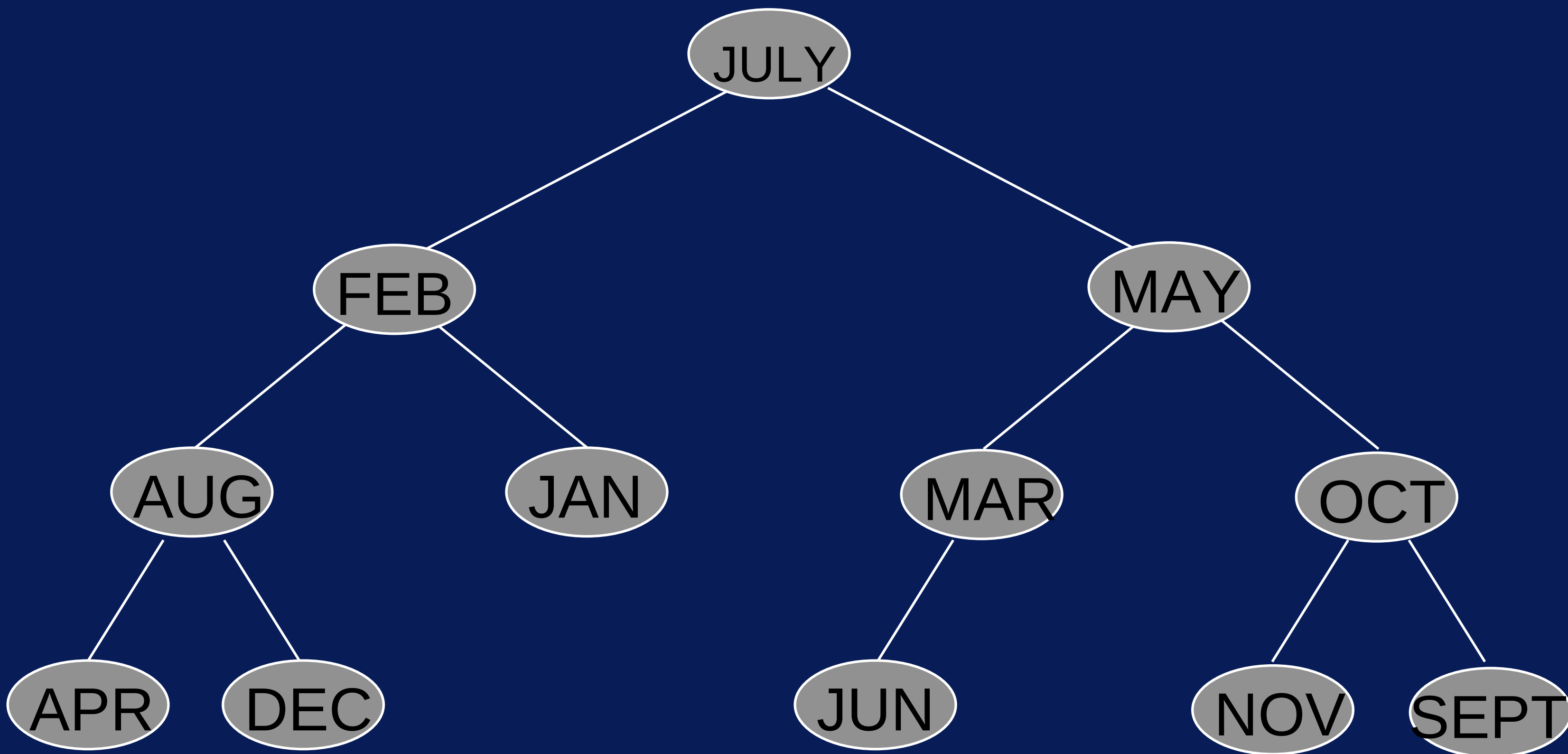
Height-Balanced Trees

AVL Trees

AVL Trees

- We know from our study of Binary Search Trees (BST) that the average search and insertion time is O(log n)
  - If there are n nodes in the binary tree it will take, on average, log2 n comparisons/probes to find a particular node (or find out that it isn't there)
- However, this is only true if the tree is 'balanced'
  - Such as occurs when the elements are inserted in random order

AVL Trees



A Balanced Tree for the Months of the Year

AVL Trees

- However, if the elements are inserted in lexicographic order (i.e. in sorted order) then the tree degenerates into a skinny tree

AVL Trees

A Degenerate Tree for the Months of the Year

AVL Trees

- If we are dealing with a dynamic tree
- Nodes are being inserted and deleted over time
  – For example, directory of files
  – For example, index of university students
- we may need to restructure - balance - the tree so that we keep it
  – Fat
  – Full
  – Complete

AVL Trees

- Adelson-Velskii and Landis in 1962 introduced a binary tree structure that is balanced with respect to the heights of its subtrees
- Insertions (and deletions) are made such that the tree
  - starts off
  - and remains
- Height-Balanced

AVL Trees

- Definition of AVL Tree
- An empty tree is height-balanced
- If T is a non-empty binary tree with left and right sub-trees T_1 and T_2, then
- T is height-balanced iff
  - T_1 and T_2 are height-balanced, and
  - $|height(T_1) - height(T_2)| \leq 1$

AVL Trees

- So, every sub-tree in a height-balanced tree is also height-balanced

Recall: Binary Tree Terminology

- The height of T is defined recursively as

  0 if T is empty and

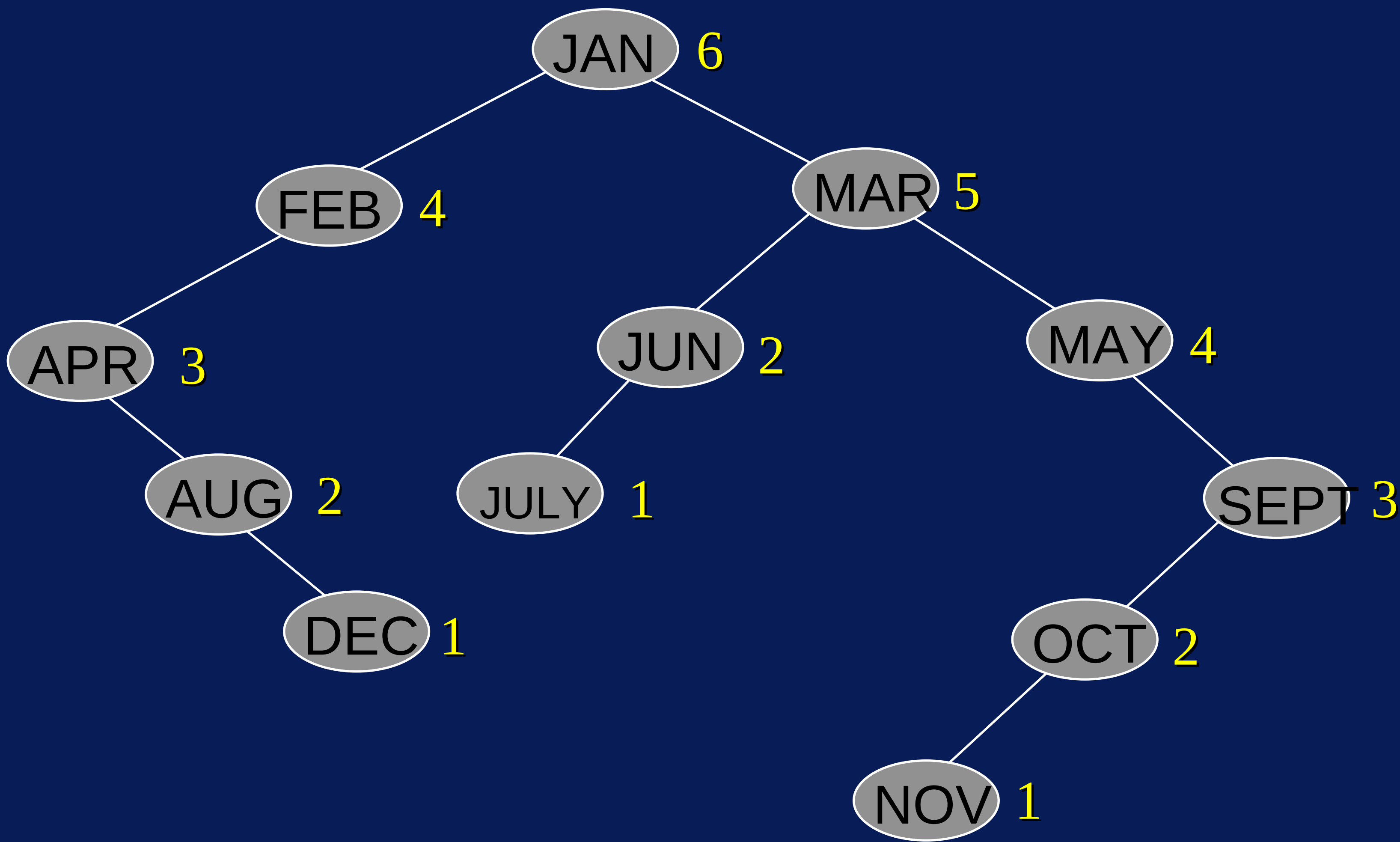
  $1 + max(height(T_1), height(T_2))$ otherwise, where T_1 and T_2 are the subtrees of the root.

- The height of a tree is the length of a longest chain of descendents

Recall: Binary Tree Terminology

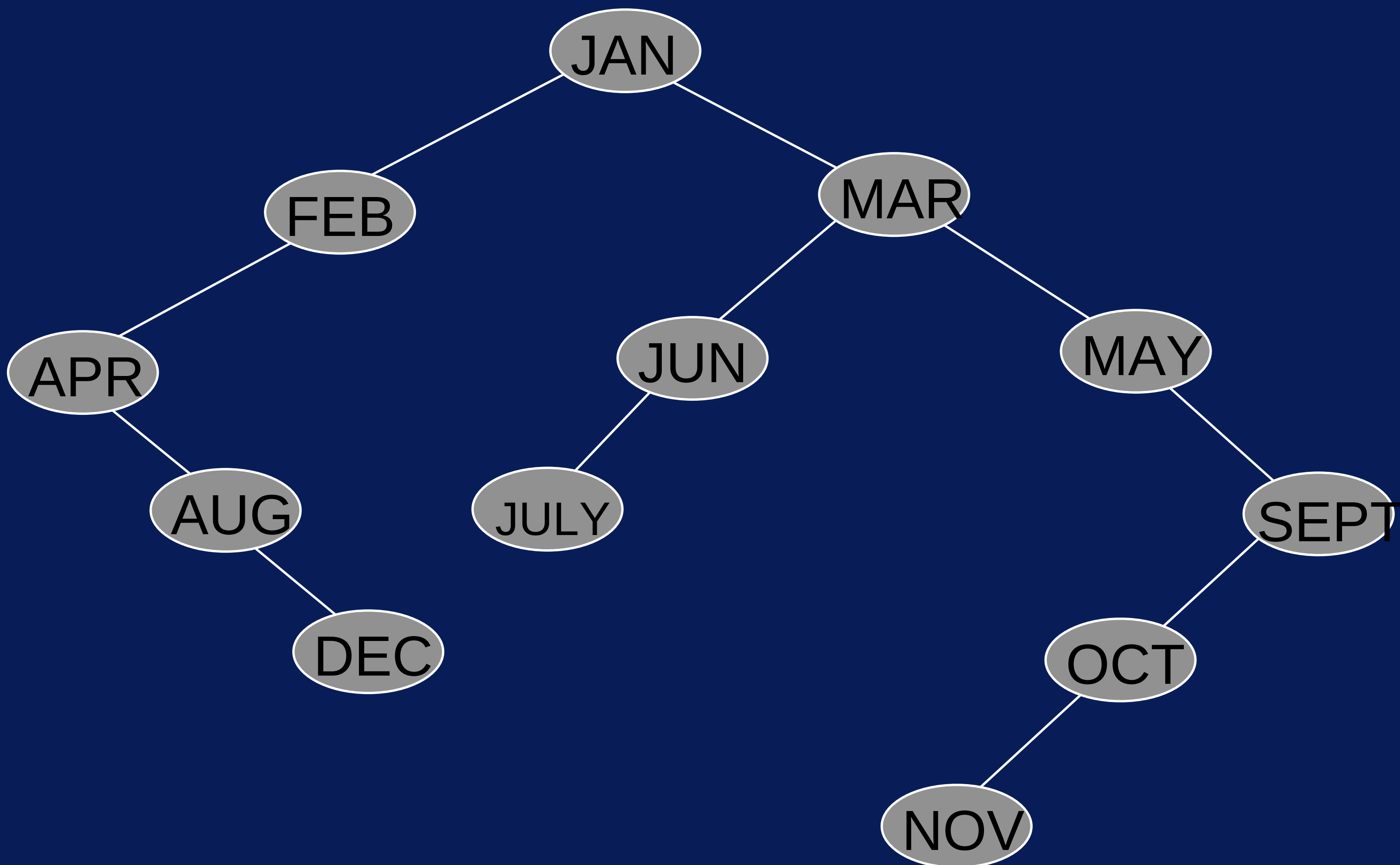
- Height Numbering
  - Number all external nodes 0
  - Number each internal node to be one more than the maximum of the numbers of its children
  - Then the number of the root is the height of *T*
- The height of a node *u* in *T* is the height of the subtree rooted at *u*

