Unit 9, Part 2

Search Trees

Computer Science S-111
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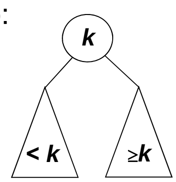
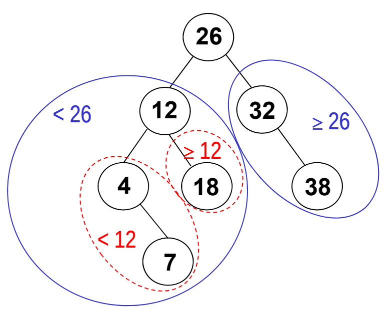
Binary *Search* Trees

- Search-tree property: for each node *k* (*k* is the key):
  - all nodes in *k*'s left subtree are < *k*
  - all nodes in *k*'s right subtree are >= *k*



- Our earlier binary-tree example is a search tree:



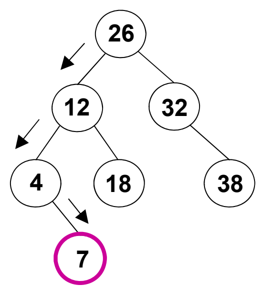
- With a search tree, an inorder traversal visits the nodes in order!
  - in order of increasing key values

Searching for an Item in a Binary Search Tree

- Algorithm for searching for an item with a key *k*:

  if *k* == the root node's key, you're done
  else if *k* < the root node's key, search the left subtree
  else search the right subtree

- Example: search for 7



Implementing Binary-Tree Search

```
public class LinkedTree {    // Nodes have keys that are ints
    …
    private Node root;

    public LLList search(int key) {      // "wrapper method"
        Node n = searchTree(root, key);  // get Node for key
        if (n == null) {
            return null;     // no such key
        } else {
            return n.data;   // return list of values for key
        }
    }

    private static Node searchTree(Node root, int key) {
        if (                ) {
                                               two base cases
        } else if (               ) {          (order matters!)

        } else if (               ) {
                                               two
        } else {                               recursive cases

        }
    }
```

Inserting an Item in a Binary Search Tree

- `public void insert(int key, Object data)`
  will add a new (key, data) pair to the tree

- Example 1: a search tree containing student records
  - key = the student's ID number (an integer)
  - data = a string with the rest of the student record
  - we want to be able to write client code that looks like this:

```
LinkedTree students = new LinkedTree();
students.insert(23, "Jill Jones,sophomore,comp sci");
students.insert(45, "Al Zhang,junior,english");
```

- Example 2: a search tree containing scrabble words
  - key = a scrabble score (an integer)
  - data = a word with that scrabble score

```
LinkedTree tree = new LinkedTree();
tree.insert(4, "lost");
```

Inserting an Item in a Binary Search Tree (cont.)

- To insert an item (*k*, *d*), we start by searching for *k*.

- If we find a node with key *k*, we add *d* to the list of data values for that node.
  - example: `tree.insert(4, "sail")`

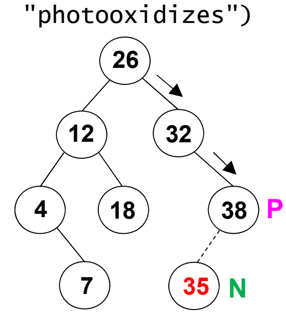
- If we don't find *k*, the last node seen in the search becomes the parent **P** of the new node **N**.
  - if *k* < **P**'s key, make **N** the left child of **P**
  - else make **N** the right child of **P**

- *Special case:* if the tree is empty, make the new node the root of the tree.

- ***Important:*** The resulting tree is still a search tree!

