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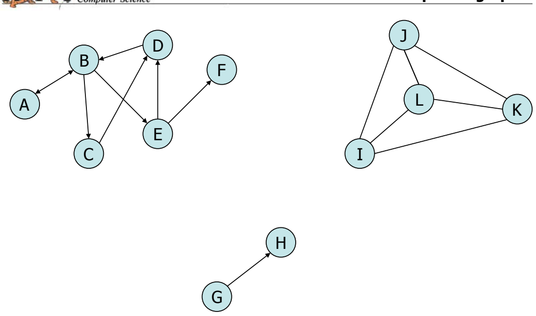
Introduction to Graphs

What's a graph?

- A graph is a collection of two types of data:
  - vertices
    - also sometimes called "points"
    - can represent geographical locations, activities, etc.
  - edges
    - connections between vertices
    - may have a direction associated with them, in which case the graph is called a *directed graph* (*digraph*)
    - if they do not have directions associated with them, the graph is called an *undirected graph*

Examples of graphs



Graphs are usually drawn using points for vertices and lines for Edges, but a graph is defined independent of its representation

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Weighting edges

- In addition to being directed or undirected, edges can be weighted or un-weighted.
  - A weighted edge has a value associated with it
  - The weight often measures the cost of using the edge to go from one node to another
- A vertex may also have data associated with it.
  - This can be the name of a city, or a task to be completed, or whatever.

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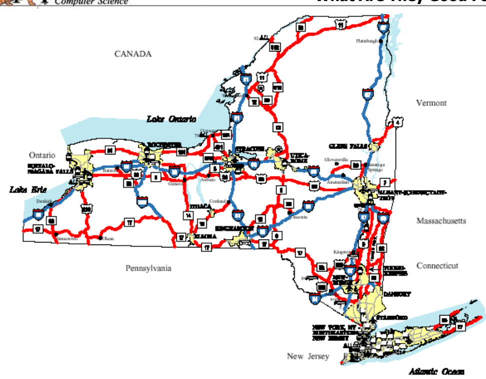
What Are They Good For?



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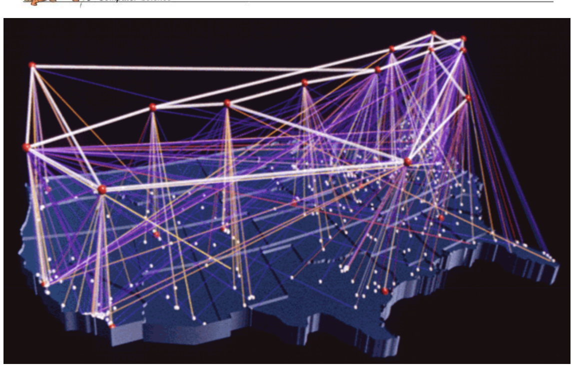
What Are They Good For?



Other examples

- Graphical representation of a UNIX file system tree
- Graph representation of a computer network
- Graph representation of a software system
- Flow graph notation for various constructs
- Relationships between people ("who knows who?")
  - Good for the Kevin Bacon game, etc.

Terminology

- Two different vertices, *x* and *y*, in a graph are said to be *adjacent* if an edge connects *x* to *y*
- A *path* is a sequence of vertices in which each vertex is adjacent to the next one
  - The *length* of a path is the number of edges in the path
  - A *simple path* is a path in which no vertex is repeated
  - A *cycle* is a path of length greater than one that begins and ends at the same vertex
    - A graph with no cycles is called a *tree*
  - A *simple cycle* is a cycle consisting of three or more distinct vertices in which no vertex is visited more than once along the cycle's path

Terminology (ctd.)

- The *degree* of a vertex *x* is the number of edges *e* in which *x* is one of the endpoints of edge *e*
- The *neighbors* of a vertex *v*, are the vertices that are directly connected to *v*

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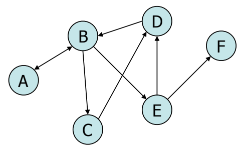
Applying the terms

- What is the length of a simple path from A to F?
- What cycles can you find?
- How many neighbors does B have?



