

cse371/math371

LOGIC

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LECTURE 3e

Chapter 3 REVIEW

Some Definitions and Problems

SOME DEFINITIONS: Part One

There are some basic **DEFINITIONS** from Chapter 3

You have to KNOW them for Q1 and MIDTERM

I could ask you to **WRITE** down a full, correct text of 1-3 of them - in EXACTLY the same form as they are presented here, or their particular case

Knowing all basic **Definitions** is the first step for understanding the material

DEFINITIONS: Propositional Extensional Semantics

Definition 1

Given a propositional language \mathcal{L}_{CON} for the set $CON = C_1 \cup C_2$, where C_1, C_2 are respectively the sets of **unary** and **binary** connectives

Let V be a non-empty set of **logical values**

Connectives $\nabla \in C_1, \circ \in C_2$ are called **extensional** iff their semantics is defined by respective functions

$$\nabla : V \longrightarrow V \quad \text{and} \quad \circ : V \times V \longrightarrow V$$

DEFINITIONS: Propositional Extensional Semantics

Definition 2

Formal definition of a **propositional extensional semantics** for a given language \mathcal{L}_{CON} consists of providing **definitions** of the following four main components:

1. Logical Connectives
2. Truth Assignment
3. Satisfaction, Model, Counter-Model
4. Tautology

CLASSICAL PROPOSITIONAL SEMANTICS

DEFINITIONS: Truth Assignment Extension v^*

Definition 3

The Language: $\mathcal{L} = \mathcal{L}_{\{\neg, \Rightarrow, \cup, \cap\}}$

Given the **truth assignment** $v : VAR \rightarrow \{T, F\}$ in **classical semantics** for the language $\mathcal{L} = \mathcal{L}_{\{\neg, \Rightarrow, \cup, \cap\}}$

We define its **extension** v^* to the set \mathcal{F} of all formulas of \mathcal{L} as $v^* : \mathcal{F} \rightarrow \{T, F\}$ such that

(i) for any $a \in VAR$

$$v^*(a) = v(a)$$

(ii) and for any $A, B \in \mathcal{F}$ we put

$$v^*(\neg A) = \neg v^*(A);$$

$$v^*((A \cap B)) = \cap(v^*(A), v^*(B));$$

$$v^*((A \cup B)) = \cup(v^*(A), v^*(B));$$

$$v^*((A \Rightarrow B)) = \Rightarrow(v^*(A), v^*(B));$$

$$v^*((A \Leftrightarrow B)) = \Leftrightarrow(v^*(A), v^*(B))$$

DEFINITIONS: Truth Assignment Extension v^* Revisited

Notation

For **binary connectives** (two argument functions) we adopt a convention to write the **symbol of the connective** (name of the 2 argument function) **between its arguments** as we do in a case **arithmetic operations**

The **condition (ii)** of the definition of the extension v^* can be hence **written** as follows

(ii) and for any $A, B \in \mathcal{F}$ we put

$$v^*(\neg A) = \neg v^*(A);$$

$$v^*((A \cap B)) = v^*(A) \cap v^*(B);$$

$$v^*((A \cup B)) = v^*(A) \cup v^*(B);$$

$$v^*((A \Rightarrow B)) = v^*(A) \Rightarrow v^*(B);$$

$$v^*((A \Leftrightarrow B)) = v^*(A) \Leftrightarrow v^*(B)$$

DEFINITIONS: Satisfaction Relation

Definition 4 Let $v : VAR \rightarrow \{T, F\}$

We say that

v **satisfies** a formula $A \in \mathcal{F}$ iff $v^*(A) = T$

Notation: $v \models A$

We say that

v **does not satisfy** a formula $A \in \mathcal{F}$ iff $v^*(A) \neq T$

Notation: $v \not\models A$

DEFINITIONS: Model, Counter-Model, Classical Tautology

Definition 5

Given a formula $A \in \mathcal{F}$ and $v : VAR \rightarrow \{T, F\}$

We say that

v is a **model** for A iff $v \models A$

v is a **counter-model** for A iff $v \not\models A$

Definition 6

A is a **tautology** iff for any $v : VAR \rightarrow \{T, F\}$ we have that $v \models A$

Notation

We write symbolically $\models A$ to denote that A is a **classical tautology**

DEFINITIONS: Restricted Truth Assignments

Notation: for any formula A , we denote by VAR_A a set of all variables that appear in A

Definition 7 Given a formula $A \in \mathcal{F}$, any function

$$v_A : VAR_A \longrightarrow \{T, F\}$$

is called a truth assignment restricted to A

DEFINITIONS: Restricted Model, Counter Model

Notation: for any formula A , we denote by VAR_A a set of all variables that appear in A

