Problem Set 13.1–13.2

Find a representation $z = z(t)$ of the straight line segment with endpoints

1. $z = 0$ and $z = 1 + 2i$
2. $z = -3 + 2i$ and $z = -4 + 5i$
3. $z = 4 + 2i$ and $z = 3 + 5i$
4. $z = 0$ and $z = 5 + 10i$
5. $z = -4i$ and $z = -7 + 38i$
6. $z = 1 - i$ and $z = 9 - 5i$

What curves are represented by the following functions?

7. $(1 + 2i)t, \quad 0 \leqq t \leqq 3$
8. $3 - it, \quad -1 \leqq t \leqq 1$
9. $1 - i - 2e^{it}, \quad 0 \leqq t \leqq \pi$
10. $2 + i + 3e^{it} \quad 0 \leqq t < 2\pi$
11. $t + 3t^2 i, \quad -1 \leqq t \leqq 2$
12. $t + 2it^3, \quad -2 \leqq t \leqq 2$
13. $\cos t + 2i \sin t, \quad -\pi < t < \pi$
14. $t + t^{-1}i, \quad \frac{1}{2} \leqq t \leqq 5$

Represent the following curves in the form $z = z(t)$.

15. $|z - 3 + 4i| = 4$
16. $|z - i| = 2$
17. $y = 1/x$ from $(1, 1)$ to $(3, \frac{1}{3})$
18. $y = x^2$ from $(0, 0)$ to $(2, 4)$
19. $x^2 + 4y^2 = 4$
20. $4(x - 1)^2 + 9(y + 2)^2 = 36$

Evaluate $\int_C f(z)\,dz$ by the method in Theorem 1 and check the result by Theorem 2:

21. $f(z) = az + b$, C the line segment from $-1 - i$ to $1 + i$
22. $f(z) = e^{2z}$, C the segment in Prob. 1
23. $f(z) = z^3$, C the semicircle $|z| = 2$ from $-2i$ to $2i$ in the right half-plane
24. $f(z) = 5z^2$, C the boundary of the triangle with vertices 0, 1, i (clockwise)

Evaluate $\int_C f(z)\,dz$, where

25. $f(z) = 2z^4 - z^{-4}$, C the unit circle (counterclockwise)
26. $f(z) = \operatorname{Re} z$, C the parabola $y = x^2$ from 0 to $1 + i$
27. $f(z) = \operatorname{Im} z$, C the circle $|z| = r$ (counterclockwise)
28. $f(z) = 4z - 3$, C the straight line segment from i to $1 + i$
29. $f(z) = (z - 1)^{-1} + 2(z - 1)^{-2}$, C the circle $|z - 1| = 4$ (clockwise)
30. $f(z) = \sin z$, C the line segment from 0 to i
31. $f(z) = e^{2z}$, C the vertical segment from πi to $2\pi i$
32. $f(z) = z \cos z^2$, C any path from 0 to πi
33. $f(z) = \cosh 3z$, C any path from $\pi i/6$ to 0
34. $f(z) = e^z$, C the boundary of the square with vertices 0, 1, $1 + i$, i (clockwise)
35. $f(z) = \operatorname{Re}(z^2)$, C the square in Prob. 34
36. $f(z) = \operatorname{Im}(z^2)$, C the square in Prob. 34
37. $f(z) = \bar{z}$, C the parabola $y = x^2$ from 0 to $1 + i$
38. $f(z) = (z - 1)^{-1} - (z - 1)^{-2}$, C the circle $|z - 1| = \frac{1}{2}$ (clockwise)
39. $f(z) = \sin^2 z$, C the semicircle $|z| = \pi$ from $-\pi i$ to πi in the right half-plane
40. $f(z) = \sec^2 z$, C any path from $\pi i/4$ to $\pi/4$ in the unit disk

Q7 $\cosh(-2+3i)$

Sol:- $f(z) = \cosh(-2+3i)$

$= \cosh(-2)\cosh(3i) + \sinh(-2)\sinh(3i)$

$= \cosh(2)\cos(3) + (-1)\sinh(2) . i\sin(3)$

$= \cosh(2)\cos(3) - i\sinh(2)\sin(3)$

$$\boxed{f(z) = -3.724 - 0.511i}$$

Ch#13

Complex Integration

Complex line Integral:- The complex line integral of a complex function $f(z)$ over a Path C is written as:

$$\int_C f(z)dz$$

If "c" is closed Path than the complex line integral can be written as:

$$\oint_C f(z)dz.$$

Note:- Path "c" can be a straight line, circle or a semicircle.

