which is convergent. Hence that Fourier series converges and the proof is complete. (Readers already familiar with uniform convergence will see that, by the Weierstrass test in Sec. 14.5, under our present assumptions the Fourier series converges uniformly, and our derivation of (6) by integrating term by term is then justified by Theorem 3 of Sec. 14.5.)

The proof of convergence in the case of a piecewise continuous function $f(x)$ and the proof that under the assumptions in the theorem the Fourier series (7) with coefficients (6) represents $f(x)$ are substantially more complicated; see, for instance, Ref. [C9]. ◀

EXAMPLE 2 Convergence at a jump as indicated in Theorem 1

The square wave in Example 1 has a jump at $x = 0$. Its left-hand limit there is $-k$ and its right-hand limit is k (Fig. 238). Hence the average of these limits is 0. The Fourier series (8) of the square wave does indeed converge to this value when $x = 0$ because then all its terms are 0. Similarly for the other jumps. This is in agreement with Theorem 1. ◀

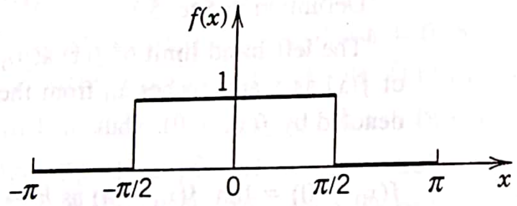
Summary. A Fourier series of a given function $f(x)$ of period 2π is a series of the form (7) with coefficients given by the Euler formulas (6). Theorem 1 gives conditions that are sufficient for this series to converge and at each x to have the value $f(x)$, except at discontinuities of $f(x)$, where the series equals the arithmetic mean of the left-hand and right-hand limits of $f(x)$ at that point.

PROBLEM SET 10.2

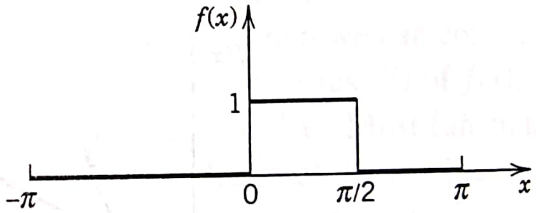
Fourier Series

Showing the details of your work, find the Fourier series of the function $f(x)$, which is assumed to have the period 2π, and plot accurate graphs of the first three partial sums, where $f(x)$ equals

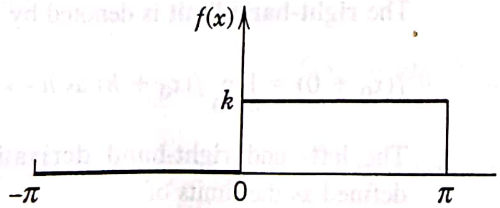
1.



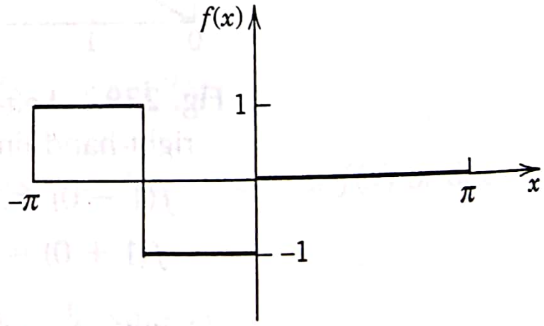
2.



3.



4.



5. $f(x) = x \quad (-\pi < x < \pi)$

6. $f(x) = x \quad (0 < x < 2\pi)$

7. $f(x) = x^2 \quad (-\pi < x < \pi)$

8. $f(x) = x^2 \quad (0 < x < 2\pi)$

9. $f(x) = x^3 \quad (-\pi < x < \pi)$

10. $f(x) = x + |x| \quad (-\pi < x < \pi)$

11. $f(x) = \begin{cases} 1 & \text{if } -\pi < x < 0 \\ -1 & \text{if } \quad 0 < x < \pi \end{cases}$

12. $f(x) = \begin{cases} -1 & \text{if } \quad 0 < x < \pi/2 \\ 0 & \text{if } \pi/2 < x < 2\pi \end{cases}$

