EXAMPLE 4 How to find a conjugate harmonic function by the Cauchy–Riemann equations

Verify that $u = x^2 - y^2 - y$ is harmonic in the whole complex plane and find a conjugate harmonic function v of u.

Solution. $\nabla^2 u = 0$ by direct calculation. Now $u_x = 2x$ and $u_y = -2y - 1$. Hence because of the Cauchy–Riemann equations a conjugate v of u must satisfy

$$v_y = u_x = 2x, \qquad v_x = -u_y = 2y + 1.$$

Integrating the first equation with respect to y and differentiating the result with respect to x, we obtain

$$v = 2xy + h(x), \qquad v_x = 2y + \frac{dh}{dx}.$$

A comparison with the second equation shows that $dh/dx = 1$. This gives $h(x) = x + c$. Hence $v = 2xy + x + c$ (c any real constant) is the most general conjugate harmonic of the given u. The corresponding analytic function is

$$f(z) = u + iv = x^2 - y^2 - y + i(2xy + x + c) = z^2 + iz + ic.$$ ◀

Example 4 illustrates that *a conjugate of a given harmonic function is uniquely determined up to an arbitrary real additive constant.*

The curves $u = const$ are called **equipotential lines** or **level curves of u.** They form a **family** of curves. Similarly for v. The two families together form an **orthogonal net.** "Orthogonal" means perpendicular. These are standard terms, which we shall use.

The Cauchy–Riemann equations are the most important equations in this chapter. Their relation to Laplace's equation opens wide ranges of engineering and physical applications, as we shall show in Chap. 16.

PROBLEM SET 12.4

Analytic Functions. Cauchy–Riemann Equations

Check for analyticity by using (1) or (7). (Show the details of your work.)

1. $f(z) = z^6$
2. $f(z) = i|z|^3$
3. $f(z) = e^x(\cos y + i \sin y)$
4. $f(z) = i/z^5$
5. $f(z) = z\bar{z}$
6. $f(z) = z + 1/z$
7. $f(z) = \ln |z| + i \operatorname{Arg} z$
8. $f(z) = 1/(1 - z^4)$
9. $f(z) = \operatorname{Re} z/\operatorname{Im} z$
10. $f(z) = \operatorname{Arg} z$
11. $f(z) = \operatorname{Re}(z^3)$
12. $f(z) = \operatorname{Re}(z^2) - i \operatorname{Im}(z^2)$

13. **(Cauchy–Riemann equations)** Derive (7) from (1).

14. **TEAM PROJECT. Conditions for $f(z) = const.$** Let $f(z)$ be analytic. Prove that each of the following conditions is sufficient for $f(z) = const.$

    (a) $\operatorname{Re} f(z) = const$ **(b)** $\operatorname{Im} f(z) = const$ **(c)** $f'(z) = 0$

    (d) $|f(z)| = const$ (see Example 3)

15. **(Formulas for the derivative)** Show that, in addition to (4) and (5),

$$\text{(11)} \qquad f'(z) = u_x - iu_y, \qquad f'(z) = v_y + iv_x.$$

16. **Formulas (4), (5), and (11)** are needed from time to time. Familiarize yourself with them by calculating $(z^3)'$ and verifying that the result is as expected.

Analytic function :-

let $w = u + iv = f(z)$ be a complex function with domain "D" This function is said to be analytic if:-

① $f(z)$ is defined at z_0.

② $f'(z)$ is defined at z_0, where z_0 is a point in domain of $f(z)$.

Cauchy - Riemann's Equation

Consider a complex function given by

$$w = f(z) = u(x,y) + iv(x,y)$$

We say that 'f' is analytic if and only if it satisfies the relation'

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \;\&\; \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \text{ or } u_x = v_y \;\&\; u_y = -v_x$$

known as Cauchy - Riemann's Equation.

Cauchy-Riemann's Equation in Polar form

If

$f(z) = u(r,\theta) + i\,v(r,\theta)$

Then Cauchy Riemann's Equation are

$$\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta}$$

& $\dfrac{\partial v}{\partial r} = -\dfrac{1}{r}\dfrac{\partial u}{\partial \theta}$ when $r > 0$.

Laplace Equation:

$f(z) = u(x,y) + i\,v(x,y)$

is analytic in domain D. Then u & v satisfy the Equations

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \text{ or } u_{xx} + u_{yy} = 0 \text{ or } \nabla^2 u = 0.$$

&

$$\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0 \text{ or } v_{xx} + v_{yy} = 0 \text{ or } \nabla^2 v = 0.$$

These are called Laplace Equations.