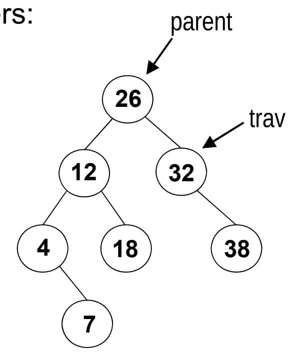
example:

```
tree.insert(35,
  "photooxidizes")
```



Implementing Binary-Tree Insertion

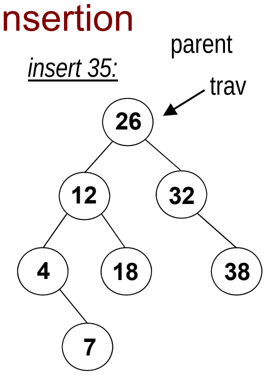
- We'll implement part of the `insert()` method together.
- We'll use iteration rather than recursion.
- Our method will use two references/pointers:
  - `trav`: performs the traversal down to the point of insertion
  - `parent`: stays one behind `trav`
    - like the `trail` reference that we sometimes use when traversing a linked list



Implementing Binary-Tree Insertion

```
public void insert(int key, Object data) {
    Node parent = null;
    Node trav = root;
    while (trav != null) {
        if (trav.key == key) {
            trav.data.addItem(data, 0);
            return;
        }
        // what should go here?



    }
    Node newNode = new Node(key, data);
    if (root == null) {   // the tree was empty
        root = newNode;
    } else if (key < parent.key) {
        parent.left = newNode;
    } else {
        parent.right = newNode;
    }
}
```

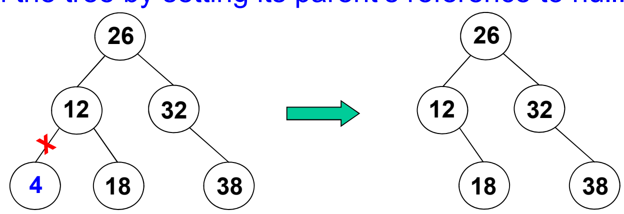
insert 35:



Deleting Items from a Binary Search Tree

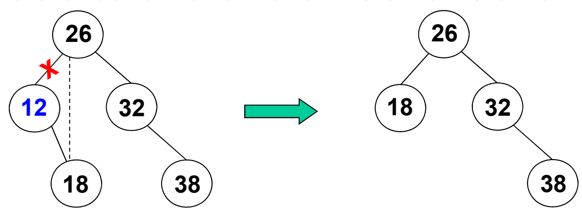
- Three cases for deleting a node *x*
- **Case 1:** *x* has no children.
  Remove *x* from the tree by setting its parent's reference to null.

ex: delete 4



- **Case 2:** *x* has one child.
  Take the parent's reference to *x* and make it refer to *x*'s child.

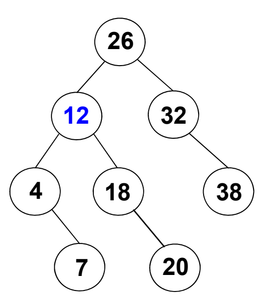
ex: delete 12



Deleting Items from a Binary Search Tree (cont.)

- **Case 3:** *x* has two children
  - we can't give both children to the parent. why?
  - instead, we leave *x's* node where it is, and we replace its key and data with those from another node
    - the replacement must maintain the search-tree inequalities

ex:
delete 12

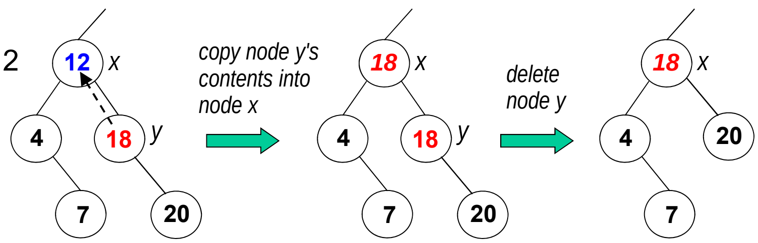


<u>two options:</u> *which ones?*

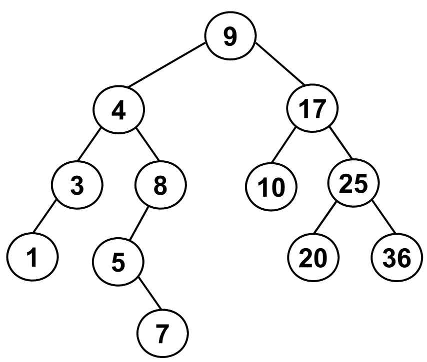
Deleting Items from a Binary Search Tree (cont.)