Definition 8 Given a formula $A \in \mathcal{F}$
Any function

$$w : VAR_A \longrightarrow \{T, F\} \quad \text{such that} \quad w^*(A) = T$$

is called a **restricted MODEL** for A

Any function

$$w : VAR_A \longrightarrow \{T, F\} \quad \text{such that} \quad w^*(A) \neq T$$

is called a **restricted Counter- MODEL** for A

DEFINITIONS: Models for Sets of Formulas

Consider $\mathcal{L} = \mathcal{L}_{\{\neg, \cup, \cap, \Rightarrow\}}$ and let $S \neq \emptyset$ be any non empty set of formulas of \mathcal{L} , i.e.

$$S \subseteq \mathcal{F}$$

Definition 9

A truth assignment $v : VAR \rightarrow \{T, F\}$ is a **model for the set** S of formulas if and only if

$$v \models A \text{ for all formulas } A \in S$$

We write

$$v \models S$$

to denote that v is a **model for the set** S of formulas

DEFINITIONS: Consistent Sets of Formulas

Definition 10

A non-empty set $\mathcal{G} \subseteq \mathcal{F}$ of **formulas** is called **consistent** if and only if \mathcal{G} **has a model**, i.e. we have that

$\mathcal{G} \subseteq \mathcal{F}$ is **consistent** if and only if
there is v such that $v \models \mathcal{G}$

Otherwise \mathcal{G} is called **inconsistent**

DEFINITIONS: Independent Statements

Definition 11

A formula A is called **independent** from a non-empty set $\mathcal{G} \subseteq \mathcal{F}$

if and only if **there are** truth assignments v_1, v_2 such that

$$v_1 \models \mathcal{G} \cup \{A\} \text{ and } v_2 \models \mathcal{G} \cup \{\neg A\}$$

i.e. we say that a formula A is **independent**

if and only if

$$\mathcal{G} \cup \{A\} \text{ and } \mathcal{G} \cup \{\neg A\} \text{ are consistent}$$

Many Valued Extensional Semantics **M**

DEFINITIONS: Semantics **M**

Definition 11

The extensional semantics **M** is defined for a non-empty set of **V** of **logical values of any cardinality**

We only **assume** that the set **V** of logical values of **M** always has a special, distinguished logical value which serves to define a **notion of tautology**

We denote this distinguished value as **T**

Formal definition of **many valued extensional semantics **M**** for the language **\mathcal{L}_{CON}** consists of giving **definitions** of the following main components:

1. **Logical Connectives** under semantics **M**
2. **Truth Assignment** for **M**
3. **Satisfaction Relation, Model, Counter-Model** under semantics **M**
4. **Tautology** under semantics **M**

Definition of **M** - Extensional Connectives

Given a propositional language \mathcal{L}_{CON} for the set $CON = C_1 \cup C_2$, where C_1 is the set of all **unary** connectives, and C_2 is the set of all **binary** connectives

Let V be a non-empty set of **logical values** adopted by the semantics **M**

Definition 12

Connectives $\nabla \in C_1$, $\circ \in C_2$ are called **M-extensional** iff their semantics **M** is defined by respective functions

$$\nabla : V \longrightarrow V \quad \text{and} \quad \circ : V \times V \longrightarrow V$$

DEFINITION: Definability of Connectives under a semantics **M**

Given a propositional language \mathcal{L}_{CON} and its **extensional semantics M**

We adopt the following definition

Definition 13

A connective $\circ \in CON$ is **definable** in terms of some connectives $\circ_1, \circ_2, \dots, \circ_n \in CON$ for $n \geq 1$ **under the semantics M** if and only if the connective \circ is a certain function composition of functions $\circ_1, \circ_2, \dots, \circ_n$ as they are **defined by the semantics M**

DEFINITION: **M** Truth Assignment Extension v^* to \mathcal{F}

Definition 14

Given the **M** truth assignment $v : VAR \rightarrow V$

We define its **M extension** v^* to the set \mathcal{F} of all formulas of \mathcal{L} as any function $v^* : \mathcal{F} \rightarrow V$, such that the following conditions are satisfied

- (i) for any $a \in VAR$

$$v^*(a) = v(a);$$

- (ii) For any connectives $\nabla \in C_1$, $\circ \in C_2$ and for any formulas $A, B \in \mathcal{F}$ we put

$$v^*(\nabla A) = \nabla v^*(A)$$

$$v^*((A \circ B)) = \circ(v^*(A), v^*(B))$$

DEFINITION: **M** Satisfaction, Model, Counter Model, Tautology

Definition 15 Let $v : VAR \rightarrow V$

Let $T \in V$ be the **distinguished logical value**

We say that

v **M satisfies** a formula $A \in \mathcal{F}$ ($v \models_M A$) iff
 $v^*(A) = T$