AVL Trees

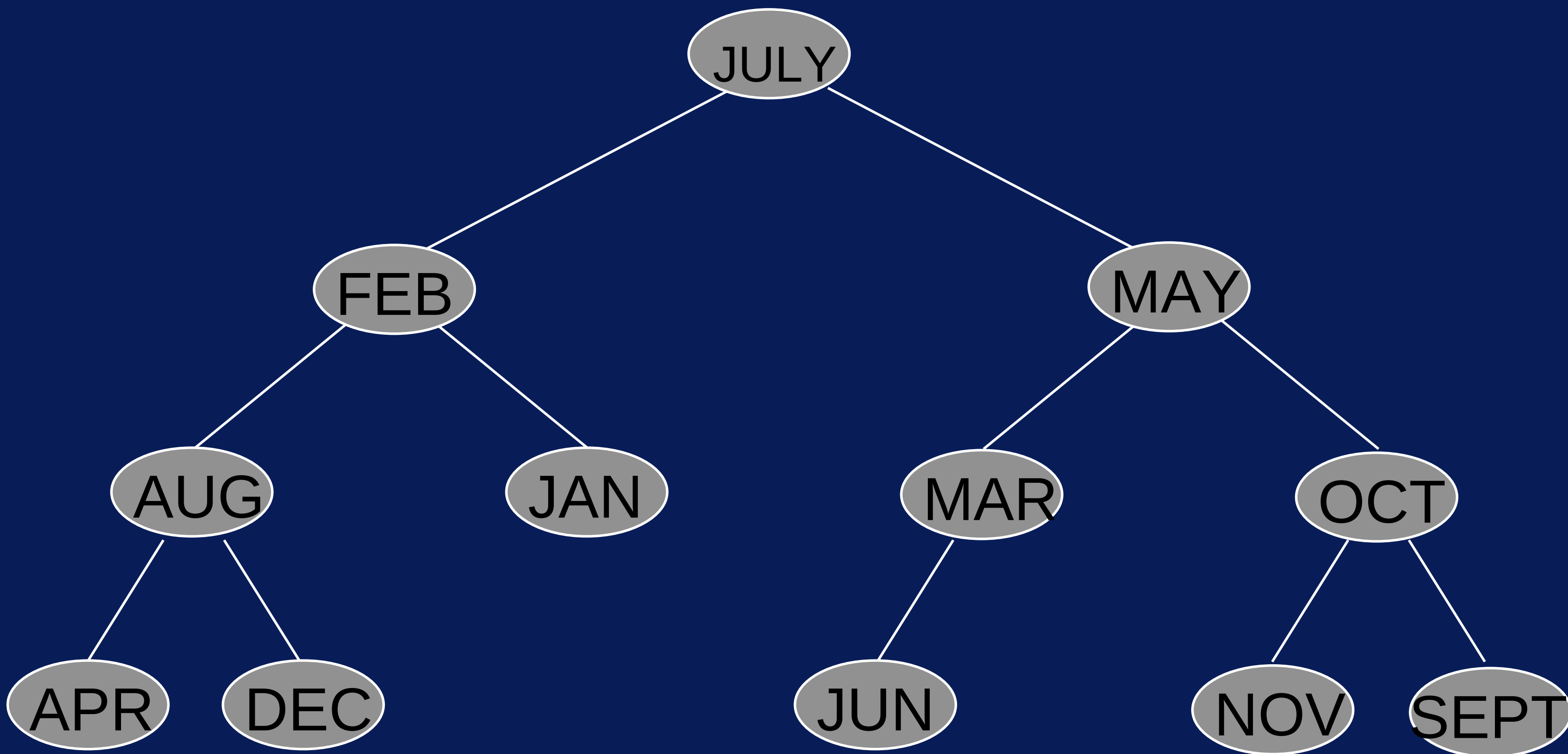


JAN 6
FEB 4
MAR 5
APR 3
JUN 2
MAY 4
AUG 2
JULY 1
SEPT 3
DEC 1
OCT 2
NOV 1

AVL Trees

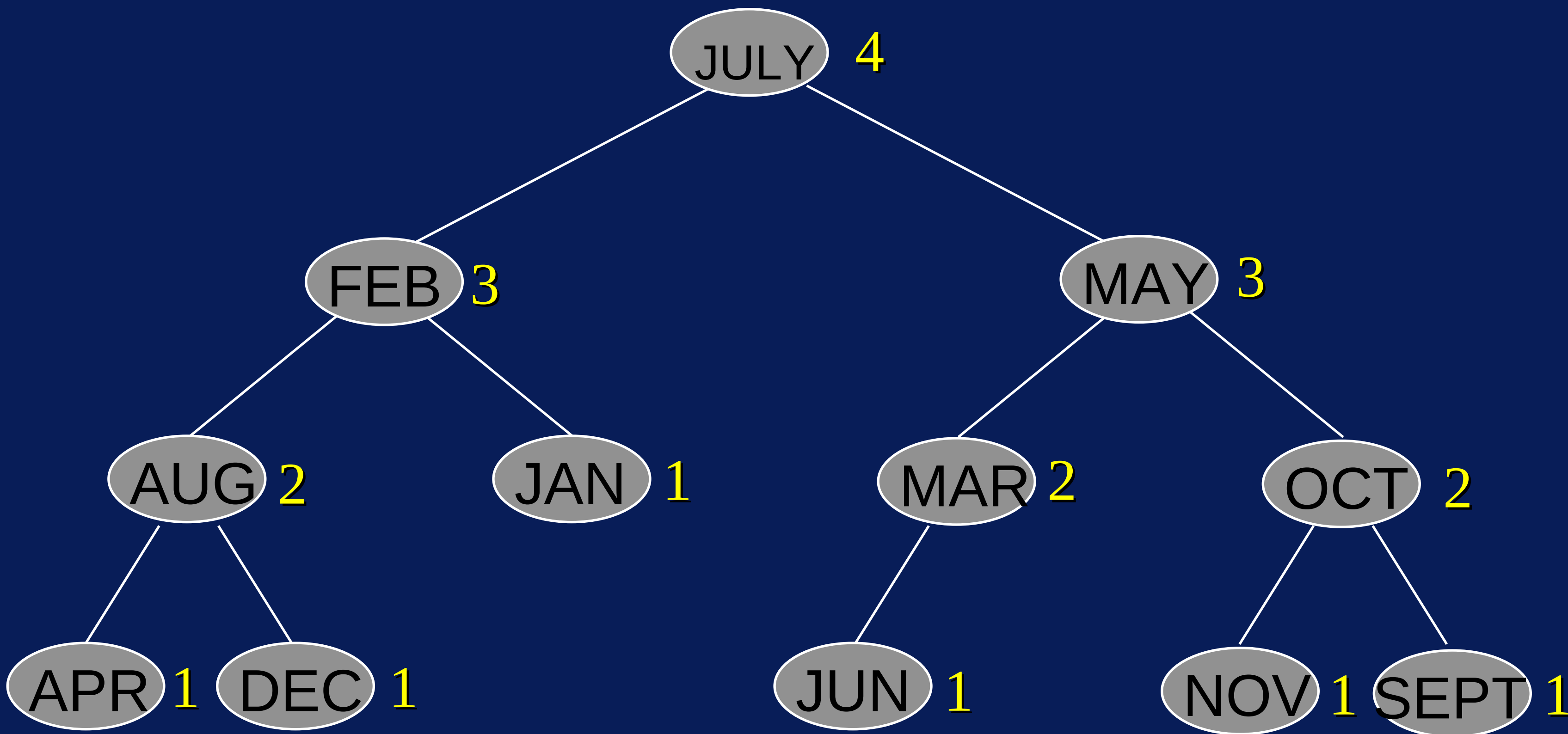


AVL Trees



A Balanced Tree for the Months of the Year

AVL Trees



A Balanced Tree for the Months of the Year

AVL Trees

- Let’s construct a height-balanced tree
- Order of insertions:

  March, May, November, August, April, January, December, July, February, June, October, September

- Before we do, we need a definition of a balance factor

AVL Trees

- Balance Factor *BF*(*T*) of a note T in a binary tree is defined to be

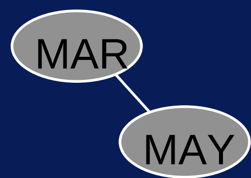
  $height(T_1) - height(T_2)$

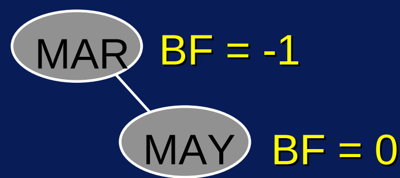
   where T_1 and T_2 are the left and right subtrees of T

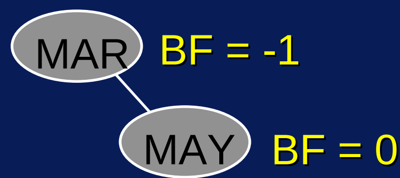
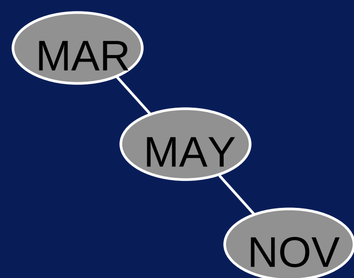
- For any node T in an AVL tree
  $BF(T) = -1, 0, +1$

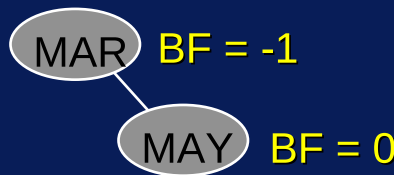
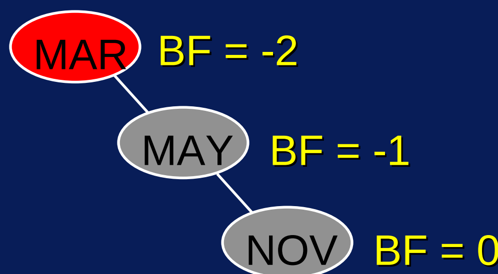
New Identifier	After Insertion	After Rebalancing
MARCH		

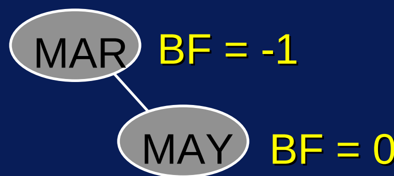
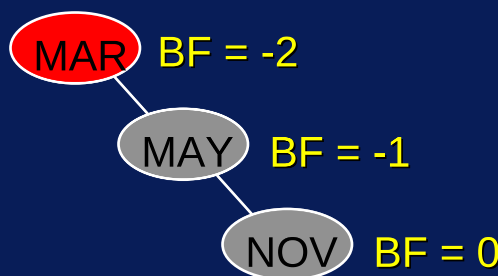
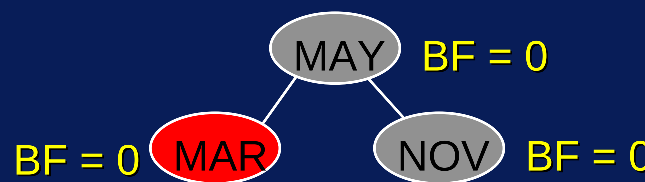
New Identifier	After Insertion	After Rebalancing
MARCH	MAR BF = 0	NO REBALANCING NEEDED

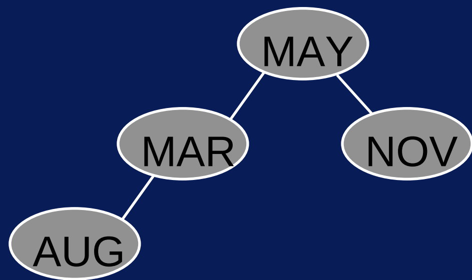
New Identifier	After Insertion	After Rebalancing
MARCH	 BF = 0	NO REBALANCING NEEDED
MAY		

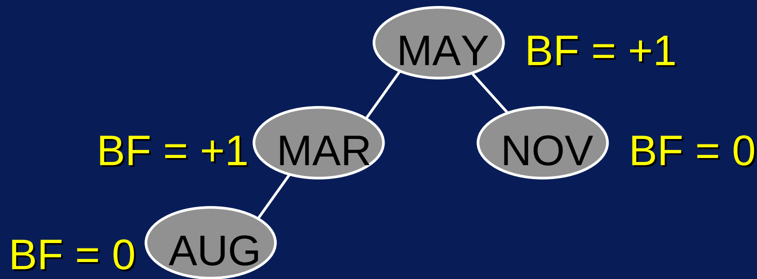
New Identifier	After Insertion	After Rebalancing
MARCH		NO REBALANCING NEEDED
MAY		NO REBALANCING NEEDED

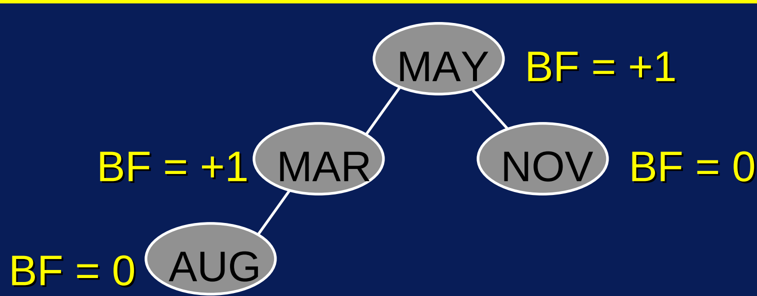
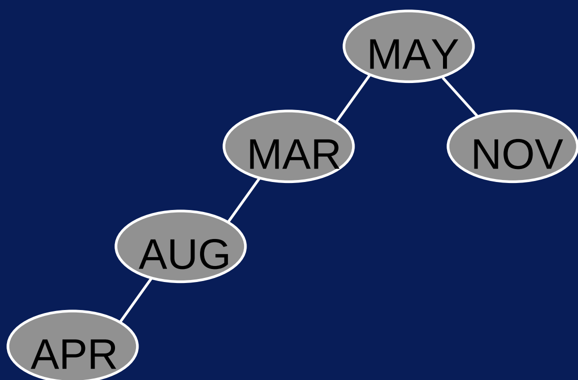
New Identifier	After Insertion	After Rebalancing
MARCH		NO REBALANCING NEEDED
MAY		NO REBALANCING NEEDED
NOVEMBER		

New Identifier	After Insertion	After Rebalancing
MARCH		NO REBALANCING NEEDED
MAY		NO REBALANCING NEEDED
NOVEMBER		

New Identifier	After Insertion	After Rebalancing
MARCH		NO REBALANCING NEEDED
MAY		NO REBALANCING NEEDED
NOVEMBER		 RR rebalancing

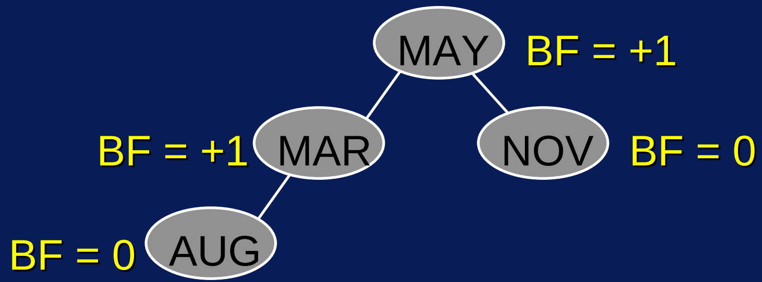
New Identifier	After Insertion	After Rebalancing
AUGUST		

New Identifier	After Insertion	After Rebalancing
AUGUST		NO REBALANCING NEEDED

New Identifier	After Insertion	After Rebalancing
AUGUST		NO REBALANCING NEEDED
APRIL		

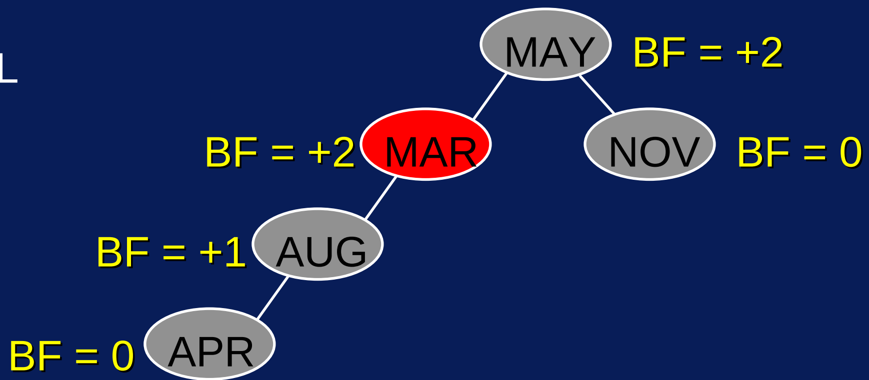
New Identifier	After Insertion	After Rebalancing

AUGUST



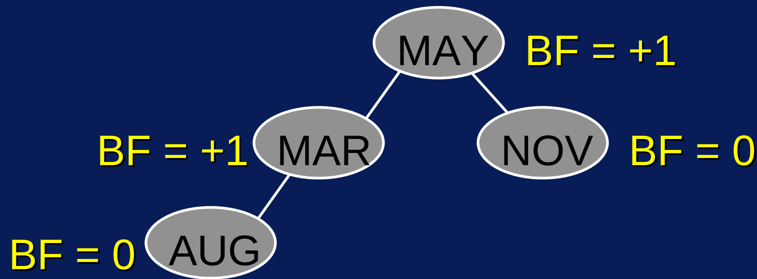
NO REBALANCING NEEDED

APRIL



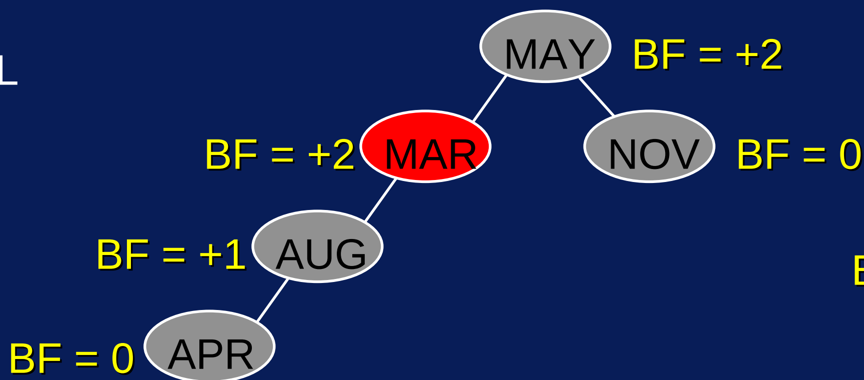
New Identifier	After Insertion	After Rebalancing

