EXAMPLE 9 Consider the linear system

$$\begin{bmatrix} 2 & 1 & 3 \\ 1 & -2 & 2 \\ 0 & 1 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}.$$

Since rank $A = \text{rank} \left[A \mid \mathbf{b} \right] = 3$, the linear system has a solution. ■

EXAMPLE 10 The linear system

$$\begin{bmatrix} 1 & 2 & 3 \\ 1 & -3 & 4 \\ 2 & -1 & 7 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}$$

has no solution because rank $A = 2$ and rank $\left[A \mid \mathbf{b} \right] = 3$ (verify). ■

We now extend our list of nonsingular equivalences.

List of Nonsingular Equivalences

The following statements are equivalent for an $n \times n$ matrix A.

1. A is nonsingular.
2. $\mathbf{x} = \mathbf{0}$ is the only solution to $A\mathbf{x} = \mathbf{0}$.
3. A is row equivalent to I_n.
4. The linear system $A\mathbf{x} = \mathbf{b}$ has a unique solution for every $n \times 1$ matrix $\mathbf{b}$.
5. $\det(A) \neq 0$.
6. A has rank n.
7. A has nullity 0.
8. The rows of A form a linearly independent set of n vectors in R^n.
9. The columns of A form a linearly independent set of n vectors in R^n.

Key Terms

Row space
Column space
Row rank
Column rank
Rank
Nonhomogeneous system

6.6 Exercises

1. Let $S = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4, \mathbf{v}_5\}$, where

$$\mathbf{v}_1 = (1, 2, 3), \quad \mathbf{v}_2 = (2, 1, 4), \quad \mathbf{v}_3 = (-1, -1, 2), \quad \mathbf{v}_4 = (0, 1, 2),$$

and $\mathbf{v}_5 = (1, 1, 1)$. Find a basis for the subspace $V = \text{span } S$ of R^3.

2. Let $S = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4, \mathbf{v}_5\}$, where

$$\mathbf{v}_1 = (1, 1, 2, 1), \quad \mathbf{v}_2 = (1, 0, -3, 1), \quad \mathbf{v}_3 = (0, 1, 1, 2), \quad \mathbf{v}_4 = (0, 0, 1, 1),$$

and $\mathbf{v}_5 = (1, 0, 0, 1)$. Find a basis for the subspace $V = \text{span } S$ of R^4.

3. Let $S = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4, \mathbf{v}_5\}$, where

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 2 \\ 1 \\ 2 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} 2 \\ 1 \\ 2 \\ 1 \end{bmatrix},$$

$$\mathbf{v}_3 = \begin{bmatrix} 3 \\ 2 \\ 3 \\ 2 \end{bmatrix}, \quad \mathbf{v}_4 = \begin{bmatrix} 3 \\ 3 \\ 3 \\ 3 \end{bmatrix}, \quad \text{and} \quad \mathbf{v}_5 = \begin{bmatrix} 5 \\ 3 \\ 5 \\ 3 \end{bmatrix}.$$

Find a basis for the subspace $V = \text{span } S$ of R^4.

4. Let $S = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4, \mathbf{v}_5\}$, where

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 2 \\ 1 \\ 1 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} 2 \\ 1 \\ 3 \\ 1 \end{bmatrix},$$

$$\mathbf{v}_3 = \begin{bmatrix} 0 \\ 2 \\ 1 \\ 2 \end{bmatrix}, \quad \mathbf{v}_4 = \begin{bmatrix} 3 \\ 2 \\ 1 \\ 4 \end{bmatrix}, \quad \text{and} \quad \mathbf{v}_5 = \begin{bmatrix} 5 \\ 0 \\ 0 \\ -1 \end{bmatrix}.$$

Find a basis for the subspace $V = \operatorname{span} S$ of R^4.

In Exercises 5 and 6, find a basis for the row space of A
(a) consisting of vectors that are not row vectors of A;
(b) consisting of vectors that are row vectors of A.

5. $A = \begin{bmatrix} 1 & 2 & -1 \\ 1 & 9 & -1 \\ [illegible] & [illegible] & 3 \\ -2 & [illegible] & 2 \end{bmatrix}$

6. $A = \begin{bmatrix} 1 & [illegible] & 1 & 3 \\ 3 & 5 & [illegible] & [illegible] \\ 0 & 1 & 2 & 1 \\ -1 & 0 & -2 & 7 \end{bmatrix}$

In Exercises 7 and 8, find a basis for the column space of A
(a) consisting of vectors that are not column vectors of A;
(b) consisting of vectors that are column vectors of A.

7. $A = \begin{bmatrix} 1 & -2 & 7 & 0 \\ 1 & -1 & 4 & 0 \\ 3 & 2 & -3 & 5 \\ 2 & 1 & -1 & 3 \end{bmatrix}$

8. $A = \begin{bmatrix} -2 & 2 & 3 & 7 & 1 \\ -2 & 2 & 4 & 8 & 0 \\ -3 & 3 & 2 & 8 & 4 \\ 4 & -2 & 1 & -5 & -7 \end{bmatrix}$

In Exercises 9 and 10, compute a basis for the row space of A, the column space of A, the row space of A^T, and the column space of A^T. Write a short description giving the relationships among these bases.

9. $A = \begin{bmatrix} 1 & -2 & 5 \\ 2 & 3 & 2 \\ 0 & -7 & 8 \end{bmatrix}$

10. $A = \begin{bmatrix} 2 & -3 & -7 & 11 \\ 3 & -1 & -7 & 13 \\ 1 & 2 & 0 & 2 \end{bmatrix}$

In Exercises 11 and 12, compute the row and column ranks of A, verifying Theorem 6.11.

11. $A = \begin{bmatrix} 1 & 2 & 3 & 2 & 1 \\ 3 & 1 & -5 & -2 & 1 \\ 7 & 8 & -1 & 2 & 5 \end{bmatrix}$

12. $A = \begin{bmatrix} 1 & 3 & 2 & 0 & 0 & 1 \\ 2 & 1 & -5 & 1 & [illegible] & 0 \\ 3 & 2 & 5 & 1 & -[illegible] & 1 \\ 5 & 8 & 9 & 1 & -2 & 2 \\ 9 & 9 & 4 & 2 & 0 & 2 \end{bmatrix}$

In Exercises 13 through 17, compute the rank and nullity of A and verify Theorem 6.12.

