Arithmetical Operations. Real and Imaginary Parts. Complex Conjugates

Let $z_1 = 4 + 3i$ and $z_2 = 2 - 5i$. Find each of the following in the form $x + iy$, showing the details of your work:

3. $z_1 z_2$ **4.** $(3z_1 - z_2)^2$ **5.** $1/z_1$ **6.** $25z_2/z_1$

7. $\mathrm{Re}\,(z_1{}^3)$, $(\mathrm{Re}\,z_1)^3$ **8.** $(z_1 - z_2)/(z_1 + z_2)$ **9.** $z_1\bar{z}_2$, $\bar{z}_1 z_2$ **10.** $1/z_1{}^2$, $1/\bar{z}_1{}^2$

11. $\bar{z}_1/\bar{z}_2$, $\overline{(z_1/z_2)}$ **12.** $z_2\bar{z}_2/(z_1\bar{z}_1)$

Let $z = x + iy$. Find (showing the details of your work)

13. $\mathrm{Im}\,(1/z)$ **14.** $\mathrm{Im}\,z^4$, $(\mathrm{Im}\,z^2)^2$ **15.** $(1 + i)^{16}$ **16.** $\mathrm{Re}\,(z/\bar{z})$ **17.** $\mathrm{Re}\,(z^2/\bar{z})$

18. (Laws of addition and multiplication) Derive the following laws for complex numbers from the corresponding laws for real numbers.

$$z_1 + z_2 = z_2 + z_1, \qquad z_1z_2 = z_2z_1 \qquad \textit{(Commutative laws)}$$

$$(z_1 + z_2) + z_3 = z_1 + (z_2 + z_3), \qquad (z_1z_2)z_3 = z_1(z_2z_3) \qquad \textit{(Associative laws)}$$

$$z_1(z_2 + z_3) = z_1z_2 + z_1z_3 \qquad \textit{(Distributive law)}$$

$$0 + z = z + 0 = z, \qquad z + (-z) = (-z) + z = 0, \qquad z \cdot 1 = z.$$

19. (Laws for conjugates) Verify (9) for $z_1 = 38 + 18i$, $z_2 = 3 + 5i$.

20. (Multiplication) If the product of two complex numbers is zero, show that at least one factor must be zero.

12.2 Polar Form of Complex Numbers Powers and Roots

We can substantially increase the usefulness of the complex plane and gain further insight into the nature of complex numbers if besides the xy-coordinates we also employ the usual polar coordinates r, θ defined by

$$x = r\cos\theta, \qquad y = r\sin\theta. \tag{1}$$

Then $z = x + iy$ takes the so-called **polar form**

$$z = r(\cos\theta + i\sin\theta). \tag{2}$$

r is called the **absolute value** or **modulus** of z and is denoted by $|z|$. Hence

$$|z| = r = \sqrt{x^2 + y^2} = \sqrt{z\bar{z}}. \tag{3}$$

Geometrically, $|z|$ is the distance of the point z from the origin (Fig. 293). Similarly, $|z_1 - z_2|$ is the distance between z_1 and z_2 (Fig. 294).

θ is called the **argument** of z and is denoted by $\arg z$. Thus (Fig. 293)

$$\theta = \arg z = \arctan\frac{y}{x} \qquad (z \neq 0). \tag{4}$$

Geometrically, θ is the directed angle from the positive x-axis to OP in Fig. 293. Here,

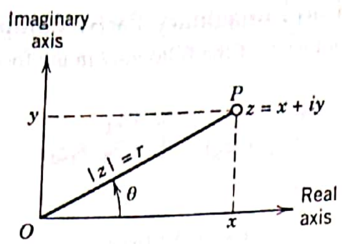

Fig. 293. Complex plane, polar form of a complex number

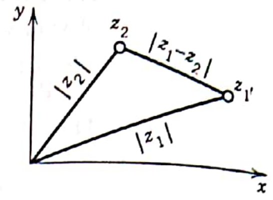

Fig. 294. Distance between two points in the complex plane

as in calculus, *all angles are measured in* **radians** *and positive in the counterclockwise sense.*

For $z = 0$ this angle θ is undefined. (Why?) For given $z \neq 0$ it is determined only up to integer multiples of 2π. The value of θ that lies in the interval $-\pi < \theta \leqq \pi$ is called the **principal value** of the argument of z ($\neq 0$) and is denoted by Arg z, with capital A. Thus, by definition $\theta = \text{Arg } z$ satisfies by definition

$$\boxed{-\pi < \text{Arg } z \leqq \pi.}$$

This principal value is important in connection with roots, the complex logarithm (Sec. 12.8), and certain integrals.

