6.3 Exercises

1. Which of the following vectors span R^2?
   (a) $(1, 2), (-1, 1)$
   (b) $(0, 0), (1, 1), (-2, -2)$
   (c) $(1, 3), (2, -3), (0, 2)$
   (d) $(2, 4), (-1, 2)$

2. Which of the following sets of vectors span R^3?
   (a) $\{(1, -1, 2), (0, 1, 1)\}$
   (b) $\{(1, 2, -1), (6, 3, 0), (4, -1, 2), (2, -5, 4)\}$
   (c) $\{(2, 2, 3), (-1, -2, 1), (0, 1, 0)\}$
   (d) $\{(1, 0, 0), (0, 1, 0), (0, 0, 1), (1, 1, 1)\}$

3. Which of the following vectors span R^4?
   (a) $(1, 0, 0, 1), (0, 1, 0, 0), (1, 1, 1, 1), (1, 1, 1, 0)$
   (b) $(1, 2, 1, 0), (1, 1, -1, 0), (0, 0, 0, 1)$
   (c) $(6, 4, -2, 4), (2, 0, 0, 1), (3, 2, -1, 2), (5, 6, -3, 2), (0, 4, -2, -1)$
   (d) $(1, 1, 0, 0), (1, 2, -1, 1), (0, 0, 1, 1), (2, 1, 2, 1)$

4. Which of the following sets of polynomials span P_2?
   (a) $\{t^2 + 1, t^2 + t, t + 1\}$
   (b) $\{t^2 + 1, t - 1, t^2 + t\}$
   (c) $\{t^2 + 2, 2t^2 - t + 1, t + 2, t^2 + t + 4\}$
   (d) $\{t^2 + 2t - 1, t^2 - 1\}$

5. Do the polynomials $t^3 + 2t + 1, t^2 - t + 2, t^3 + 2, -t^3 + t^2 - 5t + 2$ span P_3?

6. Find a set of vectors spanning the solution space of $A\mathbf{x} = \mathbf{0}$, where

$$A = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 1 & 2 & 3 & 1 \\ 2 & 1 & 3 & 1 \\ 1 & 1 & 2 & 1 \end{bmatrix}.$$

7. Find a set of vectors spanning the null space of

$$A = \begin{bmatrix} 1 & 1 & 2 & -1 \\ 2 & 3 & 6 & -2 \\ -2 & 1 & 2 & 2 \\ 0 & -2 & -4 & 0 \end{bmatrix}.$$

8. Let

$$\mathbf{x}_1 = \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}, \quad \mathbf{x}_2 = \begin{bmatrix} 4 \\ -7 \\ -1 \end{bmatrix}, \quad \mathbf{x}_3 = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}$$

belong to the solution space of $A\mathbf{x} = \mathbf{0}$. Is $\{\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3\}$ linearly independent?

9. Let

$$\mathbf{x}_1 = \begin{bmatrix} 1 \\ 2 \\ 0 \\ 1 \end{bmatrix}, \quad \mathbf{x}_2 = \begin{bmatrix} 1 \\ 0 \\ -1 \\ 1 \end{bmatrix}, \quad \mathbf{x}_3 = \begin{bmatrix} 1 \\ 6 \\ 2 \\ 0 \end{bmatrix}$$

belong to the null space of A. Is $\{\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3\}$ linearly independent?

10. Which of the following sets of vectors in R^3 are linearly dependent? For those that are, express one vector as a linear combination of the rest.
    (a) $\{(1, 2, -1), (3, 2, 5)\}$
    (b) $\{(4, 2, 1), (2, 6, -5), (1, -2, 3)\}$
    (c) $\{(1, 1, 0), (0, 2, 3), (1, 2, 3), (3, 6, 6)\}$
    (d) $\{(1, 2, 3), (1, 1, 1), (1, 0, 1)\}$

11. Consider the vector space R^4. Follow the directions of Exercise 10.
    (a) $\{(1, 1, 2, 1), (1, 0, 0, 2), (4, 6, 8, 6), (0, 3, 2, 1)\}$
    (b) $\{(1, -2, 3, -1), (-2, 4, -6, 2)\}$
    (c) $\{(1, 1, 1, 1), (2, 3, 1, 2), (3, 1, 2, 1), (2, 2, 1, 1)\}$
    (d) $\{(4, 2, -1, 3), (6, 5, -5, 1), (2, -1, 3, 5)\}$

12. Consider the vector space P_2. Follow the directions of Exercise 10.
    (a) $\{t^2 + 1, t - 2, \ + 3\}$
    (b) $\{2t^2 + 1, t^2 + 3, t\}$
    (c) $\{3t + 1, 3t^2 + 1, 2t^2 + t + 1\}$
    (d) $\{t^2 - 4, 5t^2 - 5t - 6, 3t^2 - 5t + 2\}$

13. Consider the vector space M_{22}. Follow the directions of Exercise 10.
    (a) $\left\{\begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}, \begin{bmatrix} 0 & 3 \\ 1 & 2 \end{bmatrix}, \begin{bmatrix} 2 & 6 \\ 4 & 6 \end{bmatrix}\right\}$
    (b) $\left\{\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 2 \end{bmatrix}\right\}$
    (c) $\left\{\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix}, \begin{bmatrix} 3 & 1 \\ 2 & 1 \end{bmatrix}, \begin{bmatrix} 2 & 2 \\ 1 & 1 \end{bmatrix}\right\}$

14. Let V be the vector space of all real-valued continuous functions. Follow the directions of Exercise 10.
    (a) $\{\cos t, \sin t, e^t\}$
    (b) $\{t, e^t, \sin t\}$
    (c) $\{t^2, t, e^t\}$
    (d) $\{\cos^2 t, \sin^2 t, \cos 2t\}$

15. For what values of c are the vectors $(-1, 0, -1)$, $(2, 1, 2)$, and $(1, 1, c)$ in R^3 linearly dependent?

16. For what values of λ are the vectors $t + 3$ and $2t + \lambda^2 + 2$ in P_1 linearly dependent?

17. Determine if the vectors $\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$, and $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ span B^3.

18. Determine if the vectors $\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, and $\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$ span B^3.

19. Determine if the vectors $\begin{bmatrix}1\\1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\1\\1\\0\end{bmatrix}, \begin{bmatrix}0\\0\\1\\1\end{bmatrix}$, and $\begin{bmatrix}1\\0\\0\\1\end{bmatrix}$ span B^4.

20. Determine if the vectors in Exercise 17 are linearly independent.

21. Determine if the vectors in Exercise 19 are linearly independent.

22. Show that $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$ in Example 15 are linearly dependent using Theorem 6.4.

Theoretical Exercises

T.1. Show that the vectors $\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n$ in R^n are linearly independent.

T.2. Let S_1 and S_2 be finite subsets of a vector space and let S_1 be a subset of S_2. Show that:

(a) If S_1 is linearly dependent, so is S_2.

(b) If S_2 is linearly independent, so is S_1.

T.3. Let $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}$ be a set of vectors in a vector space. Show that S is linearly dependent if and only if one of the vectors in S is a linear combination of all the other vectors in S.

T.4. Suppose that $S = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is a linearly independent set of vectors in a vector space V. Show that $T = \{\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3\}$ is also linearly independent, where $\mathbf{w}_1 = \mathbf{v}_1 + \mathbf{v}_2 + \mathbf{v}_3$, $\mathbf{w}_2 = \mathbf{v}_2 + \mathbf{v}_3$, and $\mathbf{w}_3 = \mathbf{v}_3$.

T.5. Suppose that $S = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is a linearly independent set of vectors in a vector space V. Is $T = \{\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3\}$, where $\mathbf{w}_1 = \mathbf{v}_1 + \mathbf{v}_2$, $\mathbf{w}_2 = \mathbf{v}_1 + \mathbf{v}_3$, $\mathbf{w}_3 = \mathbf{v}_2 + \mathbf{v}_3$, linearly dependent or linearly independent? Justify your answer.

T.6. Suppose that $S = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is a linearly dependent set of vectors in a vector space V. Is $T = \{\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3\}$, where $\mathbf{w}_1 = \mathbf{v}_1$, $\mathbf{w}_2 = \mathbf{v}_1 + \mathbf{v}_2$; $\mathbf{w}_3 = \mathbf{v}_1 + \mathbf{v}_2 + \mathbf{v}_3$, linearly dependent or linearly independent? Justify your answer.

