Chapter No: 14

Power Series, Taylor Series, Laurent Series

Sequence:- Arrangement of no: (complex no:) according to some rules is called a sequence

e.g (i) $2+i, 3+i, 4+i, \ldots$

(ii) $2, 4, 6, 8, \ldots$

Series:- The no: of a series written in addition form is called series.

e.g (i) $(2+i)+(3+i)+(4+i) \ldots$

(ii) $2+4+6+8+\ldots$

Types of Series:- There are two types of series.

(i) Convergent Series.

(ii) Divergent Series.

Convergent Series:- If the sum of a series is known i.e less than ∞. the series is called convergent series

Divergent Series:- If the sum of the series is not known i.e it is ∞, then then series is called Divergent Series.

Power Series :- The Power Series is given by

$$f(z) = a_0 + a_1 z + a_2 z^2 + a_3 z^3 + \cdots + a_n z^n \quad \text{--- (1)}$$

or

$$f(z) = a_0 + a_1(z-z_0) + a_2(z-z_0)^2 + a_3(z-z_0)^3 + \cdots + a_n(z-z_0)^n \quad \text{--- (2)}$$

where eq (1) is called Power Series of Z

eq (2) is called Power Series of $z-z_0$, &

$a_0, a_1, a_2, \cdots, a_n$ are constants.

Types of Power Series :-

There are three types of Power Series.

1. Taylor Series.
2. Maclaurin's Series
3. Laurents Series.

Taylor Series :- Let $f(z)$ be a function, let also $z=z_0$ is the domain of the function, so the taylor Series of the function $f(z)$ at $z=z_0$ is

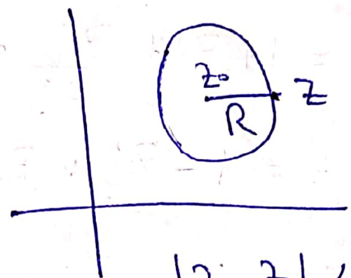
$$f(z) = f(z_0) + (z-z_0) f'(z_0) + \frac{(z-z_0)^2}{2!} f''(z_0) + \cdots$$

where $f(z_0), f'(z_0), f''(z_0)$ are constants which are to be found out.

<u>Region of convergence</u>:- For the region of convergence, the following condition should be satisfied $|z-z_0| < R$

where R = Radius of convergence

i.e $R = |z-z_0|$.



$|z-z_0| < 0$

$\Rightarrow$ convergent.

<u>Maclaurin's Series</u>:- For an analytic function $f(z)$ the taylor series at a point $z = z_0$ is:

$$f(z) = f(z_0) + (z-z_0)f'(z_0) + \frac{(z-z_0)^2}{2!}f''(z_0) + \frac{(z-z_0)^3}{3!}f'''(z_0) + \cdots\cdots \quad (1)$$

Put $z_0 = 0$

$$\text{eq}(1) \Rightarrow f(z) = f(0) + (z-0)f'(0) + \frac{(z-0)}{2!}f''(0) + \frac{(z-0)^3}{3!}f'''(0) + \cdots\cdots$$

$$f(z) = f(0) + zf'(0) + z^2\frac{f''(0)}{2!} + \frac{z^3 f'''(0)}{3!} + \cdots \quad (2)$$

eq ② is called maclaurin's Series. It is special tyles of taylor series.

Some usefull Maclaurin's Series

① $\frac{1}{1-z} = 1+z+z^2+z^3+\cdots$

② $e^z = 1+z+\frac{z^2}{2!}+\frac{z^3}{3!}+\cdots$

③ $\cos z = 1-\frac{z^2}{2!}+\frac{z^4}{4!}+\cdots$

④ $\sin z = z-\frac{z^3}{3!}+\frac{z^5}{5!}+\cdots$

⑤ $\cosh z = 1+\frac{z^2}{2!}+\frac{z^4}{4!}+\cdots$

⑥ $\sinh z = z+\frac{z^3}{3!}+\frac{z^5}{5!}+\cdots$

⑦ $\ln(1+z) = z-\frac{z^2}{2!}+\frac{z^3}{3!}+\cdots$

⑧ $\ln\left(\frac{1+z}{1-z}\right) = 2\left[z+\frac{z^3}{3!}+\frac{z^5}{5!}+\cdots\right]$

$$f'(z) = a_1 + 2a_2(z - z_0) + \cdots,$$

and more generally

$$f^{(n)}(z) = n!a_n + (n+1)n \cdots 3 \cdot 2a_{n+1}(z - z_0) + \cdots;$$

all these series converge in the disk $|z - z_0| < R$ and represent analytic functions. Hence these functions are continuous at $z = z_0$, by Theorem 1 in the last section. If we set $z = z_0$, we thus obtain

$$f(z_0) = a_0, \qquad f'(z_0) = a_1, \quad \cdots, \qquad f^{(n)}(z_0) = n!a_n, \qquad \cdots.$$

Since these formulas are identical with those in Taylor's theorem, the proof is complete. ∎

Comment. Comparison with real functions

One surprising property of complex analytic functions is that they have derivatives of all orders, and now we have discovered the other surprising property that they can always be represented by power series of the form (9). This is not true in general for ***real functions;*** there are real functions that have derivatives of all orders but cannot be represented by a power series. (Example: $f(x) = \exp(-1/x^2)$ if $x \neq 0$ and $f(0) = 0$; this function cannot be represented by a Maclaurin series since all its derivatives at 0 are zero.)

Problem Set 14.4

Find the Taylor series of the given function with the given point as center and determine the radius of convergence. (More problems of this kind follow in the next section, after the discussion of practical methods.)

1. e^{-z}, 0
2. e^{2z}, $2i$
3. $\sin \pi z$, 0
4. $\cos z$, $-\pi/2$
5. $\sin z$, $\pi/2$
6. $1/z$, 1
7. $1/(1 - z)$, -1
8. $1/(1 - z)$, i
9. $\mathrm{Ln}\, z$, 1
10. $\sinh (z - 2i)$, $2i$
11. z^5, -1
12. $z^4 - z^2 + 1$, 1
13. $\sin^2 z$, 0
14. $\cos^2 z$, 0
15. $\cos (z - \pi/2)$, $\pi/2$

Problems 16–26 illustrate how you can obtain properties of functions from their Maclaurin series.

16. Using (12), prove $(e^z)' = e^z$.
17. Derive (14) and (15) from (12). Obtain (16) from Taylor's theorem.
18. Using (14), show that $\cos z$ is even and $\sin z$ is odd.
19. Using (15), show that $\cosh z \neq 0$ for all real $z = x$.
20. Using (14), show that $\sin z \neq 0$ for all pure imaginary $z = iy \neq 0$.

Exercise 14.4

Find the taylor series of the given function with the given point as centre and determine the radius of convergence.

Q1 e^{-z}, 0

Sol:- let $f(z) = e^{-z}$ & $z_0 = 0$

As Taylor series of a function $f(z)$ at point $z = z_0$ is given by

$$f(z) = f(z_0) + (z - z_0) f'(z_0) + \frac{(z - z_0)^2}{2!} f''(z_0) + \text{---} \quad (1)$$

Now $f(z) = e^{-z}$

$f(z_0) = f(0) = e^0 = 1$

& $f'(z) = -e^{-z}$

$f'(z_0) = -e^0 = -1$

& $f''(z_0) = e^{-z}$, $f''(z_0) = e^0 = 1$

eq(1) $\Rightarrow$

$$f(z) = 1 + (z-0)(-1) + \frac{(z-0)^2}{2!}(1) + \text{---}$$

$$f(z) = 1 - z + z^2/2! - \frac{z^3}{3!} + \text{---}$$

Radius (99)

~~Region~~ of convergence

for the region of convergence

$R = |z - z_0| \longrightarrow (2)$

As $z_0 = 0$

As "z" is a singular Point of the given function

$\therefore f(z) = \infty$

$e^{-z} = \infty$

$\frac{1}{e^z} = \infty$

$e^z = 0$

$\log e^z = \log(0)$

$z = \infty$

eq (2) $\Rightarrow R = |\infty - 0|$

$\Rightarrow R = \infty$ which is the required radius of convergence.

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Q3 $f(z) = \sin \pi z$; $z_0 = 0$

Sol:- The Taylor Series of a function $f(z)$ is given by at $z = z_0$

$$f(z) = f(z_0) + (z-z_0)\frac{f'(z_0)}{1!} + (z-z_0)^2\frac{f''(z_0)}{2!} + \text{———} \quad (1)$$

Now $f(z) = \sin \pi z$ | $f(z_0) = f(0) = \sin \pi(0) = 0$

$f'(z) = \pi \cos \pi z$ | $f'(z_0) = f'(0) = \pi \cos \pi(0) = \pi$

$f''(z) = -\pi^2 \sin \pi z$ | $f''(z_0) = f''(0) = -\pi^2 \sin \pi(0) = 0$

eq(1) $\Rightarrow$

$$\sin \pi z = 0 + (z-0)(\pi) + (z-0)^2 \frac{(0)}{2!} + (z-0)^3 \frac{(-\pi^3)}{3!} + \text{———}$$

$$\sin \pi z = \pi z - \frac{1}{3!}\pi^3 z^3 + \text{———}$$

Radius of convergence

$$R = |z - z_0| \text{———} (2)$$

Now As, z is S. Point of the given function

$\therefore f(z) = \infty$

$\sin \pi z = \infty$

$\pi z = \infty$

$z = \infty$

$\therefore R = |\infty - 0| \Rightarrow \boxed{R = \infty}$

Q2 e^{2z}, $2i$

Q4 $\cos z$, $-\pi/2$

Q5 $\sin z$, $\pi/2$

Q6 $1/z$, 1

Q7 $1/(1-z)$, -1

Q8 $\frac{1}{1-z}$, i

Q9 z^5, -1

EXAMPLE 5 **Use of differential equations**

Find the Maclaurin series of $f(z) = \tan z$.

Solution. We have $f'(z) = \sec^2 z$ and, therefore, since $f(0) = 0$,

$$f'(z) = 1 + f^2(z), \qquad f'(0) = 1.$$

Observing that $f(0) = 0$, we obtain by successive differentiation

$$
\begin{aligned}
f'' &= 2ff', & f''(0) &= 0,\\
f''' &= 2f'^2 + 2ff'', & f'''(0) &= 2, & f'''(0)/3! &= 1/3,\\
f^{(4)} &= 6f'f'' + 2ff''', & f^{(4)}(0) &= 0,\\
f^{(5)} &= 6f''^2 + 8f'f''' + 2ff^{(4)}, & f^{(5)}(0) &= 16, & f^{(5)}(0)/5! &= 2/15,
\end{aligned}
$$

Hence the result is

$$\tan z = z + \frac{1}{3}z^2 + \frac{2}{15}z^5 + \frac{17}{315}z^7 + \cdots \qquad (3)$$

$\left(|z| < \frac{\pi}{2}\right)$.

EXAMPLE 6 **Undetermined coefficients**

Find the Maclaurin series of $\tan z$ by using those of $\cos z$ and $\sin z$ (Sec. 14.4).

Solution. Since $\tan z$ is odd, the desired expansion will be of the form

$$\tan z = a_1 z + a_3 z^3 + a_5 z^5 + \cdots.$$

Using $\sin z = \tan z \cos z$ and inserting those developments, we obtain

$$z - \frac{z^3}{3!} + \frac{z^5}{5!} - + \cdots = (a_1 z + a_3 z^3 + a_5 z^5 + \cdots)\left(1 - \frac{z^2}{2!} + \frac{z^4}{4!} - + \cdots\right).$$

Since $\tan z$ is analytic except at $z = \pm\pi/2, \pm 3\pi/2, \cdots$, its Maclaurin series converges in the disk $|z| < \pi/2$, and for these z we may form the Cauchy product of the two series on the right (see Sec. 14.3), that is, multiply the series term by term and arrange the resulting series in powers of z. By Theorem 2 in Sec. 14.3 the coefficient of each power of z is the same on both sides. This yields

$$1 = a_1, \qquad -\frac{1}{3!} = -\frac{a_1}{2!} + a_3, \qquad \frac{1}{5!} = \frac{a_1}{4!} - \frac{a_3}{2!} + a_5, \qquad \text{etc.}$$

Hence $a_1 = 1$, $a_3 = \frac{1}{3}$, $a_5 = \frac{2}{15}$, etc., as before.