EXAMPLE 1 Polar form of complex numbers. Principal value

$z = 1 + i$ (Fig. 295) has the polar form $z = \sqrt{2}(\cos \frac{1}{4}\pi + i \sin \frac{1}{4}\pi)$. Hence $|z| = \sqrt{2}$, $\arg z = \frac{1}{4}\pi \pm 2n\pi$ $(n = 0, 1, \cdots)$, and $\text{Arg } z = \frac{1}{4}\pi$ (the principal value).

Similarly, $z = 3 + 3\sqrt{3}i = 6(\cos \frac{1}{3}\pi + i \sin \frac{1}{3}\pi)$, $|z| = 6$, and $\text{Arg } z = \frac{1}{3}\pi$. ◀

Caution! In using (4), we must pay attention to the quadrant in which z lies, since $\tan \theta$ has period π, so that the arguments of z and $-z$ have the same tangent. *Example:* for $\theta_1 = \arg(1 + i)$ and $\theta_2 = \arg(-1 - i)$ we have $\tan \theta_1 = \tan \theta_2 = 1$.

Triangle inequality

For any complex numbers we have the important **triangle inequality**

$$\text{(5)} \qquad \boxed{|z_1 + z_2| \leqq |z_1| + |z_2|} \qquad \text{(Fig. 296)}$$

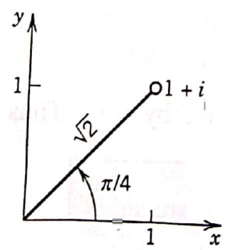

Fig. 295. Example 1

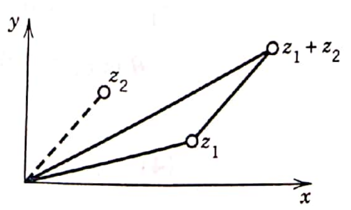

Fig. 296. Triangle inequality

which we shall use quite frequently. This inequality follows by noting that the three points 0, z_1, and $z_1 + z_2$ are the vertices of a triangle (Fig. 296) with sides $|z_1|$, $|z_2|$, and $|z_1 + z_2|$, and one side cannot exceed the sum of the other two sides. A formal proof is left to the reader (Prob. 33). (The triangle degenerates if z_1 and z_2 lie on the same straight line through the origin.)

By induction we obtain from (5) the **generalized triangle inequality**

$$\boxed{|z_1 + z_2 + \cdots + z_n| \leqq |z_1| + |z_2| + \cdots + |z_n|;} \tag{6}$$

that is, *the absolute value of a sum cannot exceed the sum of the absolute values of the terms.*

EXAMPLE 2 Triangle inequality

If $z_1 = 1 + i$ and $z_2 = -2 + 3i$, then (sketch a figure!)

$$|z_1 + z_2| = |-1 + 4i| = \sqrt{17} = 4.123 < \sqrt{2} + \sqrt{13} = 5.020.$$ ◀

Multiplication and Division in Polar Form

This will give us a "geometrical" understanding of multiplication and division. Let

$$z_1 = r_1(\cos\theta_1 + i\sin\theta_1) \qquad \text{and} \qquad z_2 = r_2(\cos\theta_2 + i\sin\theta_2).$$

Multiplication. By (3), Sec. 12.1, the product is at first

$$z_1 z_2 = r_1 r_2[(\cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2) + i(\sin\theta_1\cos\theta_2 + \cos\theta_1\sin\theta_2)].$$

The addition rules for the sine and cosine [(6) in Appendix A3.1] now yield

$$\boxed{z_1 z_2 = r_1 r_2[\cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2)].} \tag{7}$$

Taking absolute values and arguments on both sides of (7), we thus obtain the important rules

$$\boxed{|z_1 z_2| = |z_1||z_2|} \tag{8}$$

and

$$\boxed{\arg(z_1 z_2) = \arg z_1 + \arg z_2} \qquad \text{(up to multiples of } 2\pi). \tag{9}$$