T.7. Let $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$ be vectors in a vector space such that $\{\mathbf{v}_1, \mathbf{v}_2\}$ is linearly independent. Show that if $\mathbf{v}_3$ does not belong to span $\{\mathbf{v}_1, \mathbf{v}_2\}$, then $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is linearly independent.

T.8. Let A be an $m \times n$ matrix in reduced row echelon form. Show that the nonzero rows of A, viewed as vectors in R^n, form a linearly independent set of vectors.

T.9. Let $S = \{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_k\}$ be a set of vectors in a vector space, and let $T = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_m\}$, where each $\mathbf{v}_i$, $i = 1, 2, \ldots, m$, is a linear combination of the vectors in S. Show that

$$\mathbf{w} = b_1\mathbf{v}_1 + b_2\mathbf{v}_2 + \cdots + b_m\mathbf{v}_m$$

is a linear combination of the vectors in S.

T.10. Suppose that $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ is a linearly independent set of vectors in R^n. Show that if A is an $n \times n$ nonsingular matrix, then $\{A\mathbf{v}_1, A\mathbf{v}_2, \ldots, A\mathbf{v}_n\}$ is linearly independent.

T.11. Let S_1 and S_2 be finite subsets of a vector space V and let S_1 be a subset of S_2. If S_2 is linearly dependent, show by examples that S_1 may be either linearly dependent or linearly independent.

T.12. Let S_1 and S_2 be finite subsets of a vector space and let S_1 be a subset of S_2. If S_1 is linearly independent, show by examples that S_2 may be either linearly dependent or linearly independent.

T.13. Let $\mathbf{u}$ and $\mathbf{v}$ be nonzero vectors in a vector space V. Show that $\{\mathbf{u}, \mathbf{v}\}$ is linearly dependent if and only if there is a scalar k such that $\mathbf{v} = k\mathbf{u}$. Equivalently, $\{\mathbf{u}, \mathbf{v}\}$ is linearly independent if and only if one of the vectors is not a multiple of the other.

T.14. *(Uses material from Section 5.1)* Let $\mathbf{u}$ and $\mathbf{v}$ be linearly independent vectors in R^3. Show that $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{u} \times \mathbf{v}$ form a basis for R^3. [*Hint*: Form Equation (1) and take the dot product with $\mathbf{u} \times \mathbf{v}$.]

T.15. Let W be a subspace of V spanned by the vectors $\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_k$. Is there any vector $\mathbf{v}$ in V such that the span of $\{\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_k, \mathbf{v}\}$ will also be W? Describe all such vectors $\mathbf{v}$.

MATLAB Exercises

ML.1. Determine if S is linearly independent or linearly dependent.

(a) $S = \{(1, 0, 0, 1), (0, 1, 1, 0), (1, 1, 1, 1)\}$

(b) $S = \left\{\begin{bmatrix}1 & 2\\1 & 0\end{bmatrix}, \begin{bmatrix}2 & -1\\1 & 2\end{bmatrix}, \begin{bmatrix}-3 & 1\\0 & 1\end{bmatrix}\right\}$

(c) $S = \left\{\begin{bmatrix}1\\2\\1\\0\\1\end{bmatrix}, \begin{bmatrix}0\\1\\2\\-1\\1\end{bmatrix}, \begin{bmatrix}2\\1\\0\\0\\-1\end{bmatrix}, \begin{bmatrix}-2\\1\\1\\1\\1\end{bmatrix}\right\}$

Ex 6.3

Q1 Which of the following vectors span R^2?

(a) $(1,2), (-1,1)$

Sol. let $V=(a,b)$ & $V_1=(1,2)$, $V_2=(-1,1)$

Now $V=C_1V_1+C_2V_2$

$(a,b)=C_1(1,2)+C_2(-1,1)$

$(a,b)=(C_1,2C_1)+(-C_2,C_2)$

$(a,b)=(C_1-C_2, 2C_1+C_2)$

$C_1-C_2=a$ — (i)

$2C_1+C_2=b$ — (ii)

(i)+(ii)

$C_1-\cancel{C_2}=a$

$2C_1+\cancel{C_2}=b$

$3C_1=a+b \Rightarrow \boxed{C_1=\frac{a+b}{3}}$

eq (i) $\Rightarrow \frac{a+b}{3}-C_2=a$

$C_2=\frac{a+b}{3}-a=\frac{a+b-3a}{3}=\frac{-2a+b}{3}$

$\boxed{C_2=\frac{-2a+b}{3}}$

So solution exist so span R^2

~~so [ ] span R^2~~

(b) $(0,0),(1,1),(-2,-2)$

let $V=(a,b)$, $V_1=(0,0)$, $V_2=(1,1)$ $V_3=(-2,-2)$

Now $V=C_1V_1+C_2V_2+C_3V_3$

$(a,b)=C_1(0,0)+C_2(1,1)+C_3(-2-2)$

$(a,b)=(0,0)+(C_2,C_2)+(-2C_3,-2C_3)$

$(a,b)=(0+C_2-2C_3,0+C_2-2C_3)$

$0+C_2-2C_3=a$ —— ①

$0+C_2-2C_3=b$ —— ②

① - ②

$0+\cancel{C_2}-\cancel{2C_3}=a$
$0+\cancel{C_2}-\cancel{2C_3}=b$
$\underline{\quad - \qquad + \qquad - \quad}$

$0\neq a-b$

Solution doesn't exist so V_1,V_2,V_3 don't span R^2

Part c same Part b & (d) same (a)

Ex 6.3

Q2 Which of the following sets of vector span R^3?

(a) $(1,-1,2), (0,1,1)$

let $V=(a,b,c)$, $V_1=(1,-1,2)$ $V_2=(0,1,1)$.

Now $V=C_1V_1+C_2V_2$

$$(a,b,c)=C_1(1,-1,2)+C_2(0,1,1)$$

$$(a,b,c)=(C_1,-C_1,2C_1)+(0,C_2,C_2)$$

$$(a,b,c)=(C_1,-C_1+C_2,2C_1+C_2)$$

$$C_1=a \text{ --- } 1$$

$$-C_1+C_2=b \text{ --- } 2$$

$$2C_1+C_2=c \text{ --- } 3$$

2 - 3

$$-C_1+\cancel{C_2}=b$$
$$\underline{2C_1+\cancel{C_2}=c}$$
$$-3C_1=b-c$$

$$C_1=-\left(\frac{b-c}{3}\right)=\boxed{\frac{c-b}{3}=C_1}$$

There are different values of C_1 so solution does not exist Not a span.

③

Ex 6.3

Q3 Which of the following vector span R^4?

(a) $(1,0,0,1)$

$(0,1,0,0)$

$(1,1,1,1)$

$(1,1,1,0)$

let $V=(a,b,c,d)$, $V_1=(1,0,0,1)$, $V_2=(0,1,0,0)$ $V_3=(1,1,1,1)$

$V_4=(1,1,1,0)$

Now $V=C_1V_1+C_2V_2+C_3V_3+C_4V_4$

$(a,b,c,d)=C_1(1,0,0,1)+C_2(0,1,0,0)+C_3(1,1,1,1)+C_4(1,1,1,0)$

$(a,b,c,d)=(C_1,0,0,C_1)+(0,C_2,0,0)+(C_3,C_3,C_3,C_3)+(C_4,C_4,C_4,0)$

$(a,b,c,d)=(C_1+C_3+C_4,\ C_2+C_3+C_4,\ C_3+C_4,\ C_1+C_3)$

$C_1+C_3+C_4=a$ —— ①

$C_2+C_3+C_4=b$ —— ②

$C_3+C_4=c$ —— ③

$C_1+C_3=d$ —— ④

Put $c_1+c_3=d$ in eq ①

eq ① $\Rightarrow C_4+d=a \Rightarrow \boxed{C_4=a-d}$

eq ③ $\Rightarrow C_3+(a-d)=c \Rightarrow \boxed{C_3=c-a+d}$

eq ④ $\Rightarrow C_1+(c-a+d)=d \Rightarrow C_1=d-(c-a+d)$

$C_1=\cancel{d}-c+a-\cancel{d}=-c+a$

$$\boxed{c_1 = a - c}$$

eq (2) $\Rightarrow c_2 + (c - a + d) + (a - d) = b$

$$c_2 + c \cancel{- a} \cancel{+ d} \cancel{+ a} \cancel{- d} = b$$

$$\boxed{c_2 = b - c}$$

Hence solution exist so spane R^4

other parts are same as part a

Q4 (a) $\{t^2+1, t^2+t, t+1\}$

Let $P = (a, b, c)$, $P_1 = (t^2+1)$, $P_2 = (t^2+t)$, $P_3 = (t+1)$

$$P = c_1 P_1 + c_2 P_2 + c_3 P_3$$

$$(a, b, c) = c_1(t^2+1) + c_2(t^2+t) + c_3(t+1)$$

$$= (c_1 t^2 + c_1) + (c_2 t^2 + c_2 t) + (c_3 t + c_3)$$

$$= \cancel{c_1 t^2 + c_2 t^2 + c_2 t + c_3 t}$$

$$(a, b, c) = (c_1 + c_2) t^2 + (c_2 + c_3) t + (c_1 + c_3)$$

$c_1 + c_2 = a$ — (1)