Hormonic Function:-

Any function "u" which satisfies Laplace Equation is called hormonic function. i.e

$\nabla^2 U = 0$ or $U_{xx} + U_{yy} = 0$

Conjugate Harmonic function

let $f(z) = u + iv$

If "u" & "v" both satisfy CRE we say that V is the conjugate harmonic of "u"

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Exercise 12.4

Are the following function analytic

$Q_1 \xrightarrow{\text{to}} Q_{12}$

Q_1 $f(z) = z^8$

Sol:- let $z = r\cos\theta + ir\sin\theta$

$z = r[\cos\theta + i\sin\theta]$

Then $z^8 = r^8[\cos\theta + i\sin\theta]^8$

(41)

$Z = r^8 [\cos 8\theta + i \sin 8\theta]$ (D.T)

i.e

$f(z) = z^8 = r^8 \cos 8\theta + i r^8 \sin 8\theta$

$u + iv$

where

$u = r^8 \cos 8\theta \; ; \; v = r^8 \sin 8\theta$

Since a function is analytic if it satisfies the CRE

i.e iff

$$\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta} \; \& \; \frac{\partial v}{\partial r} = -\frac{1}{r}\frac{\partial u}{\partial \theta}$$

Now

$$\frac{\partial u}{\partial r} = \frac{\partial}{\partial r}[r^8 \cos 8\theta] = 8r^7 \cos 8\theta$$

Also $$\frac{\partial v}{\partial r} = \frac{\partial}{\partial r}[r^8 \sin 8\theta] = 8r^7 \sin 8\theta$$

$$\frac{\partial u}{\partial \theta} = \frac{\partial}{\partial \theta}[r^8 \cos 8\theta] = -8r^8 \sin 8\theta$$

$$\frac{\partial v}{\partial \theta} = \frac{\partial}{\partial \theta}[r^8 \sin 8\theta] = 8r^8 \cos 8\theta$$

(i) $$\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta}$$

$$8r^7 \cos 8\theta = \frac{1}{r}[8r^8 \cos 8\theta]$$

$$\boxed{8r^7 \cos 8\theta = 8r^7 \cos 8\theta}$$ Satisfied

(42)

(ii) $\frac{\partial V}{\partial r} = -\frac{1}{r}\frac{\partial U}{\partial \theta}$

$8r^7 \sin 8\theta = -\frac{1}{r}[-8r^8 \sin 8\theta]$

$\boxed{8r^7 \sin 8\theta = 8r^7 \sin 8\theta}$ Satisfied.

Hence $f(z) = z^8$ satisfied CRE so it's analytic.

Q3 $f(z) = e^x[\cos y + i \sin y]$.

Sol:- $f(z) = e^x[\cos y + i \sin y]$

Here $U = e^x \cos y$ & $V = e^x \sin y$.

Now $U_x = e^x \cos y \rightarrow A$ | $V_x = e^x \sin y \rightarrow C$

$U_y = -e^x \sin y \rightarrow B$ | $V_y = e^x \cos y \rightarrow D$

~~comparing (A) & ()~~

A & D comparing we get

$\boxed{U_x = V_y}$.

C & B comparing we get

$\boxed{U_y = -V_x}$ Hence CRE are satisfied. The given function is analytic.

(43)

Q5 $f(z) = i/z^4$

let $z = r[\cos\theta + i\sin\theta]$

$$f(z) = \frac{i}{r^4[\cos\theta + i\sin\theta]^4}$$

$$= \frac{i}{r^4[\cos 4\theta + i\sin 4\theta]} \quad \text{D.T.}$$

$$= \frac{i}{r^4[\cos 4\theta + i\sin 4\theta]} \times \frac{\cos 4\theta - i\sin 4\theta}{\cos 4\theta - i\sin 4\theta}.$$

$$= \frac{i[\cos 4\theta - i\sin 4\theta]}{r^4[\cos^2 4\theta - i^2\sin^2 4\theta]}.$$

$$= \frac{\sin 4\theta + i\cos 4\theta}{r^4(1)}.$$

$$f(z) = \frac{\sin 4\theta + i\cos 4\theta}{r^4}. = \frac{\sin 4\theta + i\cos 4\theta}{r^4}$$

Here

$$u = \frac{\sin 4\theta}{r^4} \quad ; \quad v = \frac{\cos 4\theta}{r^4}$$

(44)

$$U_r = -\frac{4\sin 4\theta}{r^5} \rightarrow (A)$$

$$U_\theta = \frac{4\cos 4\theta}{r^4} \rightarrow (B)$$

$$V_r = -\frac{4\cos 4\theta}{r^5} \rightarrow (C)$$

$$V_\theta = -\frac{4\sin 4\theta}{r^4} \rightarrow (D)$$

comparing eq (A) & eq (D) we get

$$\boxed{\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta}}$$

comparing eq (C) & eq (B) we get

$$\boxed{\frac{\partial u}{\partial \theta} = -r\frac{\partial v}{\partial r}}$$

Hence CRE are satisfied. The given function is analytic.