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Formal definitions

- A graph $G = (V, E)$, consists of a set of vertices, V, along with a set of edges, E, where the edges in E are formed from distinct pairs of vertices in V.
- In an undirected graph, each edge $e = \{ v_1, v_2 \}$ is an unordered pair of distinct vertices, which connects the two vertices v_1 and v_2, without prescribing a direction from v_1 to v_2 or from v_2 to v_1.
- In a directed graph, each edge $e = \{ v_1, v_2 \}$ is an ordered pair of vertices, which connects the pair of vertices v_1 and v_2, in the direction from v_1 to v_2. In this case we say v_1 is the origin of the edge $e = \{ v_1, v_2 \}$ and v_2 is the end of the edge e.

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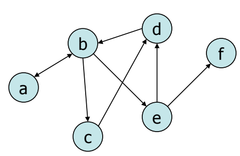
Being formal

- Write a formal definition for this directed graph.
  - G = { V, E }, where
    - V = { a, b, c, d, e, f }
    - E = { {a, b}, {b, a}, {b, c}, {c, d}, {d, b}, {b, e}, {e, d}, {e, f} }
- Write another definition, assuming that it's an undirected graph.
  - G = { V, E }, where
    - V = { a, b, c, d, e, f }
    - E = { {a, b}, {b, c}, {b, e}, {b, d}, {c, d}, {e, d}, {e, f} }



Connectivity

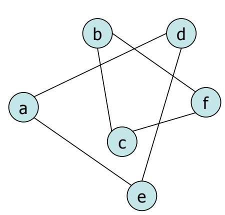
- Two vertices in a graph *G* are said to be *connected* if there is a path from the first to the second in *G*
  - If $x \in V$ and $y \in V$, where $x \neq y$, then *x* and *y* are *connected* if there exists a path, $p = v_1, v_2, \ldots, v_n$, in *G*, such that $x = v_1$ and $y = v_n$
- In the graph *G*, a *connected component* is a subset, *S*, of the vertices *V* that are all connected to one another
  - *S* is a *connected component* of *G* if, for any two distinct vertices, $x \in S$ and $y \in S$, *x* is connected to *y*
- A graph is *connected* if there is a path from every node to every other node in the graph
  - A graph that is not connected is made up of connected components

Connectivity (ctd.)

- A digraph is said to be strongly connected if for every pair of nodes, n_i and n_j, there exists a path from node n_i to node n_j.
- An undirected graph is said to be connected if for every pair of nodes, n_i and n_j, there is a path connecting the two nodes

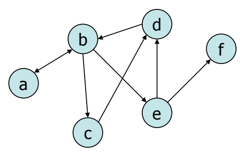
What's connected? (1)

- Given the graph to the right, what vertices are connected?
- Answer:
  - The vertices {a, d, e} and {b, c, f} form connected components in the graph



What's connected? (2)

- Given the graph to the right, what vertices are connected?
- Answer:
  - The vertices {a, b, c, d, e} form a connected component in the graph
  - The vertex f isn't part of the connected component, because you can't get <u>from</u> it to the other vertices



Edges

- If we denote the number of vertices in a graph by *V* and the number of edges by *E*, note that *E* can range anywhere from 0 to $\frac{1}{2}V(V-1)$
  - Graphs with all edges present are called *complete* graphs
  - Graphs with relatively few edges (say less than $V \log V$) are called *sparse* graphs
  - Graphs with relatively few of the possible edges missing are called *dense*

DiGraph.java (old version)

```
import java.util.*;

public interface DiGraph {

    // Methods to build the graph
    public void addVertex( Object key, Object data );
    public void addEdge( Object fromKey, Object toKey, Object data )
                throws NoSuchVertexException;

    // Operations on edges
    public boolean isEdge( Object fromKey, Object toKey ) throws NoSuchVertexException;
    public Object getEdgeData( Object fromKey, Object toKey )
                throws NoSuchVertexException;

    // Operations on vertices
    public boolean isVertex( Object key );
    public Object getVertexData( Object key ) throws NoSuchVertexException;
    public int numVertices();
    public int inDegree( Object key );
    public int outDegree( Object key );
    public Collection neighborData( Object key ) throws NoSuchVertexException;
    public Collection neighborKeys( Object key ) throws NoSuchVertexException;

    // Utility methods
    public Collection vertexData();
    public Collection vertexKeys();
    public Collection edgeData();
    public void clear();

} // DiGraph
```