Problem Set 14.5

Find the Maclaurin series of the following functions and determine the radius of convergence.

1. $\dfrac{1}{1 + z^4}$
2. $\dfrac{1}{1 - z^5}$
3. $\dfrac{z + 2}{1 - z^2}$
4. $\dfrac{4 - 3z}{(1 - z)^2}$
5. $\sin 2z^2$
6. $\dfrac{1}{(z + 3 - 4i)^2}$
7. $\dfrac{e^{z^4} - 1}{z^3}$
8. $e^{z^2}\displaystyle\int_0^z e^{-t^2}\,dt$
9. $\dfrac{2z^2 + 15z + 34}{(z + 4)^2(z - 2)}$

(102)

Ex 14.5

Find the Maclaurin series of the following functions and determine the radius of convergence.

Q1 $\frac{1}{1+z^4}$

Sol:- let $f(z) = \frac{1}{1+z^4}$ —①

using substitution method as we know that

$$\frac{1}{1-z} = 1+z+z^2+z^3+\text{——} \quad ②$$

comparing eq② with the given function

$$-z = z^4 \Rightarrow z = -z^4 \quad \text{——} ③$$

eq② $\Rightarrow$

$$\frac{1}{1-(-z^4)} = 1+(-z^4)+(-z^4)^2+\text{————}$$

$$\boxed{\frac{1}{1+z^4} = 1-z^4+z^8-z^{12}+\text{------}}$$

Radius of convergence

as $R = |z - z_0|$ ⟶ 3

By Maclaurin Series $z_0 = 0$ & "z" is a Singular Point of $f(z)$ then $f(z) = \infty$.

$$\frac{1}{1+z^4} = \infty$$

$$1 + z^4 = 0$$

$$z^4 = -1 \Rightarrow \boxed{z^2 = \pm i}$$

Q3 $f(z) = \dfrac{z+2}{1-z^2}$.

Sol:- $f(z) = (z+2) \cdot \dfrac{1}{1-z^2}$ —— ①

using substitution method as we know that

$$\frac{1}{1-z} = 1 + z + z^2 + z^3 + \cdots \text{ —— } ②$$

Comparing eq ② with $\dfrac{1}{1-z^2} \Rightarrow z = z^2$

eq ② $\Rightarrow \dfrac{1}{1-z^2} = 1+z^2+z^4+z^6+\cdots\cdots$ ③

eq ① $\Rightarrow f(z) = (z+2)\left[1+z^2+z^4+z^6+\cdots\cdots\right]$

$= (z+z^3+z^5+z^7+\cdots) + (2+2z^2+2z^4+\cdots\cdots)$

$$\boxed{f(z) = 2+z+2z^2+z^3+2z^4+\cdots\cdots}$$

Radius of convergence

As $R = |z - z_0|$ $\longrightarrow$ ④

for Maclaurin Series $z_0 = 0$ & As z is a singular point of $f(z)$

$\therefore f(z) = \infty$

$\dfrac{z+2}{1-z^2} = \infty \Rightarrow \dfrac{z+2}{\infty} = 1-z^2$

$\Rightarrow 1-z^2 = 0 \Rightarrow z^2 = 1 \Rightarrow \boxed{z = \pm 1}$

eq ④ $\Rightarrow R = |\pm 1 - 0| \Rightarrow \boxed{R = 1}$

Q_2 $\frac{1}{1-z^5}$

Q_4 $\frac{4-3z}{(1-z)^2}$

Q_5 $\sin 2z^2$

Q_6 $\frac{1}{(z+3-4i)^2}$