Date:

S M T W T F S

Name: Zainab Khalid
Reg #: 19PWCSE 1743
Section: A
Subject: CV (Assignment #2)

Question #1:

Find $\int_c f(z)dz$ where $f(z) = \operatorname{Im} z$, c is the circle $|z| = r$ (counterclockwise)

$f(z) = \operatorname{Im} z$

This function is not analytic so Cauchy's theorem can't be applied.

$f(z) = y = \operatorname{Im} z$

$\Rightarrow \quad z(t) = e^{it}$

$z'(t) = ie^{it}$

$\operatorname{Im} z = \sin t$

$$\oint_c f(z)dz = \int_0^{2\pi} \sin t \cdot i \cdot (\cos t + i \sin t)dt$$

$$= \int_0^{2\pi} i \sin t \, d(\sin t) + \int_0^{2\pi} i^2 \sin^2 t \, dt$$

$$= i \int_0^{2\pi} \sin t \, d(\sin t) + i^2 \int_0^{2\pi} \sin^2 t \, dt$$

$$= -\frac{1}{2}(2\pi)$$

$$\oint_c f(z)dz = -\pi$$

Question #2:

$f(z) = 2z^4 - \bar{z}^4$, c is unit circle (counter clock wise)

Standard form of circle is,

$$|z - z_0| = \rho \longrightarrow ①$$

Given: circle is unit circle

$$|z| = 1 \longrightarrow ②$$

Comparing ① and ②

$$z_0 = 0, \quad \rho = 1$$

The parametric form of circle is

$$z(t) = z_0 + \rho e^{it}$$

$$z(t) = 0 + 1e^{it}.$$

$$\Rightarrow z'(t) = ie^{it}$$

$$\Rightarrow f(z(t)) = 2(e^{it})^4 - (e^{it})^{-4}$$

$$= 2e^{4it} - e^{-4it}$$

According to theorem # 1;

$$\oint_c f(z)\,dz = \int_0^{2\pi} f(z(t))\, z'(t)\,dt$$

$$= \int_0^{2\pi} (2e^{4it} - e^{-4it})(ie^{it})\,dt$$

$$= 2i\int_0^{2\pi} e^{5it}\,dt - i\int e^{-3it}\,dt$$

$$= \frac{2i}{5i} e^{5it}\Big|_0^{2\pi} - \frac{i}{-3i} e^{-3it}\Big|_0^{2\pi}$$

$$= \frac{2}{5}(e^{10\pi i} - 1) + \frac{1}{3}(e^{-6\pi i} - 1)$$

$$= \frac{2}{5}\left[\cos 10\pi + i\cancel{\sin 10\pi}\right] + \frac{1}{3}\left[\cos(-6\pi) + i\cancel{\sin(-6\pi)} - 1\right]$$

$$= \frac{2}{5}[1 - 1] + \frac{1}{3}[1 - 1]$$

$$\oint_c f(z)\,dz = 0$$

Question # 3:

Find the fourier series ....

i) $f(x) = \begin{cases} -1 & \text{if } 0 < x < \pi/2 \\ 0 & \text{if } \pi/2 < x < 2\pi \end{cases}$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left[a_n \cos nx + b_n \sin nx\right] \quad -①$$

$$a_0 = \frac{1}{\pi} \int_0^{2\pi} f(x)\, dx.$$

$$= \frac{1}{\pi} \int_0^{\pi/2} (-1)\, dx + \frac{1}{\pi} \int_{\pi/2}^{2\pi} (0)\, dx$$

$$\boxed{a_0 = -1/2}$$

$$a_n = \frac{1}{\pi} \int_0^{2\pi} f(x) \cos nx\, dx$$

$$= \frac{1}{\pi} \int_0^{\pi/2} (-1) \cos nx\, dx + \frac{1}{\pi} \int_{\pi/2}^{2\pi} (0) \cdot dx$$