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DiGraph.java (new version)

```java
import java.util.*;

public interface DiGraph<VertexKey, VertexData, EdgeData> {

    // Methods to build the graph
    public void addVertex( VertexKey key, VertexData data );
    public void addEdge( VertexKey fromKey, VertexKey toKey, EdgeData data )
                throws NoSuchVertexException;

    // Operations on edges
    public boolean isEdge( VertexKey fromKey, VertexKey toKey )
                throws NoSuchVertexException;
    public EdgeData getEdgeData(VertexKey fromKey, VertexKey toKey )
                throws NoSuchVertexException;

    // Operations on vertices
    public boolean isVertex( VertexKey key );
    public VertexData getVertexData( VertexKey key ) throws NoSuchVertexException;
    public int numVertices();
    public int inDegree( VertexKey key );
    public int outDegree(VertexKey key );
    public Collection<VertexData> neighborData( VertexKey key )
                throws NoSuchVertexException;
    public Collection<VertexKey> neighborKeys(VertexKey key )
                throws NoSuchVertexException;

    // Utility methods
    public Collection<VertexData> vertexData();
    public Collection<VertexKey> vertexKeys();
    public Collection<EdgeData> edgeData();
    public void clear();

} // DiGraph
```

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Graphs for problem solving

- A problem can often be represented as a graph
- The solution to the problem is then obtained by solving a problem on the corresponding graph

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The coin game

- 3 coins are placed on a table in a row
- The goal is to get the coin row into a head-tail-head configuration in the shortest number of coin flips possible
- Rules:
  - You may flip the middle coin whenever you want to
  - You may flip one of the end coins only if the other two coins are the same as each other (both heads or both tails)
  - You are not allowed to change coins in any other way (such as shuffling them around)

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How does that fit in?

- We can used a graph to represent the possible states of the 3 coins
  - Vertices represent a given set of states for the coins
  - Edges represent a transformation from one state to another (i.e., flipping one of the coins)
- By finding a path between two states, we can find a way of getting from one combination to another
  - The shorter the path, the fewer the number of coin flips required

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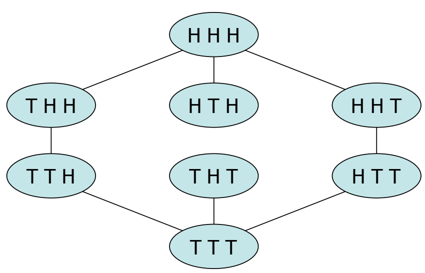
A graph for the coin game



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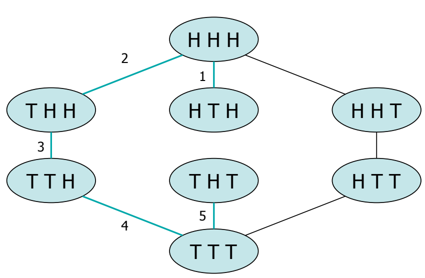
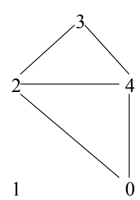
One possible path from "H T H"



Ways of representing graphs

- There are two common ways to represent a graph:
  - With an adjacency matrix
  - With an adjacency list

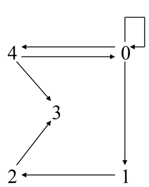
Adjacency Matrix



	0	1	2	3	4
0	F	F	T	F	T
1	F	F	F	F	F
2	T	F	F	T	T
3	F	F	T	F	T
4	T	F	T	T	F

Note: the adjacency matrix representation is often satisfactory only if the graphs to be represented are dense (most of the array will be true)

Adjacency Matrix



	0	1	2	3	4
0	T	T			T
1			T		
2				T	
3					
4	T			T	

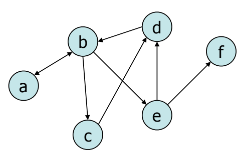
Note: sometimes folks will simply leave spots in the matrix where there is no edge empty, to improve readability....

Adjacency matrices implementation

- Usually implemented with two-dimensional arrays
  - The first (source) number is often used to denote the row, while the second (destination) denotes the column
  - Although this is just a handy convention, it is what I will assume you're using on tests, etc. unless you clearly state otherwise

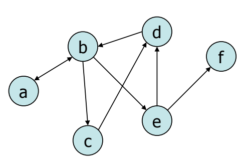
Building adjacency matrices

- What is the adjacency matrix for this graph?