13. $f(x) = \begin{cases} 1 & \text{if } -\pi/2 < x < \pi/2 \\ -1 & \text{if } \quad \pi/2 < x < 3\pi/2 \end{cases}$

14. $f(x) = \begin{cases} x & \text{if } -\pi/2 < x < \pi/2 \\ \pi - x & \text{if } \quad \pi/2 < x < 3\pi/2 \end{cases}$

15. $f(x) = \begin{cases} x & \text{if } -\pi/2 < x < \pi/2 \\ 0 & \text{if } \quad \pi/2 < x < 3\pi/2 \end{cases}$

16. $f(x) = \begin{cases} x^2 & \text{if } -\pi/2 < x < \pi/2 \\ \pi^2/4 & \text{if } \quad \pi/2 < x < 3\pi/2 \end{cases}$

17. (Discontinuity) Verify the last statement in Theorem 1 regarding discontinuities for the function in Prob. 1.

18. CAS (Orthogonality). Integrate and plot a typical integral, for instance, that of sin $3x$ sin $4x$, from $-a$ to a, as a function of a, and conclude orthogonality of sin $3x$ and sin $4x$ for $a = \pi$ from the plot.

19. CAS PROJECT. Fourier Series. (a) Write a program for obtaining any partial sum of a Fourier series (7).

(b) Using the program, list all partial sums of up to five nonzero terms of the Fourier series in Probs. 5, 11, and 15, and make three corresponding plots. Comment on the accuracy.

20. (Calculus review) Review integration techniques for integrals as they may arise from the Euler formulas, for instance, definite integrals of x sin nx, x^2 cos nx, e^{-x} sin nx, etc.

10.3 Functions of Any Period $p = 2L$

The functions considered so far had period 2π, for simplicity. Of course, in applications, periodic functions will generally have other periods. But we show that the transition from period $p = 2\pi$ to period[9] $p = 2L$ is quite simple. It amounts to a stretch (or contraction) of scale on the axis.

If a function $f(x)$ of period $p = 2L$ has a **Fourier series,** we claim that this series is

$$f(x) = a_0 + \sum_{n=1}^{\infty} \left(a_n \cos \frac{n\pi}{L} x + b_n \sin \frac{n\pi}{L} x \right) \tag{1}$$

with the **Fourier coefficients** of $f(x)$ given by the **Euler formulas**

$$\text{(2)} \quad \begin{aligned} &\text{(a)} \quad a_0 = \frac{1}{2L} \int_{-L}^{L} f(x)\, dx \\ &\text{(b)} \quad a_n = \frac{1}{L} \int_{-L}^{L} f(x) \cos \frac{n\pi x}{L}\, dx \qquad n = 1, 2, \cdots \\ &\text{(c)} \quad b_n = \frac{1}{L} \int_{-L}^{L} f(x) \sin \frac{n\pi x}{L}\, dx \qquad n = 1, 2, \cdots . \end{aligned}$$

[9] This notation is practical since in applications, L will be the length of a vibrating string (Sec. 11.2), of a rod in heat conduction (Sec. 11.5), etc.

<u>Fourier Series</u> :: Fourier Series is the approximation of function in terms of fundamental trigonometric function i.e $\cos x$ and $\sin x$. Let us assume that $f(x)$ is a Periodic function of Period 2π that can be represented by a trigonometric Series,

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx); \quad \text{—— ①}$$

that is We assume that this Series converges and has $f(x)$ as its sum. Given such a function $f(x)$ we want to determine the co-efficients a_n & b_n of the corresponding Series of equation ①

where a_n & b_n are called the fourier co-efficients and are given by.

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx \, dx.$$

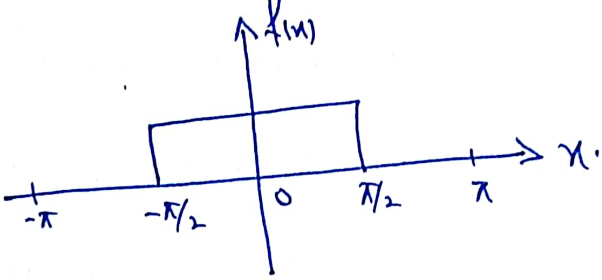
$$\& \; b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx \, dx.$$

where $n = 0, 1, 2$ ——

Ex. 10.2 (P-536).

