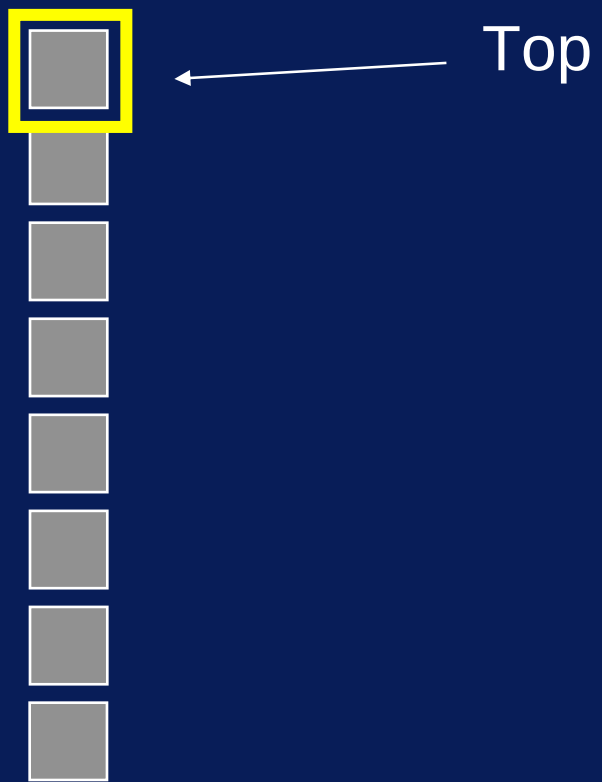
Stacks

Stacks

- A stack is a special type of list
  - all insertions and deletions take place at one end, called the top
  - thus, the last one added is always the first one available for deletion
  - also referred to as
    - » pushdown stack
    - » pushdown list
    - » LIFO list (Last In First Out)

Stacks



Stack Operations

- *Declare*: $\rightarrow S$:

  The function value of *Declare(S)* is an empty stack

Stack Operations

- *Empty*: $\rightarrow S$:

  The function *Empty* causes the stack to be emptied and it returns position *End(S)*

Stack Operations

- *IsEmpty*: S $\rightarrow$ B :

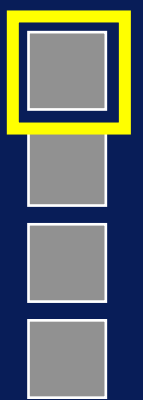
  The function value *IsEmpty*(*S*) is *true* if *S* is empty; otherwise it is *false*

Stack Operations

- *Top*: S $\rightarrow$ E :

  The function value *Top*(*S*) is the first element in the list;

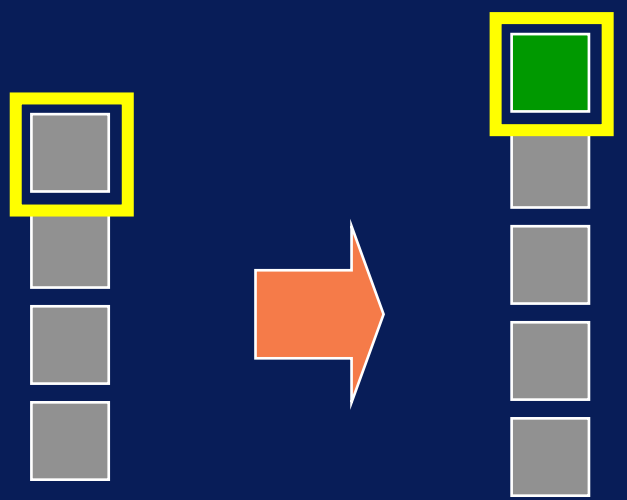
  if the list is empty, the value is undefined



Stack Operations

- *Push*: E × S → L :

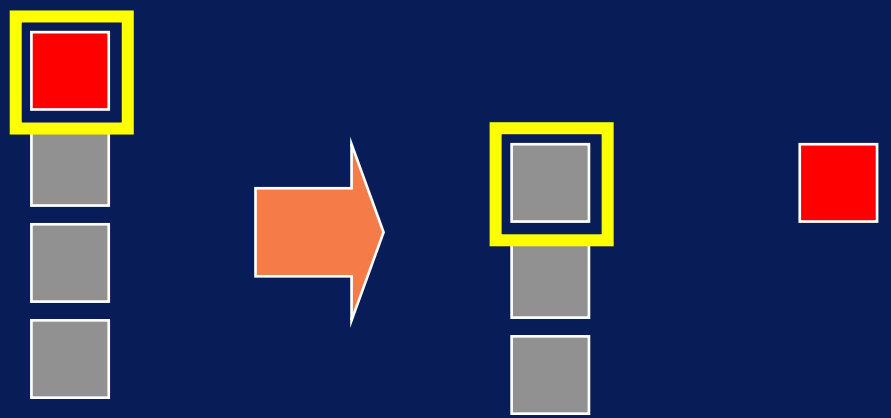
  Push(e, S)
  Insert an element *e* at the top of the stack



Stack Operations

- *Pop*: S $\rightarrow$ E :

  Pop(S)
  Remove the top element from the stack: i.e. return the top element and delete it from the stack



Stack Operations

- *All these operations can be directly implemented using the LIST ADT operations on a List S*
- *Although it may be more efficient to use a dedicated implementation*
- *It depends what you want: code efficiency or software re-use (i.e. utilization efficiency)*

Stack Operations

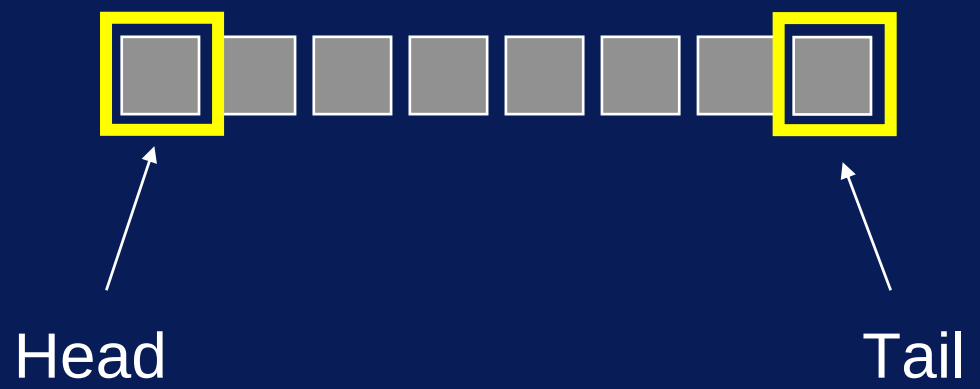
- Declare(S)
- Empty(S)
- Top(S)
  - » Retrieve(First(S), S)
- Push(e, S)
  - » Insert(e, First(S), S)
- Pop(S)
  - » Retrieve(First(S), S)
  - » Delete(First(S), S)

Queues

Queues

- A queue is another special type of list
  - insertions are made at one end, called the tail of the queue
  - deletions take place at the other end, called the head
  - thus, the last one added is always the last one available for deletion
  - also referred to as
    - » FiFO list (First In First Out)

Queues



Queue Operations

- *Declare*: $\rightarrow$ Q :

  The function value of *Declare(Q)* is an empty queue

Queue Operations

- *Empty*: $\rightarrow$ Q :

  The function *Empty* causes the queue to be emptied and it returns position *End(Q)*

Queue Operations

- *IsEmpty*: Q $\rightarrow$ B :

  The function value *IsEmpty*(*Q*) is *true* if *Q* is empty; otherwise it is *false*

Queue Operations

- *Head*: Q $\rightarrow$ E :

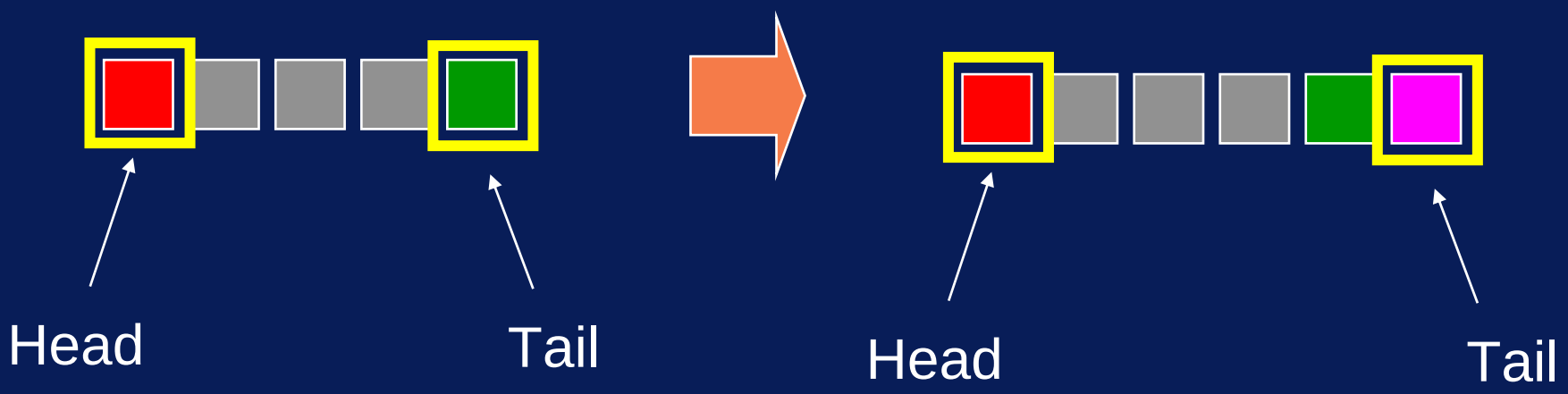
  The function value *Head*(*Q*) is the first element in the list;

  if the queue is empty, the value is undefined

Queue Operations

- *Enqueue*: $E \times Q \rightarrow Q$:

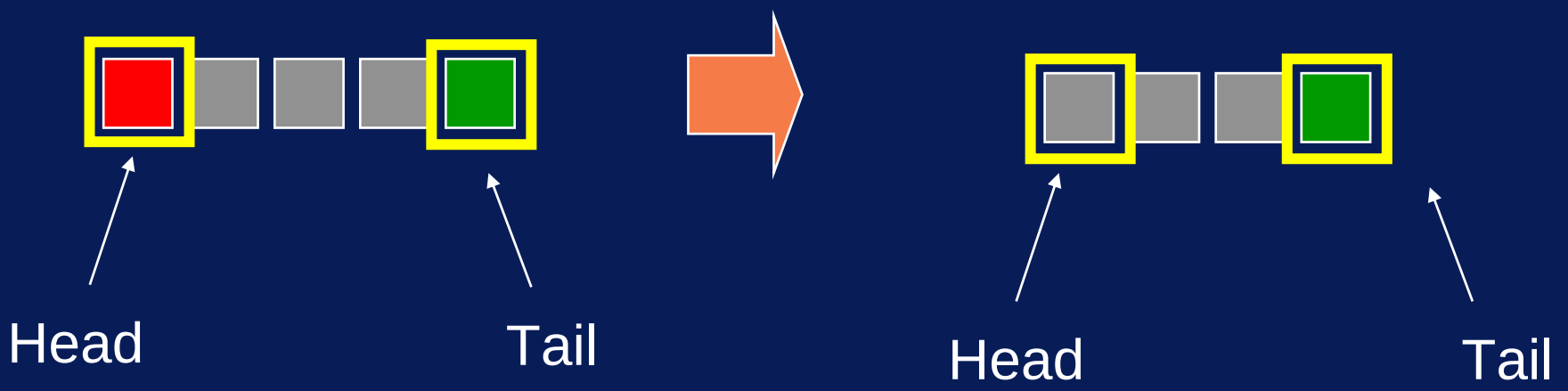
  Enqueue(e, Q)
  Add an element *e* the the tail of the queue



Queue Operations

- *Dequeue*: Q $\rightarrow$ E :

  Dequeue(Q)
  Remove the element from the head of the queue: i.e. return the first element and delete it from the queue



Queue Operations

- *All these operations can be directly implemented using the LIST ADT operations on a queue Q*
- *Again, it may be more efficient to use a dedicated implementation*
- *And, again, it depends what you want: code efficiency or software re-use (i.e. utilization efficiency)*

Queue Operations

- Declare(Q)
- Empty(Q)
- Head(Q)
  - » Retrieve(First(Q), Q)
- Enqueue(e, Q)
  - » Insert(e, End(Q), Q)
- Dequeque(Q)
  - » Retrieve(First(Q), Q)
  - » Delete(First(Q), Q)

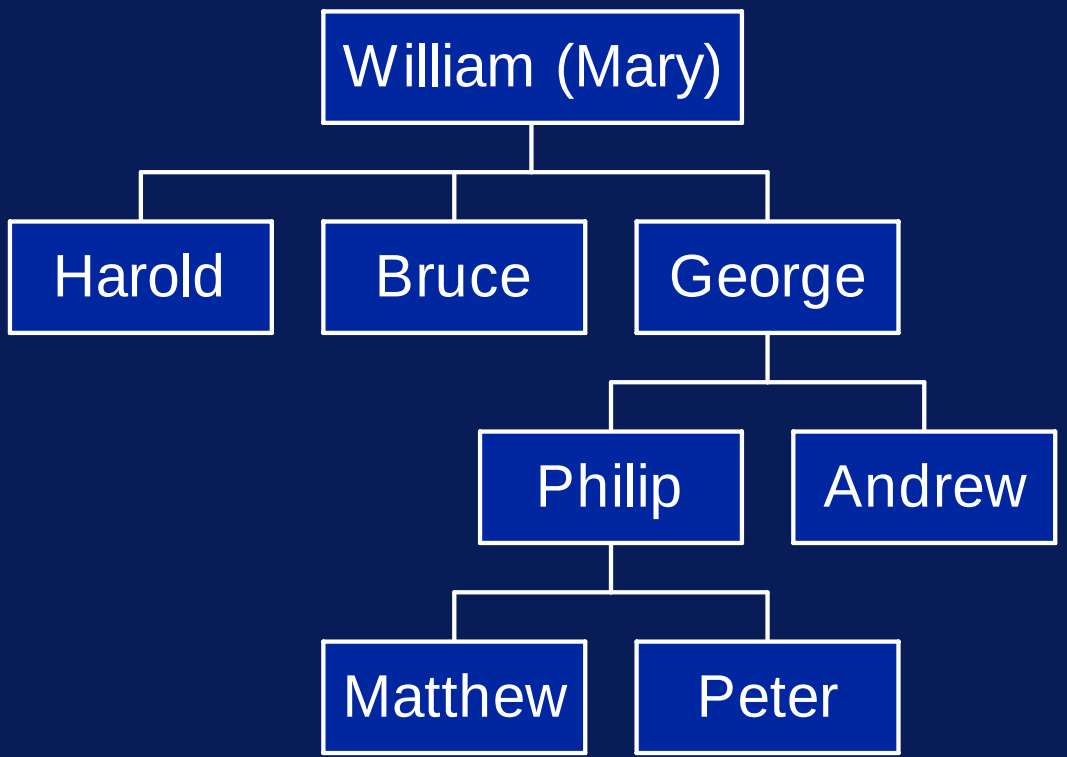
Trees

Trees

- Trees are everywhere
- Hierarchical method of structuring data
- Uses of trees:
  - genealogical tree
  - organizational tree
  - expression tree
  - binary search tree
  - decision tree