Building adjacency matrices

	a	b	c	d	e	f
a	F	**T**	F	F	F	F
b	**T**	F	**T**	F	**T**	F
c	F	F	F	**T**	F	F
d	F	**T**	F	F	F	F
e	F	F	F	**T**	F	**T**
f	F	F	F	F	F	F



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Good and bad about matrices

- Considerations:
  - Adding or removing edges
  - Checking whether or not a particular edge is present.
  - Iterating a loop that executes one time for each edge with a particular source vertex.
- What are the good things about using an adjacency matrix?
- The bad things?

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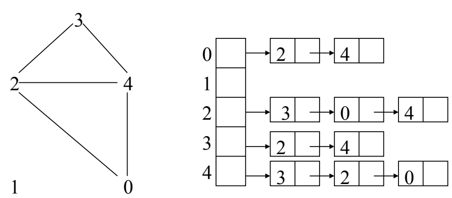
Adjacency List



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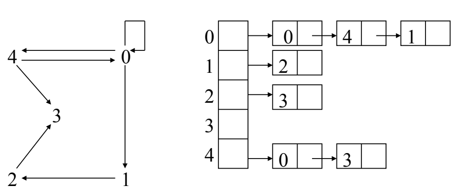
Adjacency List



Good and bad about lists

- What are the positives to using an adjacency list?
- What are some of the negatives of using an adjacency list?

Searching a graph

- Several questions arise when processing a graph
  - Is the graph connected?
  - If not, what are the connected components?
  - Does the graph have a cycle?
  - …

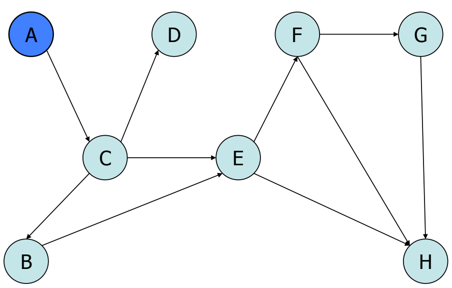
Searching various ADTs

- Trees
  - Breadth-first
  - Depth-first
    - in-order
    - pre-order
    - post-order
- Graphs
  - Breadth-first
  - Depth-first
- What do "breadth-first" and "depth-first" mean for graphs?

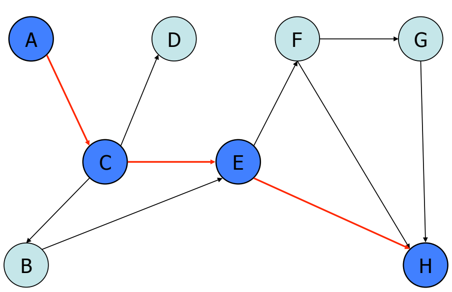
Depth-First Search

- In a depth-first search of a graph:
  - we start at a node
  - we follow whatever path we like as far as we can from that node
  - when we can go no further, we back up and find paths to other (unvisited) nodes

DFS Example



DFS Example



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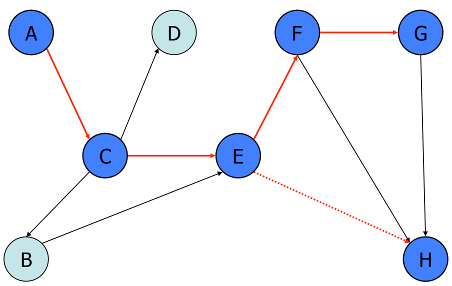
DFS Example



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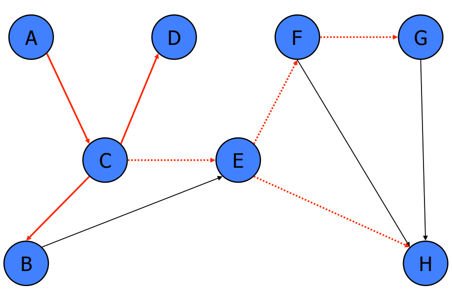
DFS Example



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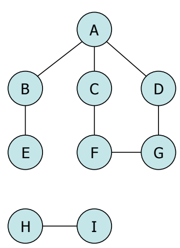
DFS Pseudo-code

- Associate with each vertex an `Integer` value
  - A value of zero indicates that the vertex has not been visited
  - A non-zero value indicates that the node has been visited
- The *pseudo-code*
  - Build the graph and initialize the values associated with each vertex to zero
  - Get a collection that contains the keys of all the vertices in the graph
  - Set an integer variable, named component, to 1
  - Iterate over the keys
    - Get the data associated with the vertex identified by the current key
    - If the *visit* value is 0 → visit(vertex, component)

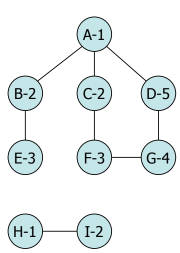
DFS Visit

- Visit(vertex *v*, Integer *component*)
  - Change the "visit value" associated with *v* to *component*
  - Get a collection that contains the keys of the neighbors of *v*
  - Iterate over the collection
    - Get the data associated with the current key
    - If the vertex has not been visited → visit(*vertex*, *component + 1*)

DFS: What values go where?