13. $A = \begin{bmatrix} 1 & 2 & 1 & 3 \\ 2 & 1 & -4 & -5 \\ 7 & 8 & -5 & -1 \\ 10 & 14 & -2 & 8 \end{bmatrix}$

14. $A = \begin{bmatrix} 1 & 2 & 1 & 3 \\ 2 & 1 & -4 & -5 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}$

15. $A = \begin{bmatrix} 1 & 2 & 3 \\ -1 & 2 & 1 \\ 3 & 1 & 2 \end{bmatrix}$ 16. $A = \begin{bmatrix} 1 & -2 & -1 \\ 2 & -1 & 3 \\ 7 & -8 & 3 \end{bmatrix}$

17. $A = \begin{bmatrix} 1 & -2 & -1 \\ 2 & -1 & 3 \\ 7 & -8 & 3 \\ 5 & -7 & 0 \end{bmatrix}$

18. If A is a 3×4 matrix, what is the largest possible value for rank A?

19. If A is a 4×6 matrix, show that the columns of A are linearly dependent.

20. If A is a 5×3 matrix, show that the rows of A are linearly dependent.

In Exercises 21 and 22, let A be a 7×3 matrix whose rank is 3.

21. Are the rows of A linearly dependent or linearly independent? Justify your answer.

22. Are the columns of A linearly dependent or linearly independent? Justify your answer.

In Exercises 23 through 25, use Theorem 6.13 to determine whether each matrix is singular or nonsingular.

23. $\begin{bmatrix} 1 & 2 & -3 \\ -1 & 2 & 3 \\ 0 & 8 & 0 \end{bmatrix}$ 24. $\begin{bmatrix} 1 & 2 & -3 \\ -1 & 2 & 3 \\ 0 & 1 & 1 \end{bmatrix}$

25. $\begin{bmatrix} 1 & 1 & 4 & -1 \\ 1 & 2 & 3 & 2 \\ -1 & 3 & 2 & 1 \\ -2 & 6 & 12 & -4 \end{bmatrix}$

In Exercises 26 and 27, use Corollary 6.3 to determine whether the linear system $A\mathbf{x} = \mathbf{b}$ has a unique solution for every 3×1 matrix $\mathbf{b}$.

26. $A = \begin{bmatrix} 1 & 2 & -2 \\ 0 & 8 & -7 \\ 3 & -2 & 1 \end{bmatrix}$

27. $A = \begin{bmatrix} 1 & -1 & 2 \\ 3 & 2 & 3 \\ 1 & -2 & 1 \end{bmatrix}$.

Use Corollary 6.4 to do Exercises 28 and 29.

28. Is

$$S = \left\{ \begin{bmatrix} 2 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 3 \end{bmatrix} \right\}$$

a linearly independent set of vectors in R^3?

29. Is

$$S = \left\{ \begin{bmatrix} 4 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 5 \\ -5 \end{bmatrix}, \begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix} \right\}$$

a linearly independent set of vectors in R^3?

In Exercises 30 through 32, find which homogeneous systems have a nontrivial solution for the given matrix A by using Corollary 6.5.

30. $A = \begin{bmatrix} 1 & 1 & 2 & -1 \\ 1 & 3 & -1 & 2 \\ 1 & 1 & 1 & 3 \\ 1 & 2 & 1 & 1 \end{bmatrix}$

31. $A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 0 \\ 1 & 0 & 3 \end{bmatrix}$

32. $A = \begin{bmatrix} 1 & 2 & -1 \\ 2 & -1 & 3 \\ 5 & -4 & 3 \end{bmatrix}$

In Exercises 33 through 36, determine which of the linear systems have a solution by using Theorem 6.14.

33. $\begin{bmatrix} 1 & 2 & 5 & -2 \\ 2 & 3 & -2 & 4 \\ 5 & 1 & 0 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

34. $\begin{bmatrix} 1 & 2 & 5 & -2 \\ 2 & 3 & -2 & 4 \\ 5 & 1 & 0 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 1 \\ -13 \\ 3 \end{bmatrix}$

35. $\begin{bmatrix} 1 & -2 & -3 & 4 \\ 4 & -1 & -5 & 6 \\ 2 & 3 & 1 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}$

36. $\begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & 1 \\ 5 & 1 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 6 \\ 2 \\ 5 \end{bmatrix}$

In Exercises 37 through 40, compute the rank of the given bit matrix.

37. $\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix}$

38. $\begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 \end{bmatrix}$

39. $\begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix}$

40. $\begin{bmatrix} 1 & 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 & 1 \end{bmatrix}$

Theoretical Exercises

T.1. Prove Corollary 6.2.

T.2. Prove Corollary 6.3.

T.3. Prove Corollary 6.4.

T.4. Let A be an $n \times n$ matrix. Show that the homogeneous system $A\mathbf{x} = \mathbf{0}$ has a nontrivial solution if and only if the columns of A are linearly dependent.

T.5. Let A be an $n \times n$ matrix. Show that rank $A = n$ if and only if the columns of A are linearly independent.

T.6. Let A be an $n \times n$ matrix. Show that the rows of A are linearly independent if and only if the columns of A span R^n.

T.7. Let A be an $m \times n$ matrix. Show that the linear system $A\mathbf{x} = \mathbf{b}$ has a solution for every $m \times 1$ matrix $\mathbf{b}$ if and only if rank $A = m$.

T.8. Let A be an $m \times n$ matrix. Show that the columns of A are linearly independent if and only if the homogeneous system $A\mathbf{x} = \mathbf{0}$ has only the trivial solution.

T.9. Let A be an $m \times n$ matrix. Show that the linear system $A\mathbf{x} = \mathbf{b}$ has at most one solution for every $m \times 1$ matrix $\mathbf{b}$ if and only if the associated homogeneous system $A\mathbf{x} = \mathbf{0}$ has only the trivial solution.

T.10. Let A be an $m \times n$ matrix with $m \neq n$. Show that either the rows or the columns of A are linearly dependent.

T.11. Suppose that the linear system $A\mathbf{x} = \mathbf{b}$, where A is $m \times n$, is consistent (has a solution). Show that the solution is unique if and only if rank $A = n$.

T.12. Show that if A is an $m \times n$ matrix such that AA^T is nonsingular, then rank $A = m$.

Ex 6.6

Q1 let $S = \{V_1, V_2, V_3, V_4, V_5\}$ where.