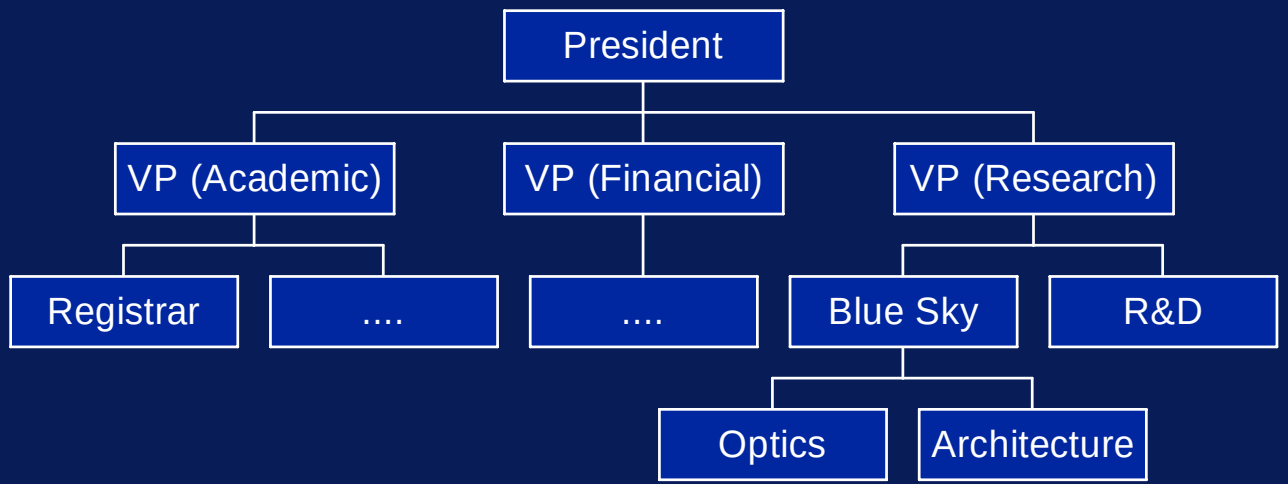
Uses of Trees

Genealogical Tree



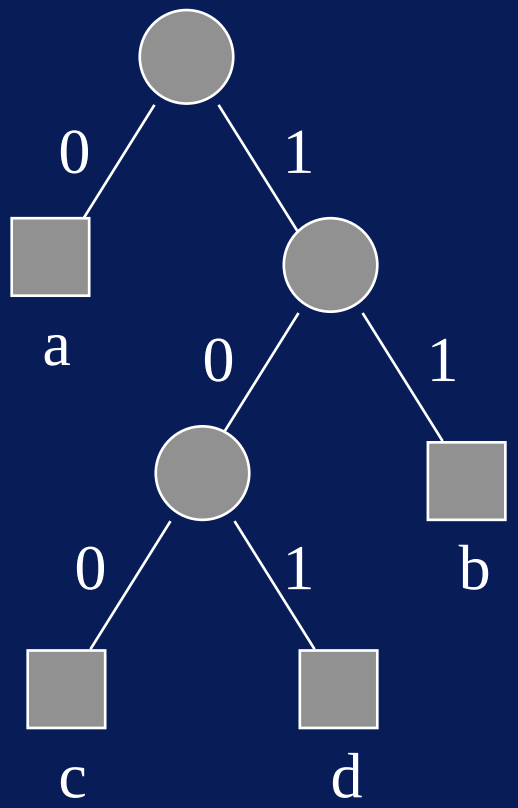
Uses of Trees

Organization Tree



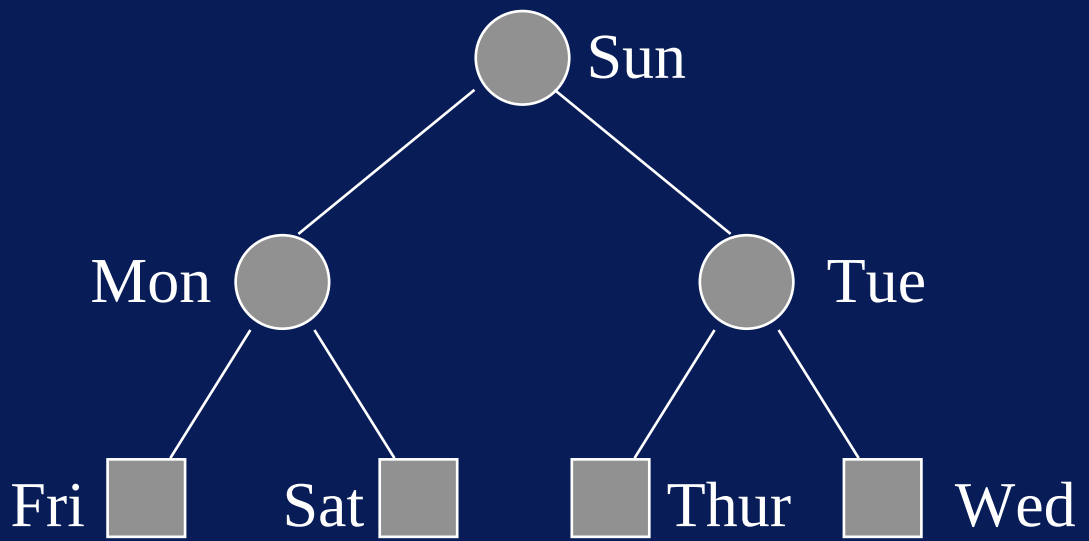
Uses of Trees

Code Tree



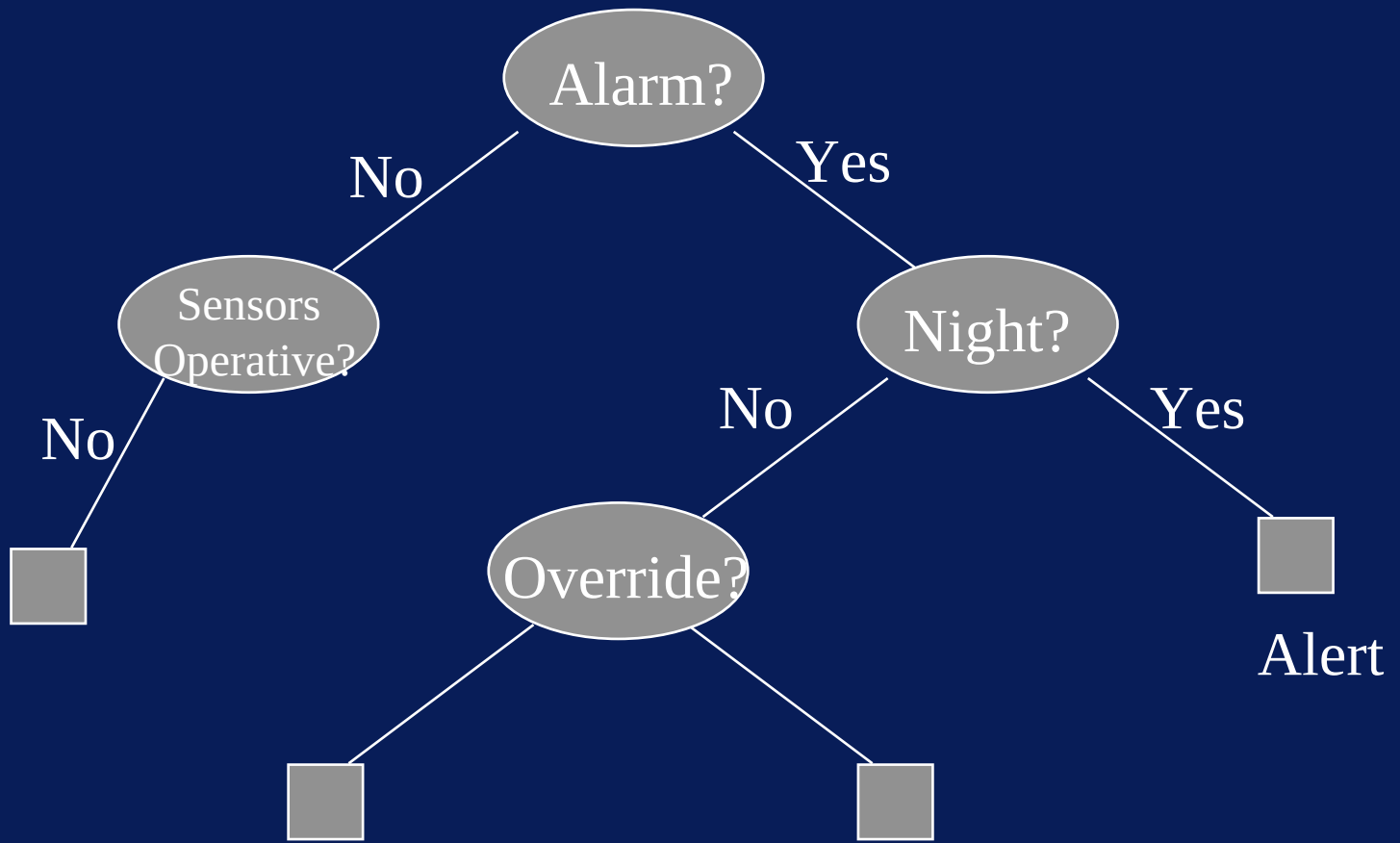
Uses of Trees

Binary Seach Tree



Uses of Trees

Decision Tree



Trees

- Fundamentals
- Traversals
- Display
- Representation
- Abstract Data Type (ADT) approach
- Emphasis on binary tree
- Also mention multi-way trees, forests, orchards

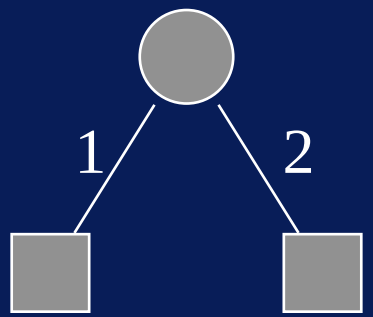
Tree Definitions

- A binary tree T of n nodes, $n \geq 0$,
  - either is empty, if $n = 0$,
  - or consists of a root node u and two binary trees $u(1)$ and $u(2)$ of n_1 and n_2 nodes, respectively, such that
    $n = n + n_1 + n_2$.
- We say that $u(1)$ is the first or left subtree of T, and $u(2)$ is the second or right subtree of T.

Binary Tree

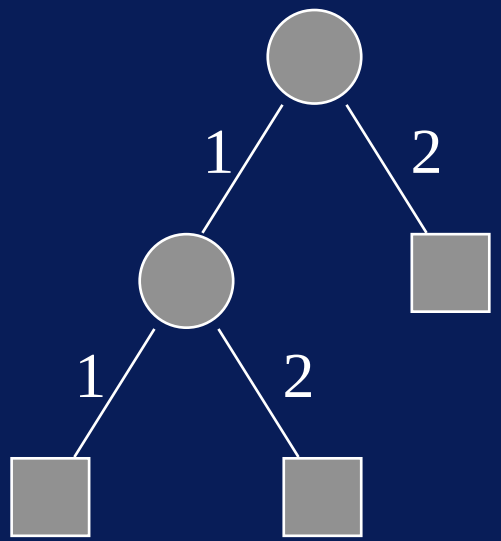
Binary Tree of zero nodes

Binary Tree



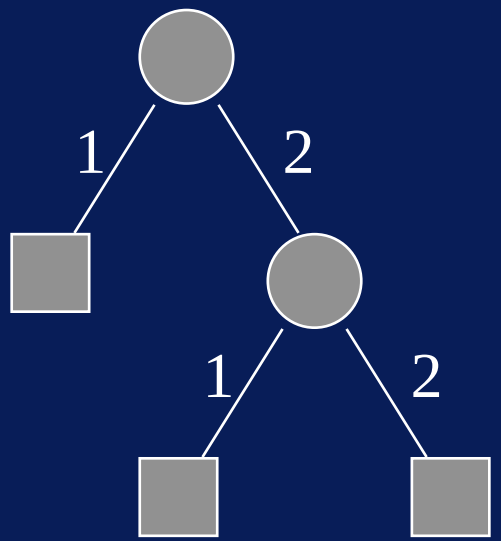
Binary Tree of one node

Binary Tree



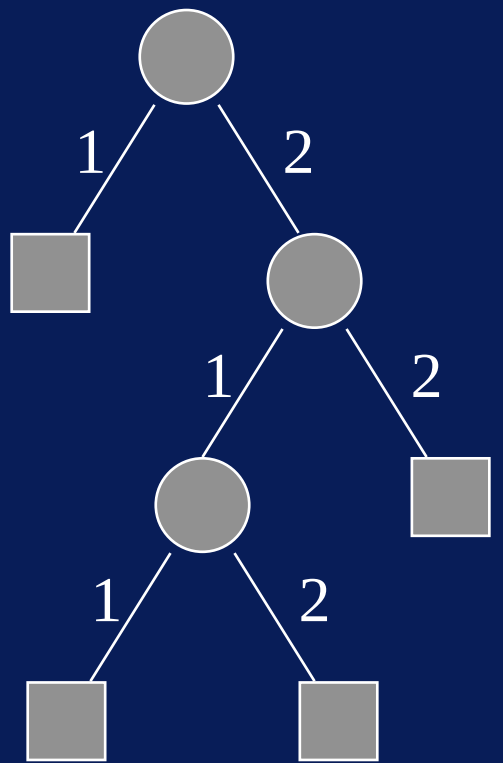
Binary Tree of two nodes

Binary Tree



Binary Tree of two nodes

Binary Tree



Binary Tree of three nodes

Binary Tree

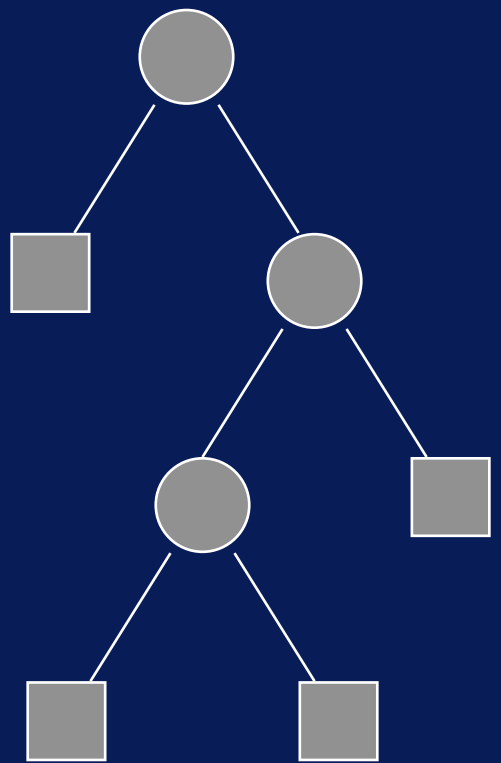
External nodes - have no subtrees

Internal nodes - always have two subtrees

Binary Tree Terminology

- Let *T* be a binary tree with root *u*
- Let *v* be any node in *T*
- If *v* is the root of either *u*(1) or *u*(2), then we say *u* is the parent of *v*,
  and that *v* is the child of *u*
- If *w* is also a child of *u*, and *w* is distinct from *v*, we say that *v* and *w* are siblings.

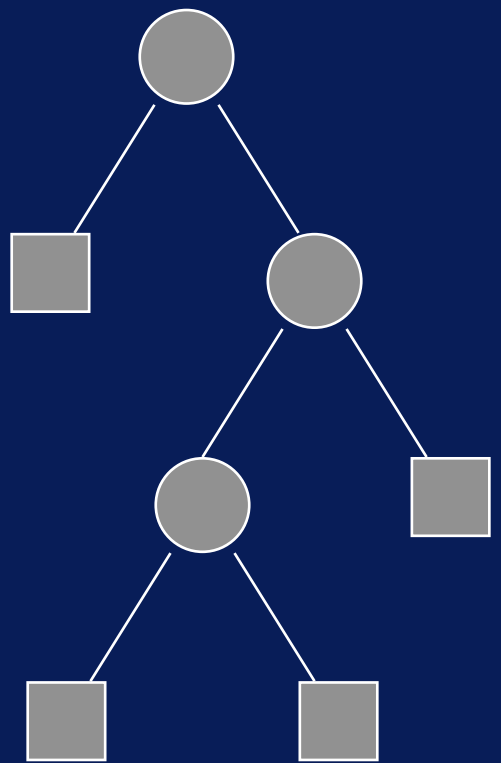
Binary Tree



Binary Tree Terminology

- If *v* is the root of *u*(*i*),
- then *v* is the *i*th child of *u*; *u*(1) is the left child and *u*(2) is the right child.
- Also have grandparents and grandchildren

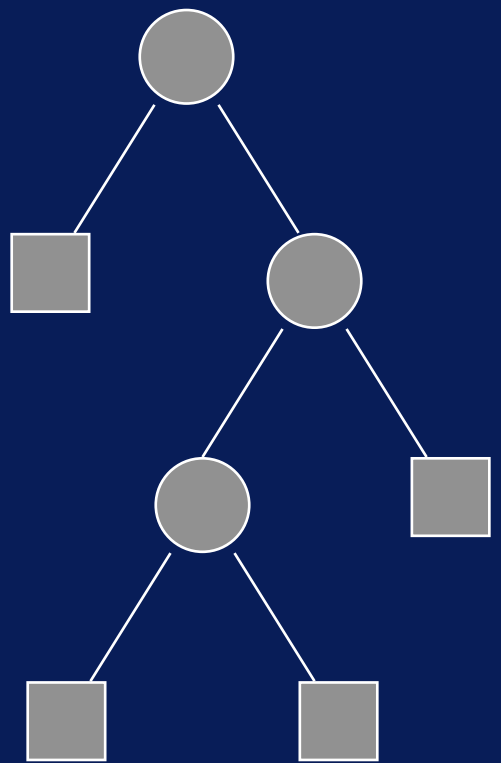
Binary Tree



Binary Tree Terminology

- Given a binary tree T of n nodes, $n \geq 0$,
- then *v* is a descendent of *u* if either
  - *v* is equal to *u*
    or
  - *v* is a child of some node *w* and *w* is a descendant of *u*.
- We write $v \; desc_T \; u$.
- v is a proper descendent of *u* if *v* is a descendant of *u* and $v \neq u$.

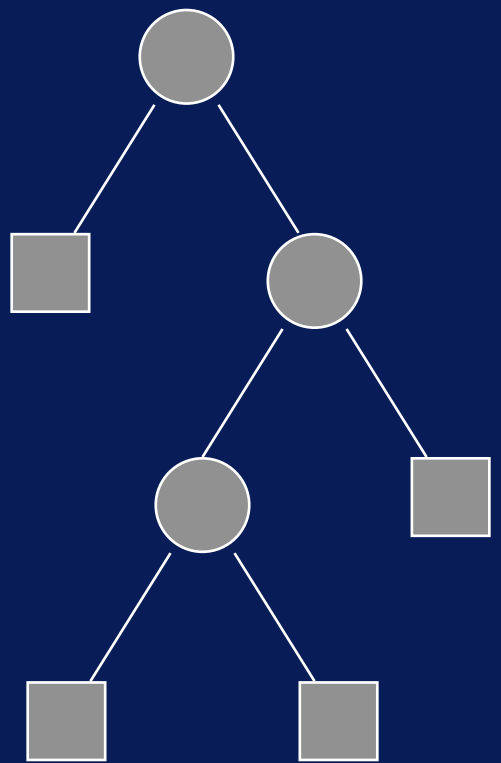
Binary Tree



Binary Tree Terminology

- Given a binary tree T of n nodes, $n \geq 0$,
- then *v* is a left descendent of *u* if either
  - *v* is equal to *u*
    or
  - *v* is a left child of some node *w* and *w* is a left descendant of *u*.
- We write $v \; ldesc_T \; u$.
- Similarly we have $v \; rdesc_T \; u$.

