers Law corresponding to the basic **predicate tautology**

$$(\forall x\, A(x) \Rightarrow \exists x\, A(x))$$

is:

$$\models_p \ (\forall_{B(x)}\, A(x) \Rightarrow (\exists x\, B(x) \Rightarrow \exists_{B(x)}\, A(x))).$$

It means that we **prove** that the corresponding proper formula of $\mathcal{L}$ obtained by the restricted quantifiers transformations rules is a predicate tautology, i.e. we **prove** that

$$\models_p (\forall x(B(x) \Rightarrow A(x)) \Rightarrow (\exists x\, B(x) \Rightarrow \exists x\, (B(x) \cap A(x))))$$

Quantifiers Laws

Another basic predicate tautology called a **dictum de omni** law is:

For any formulas $A(x), A(y)$ with free variables $x, y \in VAR$,

$$\models_p \ (\forall x\, A(x) \Rightarrow A(y))$$

The corresponding **restricted quantifiers law** is:

$$\models_p \ (\forall_{B(x)}\, A(x) \Rightarrow (B(y) \Rightarrow A(y))),$$

where $y \in VAR$

Quantifiers Laws

The next important laws are the **Distributivity Laws**

Distributivity of existential quantifier over conjunction holds only in **one direction**, namely the following is a predicate tautology.

$$\models_p \ (\exists x\,(A(x) \cap B(x)) \ \Rightarrow \ (\exists x A(x) \cap \exists x B(x))),$$

where $A(x), B(x)$ are any formulas with a free variable x

The inverse implication **is not** a predicate tautology, i.e. we have to **find** concrete **formulas** $A(x), B(x) \in \mathcal{F}$ and find a model structure $\mathbf{M} = (U, I)$ with the interpretation I of all predicate, functional, and constant symbols in the formulas $A(x), B(x)$, such that $\mathbf{M} = (U, I)$ is **counter- model** for the inverse implication

$$F: \ ((\exists x A(x) \cap \exists x B(x)) \Rightarrow \exists x\,(A(x) \cap B(x)))$$

Quantifiers Laws

Let F be a formula

$$F: \ ((\exists x A(x) \cap \exists x B(x)) \Rightarrow \exists x \ (A(x) \cap B(x)))$$

Counter - Model for F is as follows

Take $\mathbf{M} = (R, I)$ where R is the set of real numbers.

Let $A(x), B(x)$ be atomic formulas $Q(x, c), \ P(x, c)$

We define the interpretation I as $Q_I : >, \ \ P_I : <, \ \ c_I : 0$.

The formula F becomes an obviously **false** mathematical statement

$$F_I : ((\exists_{x \in R} \ x > 0 \cap \exists_{x \in R} \ x < 0) \Rightarrow \exists_{x \in R} \ (x > 0 \cap x < 0))$$

Quantifiers Laws

Distributivity of universal quantifier over disjunction holds only on **one direction**, namely the following is a predicate tautology for any formulas $A(x), B(x)$ with a free variable x.

$$\models_p \ ((\forall x A(x) \cup \forall x B(x)) \Rightarrow \forall x\ (A(x) \cup B(x))).$$

The inverse implication **is not** a predicate tautology, i.e. **there are** formulas $A(x), B(x)$ with a free variable x, such that

$$\not\models_p \ (\forall x\ (A(x) \cup B(x)) \Rightarrow (\forall x A(x) \cup \forall x B(x)))$$

Quantifiers Laws

It means that we have to find concrete formulas $A(x), B(x) \in \mathcal{F}$ and a model structure $\mathbf{M} = (U, I)$ that is a **counter- model** for the formula

$$F: \ (\forall x \ (A(x) \cup B(x)) \Rightarrow (\forall x A(x) \cup \forall x B(x))).$$

Take $\mathbf{M} = (R, I)$ where R is the set of real numbers, and $A(x), B(x)$ are atomic formulas $Q(x, c), R(x, c)$.

We define $Q_I : \geq, \ R_I : <, \ c_I : 0$.

The formula F becomes an obviously **false** mathematical statement

$$F_I : (\forall_{x \in R} \ (x \geq 0 \cup x < 0) \Rightarrow (\forall_{x \in R} \ x \geq 0 \cup \forall_{x \in R} \ x < 0)).$$

Logical Equivalence

The most frequently used laws of quantifiers have a form of a **ogical equivalence**, symbolically written as $\equiv$.

Remember that $\equiv$ not a new logical connective.

This is a very useful symbol. It says that two formulas always have the same logical value, hence it can be used in the same way we the equality symbol $=$.

Formally we define it as follows.

Definition

For any formulas $A, B \in \mathcal{F}$ of the **predicate language** $\mathcal{L}$,

$$A \equiv B \quad \text{if and only if} \quad \models_p \ (A \Leftrightarrow B).$$

We have also a similar definition for the **propositional language** and **propositional tautology**.