Find the Fourier Series of the function $f(x)$ which is assumed to have the Period 2π & plot accurate graphs of the first three Partial Sums.

Q1



Sol:-
$$f(x) = \begin{cases} 0 & \text{if } -\pi < x < -\pi/2 \text{ \& } \pi/2 < x < \pi. \\ 1 & \text{if } -\pi/2 < x < \pi/2 \end{cases}$$

Now the Fourier series corresponding to $f(x)$ is given by:

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx) \text{ ——— (1).}$$

where $a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x) \cos nx\, dx$

& $b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x) \sin nx\, dx.$

Now $a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x) \cos(0)x\, dx$

$$= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\, (1)\, dx$$

$$= \frac{1}{\pi}\int_{-\pi}^{-\pi/2} 0\, dx + \frac{1}{\pi}\int_{-\pi/2}^{\pi/2} 1\, dx + \frac{1}{\pi}\int_{\pi/2}^{\pi} 0\, dx$$

$$= 0 + \frac{1}{\pi}\int_{-\pi/2}^{\pi/2} dx + 0$$

$$a_0 = \frac{1}{\pi} x \Big|_{-\pi/2}^{\pi/2} = \frac{1}{\pi}\left[\pi/2 - (-\pi/2)\right] = \frac{1}{\cancel{\pi}}\left[\frac{\cancel{2}\cancel{\pi}}{\cancel{2}}\right] = 1$$

$$\boxed{a_0 = 1}$$

Now $a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx.$

$$= \frac{1}{\pi}\int_{-\pi}^{-\pi/2} 0\,dx + \frac{1}{\pi}\int_{-\pi/2}^{\pi/2} 1\cos nx\,dx + \frac{1}{\pi}\int_{\pi/2}^{\pi} 0\,dx$$

$$= \frac{1}{\pi}\int_{-\pi/2}^{\pi/2} \cos nx\,dx$$

$$= \frac{1}{n\pi}\left[\sin nx \Big|_{-\pi/2}^{\pi/2}\right].$$

$$= \frac{1}{n\pi}\left[\sin n\pi/2 - \sin(-n\pi/2)\right].$$

$$= \frac{1}{n\pi}\left[2\sin n\pi/2\right].$$

$$\boxed{a_n = \frac{2}{n\pi}.\sin n\pi/2}$$

Similarly for b_n

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx.$$

$$= \frac{1}{\pi}\int_{-\pi}^{-\pi/2} 0\,dx + \frac{1}{\pi}\int_{-\pi/2}^{\pi/2} 1\sin nx\,dx + \frac{1}{\pi}\int_{\pi/2}^{\pi} 0\,dx$$

$$= \frac{1}{\pi}\int_{-\pi/2}^{\pi/2} \sin nx\,dx.$$

$$= \frac{1}{\pi}\left[\frac{-\cos nx}{n} \Big|_{-\pi/2}^{\pi/2} \right].$$

$$= \frac{-1}{n\pi}\left[\cos n\frac{\pi}{2} - \cos\left(-\frac{n\pi}{2}\right) \right].$$

$$= \frac{-1}{n\pi}\left[\cancel{\cos n\pi/2} - \cancel{\cos \frac{n\pi}{2}} \right]$$

$b_n = 0$

eq (1) $\Rightarrow$

$$f(x) = \frac{1}{2} + \sum_{n=1}^{\infty} \left[\left(\frac{2}{n\pi} \sin \frac{n\pi}{2} \right) \cos nx + (0) \sin nx \right].$$

$$= \frac{1}{2} + \frac{2}{\pi} \sum_{n=1}^{\infty} \left(\frac{1}{n} \sin \frac{n\pi}{2} \right) \cos nx.$$