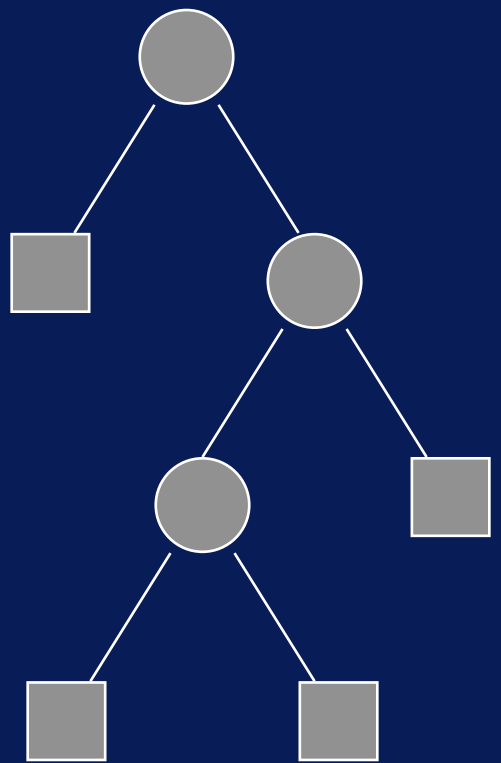
Binary Tree



Binary Tree Terminology

- $ldesc_T$ relates nodes across a binary tree rather than up and down a binary tree.
- Given two nodes *u* and *v* in a binary tree *T*, we say that *v* is to the left of *u* if there is a new node *w* in *T* such that *v* is a left descendant of *w* and *u* is a right descendant of *w*.
- We denote this relation by $left_T$ and write $v\ left_T\ u$.

Binary Tree



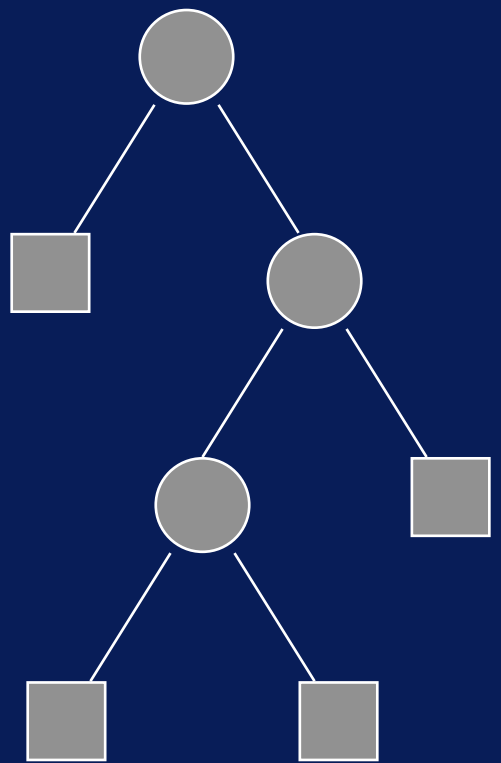
Binary Tree Terminology

- The external nodes of a tree define its frontier
- We can count the number of nodes in a binary tree in three ways:
  - Number of internal nodes
  - Number of external nodes
  - Number of internal and external nodes
- The number of internal nodes is the size of the tree

Binary Tree Terminology

- Let T be a binary tree of size n , $n \geq 0$,
- Then, the number of external nodes of T is
  $n + 1$

Binary Tree



Binary Tree Terminology

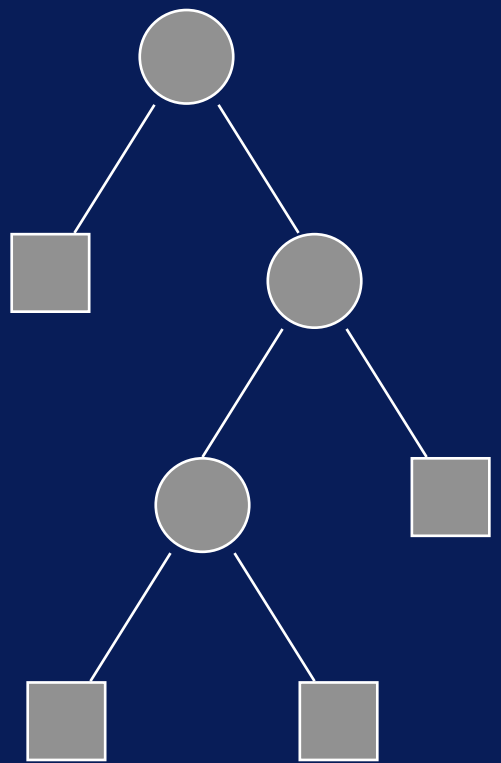
- The height of T is defined recursively as

  0 if T is empty and

  $1 + max(height(T_1), height(T_2))$ otherwise, where T_1 and T_2 are the subtrees of the root.

- The height of a tree is the length of a longest chain of descendents

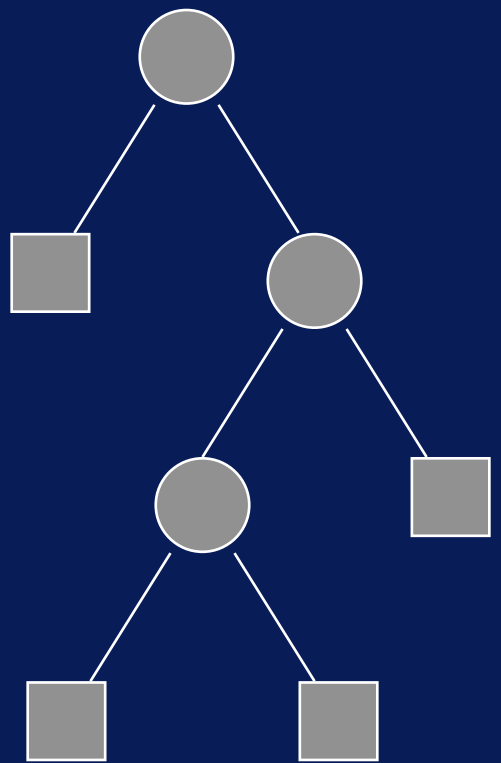
Binary Tree



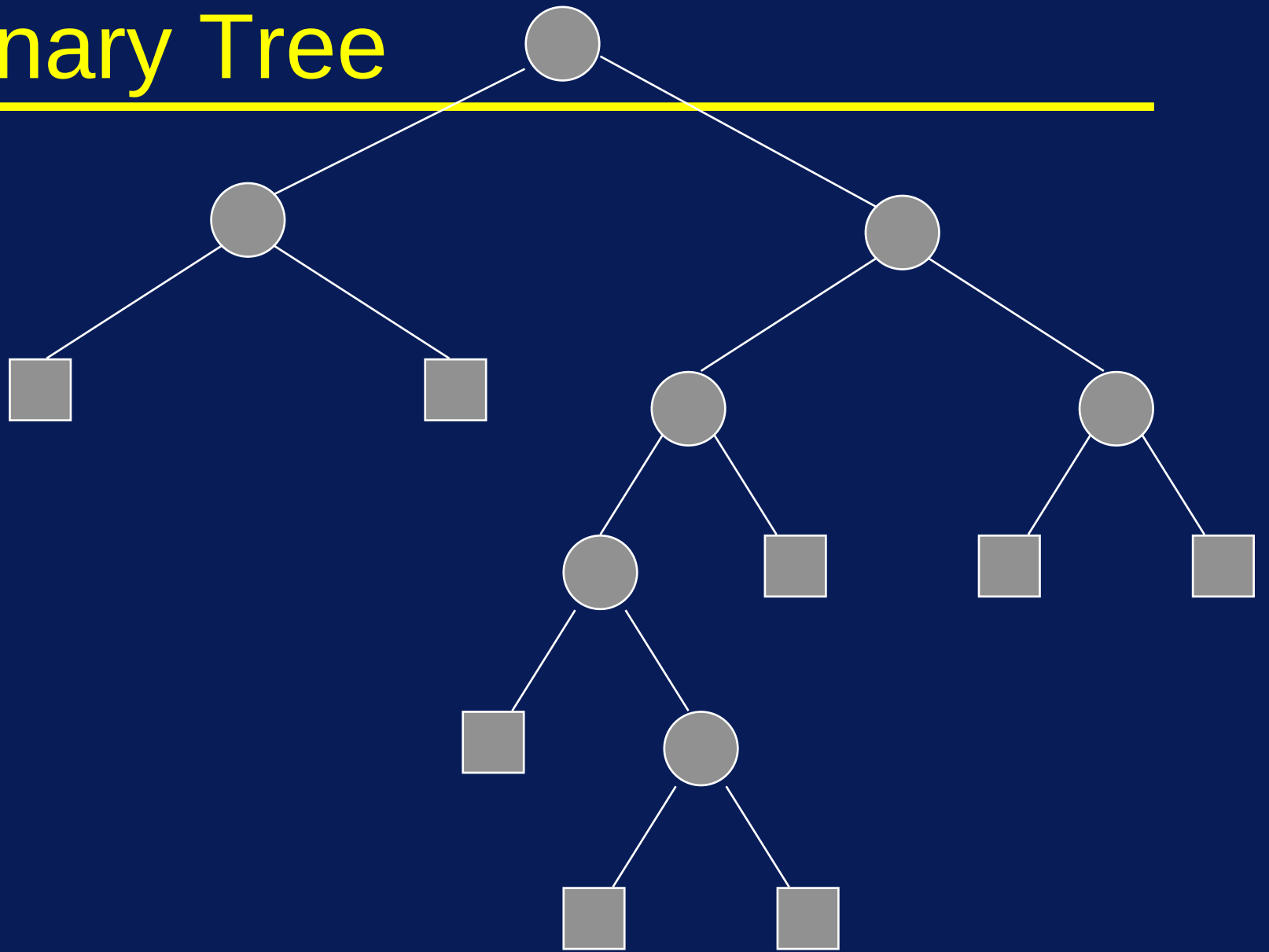
Binary Tree Terminology

- Height Numbering
  - Number all external nodes 0
  - Number each internal node to be one more than the maximum of the numbers of its children
  - Then the number of the root is the height of *T*
- The height of a node *u* in *T* is the height of the subtree rooted at *u*

Binary Tree



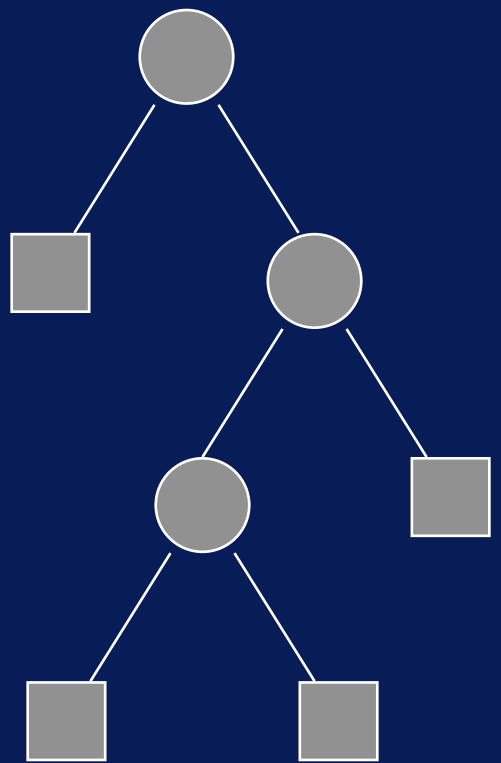
Binary Tree



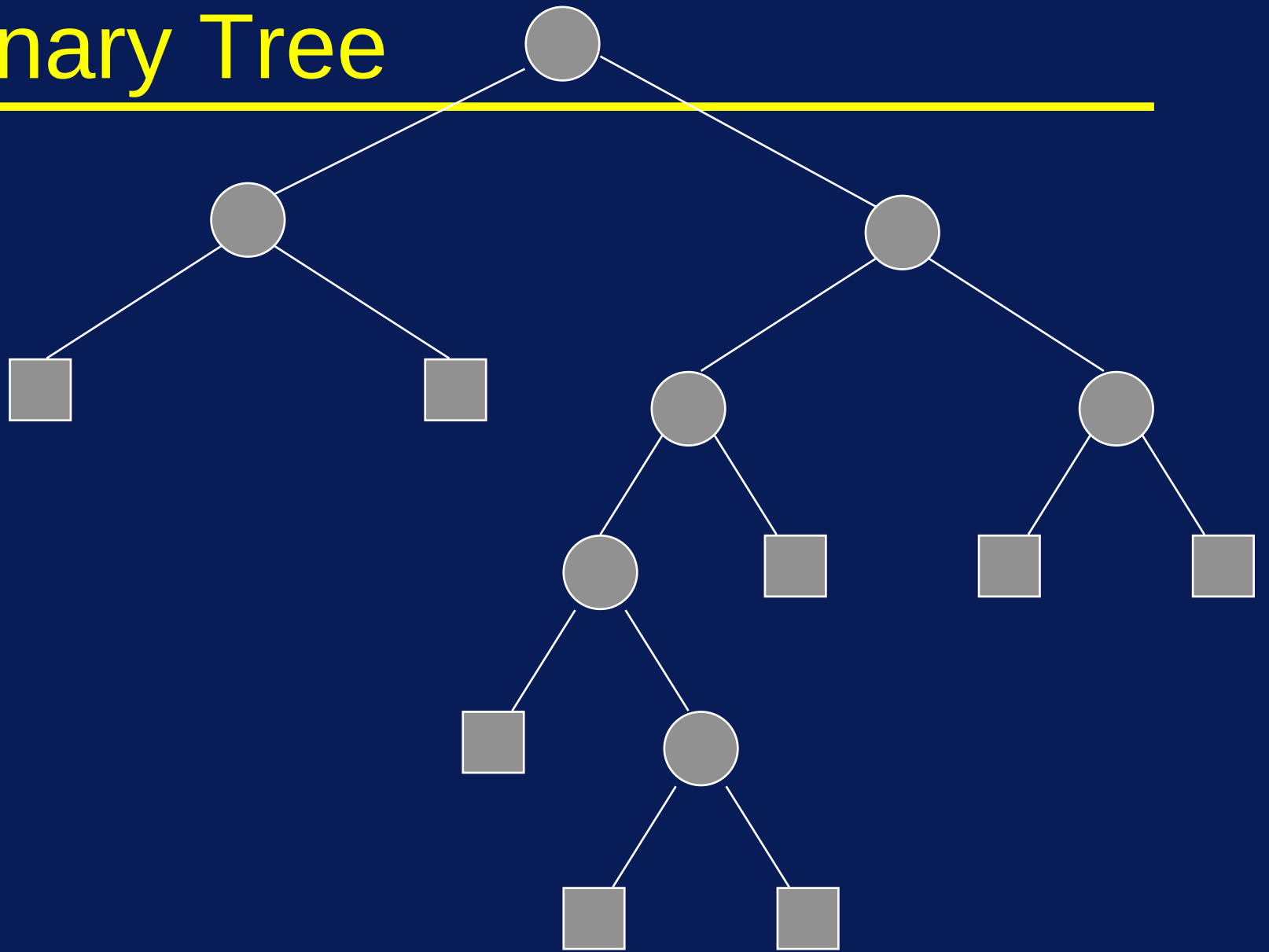
Binary Tree Terminology

- Levels of nodes
  - The level of a node in a binary tree is computed as follows
  - Number the root node 0
  - Number every other node to be 1 more than its parent
  - Then the number of a node *v* is that node's level
  - The level of *v* is the number of branches on the path from to root to *v*

