In a similar fashion we find from (2c) that $b_1 = E/2$ and $b_n = 0$ for $n = 2, 3, \cdots$. Consequently,

$$u(t) = \frac{E}{\pi} + \frac{E}{2}\sin\omega t - \frac{2E}{\pi}\left(\frac{1}{1\cdot 3}\cos 2\omega t + \frac{1}{3\cdot 5}\cos 4\omega t + \cdots\right).$$

PROBLEM SET 10.3

Fourier Series for Period $p = 2L$

Find the Fourier series of the periodic function $f(x)$, of period $p = 2L$, and sketch $f(x)$ and the first three partial sums. (Show the details of your work.)

1. $f(x) = -1 \quad (-1 < x < 0), \quad f(x) = 1 \quad (0 < x < 1), \quad p = 2L = 2$
2. $f(x) = 1 \quad (-1 < x < 0), \quad f(x) = -1 \quad (0 < x < 1), \quad p = 2L = 2$
3. $f(x) = 0 \quad (-2 < x < 0), \quad f(x) = 2 \quad (0 < x < 2), \quad p = 2L = 4$
4. $f(x) = |x| \quad (-2 < x < 2), \quad p = 2L = 4$
5. $f(x) = 2x \quad (-1 < x < 1), \quad p = 2L = 2$
6. $f(x) = 1 - x^2 \quad (-1 < x < 1), \quad p = 2L = 2$
7. $f(x) = 3x^2 \quad (-1 < x < 1), \quad p = 2L = 2$
8. $f(x) = \frac{1}{2} + x \quad (-\frac{1}{2} < x < 0), \quad f(x) = \frac{1}{2} - x \quad (0 < x < \frac{1}{2}), \quad p = 2L = 1$
9. $f(x) = 0, \quad (-1 < x < 0), \quad f(x) = x \quad (0 < x < 1), \quad p = 2L = 2$
10. $f(x) = x \quad (0 < x < 1), \quad f(x) = 1 - x \quad (1 < x < 2), \quad p = 2L = 2$
11. $f(x) = \pi \sin \pi x \quad (0 < x < 1), \quad p = 2L = 1$
12. $f(x) = \pi x^3/2 \quad (-1 < x < 1), \quad p = 2L = 2$

13. **(Periodicity)** Show that each term in (1) has the period $p = 2L$.
14. **(Rectifier)** Find the Fourier series of the periodic function that is obtained by passing the voltage $v(t) = V_0 \cos 100\pi t$ through a half-wave rectifier.
15. **(Transformation)** Obtain the Fourier series in Prob. 1 from that in Example 1, Sec. 10.2.
16. **(Transformation)** Obtain the Fourier series in Prob. 7 from that in Prob. 7, Sec. 10.2.
17. **(Transformation)** Obtain the Fourier series in Prob. 3 from that in Example 1, Sec. 10.2.
18. **(Interval of Integration)** Show that in (2) the interval of integration may be replaced by any other interval of length $p = 2L$.
19. **CAS PROJECT. Fourier Series of $2L$-Periodic Functions. (a)** Write a program for obtaining any partial sum of a Fourier series (1).
    (b) Apply the program to Probs. 5–7, plotting the first few partial sums of each of the three series on common axes. Choose the first five or more partial sums until they approximate the given function reasonably well.
20. **CAS PROJECT. Gibbs Phenomenon.** The partial sums $s_n(x)$ of a Fourier series show oscillations near a discontinuity point. These do not disappear as n increases but instead become sharp "spikes." They were explained mathematically by J. W. Gibbs.[10] Plot $s_n(x)$ in Prob. 5. When $n = 20$, say, you will see those oscillations quite distinctly. Consider two other Fourier series of your choice in a similar way.

[10]JOSIAH WILLARD GIBBS (1839—1903), American mathematician, professor of mathematical physics at Yale from 1871, one of the founders of vector calculus [another being O. Heaviside (see Sec. 5.1)], mathematical thermodynamics, and statistical mechanics. His work was of great importance to the development of mathematical physics.