$c_3 + c_2 = b$ — (2)

$c_1 + c_3 = c$ — (3)

④

Ex 63

eq ① => $c_1 = a - c_2$

eq ③ => $a - c_2 + c_3 = c$

$-c_2 + c_3 = c - a$ — ④

② + ④

$c_3 + \cancel{c_2} = b$

$c_3 - \cancel{c_2} = c - a$

$2c_3 = c - a + b$

$$\boxed{c_3 = \frac{c - a + b}{2}}$$

eq ② $c_2 = b - c_3$

$$c_2 = b - \left(\frac{c - a + b}{2}\right)$$

$$= \frac{2b - c + a - b}{2} = \frac{b - c + a}{2}$$

$$\boxed{c_2 = \frac{b - c + a}{2}}$$

eq ① => $c_1 = a - c_2$

$$= a - \left(\frac{b + a - c}{2}\right) = \frac{2a - b - a + c}{2}$$

$$\boxed{c_1 = \frac{a - b + c}{2}}$$

So the Solution Exist hence a Polyno: Span P_2

Q5 is same as Q4.

Q6 Augmented form

$$\left[\begin{array}{cccc|c} 1 & 0 & 1 & 0 & 0 \\ 1 & 2 & 3 & 1 & 0 \\ 2 & 1 & 3 & 1 & 0 \\ 1 & 1 & 2 & 1 & 0 \end{array}\right]$$

$$\sim \left[\begin{array}{cccc|c} 1 & 0 & 1 & 0 & 0 \\ 0 & 2 & 2 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 \end{array}\right] \begin{array}{l} R_2-R_1 \\ -2R_3-2R_1 \\ R_3-R_1 \end{array}$$

$$\sim \left[\begin{array}{cccc|c} 1 & 0 & 1 & 0 & 0 \\ 0 & 2 & 2 & 1 & 0 \\ 0 & 0 & 0 & 1/2 & 0 \\ 0 & 0 & 0 & 1/2 & 0 \end{array}\right] \begin{array}{l} R_3-1/2R_2 \\ R_4-1/2R_2 \end{array}$$

$$\sim \left[\begin{array}{cccc|c} 1 & 0 & 1 & 0 & 0 \\ 0 & 2 & 2 & 1 & 0 \\ 0 & 0 & 0 & 1/2 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] R_4-R_3$$

$\frac{1}{2}C_4=0 \Rightarrow C_4=0$

$2C_2+2C_3+C_4=0$

$2C_2+2C_3+0=0$

$2C_2+2C_3=0$

$$2(c_2+c_3)=0$$

$$c_2+c_3=0$$

$$c_2=-c_3$$ let $c_3=r$ (any real no)

$$c_2=-r$$

$$c_1+c_3=0$$

$$c_1=-c_3$$

$$c_1=-r$$

So span exist has infinite many solution

Q7 is same Q6

Q9 $X_1=\begin{bmatrix}1\\2\\0\\1\end{bmatrix}$, $X_2=\begin{bmatrix}1\\0\\-1\\1\end{bmatrix}$, $X_3=\begin{bmatrix}1\\6\\2\\0\end{bmatrix}$, $X=\begin{bmatrix}0\\0\\0\\0\end{bmatrix}$

Now

$$X=c_1X_1+c_2X_2+c_3X_3$$

$$\begin{bmatrix}0\\0\\0\\0\end{bmatrix}=c_1\begin{bmatrix}1\\2\\0\\1\end{bmatrix}+c_2\begin{bmatrix}1\\0\\-1\\1\end{bmatrix}+c_3\begin{bmatrix}1\\6\\2\\0\end{bmatrix}.$$

$$\begin{bmatrix}0\\0\\0\\0\end{bmatrix}=\begin{bmatrix}c_1+c_2+c_3\\2c_1+0+6c_3\\0-c_2+2c_3\\c_1+c_2+0\end{bmatrix}$$

$c_1+c_2+c_3=0$ — ①

$2c_1+6c_3=0$ — ②

$-c_2+2c_3=0$ — ③

$c_1+c_2=0$ — ④

Put c_1+c_2 values in eq ①

eq ① $\Rightarrow 0+c_3=0 \Rightarrow \boxed{c_3=0}$

eq ③ $\Rightarrow -c_2+2(0)=0 \Rightarrow \boxed{c_2=0}$

eq ④ $\Rightarrow c_1+0=0 \Rightarrow \boxed{c_1=0}$

Hence solution exist so $[x_1, x_2, x_3]$ is L.I.

———X———

Q8 is same as Q9.

———X———

Q10 $V_1=(1,2,-1)$ & $V_2=(3,2,5)$

Now $c_1V_1+c_2V_2=0$

$c_1(1,2,-1)+c_2(3,2,5)=0$

$c_1+3c_2=0$ — ①

$2c_1+2c_2=0$ — ②

$-c_1+5c_2=0$ — ③

Scanned with CamScanner

(1) - (3)

$$\cancel{c_1} + 3c_2 = 0$$
$$-\cancel{c_1} + 5c_2 = 0$$

$$8c_2 = 0$$

$$\boxed{c_2 = 0}$$

eq (1) $\Rightarrow c_1 + 3(0) = 0 \Rightarrow \boxed{c_1 = 0}$

Hence linearly independent.

(b), (c), (d) is similarly.

Q11 is similarly to Q10

Q12 (a) $P_1 = t^2+1$, $P_2 = t-2$, $P_3 = t+3$.

Now $c_1P_1 + c_2P_2 + c_3P_3 = 0$

$c_1(t^2+1) + c_2(t-2) + c_3(t+3) = 0$

$(c_1t^2 + c_1) + (c_2t - c_2) + (c_3t + c_3) = 0$

$(c_1)t^2 + (c_2 + c_3)t + (c_1 - c_2 + c_3) = 0$.

$c_1 = 0$ — (1) $\Rightarrow \boxed{c_1 = 0}$

$c_2 + c_3 = 0$ — (2)

$c_1 - c_2 + c_3 = 0$ — (3)

Scanned with CamScanner

Put $c_1 = 0$ in eq ③

eq ③ $\Rightarrow 0 - c_2 + c_3 = 0$

$-c_2 + c_3 = 0$ —— ④

② + ④

$$\cancel{c_2} + c_3 = 0$$
$$-\cancel{c_2} + c_3 = 0$$
$$2c_3 = 0$$
$$\boxed{c_3 = 0}$$

eq ② $\Rightarrow c_2 + 0 = 0$

$$\boxed{c_2 = 0}$$

$c_1 = 0, c_2 = 0, c_3 = 0$ so linearly independent.