Binary Tree



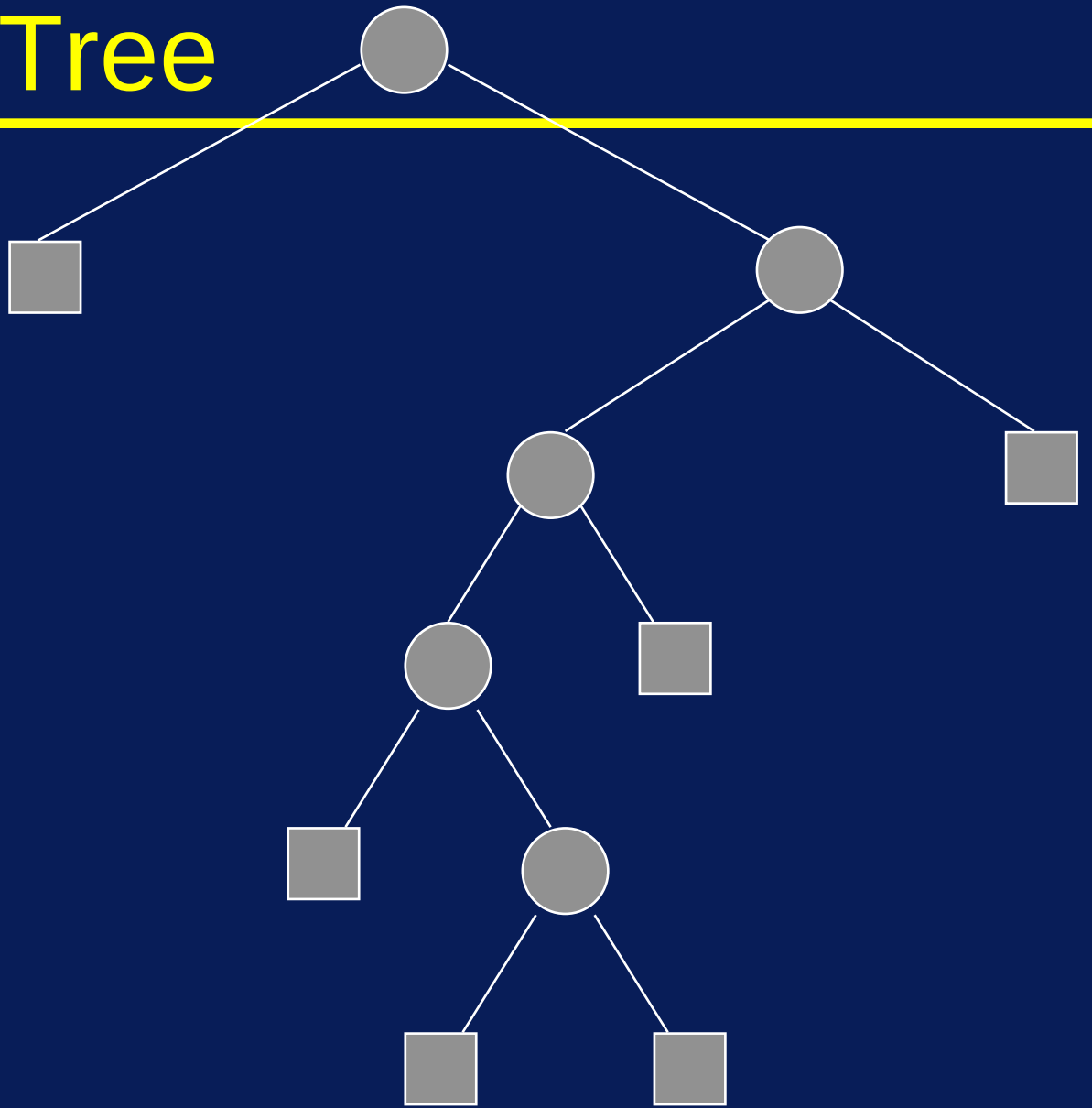
Binary Tree



Binary Tree Terminology

- Skinny Trees
  - every internal node has at most one internal child

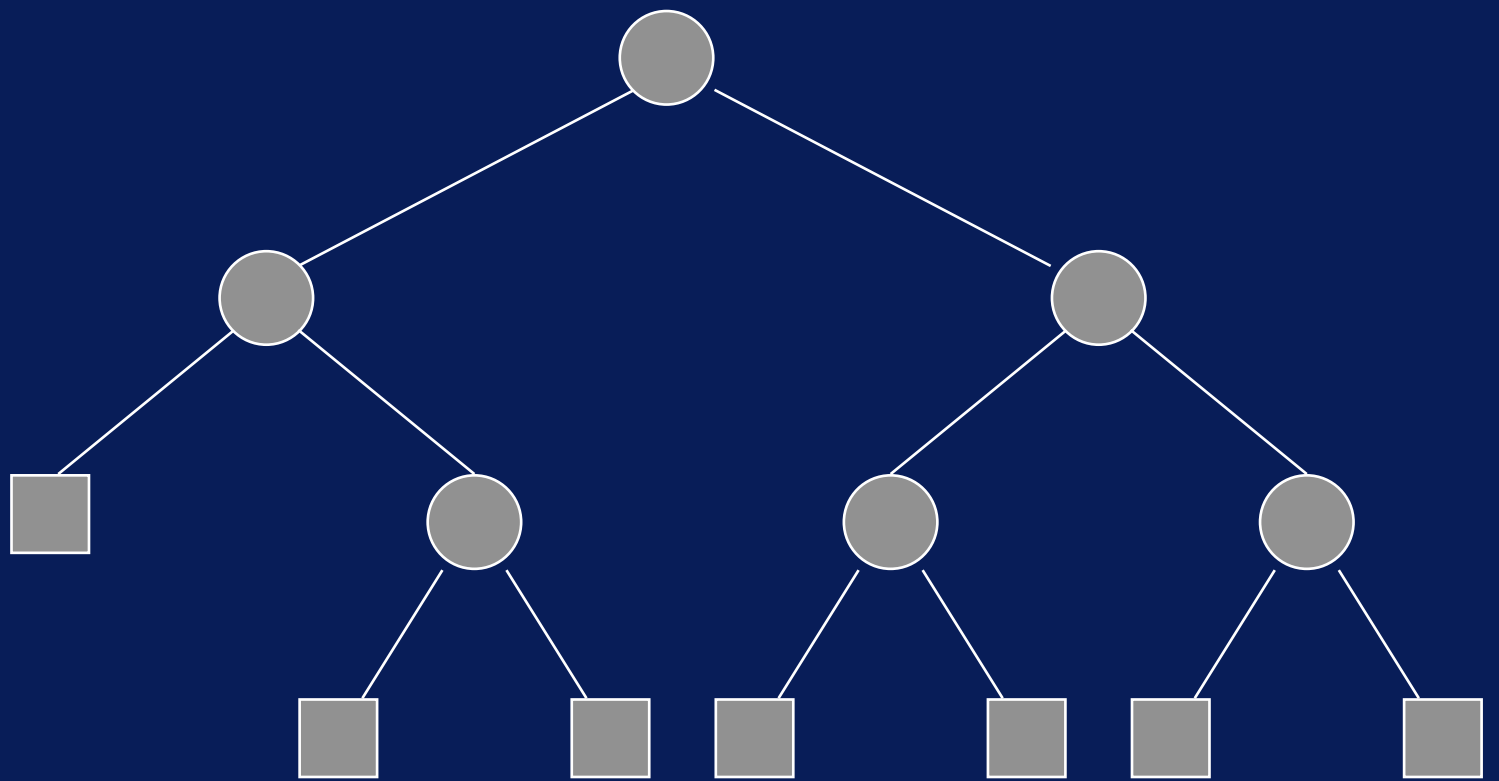
Skinny Tree



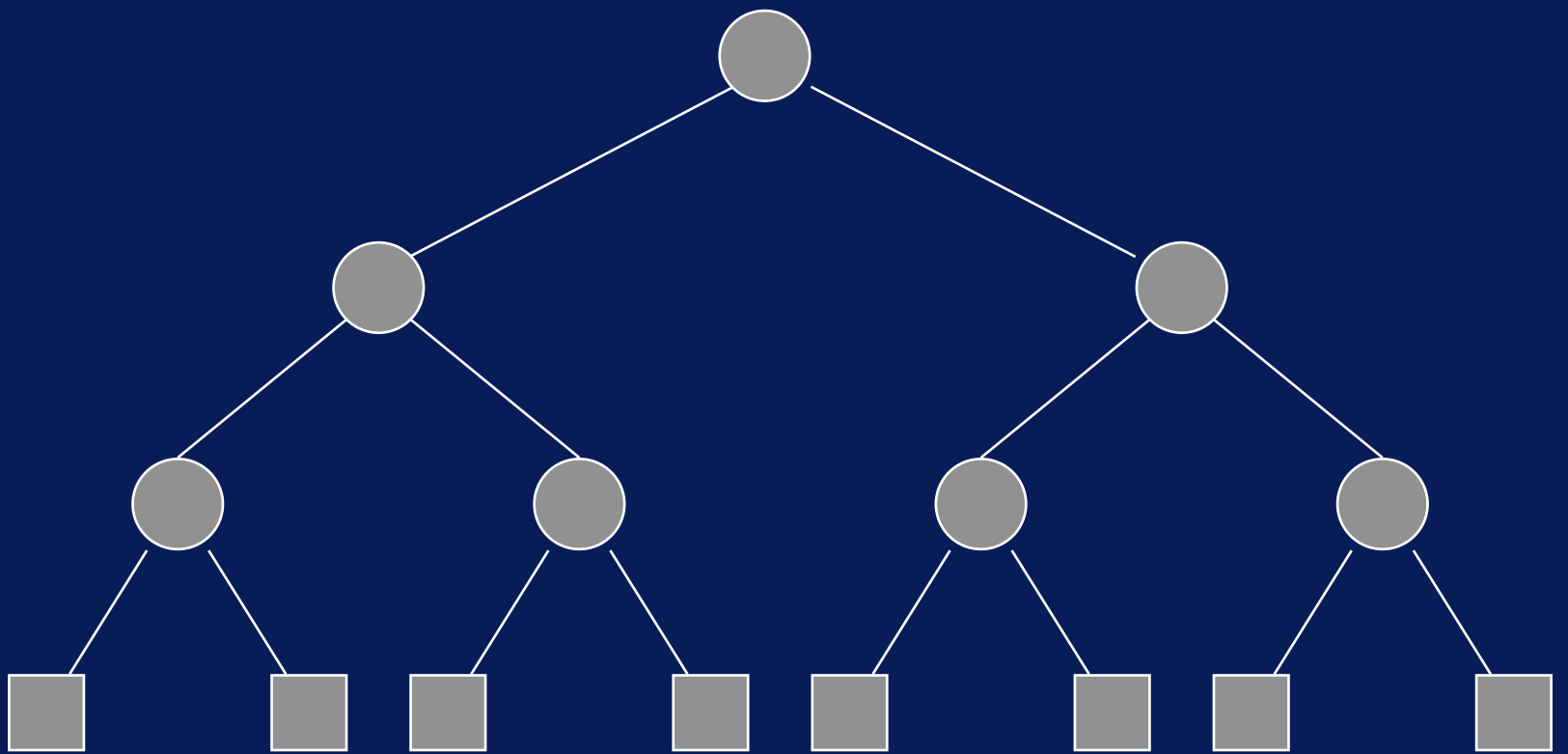
Binary Tree Terminology

- Complete Binary Trees (Fat Trees)
  - the external nodes appear on at most two adjacent levels
  - Perfect Trees: complete trees having all their external nodes on one level
  - Left-complete Trees: the internal nodes on the lowest level is in the leftmost possible position.
  - Skinny trees are the highest possible trees
  - Complete trees are the lowest possible trees

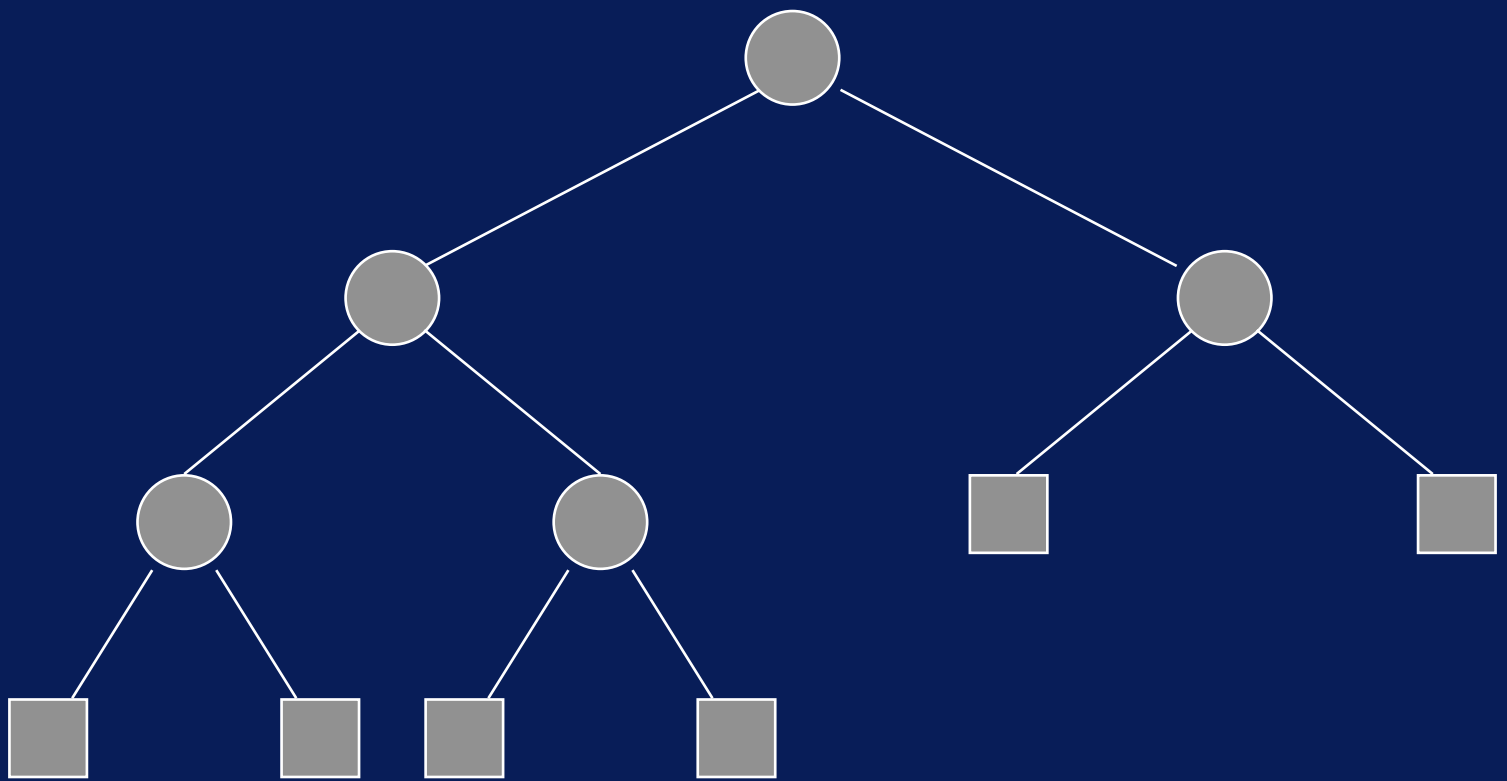
Complete Tree



Perfect Tree



Left-Complete Tree



Binary Tree Terminology

- A binary tree of height $h \geq 0$ has size at least h
- A binary tree of height at most $h \geq 0$ has size at most $2^h - 1$
- A binary tree of size $n \geq 0$ has height at most n
- A binary tree of size $n \geq 0$ has height at least $\lceil \log (n + 1) \rceil$

Multiway Trees

- Multiway trees are defined in a similar way to binary trees, except that the degree (the maximum number of children) is no longer restricted to the value 2

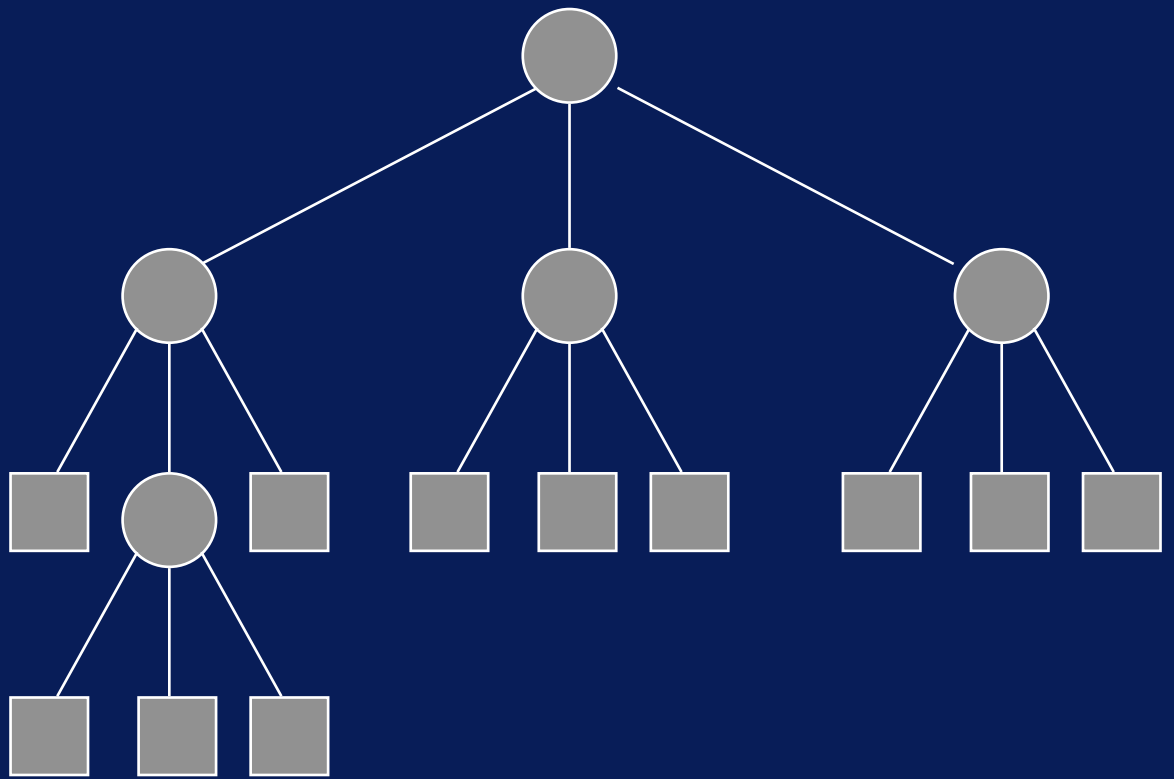
Multiway Trees

- A multiway tree T of n internal nodes, $n \geq 0$,
  - either is empty, if $n = 0$,
  - or consists of
    - » a root node u,
    - » an integer $d_u \geq 1$, the degree of u,
    - » and multiway trees $u(1)$ of n_1 nodes, ..., $u(d_u)$ of n_{d_u} nodes such that $n = 1 + n_1 + ... + n_{d_u}$

Multiway Trees

- A multiway tree T is a d-ary tree, for some $d > 0$, if $d_u = d$, for all internal nodes u in T

d-ary Tree



Multiway Trees

- A multiway tree T is a (a, b)-tree, if $1 \leq a \leq d_u \leq b$, for all u in T
- Every binary tree is a (2, 2)-tree, and vice versa