Definition 16

Given a formula $A \in \mathcal{F}$ and $v : VAR \rightarrow V$

Any v such that $v \models_M A$ is called a **M model** for A

Any v such that $v \not\models_M A$ is called a **M counter model** for A

A is a **M tautology** ($\models_M A$) iff $v \models_M A$, for all
 $v : VAR \rightarrow V$

CHAPTER 3: Some Questions

Chapter 3: Question 1

Question 1

1. Find a **restricted model** for formula **A**, where

$$A = (\neg a \Rightarrow (\neg b \cup (b \Rightarrow \neg c)))$$

You **can't use** **short-hand notation**

Show each step of solution

Solution

For any formula **A**, we denote by VAR_A a set of **all variables that appear in A**

In our case we have $VAR_A = \{a, b, c\}$

Any function $v_A : VAR_A \longrightarrow \{T, F\}$ is called a **truth assignment restricted to A**

Chapter 3: Question 1

Let $v : VAR \rightarrow \{T, F\}$ be any truth assignment such that

$$v(a) = v_A(a) = T, v(b) = v_A(b) = T, v(c) = v_A(c) = F$$

We evaluate the value of the **extension** v^* of v on the formula A as follows

$$\begin{aligned} v^*(A) &= v^*((\neg a \Rightarrow (\neg b \cup (b \Rightarrow \neg c)))) \\ &= v^*(\neg a) \Rightarrow v^*((\neg b \cup (b \Rightarrow \neg c))) \\ &= \neg v^*(a) \Rightarrow (v^*(\neg b) \cup v^*((b \Rightarrow \neg c))) \\ &= \neg v(a) \Rightarrow (\neg v(b) \cup (v(b) \Rightarrow \neg v(c))) \\ &= \neg v_A(a) \Rightarrow (\neg v_A(b) \cup (v_A(b) \Rightarrow \neg v_A(c))) \\ &(\neg T \Rightarrow (\neg T \cup (T \Rightarrow \neg F))) = F \Rightarrow (F \cup T) = F \Rightarrow T = T, \text{ i.e.} \end{aligned}$$

$$v_A \models A \quad \text{and} \quad v \models A$$

Chapter 3: Question 2

Question 2

1. Find a **restricted model** and a **restricted counter-model** for **A**, where

$$A = (\neg a \Rightarrow (\neg b \cup (b \Rightarrow \neg c)))$$

You **can use short-hand notation**. Show work

Solution

Notation: for any formula **A**, we denote by **VAR_A** a set of **all variables that appear in A**

In our case we have **VAR_A = {a, b, c}**

Any function **v_A : VAR_A → {T, F}** is called a **truth assignment restricted to A**

We define now **v_A(a) = T, v_A(b) = T, v_A(c) = F**, in shorthand: **a = T, b = T, c = F** and evaluate

(¬T ⇒ (¬T ∪ (T ⇒ ¬F))) = F ⇒ (F ∪ T) = F ⇒ T = T, i.e.

$$v_A \models A$$

Chapter 3: Question 2

Observe that

$(\neg a \Rightarrow (\neg b \cup (b \Rightarrow \neg c))) = T$ when $a = T$ and b, c any truth values as by definition of implication we have that $F \Rightarrow \text{anything} = T$

Hence $a = T$ gives us 4 models as we have 2^2 possible values on b and c

Chapter 3: Question 2

We take as a **restricted counter-model**: $a=F$, $b=T$ and $c=T$

Evaluation: observe that

$(\neg a \Rightarrow (\neg b \cup (b \Rightarrow \neg c))) = F$ if and only if

$\neg a = T$ and $(\neg b \cup (b \Rightarrow \neg c)) = F$ if and only if

$a = F$, $\neg b = F$ and $(b \Rightarrow \neg c) = F$ if and only if

$a = F$, $b = T$ and $(T \Rightarrow \neg c) = F$ if and only if

$a = F$, $b = T$ and $\neg c = F$ if and only if

$a = F$, $b = T$ and $c = T$

The above proves also that $a=F$, $b=T$ and $c=T$ is **the only restricted counter-model** for **A**

Chapter 3: Question 3

Question 3 Justify whether the following statements **true** or **false**

S1 There are more than 3 possible restricted counter-models for A

S2 There are more than 2 possible restricted models of A

Solution

Statement: There are more than 3 possible restricted counter-models for A is **false**

We have just proved that there is only one possible restricted counter-model for A

