EXAMPLE 5 Polynomials, rational functions

The nonnegative integer powers $1, z, z^2, \cdots$ are analytic in the entire complex plane, and so are **polynomials**, that is, functions of the form

$$f(z) = c_0 + c_1 z + c_2 z^2 + \cdots + c_n z^n$$

where $c_0, \cdots, c_n$ are complex constants.

The quotient of two polynomials $g(z)$ and $h(z)$,

$$f(z) = \frac{g(z)}{h(z)},$$

is called a **rational function.** This f is analytic except at the points where $h(z) = 0$; here we assume that common factors of g and h have been canceled. ◀

The concepts discussed in this section extend familiar concepts of calculus. Most important is the concept of an analytic function. Indeed, complex analysis is concerned exclusively with analytic functions, and although many simple functions are not analytic, the large variety of remaining functions will yield a branch of mathematics that is very useful for practical purposes and most beautiful from a theoretical viewpoint.

PROBLEM SET 12.3

Regions of Practical Interest

Determine and sketch or plot the sets in the complex plane given by

1. $|z + 2 + 5i| \leqq \frac{1}{2}$ **2.** $\frac{1}{2} < |z - 4 + 2i| < 2$ **3.** $0 < |z| < 1$ **4.** $0 < |z - 1 - i| < \sqrt{2}$

5. $-\pi < \operatorname{Im} z < \pi$ **6.** $\operatorname{Re}(1/z) < 1$ **7.** $\operatorname{Re}(z^2) \leqq 1$ **8.** $|\arg z| < \pi/4$

9. WRITING PROJECT. Concepts on Sets. Extend the part of the text on sets in the complex plane by formulating everything in your own words, including examples of your own and comparing with calculus when applicable.

Functions and Their Derivatives

Function Values. Find the values of Re f and Im f at the indicated point.

10. $f = z^2 + 2z + 2$ at $1 - i$ **11.** $f = 1/(1 - z)$ at $7 + 2i$ **12.** $f = (z - 2)/(z + 2)$ at $4i$

Continuity. Find out (and give reason) whether $f(z)$ is continuous at $z = 0$ if $f(0) = 0$ and for $z \neq 0$ the function f is equal to

13. $(\operatorname{Im} z)/|z|$ **14.** $(\operatorname{Re} z^2)/|z|$ **15.** $(\operatorname{Re} z)/(1 + |z|)$ **16.** $(\operatorname{Re} z - \operatorname{Im} z)/|z|^2$

Derivative. Find the value of the derivative of

17. $(z - i)/(z + i)$ at i **18.** $(z - 4i)^8$ at $5 + 4i$ **19.** $(5 + 3i)/z^3$ at $2 + i$

20. $(3z^2 + iz)^2$ at $1 + i$ **21.** $z^4 + 1/z^4$ at $-1 - i$ **22.** $(iz^3 + 3z^2)^3$ at $2i$

23. $(iz + 2)/(3z - 6i)$ at any z. (Explain the result.)

24. TEAM PROJECT. Limit, Continuity, Derivative. (a) Limit. Prove that (1) is equivalent to the pair of relations

$$\lim_{z \to z_0} \operatorname{Re} f(z) = \operatorname{Re} l, \qquad \lim_{z \to z_0} \operatorname{Im} f(z) = \operatorname{Im} l.$$

(b) Limit. If $\lim_{z \to z_0} f(z)$ exists, show that this limit is unique.

(c) Continuity. If $z_1, z_2, \cdots$ are complex numbers for which $\lim_{n \to \infty} z_n = a$, and if $f(z)$ is continuous at $z = a$, show that

$$\lim_{n \to \infty} f(z_n) = f(a).$$

Complex Function: Let "S" be a set of complex numbers. Any rule "f" that assigns to every element "z" of "S" a complex number "w" is called complex function. where "w" is a complex number. and value of the function at "z"

Let
$$w = f(z) = u + iv = u(x,y) + iv(x,y).$$

Thus $\operatorname{Re} f(z) = u(x,y)$

& $\operatorname{Im} f(z) = v(x,y)$.

Limit of a Function:

A function $f(z)$ is said to have the limit "l" as "z" approaches "z_0" when $f(z)$ gets closer & closer to "l" i.e if for any +ve real number "ε" however small, we can find a +ve real no: $\delta > 0$ such that whenever

i.e $|f(z) - l| < \varepsilon \quad \forall \quad |z - z_0| < \delta$

or $\lim\limits_{z \to z_0} f(z) = l$

Continuity:

A function $f(z)$ is said to be continuous at $z = z_0$ if it satisfied the following conditions.

1. $f(z)$ is defined at z_0
2. $\lim_{z \to z_0} f(z)$ exists
3. $\lim_{z \to z_0} f(z) = f(z_0)$.

Derivative: let $f(z)$ be any complex function. The derivative of $f(z)$ at z_0, written as $f'(z_0)$ is defined as:

$$f'(z_0) = \lim_{\Delta z \to 0} \frac{f(z+\Delta z) - f(z)}{\Delta z}$$

if this limit exists.