$V_1 = (1, 2, 3)$'
$V_2 = (2, 1, 4)$'
$V_3 = (-1, -1, 2)$.
$V_4 = (0, 1, 2)$. & $V_5 = (1, 1, 1)$

Note that V is the row space of matrix A whose rows are the given vectors

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 1 & 4 \\ -1 & -1 & 2 \\ 0 & 1 & 2 \\ 1 & 1 & 1 \end{bmatrix}.$$

$$\sim \begin{bmatrix} 1 & 2 & 3 \\ 0 & -3 & -2 \\ 0 & 1 & 5 \\ 0 & 1 & 2 \\ 0 & -1 & -2 \end{bmatrix} \begin{array}{l} \\ -2R_1 + R_2 \\ R_1 + R_3 \\ -R_1 + R_5 \\ \\ \end{array}$$

$$\sim \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 5 \\ 0 & -3 & -2 \\ 0 & 1 & 2 \\ 0 & -1 & -2 \end{bmatrix} R_{23}$$

$$\sim \begin{bmatrix} 1 & 0 & -7 \\ 0 & 1 & 5 \\ 0 & 0 & 4 \\ 0 & 1 & 2 \\ 0 & -1 & -2 \end{bmatrix} \begin{array}{l} -2R_2+R_1 \\ 3R_4+R_3 \end{array}$$

$$\sim \begin{bmatrix} 1 & 0 & -7 \\ 0 & 1 & 5 \\ 0 & 0 & 4 \\ 0 & 0 & -3 \\ 0 & 0 & 0 \end{bmatrix} \begin{array}{l} -R_2+R_4 \\ R_4+R_5 \end{array}$$

$$\sim \begin{bmatrix} 1 & 0 & -7 \\ 0 & 1 & 5 \\ 0 & 0 & 1 \\ 0 & 0 & -3 \\ 0 & 0 & 0 \end{bmatrix} R_3/4$$

$$\sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{array}{l} -5R_3+R_2 \\ 7R_3+R_1 \\ 3R_3+R_4 \end{array}$$

Which is reduced row echelon form. ~~the vectors~~

~~Now~~ Here $w_1 = (1,0,0)$
$w_2 = (0,1,0)$
$w_3 = (0,0,1)$ from a basis ~~of~~ for V.

Ex. 6.6

Q_2 is Similarly to Q_1

$\underline{Q_3}$

$V_1 = \begin{bmatrix} 1 \\ 2 \\ 1 \\ 2 \end{bmatrix} \quad V_3 = \begin{bmatrix} 3 \\ 2 \\ 3 \\ 2 \end{bmatrix} \quad V_5 = \begin{bmatrix} 5 \\ 3 \\ 5 \\ 3 \end{bmatrix}$

$V_2 = \begin{bmatrix} 2 \\ 1 \\ 2 \\ 1 \end{bmatrix} \quad V_4 = \begin{bmatrix} 3 \\ 3 \\ 3 \\ 3 \end{bmatrix}$

Now

$$A = \begin{bmatrix} 1 & 2 & 1 & 2 \\ 2 & 1 & 2 & 1 \\ 3 & 2 & 3 & 2 \\ 3 & 3 & 3 & 3 \\ 5 & 3 & 5 & 3 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 2 & 3 & 2 \\ 0 & -3 & 0 & -3 \\ 0 & -4 & 0 & -4 \\ 0 & -3 & 0 & -3 \\ 0 & -7 & 0 & -7 \end{bmatrix} \begin{matrix} -2R_1+R_2 \\ -3R_1+R_3 \\ -3R_1+R_4 \\ -5R_1+R_5 \end{matrix}$$

$$\sim \begin{bmatrix} 1 & 2 & 3 & 2 \\ 0 & 1 & 0 & 1 \\ 0 & -4 & 0 & -4 \\ 0 & -3 & 0 & -3 \\ 0 & -7 & 0 & -7 \end{bmatrix} -\tfrac{1}{3}R_2$$

$$\sim \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{matrix} -2R_2+R_1 \\ 4R_2+R_3 \\ 3R_2+R_4 \\ 7R_2+R_5 \end{matrix}$$

Hence Possible solution $\left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix} \right\}$

Q4

$$A = \begin{bmatrix} 1 & 2 & 1 & 1 \\ 2 & 1 & 3 & 2 \\ 0 & 2 & 1 & 2 \\ 3 & 2 & 1 & 4 \\ 5 & 0 & 0 & -1 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 2 & 1 & 1 \\ 0 & -3 & 1 & 0 \\ 0 & 2 & 1 & 2 \\ 0 & -4 & -2 & 1 \\ 0 & -10 & -5 & -6 \end{bmatrix} \quad \begin{matrix} R_2 - 2R_1 \\ R_4 - 3R_1 \\ R_5 - 5R_1 \end{matrix}$$

$$\sim \begin{bmatrix} 1 & 2 & 1 & 1 \\ 0 & 1 & -2 & -2 \\ 0 & 2 & 1 & 2 \\ 0 & -4 & -2 & 1 \\ 0 & -10 & -5 & -6 \end{bmatrix} \quad -(R_2 + R_3)$$

$$\sim \begin{bmatrix} 1 & 0 & 5 & 5 \\ 0 & 1 & -2 & -2 \\ 0 & 0 & 5 & 6 \\ 0 & 0 & -10 & -7 \\ 0 & 0 & -25 & -26 \end{bmatrix} \quad \begin{matrix} R_1 - 2R_2 \\ R_3 - 2R_2 \\ R_4 + 4R_2 \\ R_5 + 10R_1 \end{matrix}$$

$$\sim \begin{bmatrix} 1 & 0 & 5 & 5 \\ 0 & 1 & -2 & -2 \\ 0 & 0 & 1 & 6/5 \\ 0 & 0 & -10 & -7 \\ 0 & 0 & -25 & -26 \end{bmatrix} \quad R_3/5$$

Ex 6.6

$$\sim \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 2/5 \\ 0 & 0 & 1 & 6/5 \\ 0 & 0 & 0 & 5 \\ 0 & 0 & 0 & 4 \end{bmatrix} \begin{matrix} R_4+10R_3 \\ R_5+25R_3 \\ R_2+2R_3 \\ R_1-5R_3 \end{matrix}$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 2/5 \\ 0 & 0 & 1 & 6/5 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 4 \end{bmatrix} \tfrac{1}{5}R_4$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{matrix} R_5-4R_4 \\ R_3-6/5R_4 \\ R_2-2/5R_4 \\ R_1+R_4 \end{matrix}$$

So $w_1=(1,0,0,0)$

$w_2=(0,1,0,0)$

$w_3=(0,0,1,0)$

$w_4=(0,0,0,1)$ form a basis for V

Q5 (a) consisting of vector that are not row vectors of A

$$A = \begin{bmatrix} 1 & 2 & -1 \\ 1 & 9 & -1 \\ -3 & 8 & 3 \\ -2 & 3 & 2 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 2 & -1 \\ 0 & 7 & 0 \\ 0 & 14 & 0 \\ 0 & 7 & 0 \end{bmatrix} \begin{matrix} -R_1+R_2 \\ 3R_1+R_3 \\ 2R_1+R_4 \end{matrix}$$

$$\sim \begin{bmatrix} 1 & 2 & -1 \\ 0 & 1 & 0 \\ 0 & 14 & 0 \\ 0 & 7 & 0 \end{bmatrix} \tfrac{1}{7}R_2$$

$$\sim \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{matrix} -2R_2+R_1 \\ -14R_2+R_3 \\ -7R_2+R_1 \end{matrix}$$

Then $w_1 = (1, 0, -1)$

$w_2 = (0, 1, 0)$ is basis for A.