DFS: Solution (assuming alphabetical ordering)



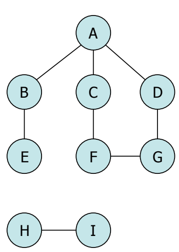
DFS Visit – Rewritten as non-recursive

- Visit(Stack *s*, vertex *v*)
  - Set *v*'s "visit value" to 1
  - Push *v* onto the stack *s*
  - While *s* is not empty
    - Pop the stack and make the vertex the current vertex
    - Get a collection that contains the keys of the neighbors of the current vertex
    - Iterate over the collection
      - If a neighbor's "visit value" is equal to 0 (i.e., it hasn't been visited)
        - » Change its "visit value" to *v.key* + 1
        - » Push the neighboring vertex onto the stack

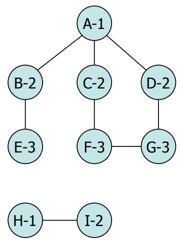
Breadth-First Search (BFS)

- What if we changed the stack in the non-recursive DFS-visit to a queue?
- Visit(Queue *q*, vertex *v*)
  - Set *v*'s "visit value" to 1
  - Enqueue *v* into the queue *q*
  - While the *q* is not empty
    - Dequeue and make the vertex the current vertex
    - Get a collection that contains the keys of the neighbors of the current vertex
    - Iterate over the collection
      - If a neighbor's "visit value" is equal to 0 (i.e., it hasn't been visited)
        - » Change its "visit value" to *v.visitValue*+ 1
        - » Enqueue the current vertex

BFS: What values go where?



BFS: Solution (assuming alphabetical ordering)



Common graph algorithms

- Some common problems/algorithms involving graphs are:
  - Graph coloring
  - Shortest path identification
    - Dijkstra's Algorithm
  - Identifying minimum spanning trees
    - Kruskal's Algorithm

So that's the basics of graphs....

- Any questions?

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Graph coloring

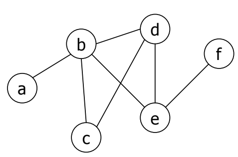
One common problem....

Graph coloring

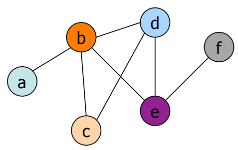
- A very common and challenging problem in graph theory is that of "coloring" a graph.
  - In this context "coloring" means assigning a color (or a number) to each node such that none of its neighbors has that same color (number).
- For a given graph, it's possible that lots of colorings may exist
  - We typically want the one that takes the least # of colors

Coloring a simple graph

- Consider this graph, for instance

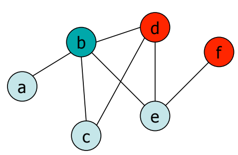


- One possibility would just be to assign a different color to *every* vertex



Number of colors: 6

- A better (optimal) solution might be to re-use colors from some nodes:



Number of colors: 3

- Graph coloring can be used
  - to coloring on a map of nations (or other entities), providing distinctive appearances for each
  - to solve problems involving scheduling and assignments, or other resource allocation issues

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Sample problem #1

- Suppose you want to schedule final exams for 7 courses, and you want to avoid having a student do more than one exam a day.
  - We shall call the courses 1,2,3,4,5,6,7.
  - In the table below a star in entry ij means that course i and j have at least one student in common so you can't have them on the same day.
  - What is the least number of days you need to schedule all the exams? Show how you would schedule the exams.

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Problem #1 data

	1	2	3	4	5	6	7
1		*	*	*		*	*
2	*		*				*
3	*	*		*			
4	*		*		*	*	
5				*		*	
6	*			*	*		*
7	*	*				*	

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Sample problem #2

- Suppose you run a day care for an office building, and you need to assign a locker where each child's parent can put the child's food.
  - There are seven children A,B,C,D,E,F,G.
  - The children come and leave so they are not all there at the same time.
  - You have 1 hour time slots starting 7:00 a.m. to 12:00 noon.
  - A star in the table means a child is present at that time.
  - What is the minimum number of lockers necessary? Show how you would assign the lockers.