Equational Laws for Quantifiers

De Morgan

For any formula $A(x) \in \mathcal{F}$ with a free variable x,

$$\neg\forall x A(x) \equiv \exists x \neg A(x), \qquad \neg\exists x A(x) \equiv \forall x \neg A(x)$$

Definability

For any formula $A(x) \in \mathcal{F}$ with a free variable x,

$$\forall x A(x) \equiv \neg\exists x \neg A(x), \quad \exists x A(x) \equiv \neg\forall x \neg A(x)$$

Equational Laws for Quantifiers

Renaming the Variables

Let $A(x)$ be any formula with a free variable x

and let y be a variable that **does not occur** in A(x).

Let $A(x/y)$ be a result of **replacement** of each occurrence of x by y, then the following holds.

$$\forall x A(x) \equiv \forall y A(y), \quad \exists x A(x) \equiv \exists y A(y)$$

Alternations of Quantifiers

Let $A(x, y)$ be any formula with a free variables x and y.

$$\forall x \forall y\ (A(x, y) \ \equiv\ \forall y \forall x\ (A(x, y),$$

$$\exists x \exists y\ (A(x, y) \ \equiv\ \exists y \exists x\ (A(x, y)$$

Equational Laws for Quantifiers

Introduction and Elimination Laws

If B is a formula such that B **does not contain** any free occurrence of x, then the following logical equivalences hold.

$$\forall x(A(x) \cup B) \equiv (\forall x A(x) \cup B),$$

$$\exists x(A(x) \cup B) \equiv (\exists x A(x) \cup B),$$

$$\forall x(A(x) \cap B) \equiv (\forall x A(x) \cap B),$$

$$\exists x(A(x) \cap B) \equiv (\exists x A(x) \cap B)$$

Equational Laws for Quantifiers

Introduction and Elimination Laws

If B is a formula such that B **does not contain** any free occurrence of x, then the following logical equivalences hold.

$$\forall x(A(x) \Rightarrow B) \equiv (\exists x A(x) \Rightarrow B),$$

$$\exists x(A(x) \Rightarrow B) \equiv (\forall x A(x) \Rightarrow B),$$

$$\forall x(B \Rightarrow A(x)) \equiv (B \Rightarrow \forall x A(x)),$$

$$\exists x(B \Rightarrow A(x)) \equiv (B \Rightarrow \exists x A(x))$$

Equational Laws for Quantifiers

Distributivity Laws

Let $A(x), B(x)$ be any formulas with afree variable x.

Distributivity of **universal quantifier** over **conjunction**.

$$\forall x\,(A(x) \cap B(x)) \equiv (\forall x A(x) \cap \forall x B(x))$$

Distributivity of **existential quantifier** over **disjunction**.

$$\exists x\,(A(x) \cup B(x)) \equiv (\exists x A(x) \cup \exists x B(x))$$

Equational Laws for Quantifiers

We also define the notion of logical equivalence $\equiv$ for the formulas of the **propositional language** and its semantics

For any formulas $A, B \in \mathcal{F}$ of the **propositional language** $\mathcal{L}$,

$$A \equiv B \quad \text{if and only if} \quad \models (A \Leftrightarrow B)$$

Moreover, we prove that any substitution of **propositional tautology** by a formulas of the predicate language is a **predicate tautology**

The same holds for the logical equivalence

Equational Laws for Quantifiers

In particular, we transform the **propositional tautologies** into the following corresponding predicate equivalences.

For any formulas A, B of the **predicate language** $\mathcal{L}$,

$$(A \Rightarrow B) \equiv (\neg A \cup B),$$

$$\neg\neg A \equiv A$$

We use them to prove the following De Morgan Laws for **restricted quantifiers**.

Equational Laws for Quantifiers

Restricted De Morgan

For any formulas $A(x), B(x) \in \mathcal{F}$ with a free variable x,

$$\neg\forall_{B(x)} A(x) \equiv \exists_{B(x)} \neg A(x), \qquad \neg\exists_{B(x)} A(x) \equiv \forall_{B(x)} \neg A(x).$$

Here is a poof of first equality. The proof of the second one is similar and is left as an exercise.

$$\neg\forall_{B(x)} A(x) \equiv \neg\forall x\,(B(x) \Rightarrow A(x))$$

$$\equiv \neg\forall x\,(\neg B(x) \cup A(x))$$

$$\equiv \exists x\,\neg(\neg B(x) \cup A(x)) \equiv \exists x\,(\neg\neg B(x) \cap \neg A(x))$$

$$\equiv \exists x\,(B(x) \cap \neg A(x)) \equiv \exists_{B(x)} \neg A(x).$$