Part b, c, d is similarly

Ex 6.3

Q13 Consider the vector space M_{22}. Follow the direction of Exercise 10

(c) $\left\{\begin{bmatrix}1&1\\1&1\end{bmatrix}, \begin{bmatrix}2&3\\1&2\end{bmatrix}, \begin{bmatrix}3&1\\2&1\end{bmatrix}, \begin{bmatrix}2&2\\1&1\end{bmatrix}\right\}$

Now $C_1V_1 + C_2V_2 + C_3V_3 + C_4V_4 = 0$

$$C_1\begin{bmatrix}1&1\\1&1\end{bmatrix} + C_2\begin{bmatrix}2&3\\1&2\end{bmatrix} + C_3\begin{bmatrix}3&1\\2&1\end{bmatrix} + C_4\begin{bmatrix}2&2\\1&1\end{bmatrix} = 0$$

$C_1 + 2C_2 + 3C_3 + 2C_4 = 0$ —— ①

$C_1 + 3C_2 + C_3 + 2C_4 = 0$ —— ②

$C_1 + C_2 + 2C_3 + C_4 = 0$ —— ③

$C_1 + 2C_2 + C_3 + C_4 = 0$ —— ④

$$\left[\begin{array}{cccc|c}1&2&3&2&0\\1&3&1&2&0\\1&1&2&1&0\\1&2&1&1&0\end{array}\right]$$

$$\sim \left[\begin{array}{cccc|c}1&2&3&2&0\\0&1&-2&0&0\\0&-1&-1&-1&0\\0&0&-2&-1&0\end{array}\right] \begin{array}{l}R_2-R_1\\R_3-R_1\\R_4-R_1\end{array}$$

$$\sim \left[\begin{array}{cccc|c}1&2&3&2&0\\0&1&-2&0&0\\0&0&-3&-1&0\\0&0&-2&-1&0\end{array}\right] R_2+R_3$$

$$\sim \left[\begin{array}{cccc|c} 1 & 2 & 3 & 2 & 0 \\ 0 & 1 & -2 & 0 & 0 \\ 0 & 0 & 1 & 1/3 & 0 \\ 0 & 0 & -2 & -1 & 0 \end{array}\right] -\tfrac{1}{3}R_3$$

$$\sim \left[\begin{array}{cccc|c} 1 & 2 & 3 & 2 & 0 \\ 0 & 1 & -2 & 0 & 0 \\ 0 & 0 & 1 & 1/3 & 0 \\ 0 & 0 & 0 & -1/3 & 0 \end{array}\right] 2R_3+R_4$$

$-\tfrac{1}{3}C_4 = \boxed{C_4 = 0}$

$C_3 + \tfrac{1}{3}C_4 = 0$

$C_3 + \tfrac{1}{3}(0) = 0$

$\boxed{C_3 = 0}$

$C_2 - 2C_3 + 0C_4 = 0$

$C_2 - 2(0) + 0 = 0$

$\boxed{C_2 = 0}$

$C_1 + 2C_2 + 3C_3 + 2C_4 = 0$

$C_1 + 2(0) + 3(0) + 2(0) = 0$

$\boxed{C_1 = 0}$

$C_1 = C_2 = C_3 = C_4 = 0$

so it's linearly Independent.

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Ex 6.3

Q4 (a) $\{\cos t, \sin t, e^t\}$.

$V_1 = \cos t,\ V_2 = \sin t,\ V_3 = e^t$

Now $C_1V_1 + C_2V_2 + C_3V_3 = 0$

$c_1\cos t + c_2\sin t + c_3e^t = 0$

$c_1 = c_2 = c_3 = 0$ linearly Independent.

(b) $\{t, e^t \sin t\}$.

$V_1 = t,\ V_2 = e^t,\ V_3 = \sin t$

Now $C_1V_1 + C_2V_2 + C_3V_3 = 0$

$c_1t + c_2e^t + c_3\sin t = 0$

$c_1 = c_2 = c_3 = 0$ L. Independent.

(d) $\{\cos^2 t, \sin^2 t, \cos 2t\}$

$V_1 = \cos^2 t,\ V_2 = \sin^2 t,\ V_3 = \cos 2t = \cos^2 t - \sin^2 t$

Now $C_1V_1 + C_2V_2 + C_3V_3 = 0$

$c_1\cos^2 t + c_2\sin^2 t + c_3(\cos^2 t - \sin^2 t) = 0.$

$(c_1 + c_3)\cos^2 t + (c_2 - c_3)\sin^2 t = 0$

$c_1 + c_3 = 0$ — ①

$c_2 - c_3 = 0$ — ②

let $\boxed{C_3 = \gamma}$

eq ② $\Rightarrow C_2 - \gamma = 0 \Rightarrow \boxed{C_2 = +\gamma}$

eq ① $\Rightarrow C_1 + \gamma = 0 \Rightarrow \boxed{C_1 = -\gamma}$

Which is linearly dependent as non-trivial.

Q15 let

$V_1 = (-1, 0, -1)$

$V_2 = (2, 1, 2)$

$V_3 = (1, 1, c)$

Now $C_1 V_1 + C_2 V_2 + C_3 V_3 = 0$

$C_1(-1, 0, -1) + C_2(2, 1, 2) + C_3(1, 1, c) = 0$

$-C_1 + 2C_2 + C_3 = 0$ — ①

$+C_2 + C_3 = 0$ — ②

$-C_1 + 2C_2 + cC_3 = 0$ — ③

given L. Dependent so the solution is non-trivial

then det $= 0$

$$\begin{vmatrix} -1 & 2 & 1 \\ 0 & 1 & 1 \\ -1 & 2 & c \end{vmatrix} = 0$$

$$-1\begin{vmatrix} 1 & 1 \\ 2 & c \end{vmatrix} - 2\begin{vmatrix} 0 & 1 \\ -1 & c \end{vmatrix} + 1\begin{vmatrix} 0 & 1 \\ -1 & 2 \end{vmatrix} = 0$$

$$-1(c-2) - 2(0+1) + 1(0+1) = 0$$

$$-c + \cancel{2} - \cancel{2} + 1 = 0$$

$$-c + 1 = 0$$

$$\Rightarrow \boxed{c = 1}$$

Q16 let $V_1 = t + 3$

$V_2 = 2t + \lambda^2 + 2$

Now $C_1 V_1 + C_2 V_2 = 0$

$$c_1(t+3) + c_2(2t + \lambda^2 + 2) = 0$$

$$c_1 t + 3c_1 + 2c_2 t + c_2(\lambda^2 + 2) = 0$$

$$c_1 t + 2c_2 t = 0 \quad (1)$$

$$3c_1 + c_2(\lambda^2 + 2) = 0 \quad (2)$$

Det = 0 ∵ for non-trival solution

$$\begin{vmatrix} 1 & 2 \\ 3 & \lambda^2 + 2 \end{vmatrix} = 0 \qquad \lambda^2 + 2 - 6 = 0$$

$$\lambda^2 - 4 = 0 \Rightarrow \lambda^2 = 4 \Rightarrow \boxed{\lambda = \pm 2}$$

Q17 $V_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, V_2 \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, V_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ & $V = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$.

$c_1V_1 + c_2V_2 + c_3V_3 = V$

$c_1 \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + c_3 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$

$\begin{bmatrix} c_1 \\ c_1 \\ 0 \end{bmatrix} + \begin{bmatrix} c_2 \\ 0 \\ c_2 \end{bmatrix} + \begin{bmatrix} c_3 \\ c_3 \\ c_3 \end{bmatrix} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$

$\begin{bmatrix} c_1 + c_2 + c_3 \\ c_1 + 0 + c_3 \\ 0 + c_2 + c_3 \end{bmatrix} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$

$c_1 + c_2 + c_3 = a$ —— 1

$c_1 + 0 + c_3 = b$ —— 2

$0 + c_2 + c_3 = c$ —— 3

1 − 2

$\cancel{c_1} + c_2 + \cancel{c_3} = a$

$\cancel{c_1} + 0 + c_3 = b$

$c_2 = a - b \Rightarrow \boxed{c_2 = a - b} = \boxed{c_2 = a + b}$ For B.M

1 − 3

$c_1 + \cancel{c_2} + \cancel{c_3} = a$

$0 + \cancel{c_2} + \cancel{c_3} = c$

$c_1 = a - c \Rightarrow \boxed{c_1 = a - c} \Rightarrow \boxed{c_1 = a + c}$ For B.M

eq 1 $\Rightarrow a + c + a + b + c_3 = a$

$(1+1)a + c_3 = a + c + b = 0(a) + c_3 = a + c + b$ For B.M.

$c_3 = a + c + b - 2a = \boxed{c + b + a = c_3}$

$c_3 = a + c + b$

$c_3 = c_1 + b$

$c_3 - c_1 \pm b$

$c_3 + c_1 = b$ (For Bit m).

$c + b + a + a + c = b$

$c(1+1) + (1+1)a + b = b$

$c(0) + (0)a + b = b$

$0 + 0 + b = b$

$b = b$ verified.

So it span B^3

Q18 is same as Q17.

Q19 is same.

Q20 is same as Q8

Q21 is similarly

Q22 " "

2. Find a spanning set of the solution space of $A\mathbf{x} = \mathbf{0}$, where

$$A = \begin{bmatrix} 1 & 2 & 0 & 1 \\ 1 & 1 & 1 & 2 \\ 2 & -1 & 5 & 7 \\ 0 & 2 & -2 & -2 \end{bmatrix}.$$

4 BASIS AND DIMENSION

In this section we continue our study of the structure of a vector space V by determining a smallest set of vectors in V that completely describes V.