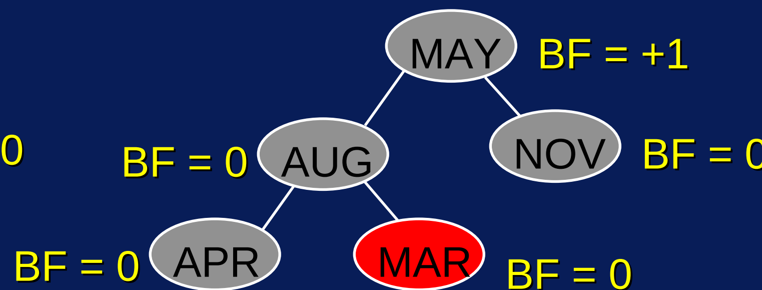
AUGUST



NO REBALANCING NEEDED

APRIL

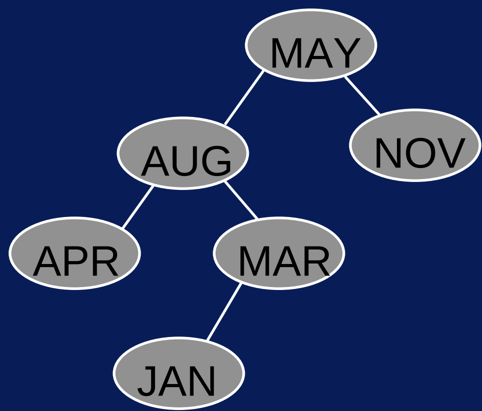




LL rebalancing

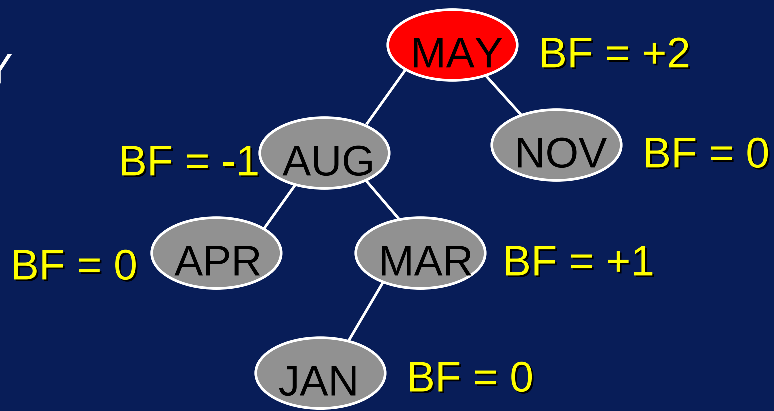
New Identifier | After Insertion | After Rebalancing

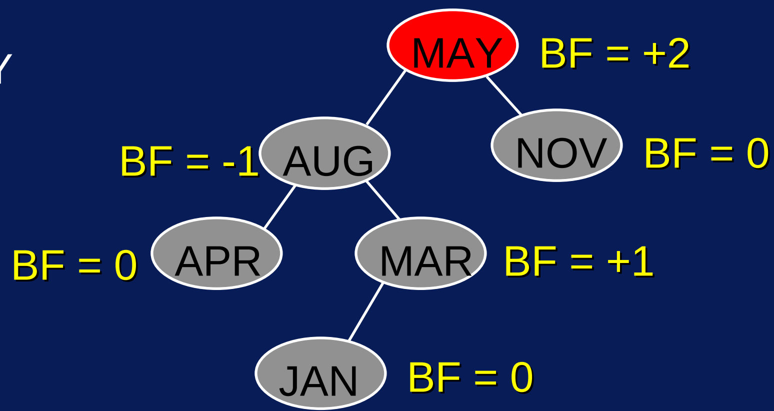
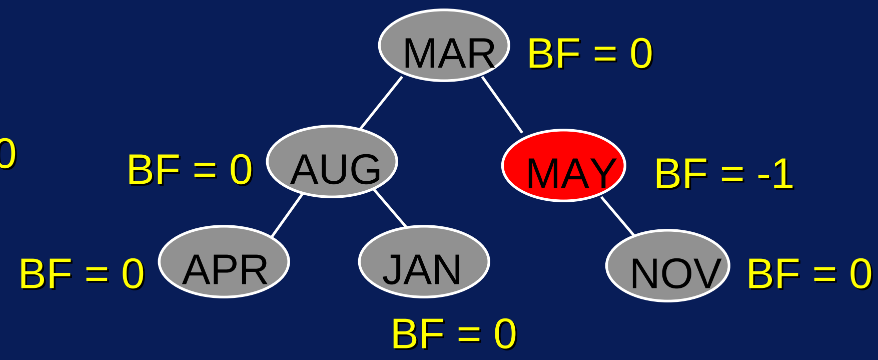
JANUARY



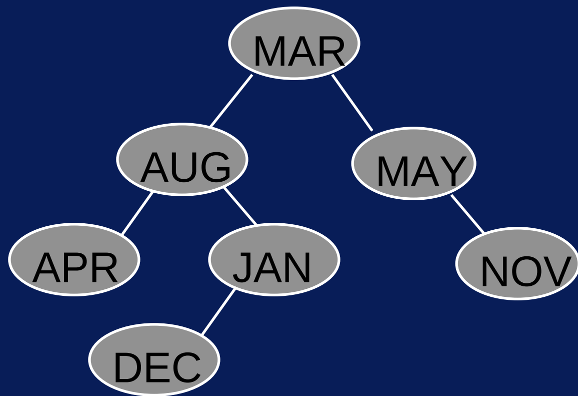
New Identifier | After Insertion | After Rebalancing

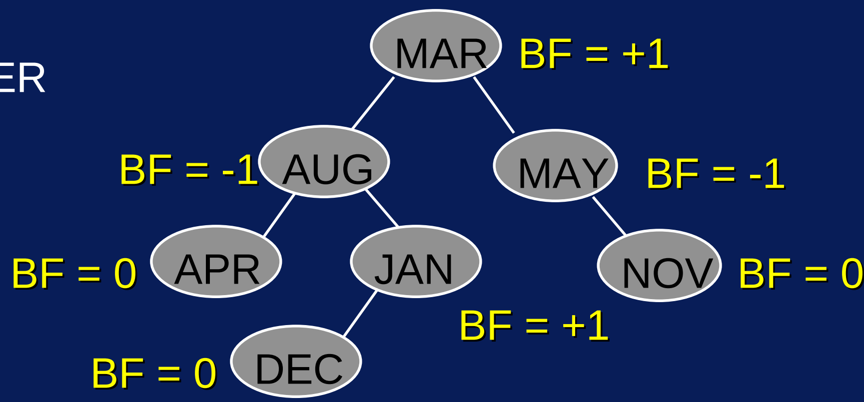
JANUARY

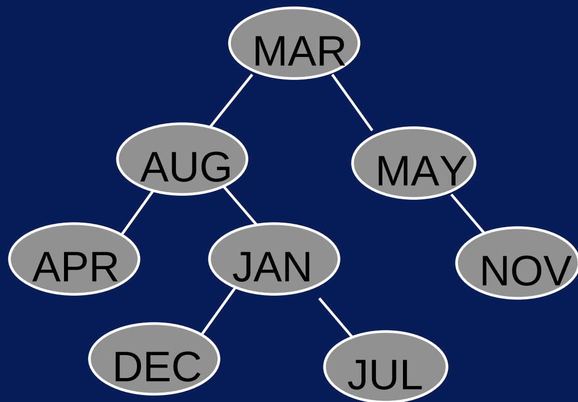


New Identifier	After Insertion	After Rebalancing
JANUARY		

LR rebalancing

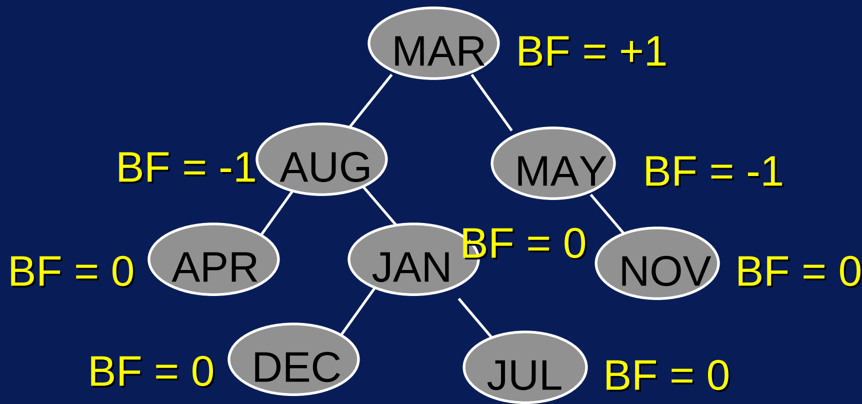
New Identifier	After Insertion	After Rebalancing
DECEMBER		

New Identifier	After Insertion	After Rebalancing
DECEMBER		NO REBALANCING NEEDED

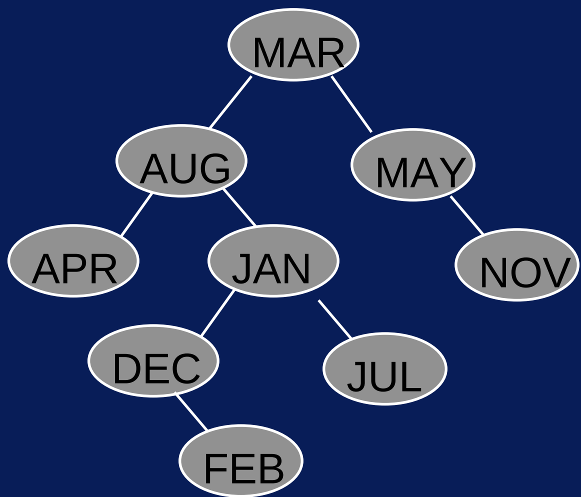
New Identifier	After Insertion	After Rebalancing
JULY		

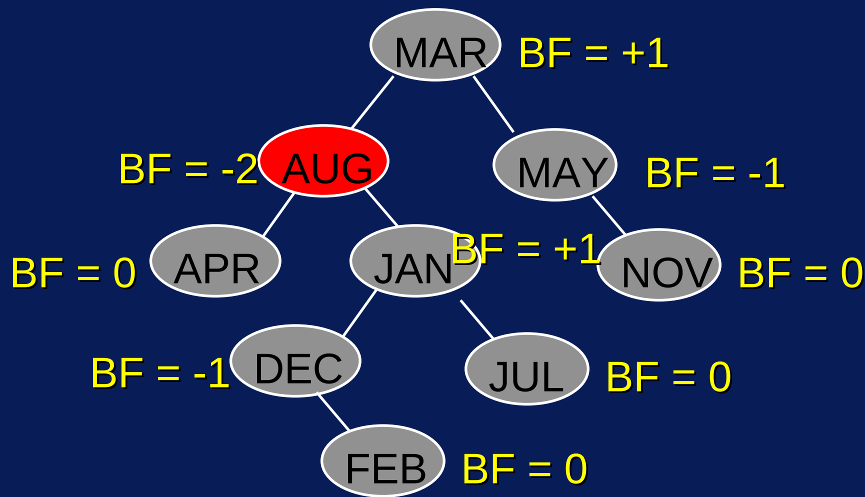
New Identifier | After Insertion | After Rebalancing

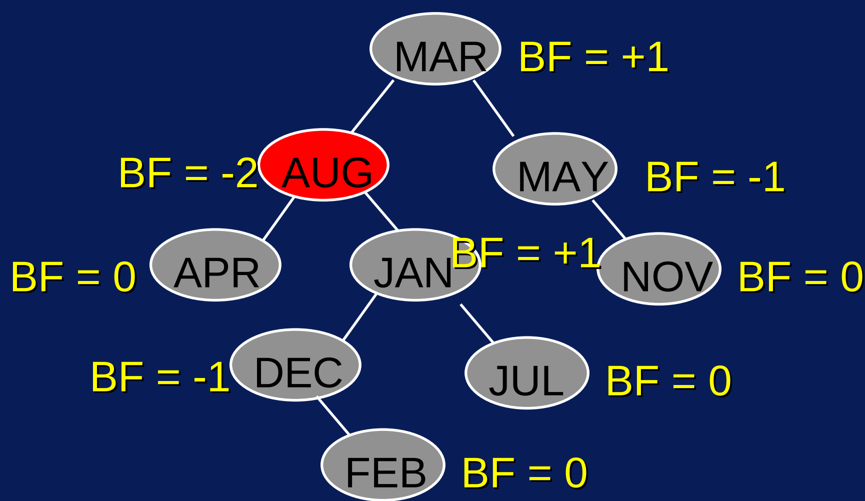
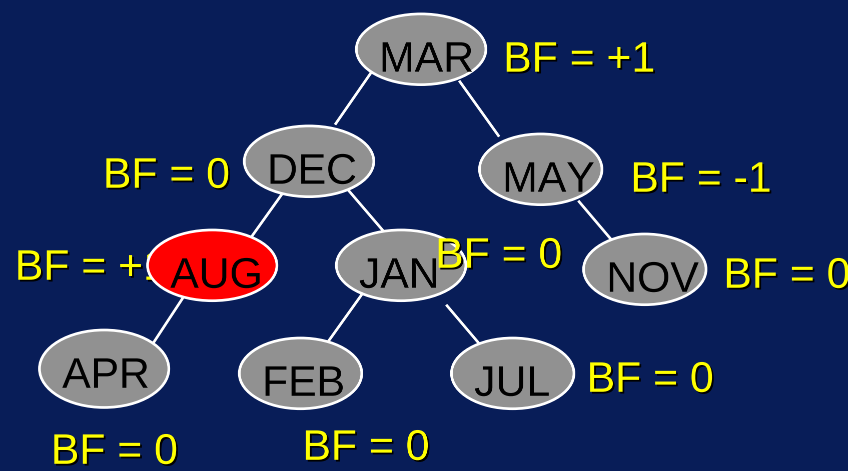
JULY



NO REBALANCING NEEDED

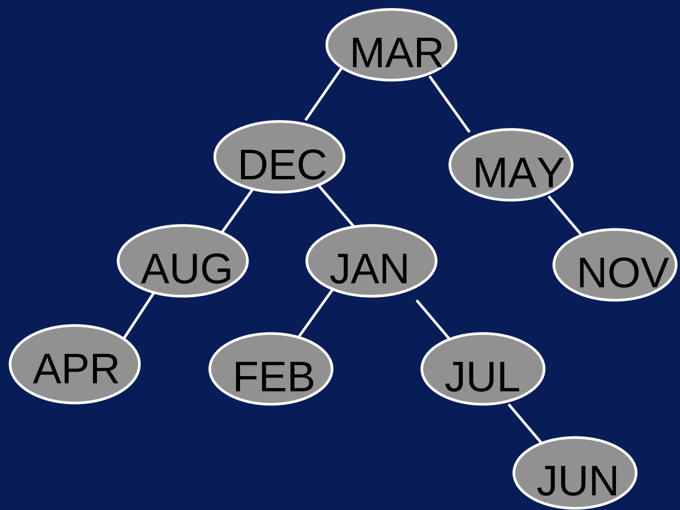
New Identifier	After Insertion	After Rebalancing
FEBRUARY		