Evaluate $\int_C f(z)\,dz$, where

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Q25 $f(z) = 2z^4 - z^{-4}$, C the unit circle (counter clock wise).

Sol: Since the standard form of a circle is

$|z - z_0| = \rho$ —(1)

Given unit circle

$|z| = 1$ —(2)

comparing eq(1) & eq(2) we get

$z_0 = 0, \quad \rho = 1$

∴ The Parameteric form of a circle is given by

$z(t) = z_0 + \rho e^{it} \qquad 0 \le t \le 2\pi$ (Full circle)

$z(t) = 0 + e^{it}$

Diff w.r.t t B.s

$z'(t) = i e^{it}$

& $f(z(t)) = 2(e^{it})^4 - (e^{it})^{-4}$

$= 2e^{4it} - e^{-4it}$

According to theorem # 1

$$\int_C f(z)\,dz = \int_0^{2\pi} f[z(t)]\, z'(t)\, dt$$

$$= \int_0^{2\pi} (2e^{4it} - e^{-4it})(ie^{it})\, dt$$

$$= 2i\int_0^{2\pi} e^{5it}\,dt - i\int_0^{2\pi} e^{-3it}\,dt$$

$$= \frac{2i}{5i}\, e^{5it}\Big|_0^{2\pi} - \frac{i}{-3i}\, e^{-3it}\Big|_0^{2\pi}$$

$$= \frac{2}{5}\left[e^{10\pi i} - 1\right] + 1/3\left[e^{-6\pi i} - 1\right]$$

$$= 2/5\left[\cos 10\pi + i\,\cancel{\sin 10\pi}^{\,0}\ \underset{-1}{}\right] + 1/3\left[\cos(-6\pi) + i\,\cancel{\sin(-6\pi)}^{\,0} - 1\right]$$

$$= 2/5\left[\cancel{1} - \cancel{1}\right] + 1/3\left[\cancel{1} - 1\right]$$

$$\int_C f(z)dz = 0$$

Q26 $f(z) = \operatorname{Re} z$, C the Parabola $y = x^2$ from 0 to $1+i$

Sol:- The general Palameteric form of a circle is:

$$Z(t) = x(t) + iy(t) \quad \text{—①}$$

let $x(t) = t$

$\Rightarrow y(t) = x^2(t)$

$\Rightarrow y = t^2$

$\because z(t) = t + it^2$

Diff: W.r.t "t" B.S

$z'(t) = 1 + 2it$

$f[z(t)] = Re[z] = t$

using theorem # 1

$$\int_c f(z)dz = \int_a^b f[z(t)] \cdot z'(t) dt.$$

Limits

$(0,0)$ to $(1,1)$

when $x=0$; $t=0$ } As $x(t)=t$
when $x=1$; $t=1$

$$\int_c f(z)dz = \int_0^1 t(1+2it)\,dt$$

$$= \int_0^1 t\,dt + 2i\int_0^1 t^2 dt = \frac{t^2}{2}\Big|_0^1 + 2i\frac{t^3}{3}\Big|_0^1$$

$$\boxed{\int_c f(z)dz = \frac{1}{2} + \frac{2}{3}i}$$

Q27 $f(z) = \operatorname{Im} z$; C the circle $|z| = r$ (counter clockwise).

Sol:- The genral eq of circle is

$$|z - z_0| = \rho \quad \text{—} ①$$

Also given $|z| = r$ — ②

comparing eq ① & eq ② we get

$$z_0 = 0, \quad \rho = r$$

As Parametric eq:- of circle is

$$z(t) = z_0 + \rho e^{it} \quad 0 \le t \le 2\pi.$$

$$z(t) = 0 + r e^{it}$$

$$z(t) = r e^{it}$$

Diff w.r.t "t" B.S

$$z'(t) = r i e^{it}$$

Now

$$f[z(t)] = \operatorname{Im}[r e^{it}]$$

$$= \operatorname{Im}[r\cos t + i r \sin t]$$

$$= r \sin t$$

using Theorem # 1

$$\int_C f(z)dz = \int_a^b f[z(t)]\, z'(t)\, dt.$$

$$= \int_0^{2\pi} r\sin t\,(r i e^{it})\,dt$$

$$= i r^2 \int_0^{2\pi} \sin t\, e^{it}\,dt$$

$$= i r^2 \int_0^{2\pi} \sin t\,[\cos t + i\sin t]\,dt$$

$$= i r^2 \int_0^{2\pi} (\sin t\cos t + i\sin^2 t)\,dt$$

$$= i r^2 \left(\left[\cancel{\frac{\sin^2 t}{2}} \right]_0^{2\pi} + i\frac{1}{2} t \Big|_0^{2\pi} \right)$$ Reduction formula.