BASIS

DEFINITION The vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k$ in a vector space V are said to form a **basis** for V if (a) $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k$ span V and (b) $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k$ are linearly independent.

Remark If $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k$ form a basis for a vector space V, then they must be nonzero (see Example 12 in Section 6.3) and distinct and so we write them as a set $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}$.

EXAMPLE 1 The vectors $\mathbf{e}_1 = (1, 0)$ and $\mathbf{e}_2 = (0, 1)$ form a basis for R^2, the vectors $\mathbf{e}_1$, $\mathbf{e}_2$, and $\mathbf{e}_3$ form a basis for R^3 and, in general, the vectors $\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n$ form a basis for R^n. Each of these sets of vectors is called the **natural basis** or **standard basis** for R^2, R^3, and R^n, respectively. ■

EXAMPLE 2 Show that the set $S = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4\}$, where $\mathbf{v}_1 = (1, 0, 1, 0)$, $\mathbf{v}_2 = (0, 1, -1, 2)$, $\mathbf{v}_3 = (0, 2, 2, 1)$, and $\mathbf{v}_4 = (1, 0, 0, 1)$ is a basis for R^4.

Solution To show that S is linearly independent, we form the equation

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3 + c_4\mathbf{v}_4 = \mathbf{0}$$

and solve for c_1, c_2, c_3, and c_4. Substituting for $\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{v}_3$, and $\mathbf{v}_4$, we obtain the linear system (verify)

$$\begin{aligned} c_1 \qquad\qquad\qquad + c_4 &= 0 \\ c_2 + 2c_3 \qquad &= 0 \\ c_1 - c_2 + 2c_3 \qquad &= 0 \\ 2c_2 + c_3 + c_4 &= 0, \end{aligned}$$

which has as its only solution $c_1 = c_2 = c_3 = c_4 = 0$ (verify), showing that S is linearly independent. Observe that the coefficient matrix of the preceding linear system consists of the vectors $\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{v}_3$, and $\mathbf{v}_4$ written in column form.

To show that S spans R^4, we let $\mathbf{v} = (a, b, c, d)$ be any vector in R^4. We now seek constants k_1, k_2, k_3, and k_4 such that

$$k_1\mathbf{v}_1 + k_2\mathbf{v}_2 + k_3\mathbf{v}_3 + k_4\mathbf{v}_4 = \mathbf{v}.$$

Substituting for $\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{v}_3$, $\mathbf{v}_4$, and $\mathbf{v}$, we find a solution (verify) for k_1, k_2, k_3, and k_4 to the resulting linear system for any a, b, c, and d. Hence S spans R^4 and is a basis for R^4.

EXAMPLE 3 Show that the set $S = \{t^2 + 1, t - 1, 2t + 2\}$ is a basis for the vector space P_2.

Solution We must show that S spans V and is linearly independent. To show that it spans V, we take any vector in V, that is, a polynomial $at^2 + bt + c$, and must find constants a_1, a_2, and a_3 such that

$$\begin{aligned} at^2 + bt + c &= a_1(t^2 + 1) + a_2(t - 1) + a_3(2t + 2) \\ &= a_1t^2 + (a_2 + 2a_3)t + (a_1 - a_2 + 2a_3). \end{aligned}$$

Since two polynomials agree for all values of t only if the coefficients of respective powers of t agree, we get the linear system

$$\begin{aligned} a_1 \qquad\qquad &= a \\ a_2 + 2a_3 &= b \\ a_1 - a_2 + 2a_3 &= c. \end{aligned}$$

Solving, we have

$$a_1 = a, \quad a_2 = \frac{a + b - c}{2}, \quad a_3 = \frac{c + b - a}{4}.$$

Hence S spans V.

To illustrate this result, suppose that we are given the vector $2t^2 + 6t + 13$. Here $a = 2$, $b = 6$, and $c = 13$. Substituting in the foregoing expressions for a_1, a_2, and a_3, we find that

$$a_1 = 2, \quad a_2 = -\tfrac{5}{2}, \quad a_3 = \tfrac{17}{4}.$$

Hence

$$2t^2 + 6t + 13 = 2(t^2 + 1) - \tfrac{5}{2}(t - 1) + \tfrac{17}{4}(2t + 2).$$

To show that S is linearly independent, we form

$$a_1(t^2 + 1) + a_2(t - 1) + a_3(2t + 2) = 0.$$

Then

$$a_1t^2 + (a_2 + 2a_3)t + (a_1 - a_2 + 2a_3) = 0.$$

Again, this can hold for all values of t only if

$$\begin{aligned} a_1 \qquad\qquad &= 0 \\ a_2 + 2a_3 &= 0 \\ a_1 - a_2 + 2a_3 &= 0. \end{aligned}$$

The only solution to this homogeneous system is $a_1 = a_2 = a_3 = 0$, which implies that S is linearly independent. Thus S is a basis for P_2. ■

The set of vectors $\{t^n, t^{n-1}, \ldots, t, 1\}$ forms a basis for the vector space P_n called the **natural basis** or **standard basis** for P_n. It has already been shown in Example 5 of Section 6.3 that this set of vectors spans P_n. Linear independence is left as an exercise (Exercise T.15).

EXAMPLE 4 Find a basis for the subspace V of P_2, consisting of all vectors of the form $at^2 + bt + c$, where $c = a - b$.

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Solution Every vector in V is of the form

$$at^2 + bt + a - b$$

which can be written as

$$a(t^2 + 1) + b(t - 1),$$

so the vectors $t^2 + 1$ and $t - 1$ span V. Moreover, these vectors are linearly independent because neither one is a multiple of the other. This conclusion could also be reached (with more work) by writing the equation

$$a_1(t^2 + 1) + a_2(t - 1) = 0$$

or

$$t^2 a_1 + t a_2 + (a_1 - a_2) = 0.$$

Since this equation is to hold for all values of t, we must have $a_1 = 0$ and $a_2 = 0$. ■

A vector space V is called **finite-dimensional** if there is a finite subset of V that is a basis for V. If there is no such finite subset of V, then V is called **infinite-dimensional.**

Almost all the vector spaces considered in this book are finite-dimensional. However, we point out that there are many infinite-dimensional vector spaces that are extremely important in mathematics and physics; their study lies beyond the scope of this book. The vector space P, consisting of all polynomials, and the vector space $C(-\infty, \infty)$, consisting of all continuous functions $f: R \to R$, are not finite-dimensional.

We now establish some results about finite-dimensional vector spaces that will tell about the number of vectors in a basis, compare two different bases, and give properties of bases.

THEOREM 6.5 *If $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ is a basis for a vector space V, then every vector in V can be written in one and only one way as a linear combination of the vectors in S.*

Proof First, every vector $\mathbf{v}$ in V can be written as a linear combination of the vectors in S because S spans V. Now let

$$\mathbf{v} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_n\mathbf{v}_n \tag{1}$$

and

$$\mathbf{v} = d_1\mathbf{v}_1 + d_2\mathbf{v}_2 + \cdots + d_n\mathbf{v}_n. \tag{2}$$

Subtracting (2) from (1), we obtain

$$\mathbf{0} = (c_1 - d_1)\mathbf{v}_1 + (c_2 - d_2)\mathbf{v}_2 + \cdots + (c_n - d_n)\mathbf{v}_n.$$

Since S is linearly independent, it follows that $c_i - d_i = 0$, $1 \le i \le n$, so $c_i = d_i$, $1 \le i \le n$. Hence there is only one way to express $\mathbf{v}$ as a linear combination of the vectors in S. ■

We can also show (Exercise T.11) that if $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ is a set of vectors in a vector space V such that every vector in V can be written in one and only one way as a linear combination of the vectors in S, then S is a basis for V.

THEOREM 6.6 *Let $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ be a set of nonzero vectors in a vector space V and let $W = \text{span } S$. Then some subset of S is a basis for W.*

Proof **Case I.** If S is linearly independent, then since S already spans W, we conclude that S is a basis for W.

Case II. If S is linearly dependent, then

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_n\mathbf{v}_n = \mathbf{0}, \tag{3}$$

where $c_1, c_2, \ldots, c_n$ are not all zero. Thus, some $\mathbf{v}_j$ is a linear combination of the preceding vectors in S (Theorem 6.4). We now delete $\mathbf{v}_j$ from S, getting a subset S_1 of S. Then, by the observation made just before Example 14 in Section 6.3, we conclude that $S_1 = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_{j-1}, \mathbf{v}_{j+1}, \ldots, \mathbf{v}_n\}$ also spans W.

If S_1 is linearly independent, then S_1 is a basis. If S_1 is linearly dependent, delete a vector of S_1 that is a linear combination of the preceding vectors of S_1 and get a new set S_2 which spans W. Continuing, since S is a finite set, we will eventually find a subset T of S that is linearly independent and spans W. The set T is a basis for W.

Alternative Constructive Proof When V is R^m, $n \geq m$. We take the vectors in S as $m \times 1$ matrices and form Equation (3). This equation leads to a homogeneous system in the n unknowns $c_1, c_2, \ldots, c_n$; the columns of its $m \times n$ coefficient matrix A are $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$. We now transform A to a matrix B in reduced row echelon form, having r nonzero rows, $1 \leq r \leq m$. Without loss of generality, we may assume that the r leading 1s in the r nonzero rows of B occur in the first r columns. Thus we have

$$B = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 & b_{1\,r+1} & \cdots & b_{1n} \\ 0 & 1 & 0 & \cdots & 0 & b_{2\,r+1} & \cdots & b_{2n} \\ 0 & 0 & 1 & \cdots & 0 & b_{3\,r+1} & \cdots & b_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & 1 & b_{r\,r+1} & \cdots & b_{rn} \\ 0 & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \end{bmatrix}.$$

Solving for the unknowns corresponding to the leading 1s, we see that $c_1, c_2, \ldots, c_r$ can be solved for in terms of the other unknowns $c_{r+1}, c_{r+2}, \ldots, c_n$. Thus

$$\begin{aligned} c_1 &= -b_{1\,r+1}c_{r+1} - b_{1\,r+2}c_{r+2} - \cdots - b_{1n}c_n, \\ c_2 &= -b_{2\,r+1}c_{r+1} - b_{2\,r+2}c_{r+2} - \cdots - b_{2n}c_n, \\ &\vdots \\ c_r &= -b_{r\,r+1}c_{r+1} - b_{r\,r+2}c_{r+2} - \cdots - b_{rn}c_n, \end{aligned} \tag{4}$$

where $c_{r+1}, c_{r+2}, \ldots, c_n$ can be assigned arbitrary real values. Letting

$$c_{r+1} = 1, \quad c_{r+2} = 0, \ldots, \quad c_n = 0$$

in Equation (4) and using these values in Equation (3), we have

$$-b_{1\,r+1}\mathbf{v}_1 - b_{2\,r+1}\mathbf{v}_2 - \cdots - b_{r\,r+1}\mathbf{v}_r + \mathbf{v}_{r+1} = \mathbf{0},$$

which implies that $\mathbf{v}_{r+1}$ is a linear combination of $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_r$. By the observation made just before Example 14 in Section 6.3, the set of vectors obtained from S by deleting $\mathbf{v}_{r+1}$ spans W. Similarly, letting

$$c_{r+1} = 0, \quad c_{r+2} = 1, \quad c_{r+3} = 0, \ldots, \quad c_n = 0,$$

we find that $\mathbf{v}_{r+2}$ is a linear combination of $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_r$ and the set of vectors obtained from S by deleting $\mathbf{v}_{r+1}$ and $\mathbf{v}_{r+2}$ spans W. Continuing in this manner, $\mathbf{v}_{r+3}, \mathbf{v}_{r+4}, \ldots, \mathbf{v}_n$ are linear combinations of $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_r$, so it follows that $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_r\}$ spans W.

We next show that $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_r\}$ is linearly independent. Consider the matrix B_D obtained by deleting from B all columns not containing a leading 1. In this case, B_D consists of the first r columns of B. Thus,

$$B_D = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & & \\ & & & \ddots & \vdots \\ 0 & 0 & & \cdots & 1 \\ 0 & 0 & & \cdots & 0 \\ \vdots & \vdots & & & \vdots \\ 0 & 0 & & \cdots & 0 \end{bmatrix}.$$

Let A_D be the matrix obtained from A by deleting the columns corresponding to the columns that were deleted in B to obtain B_D. In this case, the columns of A_D are $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_r$, the first r columns of A. Since A and B are row equivalent, so are A_D and B_D. Then the homogeneous systems

$$A_D\mathbf{x} = \mathbf{0} \quad \text{and} \quad B_D\mathbf{x} = \mathbf{0}$$

are equivalent. Recall now that the homogeneous system $B_D\mathbf{x} = \mathbf{0}$ can be written equivalently as

$$x_1\mathbf{w}_1 + x_2\mathbf{w}_2 + \cdots + x_r\mathbf{w}_r = \mathbf{0}, \tag{5}$$

where

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_r \end{bmatrix}$$

and $\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_r$ are the columns of B_D. Since the columns of B_D form a linearly independent set of vectors in R^m, Equation (5) has only the trivial solution. Hence, $A_D\mathbf{x} = \mathbf{0}$ also has only the trivial solution. Thus the columns of A_D are linearly independent. That is, $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_r\}$ is linearly independent. ■

The first proof of Theorem 6.6 leads to a simple procedure for finding a subset T of a set S so that T is a basis for span S.

Let $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ be a set of nonzero vectors in a vector space V. The procedure for finding a subset of S that is a basis for $W = \text{span } S$ is as follows.

Step 1. Form Equation (3),

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_n\mathbf{v}_n = \mathbf{0},$$

which we solve for $c_1, c_2, \ldots, c_n$. If these are all zero, then S is linearly independent and is then a basis for W.

Step 2. If $c_1, c_2, \ldots, c_n$ are not all zero, then S is linearly dependent, so one of the vectors in S — say, $\mathbf{v}_j$ — is a linear combination of the preceding vectors in S. Delete $\mathbf{v}_j$ from S, getting the subset S_1, which also spans W.

Step 3. Repeat Step 1, using S_1 instead of S. By repeatedly deleting vectors of S we obtain a subset T of S that spans W and is linearly independent. Thus T is a basis for W.

This procedure can be rather tedious, since *every time* we delete a vector from S, we must solve a linear system. In Section 6.6 we shall present a much more efficient procedure for finding a basis for $W = \text{span } S$, but the basis is *not* guaranteed to be a subset of S. In many cases this is not a cause for concern, since one basis for $W = \text{span } S$ is as good as any other basis. However, there are cases when the vectors in S have some special properties and we want the basis for $W = \text{span } S$ to have the same properties, so we want the basis to be a subset of S. If $V = R^n$, the alternative proof of Theorem 6.6 yields a very efficient procedure (see Example 5 below) for finding a basis for $W = \text{span } S$ consisting of vectors from S.

Let $V = R^m$ and let $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ be a set of nonzero vectors in V. The procedure for finding a subset of S that is a basis for $W = \text{span } S$ is as follows.

Step 1. Form Equation (3),

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_n\mathbf{v}_n = \mathbf{0}.$$

Step 2. Construct the augmented matrix associated with the homogeneous system of Equation (3), and transform it to reduced row echelon form.

Step 3. The vectors corresponding to the columns containing the leading 1s form a basis for $W = \text{span } S$.

Recall that in the alternative proof of Theorem 6.6 we assumed without loss of generality that the r leading 1s in the r nonzero rows of B occur in the first r columns. Thus, if $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_6\}$ and the leading 1s occur in columns 1, 3, and 4, then $\{\mathbf{v}_1, \mathbf{v}_3, \mathbf{v}_4\}$ is a basis for span S.

Remark In Step 2 of the procedure in the preceding box, it is sufficient to transform the augmented matrix to row echelon form (see Section 1.6).

We form the augmented matrix and use row opertions: Add row 1 to row 4, add row 2 to row 3, and add row 2 to row 4, to obtain the equivalent augmented matrix (verify)

$$\left[\begin{array}{cccc|c} 1 & 0 & 1 & 0 & a \\ 0 & 1 & 1 & 0 & b \\ 0 & 0 & 1 & 1 & b+c \\ 0 & 0 & 0 & 0 & a+b+d \end{array}\right].$$

It follows that this system is inconsistent if the choice of bits a, b, and d is such that $a+b+d \neq 0$. For example, if

$$\mathbf{w} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix},$$

then the system is inconsistent; hence S does not span B^4 and is not a basis for B^4. ■

Key Terms

Basis
Natural (or standard) basis
Finite-dimensional vector space
Infinite-dimensional vector space
Dimension

6.4 Exercises

1. Which of the following sets of vectors are bases for R^2?
   (a) $\{(1,3),(1,-1)\}$
   (b) $\{(0,0),(1,2),(2,4)\}$
   (c) $\{(1,2),(2,-3),(3,2)\}$
   (d) $\{(1,3),(-2,6)\}$
2. Which of the following sets of vectors are bases for R^3?
   (a) $\{(1,2,0),(0,1,-1)\}$
   (b) $\{(1,1,-1),(2,3,4),(4,1,-1),(0,1,-1)\}$
   (c) $\{(3,2,2),(-1,2,1),(0,1,0)\}$
   (d) $\{(1,0,0),(0,2,-1),(3,4,1),(0,1,0)\}$
3. Which of the following sets of vectors are bases for R^4?
   (a) $\{(1,0,0,1),(0,1,0,0),(1,1,1,1),(0,1,1,1)\}$
   (b) $\{(1,-1,0,2),(3,-1,2,1),(1,0,0,1)\}$
   (c) $\{(-2,4,6,4),(0,1,2,0),(-1,2,3,2),(-3,2,5,6),(-2,-1,0,4)\}$
   (d) $\{(0,0,1,1),(-1,1,1,2),(1,1,0,0),(2,1,2,1)\}$
4. Which of the following sets of vectors are bases for P_2?
   (a) $\{-t^2+t+2,\ 2t^2+2t+3,\ 4t^2-1\}$
   (b) $\{t^2+2t-1,\ 2t^2+3t-2\}$
   (c) $\{t^2+1,\ 3t^2+2t,\ 3t^2+2t+1,\ 6t^2+6t+3\}$
   (d) $\{3t^2+2t+1,\ t^2+t+1,\ t^2+1\}$
5. Which of the following sets of vectors are bases for P_3?
   (a) $\{t^3+2t^2+3t,\ 2t^3+1,\ 6t^3+8t^2+6t+4,\ t^3+2t^2+t+1\}$
   (b) $\{t^3+t^2+1,\ t^3-1,\ t^3+t^2+t\}$
   (c) $\{t^3+t^2+t+1,\ t^3+2t^2+t+3,\ 2t^3+t^2+3t+2,\ t^3+t^2+2t+2\}$
   (d) $\{t^3-t,\ t^3+t^2+1,\ t-1\}$
6. Show that the matrices
$$\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}, \quad \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}$$
form a basis for the vector space M_{22}.

In Exercises 7 and 8, determine which of the given subsets forms a basis for R^3. Express the vector $(2,1,3)$ as a linear combination of the vectors in each subset that is a basis.

7. (a) $\{(1,1,1),(1,2,3),(0,1,0)\}$
   (b) $\{(1,2,3),(2,1,3),(0,0,0)\}$
8. (a) $\{(2,1,3),(1,2,1),(1,1,4),(1,5,1)\}$
   (b) $\{(1,1,2),(2,2,0),(3,4,-1)\}$

In Exercises 9 and 10, determine which of the given subsets forms a basis for P_2. Express $5t^2 - 3t + 8$ as a linear combination of the vectors in each subset that is a basis.

9. (a) $\{t^2 + t, t - 1, t + 1\}$
   (b) $\{t^2 + 1, t - 1\}$

10. (a) $\{t^2 + t, t^2, t^2 + 1\}$
    (b) $\{t^2 + 1, t^2 - t + 1\}$

11. Let $S = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4\}$, where

$$\mathbf{v}_1 = (1, 2, 2), \qquad \mathbf{v}_2 = (3, 2, 1),$$
$$\mathbf{v}_3 = (11, 10, 7), \quad \text{and} \quad \mathbf{v}_4 = (7, 6, 4).$$

Find a basis for the subspace $W = \operatorname{span} S$ of R^3. What is $\dim W$?

12. Let $S = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4, \mathbf{v}_5\}$, where

$$\mathbf{v}_1 = (1, 1, 0, -1), \quad \mathbf{v}_2 = (0, 1, 2, 1),$$
$$\mathbf{v}_3 = (1, 0, 1, -1), \quad \mathbf{v}_4 = (1, 1, -6, -3),$$

and $\mathbf{v}_5 = (-1, -5, 1, 0)$. Find a basis for the subspace $W = \operatorname{span} S$ of R^4. What is $\dim W$?

13. Consider the following subset of P_3:

$$S = \{t^3 + t^2 - 2t + 1, t^2 + 1, t^3 - 2t, 2t^3 + 3t^2 - 4t + 3\}.$$

Find a basis for the subspace $W = \operatorname{span} S$. What is $\dim W$?

14. Let

$$S = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} \right\}.$$

Find a basis for the subspace $W = \operatorname{span} S$ of M_{22}.

15. Find a basis for M_{23}. What is the dimension of M_{23}? Generalize to M_{mn}.

16. Consider the following subset of the vector space of all real-valued functions

$$S = \{\cos^2 t, \sin^2 t, \cos 2t\}.$$

Find a basis for the subspace $W = \operatorname{span} S$. What is $\dim W$?

In Exercises 17 and 18, find a basis for the given subspaces of R^3 and R^4.

17. (a) All vectors of the form (a, b, c), where $b = a + c$
    (b) All vectors of the form (a, b, c), where $b = a$
    (c) All vectors of the form (a, b, c), where $2a + b - c = 0$

18. (a) All vectors of the form (a, b, c), where $a = 0$
    (b) All vectors of the form $(a + c, a - b, b + c, -a + b)$
    (c) All vectors of the form (a, b, c), where $a - b + 5c = 0$

In Exercises 19 and 20, find the dimensions of the given subspaces of R^4.

19. (a) All vectors of the form (a, b, c, d), where $d = a + b$
    (b) All vectors of the form (a, b, c, d), where $c = a - b$ and $d = a + b$

20. (a) All vectors of the form (a, b, c, d), where $a = b$
    (b) All vectors of the form $(a + c, -a + b, -b - c, a + b + 2c)$

21. Find a basis for the subspace of P_2 consisting of all vectors of the form $at^2 + bt + c$, where $c = 2a - 3b$.

22. Find a basis for the subspace of P_3 consisting of all vectors of the form $at^3 + bt^2 + ct + d$, where $a = b$ and $c = d$.

23. Find the dimensions of the subspaces of R^2 spanned by the vectors in Exercise 1.

24. Find the dimensions of the subspaces of R^3 spanned by the vectors in Exercise 2.

25. Find the dimensions of the subspaces of R^4 spanned by the vectors in Exercise 3.

26. Find the dimension of the subspace of P_2 consisting of all vectors of the form $at^2 + bt + c$, where $c = b - 2a$.

27. Find the dimension of the subspace of P_3 consisting of all vectors of the form $at^3 + bt^2 + ct + d$, where $b = 3a - 5d$ and $c = d + 4a$.

28. Find a basis for R^3 that includes the vectors
    (a) $(1, 0, 2)$
    (b) $(1, 0, 2)$ and $(0, 1, 3)$

29. Find a basis for R^4 that includes the vectors $(1, 0, 1, 0)$ and $(0, 1, -1, 0)$.

30. Find all values of a for which $\{(a^2, 0, 1), (0, a, 2), (1, 0, 1)\}$ is a basis for R^3.

31. Find a basis for the subspace W of M_{33} consisting of all symmetric matrices.

32. Find a basis for the subspace of M_{33} consisting of all diagonal matrices.

33. Give an example of a two-dimensional subspace of R^4.

34. Give an example of a two-dimensional subspace of P_3.

In Exercises 35 and 36, find a basis for the given plane.

35. $2x - 3y + 4z = 0$. 36. $x + y - 3z = 0$.

37. Determine if the vectors

$$\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

are a basis for B^3.

38. Determine if the vectors

$$\begin{bmatrix}1\\1\\0\end{bmatrix},\ \begin{bmatrix}1\\0\\0\end{bmatrix},\ \begin{bmatrix}0\\1\\1\end{bmatrix}$$

are a basis for B^3.

Determine if the vectors

$$\begin{bmatrix}1\\0\\0\\1\end{bmatrix},\ \begin{bmatrix}0\\0\\1\\1\end{bmatrix},\ \begin{bmatrix}0\\1\\1\\0\end{bmatrix},\ \begin{bmatrix}1\\1\\0\\0\end{bmatrix}$$

are a basis for B^4.