$$= -\frac{1}{\pi} \left. \frac{\sin nx}{n} \right|_0^{\pi/2} + 0$$

$$\boxed{a_n = \frac{-1}{n\pi} \sin n\frac{\pi}{2}}$$

Now $$b_n = \frac{1}{\pi} \int_0^{2\pi} f(x) \sin nx\, dx$$

$$= \frac{1}{\pi} \int_0^{\pi/2} (-1) \sin nx\, dx + \frac{1}{\pi} \int_{\pi/2}^{2\pi} (0)\, dx$$

$$b_n = \frac{1}{n\pi} \left[\cos \frac{n\pi}{2} - 1\right]$$

Date:

S M T W T F S

eq ① =>

$$f(x) = \frac{-1/2}{2} + \sum_{n=1}^{\infty}\left[\frac{-1}{n\pi}\sin\frac{n\pi}{2}\cos nx + \frac{1}{n\pi}\left[\cos\frac{n\pi}{2} - 1\right]\sin nx\right]$$

$$f(x) = -\frac{1}{4} + \frac{1}{\pi}\left[-\cos x - \sin x - \sin 2x - \frac{1}{3}\cos 3x - \frac{1}{3}\sin 3x + \cdots\right]$$

ii) $f(x) = \begin{cases} x^2 & \text{if } -\pi/2 < x < \pi/2 \\ \pi^2/4 & \text{if } \pi/2 < x < 3\pi/2 \end{cases}$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left[a_n \cos nx + b_n \sin nx\right] \rightarrow ①$$

$$a_0 = \frac{1}{\pi}\int_{-\pi/2}^{3\pi/2} f(x)\,dx$$

$$a_0 = \frac{1}{\pi}\int_{-\pi/2}^{\pi/2}(x^2)\,dx + \frac{1}{\pi}\int_{\pi/2}^{3\pi/2}(\pi^2/4)\,dx$$

$$a_0 = \frac{1}{\pi}\left(\frac{x^3}{3}\right)\Big|_{-\pi/2}^{\pi/2} + \frac{\pi}{4}(x)\Big|_{\pi/2}^{3\pi/2}$$

$$a_0 = \frac{1}{3\pi}\left(\left(\frac{\pi}{2}\right)^3 - \left(-\frac{\pi}{2}\right)^3\right) + \frac{\pi}{4}\left(\frac{3\pi}{2} - \frac{\pi}{2}\right)$$

$$a_0 = \left(\frac{1}{3\pi}\left(\frac{\pi^3}{8} - \left(-\frac{\pi^3}{8}\right)\right) + \frac{\pi}{4}\left(\frac{2\pi}{2}\right)\right.$$

$$a_0 = \frac{1}{12}\pi^2 + \frac{\pi^2}{4}$$

STAR NOTES

Question 4 (i):

$f(x) = x \quad (-1 < x < 1) \quad P = 2L = 2.$

$f(x) = x \quad (-1 < x < 1)$

$P = 2L = 2 \Rightarrow L = 1.$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left[a_n \cos\frac{n\pi x}{L} + b_n \sin\frac{n\pi x}{L} \right] \rightarrow ①$$

$$a_0 = \frac{1}{L} \int_{-L}^{L} f(x)\,dx$$

$$a_0 = \frac{1}{1} \int_{-1}^{1} x\,dx \Rightarrow a_0 = \frac{x^2}{2} \Big|_{-1}^{1}$$

$$a_0 = \frac{1}{2}\left((+1)^2 - (-1)^2 \right)$$

$$a_0 = 0$$

$$a_n = \frac{1}{L} \int_{-L}^{L} f(x) \cos\frac{n\pi x}{L} = \frac{1}{1} \int_{-1}^{1} x \cos\frac{n\pi x}{1}\,dx$$

$$= \left(x\frac{\sin n\pi x}{n\pi} - \int \frac{\sin n\pi x}{(n\pi)} \right) \Big|_{-1}^{1}$$

$$= \left(\frac{x \sin n\pi x}{n\pi} + \frac{1}{n\pi} \frac{\cos n\pi x}{n\pi} \right) \Big|_{-1}^{1}$$

$$= \frac{x \sin n\pi x}{n\pi} \Big|_{-1}^{1} + \frac{1}{n^2\pi^2} \cos n\pi x \Big|_{-1}^{1}$$

$$a_n = \left[\frac{1}{n\pi} (1) \sin n\pi(1) - \frac{(-1) \sin n\pi (-1)}{n\pi} \right]$$