~~$w_3 =$~~

(b) consisting of vector that are row vector of A

(4)

Ex 6.6

whose augmented matrix is

$$\left[\begin{array}{cccc|c} 1 & 1 & -3 & -2 & 0 \\ 2 & 9 & 8 & 3 & 0 \\ -1 & -1 & 3 & 2 & 0 \end{array}\right] = [A^T \vdots 0].$$

i.e. the coefficient matrix is A^T. Transforming the augmented matrix $[A^T \vdots 0]$ is (as) to reduced row echelon form we obtain

$$\sim \left[\begin{array}{cccc|c} 1 & 1 & -3 & -2 & 0 \\ 0 & 7 & 14 & 7 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \begin{array}{l} -2R_1+R_2 \\ R_1+R_3 \end{array}$$

$$\sim \left[\begin{array}{cccc|c} 1 & 1 & -3 & -2 & 0 \\ 0 & 1 & 2 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] R_2/7$$

$$\sim \left[\begin{array}{cccc|c} 1 & 0 & -5 & -3 & 0 \\ 0 & 1 & 2 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] -R_2+R_1$$

Since the leading 1s in colum 1, 2, we conclude that the 1st two rows of A form a basis for the row space A that is $\{(1,2,-1),(1,9,-1)\}$.

Q6 is similarly to Q5.

Q7 $A = \begin{bmatrix} 1 & -2 & 7 & 0 \\ 1 & -1 & 4 & 0 \\ 3 & 2 & -3 & 5 \\ 2 & 1 & -1 & 3 \end{bmatrix}$

(a) $\begin{bmatrix} 1 & 1 & 3 & 2 \\ -2 & -1 & 2 & 1 \\ 7 & 4 & -3 & -1 \\ 0 & 0 & 5 & 3 \end{bmatrix}$

$\sim \begin{bmatrix} 1 & 1 & 3 & 2 \\ 0 & 1 & 8 & 5 \\ 7 & 4 & -3 & -1 \\ 0 & 0 & 5 & 3 \end{bmatrix} 2R_1+R_2$

$\sim \begin{bmatrix} 1 & 1 & 3 & 2 \\ 0 & 1 & 8 & 5 \\ 0 & -3 & -24 & -15 \\ 0 & 0 & 5 & 3 \end{bmatrix} -7R_1+R_3$

$\sim \begin{bmatrix} 1 & 0 & -5 & -3 \\ 0 & 1 & 8 & 5 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 3/5 \end{bmatrix} \begin{matrix} -R_2+R_1 \\ 3R_2+R_3 \\ 1/5\, R_4. \end{matrix}$

$\sim \begin{bmatrix} 1 & 0 & -5 & -3 \\ 0 & 1 & 0 & 1/5 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 3/5 \end{bmatrix} -8R_4+R_2$

Scanned with CamScanner

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1/5 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 3/5 \end{bmatrix} 5R_4 + R_1$$

Hence, $\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1/5 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 3/5 \end{bmatrix} \right\}$ Ans.

is the basis for the column space A.

(b) when augmented matrix is

$$\left[\begin{array}{cccc|c} 1 & -2 & 7 & 0 & 0 \\ 1 & -1 & 4 & 0 & 0 \\ 3 & 2 & -3 & 5 & 0 \\ 2 & 1 & -1 & 3 & 0 \end{array}\right] = [A^T \,\vdots\, 0]$$

i.e the coefficient matrix is A^T. Transforming the augmented matrix $[A^T \vdots 0]$ is (3) to reduced row echelon from we obtain

$$\sim \left[\begin{array}{cccc|c} 1 & -2 & 7 & 0 & 0 \\ 0 & 1 & -3 & 0 & 0 \\ 0 & 8 & -24 & 5 & 0 \\ 0 & 5 & -15 & 3 & 0 \end{array}\right] \begin{array}{l} -R_1+R_2 \\ -3R_1+R_3 \\ -2R_1+R_4 \end{array}$$

$$\sim \left[\begin{array}{cccc|c} 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & -3 & 0 & 0 \\ 0 & 0 & 0 & 5 & 0 \\ 0 & 0 & 0 & 3 & 0 \end{array}\right] \begin{array}{l} -8R_2+R_3 \\ -5R_2+R_4 \end{array}$$

$$\sim \left[\begin{array}{cccc|c} 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & -3 & 0 & 0 \\ 0 & 0 & 0 & 5 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{array}\right] \quad \tfrac{1}{3}R_4$$

$$\sim \left[\begin{array}{cccc|c} 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & -3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{array}\right] \quad -5R_4+R_3$$

Since the leading 1's in colum 1, 2 & 4 we conclude that the 1st two rows of A form a basis for the row space A that is

$$\left\{ \begin{bmatrix} 1 \\ 1 \\ 3 \\ 2 \end{bmatrix}, \begin{bmatrix} -2 \\ -1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 5 \\ 3 \end{bmatrix} \right\}$$

Q8 is similarly to Q7

— X —

Jamal Nasir

Lecturer U.E.T Peshawar.

Scanned with CamScanner

Ex 6.6

Q9 $A = \begin{bmatrix} 1 & -2 & 5 \\ 2 & 3 & 2 \\ 0 & -7 & 8 \end{bmatrix}$

(a) basis for row space of A

$$\sim \begin{bmatrix} 1 & -2 & 5 \\ 0 & 7 & -8 \\ 0 & -7 & 8 \end{bmatrix} R_2 - 2R_1$$

$$\sim \begin{bmatrix} 1 & -2 & 5 \\ 0 & 1 & -8/7 \\ 0 & -7 & 8 \end{bmatrix} \frac{1}{7}R_2$$

$$\sim \begin{bmatrix} 1 & 0 & 19/7 \\ 0 & 1 & -8/7 \\ 0 & 0 & 0 \end{bmatrix} \begin{matrix} R_1 + 2R_2 \\ R_3 + 7R_2 \end{matrix}$$

Basis for row space of A

$A = \{[1, 0, 19/7], [0, 1, -8/7]\}$.