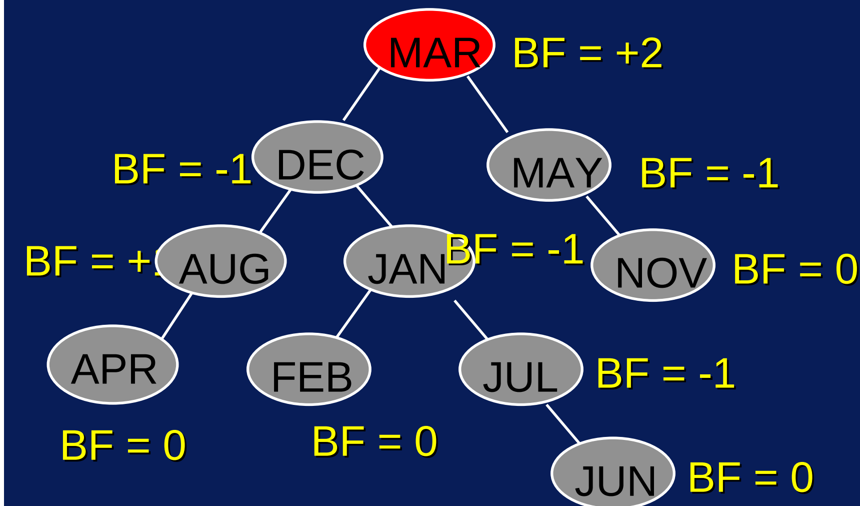
New Identifier	After Insertion	After Rebalancing
FEBRUARY		

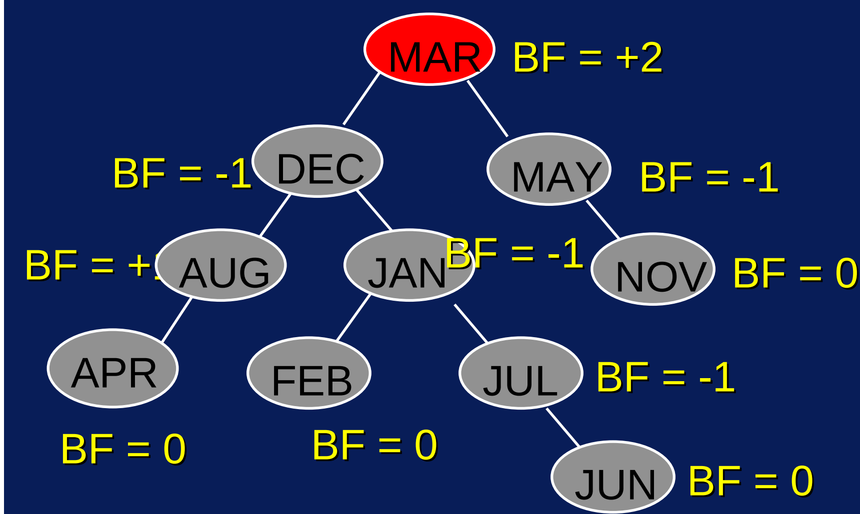
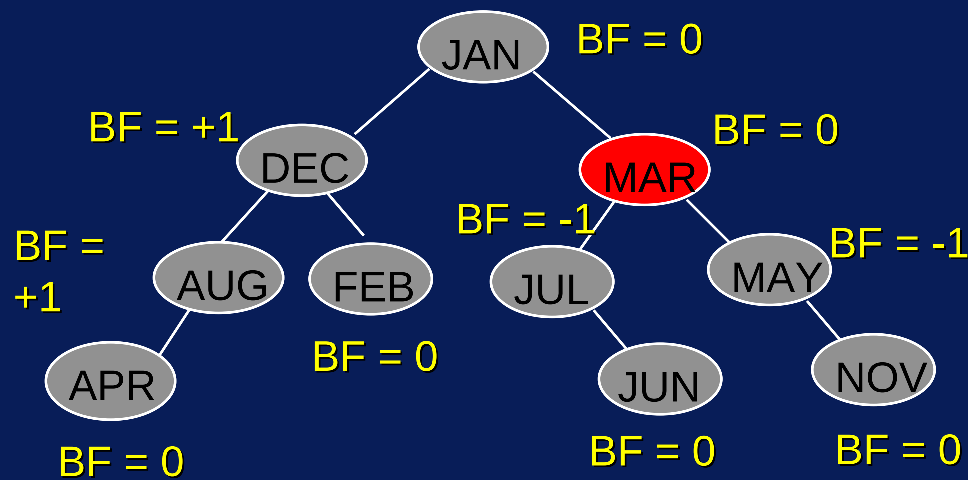
New Identifier	After Insertion	After Rebalancing
FEBRUARY		

RL rebalancing

New Identifier	After Insertion	After Rebalancing
JUNE		



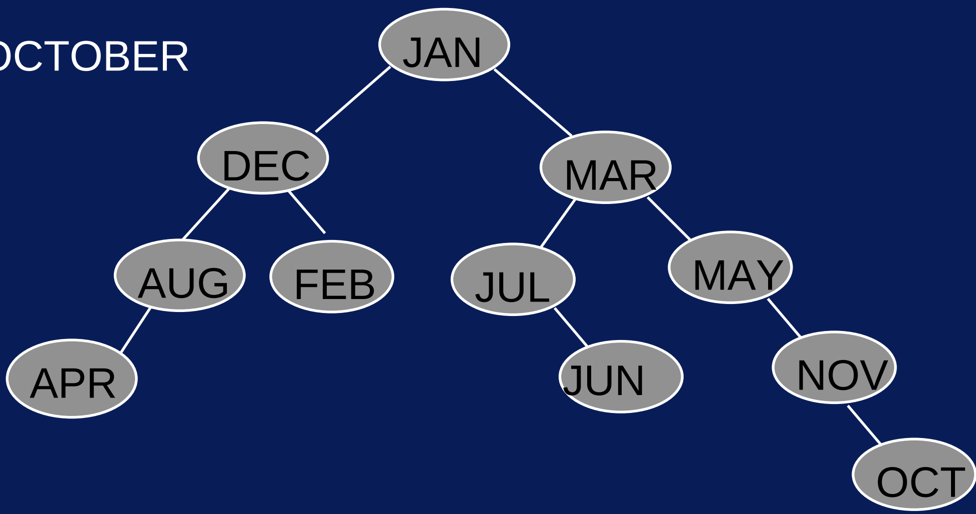
New Identifier	After Insertion	After Rebalancing
JUNE		

New Identifier	After Insertion	After Rebalancing
JUNE		

LR rebalancing

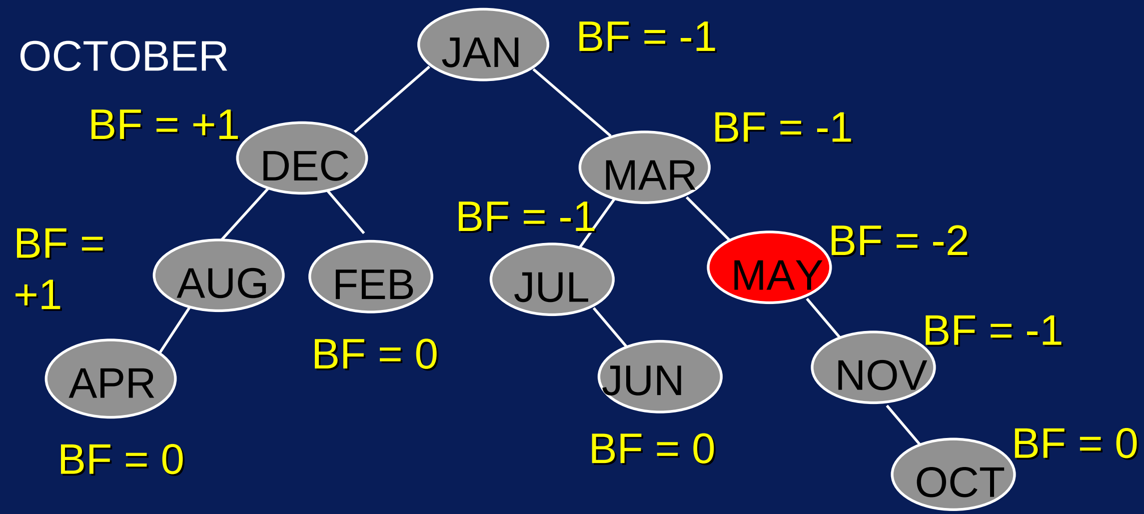
New Identifier | After Insertion | After Rebalancing

OCTOBER

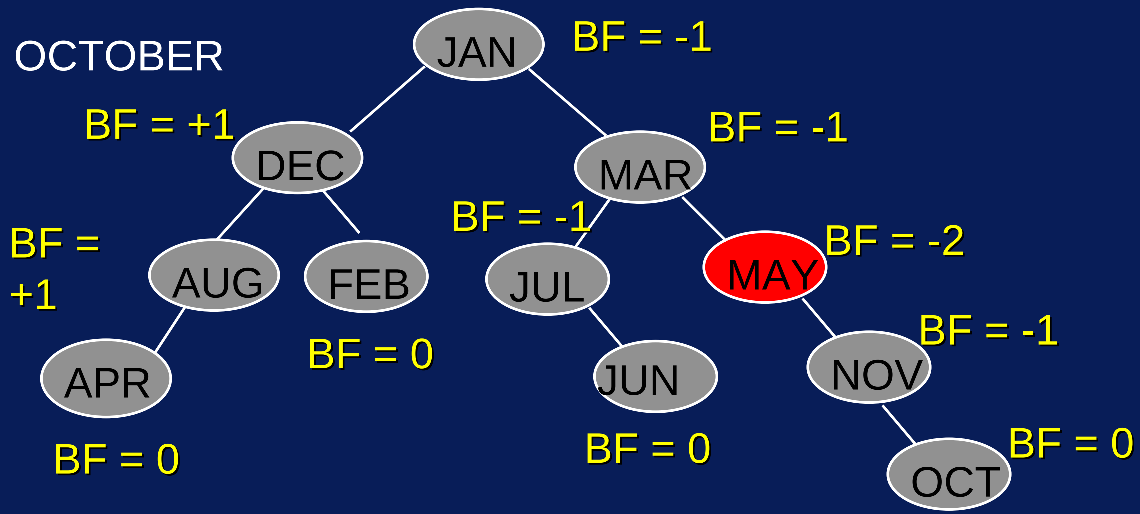


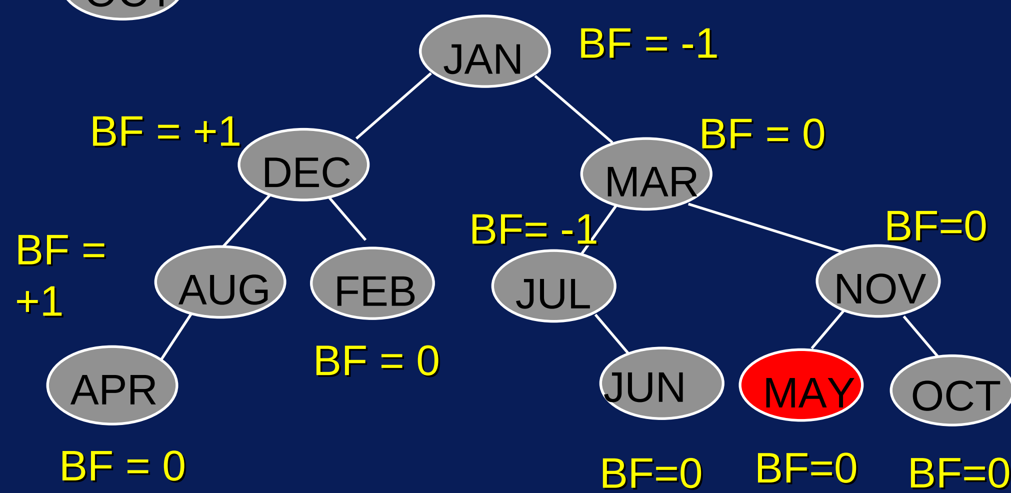
New Identifier	After Insertion	After Rebalancing

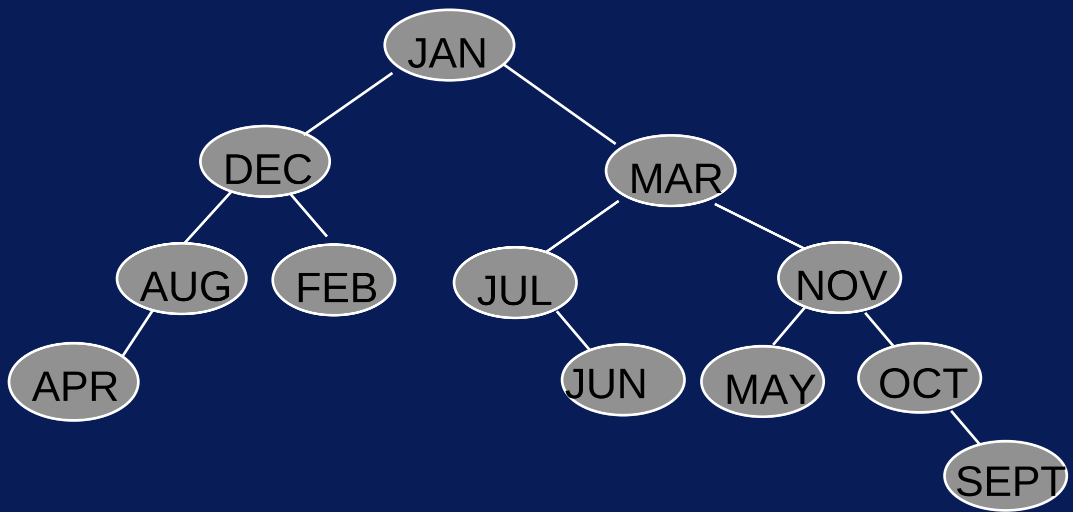
OCTOBER

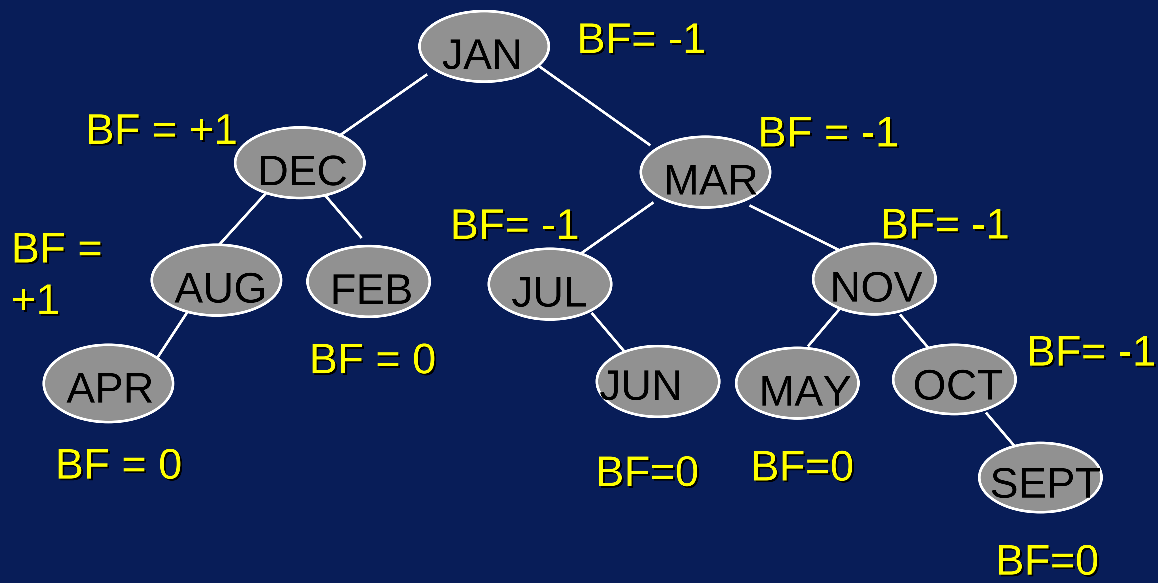


New Identifier	After Insertion	After Rebalancing
OCTOBER		RR rebalancing





New Identifier	After Insertion	After Rebalancing
SEPTEMBER		

New Identifier	After Insertion	After Rebalancing
SEPTEMBER		NO REBALANCING NEEDED

AVL Trees

- All re-balancing operations are carried out with respect to the closest ancestor of the new node having balance factor +2 or -2
- There are 4 types of re-balancing operations (called rotations)
  - RR
  - LL (symmetric with RR)
  - RL
  - LR (symmetric with RL)