Date: ________ S M T W T F S

$$+\left[\frac{1}{n^2\pi^2}\cos n\pi(1) - \frac{1}{n^2\pi^2}\cos n\pi(-1)\right]$$

$$a_n = (0) + (0)$$

$$a_n = 0$$

$$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\frac{n\pi x}{L}\,dx$$

$$\Rightarrow b_n = \int_{-1}^{1} (x)\sin n\pi x\,dx$$

$$b_n = x\left(\frac{-\cos n\pi x}{n\pi}\right)\Big|_{-1}^{1} - \int_{-1}^{1}\left(\frac{-\cos n\pi x}{n\pi}\right)dx$$

$$b_n = -\frac{x\cos n\pi x}{n\pi}\Big|_{-1}^{1} + \frac{\sin n\pi x}{n^2\pi^2}\Big|_{-1}^{1}$$

$$b_n = \frac{-1}{n\pi}\left[(1)\cos n\pi(1) - (-1)\cos n\pi(-1)\right]$$

$$+\frac{1}{n^2\pi^2}\left[\sin n\pi(1) - \sin n\pi(-1)\right]$$

$$b_n = -\frac{\cos n\pi}{n\pi} - \frac{\cos n\pi}{n\pi} + \frac{\sin n\pi}{n^2\pi^2} + \frac{\sin n\pi}{n^2\pi^2}$$

$$b_n = \frac{-2\cos n\pi}{n\pi} + \frac{2\sin n\pi}{n^2\pi^2}$$

STAR NOTES

eq (1) $\Rightarrow$

$$f(x) = \frac{0}{2} + \sum_{n=1}^{\infty} \left[0 + \left(-\frac{2\cos n\pi}{n\pi} + \frac{2\sin \pi n}{\pi^2 n^2} \right) \cdot \frac{\sin n\pi x}{L} \right]$$

$$f(x) = \sum_{n=1}^{\infty} \left[-\frac{2\cos n\pi}{n\pi} \cdot \sin n\pi x + \frac{2\sin \pi n}{n^2\pi^2} \cdot \sin n\pi x + \cdots \right]$$

$$f(x) = \left[-\frac{2\cos n\pi}{n\pi} \cdot \sin n\pi + \frac{2\sin n\pi}{n^2\pi^2} \cdot \sin n\pi + \cdots \right]$$

Question # 4.

Find fourier series of periodic function $f(x)$ of period $P=2L$, ---

i) $f(x) = -1 \; (-1 < x < 0)$ $f(x) = 1$
$(0 < x < 1)$ $P = 2L = 2$

$P = 2L = 2 \Rightarrow L = 1$

The F.S of $f(x)$ is.

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left[a_n \cos\frac{n x \pi}{L} + b_n \sin\frac{n x \pi}{L}\right] \to (1)$$

$$a_0 = \int_{-1}^{1} f(x)\, dx.$$

$$= \int_{-1}^{0} (-1)\, dx + \int_{0}^{1} (1)\, dx$$

$$a_0 = 0$$

$$a_n = \frac{1}{L}\int_{-1}^{1} f(x) \cos\frac{n\pi x}{L}\, dx$$

$$= \frac{1}{1}\int_{-1}^{0} \cos\frac{n\pi x}{1}\, dx + \int_{0}^{1} 1.\cos\frac{n\pi x}{1}\, dx$$

$$= \frac{-1}{n\pi} \sin n\pi x \Big|_{-1}^{0} + \frac{1}{n\pi} \sin n\pi x \Big|_{0}^{1}$$