Division. The quotient $z = z_1/z_2$ is the number z satisfying $zz_2 = z_1$. Hence $|zz_2| = |z|\,|z_2| = |z_1|$, $\arg(zz_2) = \arg z + \arg z_2 = \arg z_1$. This yields

$$\left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|} \qquad (z_2 \neq 0) \tag{10}$$

and

$$\arg\frac{z_1}{z_2} = \arg z_1 - \arg z_2 \qquad \text{(up to multiples of } 2\pi). \tag{11}$$

By combining formulas (10) and (11) we also have

$$\frac{z_1}{z_2} = \frac{r_1}{r_2}\,[\cos(\theta_1 - \theta_2) + i\sin(\theta_1 - \theta_2)]. \tag{12}$$

EXAMPLE 3 Illustration of formulas (8)–(11)

Let $z_1 = -2 + 2i$ and $z_2 = 3i$. Then $z_1z_2 = -6 - 6i$, $z_1/z_2 = 2/3 + (2/3)i$. Hence (make a sketch)

$$|z_1z_2| = 6\sqrt{2} = 3\sqrt{8} = |z_1||z_2|, \qquad |z_1/z_2| = 2\sqrt{2}/3 = |z_1|/|z_2|,$$

and for the arguments we obtain $\operatorname{Arg} z_1 = 3\pi/4$, $\operatorname{Arg} z_2 = \pi/2$,

$$\operatorname{Arg} z_1z_2 = -\frac{3\pi}{4} = \operatorname{Arg} z_1 + \operatorname{Arg} z_2 - 2\pi, \qquad \operatorname{Arg}(z_1/z_2) = \frac{\pi}{4} = \operatorname{Arg} z_1 - \operatorname{Arg} z_2.$$ ◀

EXAMPLE 4 Integer powers. De Moivre's formula

From (8) and (9) with $z_1 = z_2 = z$ we obtain by induction for $n = 0, 1, 2, \cdots$

$$\boxed{z^n = r^n(\cos n\theta + i\sin n\theta).} \tag{13}$$

Similarly, (12) with $z_1 = 1$ and $z_2 = z^n$ gives (13) for $n = -1, -2, \cdots$. For $|z| = r = 1$, formula (13) becomes **De Moivre's formula**[5]

$$\boxed{(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta.} \tag{13*}$$

We can use this to express $\cos n\theta$ and $\sin n\theta$ in terms of powers of $\cos\theta$ and $\sin\theta$. For instance, for $n = 2$ we have on the left $\cos^2\theta + 2i\cos\theta\sin\theta - \sin^2\theta$. Taking the real and imaginary parts on both sides of (13*) with $n = 2$ gives the familiar formulas

$$\cos 2\theta = \cos^2\theta - \sin^2\theta, \qquad \sin 2\theta = 2\cos\theta\sin\theta.$$

This shows that *complex* methods often simplify the derivation of *real* formulas. Try $n = 3$. ◀

Roots

If $z = w^n$ ($n = 1, 2, \cdots$), then to each value of w there corresponds one value of z. We shall immediately see that, conversely, to a given $z \neq 0$ there correspond precisely n distinct values of w. Each of these values is called an ***n*th root** of z, and we write

$$w = \sqrt[n]{z}. \tag{14}$$

Hence this symbol is *multivalued,* namely, *n-valued,* in contrast to the usual conventions made in *real* calculus. The n values of $\sqrt[n]{z}$ can easily be obtained as follows. In terms of polar forms

$$z = r(\cos\theta + i\sin\theta) \quad \text{and} \quad w = R(\cos\phi + i\sin\phi)$$

the equation $w^n = z$ becomes

$$w^n = R^n(\cos n\phi + i\sin n\phi) = z = r(\cos\theta + i\sin\theta).$$

[5]ABRAHAM DE MOIVRE (1667—1754), French mathematician, who introduced imaginary quantities in trigonometry and contributed to probability theory (see Sec. 22.8).