AVL Trees

- Let’s refer to the node inserted as Y
- Let’s refer to the nearest ancestor having balance factor +2 or -2 as A

AVL Trees

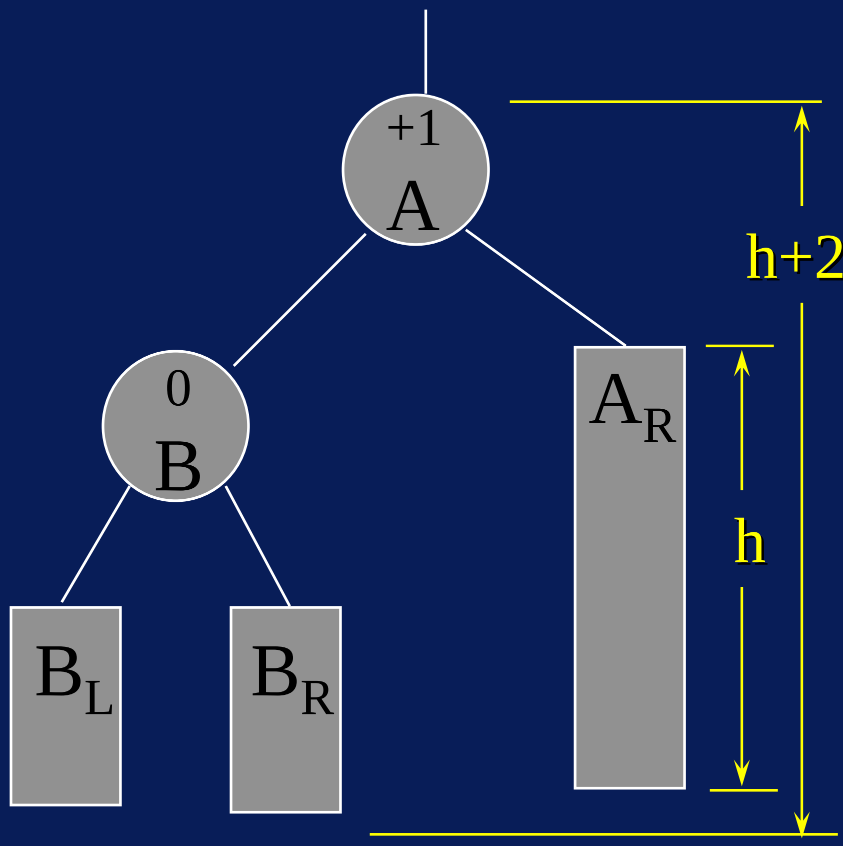
- LL: Y is inserted in the Left subtree of the Left subtree of A
  - LL: the path from A to Y
  - Left subtree then Left subtree
- LR: Y is inserted in the Right subtree of the Left subtree of A
  - LR: the path from A to Y
  - Left subtree then Right subtree

AVL Trees

- RR: Y is inserted in the Right subtree of the Right subtree of A
  - RR: the path from A to Y
  - Right subtree then Right subtree
- RL: Y is inserted in the Left subtree of the Right subtree of A
  - LL: the path from A to Y
  - Right subtree then Left subtree