Problem #2 data

	A	B	C	D	E	F	G
7:00	*			*	*		
8:00	*	*	*				
9:00	*		*	*		*	
10:00	*		*			*	*
11:00	*					*	*
12:00	*				*		

Complexity

- Graph coloring is hard to do in an efficient fashion
  - Graph coloring is considered to be an "NP-complete" problem
    - Simply *proving* that an answer to a graph coloring problem is correct is $O(n^2)$
    - *Finding* the answer is even tougher (probably a lot tougher)
  - "Greedy" algorithms exist, which try to optimize for performance by providing an approximation ("best guess") of the best coloring, but even these are $O(n^2)$

A greedy coloring algorithm

- Local variables: "numColors", "curColor"
- Set the color for all vertices in the graph to 0 ("undefined")
- Set numColors to 0
- For each vertex in the graph:
  - Set curColor to 1
  - While curColor <= numColors and one of the neighbors of the current vertex is colored with "curColor", increment curColor
  - If curColor > numColors, then numColors = curColor
  - Set the color for the current vertex to curColor

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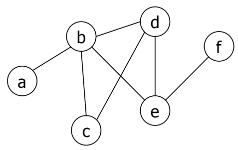
Graph coloring example

Try coloring the following graph, working in alphabetic order.
Use color ordering: red, green, blue, yellow, purple, gray.



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Graph coloring example

Try coloring the following graph, working in alphabetic order.
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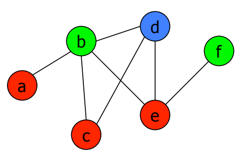

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Shortest paths in graphs

Another common problem....

Dijkstra's Shortest Path Algorithm

- Each node is labeled with its distance from the source node along the best path
- Initially no paths are known, so the values are infinity
- The algorithm starts at the source node and explores possible paths, one hop at a time
- When a label is marked permanent, its label will not change

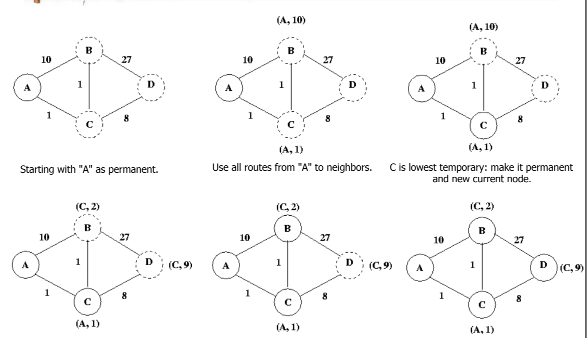
The Algorithm

- Make the source node permanent; the source node is the first working node
- Examine each non-permanent node adjacent to the working node
  - if it is not labeled, label with the distance from the source and the name of the working node
  - if it is labeled, see if the cost computed using the working node is better than the cost in the label; if so change the label to reflect the better path

The Algorithm (ctd)

- Find the non-permanent node with the smallest label, and make it permanent
  - If all the nodes are marked permanent, the algorithm terminates
  - Otherwise, the node just made permanent becomes the working node
- When the algorithm is complete, the path is found (in reverse) by reading the labels from the destination node back to the source

Example: shortest paths from "A" to other nodes



Starting with "A" as permanent.

Use all routes from "A" to neighbors.

C is lowest temporary: make it permanent and new current node.

Update C's non-permanent neighbors.

B is lowest temporary: make it permanent and new current node.

Final state (no changes from "B").

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Any questions?

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- Some additional algorithms of note:
  - Bellman-Ford Algorithm
    - computes single-source shortest paths in a weighted digraph (where some of the edge weights may be negative, unlike in Dijkstra's algorithm)
  - Floyd-Warshall Algorithm
    - a graph analysis algorithm for finding shortest paths in a weighted, directed graph
  - Prim's Algorithm
    - finds a minimum spanning tree for a connected weighted graph
  - Ford-Fulkerson algorithm
    - for computing the *Maximum Flow* within a Network Graph
  - Edmonds-Karp algorithm
    - an alternative approach for computing maximum flow in a network graph