Statement: There are more than 2 possible restricted models of A is **true**

There are 7 possible restricted models for A

Justification: $2^3 - 1 = 7$

Chapter 3: Question 3

Question 3

1. List **3 models** and **2 counter-models** for **A** from **Question 2**, i.e. for formula

$$A = (\neg a \Rightarrow (\neg b \cup (b \Rightarrow \neg c)))$$

that are **extensions** to the set **VAR** of all variables of **one** the **restricted models** and of **one** of the **restricted counter-models** that you have found in **Questions 1, 2**

Chapter 3: Question 3

Solution

One of the **restricted models** is, for example a function

$v_A : \{a, b, c\} \longrightarrow \{T, F\}$ such that

$v_A(a) = T, v_A(b) = T, v_A(c) = F$

We **extend** v_A to the set of all propositional variables **VAR** to obtain a (non restricted) **models** as follows

Chapter 3: Question 3

Model w_1 is a function

$w_1 : VAR \longrightarrow \{T, F\}$ such that

$w_1(a) = v_A(a) = T$, $w_1(b) = v_A(b) = T$,

$w_1(c) = v_A(c) = F$, and $w_1(x) = T$, for all

$x \in VAR - \{a, b, c\}$

Model w_2 is defined by a formula

$w_2(a) = v_A(a) = T$, $w_2(b) = v_A(b) = T$,

$w_2(c) = v_A(c) = F$, and $w_2(x) = F$, for all

$x \in VAR - \{a, b, c\}$

Chapter 3: Question 3

Model w_3 is defined by a formula

$$w_3(a) = v_A(a) = T, w_3(b) = v_A(b) = T, w_3(c) = v(c) = F, \\ w_3(d) = F \text{ and } w_3(x) = T \text{ for all } x \in VAR - \{a, b, c, d\}$$

There is **as many** of such models, as extensions of v_A to the set VAR , i.e. as many as **real numbers**

Chapter 3: Question 3

A counter-model for a formula $A = (\neg a \Rightarrow (\neg b \cup (b \Rightarrow \neg c)))$ is, by **definition** any function

$$v : VAR \longrightarrow \{T, F\}$$

such that $v^*(A) = F$

A **restricted counter-model** for the formula A , the only one, as already proved in is a function

$$v_A : \{a, b\} \longrightarrow \{T, F\}$$

such that

$$v_A(a) = F, \quad v_A(b) = T, \quad v_A(c) = T$$

Chapter 3: Question 3

We extend v_A to the set of all propositional variables VAR to obtain (non restricted) some counter-models.

Here are **two** of such **extensions**

Counter- model w_1 :

$$w_1(a) = v_A(a) = F, \quad w_1(b) = v_A(b) = T,$$

$$w_1(c) = v(c) = T, \quad \text{and} \quad w_1(x) = F, \quad \text{for all} \\ x \in VAR - \{a, b, c\}$$

Counter- model w_2 :

$$w_2(a) = v_A(a) = T, \quad w_2(b) = v_A(b) = T,$$

$$w_2(c) = v(c) = T, \quad \text{and} \quad w_2(x) = T \quad \text{for all} \\ x \in VAR - \{a, b, c\}$$

There is **as many** of such **counter- models**, as extensions of v_A to the set VAR , i.e. **as many as real numbers**

Chapter 3: Models for Sets of Formulas

Definition

A truth assignment v is a **model for a set** $\mathcal{G} \subseteq \mathcal{F}$
of formulas of a given language $\mathcal{L} = \mathcal{L}_{\{\neg, \Rightarrow, \cup, \cap\}}$
if and only if

$$v \models B \quad \text{for all} \quad B \in \mathcal{G}$$

We denote it by $v \models \mathcal{G}$

Observe that the set $\mathcal{G} \subseteq \mathcal{F}$ can be **finite** or **infinite**

Chapter 3: Consistent Sets of Formulas

Definition

A set $\mathcal{G} \subseteq \mathcal{F}$ of **formulas** is called **consistent** if and only if \mathcal{G} **has a model**, i.e. we have that

$\mathcal{G} \subseteq \mathcal{F}$ is **consistent** if and only if **there is** v such that $v \models \mathcal{G}$

Otherwise \mathcal{G} is called **inconsistent**

Chapter 3: Independent Statements

Definition

A formula A is called **independent** from a set $\mathcal{G} \subseteq \mathcal{F}$ if and only if **there are** truth assignments v_1, v_2 such that

$$v_1 \models \mathcal{G} \cup \{A\} \text{ and } v_2 \models \mathcal{G} \cup \{\neg A\}$$

i.e. we say that a formula A is **independent** if and only if

$$\mathcal{G} \cup \{A\} \text{ and } \mathcal{G} \cup \{\neg A\} \text{ are } \mathbf{consistent}$$