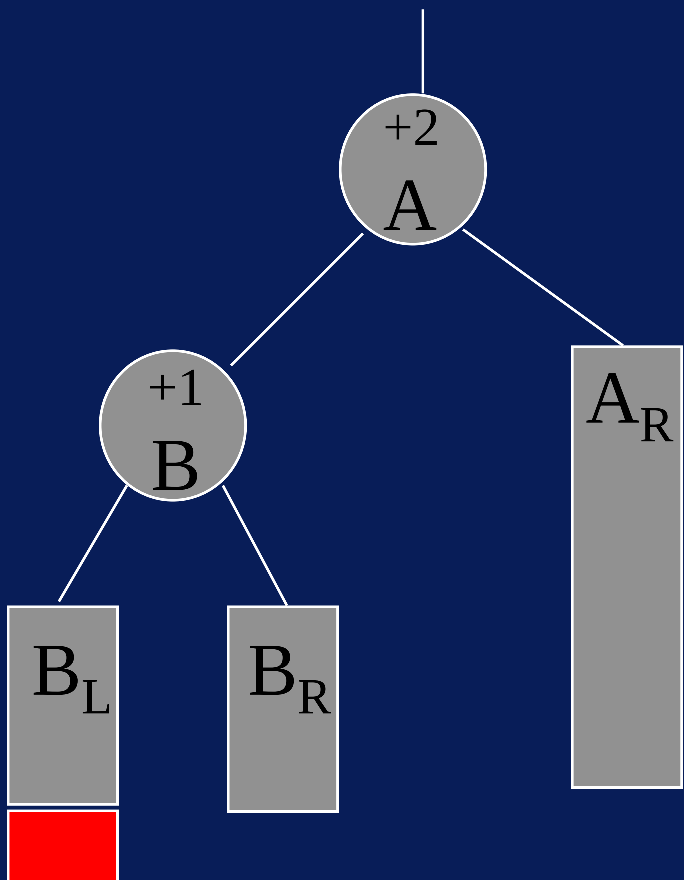
AVL Trees

Balanced Subtree



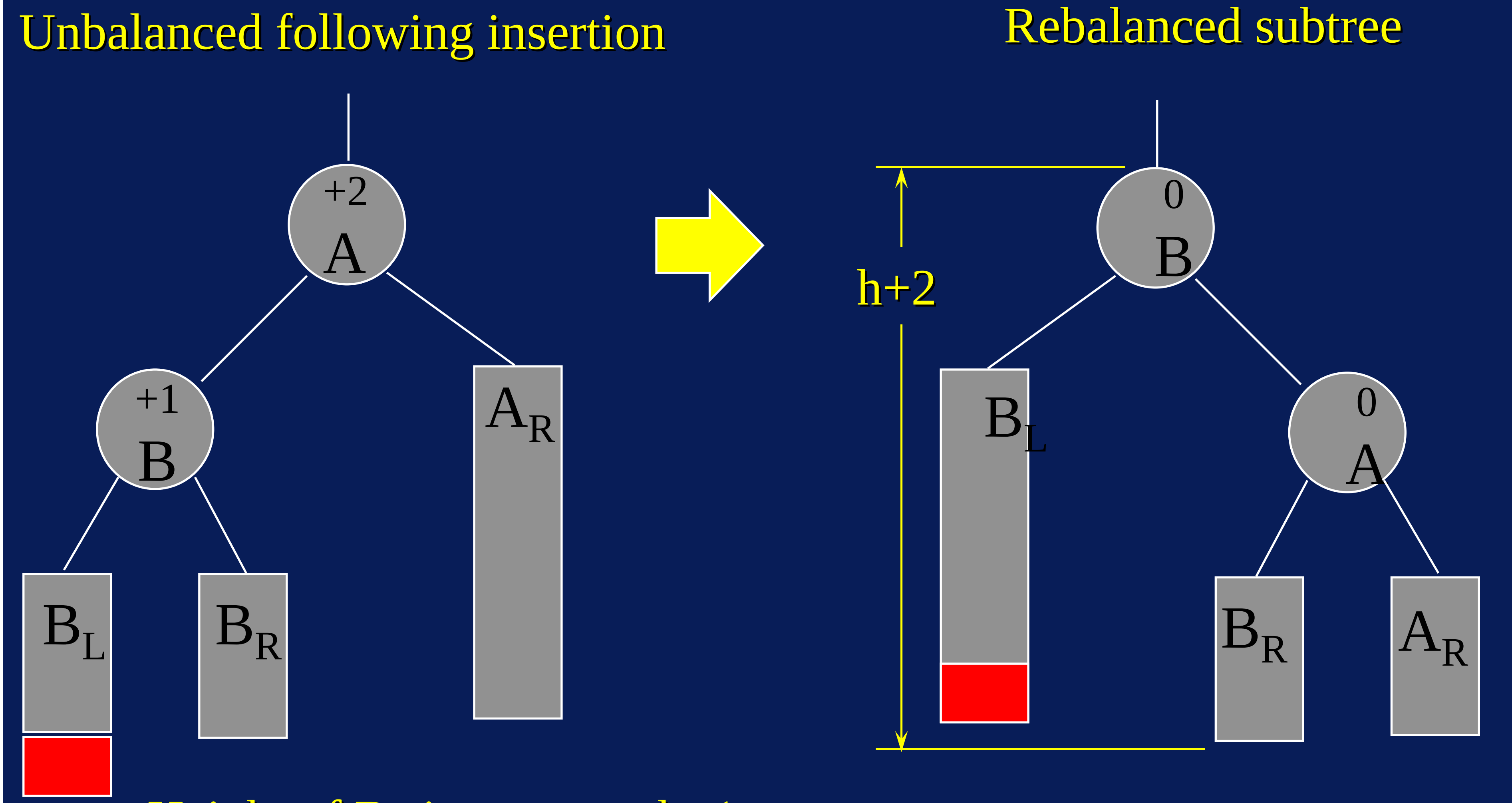
AVL Trees

Unbalanced following insertion



Height of B_L inceases to h+1

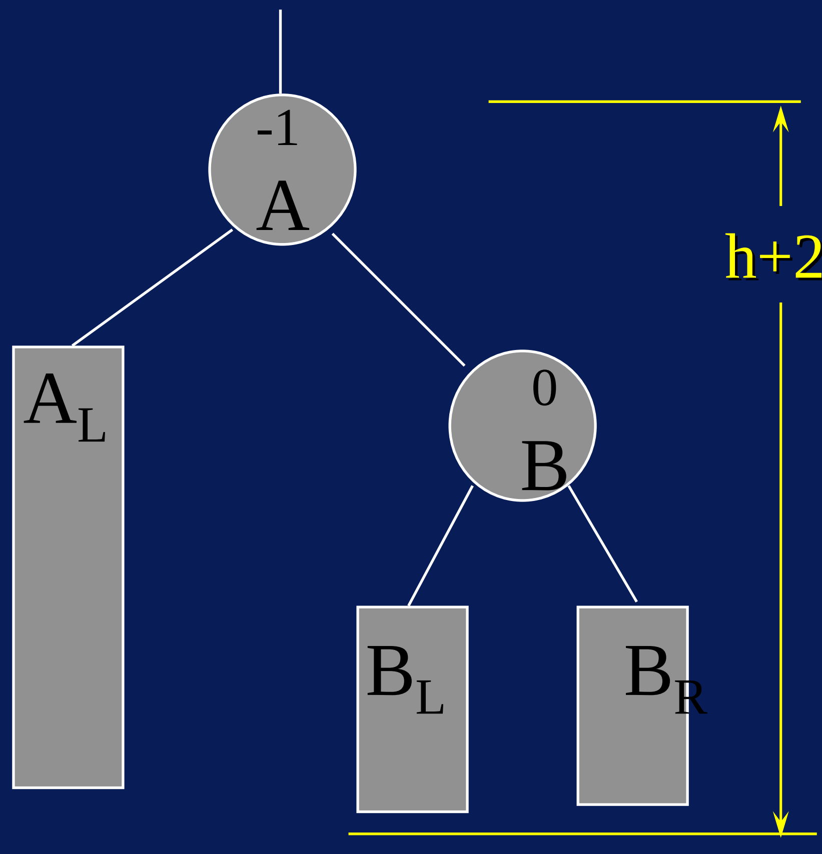
AVL Trees - LL rotation



Height of B_L inceases to h+1

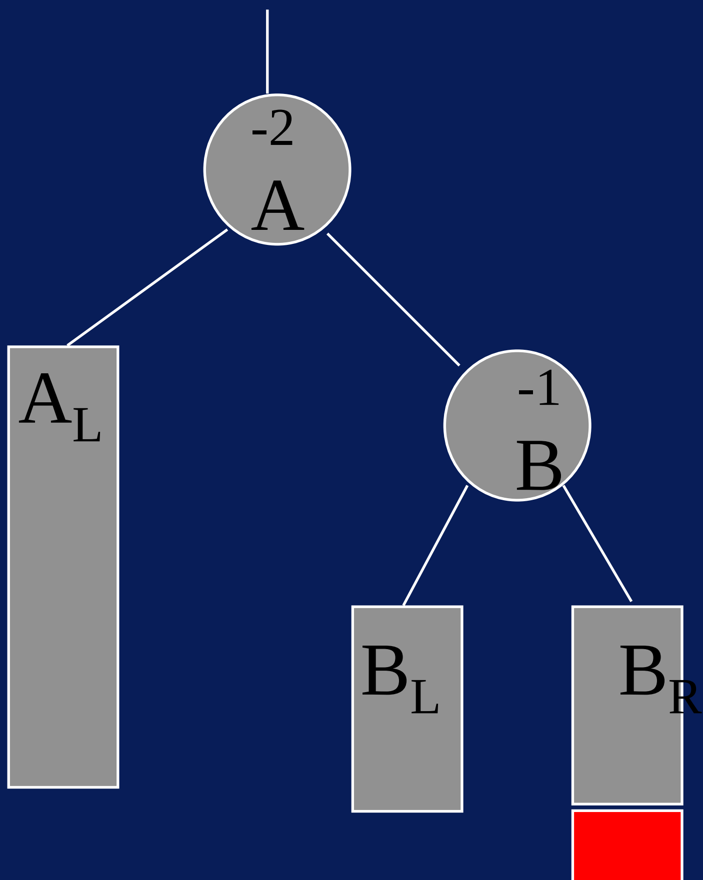
AVL Trees

Balanced Subtree



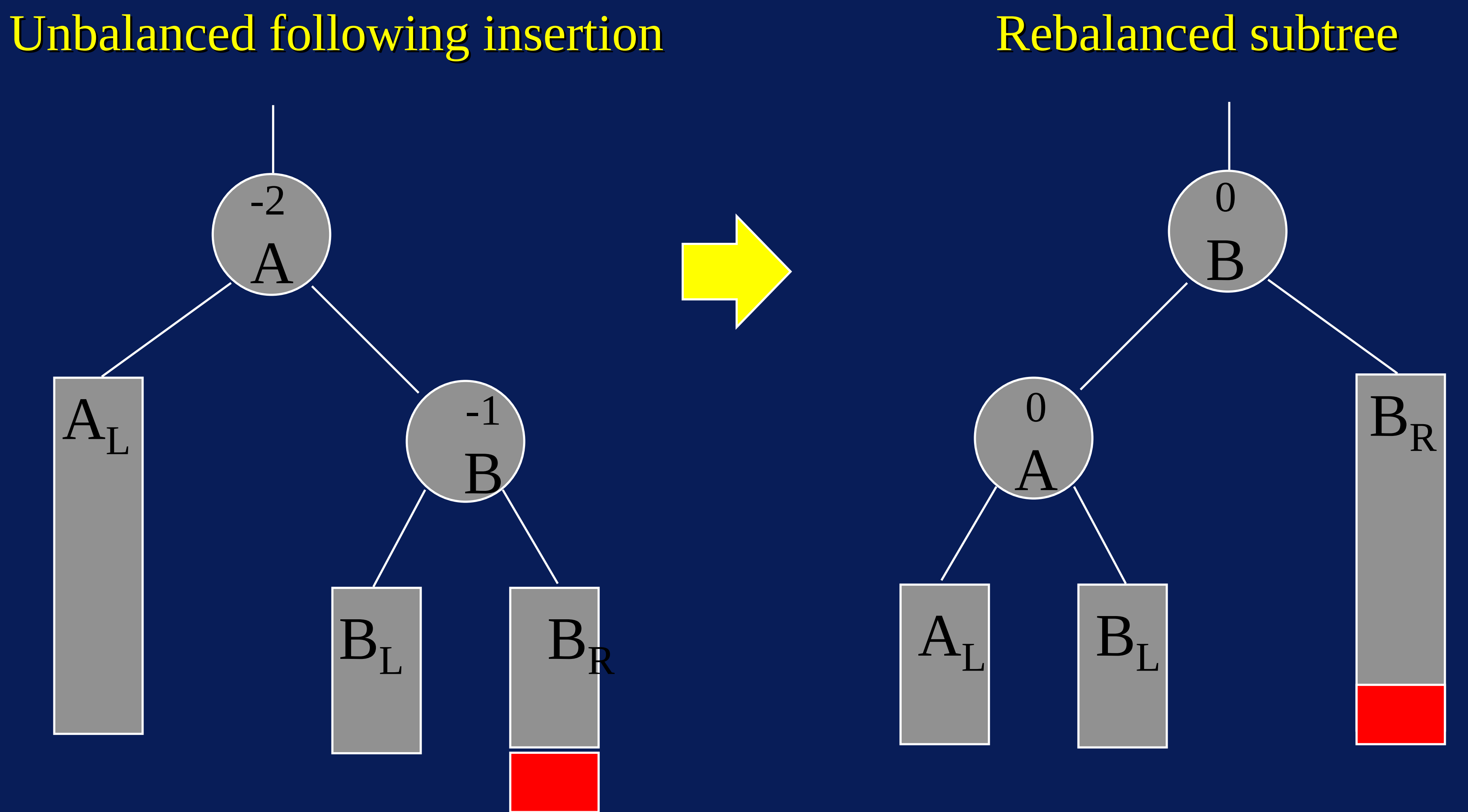
AVL Trees

Unbalanced following insertion



Height of B_R inceases to h+1

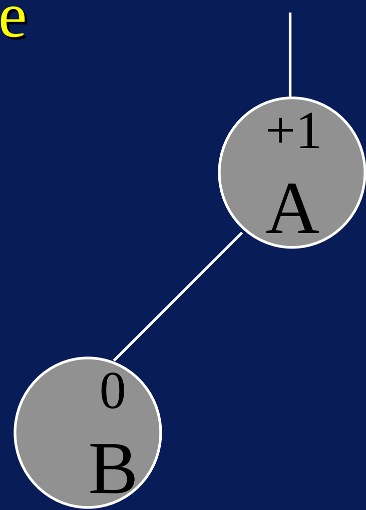
AVL Trees - RR Rotation



Height of B_R inceases to h+1

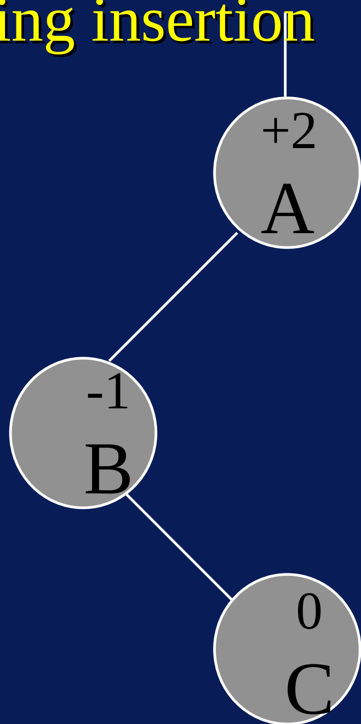
AVL Trees

Balanced Subtree

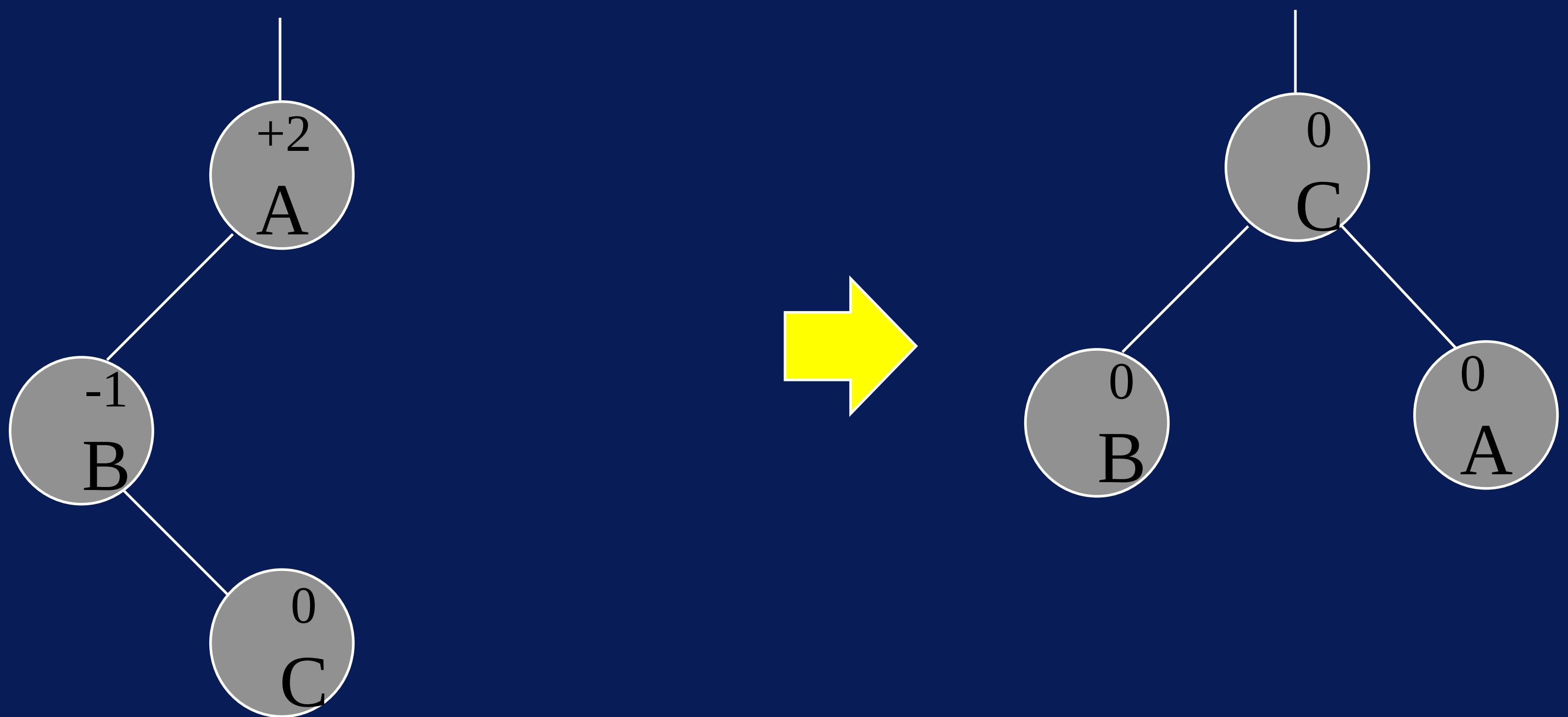


AVL Trees

Unbalanced following insertion

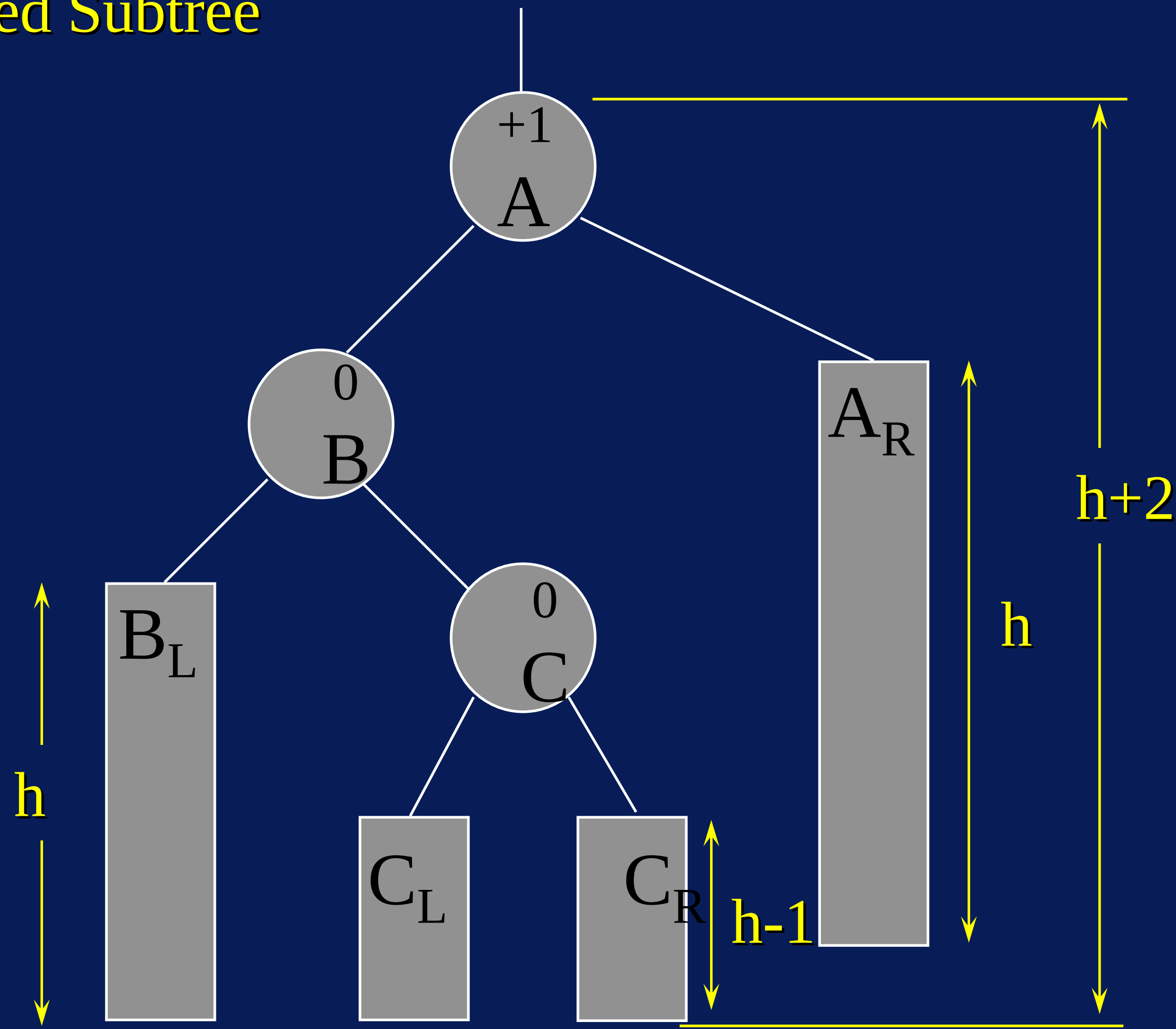


AVL Trees - LR rotation (a)