By equating the absolute values on both sides we have

$$R^n = r, \qquad \text{thus} \qquad R = \sqrt[n]{r}$$

where the root is real positive and thus uniquely determined. By equating the arguments we obtain

$$n\phi = \theta + 2k\pi, \qquad \text{thus} \qquad \phi = \frac{\theta}{n} + \frac{2k\pi}{n}$$

where k is an integer. For $k = 0, 1, \cdots, n - 1$ we get n *distinct* values of w. Further integers of k would give values already obtained. For instance, $k = n$ gives $2k\pi/n = 2\pi$, hence the w corresponding to $k = 0$, etc. Consequently, $\sqrt[n]{z}$, for $z \neq 0$, has the n distinct values

$$\text{(15)} \qquad \boxed{\sqrt[n]{z} = \sqrt[n]{r}\left(\cos\frac{\theta + 2k\pi}{n} + i\sin\frac{\theta + 2k\pi}{n}\right)}$$

where $k = 0, 1, \cdots, n - 1$. These n values lie on a circle of radius $\sqrt[n]{r}$ with center at the origin and constitute the vertices of a regular polygon of n sides. The value of $\sqrt[n]{z}$ obtained by taking the principal value of arg z and $k = 0$ in (15) is called the **principal value** of $w = \sqrt[n]{z}$.

In particular, taking $z = 1$, we have $|z| = r = 1$ and Arg $z = 0$. Then (15) gives

$$\text{(16)} \qquad \boxed{\sqrt[n]{1} = \cos\frac{2k\pi}{n} + i\sin\frac{2k\pi}{n},} \qquad k = 0, 1, \cdots, n - 1.$$

These n values are called the ***n*th roots of unity.** They lie on the circle of radius 1 and center 0, briefly called the **unit circle** (and used quite frequently!). Figures 297–299 show $\sqrt[3]{1} = 1, -\frac{1}{2} \pm \frac{1}{2}\sqrt{3}\,i$, $\sqrt[4]{1} = \pm 1, \pm i$, and $\sqrt[5]{1}$. If ω denotes the value corresponding to $k = 1$ in (16), then the n values of $\sqrt[n]{1}$ can be written as

$$1, \omega, \omega^2, \cdots, \omega^{n-1}.$$

Similarly, if w_1 is any nth root of an arbitrary complex number z, then the n values of $\sqrt[n]{z}$ in (15) are

$$w_1, \qquad w_1\omega, \qquad w_1\omega^2, \qquad \cdots, \qquad w_1\omega^{n-1}$$

because multiplying w_1 by ω^k corresponds to increasing the argument of w_1 by $2k\pi/n$.

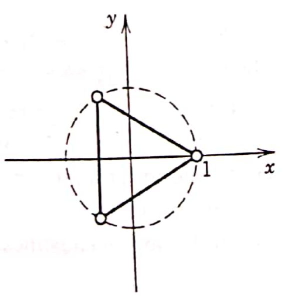

Fig. 297. $\sqrt[3]{1}$

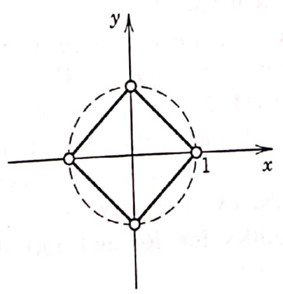

Fig. 298. $\sqrt[4]{1}$

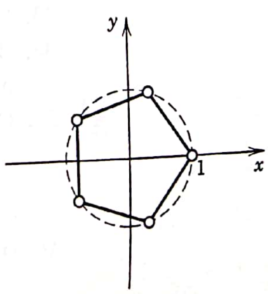

Fig. 299. $\sqrt[5]{1}$

PROBLEM SET 12.2

Polar Form. Principal Value. Conversion to $x + iy$

Polar forms will be needed frequently, so do these problems with great care. Represent each of the following in polar form and plot in the complex plane (showing the details of your work):

1. $1 + i$ **2.** $-2 + 2i$ **3.** $-3 - 4i$ **4.** -10 **5.** $3i, -3i$

6. $\dfrac{1 - i}{1 + i}$ **7.** $\left(\dfrac{6 + 8i}{4 - 3i}\right)^2$ **8.** $\dfrac{i}{3 + 3i}$ **9.** $\dfrac{2 + i}{5 - 3i}$ **10.** $\dfrac{7 - 5i}{4i}$

Determine the principal value of the argument:

11. $1 - i$ **12.** $-10, -10 - i$ **13.** $3 \pm 4i$ **14.** $-5 + 5i$ **15.** $-\pi - \pi i$

Represent each of the following in the form $x + iy$ and plot in the complex plane:

16. $\cos \frac{1}{2}\pi + i \sin \frac{1}{2}\pi$ **17.** $\sqrt{8}(\cos \frac{1}{4}\pi + i \sin \frac{1}{4}\pi)$