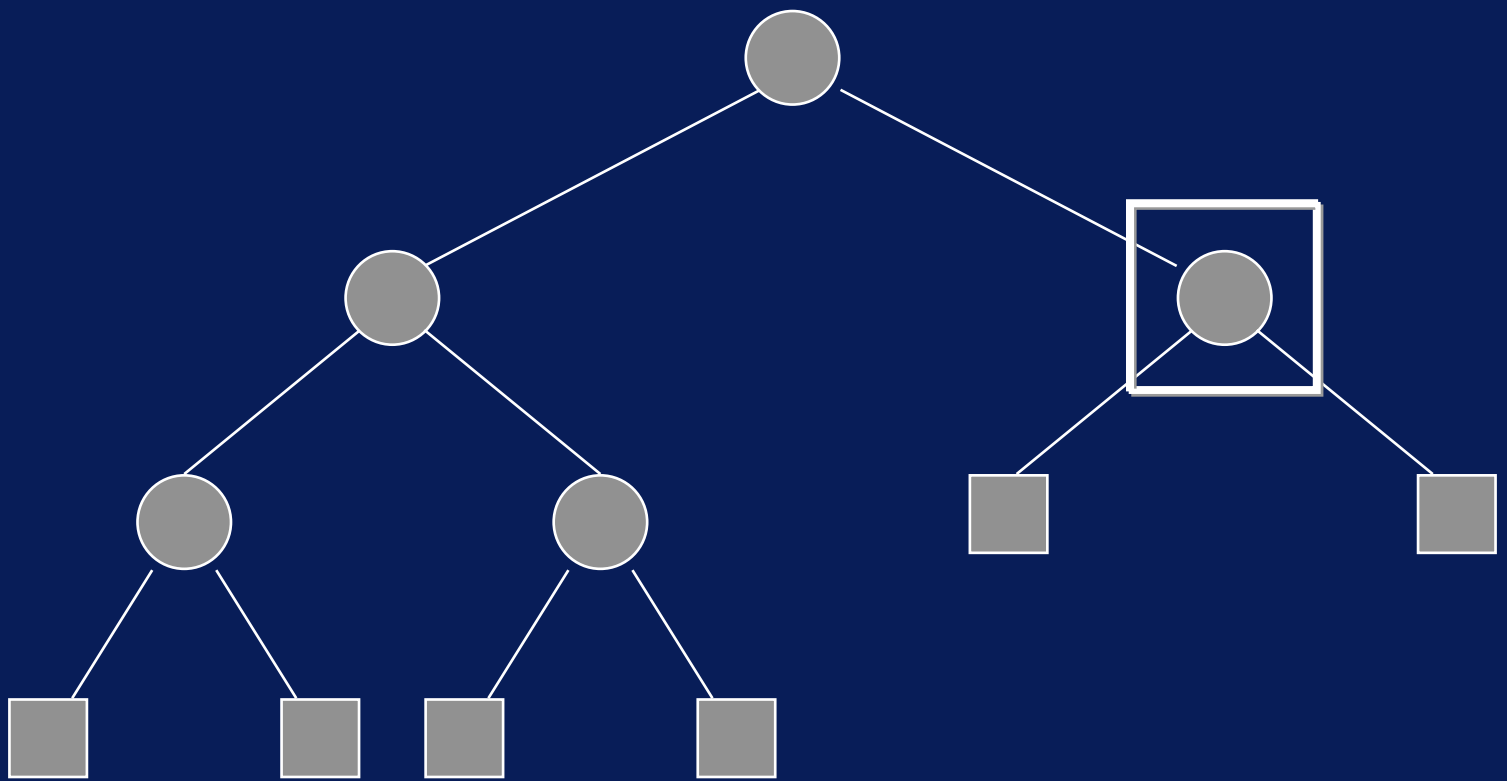
BINARY_TREE & TREE Specification

- So far, no values associated with the nodes of a tree
- Now want to introduce an ADT called BINARY_TREE, which
  - has value of type *intelementtype* associated with the internal nodes
  - has value of type *extelementtype* associated with the external nodes
- These value don’t have any effect on BINARY_TREE operations

BINARY_TREE & TREE Specification

- BINARY_TREE has explicit windows and window-manipulation operations
- A window allows us to ‘see’ the value in a node (and to gain access to it)
- Windows can be positioned over any internal or external node
- Windows can be moved from parent to child
- Windows can be moved from child to parent

Window



BINARY_TREE & TREE Specification

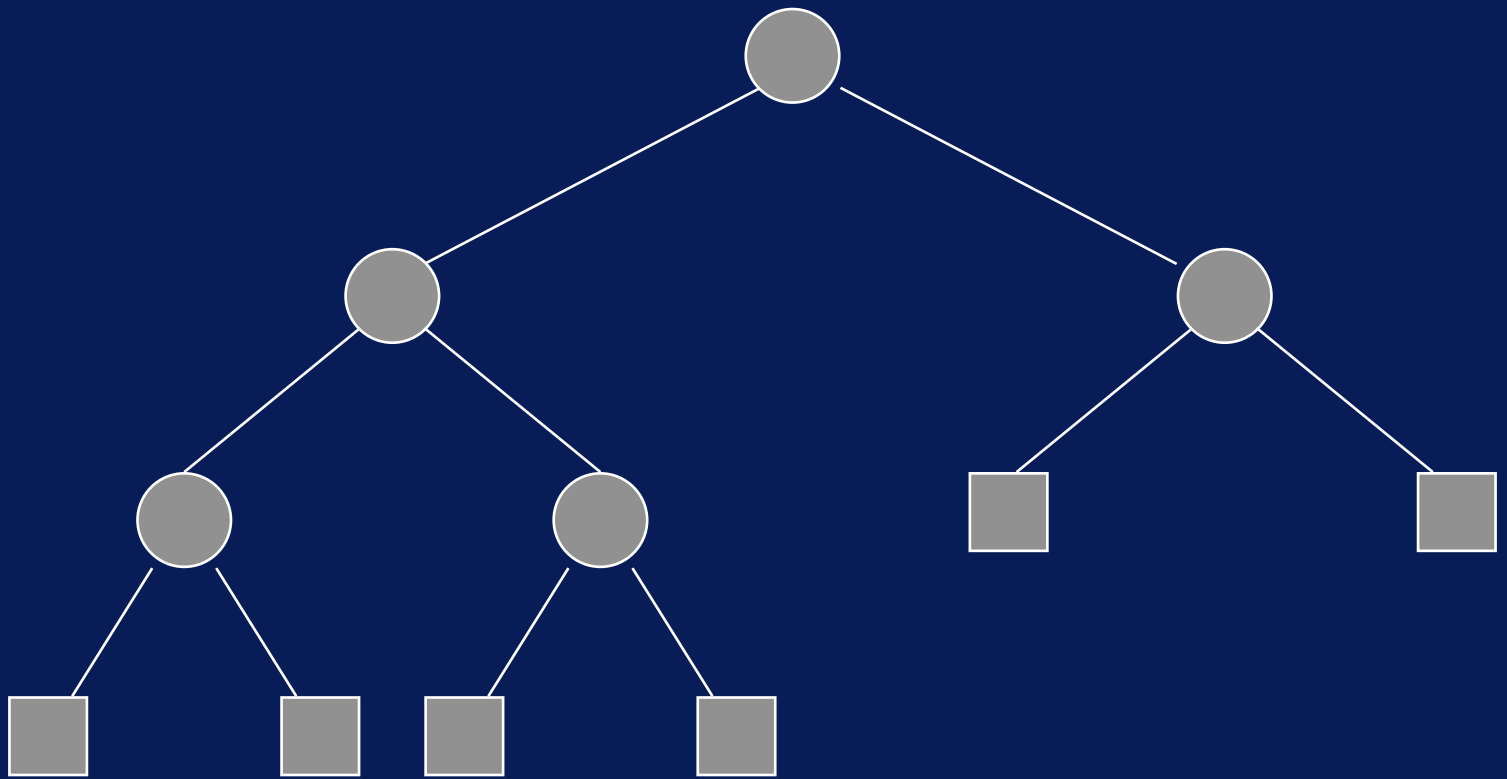
- Let BT denote denote the set of values of BINARY_TREE of *elementtype*
- Let E denote the set of values of type *elementtype*
- Let W denote the set of values of type *windowtype*
- Let B denote the set of Boolean values *true* and *false*

BINARY_TREE Operations

- *Empty*: BT $\rightarrow$ BT :
  The function *Empty(T)* is an empty binary tree; if necessary, the tree is deleted

- *IsEmpty*: BT $\rightarrow$ B :
  The function value *IsEmpty*(*T*) is *true* if *T* is empty; otherwise it is false

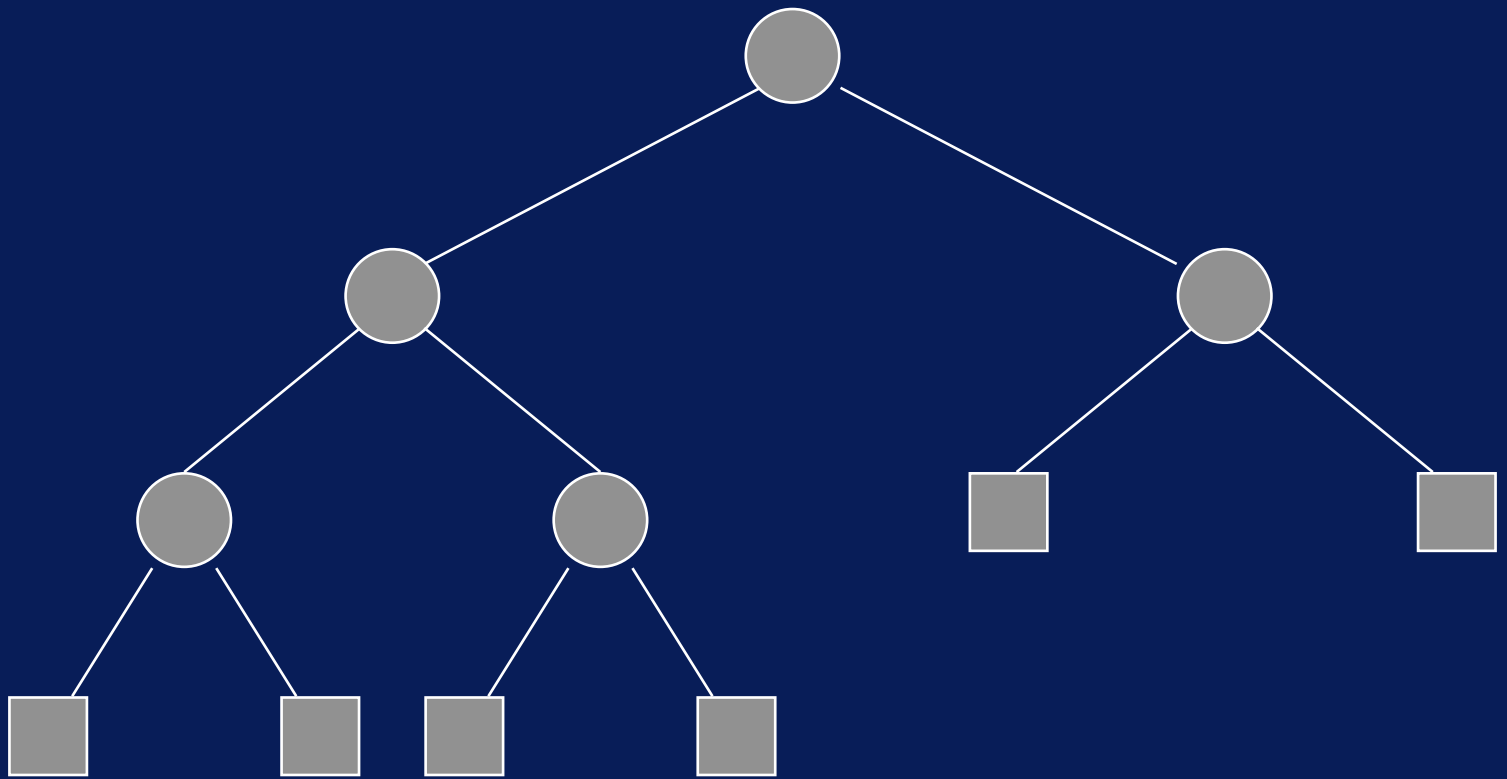
Example



BINARY_TREE Operations

- *Root*: BT $\rightarrow$ W :
  The function value *Root*(*T*) is the window position of the single external node if *T* is empty; otherwise it is the window position of the root of *T*

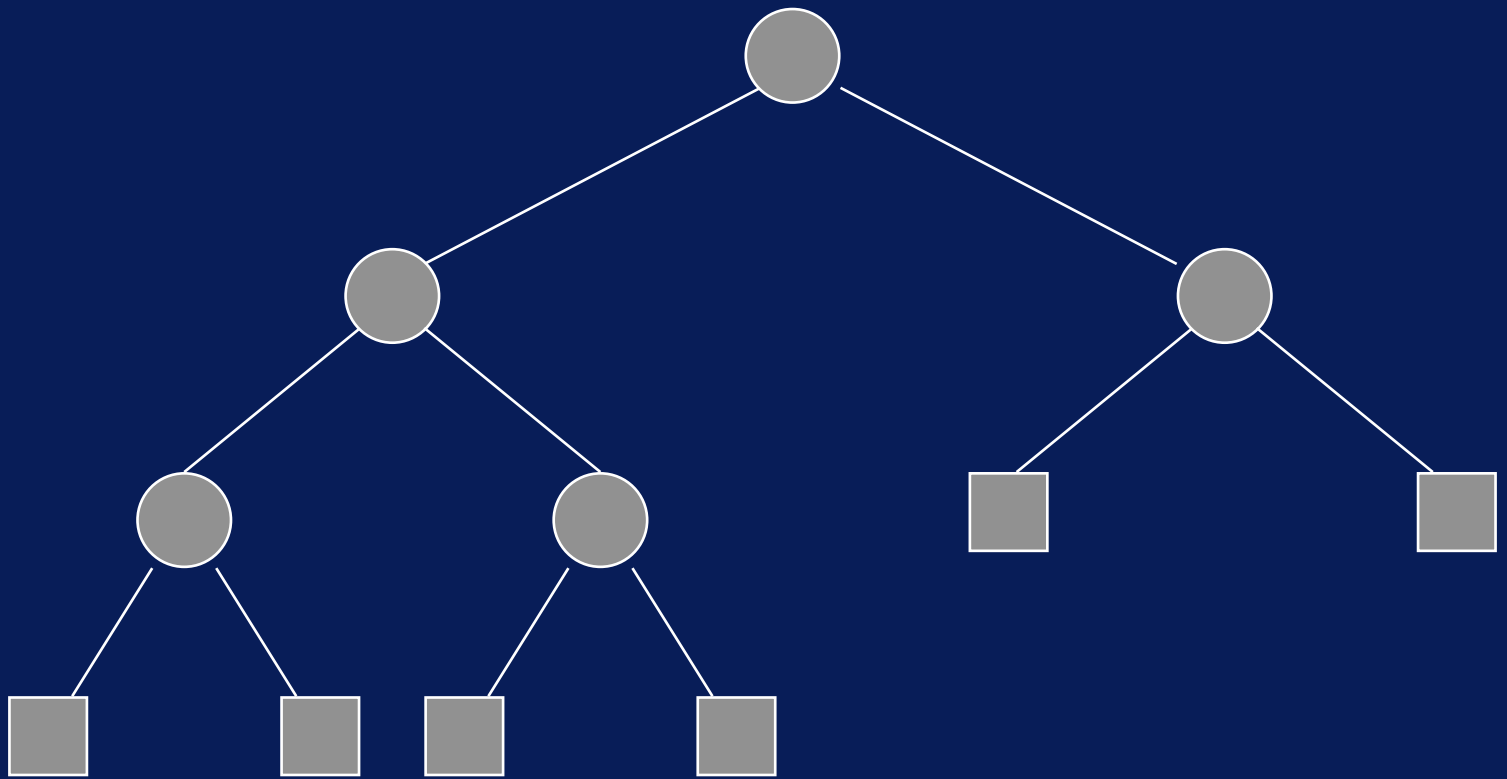
Example



BINARY_TREE Operations

- *IsRoot*: $\mathrm{W} \times \mathrm{BT} \rightarrow \mathrm{B}$:
  The function value *IsRoot*(*w*, *T*) is *true* if the window *w* is over the root; otherwise it is *false*