Function of any Period $P = 2L$

The function considered so far had Period 2π, for simplicity. of course, in applications. Periodic function will generally have other Periods. But we show that the transition from Period $P = 2\pi$ to Period $p = 2L$ is quite simple. it amounts to a stretch (or contraction) of scale on the axis.

If a function $f(x)$ of Period $p = 2L$ has a Fourier Series we claim that that this Series is

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos \frac{n\pi}{L} x + b_n \sin \frac{n\pi}{L} x \right).$$

With the Fourier co-efficients of $f(x)$ given by the Euler formulas:

$$a_0 = \frac{1}{\cancel{2}L} \int_{-L}^{L} f(x)\,dx.$$

$$a_n = \frac{1}{L} \int_{-L}^{L} f(x) \cos \frac{n\pi x}{L}\,dx \quad n = 1, 2, 3 \text{———}$$

$$b_n = \frac{1}{L} \int_{-L}^{L} f(x) \sin \frac{n\pi x}{L}\,dx \quad n = 1, 2, 3 \text{———}$$

Ex 10.3 Q1 — to — Q12

Q1 $f(x) = -1 \quad (-1 < x < 0)$

$f(x) = 1 \quad (0 < x < 1).$

$P = 2L = 2.$

Sol:- $f(x) = \begin{cases} -1 & -1 < x < 0 \\ 1 & 0 < x < 1. \end{cases}$

$P = 2L = 2 \Rightarrow L = 1.$

The F.S of $f(x)$ is

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos\frac{n\pi x}{L} + b_n \sin\frac{n\pi x}{L}\right) \quad \text{— (1)}$$

$$a_0 = \int_{-1}^{1} f(x)\,dx$$

$$= \int_{-1}^{0} -1\,dx + \int_{0}^{1} 1\,dx$$

$$\boxed{a_0 = 0}$$

$$a_n = \frac{1}{L}\int_{-1}^{1} f(x)\cos\frac{n\pi x}{L}\,dx$$

$$= \frac{1}{1}\int_{-1}^{0} -\cos\frac{n\pi x}{1}\,dx + \int_{0}^{1} 1\cdot\cos\frac{n\pi x}{1}\,dx.$$

$$= -\frac{1}{n\pi}\sin n\pi x\Big|_{-1}^{0} + \frac{1}{n\pi}\sin n\pi x\Big|_{0}^{1}$$

$$a_n = -\frac{1}{n\pi}\left[\sin(0) - \sin n(-1)\pi\right] + \frac{1}{n\pi}\left[\sin n\pi(1) - 0\right]$$

$$= \frac{1}{n\pi}\left[\cancel{\sin n\pi} - \cancel{\sin n\pi}\right]$$

$$\boxed{a_n = 0}$$

$$b_n = \frac{1}{1}\int_{-1}^{0} -\sin n\pi x\,dx + \int_{0}^{1} \sin n\pi x\,dx$$

$$= \frac{1}{n\pi}\cos n\pi x\Big|_{-1}^{0} + \frac{1}{n\pi}\cos n\pi x\Big|_{0}^{1}$$

$$= \frac{1}{n\pi}\left[\cos(0) - \cos n\pi(-1)\right] - \frac{1}{n\pi}\left[\cos n\pi - \cos(0)\right]$$

$$= \frac{1}{n\pi}\left[1 - \cos n\pi\right] + \frac{1}{n\pi}\left[1 - \cos n\pi\right]$$

$$b_n = \frac{2}{n\pi}\left[1 - \cos n\pi\right]$$

eq ① $\Rightarrow$

$$f(x) = \frac{0}{2} + \sum_{n=1}^{\infty}\left[0 \cdot \frac{\cos n\pi x}{1} + \frac{2}{n\pi}(1 - \cos n\pi) \cdot \sin n\pi x\right]$$

$$f(x) = \frac{4}{\pi}\left[\sin \pi x + \frac{1}{3}\sin 3\pi x + \frac{1}{5}\sin 5\pi x + \cdots\right]$$

Q3 $f(x) = \begin{cases} 0 & (-2<x<0) \\ 2 & (0<x<2) \end{cases}$

$P = 2L = 4$

Sol:- Given that $P = 2L = 4 \Rightarrow L = 2$.