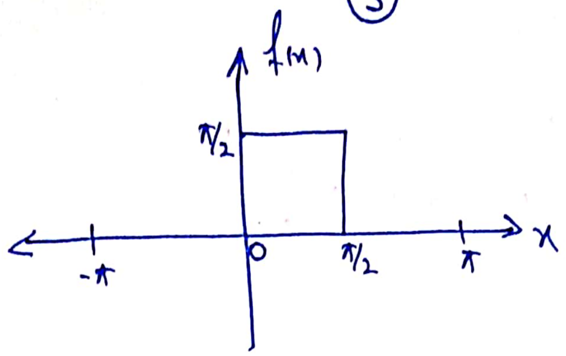
$$= \frac{1}{2} + \frac{2}{\pi} \left[\cos x + 0 - \frac{1}{3}\cos 3x + 0 + \frac{1}{5}\cos 5x + 0 - \frac{1}{7}\cos 7x + \cdots \right].$$

$$f(x) = \frac{1}{2} + \frac{2}{\pi} \left[\cos x - \frac{1}{3}\cos 3x + \frac{1}{5}\cos 5x - \frac{1}{7}\cos 7x + \cdots \right]$$

~ 0 ~

③

Q2



Sol:- $f(x) = \begin{cases} 0 & \text{if } -\pi < x < 0 \text{ \& } \pi/2 < x < \pi \\ \pi/2 & \text{if } 0 < x < \pi/2. \end{cases}$

Now the fourier series corresponding to $f(x)$ is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx) \text{ ——— } ①.$$

where $a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx \, dx$

$\&$ $b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx \, dx.$

Now $a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos(0) dx$

$$= \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) dx$$

$$= \frac{1}{\pi} \int_{-\pi}^{0} 0 \, dx + \frac{1}{\pi} \int_{0}^{\pi/2} \pi/2 \, dx + \frac{1}{\pi} \int_{\pi/2}^{\pi} 0 \, dx$$

$$= \frac{1}{\cancel{\pi}} \cdot \frac{\cancel{\pi}}{2} \int_{0}^{\pi/2} dx$$

$$= \frac{1}{2} x \Big|_{0}^{\pi/2} = \frac{1}{2} \cdot \frac{\pi}{2}$$

$$\boxed{a_0 = \pi/4}$$

Now $a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\, dx$

$$= \frac{1}{\pi}\int_{-\pi}^{0} 0\, dx + \frac{1}{\pi}\int_{0}^{\pi/2} \pi/2 \cos nx\, dx + \frac{1}{\pi}\int_{\pi/2}^{\pi} 0\, dx.$$

$$= \frac{1}{\pi}\cdot \pi/2 \int_{0}^{\pi/2} \cos nx\, dx$$

$$= \frac{1}{2}\left[\frac{\sin nx}{n}\right]_{0}^{\pi/2} \Rightarrow \frac{1}{2n}\left[\sin\frac{n\pi}{2} - \sin(0)\right].$$

$$\boxed{a_n = \frac{1}{2n}\sin n\pi/2}$$

Similarly.

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\, dx$$

$$= \frac{1}{\pi}\int_{-\pi}^{0} 0\, dx + \frac{1}{\pi}\int_{0}^{\pi/2} \pi/2 \sin nx\, dx + \frac{1}{\pi}\int_{\pi/2}^{\pi} 0\, dx.$$

$$= \frac{1}{\pi}\cdot \pi/2 \left[-\frac{\cos nx}{n}\Big|_{0}^{\pi/2}\right]$$

$$= -\frac{1}{2n}\left[\cos n\pi/2 - \cos 0\right] = \frac{1}{2n}\left[1 - \cos n\pi/2\right]$$

$$\boxed{b_n = \frac{1}{2n}\left[1 - \cos n\pi/2\right]}$$

eq (1) $\Rightarrow$

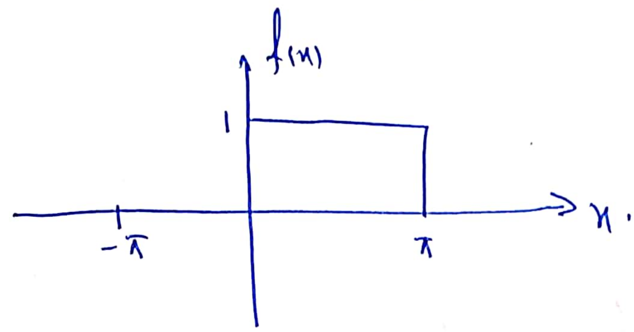
$$f(x) = \frac{\pi/4}{2} + \sum_{n=1}^{\infty}\left[\left(\frac{1}{2n}\sin\frac{n\pi}{2}\right)\cos nx + \frac{1}{2n}\left(1 - \cos n\pi/2\right)\sin nx\right].$$