18. $6(\cos \frac{1}{3}\pi + i \sin \frac{1}{3}\pi)$ **19.** $\sqrt{18}(\cos \frac{3}{4}\pi + i \sin \frac{3}{4}\pi)$

Roots, Equations

20. TEAM PROJECT. Square Root. **(a)** Show that $w = \sqrt{z}$ has the values

$$(17) \quad w_1 = \sqrt{r}\left(\cos \frac{\theta}{2} + i \sin \frac{\theta}{2}\right), \quad w_2 = \sqrt{r}\left[\cos\left(\frac{\theta}{2} + \pi\right) + i \sin\left(\frac{\theta}{2} + \pi\right)\right] = -w_1.$$

(b) Obtain from (17) the often more practical formula

$$(18) \qquad \sqrt{z} = \pm\left[\sqrt{\tfrac{1}{2}(|z| + x)} + (\operatorname{sign} y)i\sqrt{\tfrac{1}{2}(|z| - x)}\right]$$

where sign $y = 1$ if $y \geqq 0$, sign $y = -1$ if $y < 0$, and all square roots of positive numbers are taken with the positive sign. *Hint.* Use (10) in Appendix 3 with $x = \theta/2$.

(c) Find the square roots of $4i$, $20 + 48i$, and $23 - 5\sqrt{8}\,i$ by both (17) and (18) and comment on the work involved.

(d) Do some further examples of your own and apply a method of checking your results.

Find and plot all roots:

21. $\sqrt[3]{1 + i}$ **22.** $\sqrt[3]{8i}$ **23.** $\sqrt[3]{216}$ **24.** $\sqrt[4]{-4}$ **25.** $\sqrt{-7 + 24i}$

26. $\sqrt[4]{-7 + 24i}$ **27.** $\sqrt[8]{1}$

Solve the equations:

28. $z^2 - (5 + i)z + 8 + i = 0$ **29.** $z^2 - (7 + i)z + 24 + 7i = 0$

30. $z^4 - (3 + 6i)z^2 - 8 + 6i = 0$ **31.** $z^2 + z + 1 - i = 0$

32. (Triangle inequality) Verify (5) for $z_1 = 4 - 6i$, $z_2 = 2 + 2.5i$.

33. (Triangle inequality) Prove (5).

34. (Parallelogram equality) Show that $|z_1 + z_2|^2 + |z_1 - z_2|^2 = 2(|z_1|^2 + |z_2|^2)$. This is called the *parallelogram equality*. Can you see why?

35. CAS PROJECT. Roots of Unity and Their Plots. Write a program for calculating these roots and for plotting them as points on the unit circle. Apply the program to $z^n = 1$ with $n = 2, 3, \cdots, 10$. Then extend the program to one for arbitrary roots, using an idea near the end of the text, and apply it to examples of your own choice.

36. (Inequalities for Re and Im) Prove the following inequalities, which we shall need occasionally.

$$(19) \qquad |\operatorname{Re} z| \leqq |z|, \qquad |\operatorname{Im} z| \leqq |z|.$$

Exercise 12.2 P/662

Represent in Polar form

· Q1 —— to —— Q10

Q1 $1+i$

Sol:- $Z = 1+i = x+iy$

where $x=1,\ y=1$

Since Polar form of complex No:

$Z = r(\cos\theta + i\sin\theta) \longrightarrow ①$

Also $r = \sqrt{x^2+y^2} \Rightarrow r = \sqrt{1+1} = \sqrt{2}$

& $\theta = \tan^{-1} y/x = \tan^{-1}(1/1) = \pi/4$

eq① $\Rightarrow$ $Z = \sqrt{2}\left[\cos\pi/4 + i\sin\pi/4\right]$

or

$Z = \sqrt{2}\, e^{i\pi/4}$

—— x ——

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Q2 $2i, -2i$

Sol:- (i) $Z = 2i = 0 + 2i \Rightarrow x + iy$

where $x = 0$, $y = 2$

Since Polar form of complex No:

$Z = r(\cos\theta + i\sin\theta) \longrightarrow ①$

Also $r = \sqrt{x^2 + y^2} \Rightarrow r = \sqrt{0 + 4} = 2$

$\boxed{r = 2}$

& $\theta = \tan^{-1}(y/x) = \tan^{-1}\left(\frac{2}{0}\right) = \pi/2$

eq ① $\Rightarrow Z = 2(\cos\pi/2 + i\sin\pi/2)$

$\boxed{Z = 2e^{i\pi/2}}$

(ii) $Z = -2i = 0 - 2i \Rightarrow x + iy$

$x = 0$, $y = -2$

Since $Z = r(\cos\theta + i\sin\theta)$ —— ①

$r = \sqrt{0 + 4} = \boxed{r = 2}$ & $\theta = \tan^{-1}\left(\frac{-2}{0}\right)$

$\theta = ?$ to calculate difficult so use another method.