Chapter 3: Question 4

Question 4

Given a set

$$\mathcal{G} = \{((a \cap b) \Rightarrow b), (a \cup b), \neg a\}$$

Show that \mathcal{G} is **consistent**

Solution

We have to find $v : VAR \rightarrow \{T, F\}$ such that

$$v \models \mathcal{G}$$

It means that we need to find v such that

$$v^*((a \cap b) \Rightarrow b) = T, \quad v^*(a \cup b) = T, \quad v^*(\neg a) = T$$

Chapter 3: Question 4

Observe that $\models ((a \cap b) \Rightarrow b)$, hence we have that

1. $v^*((a \cap b) \Rightarrow b) = T$ for any v

$$v^*(\neg a) = \neg v^*(a) = \neg v(a) = T$$

only when $v(a) = F$ hence

2. $v(a) = F$

$$v^*(a \cup b) = v^*(a) \cup v^*(b) = v(a) \cup v(b) = F \cup v(b) = T$$

only when $v(b) = T$ so we get

3. $v(b) = T$

This **means** that for any $v : VAR \rightarrow \{T, F\}$ such that

$$v(a) = F, v(b) = T, v \models \mathcal{G}$$

and we **proved** that \mathcal{G} is **consistent**

Chapter 3: Question 5

Question 5

Show that a formula $A = (\neg a \wedge b)$ is **not independent** of

$$\mathcal{G} = \{((a \wedge b) \Rightarrow b), (a \vee b), \neg a\}$$

Solution

We have to show that **it is impossible** to construct v_1, v_2 such that

$$v_1 \models \mathcal{G} \cup \{A\} \text{ and } v_2 \models \mathcal{G} \cup \{\neg A\}$$

Observe that we have just proved that any v such that $v(a) = F$, and $v(b) = T$ is **the only** model restricted to the set of variables $\{a, b\}$ for \mathcal{G} so we have to check now if it is **possible** that $v \models A$ and $v \models \neg A$

Chapter 3: Question 5

We have to evaluate $v^*(A)$ and $v^*(\neg A)$ for

$v(a) = F$, and $v(b) = T$

$$v^*(A) = v^*(\neg a \wedge b) = \neg v(a) \wedge v(b) = \neg F \wedge T = T \wedge T = T$$

and so $v \models A$

$$v^*(\neg A) = \neg v^*(A) = \neg T = F$$

and so $v \not\models \neg A$

This **ends the proof** that A is **not independent** of \mathcal{G}

Chapter 3: Question 6

Question 6

Find an **infinite number of formulas** that are **independent** of

$$\mathcal{G} = \{((a \cap b) \Rightarrow b), (a \cup b), \neg a\}$$

This **my solution** - there are many others, but this one seemed to me to be the **simplest**

Solution

We just proved that any v such that $v(a) = F$, $v(b) = T$ is **the only** model restricted to the set of variables $\{a, b\}$ and so all other possible models for \mathcal{G} must be **extensions** of v

Chapter 3: Question 6

We **define** a **countably infinite** set of formulas (and their negations) and corresponding **extensions** of \mathbf{v} (restricted to to the set of variables $\{a, b\}$) such that $\mathbf{v} \models \mathcal{G}$ as follows

Observe that **all extensions** of \mathbf{v} restricted to to the set of variables $\{a, b\}$ have as domain the **infinitely countable** set

$$VAR - \{a, b\} = \{a_1, a_2, \dots, a_n, \dots\}$$

We take as a set of formulas (to be **proved** to be independent) the set of **atomic formulas**

$$\mathcal{F}_0 = VAR - \{a, b\} = \{a_1, a_2, \dots, a_n, \dots\}$$

Chapter 3: Question 6

proof of independence of any formula of \mathcal{F}_0

Let $c \in \mathcal{F}_0$

We define truth assignments $v_1, v_2 : \text{VAR} \rightarrow \{T, F\}$
such that

$$v_1 \models \mathcal{G} \cup \{c\} \text{ and } v_2 \models \mathcal{G} \cup \{\neg c\}$$

as follows

$$v_1(a) = v(a) = F, \quad v_1(b) = v(b) = T \text{ and } v_1(c) = T$$

for all $c \in \mathcal{F}_0$

$$v_2(a) = v(a) = F, \quad v_2(b) = v(b) = T \text{ and } v_2(c) = F$$

for all $c \in \mathcal{F}_0$

CHAPTER 3

Some Extensional Many Valued Semantics

Chapter 3: Question 7

Question 7

We **define** a 4 valued \mathbf{H}_4 logic semantics as follows

The language is $\mathcal{L} = \mathcal{L}_{\{\neg, \Rightarrow, \cup, \cap\}}$

The logical connectives $\neg, \Rightarrow, \cup, \cap$ of \mathbf{H}_4 are operations in the set $\{F, \perp_1, \perp_2, T\}$, where $\{F < \perp_1 < \perp_2 < T\}$ and are defined as follows