$$f(x) = \frac{\pi}{8} + \frac{1}{2}\left[(\cos x + \sin x) + \sin 2x + \left(-\frac{1}{3}\cos 3x + \frac{1}{3}\sin 3x\right) + 0 + \left(\frac{1}{5}\cos 5x + \frac{1}{5}\sin 5x\right) + \cdots\right].$$

$$= \frac{\pi}{8} + \frac{1}{2}\left[\cos x + \sin x + \sin 2x - \frac{1}{3}\cos 3x + \frac{1}{3}\sin 3x + \frac{1}{5}\cos 5x + \frac{1}{5}\sin 5x + \cdots\right]$$

~ 0 ~

Q3



Sol:-

$$f(x) = \begin{cases} 0 & \text{if } -\pi < x < 0 \\ 1 & \text{if } 0 < x < \pi. \end{cases}$$

The fourier series corresponding to $f(x)$ is given by.

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left[a_n \cos nx + b_n \sin nx\right] \quad \text{——— (1)}$$

where $a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx$ & $b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx$.

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(0)\,dx$$

$$= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cdot 1\,dx = \frac{1}{\pi}\int_{-\pi}^{0} 0\,dx + \frac{1}{\pi}\int_{0}^{\pi} 1\,dx = 1$$

$$\boxed{a_0 = 1}$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx$$

$$= \frac{1}{\pi}\int_{-\pi}^{0} 0\cos nx\,dx + \frac{1}{\pi}\int_{0}^{\pi} 1\cdot\cos nx\,dx.$$

$$= \frac{1}{\pi}\,\frac{\sin nx}{n}\Big|_{0}^{\pi} = \frac{1}{n\pi}\sin \pi n$$

$$\boxed{a_n = \frac{1}{n\pi}\sin \pi n}$$

Similarly:

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx.$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{0} 0\,dx + \frac{1}{\pi}\int_{0}^{\pi} 1\cdot\sin nx\,dx.$$

$$b_n = \frac{1}{\pi}\left[-\frac{\cos nx}{n}\right|_{0}^{\pi}$$

$$\boxed{b_n = \frac{1}{n\pi}\left[1-\cos n\pi\right]}$$

eq(1) $\Rightarrow$

$$f(x) = \frac{1}{2} + \sum_{n=1}^{\infty}\left[\left(\frac{1}{n\pi}\sin n\pi\right)\cos nx + \frac{1}{n\pi}\left[1-\cos n\pi\right]\sin nx\right].$$

$$= \frac{1}{2} + \frac{1}{\pi}\sum_{n=1}^{\infty}\left[\frac{1}{n}\sin n\pi\cos nx + \frac{1}{n}\sin nx - \frac{1}{n}\cos n\pi\sin nx\right].$$

$$= \frac{1}{2} + \frac{1}{\pi}\Big[(\sin x + \sin x) + (\tfrac{1}{2}\cancel{\sin 2x} - \tfrac{1}{2}\cancel{\sin 2x})$$

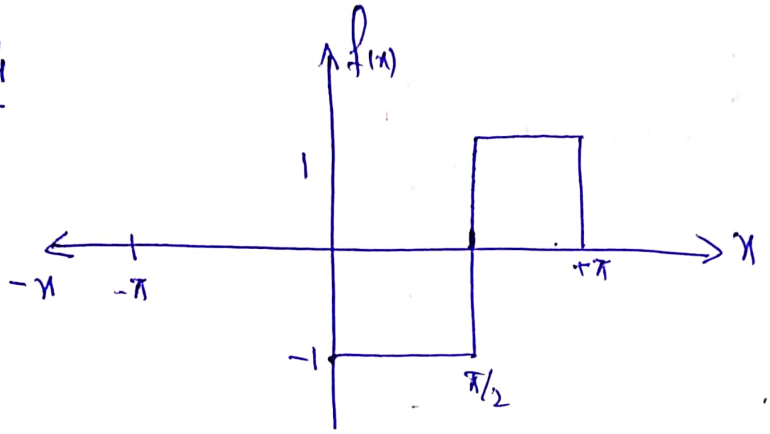
$$+ (\tfrac{1}{3}\sin 3x + \tfrac{1}{3}\sin 3x) + (\tfrac{1}{4}\cancel{\sin 4x} - \tfrac{1}{4}\cancel{\sin 4x}) + \cdots\Big]$$