(b) Basis for column space of A

$$A^T = \begin{bmatrix} 1 & 2 & 0 \\ -2 & 3 & -7 \\ 5 & 2 & 8 \end{bmatrix}.$$

Augmented matrix is

$$\left[\begin{array}{ccc|c} 1 & -2 & 5 & 0 \\ 2 & 3 & 2 & 0 \\ 0 & -7 & 8 & 0 \end{array}\right] \quad [A : 0]$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 19/7 & 0 \\ 0 & 1 & -8/7 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$ (From 1st part(a))

As leading 1s occur in col 1 & 2 so 1st & 2nd col of A^T form basis

$$\begin{bmatrix} 1 \\ 0 \\ 19/7 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 8/9 \end{bmatrix}$$

(c) Basis for row space of A^T

$$\left[\begin{array}{ccc|c} 1 & 2 & 0 & 0 \\ -2 & 3 & -7 & 0 \\ 5 & 2 & 8 & 0 \end{array}\right] \quad [A^T \,|\, 0]$$

$$\sim \left[\begin{array}{ccc|c} 1 & 2 & 0 & 0 \\ 0 & 7 & -7 & 0 \\ 0 & -8 & 8 & 0 \end{array}\right] \begin{matrix} R_2+2R_1 \\ R_3-5R_1 \end{matrix}$$

$$\sim \left[\begin{array}{ccc|c} 1 & 2 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & -8 & 8 & 0 \end{array}\right] \frac{1}{7}R_2$$

$$\sim \left[\begin{array}{ccc|c} 1 & 0 & 2 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \begin{matrix} R_1-2R_2 \\ R_3+8R_2 \end{matrix}$$

$\{[1 . 0\ 2], [0\ 1\ -1]\}$ form Basis

Ex 6.6

(d) colum space of A

$$\begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix} : [A:0]$$

Basis for col space of A

$\left\{ \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} \right\}$ form Basis

---

Q10 is similarly to Q 9.

Q11 $A = \begin{bmatrix} 1 & 2 & 3 & 2 & 1 \\ 3 & 1 & -5 & -2 & 1 \\ 7 & 8 & -1 & 2 & 5 \end{bmatrix}$

augmented matrix is

$$\left[\begin{array}{ccc|c} 1 & 3 & 7 & 0 \\ 2 & 1 & 8 & 0 \\ 3 & -5 & -1 & 0 \\ 2 & -2 & 2 & 0 \\ 1 & 1 & 5 & 0 \end{array}\right] = [A^T \mid 0]$$

$$\sim \left[\begin{array}{ccc|c} 1 & 3 & 7 & 0 \\ 0 & -5 & -6 & 0 \\ 0 & -14 & -22 & 0 \\ 0 & -8 & -12 & 0 \\ 0 & -2 & -2 & 0 \end{array}\right] \begin{array}{l} \\ -2R_1+R_2 \\ -3R_1+R_3 \\ -2R_1+R_4 \\ -R_1+R_5 \end{array}$$

$$\sim \left[\begin{array}{ccc|c} 1 & 3 & 7 & 0 \\ 0 & 1 & 6/5 & 0 \\ 0 & -14 & -22 & 0 \\ 0 & -8 & -12 & 0 \\ 0 & -2 & -2 & 0 \end{array}\right] -\tfrac{1}{5}R_2$$

$$\sim \left[\begin{array}{ccc|c} 1 & 0 & 17/5 & 0 \\ 0 & 1 & 6/5 & 0 \\ 0 & 0 & -8 & 0 \\ 0 & -8 & -12 & 0 \\ 0 & -2 & -2 & 0 \end{array}\right] \begin{array}{l} -3R_2+R_1 \\ -7R_5+R_3 \end{array}$$

$$\sim \left[\begin{array}{ccc|c} 1 & 0 & 17/5 & 0 \\ 0 & 1 & 6/5 & 0 \\ 0 & 0 & -8 & 0 \\ 0 & 0 & -4 & \\ 0 & -2 & -2 & 0 \end{array}\right] -4R_5+R_4$$

$$\sim \left[\begin{array}{ccc|c} 1 & 0 & 17/5 & 0 \\ 0 & 1 & 6/5 & 0 \\ 0 & 0 & -8 & 0 \\ 0 & 0 & -4 & 0 \\ 0 & 0 & 2/5 & 0 \end{array}\right] 2R_3+R_5$$

$$\sim \left[\begin{array}{ccc|c} 1 & 0 & 17/5 & 0 \\ 0 & 1 & 6/5 & 0 \\ 0 & 0 & -8 & 0 \\ 0 & 0 & -4 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right] \tfrac{5}{2}R_5$$

$$\sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right] \begin{array}{l} 4R_5+R_4 \\ 8R_5+R_3 \\ -6/5\,R_5+R_2 \\ -17/5\,R_5+R_1 \end{array}$$

Since the non-zero row is 3 so rank = 3.

(b) whose augmented matrix is $[A \mid 0]$

$$\left[\begin{array}{ccccc|c} 1 & 2 & 3 & 2 & 1 & 0 \\ 3 & 1 & -5 & -2 & 1 & 0 \\ 7 & 8 & -1 & 2 & 5 & 0 \end{array}\right] = [A \mid 0]$$

$$\sim \left[\begin{array}{ccccc|c} 1 & 2 & 3 & 2 & 1 & 0 \\ 0 & -5 & -11 & -8 & -2 & 0 \\ 0 & -6 & -22 & -12 & -2 & 0 \end{array}\right] \begin{array}{l} -3R_1+R_2 \\ -7R_1+R_3 \end{array}$$

$$\sim \left[\begin{array}{ccccc|c} 1 & 2 & 3 & 2 & 1 & 0 \\ 0 & 1 & 11/5 & 8/5 & 2/5 & 0 \\ 0 & -6 & -22 & -12 & -2 & 0 \end{array}\right] -1/5\,R_2$$

$$\sim \left[\begin{array}{ccccc|c} 1 & 0 & -7/5 & -6/5 & 1/5 & 0 \\ 0 & 1 & 11/5 & 8/5 & 2/5 & 0 \\ 0 & -6 & -22 & -12 & -2 & 0 \end{array}\right] -2R_2+R_1$$

$$\sim \left[\begin{array}{ccccc|c} 1 & 0 & -7/5 & -6/5 & 1/5 & 0 \\ 0 & 1 & 11/5 & 8/5 & 2/5 & 0 \\ 0 & 0 & -45/5 & -12/5 & 2/5 & 0 \end{array}\right] \quad 6R_2+R_3$$

$$\sim \left[\begin{array}{ccccc|c} 1 & 0 & -7/5 & -6/5 & 1/5 & 0 \\ 0 & 1 & 11/5 & 8/5 & 2/5 & 0 \\ 0 & 0 & 1 & 12/45 & -2/45 & 0 \end{array}\right] \quad -5/45\, R_3$$

$$\sim \left[\begin{array}{ccccc|c} 1 & 0 & 0 & -62/75 & 31/225 & 0 \\ 0 & 1 & 0 & 76/75 & 68/225 & 0 \\ 0 & 0 & 1 & 12/45 & -2/45 & 0 \end{array}\right] \quad \begin{array}{l} -\frac{11}{5}R_3+R_2 \\ \frac{7}{5}R_3+R_1 \end{array}$$

So non-zero row = 3 rank = 3

Row rank = column rank = 3.