Conjunction \cap is a function

$\cap : \{F, \perp_1, \perp_2, T\} \times \{F, \perp_1, \perp_2, T\} \longrightarrow \{F, \perp_1, \perp_2, T\}$,

such that for any $x, y \in \{F, \perp_1, \perp_2, T\}$

$$x \cap y = \min\{x, y\}$$

Chapter 3: Question 7

Disjunction \cup is a function

$$\cup : \{F, \perp_1, \perp_2, T\} \times \{F, \perp_1, \perp_2, T\} \longrightarrow \{F, \perp_1, \perp_2, T\},$$

such that for any $x, y \in \{F, \perp_1, \perp_2, T\}$

$$x \cup y = \max\{x, y\}$$

Implication \Rightarrow is a function

$$\Rightarrow : \{F, \perp_1, \perp_2, T\} \times \{F, \perp_1, \perp_2, T\} \longrightarrow \{F, \perp_1, \perp_2, T\},$$

such that for any $x, y \in \{F, \perp_1, \perp_2, T\}$,

$$x \Rightarrow y = \begin{cases} T & \text{if } x \leq y \\ y & \text{otherwise} \end{cases}$$

Negation: for any $x, y \in \{F, \perp_1, \perp_2, T\}$

$$\neg x = x \Rightarrow F$$

Chapter 3: Question 7

Part 1 Write **Truth Tables** for IMPLICATION and NEGATION in **H₄**

Solution

H₄ Implication

\Rightarrow	F	\perp_1	\perp_2	T
F	T	T	T	T
\perp_1	F	T	T	T
\perp_2	F	\perp_1	T	T
T	F	\perp_1	\perp_2	T

H₄ Negation

\neg	F	\perp_1	\perp_2	T
	T	F	F	F

Chapter 3: Question 7

Part 2 Verify whether

$$\models_{\mathbf{H}_4} ((a \Rightarrow b) \Rightarrow (\neg a \cup b))$$

Solution

Take any \mathbf{v} such that

$$\mathbf{v}(a) = \perp_1 \quad \mathbf{v}(b) = \perp_2$$

Evaluate

$$\begin{aligned} \mathbf{v} * ((a \Rightarrow b) \Rightarrow (\neg a \cup b)) &= (\perp_1 \Rightarrow \perp_2) \Rightarrow (\neg \perp_1 \cup \perp_2) = \\ T \Rightarrow (F \cup \perp_2) &= T \Rightarrow \perp_2 = \perp_2 \end{aligned}$$

This proves that our \mathbf{v} is a **counter-model** and hence

$$\not\models_{\mathbf{H}_4} ((a \Rightarrow b) \Rightarrow (\neg a \cup b))$$

Chapter 3: Question 8

Question 8

Show that (can't use TTables!)

$$\models ((\neg a \cup b) \Rightarrow (((c \cap d) \Rightarrow \neg d) \Rightarrow (\neg a \cup b)))$$

Solution

Denote $A = (\neg a \cup b)$, and $B = ((c \cap d) \Rightarrow \neg d)$

Our formula becomes a substitution of a **basic tautology**

$$(A \Rightarrow (B \Rightarrow A))$$

and hence is a **tautology**

Chapter 3: Challenge Exercise

1. **Define** *your own* propositional language \mathcal{L}_{CON} that contains also **different connectives** that the standard connectives $\neg, \cup, \cap, \Rightarrow$

Your language \mathcal{L}_{CON} **does not need** to include all (if any!) of the standard connectives $\neg, \cup, \cap, \Rightarrow$

2. **Describe** intuitive meaning of the new connectives of your language

3. Give some *motivation* for **your own semantic**

4. **Define** formally *your own extensional semantics* **M** for your language \mathcal{L}_{CON} - it means
write carefully all **Steps 1- 4** of the definition of your **M**

Chapter 3: Question 9

Question 9

Definition

Let S_3 be a 3-valued semantics for $\mathcal{L}_{\{\neg, \cup, \Rightarrow\}}$ defined as follows:

$V = \{F, U, T\}$ is the set of logical values with the distinguished value T

$$x \Rightarrow y = \neg x \cup y \quad \text{for any } x, y \in \{F, U, T\}$$

$$\neg F = T, \quad \neg U = F, \quad \neg T = U$$

and

U	F	U	T
F	F	U	T
U	U	U	U
T	T	U	T

Question 9

Part 1

Consider the following classical tautologies:

$$A_1 = (a \cup \neg a), \quad A_2 = (a \Rightarrow (b \Rightarrow a))$$

Find S_3 **counter-models** for A_1, A_2 , if exist

You **can't use shorthand** notation

Solution

Any v such that $v(a) = v(b) = U$ is a **counter-model** for both A_1 and A_2 , as

$$v^*(a \cup \neg a) = v^*(a) \cup \neg v^*(b) = U \cup \neg U = U \cup F = U \neq T$$

$$v^*(a \Rightarrow (b \Rightarrow a)) = v^*(a) \Rightarrow (v^*(b) \Rightarrow v^*(a)) = U \Rightarrow (U \Rightarrow U) = U \Rightarrow U = \neg U \cup U = F \cup U = U \neq T$$

Question 9

Part 2

Consider the following classical tautologies:

$$A_1 = (a \cup \neg a), \quad A_2 = (a \Rightarrow (b \Rightarrow a))$$

Define your own 2-valued semantics S_2 for \mathcal{L} , such that **none of** A_1, A_2 is a S_2 **tautology**

Verify your results. You **can use shorthand** notation.

Solution

This is not the only solution, but it is the simplest and most obvious I could think of! Here it is.

We define S_2 **connectives** as follows

$$\neg x = F, \quad x \Rightarrow y = F, \quad x \cup y = F \quad \text{for all } x, y \in \{F, T\}$$

Obviously, for any v ,

$$v^*(a \cup \neg a) = F \quad \text{and} \quad v^*(a \Rightarrow (b \Rightarrow a)) = F$$

Chapter 3: Question 10

Question 10

Prove using proper **classical** logical equivalences (list them at each step) that for any formulas A, B of language $\mathcal{L}_{\{\neg, \cup, \Rightarrow\}}$

$$\neg(A \Leftrightarrow B) \equiv ((A \cap \neg B) \cup (\neg A \cap B))$$

Solution

$$\begin{aligned}\neg(A \Leftrightarrow B) &\equiv^{def} \neg((A \Rightarrow B) \cap (B \Rightarrow A)) \\ &\equiv^{deMorgan} (\neg(A \Rightarrow B) \cup \neg(B \Rightarrow A)) \\ &\equiv^{negimpl} ((A \cap \neg B) \cup (B \cap \neg A)) \equiv^{commut} ((A \cap \neg B) \cup (\neg A \cap B))\end{aligned}$$

Question 11

Question 11

Prove using proper **classical** logical equivalences (list them at each step) that for any formulas A, B of language $\mathcal{L}_{\{\neg, \cup, \Rightarrow\}}$

$$((B \cap \neg C) \Rightarrow (\neg A \cup B)) \equiv ((B \Rightarrow C) \cup (A \Rightarrow B))$$

Solution

$$\begin{aligned} & ((B \cap \neg C) \Rightarrow (\neg A \cup B)) \\ & \equiv^{impl} (\neg(B \cap \neg C) \cup (\neg A \cup B)) \\ & \equiv^{deMorgan} ((\neg B \cup \neg\neg C) \cup (\neg A \cup B)) \\ & \equiv^{dneg} ((\neg B \cup C) \cup (\neg A \cup B)) \equiv^{impl} ((B \Rightarrow C) \cup (A \Rightarrow B)) \end{aligned}$$

Question 12

Question 12

We **define** \vdash connectives for $\mathcal{L}_{\{\neg, \cup, \Rightarrow\}}$ as follows

\vdash **Negation** \neg is a **function**:

$$\neg : \{T, \perp, F\} \longrightarrow \{T, \perp, F\}$$

such that $\neg \perp = \perp$, $\neg T = F$, $\neg F = T$

\vdash **Conjunction** \cap is a **function**:

$$\cap : \{T, \perp, F\} \times \{T, \perp, F\} \longrightarrow \{T, \perp, F\}$$

such that $x \cap y = \min\{x, y\}$ for all $x, y \in \{T, \perp, F\}$

Remember that we assumed: $F < \perp < T$

Question 12

\vdash Implication \Rightarrow is a function:

$$\Rightarrow: \{T, \perp, F\} \times \{T, \perp, F\} \longrightarrow \{T, \perp, F\}$$

such that

$$x \Rightarrow y = \begin{cases} \neg x \cup y & \text{if } x > y \\ T & \text{otherwise} \end{cases}$$

Given a formula $((a \cap b) \Rightarrow \neg b) \in \mathcal{F}$ of $\mathcal{L}_{\{\neg, \cup, \Rightarrow\}}$

Use the fact that $v: \text{VAR} \longrightarrow \{F, \perp, T\}$ is such that

$$v^*((a \cap b) \Rightarrow \neg b) = \perp \quad \text{under } \vdash \text{ semantics to evaluate} \\ v^*((b \Rightarrow \neg a) \Rightarrow (a \Rightarrow \neg b)) \cup (a \Rightarrow b)$$

You **can** use shorthand notation

Question 12 Solution

Solution

The formula $((a \cap b) \Rightarrow \neg b) = \perp$ in \mathcal{L} connectives semantics in

two cases written is the shorthand notation as

C1 $(a \cap b) = \perp$ and $\neg b = F$

C2 $(a \cap b) = T$ and $\neg b = \perp$.