$$f(x) = \frac{1}{2} + \frac{1}{\pi}\left[2\sin x + \frac{2}{3}\sin 3x + \frac{2}{5}\sin 5x + \cdots\right].$$

$$= \frac{1}{2} + \frac{2}{\pi}\left[\sin x + \frac{1}{3}\sin 3x + \frac{1}{5}\sin 5x + \cdots\right].$$

~ 0 ~

Q4



Sol: $f(x) = \begin{cases} 0 & \text{if } -\pi < x < 0 \\ -1 & \text{if } 0 < x < \pi/2 \\ 1 & \text{if } \pi/2 < x < \pi. \end{cases}$

Now $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left[a_n \cos nx + b_n \sin nx\right]$ — ①

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(0)\,dx$$

$$a_0 = \frac{1}{\pi}\int_{\pi}^{0} 0\,dx + \frac{1}{\pi}\int_{0}^{\pi/2} -1\,dx + \frac{1}{\pi}\int_{\pi/2}^{\pi} 1\,dx$$

$$\boxed{a_0 = 0}$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx.$$

$$= \frac{1}{\pi}\int_{-\pi}^{0} 0\,dx + \frac{1}{\pi}\int_{0}^{\pi/2} -1 \cos nx\,dx + \frac{1}{\pi}\int_{\pi/2}^{\pi} (1)\cos nx\,dx.$$

$$\boxed{a_n = -\frac{2}{n\pi}\sin\frac{n\pi}{2} + \frac{1}{n\pi}\sin n\pi}$$

Now

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx$$

$$= \frac{1}{\pi}\int_{-\pi}^{0} 0\,dx + \frac{1}{\pi}\int_{0}^{\pi/2} (-1)\sin nx\,dx + \frac{1}{\pi}\int_{\pi/2}^{\pi} 1\sin nx\,dx.$$

$$\boxed{b_n = \frac{2}{n\pi}\cos\frac{n\pi}{2} - \frac{1}{n\pi}\cos n\pi - \frac{1}{n\pi}}$$

eq ① $\Rightarrow$

$$f(x) = \frac{0}{2} + \sum_{n=1}^{\infty}\left[\left(\frac{-2}{n\pi}\sin\frac{n\pi}{2} + \frac{1}{n\pi}\sin n\pi\right)\cos nx + \left(\frac{2}{n\pi}\cos\frac{n\pi}{2} - \frac{1}{n\pi}\cos n\pi - \frac{1}{n\pi}\right)\sin nx\right].$$

$$= \frac{1}{\pi}\left[(-2\cos x) + (-2\sin 2x) + \left(\frac{2}{3}\cos 3x\right) + \cdots\right].$$

$$f(x) = \frac{1}{\pi}\left[-2\cos x - 2\sin 2x + \frac{2}{3}\cos 3x + \cdots\right]$$

Q5 $f(x) = x \ (-\pi < x < \pi)$.

Sol:- The F.S of $f(x)$ is given as:

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} [a_n \cos nx + b_n \sin nx] \quad \text{--- (1)}$$

where $a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)dx$

$$= \frac{1}{\pi}\int_{-\pi}^{\pi} x\,dx = \frac{1}{\pi}\left[\frac{x^2}{2} \Big|_{-\pi}^{\pi} \right] = \frac{1}{2\pi}\left[\cancel{\pi^2} - \cancel{\pi^2}\right]$$

$$\boxed{a_0 = 0}$$

Now $a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx$

$$= \frac{1}{\pi}\int_{-\pi}^{\pi} x\cos nx\,dx.$$

$$= \frac{1}{\pi}\left[x\frac{\sin nx}{n}\Big|_{-\pi}^{\pi} - 1/n\left(-\frac{\cos nx}{n}\right)\Big|_{-\pi}^{\pi} \right].$$