- **Case 3:** *x* has two children (continued):
  - replace *x*'s key and data with those from the smallest node in *x*'s right subtree—call it *y*
  - we then delete *y*
    - it will either be a leaf node or will have one right child. why?
    - thus, we can delete it using case 1 or 2

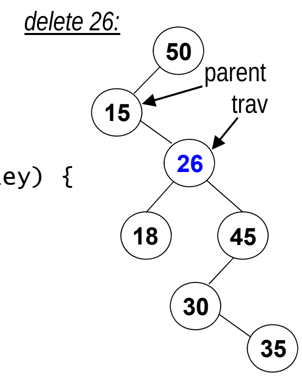
ex:
delete 12



Which Nodes Could We Use To Replace 9?



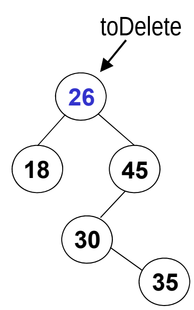
Implementing Deletion




```
public LLList delete(int key) {
    // Find the node and its parent.
    Node parent = null;
    Node trav = root;
    while (trav != null && trav.key != key) {
        parent = trav;
        if (key < trav.key) {
            trav = trav.left;
        } else {
            trav = trav.right;
        }
    }

    // Delete the node (if any) and return the removed items.
    if (trav == null) {   // no such key
        return null;
    } else {
        LLList removedData = trav.data;
        deleteNode(trav, parent);   // call helper method
        return removedData;
    }
}
```

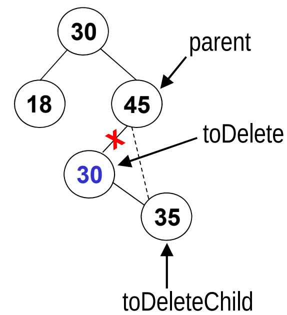
Implementing Case 3

```
private void deleteNode(Node toDelete, Node parent) {
    if (toDelete.left != null && toDelete.right != null) {
        // Find a replacement - and
        // the replacement's parent.
        Node replaceParent = toDelete;

        // Get the smallest item
        // in the right subtree.
        Node replace = toDelete.right;
        // what should go here?



        // Replace toDelete's key and data
        // with those of the replacement item.
        toDelete.key = replace.key;
        toDelete.data = replace.data;

        // Recursively delete the replacement
        // item's old node. It has at most one
        // child, so we don't have to
        // worry about infinite recursion.
        deleteNode(replace, replaceParent);
    } else {
        ...
}
```

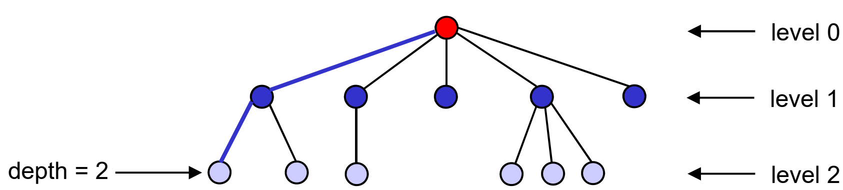


Implementing Cases 1 and 2

```
private void deleteNode(Node toDelete, Node parent) {
    if (toDelete.left != null && toDelete.right != null) {
        ...
    } else {
        Node toDeleteChild;
        if (toDelete.left != null)
            toDeleteChild = toDelete.left;
        else
            toDeleteChild = toDelete.right;
        // Note: in case 1, toDeleteChild
        // will have a value of null.

        if (toDelete == root)
            root = toDeleteChild;
        else if (toDelete.key < parent.key)
            parent.left = toDeleteChild;
        else
            parent.right = toDeleteChild;
    }
}
```



Recall: Path, Depth, Level, and Height



- There is exactly one *path* (one sequence of edges) connecting each node to the root.
- *depth* of a node = # of edges on the path from it to the root
- Nodes with the same depth form a *level* of the tree.
- The *height* of a tree is the maximum depth of its nodes.
    - example: the tree above has a height of 2