$$= i r^2 [\pi i]$$

$$\boxed{\int_C f(z)\,dz = -\pi r^2}$$

Q28 $f(z) = 4z-3$, c the straight line segment from i to $1+i$

Sol:- Given $f(z) = 4z-3$, $z_1 = i$ & $z_2 = 1+i$

The Parametric form of a ~~straig~~ straight line is

$$z(t) = z_1 + (z_2 - z_1)t \; ; \; 0 \le t \le 1$$

$z(t) = i + (1+i-i)t$

$z(t) = i+t$

Diff: w.r.t t B.S

$z'(t) = 1$

Now

$f[z(t)] = 4[i+t] - 3$

$= 4i + 4t - 3$

$= 4t - 3 + 4i$

using theorm # 1.

$$\int_c f(z)dz = \int_a^b f[z(t)]\, z'(t)\, dt.$$

$$= \int_0^1 (4t-3+4i)\, 1\, dt$$

$$= 4t^2/2 \Big|_0^1 - 3t \Big|_0^1 + 4it \Big|_0^1$$

$$= 2 - 3(1) + 4i$$

$$\boxed{\int_c f(z)dz = -1+4i}$$

Q29 $f(z) = (z-1)^{-1} + 2(z-1)^{-2}$, C the circle $|z-1| = 4$ (clockwise).

Sol:- Since the general form of circle:

$$|z - z_0| = \rho \quad \text{—} ①$$

Given that

$$|z-1| = 4 \quad \text{—} ②$$

comparing eq① & eq② we get

$$z_0 = 1 \ ; \ \rho = 4$$

$\because$ Parametric form of circle

$$z(t) = z_0 + \rho e^{it} \ ; \ 2\pi \leq t \leq 0$$

$$z(t) = 1 + 4e^{it}$$

Diff: w.r.t "t" B.S

$$z'(t) = 4ie^{it}$$

Now

$$f[z(t)] = [\cancel{1} + 4e^{it} - \cancel{1}]^{-1} + 2[\cancel{1} + 4e^{it} - \cancel{1}]^{-2}$$

$$= \frac{1}{4} e^{-it} + \frac{2}{(4)^2} e^{-2it}$$

$$= \frac{1}{4} e^{-it} + \frac{1}{8} e^{-2it}$$

. Using theorem # 1

$$\int_C f(z)\,dz = \int_a^b f[z(t)]\,z'(t)\,dt$$

$$= \int_{2\pi}^{0} \left(\tfrac{1}{4}e^{-it} + \tfrac{1}{8}e^{-2it}\right) 4ie^{it}\,dt$$

$$= \int_{2\pi}^{0} [i + \tfrac{i}{2}e^{-it}]\,dt$$

$$= it\Big|_{2\pi}^{0} + \tfrac{i}{2}\frac{e^{-it}}{-i}\Big|_{2\pi}^{0}$$

$$= -2\pi i - \tfrac{1}{2}[1 - e^{-2\pi i}]$$

$$= -2\pi i - \tfrac{1}{2}[1 - \{\cos(-2\pi) + i\overset{0}{\sin(-2\pi)}\}]$$

$$= -2\pi i - \tfrac{1}{2}[1 - \cos 2\pi]$$

$$= -2\pi i - \tfrac{1}{2}[1 - 1]$$

$$\boxed{\int_C f(z)\,dz = -2\pi i}$$

Q30 $f(z) = \sin z$, c the line segment from 0 to i

Sol:- $z_1 = 0$; $z_2 = i$

As $z(t) = z_1 + (z_2 - z_1)t$; $0 \le t \le 1$

$z(t) = 0 + (i - 0)t$

$z(t) = it$

Diff: w.r.t "t" B.S

$\frac{d}{}$ $f[z(t)] = \sin it$

$$\int_c f(z)dz = \int_a^b f(z(t))\, z'(t)dt$$

$$= \int_0^1 \sin t \cdot i\, dt$$

$$= i\int_0^1 \sin t\, dt$$

As $\sin it = i \sinh t$

$$\int_c f(z)dz = i^2 \int_0^1 \sinh t\, dt = -[\cosh t]_0^1$$

$$= -[1.543 - 1] = -0.543$$

$$\boxed{\int_c f(z)dz = -0.543}$$

Q31 $f(z) = e^{2z}$, c the vertical segment from πi to $2\pi i$

Sol:- $z_1 = \pi i$, $z_2 = 2\pi i$

As

$$z(t) = z_1 + (z_2 - z_1)t \; ; \; 0 \le t \le 1$$

$$= \pi i + (2\pi i - \pi i)t$$

$$z(t) = \pi i + \pi i t$$

Diff:

$$z'(t) = \pi i$$

$$\text{\&} \; f[z(t)] = e^{2\pi i(1+t)}$$

As $$\int_c f(z)dz = \int_a^b f[z(t)]\, z'(t)\, dt$$

$$= \int_0^1 e^{2\pi i(1+t)} . \pi i\, dt$$

$$= \pi i \left. \frac{e^{2\pi i(1+t)}}{2\pi i} \right|_0^1$$

$$= \tfrac{1}{2}\left[e^{4\pi i} - e^{2\pi i}\right]$$

$$= \tfrac{1}{2}\left[\cos 4\pi + i\cancel{\sin 4\pi}^{0} - \cos 2\pi - i\cancel{\sin 2\pi}^{0}\right]$$

$$\int_c f(z)dz = \tfrac{1}{2}[1-1] = 0$$

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Q32 $f(z) = z\cos z^2$, C any Path from 0 to πi

Sol:- let C be a St: line

$$Z_1 = 0 \; ; \; Z_2 = \pi i$$

As $Z(t) = Z_1 + (Z_2 - Z_1)t \; ; \; 0 \le t \le 1.$

$$= 0 + (\pi i - 0)t$$

$$Z(t) = \pi i t$$

Diff:

$$Z'(t) = \pi i$$

Also $f[Z(t)] = \pi i t \cos(\pi i t)^2$

$$f[Z(t)] = \pi i t \cos \pi^2 t^2$$

using Th# 1

$$\int_C f(z)\,dz = \int_a^b f[Z(t)]\, Z'(t)\, dt.$$

$$= \int_0^1 \pi i t \cos \pi^2 t^2 \cdot \pi i \, dt$$

$$= \pi^{2} i^{2} \int_0^1 t \cos \pi^2 t^2 \, dt$$

$$= -\pi^2 \int_0^1 t \cos \pi^2 t^2 \, dt$$

Using substitution method for solving the integral

Put $t^2 = u$

$2t\,dt = du$

$t\,dt = \frac{1}{2} du$

$$\int_C f(z)dz = -\pi^2/2 \int_0^1 \cos \pi^2 u \, du$$

$$= -\pi^2/2 \left[\frac{\sin \pi^2 u}{\pi^2} \Big|_0^1 \right]$$

$$= -\frac{1}{2}\left[\sin \pi^2 - \sin 0\right]$$

$$= -\frac{1}{2}\left[\sin \pi^2 - 0\right]$$

$$\boxed{\int_C f(z)dz = -\frac{\sin \pi^2}{2}}$$

Q33 $f(z) = \cosh 3z$, C any Path from $\pi i/6$ to 0

Sol:- Let "C" be a st: line

$$Z(t) = Z_1 + (Z_2 - Z_1)t \quad 0 \le t \le 1$$

$$= \frac{\pi i}{6} + \left(0 - \frac{\pi i}{6}\right)t$$

$$= \frac{\pi i}{6} - \frac{\pi i}{6} t$$

$z'(t) = -\pi i/6$

$f[z(t)] = \cosh 3(\pi/6\, i - \frac{\pi i}{6} t)$

$= \cosh(\pi/2\, i - \frac{\pi}{2} it)$

$= \cosh i(\pi/2 - \pi/2\, t)$.

As $\cosh i\theta = \cos\theta$

$\therefore f[z(t)] = \cos[\pi/2 - \pi/2\, t]$

$= \cancel{\cos\pi/2}^{\,0} \cos\pi/2\, t + \sin\pi/2 \sin\pi/2\, t$

$f[z(t)] = \sin\pi/2\, t$

As

$$\int_c f(z)dz = \int_a^b f[z(t)]. z'(t) dt.$$

$$= \int_0^1 \sin(\pi/2\, t) \cdot (-\frac{\pi i}{6}) dt$$

$$= -\frac{\pi i}{6} \left[\frac{-\cos(\pi t/2)}{\pi/2} \right]_0^1$$

$$= i/3 (\cos\pi/2 - 1)$$

$$\boxed{\int_c f(z)dz = -i/3}$$