$$= \frac{1}{\pi}\left[\frac{1}{n}\left[\pi \cancel{\sin n\pi} - \pi \cancel{\sin n\pi}\right] + 1/n^2\left[\cancel{\cos n\pi} - \cancel{\cos n\pi}\right]\right]$$

$$\boxed{a_n = 0}$$

$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx$

$$= \frac{1}{\pi}\int_{-\pi}^{\pi} x\sin nx\,dx.$$

$$= \frac{1}{\pi}\left[-\frac{x\cos nx}{n}\Big|_{-\pi}^{\pi} + \frac{\sin nx}{n^2}\Big|_{-\pi}^{\pi} \right].$$

$$= -\frac{1}{n\pi}\left[\pi\cos n\pi + \pi\cos n\pi\right] + \frac{1}{n^2}\left[\sin n\pi + \sin n\pi\right].$$

$$\boxed{b_n = -\frac{2}{n}\cos n\pi + \frac{2}{n^2}\sin n\pi}$$

eq (1) $\Rightarrow$

$$f(x) = 0 + \sum_{n=1}^{\infty}\left[(0)\cos nx + \left(-\frac{2}{n}\cos n\pi + \frac{2}{n^2}\sin n\pi\right)\sin nx\right].$$

$$= 2\sin x - \sin 2x + 2/3 \sin 3x - 1/2 \sin 4x + \cdots$$

$$\boxed{f(x) = 2\left[\sin x - 1/2 \sin 2x + 1/3 \sin 3x - 1/4 \sin 4x + \cdots\right].}$$

Q(6) $f(x) = x \quad (0 < x < 2\pi)$.

Sol:- The F.S of $f(x)$ is

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left[a_n\cos nx + b_n\sin nx\right] \quad \text{—— (1)}$$

when $a_0 = \frac{1}{\pi}\int_0^{2\pi} f(x)\,dx$

$$= \frac{1}{\pi}\int_0^{2\pi} x\,dx$$

$$\boxed{a_0 = 2\pi}$$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} x\cos nx\,dx$$

$$= \frac{1}{\pi}\left[x\frac{\sin nx}{n}\Big|_0^{2\pi} + \frac{1}{n^2}\cos nx\Big|_0^{2\pi}\right]$$

$$a_n = \frac{1}{n\pi}\left[2\pi\sin 2n\pi - 0\right] + \frac{1}{n^2}\left[\cos 2n\pi - \cos(0)\right].$$

$$= \frac{1}{n\pi}\left[2\pi\sin 2n\pi\right] + \frac{1}{n^2}\left[\cos 2n\pi - 1\right].$$

$$\boxed{a_n = \frac{1}{n}\sin 2n\pi + \frac{1}{n^2}(\cos 2n\pi - 1)}$$

Now

$$b_n = \frac{1}{\pi}\int_0^{2\pi} x\sin nx\,dx.$$

$$= \frac{1}{\pi}\left[-x\frac{\cos nx}{n}\Big|_0^{2\pi} + \frac{1}{n^2}\sin nx\Big|_0^{2\pi}\right].$$

$$b_n = \frac{1}{\pi}\left[\frac{-2\pi}{n}\cos 2n\pi + \frac{1}{n^2}\sin 2\pi n\right].$$

eq (1) $\Rightarrow$

$$f(x) = \frac{2\pi}{2} + \sum_{n=1}^{\infty}\left[\left(\frac{1}{n}\sin 2n\pi + \frac{1}{n^2}(\cos 2n\pi - 1)\right)\cos nx\right.$$
$$\left. + \frac{1}{\pi}\left[\frac{-2\pi}{n}\cos 2n\pi + \frac{1}{n^2}\sin 2\pi n\right]\sin nx\right].$$

$$f(x) = \pi + \left[-2\sin x - \sin 2x - \frac{2}{3}\sin 3x - \frac{1}{2}\sin 4x + \cdots\right].$$

Q10 $f(x) = x + |x| \quad (-\pi < x < \pi)$.