Note :-

① $\left.\begin{array}{l}\cos\theta = \frac{\sqrt{3}}{2} \\ \sin\theta = 1/2\end{array}\right\} \Rightarrow \theta = 30^\circ$

③ $\left.\begin{array}{l}\cos\theta = -\frac{\sqrt{3}}{2} \\ \sin\theta = 1/2\end{array}\right\} \Rightarrow \theta = \pi - 30^\circ$

② $\left.\begin{array}{l}\cos\theta = \frac{\sqrt{3}}{2} \\ \sin\theta = -1/2\end{array}\right\} \Rightarrow \theta = -30^\circ$

④ $\left.\begin{array}{l}\cos\theta = -\frac{\sqrt{3}}{2} \\ \sin\theta = -1/2\end{array}\right\} \Rightarrow \theta = 30^\circ - \pi$

⑤ $\sin\theta = \sin(\theta + 2k\pi)$; $k \in \mathbb{Z}$

⑥ $\cos\theta = \cos(\theta + 2k\pi)$; $k \in \mathbb{Z}$.

⑦ $\text{cis}\,\theta = \cos\theta + i\sin\theta$

⑧ $\text{cis}(-\theta) = \cos\theta - i\sin\theta$

— X ————

(i) $Z = 2i$

$Z = 0 + 2i$

$Z = r\cos\theta + ir\sin\theta$

$Z = r[\cos\theta + i\sin\theta]$ — ①

To find r & θ

$\left.\begin{array}{l} r\cos\theta = 0 \\ r\sin\theta = 2 \end{array}\right\}$ Squaring & adding.

$r^2\cos^2\theta + r^2\sin^2\theta = 4$

$r^2[\cos^2\theta + \sin^2\theta] = 4$

$r^2 = 4 \Rightarrow \boxed{r = 2}$

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$$2\cos\theta = 0 \Rightarrow \cos\theta = 0$$
$$2\sin\theta = 2 \Rightarrow \sin\theta = 1 \Bigg\} \Rightarrow \theta = \pi/2$$

$$eq\ ① \Rightarrow Z = 2\left[\cos \pi/2 + i\sin \pi/2\right].$$

$$\boxed{Z = 2e^{i\pi/2}}$$

(ii) $Z = -2i$

$$Z = 0 - 2i$$

$$Z = r\cos\theta + i r\sin\theta$$

$$Z = r\left[\cos\theta + i\sin\theta\right] \text{ — ①}$$

To find r & θ

$$r\cos\theta = 0$$
$$r\sin\theta = -2 \Bigg\}$$ Squaring & addition

$$r^2\cos^2\theta + r^2\sin^2\theta = 4$$

$$r^2\left[\cos^2\theta + \sin^2\theta\right] = 4$$

$$r^2 = 4 \Rightarrow r = 2$$

$$2\cos\theta = 0 \Rightarrow \cos\theta = 0$$
$$2\sin\theta = -2 \Rightarrow \sin\theta = -1 \Bigg\} \Rightarrow \theta = -\pi/2$$

$$eq\ ① \Rightarrow Z\ 2\left[\cos(-\pi/2) + i\sin(-\pi/2)\right]$$

$Z = 2[\cos \pi/2 - i \sin \pi/2]$

$\boxed{Z = 2e^{-i\pi/2}}$

— X —

Q7 $\left(\frac{6+8i}{4-3i}\right)^2$

Now $\frac{6+8i}{4-3i} \times \frac{4+3i}{4+3i}$

$\frac{(6+8i)(4+3i)}{16+9} = \frac{\cancel{24}+18i+32i-\cancel{24}}{25}$

$\frac{50i}{25} = 2i$

$\left(\frac{6+8i}{4-3i}\right)^2 = (2i)^2 = -4+0i$

$Z = r\cos\theta + ir\sin\theta$

$= r[\cos\theta + i\sin\theta]$ — (1)