The F.S of $f(x)$ is given by:

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left[a_n \cos\frac{n\pi x}{L} + b_n \sin\frac{n\pi x}{L} \right] \quad \text{—— (1)}$$

$$a_0 = \frac{1}{L}\int_{-L}^{L} f(x)\,dx.$$

$$= \frac{1}{2}\left[\int_{-2}^{0} 0\,dx + \int_{0}^{2} 2\,dx \right].$$

$$\boxed{a_0 = 2}$$

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x) \cos\frac{n\pi x}{L}\,dx$$

$$= \frac{1}{2}\left[\int_{-2}^{0} 0\,dx + \int_{0}^{2} 2\cos\frac{n\pi x}{2}\,dx. \right.$$

$$= \frac{\cancel{2}}{\cancel{2}}\left[\sin\frac{n\pi x}{2} \cdot \Big|_0^2 \frac{2}{n\pi} \right]$$

$$a_n = \frac{2}{n\pi}\left[\sin\frac{\cancel{2}n\pi}{\cancel{2}} - \sin(0) \right].$$

$$\boxed{a_n = \frac{2}{n\pi}\sin n\pi}$$

$$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\frac{n\pi x}{L}\,dx.$$

$$= \frac{1}{2}\left[\int_{-2}^{0} 0\,dx + \int_{0}^{2} 2\sin\frac{n\pi x}{2}\,dx.\right.$$

$$\boxed{b_n = \frac{2}{n\pi}\left[1-\cos n\pi\right].}$$

eq(1) $\Rightarrow$

$$f(x) = \frac{2}{2} + \sum_{n=1}^{\infty}\left[\frac{2}{n\pi}\sin n\pi.\cos\frac{n\pi x}{L} + \frac{2}{n\pi}(1-\cos n\pi).\sin\frac{n\pi x}{L}\right]$$

$$f(x) = 1 + \sum_{n=1}^{\infty}\left[\frac{2}{n\pi}\sin n\pi.\cos\frac{n\pi x}{2} + \frac{2}{n\pi}(1-\cos n\pi).\sin\frac{n\pi x}{2}\right].$$

$$f(x) = 1 + \frac{4}{\pi}\left[\sin\frac{\pi x}{2} + \frac{1}{3}\sin\frac{3\pi x}{2} + \frac{1}{5}\sin\frac{5\pi x}{2} + \cdots\right)$$

G

Hint

Q4 $f(x) = |x| \quad (-2 < x < 2) \qquad P = 2L = 4$

$L = 2.$

$$f(x) = \begin{cases} -x & (-2 < x < 0) \\ x & (0 < x < 2). \end{cases}$$

S.Y.S.

Q8 $f(x) = \begin{cases} 1/2 + x & -1/2 < x < 0 \\ 1/2 - x & 0 < x < 1/2 \end{cases}$

$P = 2L = 1$

Sol:- $P = 2L = 1 \Rightarrow L = 1/2$.

The F.S of $f(x)$ is given as:

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left[a_n \cos\frac{n\pi x}{L} + b_n \sin\frac{n\pi x}{L}\right] \text{ --- (1)}$$

$$a_0 = \frac{1}{L}\int_{-L}^{L} f(x)\,dx$$

$$= \frac{1}{1/2}\left[\int_{-1/2}^{0} (1/2 + x)\,dx + \int_{0}^{1/2} (1/2 - x)\,dx\right]$$

$$= 2\left[1/2\int_{-1/2}^{0} dx + \int_{-1/2}^{0} x\,dx + \frac{1}{2}\int_{0}^{1/2} dx - \int_{0}^{1/2} x\,dx\right]$$

$$a_0 = \frac{1}{2} - 1/4 + 1/2 - 1/4$$

$$\boxed{a_0 = 1/2}$$

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\frac{n\pi x}{L}\,dx$$

$$= 1/1/2\left[\int_{-1/2}^{0} (1/2 + x)\cos(n\pi x)2\,dx + \int_{0}^{1/2} (1/2 - x)\cos 2n\pi x\,dx\right]$$