Consider case **C1**

$\neg b = F$, so $v(b) = T$, and hence $(a \cap T) = v(a) \cap T = \perp$
if and only if $v(a) = \perp$

It means that $v^*((a \cap b) \Rightarrow \neg b) = \perp$ for any v , is such that
 $v(a) = \perp$ and $v(b) = T$

Question 12 Solution

We now **evaluate** (in shorthand notation)

$$\begin{aligned} & v^*(((b \Rightarrow \neg a) \Rightarrow (a \Rightarrow \neg b)) \cup (a \Rightarrow b)) \\ &= (((T \Rightarrow \neg \perp) \Rightarrow (\perp \Rightarrow \neg T)) \cup (\perp \Rightarrow T)) = ((\perp \Rightarrow \perp) \cup T) = T \end{aligned}$$

Consider now **Case C2**

$\neg b = \perp$, i.e. $b = \perp$, and hence $(a \cap \perp) = T$ what is **impossible**, hence v from the **Case C1** is the only one

Question 13

Question 13

Use the **Definability of Conjunction** in terms of disjunction and negation **Equivalence**

$$(A \cap B) \equiv \neg(\neg A \cup \neg B)$$

to transform a formula

$$A = \neg(\neg(\neg a \cap \neg b) \cap a)$$

of the language $\mathcal{L}_{\{\cap, \neg\}}$ into a logically equivalent formula B
of the language $\mathcal{L}_{\{\cup, \neg\}}$

Question 13

Solution

$$\neg(\neg(\neg a \cap \neg b) \cap a) \equiv \neg\neg(\neg\neg(\neg a \cap \neg b) \cup \neg a)$$

$$\equiv ((\neg a \cap \neg b) \cup \neg a) \equiv (\neg(\neg\neg a \cup \neg\neg b) \cup \neg a)$$

$$\equiv \neg(a \cup b) \cup \neg a$$

The formula **B** of $\mathcal{L}_{\{\cup, \neg\}}$ equivalent to **A** is

$$B = (\neg(a \cup b) \cup \neg a)$$

Equivalence of Languages Definition

Definition

Given two languages: $\mathcal{L}_1 = \mathcal{L}_{CON_1}$ and $\mathcal{L}_2 = \mathcal{L}_{CON_2}$, for $CON_1 \neq CON_2$

We say that they are **logically equivalent**, i.e.

$$\mathcal{L}_1 \equiv \mathcal{L}_2$$

if and only if the following conditions **C1**, **C2** hold.

C1: for any formula A of \mathcal{L}_1 , there is a formula B of \mathcal{L}_2 , such that $A \equiv B$

C2: for any formula C of \mathcal{L}_2 , there is a formula D of \mathcal{L}_1 , such that $C \equiv D$

Question 15

Question 15

Prove the logical equivalence of the languages

$$\mathcal{L}_{\{\neg, \cup\}} \equiv \mathcal{L}_{\{\neg, \Rightarrow\}}$$

Solution

We need **two definability equivalences**:

implication in terms of **disjunction** and negation

$$(A \Rightarrow B) \equiv (\neg A \cup B)$$

and **disjunction** in terms of **implication** negation,

$$(A \cup B) \equiv (\neg A \Rightarrow B)$$

and the **Substitution Theorem**

Question 16

Question 16

Prove the logical equivalence of the languages

$$\mathcal{L}_{\{\neg, \cup, \Rightarrow\}} \equiv \mathcal{L}_{\{\neg, \cup\}}$$

Solution

We need **only** the **definability of implication** in terms of **disjunction** and **negation** equivalence

$$(A \Rightarrow B) \equiv (\neg A \cup B)$$

as the **Substitution Theorem** for any formula A of $\mathcal{L}_{\{\neg, \cup, \Rightarrow\}}$ **there is** a formula B of $\mathcal{L}_{\{\neg, \cup\}}$ such that $A \equiv B$ and the condition **C1** holds

Observe that any formula A of language $\mathcal{L}_{\{\neg, \cup\}}$ is also a formula of the language $\mathcal{L}_{\{\neg, \cup, \Rightarrow\}}$ and of course $A \equiv A$ so the condition **C2** also holds