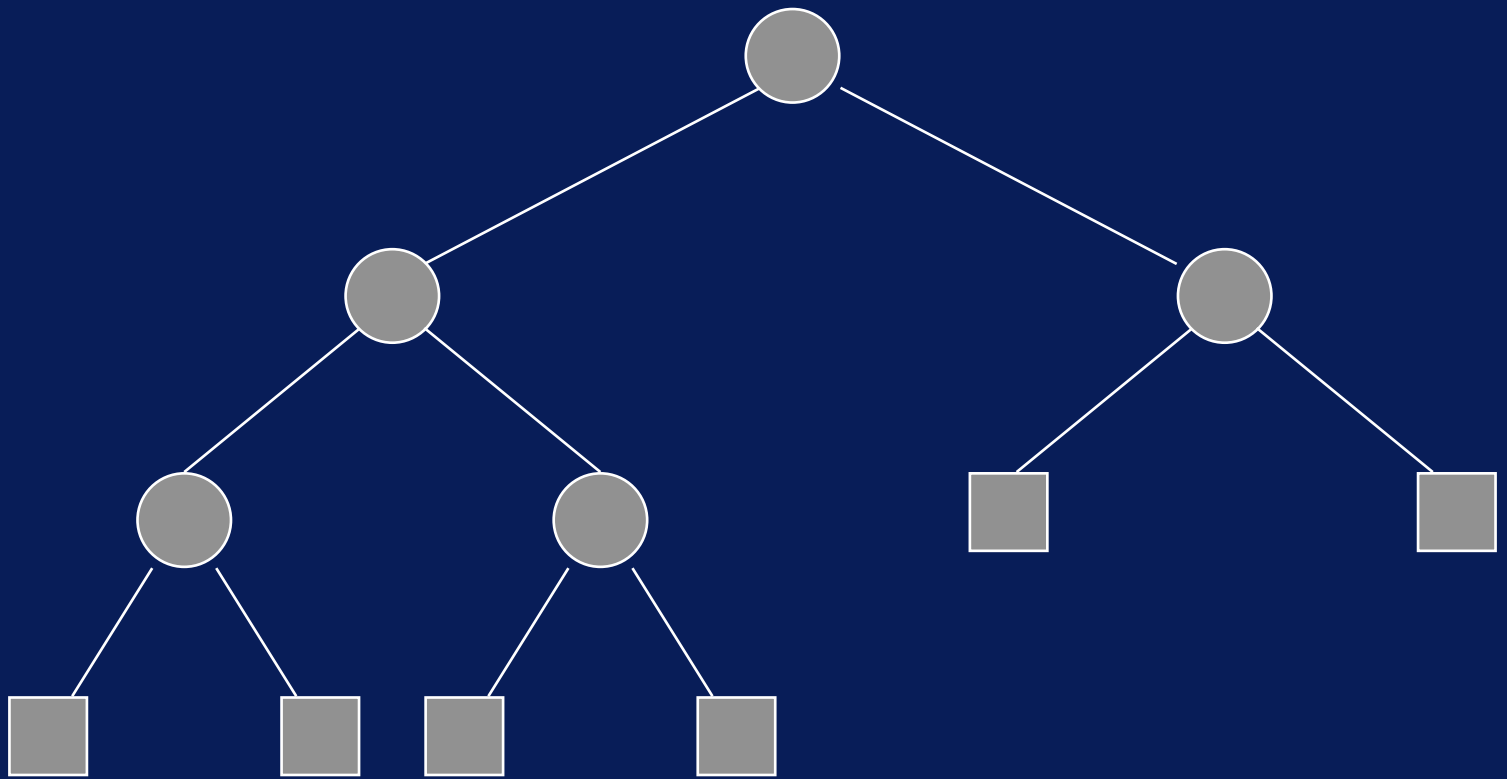
Example



BINARY_TREE Operations

- *IsExternal*: W $\times$ BT $\rightarrow$ B :
  The function value *IsExternal*(*w*, *T*) is *true* if the window *w* is over an external node of *T*; otherwise it is *false*

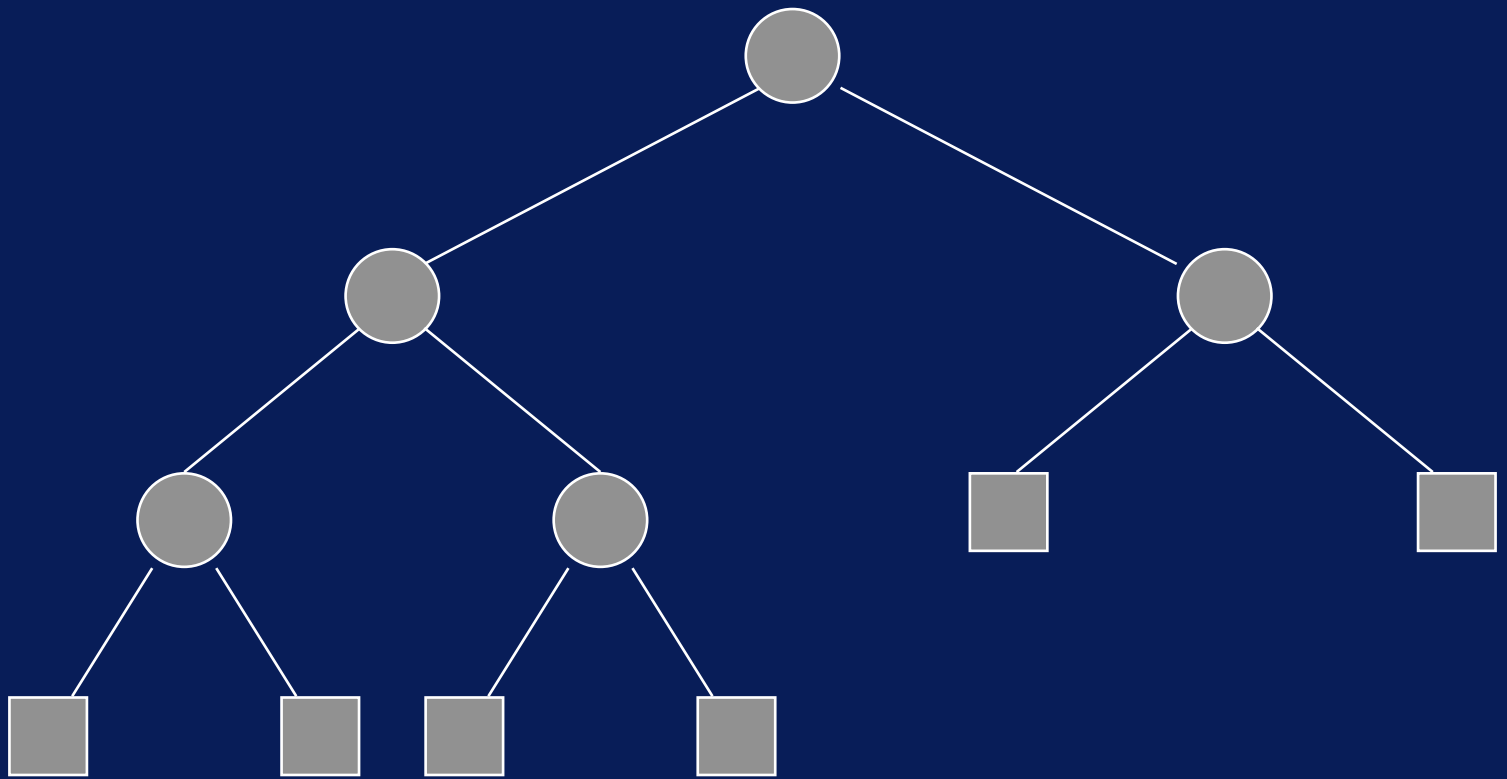
Example



BINARY_TREE Operations

- *Child*: N $\times$ W $\times$ BT $\rightarrow$ W :
  The function value *Child*(*i*, *w*, *T*) is undefined if the node in the window *W* is external or the node in *w* is internal and *i* is neither 1 nor 2; otherwise it is the *i*th child of the node in *w*

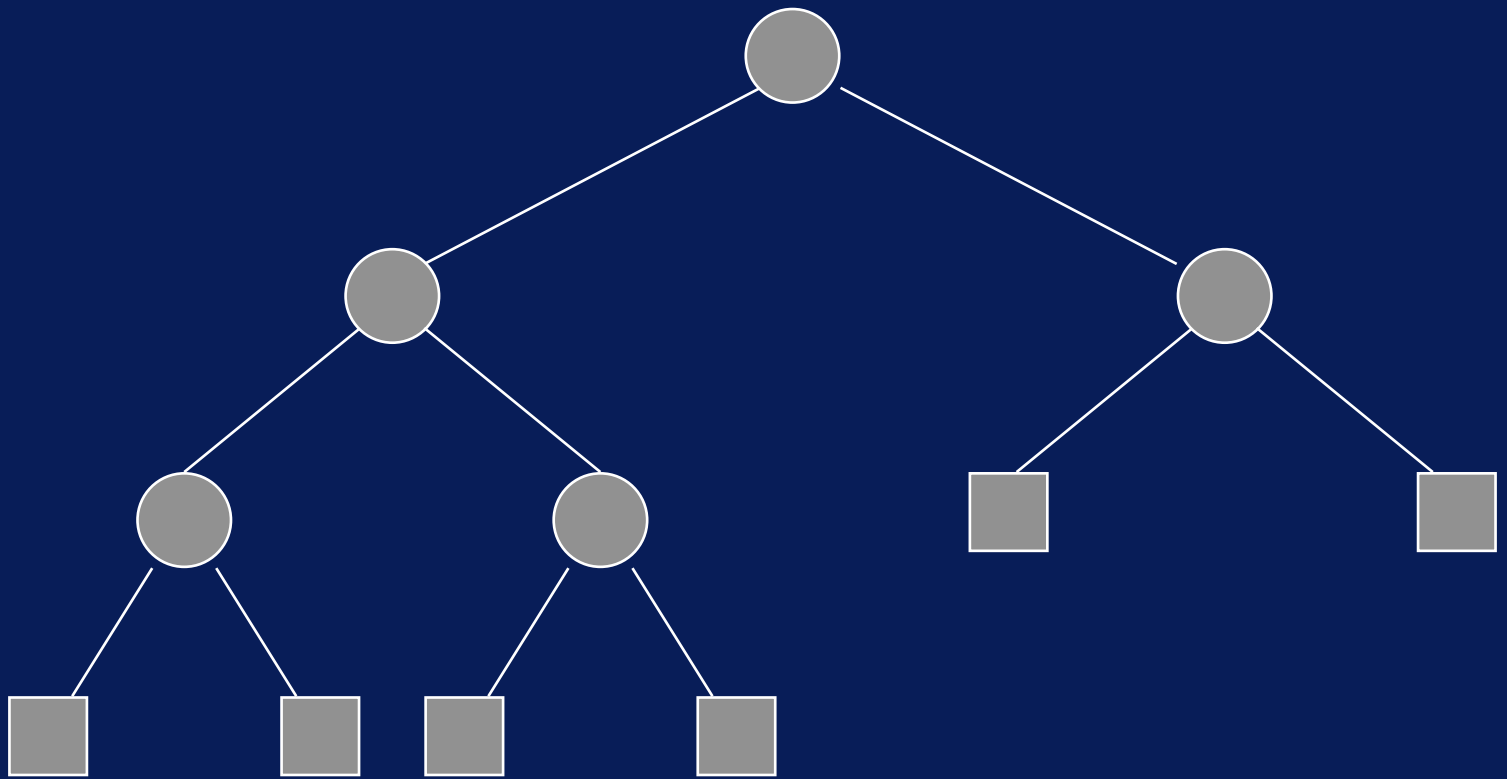
Example



BINARY_TREE Operations

- *Parent*: $\mathrm{W} \times \mathrm{BT} \rightarrow \mathrm{W}$:
  The function value *Parent*(*w*, *T*) is undefined if *T* is empty or *w* is over the root of *T*; otherwise it is the window position of the parent of the node in the window *w*

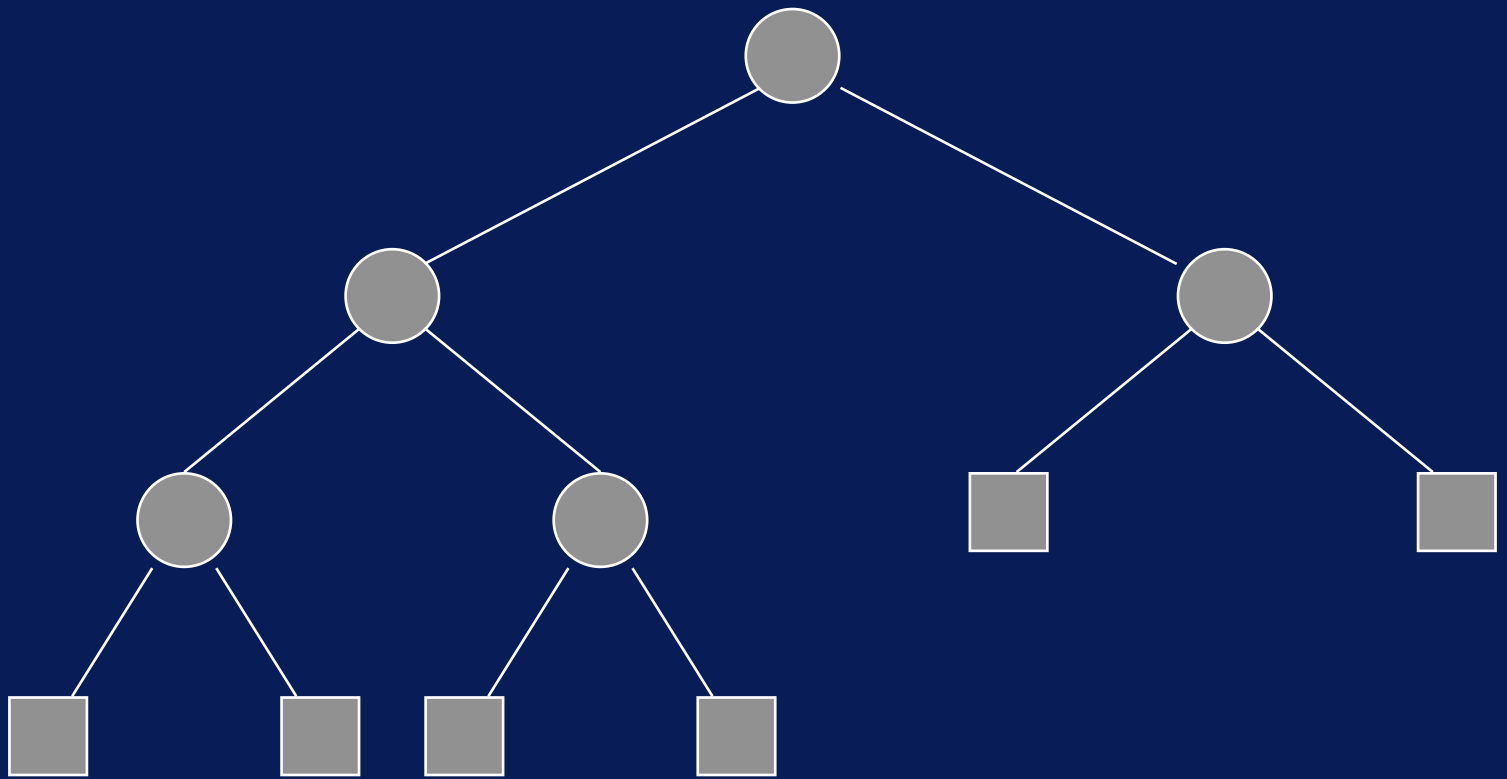
Example



BINARY_TREE Operations

- *Examine*: W $\times$ BT $\rightarrow$ *I* :
  The function value *Examine*(*w*, *T*) is undefined if *w* is over an external node; otherwise it is element at the internal node in the window *w*

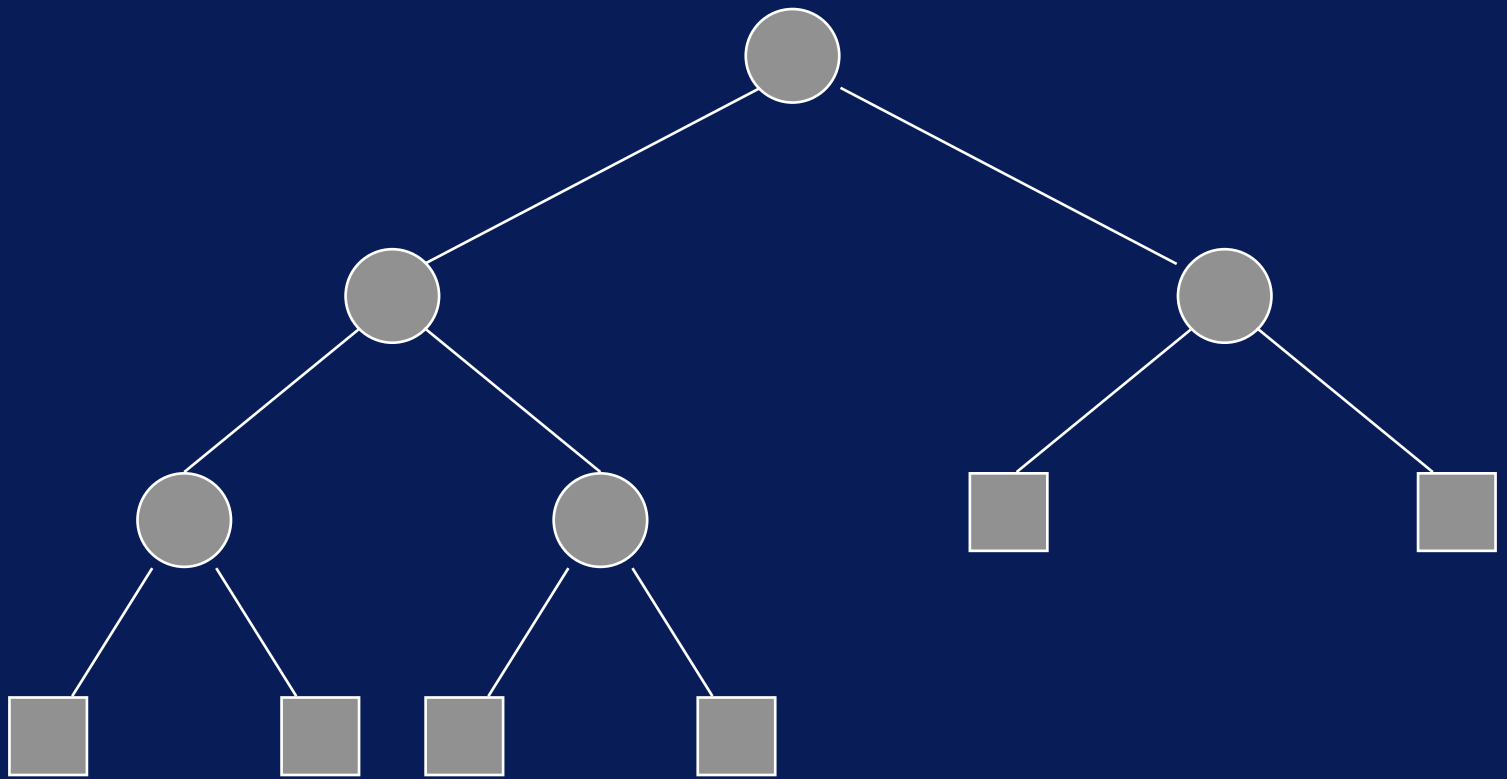
Example



BINARY_TREE Operations

- *Replace*: $E \times \mathrm{W} \times \mathrm{BT} \rightarrow \mathrm{BT}$:
  The function value *Replace*(*e*, *w*, *T*) is undefined if *w* is over an external node; otherwise it is *T*, with the element at the internal node in *w* replaced by *e*