(12)

$$= 2\left[\frac{1}{4n\pi}\sin 2n\pi x\Big|_{-1/2}^{0} + \frac{x\sin 2n\pi x}{2n\pi}\Big|_{-1/2}^{0} - \frac{1}{2n\pi}\int_{-1/2}^{0}\sin 2n\pi x\,dx\right.$$

$$\left.+\frac{1}{4n\pi}\sin 2n\pi x\Big|_{0}^{1/2} - \frac{x}{2n\pi}\sin 2n\pi x\Big|_{0}^{1/2} + \frac{1}{2n\pi}\int_{0}^{1/2}\sin 2n\pi x\,dx\right]$$

$$\boxed{a_n = \frac{1}{n^2\pi^2}(1-\cos n\pi)}$$

$$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\frac{n\pi x}{L}.$$

$$b_n = \frac{1}{1/2}\left[\int_{-1/2}^{0}(1/2+x)\sin 2n\pi x\,dx + \int_{0}^{1/2}(1/2-x)\sin 2n\pi x\,dx\right].$$

$$\boxed{b_n = 0}$$

eq(1) $\Rightarrow$.

$$f(x) = \frac{1/2}{2} + \sum_{n=1}^{\infty}\left[\frac{1}{n^2\pi^2}(1-\cos n\pi).\cos\frac{n\pi x}{1/2} + 0.\sin\frac{n\pi x}{1/2}\right].$$

$$f(x) = 1/4 + \frac{2}{\pi^2}\left(\cos 2\pi x + \frac{1}{9}\cos 6\pi x + \cdots\right).$$

6

Q12 $f(x) = \frac{\pi x^3}{2}$ $(-1 < x < 1)$.

$P = 2L = 2$

Sol:- $P = 2L = 2 \Rightarrow L = 1$

The F.S of $f(x)$ is given by:

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} [a_n \cos \frac{n\pi x}{L} + b_n \sin \frac{n\pi x}{L}] \quad \text{—} (1)$$

$$a_0 = \frac{1}{L} \int_{-L}^{L} f(x) dx.$$

$$a_0 = \frac{1}{1} \int_{-1}^{1} \frac{\pi x^3}{2} dx = 0$$

$$\boxed{a_0 = 0}$$

$$a_n = \frac{1}{L} \int_{-L}^{L} f(x) \cos \frac{n\pi x}{L} dx$$

$$a_n = \frac{1}{1} \int_{-1}^{1} \frac{\pi x^3}{2} \cos n\pi x \, dx.$$

$$a_n = \frac{\pi}{2} \left[\frac{x^3 \sin n\pi x}{n\pi} \Big|_{-1}^{1} - \frac{3}{n\pi} \int_{-1}^{1} x^2 \sin n\pi x \, dx \right].$$

$$= \frac{\pi}{2} \left[\frac{1}{n\pi} (\cancel{\sin n\pi} - \cancel{\sin n\pi}) - \frac{3}{n\pi} \left[\frac{-x^2}{n\pi} \cos n\pi \Big|_{-1}^{1} + \frac{2}{n\pi} \int_{-1}^{1} x \cos n\pi \, dx \right.\right.$$

$$\boxed{a_n = 0}$$

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$$b_n = \frac{1}{L}\int_{-L}^{L} f(x) \sin\frac{n\pi x}{L}\,dx.$$

$$b_n = \frac{1}{1}\int_{-1}^{1} \frac{\pi x^3}{2} \sin n\pi x\,dx.$$

$$= \frac{\pi}{2}\left[-\frac{x^3}{n\pi}\cos n\pi x\Big|_{-1}^{1} + \frac{3}{n\pi}\int_{-1}^{1} x^2\cos n\pi x\,dx\right].$$