$\left.\begin{array}{l} r\cos\theta = -4 \\ r\sin\theta = 0 \end{array}\right]$ Sq + adding

$r^2\cos^2\theta + r^2\sin^2\theta = 16$

$r^2[\cos^2\theta + \sin^2\theta] = 16$

$r^2 = 16 \Rightarrow \boxed{r = 4}$

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21

$4\cos\theta = -4 \Rightarrow \cos\theta = -1$
$4\sin\theta = 0 \Rightarrow \sin\theta = 0$ $\Big\} \Rightarrow \theta = \pi - 0 = \pi$

eq ① $\Rightarrow$ $Z = 4[\cos\pi + i\sin\pi]$

$$\boxed{Z = 4e^{i\pi}}$$

Determine the Principle value of the argument

Q11 ⟶ to ⟶ Q15.

Q11 $1 - i$

$Z = r\cos\theta + ir\sin\theta$

$Z = r[\cos\theta + i\sin\theta]$ — ①

$r\cos\theta = 1$
$r\sin\theta = -1$ $\Big\}$ Sq: & adding

$r^2[\cos^2\theta + \sin^2\theta] = 2$

$r = \sqrt{2}$

$\sqrt{2}\cos\theta = 1 \Rightarrow \cos\theta = 1/\sqrt{2}$
$\sqrt{2}\sin\theta = -1 \Rightarrow \sin\theta = -1/\sqrt{2}$ $\Big\}$ $\boxed{\theta = -\pi/4}$

~~eq ① Z~~ = Principle value of θ is

$$\boxed{\theta = -\pi/4}$$

Represent each of the following in the form $x+iy$ and Plot in the complex plane.

Q16 —— to —— 19

Q1 $Z = 4\left[\cos \pi/3 + i \sin \pi/3\right]$

$Z = 4\left[1/2 + i\sqrt{3}/2\right]$

$\boxed{Z = 2 + 2\sqrt{3}i}$

—— X ——

Q2 $Z = 2\sqrt{2}\left[\cos \frac{3}{4}\pi + i \sin \frac{3}{4}\pi\right]$

$Z = 2\sqrt{2}\left[-1/\sqrt{2} + i \frac{1}{\sqrt{2}}\right]$.

$\boxed{Z = -2 + 2i}$

—— X ——

Q3 $Z = 10(\cos 0.4 + i \sin 0.4)$

$= 10[0.921 + i0.389)$

$\boxed{Z = 9.21 + i3.89}$

Find and plot all roots:

Q21 ——— to ——— Q27

Q21 let $Z_k = (1+i)^{1/3}$ $k = 0,1,2$

$= (r\cos\theta + i\, r\sin\theta)^{1/3} = r^{1/3}[\cos\theta + i\sin\theta]^{1/3}$

$= r^{1/3}[\cos(\theta + 2k\pi) + i\sin(\theta + 2k\pi)]^{1/3}$

$= r^{1/3}[\cos\frac{1}{3}(\theta + 2k\pi) + i\sin 1/3(\theta + 2k\pi)]$ D.T

$Z_k = r^{1/3}\operatorname{cis} 1/3(\theta + 2k\pi)$ —— (1)

To find r + θ

$\left.\begin{array}{l} r\cos\theta = 1 \\ r\sin\theta = 1 \end{array}\right\}$ Squaring and adding

$r^2\cos^2\theta + r^2\sin^2\theta = 2$

$\boxed{r = \sqrt{2}}$

$\left.\begin{array}{l} \sqrt{2}\cos\theta = 1 \Rightarrow \cos\theta = \frac{1}{\sqrt{2}} \\ \sqrt{2}\sin\theta = 1 \Rightarrow \sin\theta = 1/\sqrt{2} \end{array}\right\} \Rightarrow \theta = 45 = \pi/2$

eq (1) $\Rightarrow Z_k = (\sqrt{2})^{1/3}\operatorname{cis} 1/3[\pi/4 + 2k\pi]$

$Z_k = 2^{1/6}\operatorname{cis}(\frac{\pi}{12} + \frac{2k\pi}{3})$

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(24)

$$= 2^{1/6}\left[\cos\left(\frac{\pi}{12}+\frac{2k\pi}{3}\right)+i\sin\left(\frac{\pi}{12}+\frac{2k\pi}{3}\right)\right] \quad k=0,1,2$$

$k=0;\ 2^{1/6}\left[\cos \pi/12 + i\sin \pi/12\right]$

$k=1;\ 2^{1/6}\left[\cos\left(\frac{3\pi}{4}\right)+i\sin\left(3\pi/4\right)\right].$

$k=2;\ 2^{1/6}\left[\cos\left(\frac{17\pi}{12}\right)+i\sin\left(\frac{17\pi}{12}\right)\right].$

———— x ————

Solve the Equations:

Q28 —— to —— Q31

Q28 $z^2-(5+i)z+(8+i)=0$.