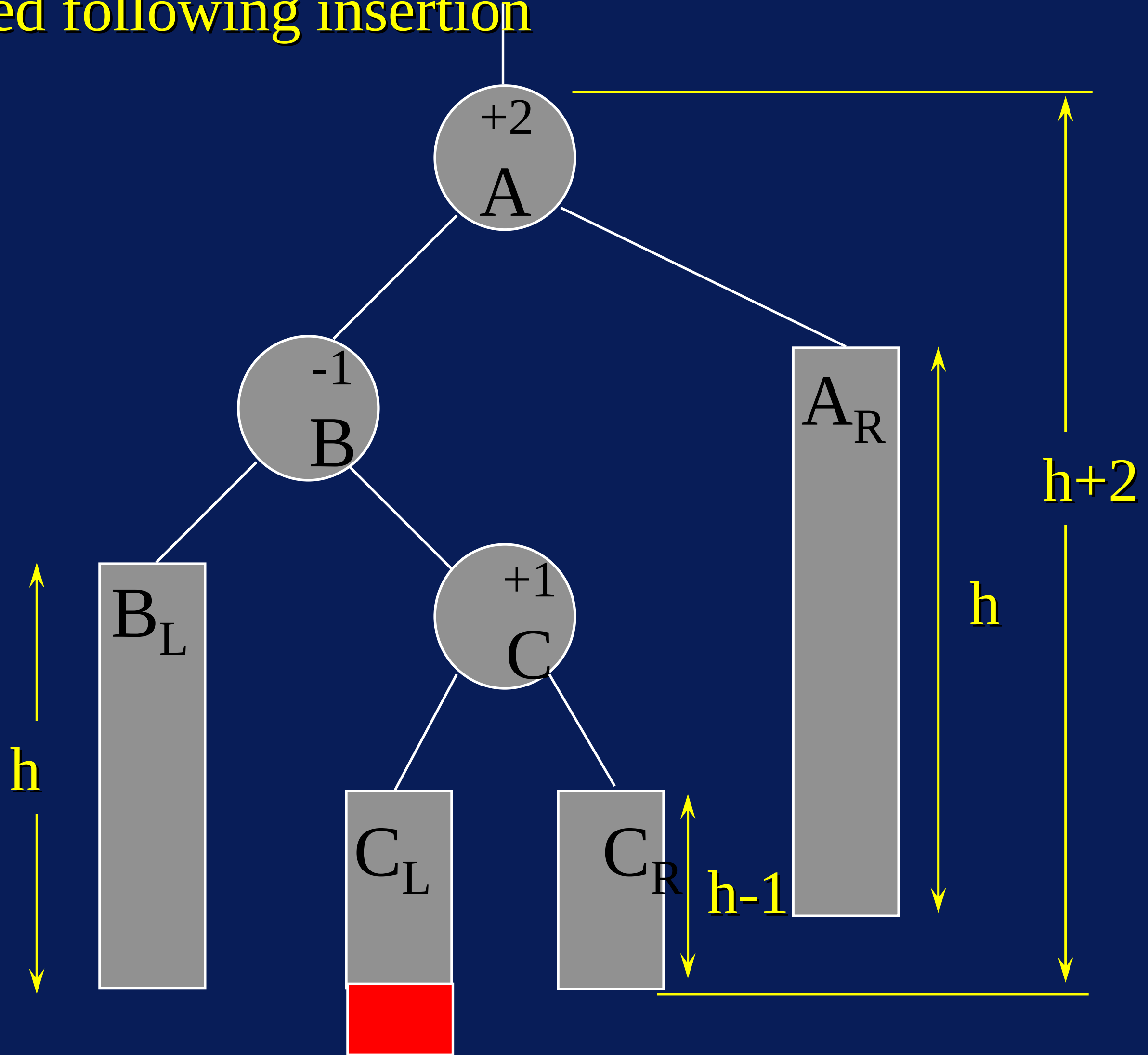
AVL Trees

Balanced Subtree

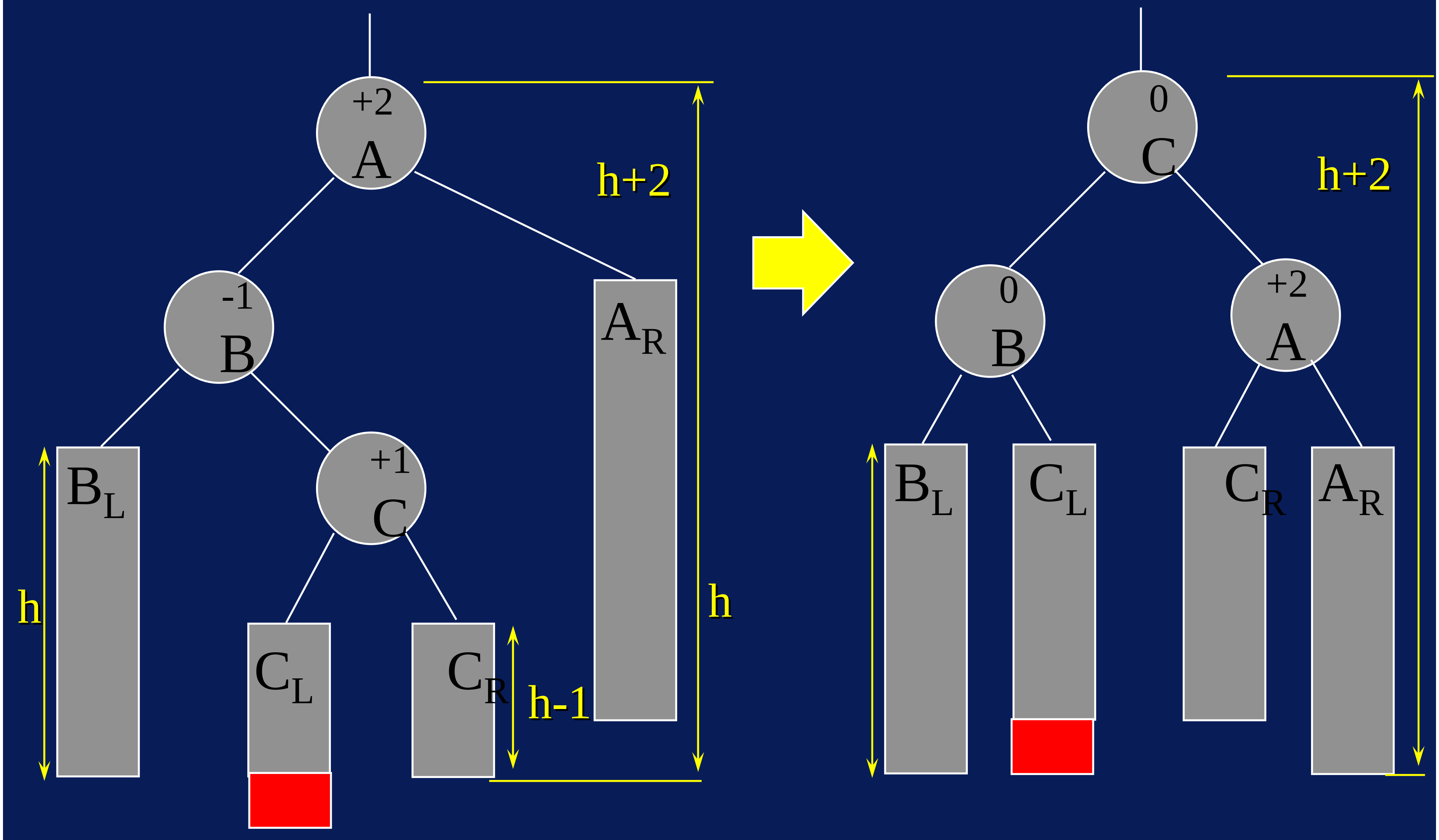


AVL Trees

Unbalanced following insertion

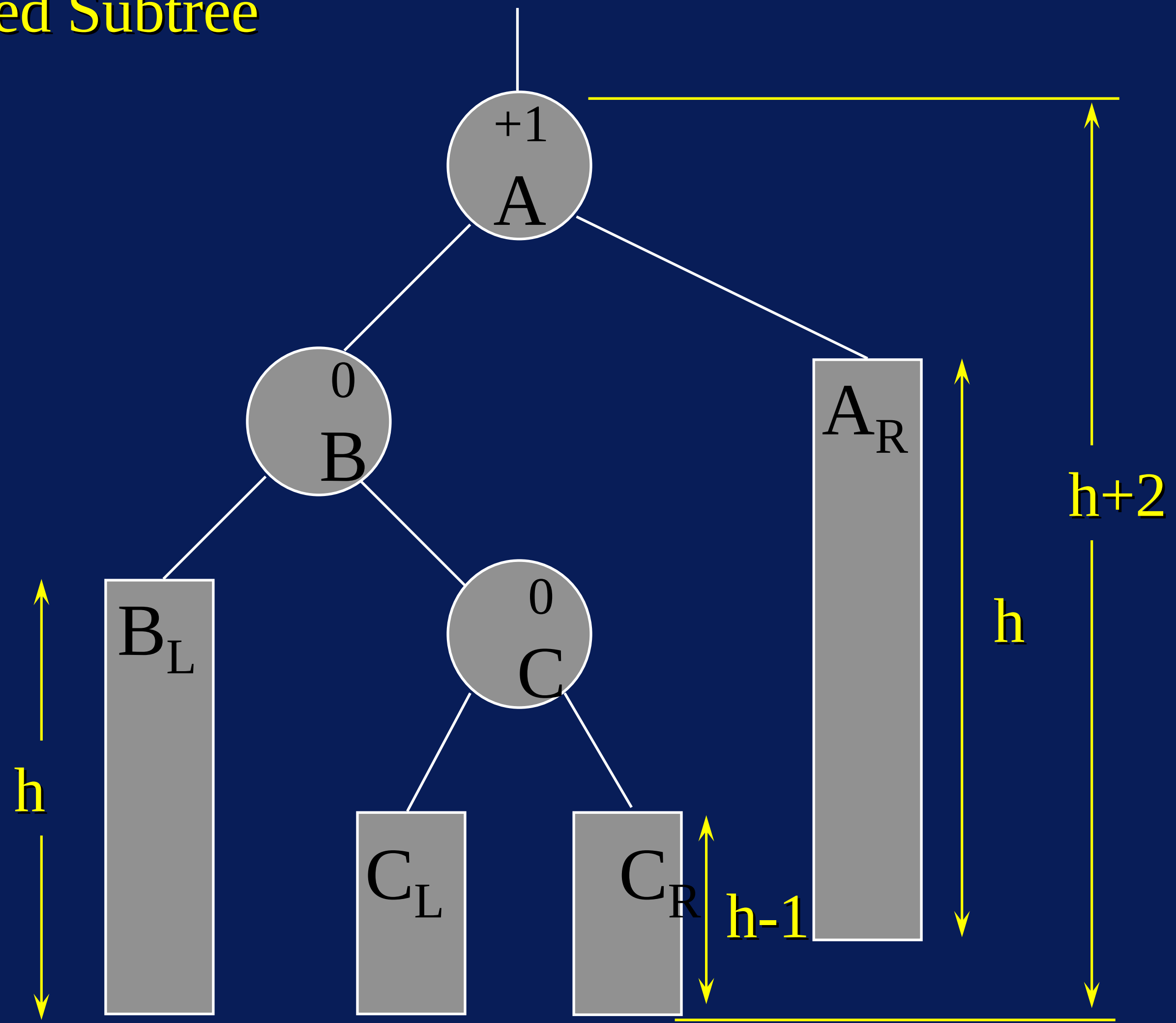


AVL Trees - LR rotation (b)



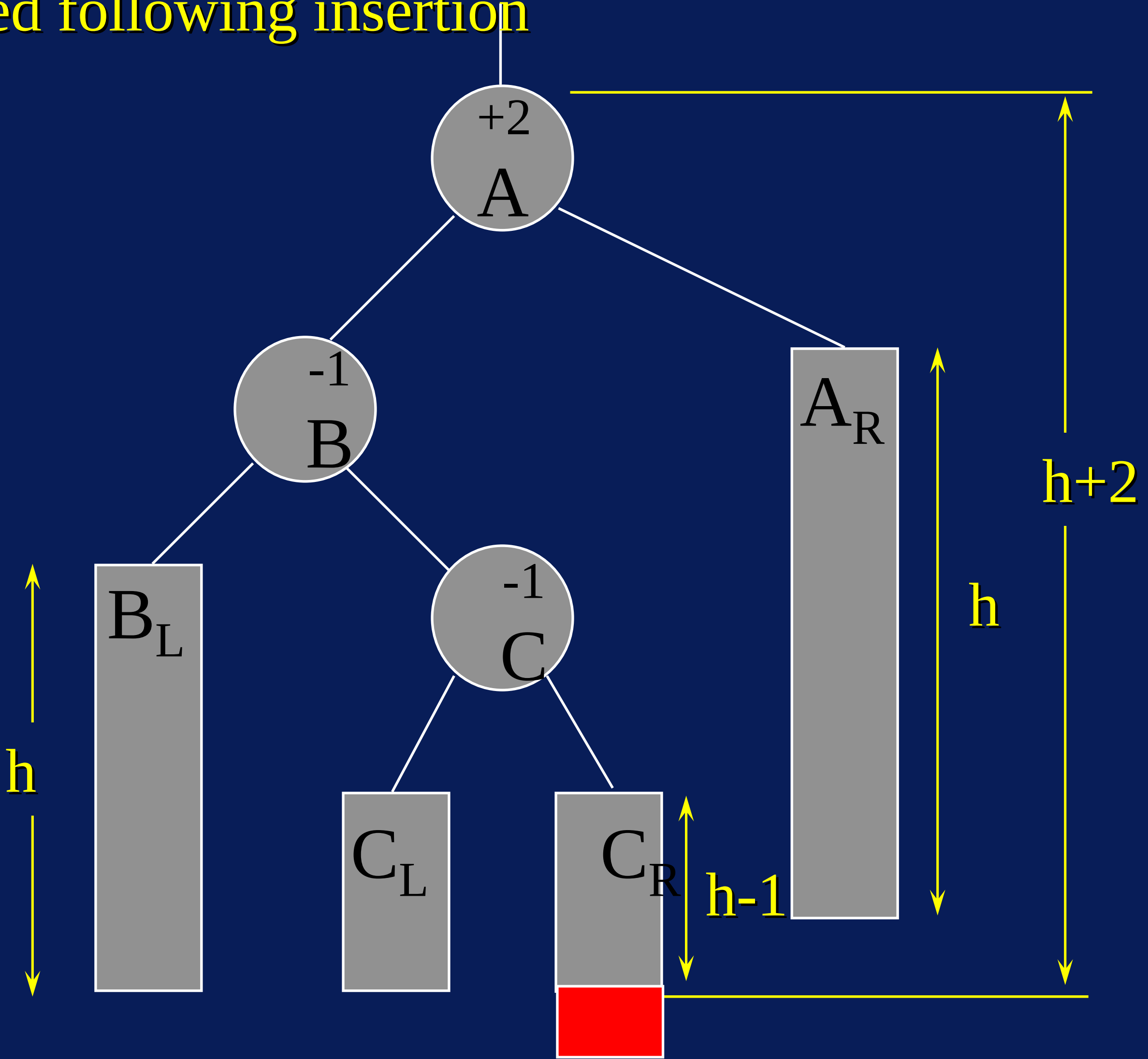
AVL Trees

Balanced Subtree

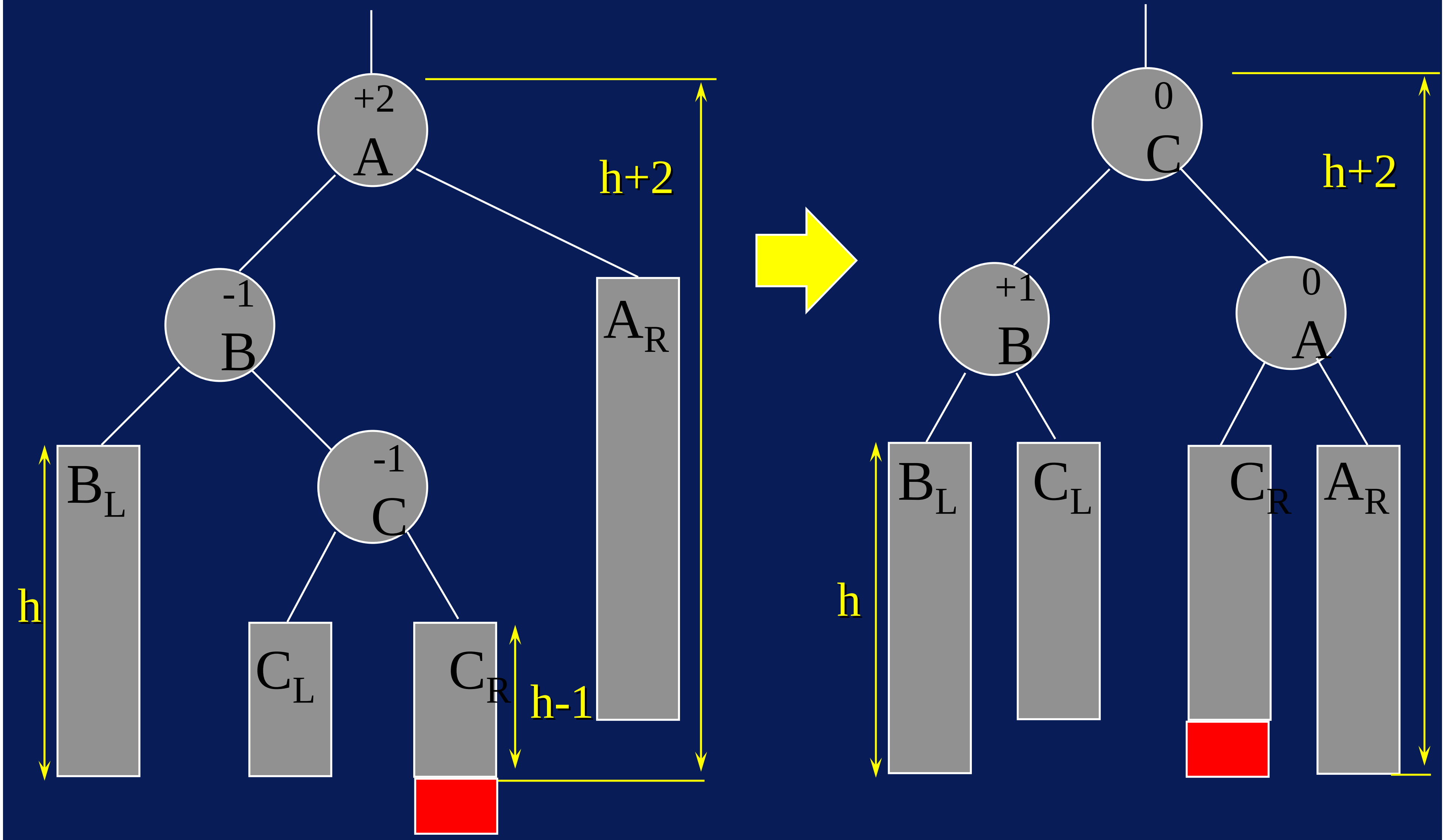


AVL Trees

Unbalanced following insertion

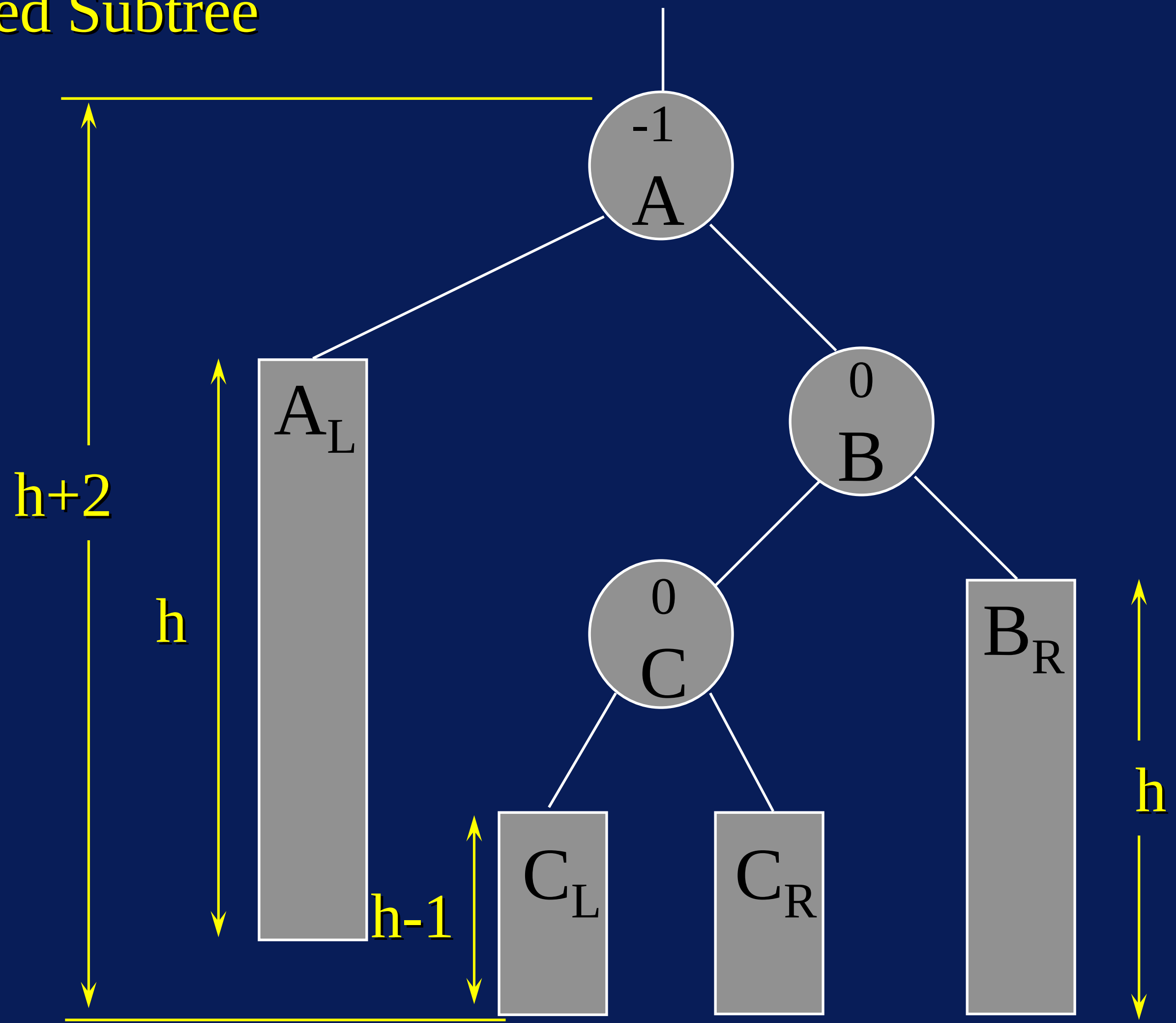


AVL Trees - LR rotation (c)