$$b_n = -\frac{1}{n}\cos n\pi + \frac{6}{n^3\pi^2}\cos n\pi.$$

eq (1) $\Rightarrow$

$$f(x) = 0 + \sum_{n=1}^{\infty}\left[0\cdot\cos n\pi x + \left(-\frac{1}{n}\cos n\pi + \frac{6}{n^3\pi^2}\cos n\pi\right)\sin n\pi x\right].$$

$$= \left(\sin n\pi x - \frac{1}{2}\sin 2\pi x + \frac{1}{3}\sin 3\pi x + \cdots\right)$$

$$- \frac{6}{\pi^2}\left(\sin\pi x - \frac{1}{2^3}\sin 2\pi x + \frac{1}{3^3}\sin 3\pi x + \cdots\right).$$

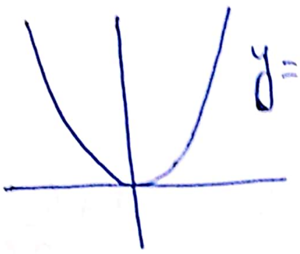
Even function: A function $y = g(x)$ is even if
$$g(-x) = g(x).$$

e.g $y = f(x) = x^2$

$f(-x) = (-x)^2 = x^2 = f(x)$

$f(-x) = f(x)$

The graph of even function is symmetric about y-axis.



$y = x^2$.

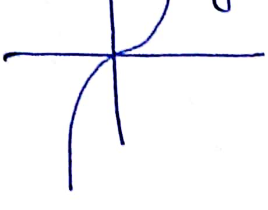
Odd function: A function $y = g(x)$ is odd if
$$g(-x) = -g(x).$$

e.g $y = f(x) = x^3$

$f(-x) = (-x)^3 = -x^3 = -f(x)$

$f(-x) = -f(x)$

The graph of odd function is symmetric about origin.



$y = x^3$.

<u>Niether even nor odd function</u>: A function is said to be niether even nor odd if the above two condition are not satisfied.

e.g $f(x) = e^x$ is niether even nor odd.

$$f(-x) = e^{-x} \neq f(x)$$
$$f(-x) = e^{-x} \neq -f(x).$$

<u>Theorem 1</u> (Fourier cosine Series, Fourier Sine Series)
P-542

The fourier series of an even function of period $2L$ is a Fourier cosine series.

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi}{L}x \quad —(1)$$

With co-efficients.

$$a_0 = \frac{1}{L}\int_0^L f(x)dx, \quad a_n = \frac{2}{L}\int_0^L f(x)\cos\frac{n\pi x}{L}dx \quad n=1,2,3\cdots$$

The Fourier Series of an odd function of period $2L$ is a Fourier Sine series

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi}{L}x \quad —(2)$$

with co-efficients: $b_n = \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{L}dx.$

Now in case of Period 2π. In this case Theorem 1 gives for an even function simply.

eq ① $\Rightarrow$ $$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos nx.$$

With co-efficients.

$$a_0 = \frac{1}{\pi}\int_0^{\pi} f(x)\,dx \quad , \quad a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\,dx.$$

Similarly for an odd 2π-Period function we simply have

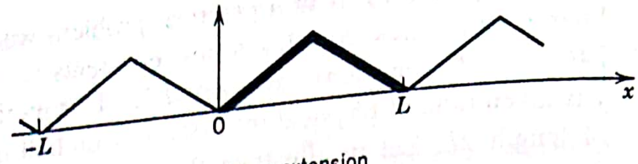
eq ② $\Rightarrow$ $$f(x) = \sum_{n=1}^{\infty} b_n \sin nx$$

With co-efficients

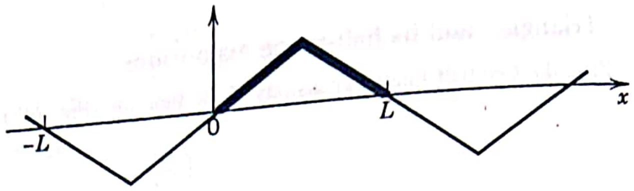
$$b_n = \frac{2}{\pi}\int_0^{\pi} f(x)\sin nx\,dx.$$

Note:- ① Product of two odd function is even.

② Product of two even function is even.



(a) Even extension



(b) Odd extension

Fig. 248. Periodic extensions of $f(x)$ in Example 3

and $a_n = 0$ if $n \neq 2, 6, 10, 14, \cdots$. Hence the first half-range expansion of $f(x)$ is

$$f(x) = \frac{k}{2} - \frac{16k}{\pi^2}\left(\frac{1}{2^2}\cos\frac{2\pi}{L}x + \frac{1}{6^2}\cos\frac{6\pi}{L}x + \cdots\right).$$

This Fourier cosine series represents the even periodic extension of the given function $f(x)$, of period $2L$, shown in Fig. 248a.

(b) Odd periodic extension. Similarly, from (6) we obtain

$$b_n = \frac{8k}{n^2\pi^2}\sin\frac{n\pi}{2}. \tag{7}$$

Hence the other half-range expansion of $f(x)$ is

$$f(x) = \frac{8k}{\pi^2}\left(\frac{1}{1^2}\sin\frac{\pi}{L}x - \frac{1}{3^2}\sin\frac{3\pi}{L}x + \frac{1}{5^2}\sin\frac{5\pi}{L}x - + \cdots\right).$$

This series represents the odd periodic extension of $f(x)$, of period $2L$, shown in Fig. 248b.
Basic applications of these results will be shown in Secs. 11.3 and 11.5. ◀

PROBLEM SET 10.4

Even and Odd Functions

Are the following functions odd, even, or neither odd nor even?

1. $|x^3|$, $x\cos nx$, $x^2\cos nx$, $\cosh x$, $\sinh x$, $\sin x + \cos x$, $x|x|$

2. $x + x^2$, $|x|$, e^x, e^{x^2}, $\sin^2 x$, $x\sin x$, $\ln x$, $x\cos x$, $e^{-|x|}$

Are the following functions $f(x)$, which are assumed to be periodic, of period 2π, even, odd or neither even nor odd?

3. $f(x) = x^2 \quad (0 < x < 2\pi)$

4. $f(x) = x^4 \quad (0 < x < 2\pi)$

5. $f(x) = e^{-|x|} \quad (-\pi < x < \pi)$

6. $f(x) = |\sin 5x| \quad (-\pi < x < \pi)$

7. $f(x) = \begin{cases} 0 & \text{if } 2 < x < 2\pi - 2 \\ x & \text{if } -2 < x < 2 \end{cases}$

8. $f(x) = \begin{cases} \cos^2 x & \text{if } -\pi < x < 0 \\ \sin^2 x & \text{if } 0 < x < \pi \end{cases}$

9. $f(x) = x^3 \quad (-\pi/2 < x < 3\pi/2)$

10. PROJECT. Even and Odd Functions. (a) Are the following expressions even or odd? Sums and products of even functions and of odd functions. Products of even times odd functions. Absolute values of odd functions. $f(x) + f(-x)$ and $f(x) - f(-x)$ for arbitrary $f(x)$.

(b) Write e^{kx}, $1/(1-x)$, $\sin(x+k)$, $\cosh(x+k)$ as sums of an even and an odd function.

(c) Find all functions that are both even and odd.

(d) Is $\cos^3 x$ even or odd? $\sin^3 x$? Find the Fourier series of these two functions. Do you recognize familiar identities?

Fourier Series of Even and Odd Functions

State whether the given function is even or odd. Find its Fourier series. Sketch the function and some partial sums. (Show the details of your work.)

11. $f(x) = \begin{cases} k & \text{if } -\pi/2 < x < \pi/2 \\ 0 & \text{if } \pi/2 < x < 3\pi/2 \end{cases}$

12. $f(x) = \begin{cases} -2x & \text{if } -\pi < x < 0 \\ 2x & \text{if } 0 < x < \pi \end{cases}$

13. $f(x) = \begin{cases} x & \text{if } -\pi/2 < x < \pi/2 \\ \pi - x & \text{if } \pi/2 < x < 3\pi/2 \end{cases}$

14. $f(x) = \begin{cases} x & \text{if } 0 < x < \pi \\ \pi - x & \text{if } \pi < x < 2\pi \end{cases}$

15. $f(x) = x^2/2 \quad (-\pi < x < \pi)$

16. $f(x) = 3x(\pi^2 - x^2) \quad (-\pi < x < \pi)$

Show that

17. $1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + - \cdots = \frac{\pi}{4}$ (Use Prob. 11.)

18. $1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \frac{1}{25} + \cdots = \frac{\pi^2}{6}$ (Use Prob. 15.)

19. $1 - \frac{1}{4} + \frac{1}{9} - \frac{1}{16} + - \cdots = \frac{\pi^2}{12}$ (Use Prob. 15.)

Half-Range Expansions

Find the Fourier cosine series as well as the Fourier sine series. Sketch $f(x)$ and its two periodic extensions. (Show the details.)

20. $f(x) = 1 \quad (0 < x < L)$

21. $f(x) = x \quad (0 < x < L)$

22. $f(x) = x^2 \quad (0 < x < L)$

23. $f(x) = \pi - x \quad (0 < x < \pi)$

24. $f(x) = x^3 \quad (0 < x < L)$

25. $f(x) = e^x \quad (0 < x < L)$

10.5 Complex Fourier Series. *Optional*

In this optional section we show that the Fourier series

$$f(x) = a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx) \tag{1}$$

can be written in complex form, which sometimes simplifies calculations (see Example 1, below). This is done by the Euler formula (5), Sec. 2.3, with nx instead of x, that is,

$$e^{inx} = \cos nx + i \sin nx, \tag{2}$$

$$e^{-inx} = \cos nx - i \sin nx. \tag{3}$$

Ex 10.4

Are the following function odd, even, or neither odd nor even?

Q1 $|x^3| = x^3$

$f(x) = x^3$

$f(-x) = (-x)^3 = -x^3 = -f(x)$

$f(-x) = -f(x)$.

Hence the function is odd.

Q(ii) $f(x) = x\cos nx$.

$f(-x) = -x\cos n(-x)$

$= -x\cos nx = -f(x)$

$f(-x) = -f(x)$

Hence the function is odd.

Q2 is same as Q1.

Fourier Series of Even and Odd function

State whether the given function is even or odd. Find its Fourier Series. Sketch the functions and some Partial sums. Which are assume to have the Period 2π.

Q11 $f(x) = \begin{cases} k & \text{if } -\pi/2 < x < \pi/2 \\ 0 & \text{if } \pi/2 < x < 3\pi/2 \end{cases}$

Sol:- $f(x) = k \Rightarrow f(-x) = k = f(x)$

& $f(x) = 0 \Rightarrow f(-x) = 0 = f(x)$.

Hence the function is even

The fourier Series for an even function is

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{L} \quad \text{——} (1)$$

$\cancel{2}L = \cancel{2}\pi$

$\therefore L = \pi$

$$a_0 = \frac{2}{L} \int_0^L f(x)\,dx$$

$$= \frac{2}{\pi} \left[\int_0^{\pi/2} k\,dx + \int_{\pi/2}^{\pi} 0\,dx \right]$$

$$\boxed{a_0 = k}$$

$$a_n = \frac{2}{L} \int_0^L f(x) \cos \frac{n\pi x}{L}\,dx$$

$$= \frac{2}{\pi} \int_0^{\pi} f(x) \cos \frac{n\cancel{\pi} x}{\cancel{\pi}}\,dx$$

$$= \frac{2}{\pi}\left[\int_0^{\pi/2} k\cos nx\,dx + \int_{\pi/2}^{\pi} 0\,dx\right].$$

$$a_n = \frac{2k}{n\pi}\sin n\pi/2$$

$$eq(1) \Rightarrow k/2 + \sum_{n=1}^{\infty}\left[\frac{2k}{n\pi}\sin\frac{n\pi}{2}\cdot\cos\frac{n\pi}{\pi}x\right]$$

$$f(x) = k/2 + \sum_{n=1}^{\infty}\left[\frac{2k}{n\pi}\sin\frac{n\pi}{2}\cdot\cos nx\right]$$

$$f(x) = k/2 + \frac{2k}{\pi}\left[\cos x + 1/3\cos 3x + \text{-----}\right].$$

Q13 $f(x) = \begin{cases} x & \text{if } -\pi/2 < x < \pi/2 \\ \pi - x & \text{if } \pi/2 < x < 3\pi/2 \end{cases}$.

Sol:- $f(x) = x$

$f(-x) = -x = -f(x)$.

& $f(x) = \pi - x$

$f(-x) = \pi + x \neq f(x)$

So the given function is odd.

The F. Series for an odd function is

$$f(x) = \sum_{n=1}^{\infty} b_n \sin nx \quad \text{--- (1)}$$

where

$$b_n = \frac{2}{L}\int_0^L f(x)\sin nx\,dx$$

$$b_n = \frac{2}{\pi}\left[\int_0^{\pi/2} x\sin nx\,dx + \int_{\pi/2}^{\pi}(\pi-x)\sin nx\,dx\right.$$

$$b_n = \frac{4}{n^2\pi}\sin n\pi/2$$

eq (B) $\Rightarrow$ $$f(x) = \sum_{n=1}^{\infty}\frac{4}{n^2\pi}\sin\frac{n\pi}{2}\sin nx.$$

$$f(x) = \frac{4}{\pi}\left[\sin x - \frac{1}{3^2}\sin 3x + \frac{1}{5^2}\sin 5x + \dots\right].$$

Q15 $f(x) = \frac{x^2}{4}$ $(-\pi < x < \pi)$.

Sol:- $f(-x) = \frac{(-x)^2}{4} = \frac{x^4}{4} = f(x)$

Hence even function Now their F. Series is

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n\cos nx) \quad \text{--- (1)}$$

$$a_0 = \frac{2}{\pi}\int_0^{\pi}\frac{x^2}{4}dx$$

$$\boxed{a_0 = \pi^2/6}$$

& $$a_n = \frac{2}{\pi}\int_0^{\pi}\frac{x^2}{4}\cos nx\,dx.$$

$$\boxed{a_n = \frac{1}{n^2}\cos n\pi}$$

eq (1) =>

$$f(x) = \frac{\pi^2/6}{2} + (a_1\cos x + a_2\cos 2x + a_3\cos 3x + \cdots)$$

$$= \frac{\pi^2}{12} - \left[\cos x - \frac{\cos 2x}{(2)^2} + \frac{\cos 3x}{(3)^2} - \frac{\cos 4x}{(4)^2} + \cdots\right].$$

———o———

Q16 $f(x) = x(\pi^2 - x^2) \quad (-\pi < x < \pi)$.

Sol:- $f(x) = x(\pi^2 - x^2)$.

$f(-x) = -x(\pi^2 - x^2) = -f(x)$

The function is odd Now their F. Series is

$$f(x) = \sum_{n=1}^{\infty}(b_n \sin nx) \quad \text{—— (1)}$$

$$b_n = \frac{2}{\pi}\int_0^{\pi}(x\pi^2 - x^3)\sin nx\,dx.$$

$$\boxed{b_n = \frac{-12}{n^3}\cos n\pi}$$

eq (B) $\Rightarrow$

$$f(x) = b_1 \sin x + b_2 \sin 2x + b_3 \sin 3x + \cdots$$

$$f(x) = 12 \sin x - \frac{3}{2} \sin 2x + \frac{4}{9} \sin 3x - \frac{3}{16} \sin 4x + \cdots$$