Also we say that $f(z)$ is differentiable at z_0.

~~If~~ let $z = z_0 + \Delta z$

Then from above, we may write as:

$$f'(z_0) = \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}$$

Note:- A function which is differentiable at a Point is said to be "Analytic" there

Exercise 12.3

Find the value of ~~Ref~~ Ref and Im f at the indicated Point. Q10 —→ Q12

Q10 $f = z^2 + 2z + 2$ at $z = 1 - i$

$f = (1-i)^2 + 2(1-i) + 2$

$= 1 + i^2 - 2i + 2 - 2i + 2$

$= \cancel{1} - \cancel{1} - 2i + 2 - 2i + 2$

$f = +4 - 4i$

Re f: 4

Im f = -4.

Q11 $f = \frac{1}{1-z}$ at $7+2i$

$$f = \frac{1}{1-(7+2i)} = \frac{1}{1-7-2i} = \frac{1}{-6-2i}$$

$$= \frac{1}{-6-2i} \times \frac{-6+2i}{-6+2i}$$

$$= \frac{-6+2i}{36+4} = -\frac{6}{40} + \frac{2i}{40} = -3/20 + i/20$$

$$\boxed{\text{Re} f = -3/20, \ \text{Im} f = 1/20}$$

Continuity. Find out (and give reason) whether $f(z)$ is continous at $z=0$ if $f(0)=0$ and for $z \neq 0$ the function f is equal to: Q13 ⟶ ⟶ Q16

Q13 $(\text{Im} z)/|z|$.

Sol:- ① Given that $f(z)=0$

i.e $f(z)$ is defined at "0"

Also $f(z) = \frac{\text{Im} z}{|z|}$ for $z \neq 0$.

(2) let $z = x+iy$

$$\lim_{z\to 0} f(z) = \lim_{z\to 0} \frac{\operatorname{Im} z}{|z|}$$

$$\lim_{x+iy\to 0} = \lim_{z\to 0} \frac{y}{\sqrt{x^2+y^2}}$$

or

$$\lim_{\substack{x\to 0\\ y\to 0}} \frac{y}{\sqrt{x^2+y^2}}$$

Here we have two Possibilities.

Case 1 1st $x\to 0$ then $y\to 0$

$$\lim_{z\to 0} f(z) = \lim_{y\to 0}\left(\lim_{x\to 0} \frac{y}{\sqrt{x^2+y^2}}\right) = \lim_{y\to 0}\left(\frac{y}{y}\right) = 1 \quad -(1)$$

Case 2 1st $y\to 0$ then $x\to 0$.

$$\lim_{z\to 0} f(z) = \lim_{x\to 0}\left(\lim_{y\to 0} \frac{y}{\sqrt{x^2+y^2}}\right) = \lim_{x\to 0}(0) = 0 \quad -(2)$$

from eq (1) & eq (2) we have

$$\lim_{y\to 0}\left(\lim_{x\to 0} f(z)\right) \neq \lim_{x\to 0}\left(\lim_{y\to 0} f(z)\right)$$

Hence limit does not exist.

Thus $f(z) = \frac{Im(z)}{|z|}$ is not continuous at origin.

———— o ~~ o ~~ o

Q14 $(Re\, z^2)/|z|$.

Sol:- (1) Given that $f(z) = 0$
i.e $f(z)$ is defined at "0"

Also $f(z) = \frac{(Re\, z^2)}{|z|}$ for $z \neq 0$.

(2) let $z = x+iy \Rightarrow z^2 = (x+iy)(x+iy) = x^2 - y^2 + 2ixy$

$\cancel{|z^2| = \sqrt{(x^2-y^2)^2+(2xy)^2}}$

$|z| = \sqrt{x^2+y^2}$

$(Re\, z^2) = x^2 - y^2$

Now $f(z) = \frac{(Re\, z^2)}{|z|} = \frac{x^2-y^2}{\sqrt{x^2+y^2}}$.

$$\lim_{z \to 0} f(z) = \lim_{(x+iy) \to 0} \frac{x^2-y^2}{\sqrt{x^2+y^2}}$$

Case 1 $\lim_{y \to 0} \left[\lim_{x \to 0} \frac{x^2 - y^2}{\sqrt{x^2 + y^2}} \right]$.

$= \lim_{y \to 0} \left(- \frac{y^{\cancel{2}}}{\cancel{y}} \right) = 0$ — ①

Case 2 $\lim_{x \to 0} \left[\lim_{y \to 0} \frac{x^2 - y^2}{\sqrt{x^2 + y^2}} \right]$.

$\lim_{x \to 0} \left(\frac{x^{\cancel{2}}}{\cancel{x}} \right) = 0$ — ②

from eq① & eq② we have.

$\lim_{x \to 0} \left[\lim_{y \to 0} f(z) \right] = \lim_{y \to 0} \left[\lim_{x \to 0} f(z) \right]$

Hence the limit exist i.e

$\lim_{z \to 0} f(z) = 0$

③ $f(z) = 0 = \lim_{z \to 0} f(z)$

Hence it's satisfied all three ~~cases~~ condition so the function is continous at origin.