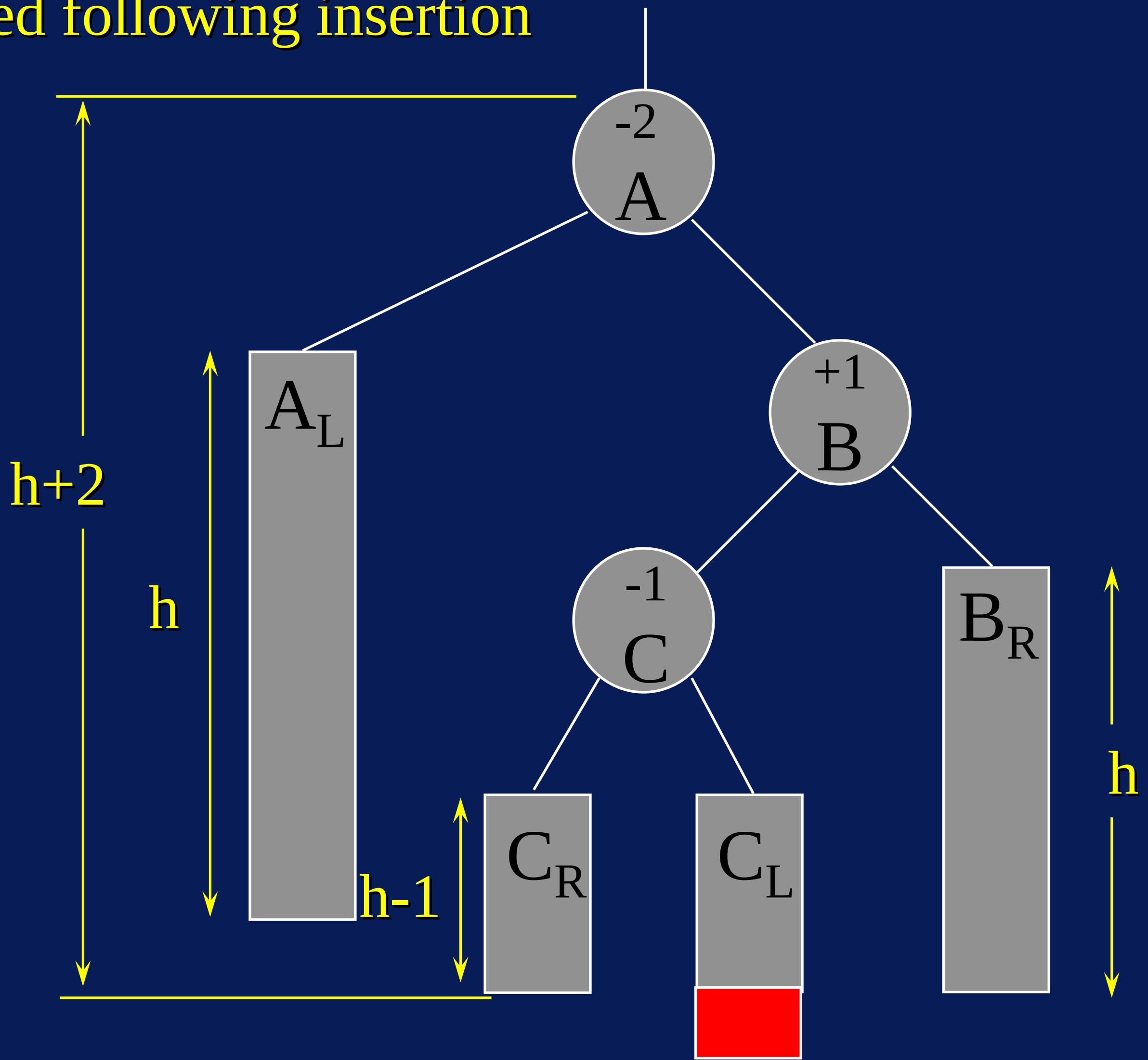
AVL Trees

Balanced Subtree

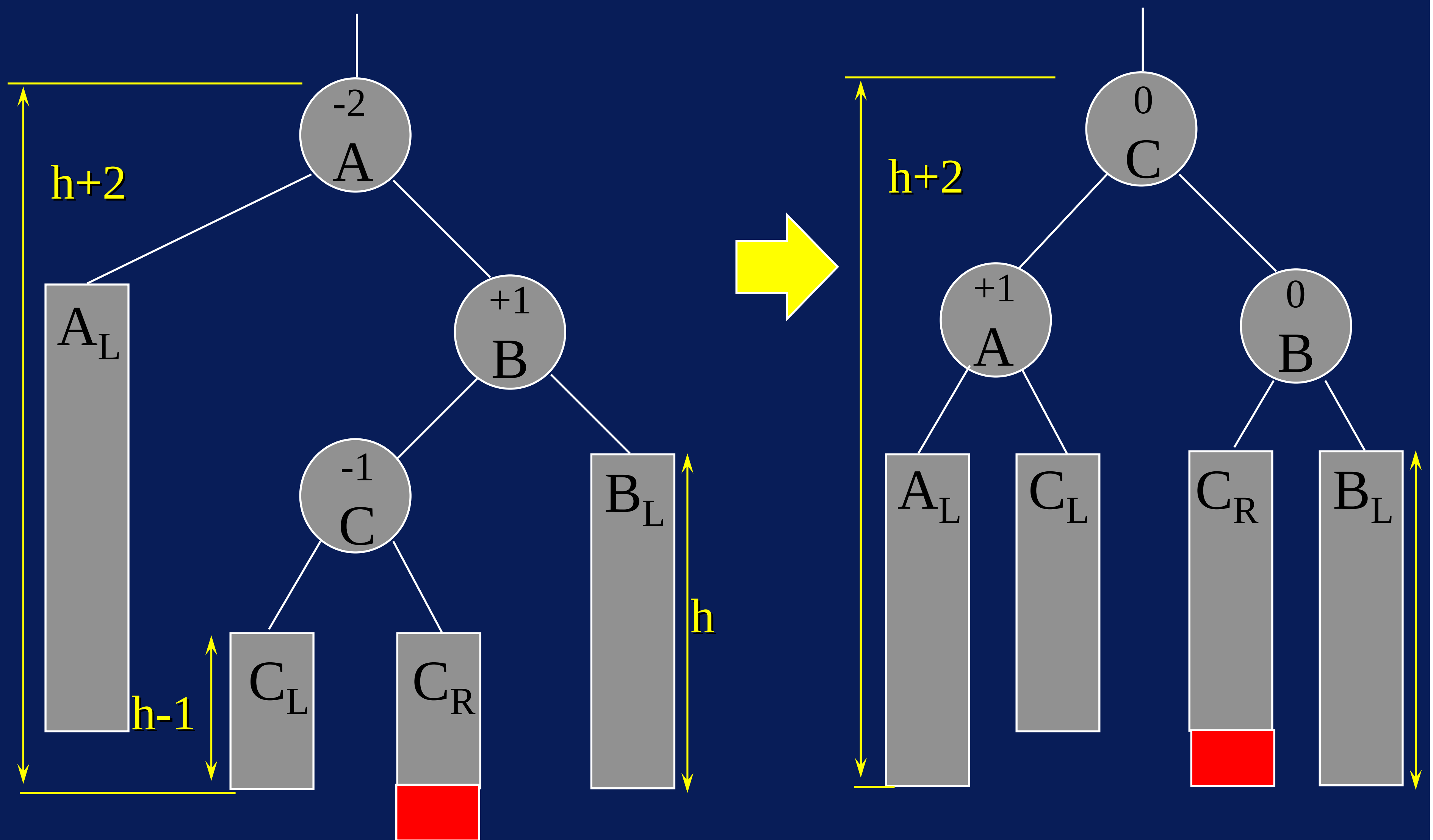


AVL Trees

Unbalanced following insertion



AVL Trees - RL rotation



AVL Trees

- To carry out this rebalancing we need to locate A, i.e. to window A
  - A is the nearest ancestor to Y whose balance factor becomes +2 or -2 following insertion
  - Equally, A is the nearest ancestor to Y whose balance factor was +1 or -1 before insertion
- We also need to locate F, the parent of A
  - This is where our complex window variable

AVL Trees

- Note in pasing that, since A is the nearest ancestor to Y whose balance factor was +1 or -1 before insertion, the balance factor of all other nodes on the part from A to Y must be 0
- When we re-balance the tree, the balance factors change (see diagrams above)
  - But changes only occur in subtree which is being rebalanced

AVL Trees

- The balance factors also change following an insertion which requires no rebalancing
- BF(A) is +1 or -1 before insertion
- Insertion causes height of one of A’s subtrees to increase by 1
- Thus, BF(A) must be 0 after insertion (since, in this case, it’s not +2 or -2)

Implementation of AVL_Insert()

```
PROCEDURE AVL_insert(e:elementtype; w:windowtype;
                  T: BINTREE);

(* We assume that variables of element type have two *)
(* data fields: the information field and a balance  *)
(* factor                                            *)
(* Assume also existence of two ADT functions to     *)
(* examine these fields:                             *)
(*                      Examine_BF(w, T)             *)
(*                      Examine_data(w, T)           *)
(* and one to modify the balance factor field        *)
(*                      Replace_BF(bf, w, T)         *)
var newnode: linktype;
begin
```

Implementation of AVL_Insert()

```
IF IsEmpty(T) (* special case *)
   THEN
      Insert(e, w, T);  (*insert as before *)
      Replace_BF(0, w, T)
   ELSE
      (* Phase 1: locate insertion point        *)
      (* A keeps track of most recent node with *)
      (* balance factor +1 or -1                *)
      A := w;
      WHILE ((NOT IsExternal(w, T)) AND
             (NOT (e.data = Examine_Data(w, T))) DO
         IF Examine_BF(w, T) <> 0 (* non-zero BF *)
            THEN
               A := w;
         ENDIF;
```

Implementation of AVL_Insert()

```
        IF (e.data < Examine_Data(w, T) )
           THEN
              Child(0, w, T)
           ELSE IF (e.data > Examine_Data(w, T) )
              Child(1, w, T)
           ENDIF
        ENDIF
     ENDWHILE
     (* If not found, then embark on Phase 2: *)
     (* insert & rebalance                    *)
     IF IsExternal(w, T)
        THEN
           Insert(e, w, T);  (*insert as before *)
           Replace_BF(0, w, T)
     ENDIF
```

Implementation of AVL_Insert()

```
(* adjust balance factors of nodes on path      *)
(* from A to parent of newly-inserted node      *)
(* By definition, they will have had BF=0       *)
(* and so must now change to +1 or -1           *)
(* Let d = this change,                         *)
(* d = +1 ... insertion in A’s left subtree     *)
(* d = -1 ... insertion in A’s right subtree    *)

IF (e.data < Examine_Data(A, T) )
   THEN
      v:= A;
      Child(0, v, T)
      B:= v;
      d := +1
   ELSE
```

Implementation of AVL_Insert()

```
        ELSE
           v:= A; Child(1, v, T)
           B:= v;
           d := -1
     ENDIF
     WHILE ((NOT IsEqual(w, v))) DO
        IF (e.data < Examine_Data(v, T) )
           THEN
              ReplaceBF(+1, v, T);
              Child(0, v, T) (* height of Left ^ *)
           ELSE
              ReplaceBF(-1, v, T);
              Child(1, v, T) (* height of Right ^ *)
        ENDIF
     ENDWHILE
```

Implementation of AVL_Insert()

```
(* check to see if tree is unbalanced *)

IF (ExamineBF(A, T) = 0 )
   THEN
      ReplaceBF(d, A, T) (* still balanced *)
   ELSE
      IF ((ExamineBF(A, T) + d) = 0)
         THEN
            ReplaceBF(0, A, T)(*still balanced*)
         ELSE

            (* Tree is unbalanced      *)
            (* determine rotation type *)
```

Implementation of AVL_Insert()

```
(* Tree is unbalanced      *)
(* determine rotation type *)

IF d = +1
   THEN  (* left imbalance *)
      IF ExamineBF(B) = +1
         THEN (* LL Rotation *)
            (* replace left subtree of A *)
            (* with right subtree of B   *)
            temp := B; Child(1, temp, T);
            ReplaceChild(0, A, T,  temp);

            (* replace right subtree of B with A *)
            ReplaceChild(1, B, T, A);
```

Implementation of AVL_Insert()

```
        (* replace right subtree of B with A *)
        ReplaceChild(1, B, T, A);

        ReplaceBF(0, A, T);
        ReplaceBF(0, B, T);
     ELSE (* LR Rotation *)
        C := B; Child(1, C, T);
        C_L := C; Child(0, C_L, T);
        C_R := C; Child(1, C_R, T);
        ReplaceChild(1, B, T, C_L);
        ReplaceChild(0, A, T, C_R);
        ReplaceChild(0, C, T, B);
        ReplaceChild(1, C, T, A);
```

Implementation of AVL_Insert()

```
IF ExamineBF(C) = +1 (* LR(b) *)
   THEN
      ReplaceBF(-1, A, T);
      ReplaceBF(0, B, T);
   ELSE
      IF ExamineBF(C) = -1 (* LR(c) *)
         THEN
            ReplaceBF(+1, B, T);
            ReplaceBF(0, A, T);
         ELSE                  (* LR(a) *)
            ReplaceBF(0, A, T);
            ReplaceBF(0, B, T);
      ENDIF
ENDIF
```

Implementation of AVL_Insert()

```
            (* B is new root *)
            ReplaceBF(0, C, T);
            B := C
      ENDIF (* LR rotation *)
   ELSE (* right imbalance *)

      (* this is symmetric to left imbalance *)
      (* and is left as an exercise!         *)

ENDIF (* d = +1 *)
```

Implementation of AVL_Insert()

```
                    (* the subtree with root B has been *)
                    (* rebalanced and it now replaces   *)
                    (* A as the root of the originally  *)
                    (* unbalanced tree                  *)

                    ReplaceTree(A, T, B)
                    (* Replace subtree A with B in T     *)
                    (* Note: this is a trivial operation *)
                    (* since we are using a complex      *)
                    (* window variable                   *)
                 ENDIF
           ENDIF
     ENDIF
END (* AVL_Insert() *)
```