Q12 is similarly to Q11

Jamal Nasir

Lecturer U.E.T Peshawar

Ex 6.6

Q15 $A = \begin{bmatrix} 1 & 2 & 3 \\ -1 & 2 & 1 \\ 3 & 1 & 2 \end{bmatrix}$

$\sim \begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 4 \\ 0 & -5 & -7 \end{bmatrix}$ R_2+R_1, R_3-3R_1

$\sim \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 1 \\ 0 & -5 & -7 \end{bmatrix}$ $\frac{1}{4}R_2$

$\sim \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 1 \\ 0 & 0 & -2 \end{bmatrix}$ R_3+5R_2

$\sim \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & 1 \\ 0 & 0 & -2 \end{bmatrix}$ R_1-2R_2

Rank of A = No: of non-row = 3.

So rank = 3.

For nullity $Ax = 0$

$\left[\begin{array}{ccc|c} 1 & 0 & 3 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & -2 & 0 \end{array}\right]$

$-2x_3 = 0 \Rightarrow \boxed{x_3 = 0}$

$x_2 + x_3 = 0 \Rightarrow \boxed{x_2 = 0}$

$x_1 + 3x_3 = 0 \Rightarrow \boxed{x_1 = 0}$

Q13, Q14, Q16, Q_{17} is similarly to Q_{15}

Q26 If Rank $A = n = 3$
Then there is unique solution

$$A = \begin{bmatrix} 1 & 2 & -2 \\ 0 & 8 & -7 \\ 3 & -2 & 1 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 2 & -2 \\ 0 & 8 & -7 \\ 0 & -8 & 7 \end{bmatrix} R_3 - 3R_1$$

$$\sim \begin{bmatrix} 1 & 2 & -2 \\ 0 & 1 & -7/8 \\ 0 & -8 & 7 \end{bmatrix} R_2/8$$

$$\sim \begin{bmatrix} 1 & 2 & -2 \\ 0 & 1 & -7/8 \\ 0 & 0 & 0 \end{bmatrix} R_3 + 8R_2$$

Non-zero row is 2

So rank = 2

No unique solution

Q 27 is same as Q 26

Ex 6.6

Q28. Linearly Independent holds when det $\neq 0$

$$\begin{vmatrix} 2 & 1 & 0 \\ 2 & 0 & 1 \\ 3 & 2 & 3 \end{vmatrix}$$

$2(0-2)-1(6-3)+0(4-0)$

$-4-(3)+0$

$-4-3$

$-7 \neq 0$ so linearly independent

Q29 is same as Q28

Q31 Non-trivial solution occurs when rank $< n$.

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 0 \\ 1 & 0 & 3 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 0 \\ 0 & -2 & 0 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \; 2R_2+R_3$$

Rank A < 3 so non-trivial.

Q 30 & Q 32 is same as Q 31

Q33 Solution exist when

Rank A = Rank of [A:b].

$$\left[\begin{array}{cccc|c} 1 & 2 & 5 & -2 & 0 \\ 2 & 3 & -2 & 4 & 0 \\ 5 & 1 & 0 & 2 & 0 \end{array}\right]$$

$$\sim \left[\begin{array}{cccc|c} 1 & 2 & 5 & -2 & 0 \\ 0 & -1 & -12 & 8 & 0 \\ 0 & -9 & -25 & 12 & 0 \end{array}\right] \begin{array}{l} R_2-2R_1 \\ R_3-5R_1 \end{array}$$

$$\sim \left[\begin{array}{cccc|c} 1 & 2 & 5 & -2 & 0 \\ 0 & 1 & 12 & -8 & 0 \\ 0 & -9 & -25 & 12 & 0 \end{array}\right] -R_2$$

$$\sim \left[\begin{array}{cccc|c} 1 & 0 & -19 & 14 & 0 \\ 0 & 1 & 12 & -8 & 0 \\ 0 & 0 & 83 & -60 & 0 \end{array}\right] \begin{array}{l} R_3+9R_2 \\ R_1-2R_2 \end{array}$$

Rank of A = Rank of [A:0]

$\Rightarrow$ So it has Solution

(11)

Ex 6.6

Q35 solution exist when

Rank of A = Rank of [A:b]

$$\left[\begin{array}{cccc|c} 1 & -2 & -3 & 4 & 1 \\ 4 & -1 & -5 & 6 & 2 \\ 2 & 3 & 1 & -2 & 2 \end{array}\right]$$

$$\sim \left[\begin{array}{cccc|c} 1 & -2 & -3 & 4 & 1 \\ 0 & 7 & 7 & -10 & -2 \\ 0 & 7 & 7 & -10 & 0 \end{array}\right] \begin{array}{l} R_2 - 4R_1 \\ R_3 - 2R_1 \end{array}$$

$$\sim \left[\begin{array}{cccc|c} 1 & -2 & -3 & 4 & 1 \\ 0 & 7 & 7 & -10 & -2 \\ 0 & 0 & 0 & 0 & 2 \end{array}\right] R_3 - R_1$$

Rank of A = 3 & Rank of [A:b] = 2

Rank of A $\neq$ Rank of [A:b]

$\Rightarrow$ so it's has no solutio

Q 34, Q 36 is similarly

Q37

bit matrix

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix} R_3 + R_1$$

$$\sim \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} R_3 + R_2$$

Non-zero row = 2

So rank = 2.

Q38

$$\begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \end{bmatrix} R_4 + R_1$$

$$\sim \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix} \begin{matrix} R_3 + R_2 \\ R_4 + R_2 \end{matrix}$$

$$\sim \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} R_4 + R_3$$

non-zero row = 3 $\Rightarrow$ rank = 3

Q39 & Q40 similarly.