Sol: Here $a=1,\ b=-(5+i),\ c=(8+i)$

$$z=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$$

$$z=\frac{(5+i)\pm\sqrt{(5+i)^2-4(1)(8+i)}}{2(1)}$$

$$z=\frac{(5+i)\pm\sqrt{-8+6i}}{2} \quad \text{—(1)}$$

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Formula (25)

$$\boxed{\sqrt{z} = \pm\left[\sqrt{\tfrac{1}{2}(|z|+x)} + (\text{Sign}\,y)\,i\sqrt{\tfrac{1}{2}(|z|-x)}\right]}$$

If Sign $y = 1$ for $y \geq 0$ & Sign $y = -1$ if $y < 0$

Now $z = -8+6i = x+iy$

$|z| = |-8+6i| = \sqrt{64+36} = 10$

$\sqrt{-8+6i} = \pm\left[\sqrt{\tfrac{1}{2}(10-8)} + (1)\,i\sqrt{\tfrac{1}{2}(10+8)}\right]$

$= \pm\left[\sqrt{\tfrac{2}{2}} + i\sqrt{18/2}\right]$

$= \pm[1+i3]$

eq (1) $\Rightarrow$

$z = \dfrac{(5+i) \pm (1+3i)}{2}$

$= \dfrac{(5+i)+(1+3i)}{2}$ | $\dfrac{(5+i)-(1+3i)}{2}$

$\dfrac{6+4i}{2}$ | $\dfrac{4-2i}{2}$

$3+2i$ | $2-i$

$$\boxed{z = 3+2i,\ 2-i}$$

(26)

Q 30 $z^4 - 3(1+2i)z^2 - 8 + 6i = 0$

let $z_1 = z^2 \Rightarrow z_1^2 = z^4$

Now the given eq: becomes:

$z_1^2 - 3(1+2i)z_1 + (-8+6i) = 0$

$$z_1 = \frac{3(1+2i) \pm \sqrt{9(1+2i)^2 - 4(1)(-8+6i)}}{2(1)}$$

$$= \frac{(3+6i) \pm \sqrt{5+12i}}{2} \quad \text{—— } (1)$$

$z_2 = 5+12i$

$|z_2| = |5+12i| = \sqrt{25+144} = 13$

$$\sqrt{z_2} = \pm\left(\sqrt{\tfrac{1}{2}[|z_2|+x]} + (\text{Sign})\, i \sqrt{\tfrac{1}{2}[|z_2|-x]}\right)$$

$$= \pm\left(\sqrt{\tfrac{1}{2}(13+5)} + i\sqrt{\tfrac{1}{2}(13-5)}\right)$$

$\sqrt{z_2} = 3+2i$

eq (1) $\Rightarrow$

$$z_1 = \frac{(3+6i) \pm (3+2i)}{2}$$

$$z_1 = \left\{ \begin{matrix} 3+4i \\ 2i \end{matrix} \right\}$$

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(27)

Since $z_1 = z^2$

$z^2 = \{3+4i, 2i\}$

$z = \{\sqrt{3+4i}, \sqrt{2i}\}$

Now

$|z| = |3+4i| = \sqrt{9+16} = 5$

$\sqrt{3+4i} = \pm\left[\sqrt{\frac{1}{2}(5+3)} + i\sqrt{\frac{1}{2}(5-3)}\right]$

$\sqrt{3+4i} = \pm(2+i)$

Also

$|z| = |2i| = \sqrt{(0)^2+(2)^2} = 2$

$\sqrt{2i} = \pm\left[\sqrt{\frac{1}{2}(2)} + i\sqrt{\frac{1}{2}(2)}\right]$

$\sqrt{2i} = \pm(1+i)$

$z = \{2+i, -2-i, 1+i, -1-i\}$