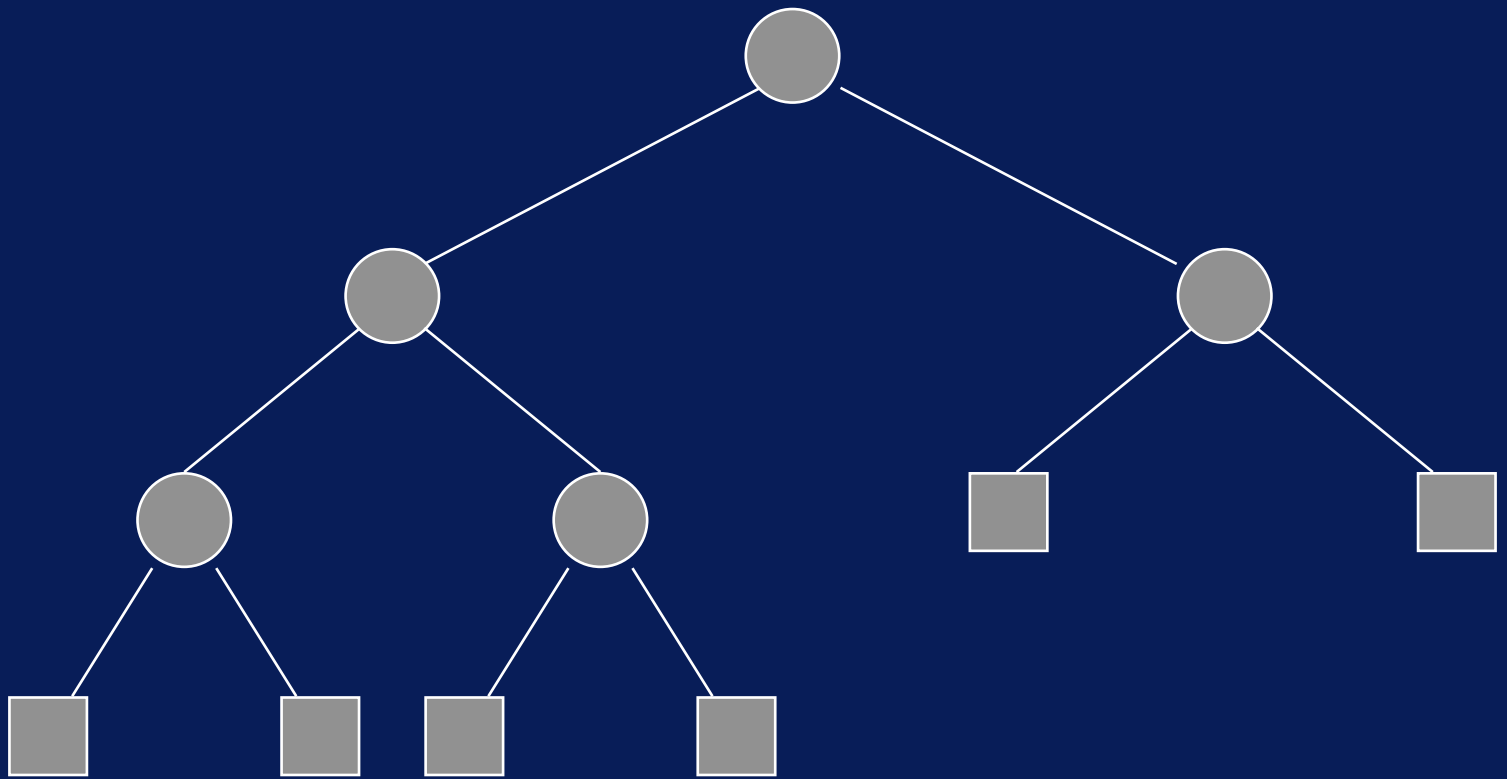
Example



BINARY_TREE Operations

- *Insert*: $E \times \mathrm{W} \times \mathrm{BT} \rightarrow \mathrm{W} \times \mathrm{BT}$:
  The function value *Insert*(*e*, *w*, *T*) is undefined if *w* is over an internal node; otherwise it is *T*, with the external node in *w* replaced by a new internal node with two external children.
  - Furthermore, the new internal node is given the value *e* and the window is moved over the new internal node.

Example



BINARY_TREE Operations

- *Delete*: $W \times BT \rightarrow W \times BT$:
  - The function value *Delete*(*w*, *T*) is undefined if *w* is over an external node;
  - If w is over a leaf node (both its children are external nodes), then the function value is T with the internal node to be deleted replaced by its left external node

BINARY_TREE Operations

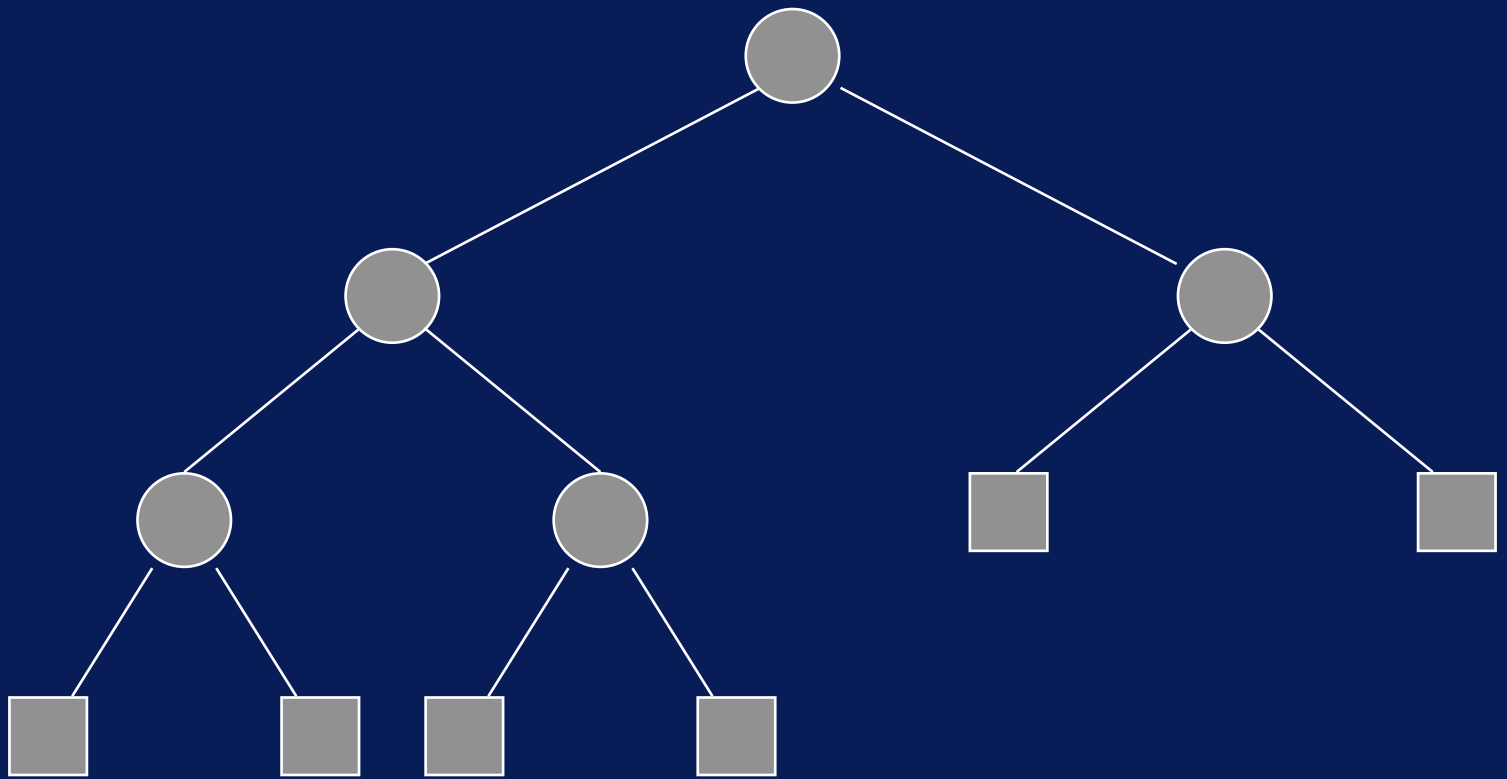
- *Delete*: $W \times BT \rightarrow W \times BT$:

  If w is over an internal node with just one internal node child, then the function value is T with the internal node to be deleted replaced by its child

BINARY_TREE Operations

- *Delete*: $W \times BT \rightarrow W \times BT$:
  - if w is over an internal node with two internal node children, then the function value is T with the internal node to be deleted replaced by the leftmost internal node descendent in its right sub-tree
  - In all cases, the window is moved over the replacement node.

Example

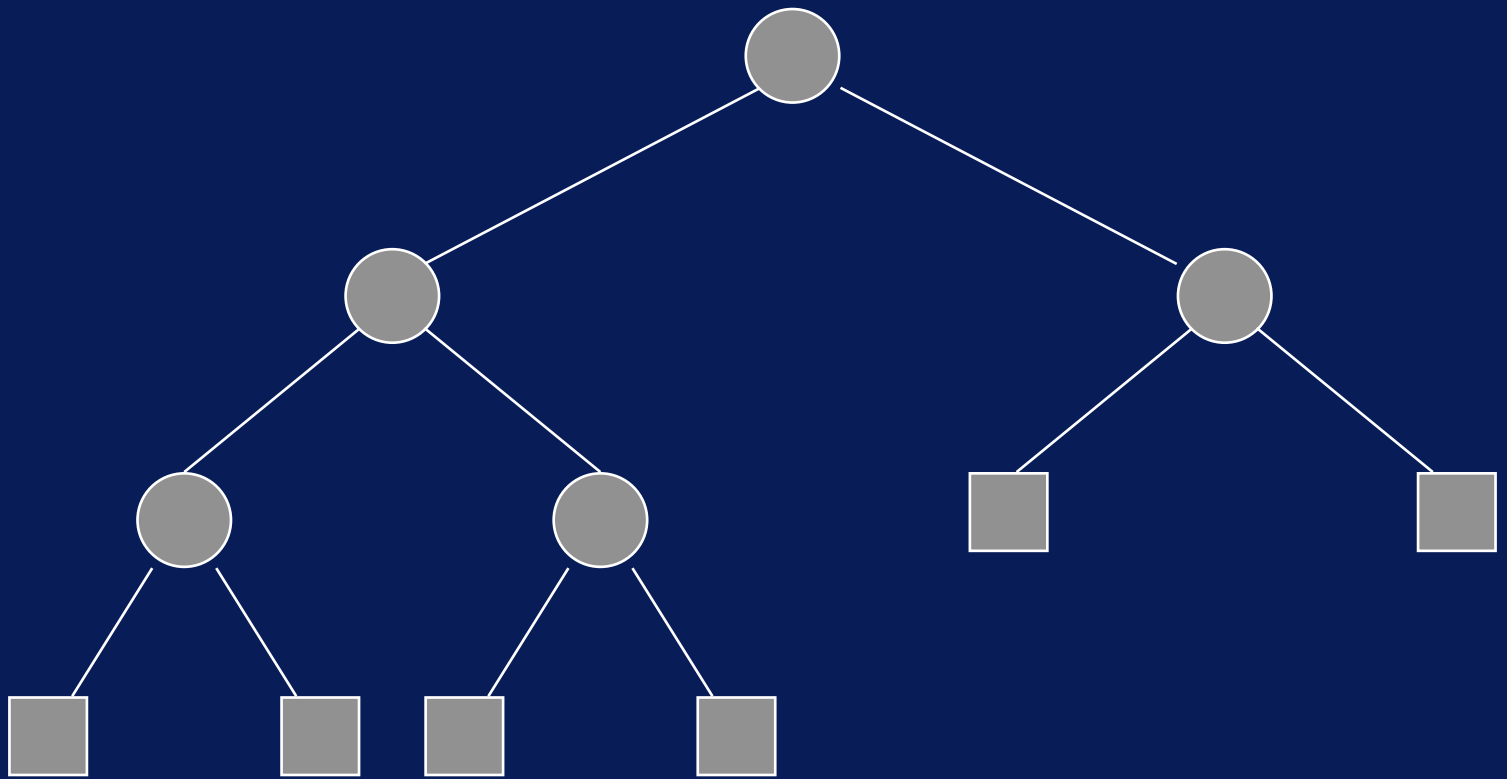


BINARY_TREE Operations

- *Left*: W $\times$ BT $\rightarrow$ W :

  The function value *Left*(*w*, *T*) is undefined if *w* is over an external node; otherwise it is the window position of the left (or first) child of the node *w*

Example

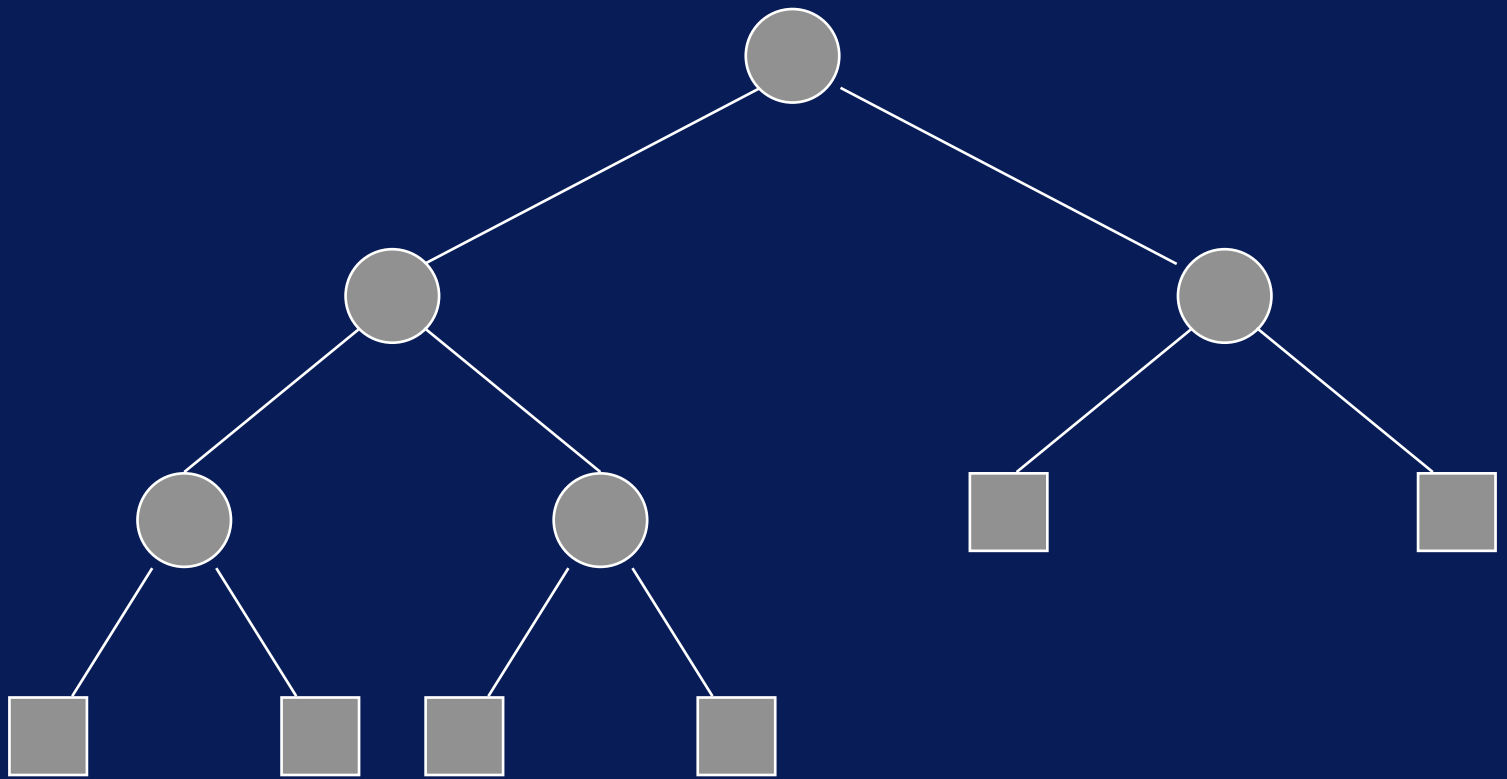


BINARY_TREE Operations

- *Right*: W $\times$ BT $\rightarrow$ W :

  The function value *Right*(*w*, *T*) is undefined if *w* is over an external node; otherwise it is the window position of the right (or second) child of the node *w*

Example

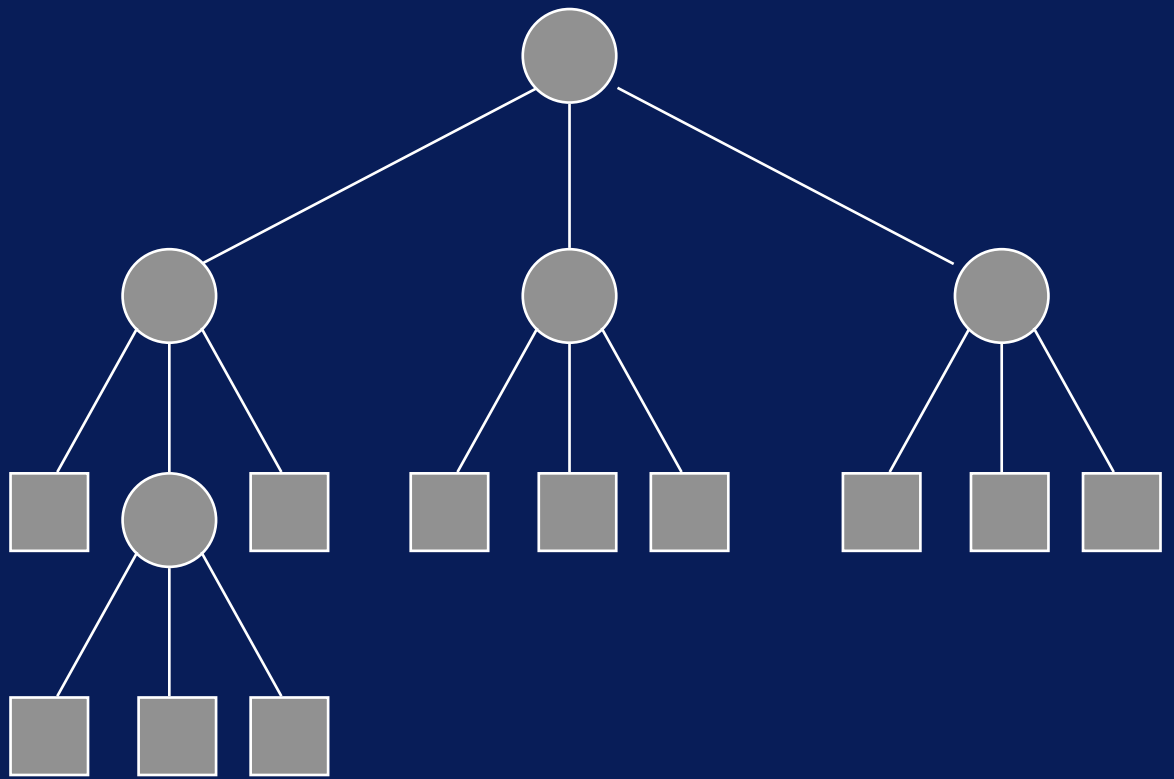


TREE Operations

- *Degree*: $W \times T \rightarrow I$:

  The function value *Degree*(*w*, *T*) is the degree of the node in the window *w*

d-ary Tree

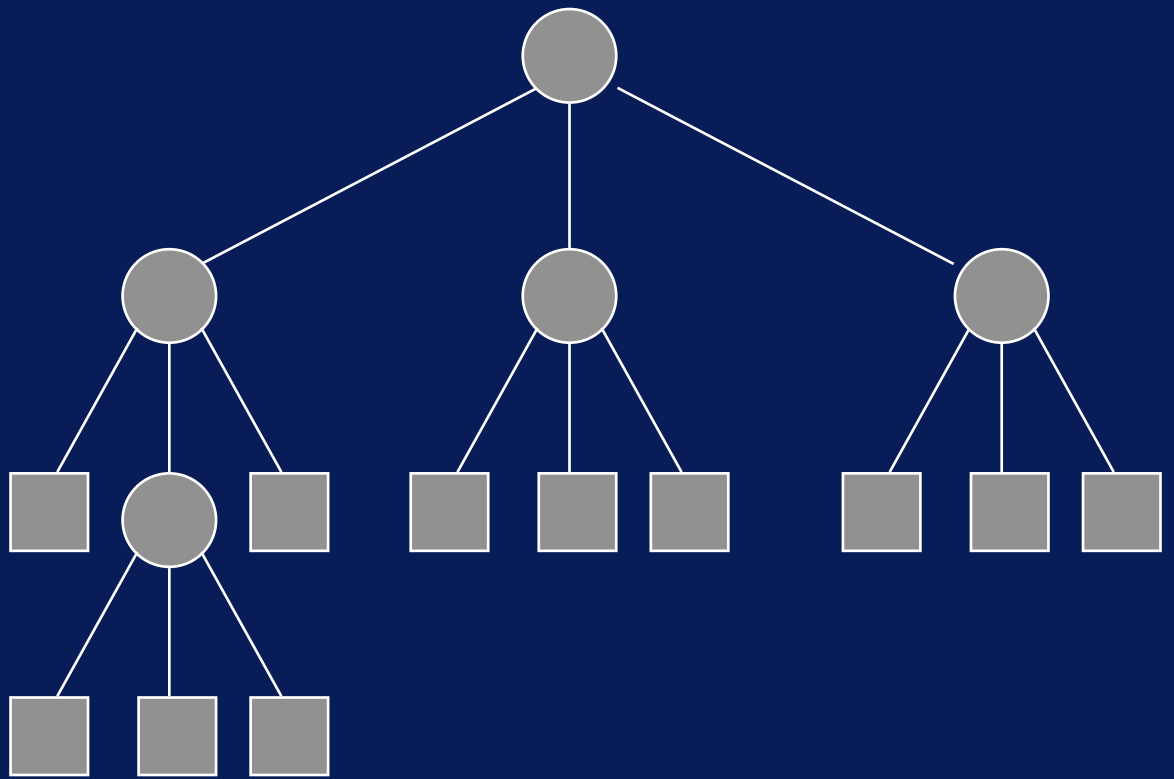


TREE Operations

- *Child*: $\mathbb{N} \times \mathbb{W} \times \mathbb{T} \rightarrow \mathbb{W}$:

  The function value *Child*(*i*, *w*, *T*) is undefined if the node in the window *w* is external, or if the node in *w* is internal and *i* is outside the range 1..*d*, where *d* is the degree of the node; otherwise it is the *i*th child of the node in *w*

d-ary Tree



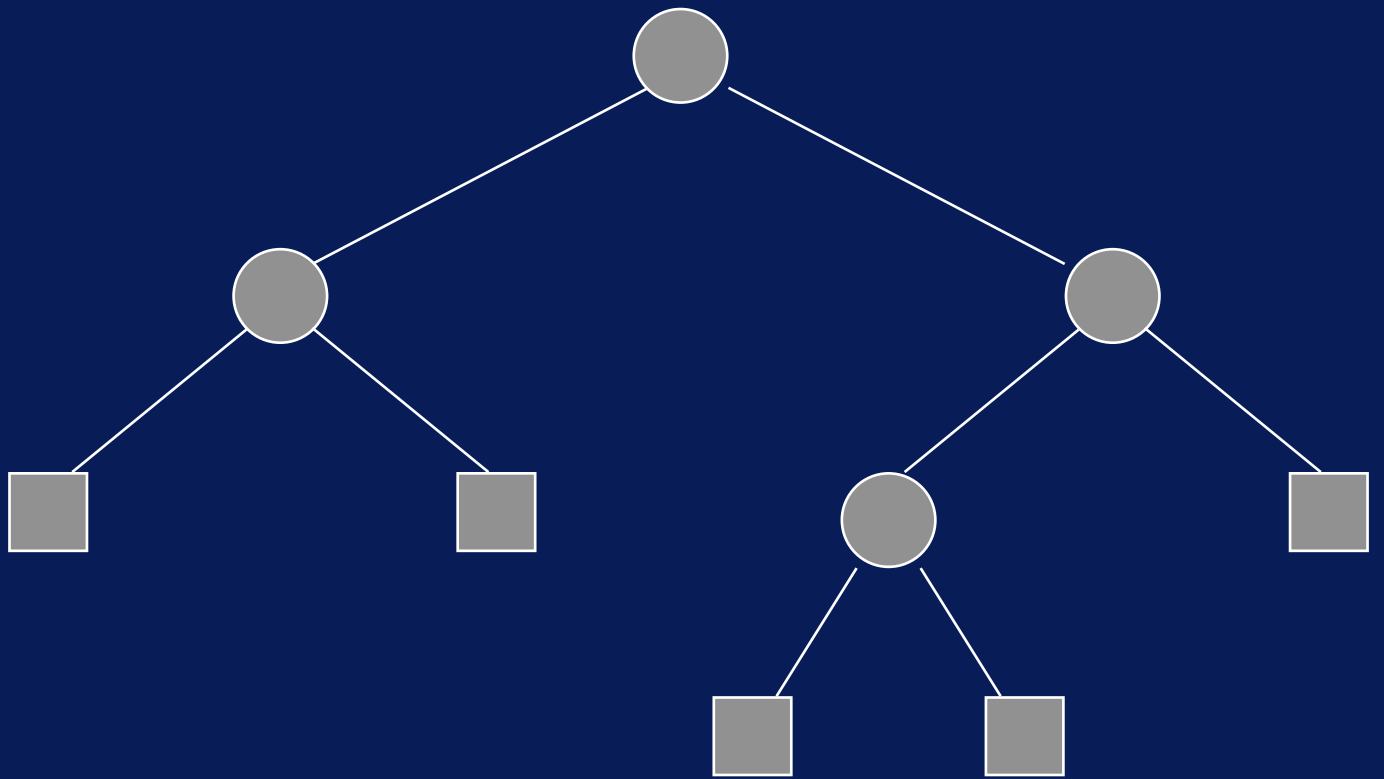
BINARY_TREE Representation

```
/* pointer implementation of BINART_TREE ADT */

#include <stdio.h>
#include <math.h>
#include <string.h>

#define FALSE 0
#define TRUE 1

typedef struct {
          int number;
          char *string;
       } ELEMENT_TYPE;
```

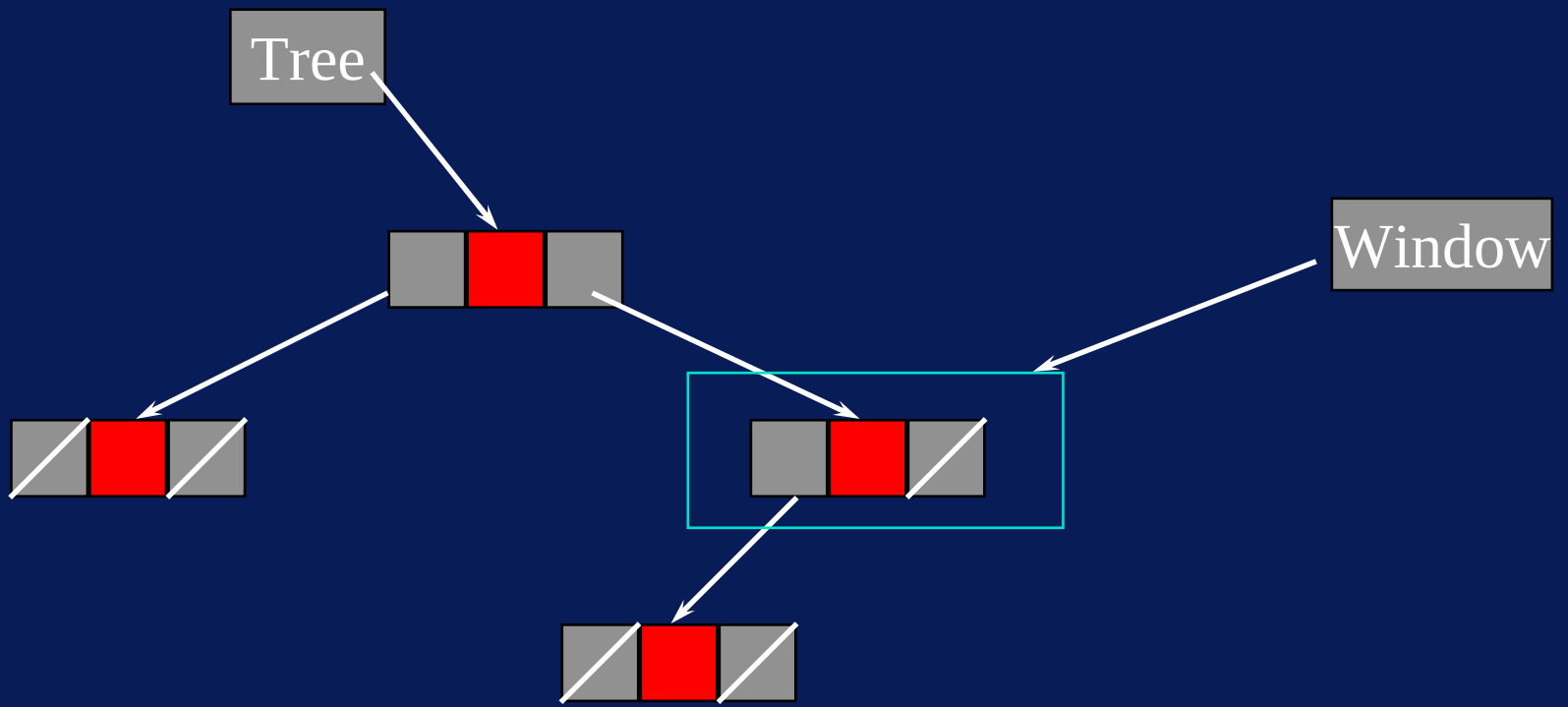
BINARY_TREE Representation

```
typedef struct node *NODE_TYPE;

typedef struct node{
          ELEMENT_TYPE element;
          NODE_TYPE left, right;
        } NODE;

typedef NODE_TYPE BINARY_TREE_TYPE;
typedef NODE_TYPE WINDOW_TYPE;
```

BINARY_TREE Representation



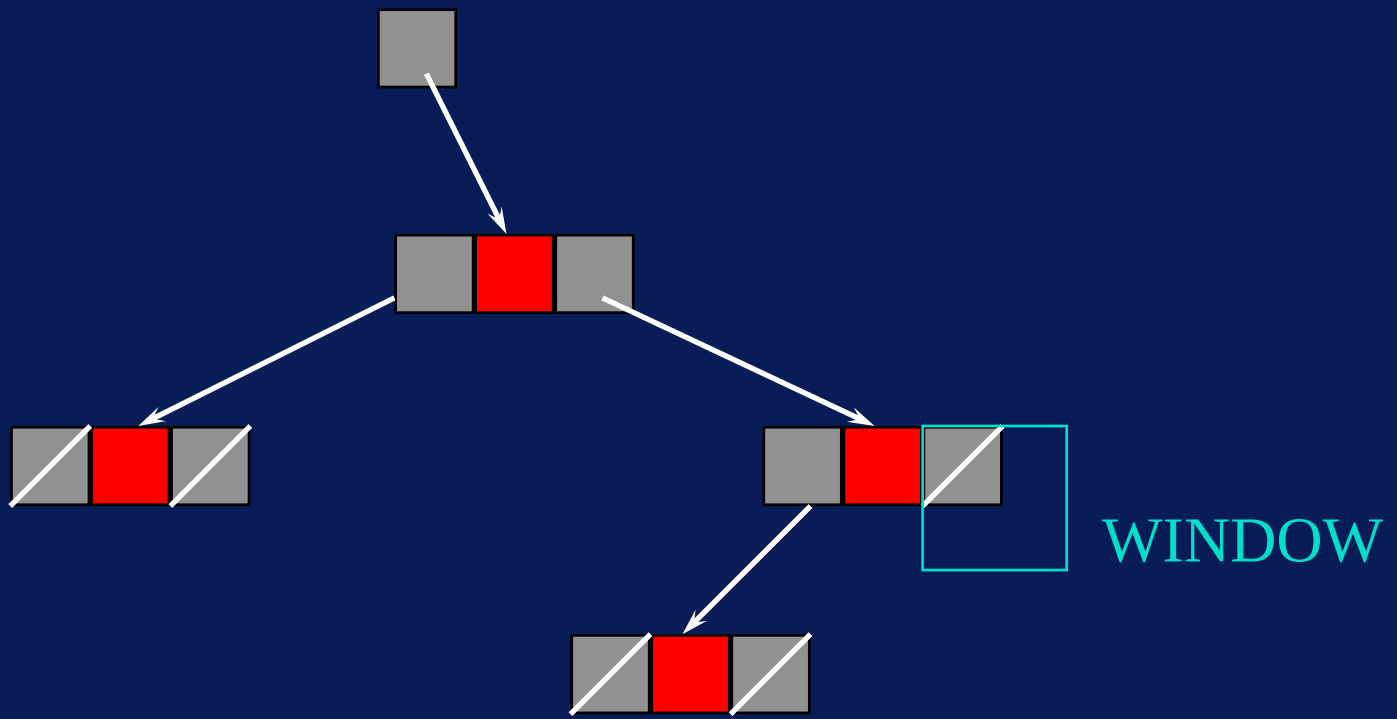
BINARY_TREE Representation



BINARY_TREE Representations

- This implementation assumes that we are going to represent external nodes as NULL links
- For many ADT operations, we need to know if the window is over an internal or an external node
  - we are over an external node if the window is NULL

BINARY_TREE Representation



BINARY_TREE Representations

- Whenever we insert an internal node (remember we can only do this if the window is over an external node) we simply make its two children NULL