Key Terms

Eigenvalue
Eigenvector
Proper value
Characteristic value
Latent value
Characteristic polynomial
Characteristic equation
Roots of the characteristic polynomial
Eigenspace
Leslie matrix
Stable age distribution
Invariant subspace

8.1 Exercises

1. Let $A = \begin{bmatrix} 3 & -1 \\ -2 & 2 \end{bmatrix}$.

(a) Verify that $\lambda_1 = 1$ is an eigenvalue of A and $\mathbf{x}_1 = \begin{bmatrix} r \\ 2r \end{bmatrix}$, $r \neq 0$, is an associated eigenvector.

(b) Verify that $\lambda_2 = 4$ is an eigenvalue of A and $\mathbf{x}_2 = \begin{bmatrix} r \\ -r \end{bmatrix}$, $r \neq 0$, is an associated eigenvector.

2. Let $A = \begin{bmatrix} 2 & 2 & 3 \\ 1 & 2 & 1 \\ 2 & -2 & 1 \end{bmatrix}$.

(a) Verify that $\lambda_1 = -1$ is an eigenvalue of A and $\mathbf{x}_1 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$ is an associated eigenvector.

(b) Verify that $\lambda_2 = 2$ is an eigenvalue of A and $\mathbf{x}_2 = \begin{bmatrix} -2 \\ -3 \\ 2 \end{bmatrix}$ is an associated eigenvector.

(c) Verify that $\lambda_3 = 4$ is an eigenvalue of A and $\mathbf{x}_3 = \begin{bmatrix} 8 \\ 5 \\ 2 \end{bmatrix}$ is an associated eigenvector.

In Exercises 3 through 7, find the characteristic polynomial of each matrix.

3. $\begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 2 \\ -1 & 3 & 2 \end{bmatrix}$ **4.** $\begin{bmatrix} 2 & 1 \\ -1 & 3 \end{bmatrix}$

5. $\begin{bmatrix} 4 & -1 & 3 \\ 0 & 2 & 1 \\ 0 & 0 & 3 \end{bmatrix}$ **6.** $\begin{bmatrix} 4 & 2 \\ 3 & 3 \end{bmatrix}$

7. $\begin{bmatrix} 2 & 1 & 2 \\ 2 & 2 & -2 \\ 3 & 1 & 1 \end{bmatrix}$

In Exercises 8 through 15, find the characteristic polynomial, eigenvalues, and eigenvectors of each matrix.

8. $\begin{bmatrix} 0 & 1 & 2 \\ 0 & 0 & 3 \\ 0 & 0 & 0 \end{bmatrix}$ **9.** $\begin{bmatrix} 1 & 0 & 0 \\ -1 & 3 & 0 \\ 3 & 2 & -2 \end{bmatrix}$

10. $\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ **11.** $\begin{bmatrix} 1 & -1 \\ 2 & 4 \end{bmatrix}$

12. $\begin{bmatrix} 2 & -2 & 3 \\ 0 & 3 & -2 \\ 0 & -1 & 2 \end{bmatrix}$ **13.** $\begin{bmatrix} 2 & 2 & 3 \\ 1 & 2 & 1 \\ 2 & -2 & 1 \end{bmatrix}$

14. $\begin{bmatrix} 2 & 0 & 0 \\ 3 & -1 & 0 \\ 0 & 4 & 3 \end{bmatrix}$ **15.** $\begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & -1 & 3 & 2 \\ 0 & 0 & 3 & 3 \\ 0 & 0 & 0 & 2 \end{bmatrix}$

16. Find the characteristic polynomial, the eigenvalues and associated eigenvectors of each of the following matrices.

(a) $\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ (b) $\begin{bmatrix} -2 & -4 & -8 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$

(c) $\begin{bmatrix} 2-i & 2i & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$ (d) $\begin{bmatrix} 5 & 2 \\ -1 & 3 \end{bmatrix}$

17. Find all the eigenvalues and associated eigenvectors of each of the following matrices.

(a) $\begin{bmatrix} -1 & -1+i \\ 1 & 0 \end{bmatrix}$ (b) $\begin{bmatrix} i & 1 & 0 \\ 1 & i & 0 \\ 0 & 0 & 1 \end{bmatrix}$

(c) $\begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$ (d) $\begin{bmatrix} 0 & 0 & -9 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$

In Exercises 18 and 19, find bases for the eigenspaces (see Exercise T.1) associated with each eigenvalue.

18. $\begin{bmatrix} 2 & 3 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$ **19.** $\begin{bmatrix} 2 & 2 & 3 & 4 \\ 0 & 2 & 3 & 2 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

In Exercises 20 through 23, find a basis for the eigenspace (see Exercise T.1) associated with λ.

20. $\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$, $\lambda = 1$ **21.** $\begin{bmatrix} 2 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 2 \end{bmatrix}$, $\lambda = 2$

22. $\begin{bmatrix} 3 & 0 & 0 \\ -2 & 3 & -2 \\ 2 & 0 & 5 \end{bmatrix}$, $\lambda = 3$

23. $\begin{bmatrix} 4 & 2 & 0 & 0 \\ 3 & 3 & 0 & 0 \\ 0 & 0 & 2 & 5 \\ 0 & 0 & 0 & 2 \end{bmatrix}$, $\lambda = 2$

Exer: 8.1

Q1 let $A = \begin{bmatrix} 3 & -1 \\ -2 & 2 \end{bmatrix}$.

For eigenvalues

$\det(\lambda I_n - A) = 0$

$\det\left(\lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 3 & -1 \\ -2 & 2 \end{bmatrix}\right) = 0$

$\det\left(\begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} - \begin{bmatrix} 3 & -1 \\ -2 & 2 \end{bmatrix}\right) = 0$

$\det\left(\begin{bmatrix} \lambda-3 & 1 \\ 2 & \lambda-2 \end{bmatrix}\right) = 0$

$(\lambda-3)(\lambda-2) - 2 = 0$

$\lambda^2 - 2\lambda - 3\lambda + 6 - 2 = 0$

$\lambda^2 - 5\lambda + 4 = 0$

$\boxed{\lambda = 1}$ $\boxed{\lambda = 4}$

For eigenvector

$(\lambda I_n - A)X = 0$

For $\lambda = 1$

$1\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 3 & -1 \\ -2 & 2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 3 & -1 \\ -2 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

$$\begin{bmatrix} -2 & 1 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

$-2x_1 + x_2 = 0$

$2x_1 - x_2 = 0$

$x_1 = \frac{1}{2} x_2$ $\qquad X = \begin{bmatrix} \frac{1}{2}r \\ r \end{bmatrix}.$

$x_2 = r$ (Any real no's)

$\boxed{\text{For } \lambda = 4}$

$$4\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 3 & -1 \\ -2 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

$$\begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix} - \begin{bmatrix} 3 & -1 \\ -2 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

$$\begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

$x_1 + x_2 = 0$

$2x_1 + 2x_2 = 0.$

$x_1 = -x_2$

$x_2 = r$ (any real no's)

$$X = \begin{bmatrix} -r \\ r \end{bmatrix}$$

Ex 8.1

Q3 let $A = \begin{bmatrix} 2 & 2 & 3 \\ 1 & 2 & 1 \\ 2 & -2 & 1 \end{bmatrix}$.