Sol:- $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} [a_n \cos nx + b_n \sin nx]$ — ①

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} (x + |x|)\,dx$$

$$= \frac{1}{\pi}\frac{x^2}{2}\Big|_{-\pi}^{\pi} + \frac{1}{\pi}\frac{x^2}{2}\Big|_{-\pi}^{\pi}$$

$$= \frac{1}{2\pi}(\cancel{\pi^2} - \cancel{\pi^2}) + \frac{1}{2\pi}(\cancel{\pi^2} - \cancel{\pi^2})$$

$$\boxed{a_0 = 0}$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} (x + |x|)\cos nx\,dx$$

$$= \frac{1}{\pi}\left[x\frac{\sin nx}{n}\Big|_{-\pi}^{\pi} + \frac{1}{\pi}\frac{\cos nx}{n^2}\Big|_{-\pi}^{\pi} + \frac{1}{\pi}|x|\cdot\frac{\sin nx}{n}\Big|_{-\pi}^{\pi} + \frac{1}{\pi}\frac{\cos nx}{n^2}\Big|_{-\pi}^{\pi}\right.$$

$$\boxed{a_n = \frac{2}{n}\sin n\pi}$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} (x + |x|)\sin nx\,dx$$

$$= \frac{1}{\pi}\left[-x\frac{\cos nx}{n}\Big|_{-\pi}^{\pi} + \frac{1}{n^2}\sin nx\Big|_{-\pi}^{\pi} - |x|\frac{\cos nx}{n}\Big|_{-\pi}^{\pi} + \frac{1}{n^2}\sin n\pi\Big|_{-\pi}^{\pi}\right].$$

$$\boxed{b_n = \frac{1}{\pi}\left[-\frac{2\pi}{n}\cos n\pi + \frac{2}{n^2}\sin n\pi + \frac{2}{n^2}\sin n\pi\right]}$$

e) Q $\Rightarrow$

$$f(x) = \frac{0}{2} + \sum_{n=1}^{\infty} \left[\left(\frac{2}{n}\sin n\pi\right)\cos nx + \frac{1}{\pi}\left[\frac{-2\pi}{n}\cos n\pi + \frac{2}{n^2}\sin n\pi + \frac{2}{n^2}\sin n\pi\right]\sin nx\right].$$

$$f(x) = 2\sin x - \sin 2x + \frac{2}{3}\sin 3x - \frac{2}{4}\sin 4x + \cdots$$

$$f(x) = 2\left(\sin x - \tfrac{1}{2}\sin 2x + \tfrac{1}{3}\sin 3x - \tfrac{1}{4}\sin 4x + \cdots\right).$$

Q12 $f(x) = \begin{cases} -1 & \text{if } 0 < x < \pi/2 \\ 0 & \text{if } \pi/2 < x < 2\pi \end{cases}$

Sol:- $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$ —— (1)

$$a_0 = \frac{1}{\pi}\int_0^{2\pi} f(x)\,dx$$

$$= \frac{1}{\pi}\int_0^{\pi/2} -1\,dx + \frac{1}{\pi}\int_{\pi/2}^{2\pi} 0\,dx$$

$$\boxed{a_0 = -\tfrac{1}{2}}$$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx\,dx$$

$$= \frac{1}{\pi}\int_0^{\pi/2} (-1)\cos nx\,dx + \frac{1}{\pi}\int_{\pi/2}^{2\pi} 0\,dx$$

$$= -\frac{1}{\pi}\left.\frac{\sin nx}{n}\right|_0^{\pi/2} + 0$$

$$\boxed{a_n = -\frac{1}{n\pi}\sin n\pi/2}$$

$$b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx\,dx.$$

$$= \frac{1}{\pi}\int_0^{\pi/2}(-1)\sin nx\,dx + \frac{1}{\pi}\int_{\pi/2}^{2\pi} 0\,dx$$

$$\boxed{b_n = \frac{1}{n\pi}\left[\cos\frac{n\pi}{2} - 1\right]}$$

eq ① $\Rightarrow$

$$f(x) = \frac{-1/2}{2} + \sum_{n=1}^{\infty}\left[\frac{-1}{n\pi}\sin\frac{n\pi}{2}\cos nx + \frac{1}{n\pi}\left[\cos\frac{n\pi}{2} - 1\right]\sin nx\right].$$

$$= -1/4 + \frac{1}{\pi}\left[-\cos x - \sin x - \sin 2x - 1/3\cos 3x - 1/3\sin 3x + \cdots\right].$$

~ 0 ~