$$a_n = \frac{-1}{n\pi}\left[\sin(0) - \sin n(-1)\pi\right]$$
$$+ \frac{1}{n\pi}\left[\sin n\pi(1) - 0\right]$$

$$= \frac{1}{n\pi}\left[\sin n\pi - \sin n\pi\right]$$

$$\boxed{a_n = 0}$$

$$b_n = \frac{1}{1}\int_{-1}^{0} -\sin n\pi x\,dx + \int_{0}^{1} \sin n\pi x\,dx$$

$$= \frac{1}{n\pi}\cos n\pi x\Big|_{-1}^{0} - \frac{1}{n\pi}\cos n\pi x\Big|_{0}^{1}$$

$$= \frac{1}{n\pi}\left[\cos(0) - \cos n\pi(-1)\right] \frac{-1}{n\pi}\left[\cos n\pi - \cos(0)\right]$$

$$= \frac{1}{n\pi}\left[1 - \cos n\pi\right] + \frac{1}{n\pi}\left[1 - \cos n\pi\right]$$

$$b_n = \frac{2}{n\pi}\left[1 - \cos n\pi\right]$$

eq ① =>

$$f(x) = \frac{0}{2} + \sum_{n=1}^{\infty}\left[0 \cdot \frac{\cos n\pi x}{1} + \frac{2}{n\pi}(1-\cos n\pi) \cdot \sin n\pi x\right]$$

$$f(x) = \frac{4}{\pi}\left[\sin \pi x + \frac{1}{3}\sin 3\pi x + \frac{1}{5}\sin 5\pi x + \dots\right]$$

(iii) $f(x) = \pi x^3/2 \quad (-1 < x < 1)$, $P = 2L = 2$

$P = 2L = 2$

$\Rightarrow L = 1$

The F.S of $f(x)$ is given by;

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left[a_n \cos\frac{n\pi x}{L} + b_n \sin\frac{n\pi x}{L}\right] \rightarrow (1)$$

$$a_0 = \frac{1}{L}\int_{-L}^{L} f(x)\,dx$$

$$a_0 = \frac{1}{1}\int_{-1}^{1} \frac{\pi x^3}{2}\,dx = 0$$

$$\boxed{a_0 = 0}$$

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x) \cos\frac{n\pi x}{L}\,dx$$

$$a_n = \frac{1}{1}\int_{-1}^{1}\left(\frac{\pi x^3}{2}\right) \cos\frac{n\pi x}{1}\,dx$$

$$a_n = \frac{\pi}{2}\left[x^3 \frac{\sin n\pi x}{n\pi}\Big|_{-1}^{1} - \frac{3}{n\pi}\int_{-1}^{1} x^2 \sin n\pi x\,dx\right]$$

$$= \frac{\pi}{2}\left[\frac{1}{n\pi}\left[\sin n\pi - \sin n\pi\right] - \frac{3}{n\pi}\left[\frac{-x^2 \cos n\pi}{n\pi}\Big|_{-1}^{1}\right.\right.$$

$$+ \frac{2}{n\pi}\int_{-1}^{1} x \cos n\pi\,dx$$

$$\boxed{a_n = 0}$$

$$b_n = \frac{1}{L}\int_{-L}^{L} f(x) \sin\frac{n\pi x}{L}\,dy$$

$$b_n = \frac{1}{1}\int_{-1}^{1} \frac{\pi x^3}{2} \sin n\pi x\,dx$$

$$= \frac{\pi}{2}\left[\frac{-x^3}{n\pi}\cos\pi x\Big|_{-1}^{1} + \frac{3}{n\pi}\int_{-1}^{1} x^2 \cos n\pi x\,dx\right]$$

$$b_n = \frac{-1}{n}\cos n\pi + \frac{6}{n^3\pi^2}\cos n\pi$$

eq ① =>

$$f(x) = 0 + \sum_{n=1}^{\infty}\left[0 \cdot \cos n\pi x + \left(\frac{-1}{n}\cos n\pi + \frac{6}{n^3\pi^2}\cos n\pi\right) \cdot \sin n\pi x\right]$$

$$= \left(\sin n\pi x - \frac{1}{2}\sin 2\pi x + \frac{1}{3}\sin 3\pi x + \cdots\right)$$

$$= \frac{-6}{\pi^2}\left(\sin \pi x - \frac{1}{2^3}\sin 2\pi x + \frac{1}{3^3}\sin 3\pi x + \cdots\right)$$

Question # 5:

State whether the given function is even or odd - - - .

i- $f(x) = \begin{cases} k & \text{if } -\pi/2 < x < \pi/2 \\ 0 & \text{if } \pi/2 < x < 3\pi/2 \end{cases}$

$f(x) = k$

$f(-x) = k = f(x)$

and $f(x) = 0$

$f(-x) = 0 = f(x)$

As from above These functions are even coz $f(-x) = f(x)$

Date:

S M T W T F S

The fourier series for an even function is

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{L} \rightarrow ①$$

$$a_0 = \frac{2}{L} \int_0^L f(x)\,dx$$

$2L = 2\pi$

$\therefore L = \pi$

$$a_0 = \frac{2}{\pi} \left[\int_0^{\pi/2} k\,dx + \int_{\pi/2}^{\pi} 0\,dx \right]$$

$$\boxed{a_0 = k}$$

$$a_n = \frac{2}{L} \int_0^L f(x) \cos \frac{n\pi x}{L}\,dx$$

$$= \frac{2}{\pi} \int_0^{\pi} f(x) \cos \frac{n\cancel{\pi} x}{\cancel{\pi}}\,dx$$

$$= \frac{2}{\pi} \left[\int_0^{\pi/2} k \cos nx\,dx + \int_{\pi/2}^{\pi} 0\,dx \right.$$

$$a_n = \frac{2k}{n\pi} \sin n\pi/L$$

eq ① $\Rightarrow$

$$f(x) = \frac{k}{2} + \sum_{n=1}^{\infty} \left[\frac{2k}{n\pi} \sin \frac{n\pi}{2} \cdot \cos \frac{n\pi x}{\pi} \right]$$

$$= \frac{k}{2} + \sum_{n=1}^{\infty} \left[\frac{2k}{n\pi} \sin \frac{n\pi}{2} \cdot \cos nx \right]$$

$$f(x) = \frac{k}{2} + \frac{2k}{\pi} \left[\cos x + \frac{1}{3} \cos 3x + \cdots \right]$$

Date:

Q#6:

Find Taylor series of the given function ------.

i) e^{2z}, $2i$

The Taylor's series of a function $f(z)$ is given by at $z = z_0$

$$f(z) = f(z_0) + (z - z_0)\frac{f'(z_0)}{1!} + (z - z_0)^2 \frac{f''(z_0)}{2!} + \text{----} \quad (1)$$

Now.

$z = z_0 = 2i$

$f(z) = e^{2z}$ $\qquad$ $f(z_0) = e^{2(2i)} = e^{4i}$

$f'(z) = 2e^{2z}$ $\qquad$ $f'(z_0) = 2e^{2(2i)} = 2e^{4i}$

$f''(z) = 4e^{2z}$ $\qquad$ $f''(z_0) = 4e^{2(2i)} = 4e^{4i}$

eq (1) $\Rightarrow$

$$f(z) = e^{2z} = e^{4i} + \frac{(z-2i).2e^{4i}}{1!} + \frac{(z-2i)^2.4e^{4i}}{2!} + \text{---}$$

$$e^{2z} = e^{4i} + (z-2i).2e^{4i} + \frac{(z-2i)^2\, 4e^{4i}}{2} + \text{---}$$

$$e^{2z} = e^{4i} + (z-2i)\,2e^{4i} + (z-2i)^2.\,2e^{4i} + \text{---}$$

Date:

S M T W T F S

Radius of Convergence :-

$$R = |z - z_0| \rightarrow ②$$

As z is a s.point of given function

$\therefore f(z) = \infty$

$$e^{2z} = \infty$$

$$\ln \cancel{e}^{2z} = \ln \infty$$

$$2z = \infty \Rightarrow z = \infty$$

$$\therefore R = |\infty - 0i|.$$

$\Rightarrow R = \infty$ is the radius of convergence.

ii) $\sin \pi z, 0$

Taylor's series of a given function is,

$$f(z) = f(z_0) + (z - z_0)\frac{f'(z_0)}{1!} + (z - z_0)^2\frac{f''(z_0)}{2!} + \cdots$$

$f(z) = \sin \pi z$ $\qquad$ $f(z_0) = f(0) = \sin \pi(0) = 0$

$f'(z) = \pi \cos(\pi z)$ $\qquad$ $f'(z_0) = f'(0) = \pi \cos \pi(0) = \pi$

$f''(z) = -\pi^2 \sin \pi z$ $\qquad$ $f''(z_0) = f''(0) = -\pi^2 \sin \pi(0) = 0$

equation ① $\Rightarrow$

$$\sin \pi z = 0 + (z-0)(\pi) + (z-0)^2\frac{(0)}{2!} + (z-0)^3\frac{(-\pi^3)}{3!} +$$

Date:

$$\sin \pi z = \pi z - \frac{1}{3!} \pi^3 z^3 + \text{———}$$

Radius of Convergence :-

$R = |z - z_0| \rightarrow (2)$

Now As z is a singular point of given function,

$\therefore \quad f(z) = \infty$

$\sin \pi z = \infty \quad \Rightarrow \quad \pi z = \infty$

$z = \infty$

$\therefore \quad R = |\infty - 0| \Rightarrow R = \infty$ is the radius of convergence.

iii) $1/z$, 1

The Taylor's series is given by ;

$$f(z) = f(z_0) + (z - z_0) \frac{f'(z_0)}{1!} + (z - z_0)^2 \frac{f''(z)}{2!} + \text{—} \quad (1)$$

$f(z) = 1/z = z^{-1}$ $\qquad f(1) = 1/1 = 1$

$f'(z) = -1 z^{-2} = -1/z^2$ $\qquad f'(1) = -1/(1)^2 = -1/(1) = -1$

$f''(z) = 2z^{-3} = \frac{2}{z^3}$ $\qquad f''(1) = 2/(1)^3 = 2$.

equation (1) $\Rightarrow$

$$f(z) = 1/z = 1 + (z-1)\frac{(-1)}{1!} + (z-1)^2 \frac{2}{2!} + \text{– –}$$

$$f(z) = 1/z = 1 - (z-1) + (z-1)^2 + \cdots$$

Radius of Convergence :-

$$R = |z - z_\circ| \rightarrow ②$$

As z is a singular point of given function,

$$\therefore f(z) = \infty$$

$$\frac{1}{z} = \infty \qquad \Rightarrow z = 0$$

$$\therefore R = |0 - 1| \Rightarrow R = 1$$

is the radius of convergence.

Question # 7:

Find following functions in Laurent Series ------

i) $\dfrac{e^z}{z^2} \rightarrow ①$

For singular point,

$$\frac{e^z}{z^2} = \infty \Rightarrow z^2 = 0 \Rightarrow z = 0$$

here no point is given, take $z_\circ = 0$

let $z - z_\circ = u$

$$z - 0 = u \Rightarrow z = u \rightarrow ②$$

equation ① $\Rightarrow$

$$f(u) = \frac{e^u}{u^2}$$

$$= \frac{1}{u^2}\left[1 + u + \frac{u^2}{2!} + \frac{u^3}{3!} + \cdots\right]$$

$$= \frac{1}{u^2} + \frac{u}{u^2} + \frac{\cancel{u^2}}{2\cancel{u^2}} + \frac{u^{\cancel{3}}}{6\cancel{u^2}} + \cdots$$

$$= \frac{1}{u^2} + \frac{1}{u} + \frac{1}{2} + \frac{u}{6} + \cdots$$

$$= \frac{1}{2} + \frac{u}{6} + \cdots + \frac{1}{u} + \frac{1}{u^2} + \cdots \quad (3)$$

Putting $u = z$ in eq (3),

$$\Rightarrow f(z) = \frac{1}{2} + \frac{z}{6} + \cdots + \frac{1}{z} + \frac{1}{z^2} + \cdots$$

$$\text{or } f(z) = \frac{1}{2!} + \frac{z}{3!} + \cdots + \frac{1}{z} + \frac{1}{z^2} + \cdots \quad (4)$$

The Lauren Series is given by;

$$f(z) = a_0 + a_1(z - z_0) + a_2(z - z_0)^2 + \cdots + \frac{b_1}{z - z_0} + \frac{b_2}{(z - z_0)^2} + \cdots \quad (5)$$

Comparing eq (4) with eq (5),

$$a_0 = \frac{1}{2!}, \quad a_1 = \frac{1}{3!}, \quad \cdots$$

$$b_1 = 1, \quad b = 1, \quad b_3 = b_4 = \cdots b_n = 0$$

and $z_0 = 0$

Hence equation (4) ⇒ gives the required Laurent Series.

ii) $\frac{\sin 4z}{z^4}$

$$f(z) = \frac{\sin 4z}{z^4} \longrightarrow ①$$

The Laurent Series of $f(z)$ is given by,

$$f(z) = a_0 + a_1(z-z_0) + a_2(z-z_0)^2 + \cdots + \frac{b_1}{z-z_0} + \frac{b_2}{(z-z_0)^2} + \cdots$$

For a singular point,

$\therefore f(z) = \infty$

$$\Rightarrow \frac{\sin 4z}{z^4} = \infty, \quad z^4 = 0$$

$$z = 0 \quad \text{or} \quad z_0 = 0$$

Now put $z - z_0 = u$

$$z - 0 = u \Rightarrow z = u$$

equation ① $\Rightarrow$

$$f(u) = \frac{\sin 4u}{u^4} = \frac{1}{u^4}\left[4u - \frac{(4u)^3}{3!} + \frac{(4u)^5}{5!} + \cdots\right]$$

$$= \frac{1}{u^4}\left[4u - \frac{(4)^3(u)^3}{3!} + \frac{(4)^5(u)^5}{5!} + \cdots\right]$$

$$= \frac{4u}{u^4} - \frac{64u^3}{3!.u^4} + \frac{1024u^5}{5!.u^4} - \frac{16384u^7}{7!.u^4} + \cdots$$

$$= \frac{4}{u^3} - \frac{64}{3!}.\frac{1}{u} + \frac{1024}{5!}.u - \frac{16384u^3}{7!} + \cdots$$

$$= \frac{1024u}{5!} + \frac{16384}{7!}u^3 + \cdots + \frac{64}{3!\,u} + \frac{4}{u^3} + \cdots$$

Put $u = z$

$$= \frac{1024}{5!}z - \frac{16384}{7!}z^3 + \cdots - \frac{64}{3!\,z} + \frac{4}{z^3} + \cdots$$

This is the required Laurent Series of given function.

iii) $\frac{\cosh 2z}{z}$

$$f(z) = \frac{\cosh 2z}{z} \rightarrow ①$$

The Laurent Series of $f(z)$ at z_0 is given by;

$$f(z) = a_0 + (z-z_0)a_1 + a_2(z-z_0)^2 + \cdots + \frac{b_1}{z-z_0} + \frac{b_2}{(z-z_0)^2} + \cdots$$

For singular point,

$$f(z) = \infty \Rightarrow \frac{\cosh 2z}{z} = \infty$$

$$z = 0$$

let $z - z_0 = u \Rightarrow z = u$

eq ① ⇒

$$f(u) = \frac{\cosh 2u}{u} = \frac{1}{u}\left[1 + \frac{(2u)^2}{2!} + \frac{(2u)^4}{4!} + \cdots\right]$$

$$= \frac{1}{u} + \frac{4u^2}{2u} + \frac{16u^4}{(24)u} + \cdots$$

Date:

S M T W T F S

$$= \frac{1}{u} + \frac{2u^2}{u} + \frac{2}{3}\frac{u^4}{u} + \cdots$$

$$= \frac{1}{u} + 2u + \frac{2}{3}u^3 + \cdots$$

$$\Rightarrow f(u) = 2u + \frac{2u^3}{3} + \cdots + \frac{1}{u} + \cdots$$

~~This is in form of Laurent series~~
~~so this is required Laurent series~~
~~of given function~~.

put $u = z$

$$f(z) = 2z + \frac{2}{3}z^3 + \cdots + \frac{1}{z} + \cdots$$

This is the required Laurent series of given function.

Question #8:

Determine location and type of the singularities of following;

i) cot z

$$f(z) = \cot z \rightarrow ①$$

For singularity,

$$f(z) = \infty$$

$$\cot z = \infty$$

$$\Rightarrow \frac{\cos z}{\sin z} = \infty$$

$$\Rightarrow \sin z = 0$$

$$\Rightarrow z = n\pi \quad \text{where } n = 0, \pm 1, \pm 2 + \cdots$$

These are the singular points of the given function,

for $n = 0$, $z = 0$

Now $\lim_{z \to 0} \frac{\cos z}{\cancel{\sin z}} * \cancel{\sin z} = \cos(0)$

$= 1 = \text{constant}$

$\therefore$ $z = 0$ is pole of order '1'

Now for $n = 1$, $z = \pi$

$\lim_{z \to \pi} \frac{\cos z}{\cancel{\sin z}} \times \cancel{\sin z} = \cos(\pi) = -1 = \text{const}$.

$z = 1$ is pole of order '1'

Similarly for $n = -1, \pm 2, ---$

$z = n\pi$ are poles of order '1'

Now for $n = \infty$

$\text{Limit}_{z \to \infty} \frac{\cos z}{\cancel{\sin z}} \times \cancel{\sin z} = \cos(\infty) = \infty$

$\therefore$ $z = \infty$ is essential singularity

) $1/(z+a)^4$

$$f(z) = \frac{1}{(z+a)^4} \rightarrow (1)$$

For singularity.

$(z+a)^4 = 0 \Rightarrow z = -a$

$z = -a$ is the singular point of function (1)

$$\lim_{z \to -a} \frac{1}{(z+a)^4} \times (z+a)^4 = 1 = \text{const}.$$

$z = -4$ is pole of order '4'

Similarly at $z = \infty$

$$\lim_{z \to \infty} \frac{1}{(z+a)^4} \times (z+a)^4 = 1 = \text{const}$$

$\Rightarrow$ $z = \infty$ is also pole of order '4'

iii) $z + 1/z$

$$f(z) = z + 1/z = \frac{z^2+1}{z} \rightarrow (1)$$

For singularity,

$$\frac{z^2+1}{z} = \infty \Rightarrow z = 0$$

$z = 0$ is singular point of function (1)

$$\lim_{z \to 0} \frac{z^2+1}{\cancel{z}} \times \cancel{z} = \lim_{z \to 0} (z^2+1)$$

$$\lim_{z \to 0} \frac{z^2+1}{z} = 1 = \text{const}.$$

$z = 1$ is pole of order '1'

Similarly at $z = \infty$

$$\lim_{z \to \infty} \frac{z^2+1}{\cancel{z}} \times \cancel{z} = \lim_{z \to \infty} z^2+1 = \infty$$

$= \text{const}$

$z = \infty$ is pole of order '1'.

Scanned with CamScanner