For eigenvalues.

$\det(\lambda I_n - A) = 0$

$$\det\left(\lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 2 & 2 & 3 \\ 1 & 2 & 1 \\ 2 & -2 & 1 \end{bmatrix}\right) = 0$$

$$\det\left(\begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix} - \begin{bmatrix} 2 & 2 & 3 \\ 1 & 2 & 1 \\ 2 & -2 & 1 \end{bmatrix}\right) = 0$$

$$\det\left(\begin{bmatrix} \lambda-2 & -2 & -3 \\ -1 & \lambda-2 & -1 \\ -2 & 2 & \lambda-1 \end{bmatrix}\right) = 0$$

$$\lambda-2\left[(\lambda-1)(\lambda-2)+2\right]+2\left[-(\lambda-1)-2\right]-3\left[-2+2(\lambda-2)\right]=0$$

$$\Rightarrow (\lambda-2)(\lambda^2-3\lambda+4)+(-2\lambda-2)+(-6\lambda+18)=0$$

$$\lambda^3-5\lambda^2+2\lambda+8=0$$

$$\lambda=-1$$

S.D

$$\begin{array}{r|rrrr} -1 & 1 & -5 & 2 & 8 \\ & & -1 & 6 & -8 \\ \hline & 1 & -6 & 8 & \boxed{0} \end{array}$$

$$(\lambda+1)(\lambda^2-6\lambda+8)=0$$

$\lambda+1=0$

$\lambda=-1$

$\lambda^2-6\lambda+8=0$

$\lambda^2-4\lambda-2\lambda+8=0$

$\lambda(\lambda-4)-2(\lambda-4)=0$

$(\lambda-2)(\lambda-4)=0$

$\lambda-2=0,\ \lambda-4=0$

$\lambda=2,\ \lambda=4.$

Now $\lambda=-1,\ \lambda=2,\ \lambda=4.$

For eigenvector

$(\lambda I_n - A)X=0$

$\lambda=-1$

$$=\left(-1\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}-\begin{bmatrix}2&2&3\\1&2&1\\2&-2&1\end{bmatrix}\right)\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}.$$

$$=\left(\begin{bmatrix}-1&0&0\\0&-1&0\\0&0&-1\end{bmatrix}-\begin{bmatrix}2&2&3\\1&2&1\\2&-2&1\end{bmatrix}\right)\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}.$$

$$=\begin{bmatrix}-3&-2&-3\\-1&-3&-1\\-2&2&-2\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$$

Ex 8.1

$$(A/b) = \left[\begin{array}{ccc|c} -3 & -2 & -3 & 0 \\ -1 & -3 & -1 & 0 \\ -2 & 2 & -2 & 0 \end{array}\right]$$

$$\overset{R}{\sim} \left[\begin{array}{ccc|c} 1 & 3 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \end{array}\right]$$

$x_3 = -1$

$x_2 = 0$

$x_1 = -3x_2 - x_3$

$x_3 = -3(0) - (-1) = +1$

$$X = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}.$$

$\lambda = 2, \lambda = 4$ S.Y.S

Q3 Let $A = \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 2 \\ -1 & 3 & 2 \end{bmatrix}$

The characteristic Poly: is

$P(\lambda) = \det(\lambda I_n - A)$

$= \det\left(\begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix} - \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 2 \\ -1 & 3 & 2 \end{bmatrix} \right)$

$= \det\left(\begin{bmatrix} \lambda-1 & -2 & -1 \\ 0 & \lambda-1 & -2 \\ 1 & -3 & \lambda-3 \end{bmatrix} \right)$

$= (\lambda-1)\left[(\lambda-1)(\lambda-2)-6\right] + 2(0+2) - 1(0+1-\lambda)$

$= (\lambda-1)(\lambda^2-3\lambda-4) + 4 + \lambda - 1$

$P(\lambda) = \lambda^3 - 4\lambda^2 + 7$ which is required ch: Poly:

Q 4, Q5, Q 6, Q7 is similarly to Q3

Q8 $A = \begin{bmatrix} 0 & 1 & 2 \\ 0 & 0 & 3 \\ 0 & 0 & 0 \end{bmatrix}$

The characteristic poly. of matrix A is given by

$P(\lambda) = \det(\lambda I_3 - A)$

$= \det\left(\begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix} - \begin{bmatrix} 0 & 1 & 2 \\ 0 & 0 & 3 \\ 0 & 0 & 0 \end{bmatrix} \right)$

$= \det\left(\begin{bmatrix} \lambda & -1 & -2 \\ 0 & \lambda & -3 \\ 0 & 0 & \lambda \end{bmatrix} \right)$

$= \lambda(\lambda^2 + 0) + 1(0+0) - 2(0+0)$

$P(\lambda) = \lambda^3$

Now $P(\lambda) = \lambda^3 = 0$.

$\lambda = 0$ Eigen value.

For Eigenvector

$(\lambda I_3 - A)x = 0$

$$\left(\lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 0 & 1 & 2 \\ 0 & 0 & 3 \\ 0 & 0 & 0 \end{bmatrix} \right) \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\left(0 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 0 & 1 & 2 \\ 0 & 0 & 3 \\ 0 & 0 & 0 \end{bmatrix} \right) \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\left(0 - \begin{bmatrix} 0 & 1 & 2 \\ 0 & 0 & 3 \\ 0 & 0 & 0 \end{bmatrix}\right)\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & -1 & -2 \\ 0 & 0 & -3 \\ 0 & 0 & -0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

$$(A/b) \quad \left[\begin{array}{ccc|c} 0 & -1 & -2 & 0 \\ 0 & 0 & -3 & 0 \\ 0 & 0 & -0 & 0 \end{array}\right]$$

$$\boxed{x_3 = 0}$$

$$-x_2 - 2x_3 = 0$$

$$-x_2 = 2x_3$$

$$-x_2 = 2(0)$$

$$-x_2 = 0$$

$$\boxed{x_2 = 0}$$

let $x_1 = r$

$$X = \begin{bmatrix} x_1 \\ x_1 \\ x_1 \end{bmatrix} = \begin{bmatrix} r \\ 0 \\ 0 \end{bmatrix}.$$

Q9, Q10, Q11, Q12, Q13, Q14, Q15

Similarly: