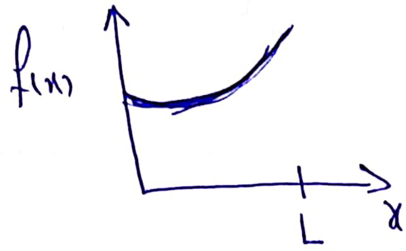
Half-Range Expansions

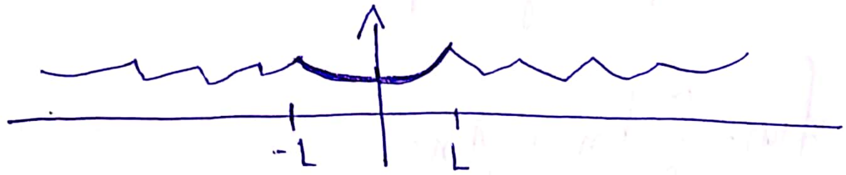
In various applications there is a Practical need to use Fourier Series in connection with function $f(x)$ that are given on some interval only, say $0 \le x \le L$. we could extend $f(x)$ Periodically with Period L and then represent the extended function by a Fourier Series, which in general would involve both cosine and Sine terms. we can do better and always get a cosine Series by 1st extending $f(x)$ from $0 \le x \le L$ as an even function on the range $-L \le x \le L$ and then extend this new function as a Periodic function of Period $2L$ and, Since it is even represent it by a Fourier cosine Series, or we can

Extend $f(x)$ from $0 \le x \le L$ as an

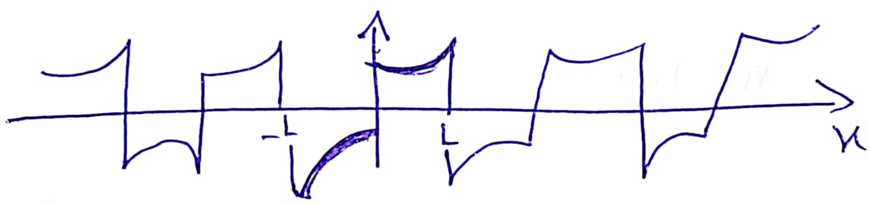
(a) The given function $f(x)$



(b) $f(x)$ Extended as an even Periodic function of Period $2L$



(c) $f(x)$ Extended as an odd Periodic function of Period $2L$



(a) Function $f(x)$ given on an interval $0 \le x \le L$.

(b) its even Extension to the full range $-L \le x \le L$ and the Periodic Extension of Period $2L$ to the x-axis.

(c) it's odd extension to $-L \le x \le L$ and the Periodic

Extension of Period 2L to the x-axis.

The cosine half-range Expansion is

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi}{L} x.$$

where

$$a_0 = \frac{1}{L} \int_0^L f(x)\, dx$$

$$\& \ a_n = \frac{2}{L} \int_0^L f(x) \cos \frac{n\pi x}{L}\, dx.$$

The Sine half-range Expansion is

$$f(x) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi}{L} x.$$

where $b_n = \frac{2}{L} \int_0^L f(x) \sin \frac{n\pi x}{L}\, dx.$

$n = 1, 2, 3 \ldots$

~ o ~

Find the Fourier cosine series as well as the Fourier sine series.

Q20 $f(x) = 1 \quad (0 < x < L)$.

The Fourier cosine series corresponding to $f(x)$ is given by:

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{L} \quad \text{—} (1).$$

$$a_0 = \frac{2}{L}\int_0^L 1\,dx$$

$$\boxed{a_0 = 2}$$

$$\& \; a_n = \frac{2}{L}\int_0^L \cos\frac{n\pi x}{L}\,dx.$$

$$a_n = \frac{L}{n\pi}\cdot\frac{2}{L}\left[\cancel{\sin n\pi}^{\,0} - 0\right]$$

$$\boxed{a_n = 0}$$

eq (1) =>

$$f(x) = \frac{\cancel{2}}{\cancel{2}} + \sum_{n=1}^{\infty} 0 \cdot \cos\frac{n\pi x}{L}$$

$$\boxed{f(x) = 1}$$

Q24 $f(x) = x^3 \quad (0 < x < L)$.

The Fourier Sine Series corresponding to $f(x)$ is given by:

$$f(x) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{L} \longrightarrow (1)$$

$$b_n = \frac{2}{L}\int_0^L x^3 \sin\frac{n\pi x}{L}\,dx$$

$$= \frac{2}{L}\left[-\frac{L}{n\pi}x^3 \cos\frac{n\pi x}{L}\Big|_0^L - \int_0^L 3x^2 \frac{L}{n\pi}\cos\frac{n\pi x}{L}\,dx\right]$$

$$= -\frac{2}{n\pi}L^3 \cos n\pi + \frac{6}{n\pi}\left[\frac{Lx^2}{n\pi}\sin\frac{n\pi x}{L}\Big|_0^L - \int_0^L \frac{2xL}{n\pi}\sin\frac{n\pi x}{L}\,dx\right]$$

$$= -\frac{2L^3}{n\pi}\cos n\pi + \frac{6L^3}{n\pi}\overset{0}{\cancel{\sin n\pi}} - \frac{12L}{n^2\pi^2}\left[-\frac{xL}{n\pi}\cos\frac{n\pi x}{L}\Big|_0^L + \frac{L^2}{n^2\pi^2}\sin\frac{n\pi x}{L}\Big|_0^L\right]$$

$$b_n = -\frac{2L^3}{n\pi}\cos n\pi + \frac{12L^3}{n^3\pi^3}\cos n\pi.$$

eq (1) $\Rightarrow$ $$f(x) = \sum_{n=1}^{\infty}\left(-\frac{2L^3}{n\pi}\cos n\pi + \frac{12L^3}{n^3\pi^3}\cos n\pi\right).$$

$$f(x) = \frac{2L^3}{\pi}\left(\sin\frac{\pi x}{L} - \frac{1}{2}\sin\frac{2\pi x}{L} + \cdots\right) + \frac{12L^3}{\pi^3}\left(-\sin\frac{\pi x}{L} + \frac{1}{8}\sin\frac{2\pi x}{L} + \cdots\right).$$

Fourier integral (F.I)

The F.I of a function "f" defined on the interval $(-\infty, \infty)$. and non-Periodic is given by.

$$f(t) = \int_0^\infty [A(w)\cos wt + B(w)\sin wt]\,dw \quad (-\infty < t < \infty).$$

Where $A(w) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(t)\cos wt\,dt.$

& $B(w) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(t)\sin wt\,dt$

Exp) Find the F.I of the function.

$$f(t) = \begin{cases} 0 & t < 0. \\ 1 & 0 < t < 2. \\ 0 & t > 2. \end{cases}$$

Sol:- $A(w) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(t)\cos wt\,dt.$

$$\pi A(w) = \int_{-\infty}^{0} f(t)\cos wt\,dt + \int_0^2 f(t)\cos wt\,dt + \int_2^\infty f(t)\cos wt\,dt.$$

$$= \int_{-\infty}^{0} 0\cos wt\,dt + \int_0^2 1\cdot\cos wt\,dt + \int_2^\infty 0\cos wt\,dt.$$

$$= 0 + \frac{\sin wt}{w}\Big|_0^2 + 0$$

$$\pi(A(w)) = \frac{\sin 2w}{w} \Rightarrow \boxed{A(w) = \frac{\sin 2w}{\pi w}}$$

$$\pi B(w) = \int_{-\infty}^{0} f(t)\sin wt\,dt + \int_{0}^{2} f(t)\sin wt\,dt + \int_{2}^{\infty} f(t)\sin wt\,dt.$$

$$= \int_{-\infty}^{0} 0\cdot \sin wt\,dt + \int_{0}^{2} 1\cdot \sin wt\,dt + \int_{2}^{\infty} 0\cdot \sin wt\,dt.$$

$$= 0 - \frac{\cos wt}{w}\Big|_{0}^{2} + 0$$

$$\pi B(w) = -\frac{\cos 2w}{w} - \left(-\frac{\cos 0t}{w}\right).$$

$$= \frac{-\cos 2w + \cos 0}{w} = \frac{1-\cos 2w}{w}$$

$$\boxed{B(w) = \frac{1-\cos 2w}{\pi w}}$$

Note 1) The Fourier cosine integral is given by:

$$f(x) = \int_{0}^{\infty} A(w)\cos wx\,dw.$$

where $A(w) = \frac{2}{\pi}\int_{0}^{\infty} f(v)\cos wv\,dv.$

Note 2) The Fourier Sine Integral is given by:

$$f(x) = \int_{0}^{\infty} B(w)\sin wx\,dw.$$

where $B(w) = \frac{2}{\pi}\int_{0}^{\infty} f(v)\sin wv\,dv.$

From this representation we see that

$$\int_0^\infty \frac{\cos wx}{k^2 + w^2}\, dw = \frac{\pi}{2k} e^{-kx} \qquad (x > 0,\ k > 0). \tag{15}$$

(b) Similarly, from (12) we have

$$B(w) = \frac{2}{\pi}\int_0^\infty e^{-kv} \sin wv\, dv.$$

By integration by parts,

$$\int e^{-kv} \sin wv\, dv = -\frac{w}{k^2 + w^2} e^{-kv}\left(\frac{k}{w}\sin wv + \cos wv\right).$$

This equals $-w/(k^2 + w^2)$ if $v = 0$, and approaches 0 as $v \to \infty$. Thus

$$B(w) = \frac{2w/\pi}{k^2 + w^2}. \tag{16}$$

From (13) we thus obtain the Fourier sine integral representation

$$f(x) = e^{-kx} = \frac{2}{\pi}\int_0^\infty \frac{w \sin wx}{k^2 + w^2}\, dw.$$

From this we see that

$$\int_0^\infty \frac{w \sin wx}{k^2 + w^2}\, dw = \frac{\pi}{2} e^{-kx} \qquad (x > 0,\ k > 0). \tag{17}$$

The integrals (15) and (17) are called the **Laplace integrals.** ◀

PROBLEM SET 10.8

Evaluation of Integrals

Using (5), (11), or (13), show that the given integrals represent the indicated functions. (Can you see that the integral tells you which formula to use? Show the details of your work.)

1. $\displaystyle\int_0^\infty \frac{\cos xw + w \sin xw}{1 + w^2}\, dw = \begin{cases} 0 & \text{if } x < 0 \\ \pi/2 & \text{if } x = 0 \\ \pi e^{-x} & \text{if } x > 0 \end{cases}$

2. $\displaystyle\int_0^\infty \frac{\sin w \cos xw}{w}\, dw = \begin{cases} \pi/2 & \text{if } 0 \leqq x < 1 \\ \pi/4 & \text{if } x = 1 \\ 0 & \text{if } x > 1 \end{cases}$

3. $\displaystyle\int_0^\infty \frac{1 - \cos \pi w}{w} \sin xw\, dw = \begin{cases} \pi/2 & \text{if } 0 < x < \pi \\ 0 & \text{if } x > \pi \end{cases}$

4. $\displaystyle\int_0^\infty \frac{\cos (\pi w/2) \cos xw}{1 - w^2}\, dw = \begin{cases} (\pi/2) \cos x & \text{if } |x| < \pi/2 \\ 0 & \text{if } |x| > \pi/2 \end{cases}$

5. $\displaystyle\int_0^\infty \frac{\cos xw}{1 + w^2}\, dw = \frac{\pi}{2} e^{-x}$ if $x > 0$

6. $\displaystyle\int_0^\infty \frac{w^3 \sin xw}{w^4 + 4}\, dw = \frac{\pi}{2} e^{-x} \cos x$ if $x > 0$

Fourier Cosine Integral Representation

Represent the following functions $f(x)$ in the form (11).

7. $f(x) = \begin{cases} 1 & \text{if } 0 < x < 1 \\ 0 & \text{if } x > 1 \end{cases}$

8. $f(x) = \begin{cases} x^2 & \text{if } 0 < x < 1 \\ 0 & \text{if } x > 1 \end{cases}$

9. $f(x) = \begin{cases} x & \text{if } 0 < x < a \\ 0 & \text{if } x > a \end{cases}$

10. $f(x) = \begin{cases} a^2 - x^2 & \text{if } 0 < x < a \\ 0 & \text{if } x > a \end{cases}$

11. $f(x) = 1/(1 + x^2)$ $[x > 0$, see (15)]

12. $f(x) = e^{-x} + e^{-2x}$ $(x > 0)$

Fourier Sine Integral Representation

Represent the following functions $f(x)$ in the form (13).

13. $f(x) = \begin{cases} 1 & \text{if } 0 < x < a \\ 0 & \text{if } x > a \end{cases}$

14. $f(x) = \begin{cases} x & \text{if } 0 < x < a \\ 0 & \text{if } x > a \end{cases}$

15. $f(x) = \begin{cases} \sin x & \text{if } 0 < x < \pi \\ 0 & \text{if } x > \pi \end{cases}$

16. $f(x) = \begin{cases} \pi - x & \text{if } 0 < x < \pi \\ 0 & \text{if } x > \pi \end{cases}$

17. $f(x) = \begin{cases} e^{x} & \text{if } 0 < x < 1 \\ 0 & \text{if } x > 1 \end{cases}$

18. $f(x) = \begin{cases} e^{-x} & \text{if } 0 < x < 1 \\ 0 & \text{if } x > 1 \end{cases}$

19. **(CAS. Sine integral)** Plot Si(u) for positive u. Does the sequence of the maximum and minimum values make the impression that it converges and has the limit $\pi/2$? Investigate the Gibbs phenomenon graphically.

20. **PROJECT. Properties of Fourier Integrals. (a) Fourier Cosine Integral.** Show that (11) implies

$$\text{(a1)} \quad f(ax) = \frac{1}{a}\int_0^\infty A\left(\frac{w}{a}\right) \cos xw \, dw \quad (a > 0)$$

$$\text{(a2)} \quad xf(x) = \int_0^\infty B^*(w) \sin xw \, dw, \quad B^* = -\frac{dA}{dw}, \quad A \text{ as in (10)}$$

$$\text{(a3)} \quad x^2 f(x) = \int_0^\infty A^*(w) \cos xw \, dw, \quad A^* = -\frac{d^2A}{dw^2}.$$

(b) Solve Prob. 8 by applying (a3) to the result of Prob. 7.

(c) Verify (a2) for $f(x) = 1$ if $0 < x < a$ and $f(x) = 0$ if $x > a$.

(d) Fourier Sine Integral. Find formulas for the Fourier sine integral similar to those in (a).

10.9 Fourier Cosine and Sine Transforms

An **integral transform** is a transformation that produces from given functions new functions that depend on a different variable and appear in the form of an integral. These transformations are of interest mainly as tools in solving ordinary differential equations, partial differential equations, and integral equations, and they often also help in handling and applying special functions. The **Laplace transform** (Chap. 5) is of this kind and is by far the most important integral transform in engineering. From the viewpoint of applications, the next in order of importance are perhaps the **Fourier transforms,** although

(21)

Ex 10.8

using the Fourier integral formula representation, show that

Q1 $$\int_0^\infty \frac{\cos xw + w\sin xw}{1+w^2}\,dw = \begin{cases} 0 & \text{if } x<0 \\ \pi/2 & \text{If } x=0 \\ \pi e^{-x} & \text{if } x>0 \end{cases}$$

Sol:- R.H.S

$$f(x) = \begin{cases} 0 & \text{if } x<0 \\ \pi/2 & \text{If } x=0 \\ \pi e^{-x} & \text{if } x>0 \end{cases} \longrightarrow (*)$$

The Fourier integral ref $f(x)$ is given by.

$$f(x) = \int_0^\infty [A(w)\cos wx + B(w)\sin wx]\,dw \longrightarrow (1)$$

where $A(w) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(v)\cos wv\,dv$

$$= \frac{1}{\pi}\Big[\int_{-\infty}^{0} 0\times\cos wv\,dv + \int_0^0 \frac{\pi}{2}\cos wv\,dv + \int_0^\infty \pi e^{-v}\cos wv\,dv\Big].$$

$$= \frac{1}{\pi}\Big[0 + 0 + \int_0^\infty \pi e^{-v}\cos wv\,dv\Big].$$

$$A(w) = \frac{1}{\pi}\int_0^\infty \pi \underset{\text{I}}{e^{-v}}\cos wv\,dv$$

By Part

$$\boxed{A(w) = \frac{1}{1+w^2}}$$

Now $B(w) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(v).\sin wv\,dv$

$$= \frac{1}{\pi}\left[\int_{-\infty}^{0} 0.\sin wv\,dv + \int_{0}^{0} \frac{\pi}{2}\sin wv\,dv + \int_{0}^{\infty} \pi e^{-v}\sin wv\,dv\right]$$

$$B(w) = \frac{1}{\pi}\left[0 + 0 + \pi\int_{0}^{\infty} e^{-v}\sin wv\,dv\right].$$

$$B(w) = \int_{0}^{\infty} e^{-v}\sin wv\,dv$$

By Part

$$\boxed{B(w) = \frac{w}{1+w^2}}$$

eq (1) $\Rightarrow$

$$f(x) = \int_{0}^{\infty}\left[\left[\frac{1}{1+w^2}\cos wx + \frac{w}{1+w^2}\sin wx\right]dw\right].$$

$$f(x) = \int_{0}^{\infty} \frac{\cos wx + w\sin wx}{1+w^2}\,dw \longrightarrow (**)$$

from eq (*) & eq (**) we get

$$\int_{0}^{\infty} \frac{\cos wx + w\sin wx}{1+w^2}\,dw = \begin{cases} 0 & \text{if } x < 0 \\ \pi/2 & \text{if } x = 0 \\ \pi e^{-x} & \text{if } x > 0 \end{cases}$$

Q2 $\displaystyle\int_0^\infty \frac{\sin w \cos wx}{w}\,dw = \begin{cases} \pi/2 & \text{if } 0 \le x < 1 \\ \pi/4 & \text{if } x = 1 \\ 0 & \text{if } x > 1 \end{cases}$

Sol:- $f(x) = \begin{cases} \pi/2 & \text{if } 0 \le x < 1 \\ \pi/4 & \text{if } x = 1 \\ 0 & \text{if } x > 1 \end{cases} \longrightarrow (*)$

The Fourier integral of $f(x)$ is given by.

$$f(x) = \int_0^\infty A(w) \cos wx \, dw \longrightarrow (1).$$

where $A(w) = \frac{2}{\pi}\int_0^\infty f(v) \cos vw \, dv$

$$A(w) = \frac{2}{\pi}\left[\int_0^1 \pi/2 \cos vw \, dv + \int_1^1 \pi/4 \cos vw\, dv + \int_1^\infty 0 \, dv\right].$$

$$A(w) = \frac{2}{\pi}\left[\pi/2 \frac{\sin vw}{w}\Big|_0^1 + 0 + 0\right].$$

$$A(w) = \frac{1}{w}\sin w$$

eq (1) $\Rightarrow$ $f(x) = \displaystyle\int_0^\infty \frac{\sin w \cos wx}{w} \longrightarrow (**)$

from eq (*) & eq (**) we get

$$\int_0^\infty \frac{\sin w \cos wx}{w}\,dw = \begin{cases} \pi/2 & \text{if } 0 \le x < 1 \\ \pi/4 & \text{if } x = 1 \\ 0 & \text{if } x > 1 \end{cases}$$

Q3 $\int_0^{\infty} \frac{1-\cos\pi w}{w} \sin xw \, dw = \begin{cases} \pi/2 & \text{if } 0 < x < \pi. \\ 0 & \text{if } x > \pi. \end{cases}$

Sol:- $f(x) = \begin{cases} \pi/2 & \text{if } 0 < x < \pi \\ 0 & \text{if } x > \pi \end{cases} \longrightarrow (1)$

Now their Fourier Sine integral

$$f(x) = \int_0^{\infty} B(w) \sin wx \, dw \longrightarrow (2)$$

where $B(w) = \frac{2}{\pi} \left\{ \int_0^{\pi} \pi/2 \sin vw \, dv + \int_{\pi}^{\infty} 0 \, dv \right\}$

$$B(w) = \frac{1-\cos\pi w}{w}$$

eq (2) $\Rightarrow$

$$f(x) = \int_0^{\infty} \frac{1-\cos\pi w}{w} \sin wx \, dw \longrightarrow (3)$$

From eq (3) & eq (1) we can say that

$$\int_0^{\infty} \frac{1-\cos\pi w}{w} \sin wx \, dw = \begin{cases} \pi/2 & \text{if } 0 < x < \pi \\ 0 & \text{if } x > \pi. \end{cases}$$

Q Find the Fourier cosine integral of the given Function

Q7 $f(x) = \begin{cases} 1 & \text{if } 0 < x < 1 \\ 0 & \text{if } x > 1 \end{cases}$

Sol:- The Fourier cosine integral of $f(x)$ is given

By:

$$f(x) = \int_0^{\pi} A(w) \cos wx \, dw \longrightarrow (1)$$

where $A(w) = \frac{2}{\pi} \int_0^{\infty} f(v) \cos wv \, dv$

$$= \frac{2}{\pi} \left[\int_0^1 1 \cos wv \, dv + \int_1^{\infty} 0 \, dv \right]$$

$$A(w) = \frac{2}{\pi} \left. \frac{\sin wv}{w} \right|_0^1$$

$$A(w) = \frac{2 \sin w}{\pi w}$$

eq (1) $\Rightarrow$

$$f(x) = \int_0^{\pi} \frac{2}{\pi w} \sin w \cos wx \, dw$$

$$f(x) = \frac{2}{\pi} \int_0^{\infty} \frac{\sin w \cos wx}{w} \, dw$$

Q8 $f(x) = \begin{cases} x^2 & \text{if } 0 < x < 1 \\ 0 & \text{if } x > 1. \end{cases}$

Sol:- The Fourier cosine integral is given by:

$$f(x) = \int_0^\infty A(w) \cos wx \, dw \quad \text{—— (1)}$$

where $A(w) = \frac{2}{\pi} \left[\int_0^1 v^2 \cos wv \, dv + \int_1^\infty 0 \, dv \right].$

$$= \frac{2}{\pi} \left[\frac{v^2 \sin wv}{w} \Big|_0^1 - \frac{2}{w} \int_0^1 v \sin wv \, dv \right].$$

$$= \frac{2}{\pi} \left[\frac{\sin w}{w} - \frac{2}{w} \left\{ \frac{-v \cos wv}{w} \Big|_0^1 + \frac{1}{w^2} \sin wv \Big|_0^1 \right\} \right]$$

$$= \frac{2}{\pi} \left[\frac{\sin w}{w} + \frac{2 \cos w}{w^2} - \frac{2 \sin w}{w^2} \right].$$

$$= \frac{2}{\pi} \left[\sin w \left(\frac{w - 2}{w^2} \right) + \frac{2}{w^2} \cos w \right].$$

eq (1) $\Rightarrow$

$$f(x) = \frac{2}{\pi} \int_0^\infty \frac{1}{w} \left[\left(1 - \frac{2}{w}\right) \sin w + \frac{2}{w} \cos w \right] \cos wx \, dw.$$

Q9 $f(x) = \begin{cases} x & \text{if } 0 < x < a. \\ 0 & \text{if } x > a. \end{cases}$

Sol:- The fourier cosine integral is given by

$$f(x) = \int_0^{\infty} A(w) \cos wx \, dw \quad \text{---} (1)$$

where $A(w) = \frac{2}{\pi} \left[\int_0^{a} v \cos vw \, dv + \int_a^{\infty} 0 \, dv \right].$

$$= \frac{2}{\pi} \left[\frac{v}{w} \sin vw \Big|_0^a + \frac{1}{w^2} \cos vw \Big|_0^a \right].$$

$$= \frac{2}{\pi} \left[\frac{a \sin aw}{w} + \frac{\cos aw}{w^2} - \frac{1}{w^2} \right].$$

$$= \frac{2}{\pi} \left[\frac{a \sin aw}{w} + \frac{1}{w^2} (\cos aw - 1) \right].$$

eq (1) $\Rightarrow$

$$f(x) = \frac{2}{\pi} \int_0^{\infty} \left[\frac{a}{w} \sin aw + \frac{1}{w^2} (\cos aw - 1) \right] . \cos wx \, dw$$

Find the Fourier Sine integral of the given function.

Exp) $f(x) = \begin{cases} a^2 - x^2 & \text{if } 0 < x < a. \\ 0 & \text{if } x > a. \end{cases}$

Sol:- The Fourier Sine integral is given by:

$$f(x) = \int_0^\infty B(w) \sin wx \, dw \quad \text{—} (1)$$

where $B(w) = \frac{2}{\pi} \int_0^\infty f(v) \sin wv \, dv$.

S.Y.S.

Q13 $f(x) = \begin{cases} 1 & \text{if } 0 < x < a, \\ 0 & \text{if } x > a. \end{cases}$

Q14 $f(x) = \begin{cases} x & \text{if } 0 < x < a. \\ 0 & \text{if } x > a. \end{cases}$

Q15 $f(x) = \begin{cases} \sin x & \text{if } 0 < x < \pi \\ 0 & \text{if } x > \pi \end{cases}$

Q16 $f(x) = \begin{cases} \pi - x & \text{if } 0 < x < \pi \\ 0 & \text{if } x > \pi \end{cases}$

Q17

$f(x) = \begin{cases} e^{x} & \text{if } 0 < x < 1 \\ 0 & \text{if } x > 1 \end{cases}$

Q18

$f(x) = \begin{cases} e^{-x} & \text{if } 0 < x < 1 \\ 0 & \text{if } x > 1 \end{cases}$

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