Efficiency of a Binary Search Tree

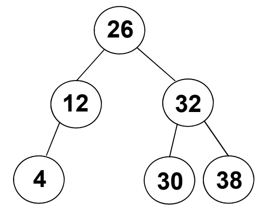
- For a tree containing *n* items, what is the efficiency of any of the traversal algorithms?
  - you process all *n* of the nodes
  - you perform *O*(1) operations on each of them
- Search, insert, and delete all have the same time complexity.
  - insert is a search followed by *O*(1) operations
  - delete involves either:
    - a search followed by *O*(1) operations (cases 1 and 2)
    - a search partway down the tree for the item, followed by a search further down for its replacement, followed by *O*(1) operations (case 3)

Efficiency of a Binary Search Tree (cont.)

- Time complexity of searching:
  - best case:
  - worst case:
    - you have to go all the way down to level *h* before finding the key or realizing it isn't there
    - along the path to level *h*, you process *h* + 1 nodes
  - average case:
- What is the height of a tree containing *n* items?

Balanced Trees

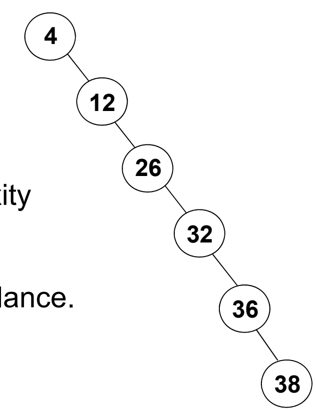
- A tree is *balanced* if, for *each* of its nodes, the node's subtrees have the same height or have heights that differ by 1.
  - example:
    - 26: both subtrees have a height of 1
    - 12: left subtree has height 0
      right subtree is empty (height = -1)
    - 32: both subtrees have a height of 0
    - all leaf nodes: both subtrees are empty



- For a balanced tree with n nodes, height = $O(\log n)$
  - each time that you follow an edge down the longest path, you cut the problem size roughly in half!
- Therefore, for a *balanced* binary search tree, the worst case for search / insert / delete is $O(h) = O(\log n)$
  - the "best" worst-case time complexity

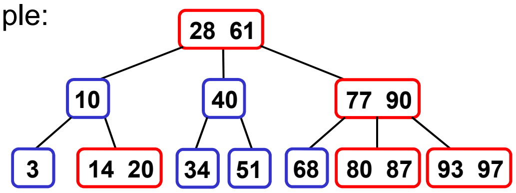
What If the Tree Isn't Balanced?

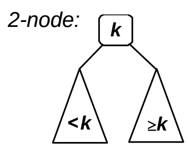
- Extreme case: the tree is equivalent to a linked list
  - height = $n - 1$
- Therefore, for a unbalanced binary search tree, the worst case for search / insert / delete is $O(h) = O(n)$
  - the "worst" worst-case time complexity
- We'll look next at search-tree variants that take special measures to ensure balance.

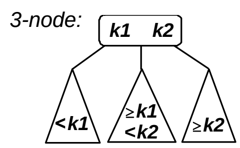


2-3 Trees

- A 2-3 tree is a balanced tree in which:
  - *all* nodes have equal-height subtrees (perfect balance)
  - each node is either
    - a **2-node**, which contains one data item and 0 or 2 children
    - a **3-node**, which contains two data items and 0 or 3 children
  - the keys form a search tree
- Example:



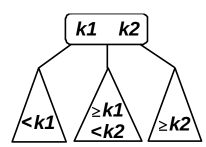




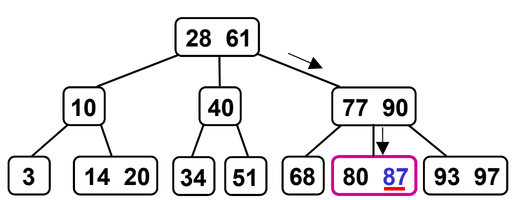
Search in 2-3 Trees

- Algorithm for searching for an item with a key *k*:

  if *k* == one of the root node's keys, you're done
  else if *k* < the root node's first key
      search the left subtree
  else if the root is a 3-node and k < its second key
      search the middle subtree
  else
      search the right subtree



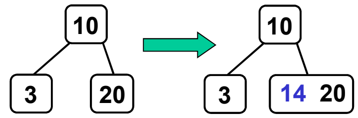
- Example: search for 87



Insertion in 2-3 Trees

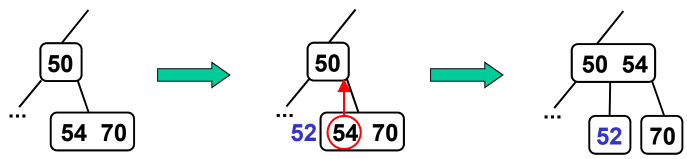
- Algorithm for inserting an item with a key *k*:

  search for *k,* but don't stop until you hit a leaf node
  let L be the leaf node at the end of the search
  if L is a 2-node
   add *k* to L, making it a 3-node



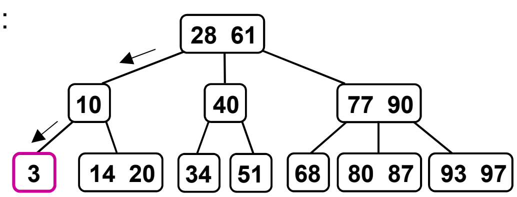
  else if L is a 3-node
   split L into two 2-nodes containing the items with the smallest and largest of: *k,* L's 1st key, L's 2nd key
   the middle item is "sent up" and inserted in L's parent

  example: add 52

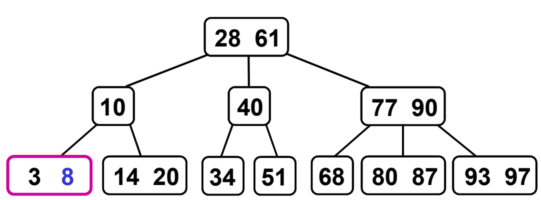


Example 1: Insert 8

- Search for 8:

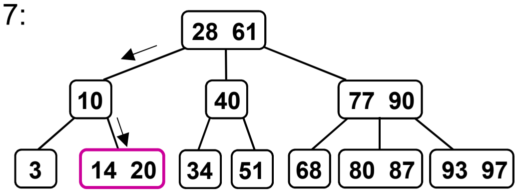


- Add 8 to the leaf node, making it a 3-node:

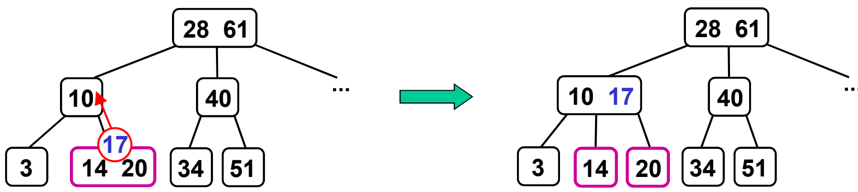


Example 2: Insert 17

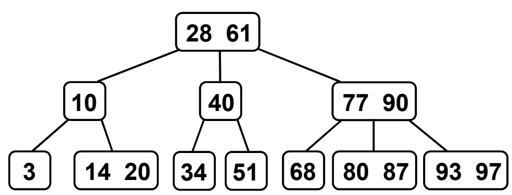
- Search for 17:



- Split the leaf node, and send up the middle of 14, 17, 20 and insert it the leaf node's parent:



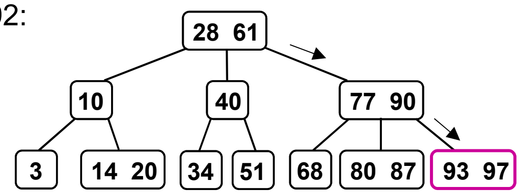
Example 3: Insert 92



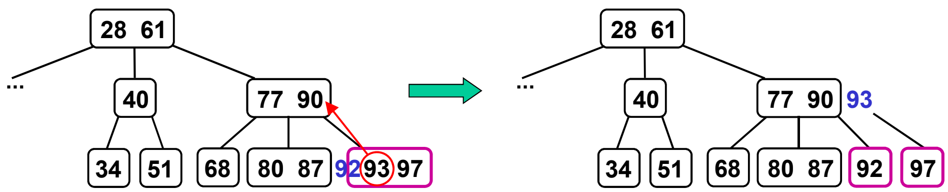
- In which node will we initially try to insert it?

Example 3: Insert 92

- Search for 92:



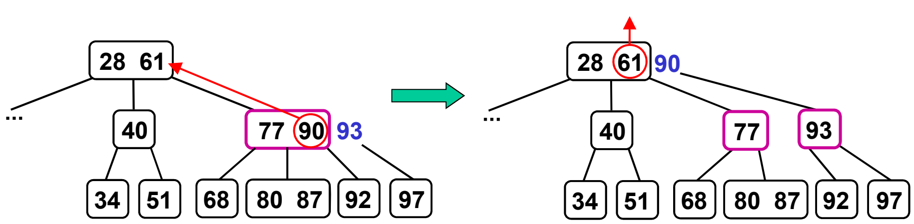
- Split the leaf node, and send up the middle of 92, 93, 97 and insert it the leaf node's parent:



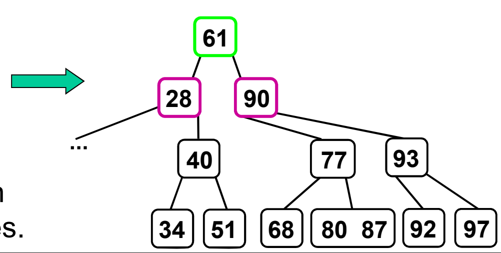
- In this case, the leaf node's parent is also a 3-node, so we need to split is as well…

Example 3 (cont.)

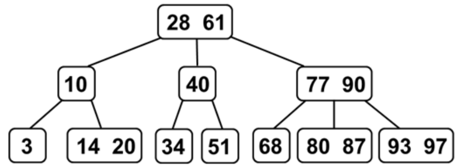
- We split the [77 90] node and we send up the middle of 77, 90, 93:
- We try to insert it in the root node, but the root is also full!



- Then we split the root, which increases the tree's height by 1, but the tree is still balanced.
- This is only case in which the tree's height increases.



Efficiency of 2-3 Trees



- A 2-3 tree containing n items has a height h <= $\log_2 n$.
- Thus, search and insertion are both $O(\log n)$.
  - search visits at most h + 1 nodes
  - insertion visits at most 2h + 1 nodes:
    - starts by going down the full height
    - in the worst case, performs splits all the way back up to the root
- Deletion is tricky – you may need to coalesce nodes!
  However, it also has a time complexity of $O(\log n)$.
- Thus, we can use 2-3 trees for a $O(\log n)$-time data dictionary!

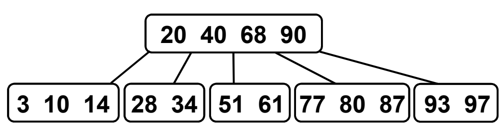
External Storage

- The balanced trees that we've covered don't work well if you want to store the data dictionary externally – i.e., on disk.
- Key facts about disks:
  - data is transferred to and from disk in units called *blocks,* which are typically 4 or 8 KB in size
  - disk accesses are slow!
    - reading a block takes ~10 milliseconds (10^{-3} sec)
    - vs. reading from memory, which takes ~10 nanoseconds
    - in 10 ms, a modern CPU can perform millions of operations!

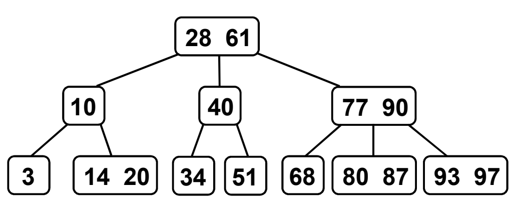
B-Trees

- A B-tree of order *m* is a tree in which each node has:
  - at most 2*m* entries (and, for internal nodes, 2*m* + 1 children)
  - at least *m* entries (and, for internal nodes, *m* + 1 children)
  - exception: the root node may have as few as 1 entry
  - a 2-3 tree is essentially a B-tree of order 1
- To minimize the number of disk accesses, we make *m* as large as possible.
  - each disk read brings in more items
  - the tree will be shorter (each level has more nodes), and thus searching for an item requires fewer disk reads
- A large value of *m* doesn't make sense for a memory-only tree, because it leads to many key comparisons per node.
- These comparisons are less expensive than accessing the disk, so large values of *m* make sense for on-disk trees.

Example: a B-Tree of Order 2



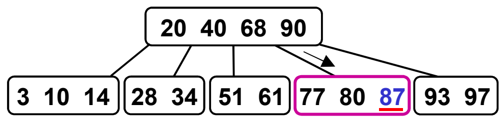
- *m* = 2: at most 2*m* = 4 items per node (and at most 5 children)
  at least m = 2 items per node (and at least 3 children)
  (except the root, which could have 1 item)
- The above tree holds the same keys this 2-3 tree:



- We used the same order of insertion to create both trees:
  51, 3, 40, 77, 20, 10, 34, 28, 61, 80, 68, 93, 90, 97, 87, 14

Search in B-Trees

- Similar to search in a 2-3 tree.
- Example: search for 87



Insertion in B-Trees

- Similar to insertion in a 2-3 tree:

  search for the key until you reach a leaf node

  if a leaf node has fewer than 2*m* items, add the item to the leaf node

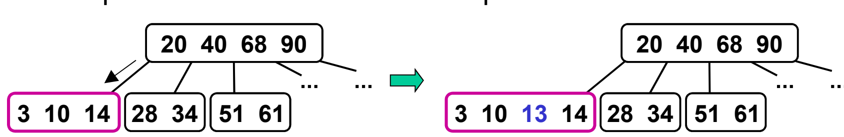
  else split the node, dividing up the 2*m* + 1 items:

    the smallest *m* items remain in the original node

    the largest *m* items go in a new node

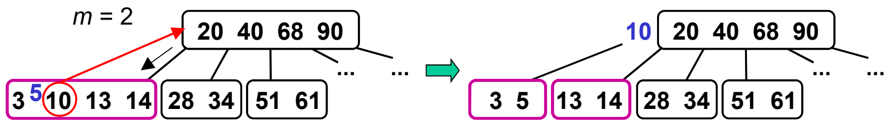
    send the middle entry up and insert it (and a pointer to the new node) in the parent

- Example of an insertion without a split: insert 13

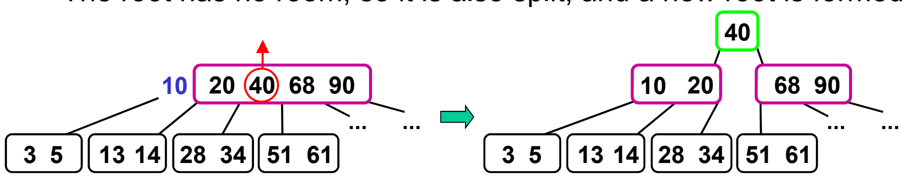


Splits in B-Trees

- Insert 5 into the result of the previous insertion:

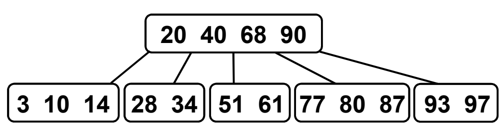


- The middle item (the 10) is sent up to the root.
  The root has no room, so it is also split, and a new root is formed:



- Splitting the root increases the tree's height by 1, but the tree is still balanced. This is only way that the tree's height increases.
- When an internal node is split, its 2*m* + 2 pointers are split evenly between the original node and the new node.

Analysis of B-Trees



- All internal nodes have at least *m* children (actually, at least *m*+1).
- Thus, a B-tree with n items has a height <= $\log_m n$, and search and insertion are both $O(\log_m n)$.
- As with 2-3 trees, deletion is tricky, but it's still logarithmic.

Search Trees: Conclusions

- Binary search trees can be $O(\log n)$, but they can degenerate to $O(n)$ running time if they are out of balance.
- 2-3 trees and B-trees are *balanced* search trees that guarantee $O(\log n)$ performance.
- When data is stored on disk, the most important performance consideration is reducing the number of disk accesses.
- B-trees offer improved performance for on-disk data dictionaries.