Q15 $\frac{\operatorname{Im} z}{(1+|z|)}$.

Sol: let $z = x+iy$ & $|z| = \sqrt{x^2+y^2}$

$$f(z) = \frac{\operatorname{Im} z}{(1+|z|)} = \frac{y}{1+\sqrt{x^2+y^2}}$$

(1) $f(z) = 0$ (Given)

i.e $f(z)$ is defined at "0"

(2) $\lim_{z\to 0} f(z) = \lim_{x+iy\to 0} \frac{y}{1+\sqrt{x^2+y^2}}$

Case 1 $\lim_{y\to 0}\left(\lim_{x\to 0} \frac{y}{1+\sqrt{x^2+y^2}}\right)$

$\lim_{y\to 0}\left[\frac{y}{1+y}\right] = 0$ —(1)

Case 2 $\lim_{x\to 0}\left(\lim_{y\to 0} \frac{y}{1+\sqrt{x^2+y^2}}\right) \Rightarrow \lim_{x\to 0}\left[\frac{0}{1+x}\right] = 0$ —(2)

from eq (1) & eq (2) we have

$$\lim_{x\to 0}\left[\lim_{y\to 0} f(z)\right] = \lim_{y\to 0}\left(\lim_{x\to 0} f(z)\right)$$

Hence limit Exist at ~~z~~ 0. origin i.e

(31)

$$\lim_{z \to 0} f(z) = 0$$

(3) $f(z) = 0 = \lim_{z \to 0} f(z)$

Hence it's satisfied all three condition so the function is continous at origin.

Derivative. Find the value of the derivative of

Q17 —— to —— Q23

Q17 $(z+i)/(z-i)$ at $-i$

Sol: Given $f(z) = \frac{z+i}{z-i}$

$$f'(z) = \frac{\cancel{z}-i-\cancel{z}-i}{(z-i)^2} = -\frac{2i}{(z-i)^2}$$

$$f'(z)\Big|_{z=-i} = -\frac{2i}{(-i-i)^2} = -\frac{2i}{(-2i)^2} = -\frac{2i}{+4i^2} = \frac{1}{2}i$$

$$\boxed{f'(z)\Big|_{z=-i} = \frac{1}{2}i}$$

(37)

Q18 $f(z) = (z-4i)^8$ at $5+4i$

$$f'(z) = 8(z-4i)^7$$

$$f'(z)\Big|_{z=5+4i} = 8\left[5+\cancel{4i}-\cancel{4i}\right]^7$$

$$= 8\left[5\right]^7$$

$$= \boxed{625000}$$

Q23 $f(z) = \frac{(iz+2)}{(3z-6i)}$

$$f'(z) = \frac{(3z-6i)i-(iz+2)3}{(3z-6i)^2}$$

$$f'(z) = \frac{3\cancel{zi}+\cancel{6}-3\cancel{iz}-\cancel{6}}{(3z-6i)^2}$$

$$\boxed{f'(z) = 0}$$

——— x ———

Analytic function :-

let $w = u + iv = f(z)$ be a complex function with domain "D" This function is said to be analytic if:

① $f(z)$ is defined at z_0.

② $f'(z)$ is defined at z_0, where z_0 is a point in domain of $f(z)$.

Cauchy-Riemann's Equation

Consider a complex function given by

$$w = f(z) = u(x,y) + i\,v(x,y)$$

We say that f is analytic if and only if it satisfies the relation:

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \& \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}, \text{ or } u_x = v_y \; \& \; u_y = -v_x$$

known as Cauchy-Riemann's Equation.

Cauchy-Riemann's Equation in Polar form

If

$$f(z) = u(r,\theta) + i\,v(r,\theta)$$

Then Cauchy Riemann's Equation are

$$\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta}$$

& $\frac{\partial v}{\partial r} = -\frac{1}{r}\frac{\partial u}{\partial \theta}$ when $r > 0$.

Laplace Equation:

$$f(z) = u(x,y) + i\,v(x,y)$$

is analytic in domain D. Then u & v satisfy the Equations

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \text{ or } u_{xx} + u_{yy} = 0 \text{ or } \nabla^2 u = 0.$$

&

$$\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0 \text{ or } v_{xx} + v_{yy} = 0 \text{ or } \nabla^2 v = 0.$$

These are called Laplace Equations.

Hormonic Function:

Any function "u" which satisfies Laplace Equation is called hormonic function. i.e

$\nabla^2 U = 0$ or $U_{xx} + U_{yy} = 0$

Conjugate Harmonic function

Let $f(z) = u + iv$

If "u" & "v" both satisfy CRE we say that V is the conjugate harmonic of "u"

X

Excercise 12.4

Are the following function analytic

Q1 to Q12

Q1 $f(z) = z^8$

Sol:- Let $z = r\cos\theta + i r\sin\theta$

$z = r[\cos\theta + i\sin\theta]$

Then $z^8 = r^8[\cos\theta + i\sin\theta]^8$

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