Q6 $f(z) = z + 1/z$

sol:- let $z = x+iy$

$$f(z) = z + 1/z$$

$$= x+iy + \frac{1}{x+iy}$$

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$$= \frac{(x+iy)^2+1}{x+iy}$$

$$= \frac{x^2-y^2+1+2ixy}{x+iy}$$

$$= \frac{(x^2-y^2+1)+i2xy}{x+iy} \times \frac{x-iy}{x-iy}$$

$$= \frac{(x^3-xy^2+x+2xy^2)+i(2x^2y-x^2y+y^3-y)}{x^2+y^2}$$

$$= \frac{x^3+x+xy^2}{x^2+y^2} + i\,\frac{x^2y+y^3-y}{x^2+y^2}$$

Here $u = \dfrac{x^3+x+xy^2}{x^2+y^2}$; $v = \dfrac{x^2y+y^3-y}{x^2+y^2}$.

$$u_x = \frac{(x^2+y^2)(3x^2+1+y^2)-(x^3+x+xy^2)(2x)}{(x^2+y^2)^2}$$

$$u_x = \frac{x^4+2x^2y^2-x^2+y^4+y^2}{(x^2+y^2)^2} \longrightarrow (1)$$

$$v_y = \frac{(x^2+y^2)(x^2+3y^2-1)-(x^2y+y^3-y)(2y)}{(x^2+y^2)^2}$$

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$$V_y = \frac{x^4+2x^2y^2-x^2+y^4+y^2}{(x^2+y^2)^2} \longrightarrow (2)$$

from eq (1) & eq (2) we have

$$\boxed{U_x = V_y}$$

Also

$$U_y = \frac{(x^2+y^2)(2xy)-(x^3+x+xy^2)(2y)}{(x^2+y^2)^2}$$

$$U_y = -2xy/(x^2+y^2)^2 \longrightarrow (3)$$

& $$V_x = \frac{(x^2+y^2)(2xy)-(x^2y+y^3-y)(2x)}{(x^2+y^2)^2}$$

$$V_x = 2xy/(x^2+y^2)^2.$$

$$-V_x = -2xy/(x^2+y^2)^2 \longrightarrow (4)$$

from eq (3) & eq (4) we have

$\boxed{U_y = -V_x}$ Hence satisfied CRE. Also the given function is analytic.

Determine wether the following functions are harmonic If your Answer is Yes, find a corresponding analytic function $f(z) = u(x,y) + i v(x,y)$.

Q 17 — to — Q 24

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Q13 $u = xy$

"u" will be hormonic iff it satisfies the L. equation

$$\frac{\partial u}{\partial x} = y \Rightarrow \frac{\partial^2 u}{\partial x^2} = 0 \longrightarrow A$$

Similarly

$$\frac{\partial u}{\partial y} = x \Rightarrow \frac{\partial^2 u}{\partial y^2} = 0 \longrightarrow B$$

from eq A & eq B we have

$$\boxed{\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0}$$

Hence the given function "u" is hormonic.

let v be a complex conjugate of u

The CRE are

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \longrightarrow (1) \quad \& \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \longrightarrow (2)$$

Since $u = xy$

$$\frac{\partial u}{\partial x} = y$$

$$eq(1) \Rightarrow \frac{\partial v}{\partial y} = y \Rightarrow \partial v = y\,\partial y$$

Integration B.S W.r.t "y"

$$v = \frac{y^2}{2} + k(x) \longrightarrow (3)$$

Now P. diff W.r.t "x"

$$\frac{\partial v}{\partial x} = k'(x) \longrightarrow (4)$$

Since $u = xy$

$$\frac{\partial u}{\partial y} = x$$

$$eq(2) \Rightarrow x = -\frac{\partial v}{\partial x} \longrightarrow (5)$$

$$\frac{\partial v}{\partial x} = -x$$

Comparing eq(4) & eq(5) we get

$$k'(x) = -x$$

Integration B.S W.r.t x, we get

$K(x) = -x^2/2 + c$

eq (8) $\Rightarrow$ $v = y^2/2 - x^2/2 + c$

Since $f(z) = u + iv$

$= xy + i\left(-x^2/2 + y^2/2 + c\right)$

$= -i^2xy - i\left(x^2/2 - y^2/2\right) + ic$

$= -i/2\left[x^2 + i^2y^2 + 2ixy\right] + c'$ $\quad \because ic = c'$

$= -i/2\left[x + iy\right]^2 + c'$

$$\boxed{f(z) = \frac{-iZ^2}{2} + c}$$

Q15 $u = x/x^2+y^2$

$$U_x = \frac{x^2+y^2-2x^2}{(x^2+y^2)^2} = \frac{y^2-x^2}{(x^2+y^2)^2}$$

$$U_{xx} = \frac{(x^4+y^4+2x^2y^2)(-2x) - (y^2-x^2)\,2(x^2+y^2)(2x)}{(x^2+y^2)^2}$$

$$U_{xx} = \frac{2x^5 - 6xy^4 - 4x^3y^2}{(x^2+y^2)^4} \rightarrow (A)$$

Similarly (50)

$$U_y = \frac{(-1)(x)(2y)}{(x^2+y^2)^2} = -\frac{2xy}{(x^2+y^2)^2}$$

$$U_{yy} = \frac{(x^4+y^4+2x^2y^2)(-2x)+2xy(2)(2y)(x^2+y^2)}{(x^2+y^2)^4}$$

$$U_{yy} = -\frac{(2x^5-6xy^4-4x^3y^2)}{(x^2+y^2)^2} \longrightarrow (B)$$

Adding eq (A) & eq (B)

$$\boxed{U_{xx}+U_{yy}=0}$$

∴ The function is harmonic

Now CRE's are

$$U_x = V_y \longrightarrow (1)$$

$$U_y = -V_x = \frac{-2xy}{(x^2+y)^2} \longrightarrow (2)$$

$$eq(1) \Rightarrow V_y = \frac{y^2-x^2}{(x^2-y^2)^2} \longrightarrow (3)$$

$$eq(2) \Rightarrow V_x = \frac{2xy}{(x^2+y^2)^2}$$

integration B.S w.r.t "x"

(51)

$$\int V_x \, \partial x = y \int (2x)(x^2+y^2)^{-2} \, \partial x$$

$$= y \frac{(x^2+y^2)^{-2+1}}{-2+1} + k(y)$$

$$V = -\frac{y}{x^2+y^2} + k(y) \longrightarrow (4)$$

P. diff w.r.t "y"

$$\frac{\partial v}{\partial y} = \frac{-x^2-y^2+2y^2}{(x^2+y^2)^2} + k'(y)$$

$$= y^2-x^2/(x^2+y^2)^2 + k'(y) \longrightarrow (5)$$

comparing eq (3) & eq (5)

$$k'(y) + y^2-x^2/\cancel{(x^2+y^2)^2} = \frac{y^2-x^2}{\cancel{(x^2+y^2)^2}}$$

$$k'(y) = 0$$

Integration w.r.t "y"

$$k(y) = c'$$

$$eq (4) \Rightarrow V = -y/x^2+y^2 + c'$$

As $f(z) = u + iv = \frac{x}{x^2+y^2} + i\left(\frac{-y}{x^2+y^2}\right) + ic'$

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$$f(z) = \frac{x-iy}{(x^2-i^2y^2)} + c$$

$$= \frac{\cancel{(x-iy)}}{(x+iy)\cancel{(x-iy)}} + c$$

$$= \frac{1}{x+iy} + c$$

$$\boxed{f(z) = 1/z + c}$$

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<u>Q22</u> $U = \sin x \cosh y$

<u>Sol</u>:- $U_x = \cos x \cosh y$

$U_{xx} = -\sin x \cosh y \longrightarrow A$

$U_y = \sin x \sinh y$

$U_{yy} = \sin x \cosh y \longrightarrow B$

A + B

$$\boxed{U_{xx} + U_{yy} = 0}$$

Hence the function is harmonic Now CRE's are

$U_x = V_y = \cos x \cosh y \longrightarrow 1$

$-U_y = V_x = -\sin x \sinh y \rightarrow (2)$

Now

$V_y = \cos x \cosh y$

Integration Both sides w.r.t 'y'

$V = +\cos x \sinh y + k(x) \rightarrow (3)$

P. Diff w.r.t 'x'

$V_x = -\sin x \sinh y + k'(x) \rightarrow (4)$

comparing eq (2) & eq (4) we get

$\cancel{-\sin x \sinh y} + k'(x) = \cancel{-\sin x \sinh y}$

$k'(x) = 0$

Integration w.r.t 'x'

$k(x) = c'$

eq (3) => $V = \cos x \sinh y + c'$

Sm $f(z) = U + iV = \sin x \cosh y + i \cos x \sinh y + ic'$

$f(z) = \sin(x+iy) + c \quad \because ic = c$

$$\boxed{f(z) = \sin z + c}$$