40. Determine if the vectors

$$\begin{bmatrix}1\\1\\0\\1\end{bmatrix},\ \begin{bmatrix}1\\0\\1\\1\end{bmatrix},\ \begin{bmatrix}0\\1\\1\\0\end{bmatrix},\ \begin{bmatrix}1\\1\\1\\1\end{bmatrix}$$

are a basis for B^4.

Theoretical Exercises

T.1. Suppose that in the nonzero vector space V, the largest number of vectors in a linearly independent set is m. Show that any set of m linearly independent vectors in V is a basis for V.

T.2. Show that if V is a finite-dimensional vector space, then every nonzero subspace W of V has a finite basis and $\dim W \leq \dim V$.

T.3. Show that if $\dim V = n$, then any $n + 1$ vectors in V are linearly dependent.

T.4. Show that if $\dim V = n$, then no set of $n - 1$ vectors in V can span V.

T.5. Prove Theorem 6.8.

T.6. Prove Theorem 6.9.

T.7. Show that if W is a subspace of a finite-dimensional vector space V and $\dim W = \dim V$, then $W = V$.

T.8. Show that the subspaces of R^3 are $\{0\}$, R^3, all lines through the origin, and all planes through the origin.

T.9. Show that if $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ is a basis for a vector space V and $c \neq 0$, then $\{c\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ is also a basis for V.

T.10. Let $S = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ be a basis for vector space V. Then show that $T = \{\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3\}$, where

$$\mathbf{w}_1 = \mathbf{v}_1 + \mathbf{v}_2 + \mathbf{v}_3,$$
$$\mathbf{w}_2 = \mathbf{v}_2 + \mathbf{v}_3,$$

and

$$\mathbf{w}_3 = \mathbf{v}_3,$$

is also a basis for V.

T.11. Let

$$S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$$

be a set of nonzero vectors in a vector space V such that every vector in V can be written in one and only one way as a linear combination of the vectors in S. Show that S is a basis for V.

T.12. Suppose that

$$\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$$

is a basis for R^n. Show that if A is an $n \times n$ nonsingular matrix, then

$$\{A\mathbf{v}_1, A\mathbf{v}_2, \ldots, A\mathbf{v}_n\}$$

is also a basis for R^n. (*Hint*: See Exercise T.10 in Section 6.3.)

T.13. Suppose that

$$\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$$

is a linearly independent set of vectors in R^n and let A be a singular matrix. Prove or disprove that

$$\{A\mathbf{v}_1, A\mathbf{v}_2, \ldots, A\mathbf{v}_n\}$$

is linearly independent.

T.14. Show that the vector space P of all polynomials is not finite-dimensional. [*Hint*: Suppose that

$$\{p_1(t), p_2(t), \ldots, p_k(t)\}$$

is a finite basis for P. Let $d_j = \text{degree } p_j(t)$. Establish a contradiction.]

T.15. Show that the set of vectors

$$\{t^n, t^{n-1}, \ldots, t, 1\}$$

in P_n is linearly independent.

T.16. Show that if the sum of the vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ from B^n is $\mathbf{0}$, then these vectors cannot form a basis for B^n.

T.17. Let $S = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ be a set of vectors in B^3.

(a) Find linearly independent vectors $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ such that $\mathbf{v}_1 + \mathbf{v}_2 + \mathbf{v}_3 \neq \mathbf{0}$.

(b) Find linearly dependent vectors $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ such that $\mathbf{v}_1 + \mathbf{v}_2 + \mathbf{v}_3 \neq \mathbf{0}$.

(1)

Exercise : 6.4

Question No. 1:-

$\{(1,2),(2,-3),(3,2)\}$. Is this bases for R^2?

Solution:-

The vectors $v_1, v_2, \dots, v_k$ in a vector space V are basis for V if.

(a) $v_1, v_2 \cdots, v_k$ span V &

(b) $v_1, v_2, \dots, v_k$ are linearly independent.

So,

(a) Linear Independence:-

$$c_1v_1 + c_2v_2 + c_3v_3 = 0$$

$$c_1\begin{bmatrix}1\\2\end{bmatrix} + c_2\begin{bmatrix}2\\-3\end{bmatrix} + c_3\begin{bmatrix}3\\2\end{bmatrix} = 0$$

$$\begin{bmatrix}c_1\\2c_1\end{bmatrix} + \begin{bmatrix}2c_2\\-3c_2\end{bmatrix} + \begin{bmatrix}3c_3\\2c_3\end{bmatrix} = 0$$

$$c_1 + 2c_2 + 3c_3 = 0$$
$$2c_1 - 3c_2 + 2c_3 = 0$$

$$\left[\begin{array}{ccc|c}1 & 2 & 3 & 0\\2 & -3 & 2 & 0\end{array}\right] \overset{R}{\sim} \left[\begin{array}{ccc|c}1 & 2 & 3 & 0\\0 & -7 & -4 & 0\end{array}\right] -2R_1+R_2$$

$$\overset{R}{\sim} \left[\begin{array}{ccc|c}1 & 2 & 3 & 0\\0 & 1 & 4/7 & 0\end{array}\right] -1/7\,R_2$$

$$\overset{R}{\sim} \left[\begin{array}{ccc|c}1 & 0 & 13/7 & 0\\0 & 1 & 4/7 & 0\end{array}\right] -2R_2+R_1$$

$$c_2 + 4/7\,c_3 = 0 \longrightarrow (i)$$
$$c_1 + 13/7\,c_3 = 0 \longrightarrow (ii)$$

from (i) $c_2 = -4/7\,c_3$

from (ii) $c_1 = -13/7\,c_3$

Let $c_3 = -7$ then $c_2 = -4$ and $c_1 = -13$

As $c_1 \neq c_2 \neq c_3$, so v_1, v_2 & v_3 are not linearly independent. So as one condition is not satisfied so it doesnot form basis for R^2.

Question No. 2:-

$\{(1,2,0),(0,1,-1)\}$. Is this basis for R^3?

Solution:-

For basis they must be

(a) linear independent &

(b) span.

(a) **Linear Independent:-**

$$c_1V_1 + c_2V_2 = 0$$

$$c_1\begin{bmatrix}1\\2\\0\end{bmatrix} + c_2\begin{bmatrix}0\\1\\-1\end{bmatrix} = 0$$

$$\begin{bmatrix}c_1\\2c_1\\0\end{bmatrix} + \begin{bmatrix}0\\c_2\\-c_2\end{bmatrix} = 0$$

$c_1 = 0 \rightarrow (i)$

$2c_1 + c_2 = 0 \rightarrow (ii)$

$-c_2 = 0 \rightarrow (iii) \Rightarrow c_2 = 0$

As $c_1 = c_2 = 0$, so they are linear independent.

(b) **Span:-**

$$V = c_1V_1 + c_2V_2$$

let $v = (a, b, c)$

$$c_1\begin{bmatrix}1\\2\\0\end{bmatrix} + c_2\begin{bmatrix}0\\1\\-1\end{bmatrix} = \begin{bmatrix}a\\b\\c\end{bmatrix}$$

$$\begin{bmatrix}c_1\\2c_1\\0\end{bmatrix} + \begin{bmatrix}0\\c_2\\-c_2\end{bmatrix} = \begin{bmatrix}a\\b\\c\end{bmatrix}$$

$c_1 = a \rightarrow (iv)$ $\quad 2c_1 + c_2 = b \rightarrow (v)$

$-c_2 = c \rightarrow (vi)$

from (vi) $\quad c_2 = -c$

from (iv) $\quad c_1 = a$

from (v) $\quad 2c_1 + c_2 = b$

Put c_1 and c_2 in (v)

$2(+a) + (-c) = b \quad \Rightarrow 2a - c \neq b$

As V_1, V_2 donot form span, so they don't form basis.

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(2)

Question No. 3:-

$\{(1,0,0,1),(0,1,0,0),(1,1,1,1),(0,1,1,1)\}$. Is this vector basis for R^4.

Solution:-

For basis the vector must be
(a) linear independent &
(b) form span.

(a) Linear independent:-

$$c_1V_1 + c_2V_2 + c_3V_3 + c_4V_4 = 0$$

$$c_1\begin{bmatrix}1\\0\\0\\1\end{bmatrix} + c_2\begin{bmatrix}0\\1\\0\\0\end{bmatrix} + c_3\begin{bmatrix}1\\1\\1\\1\end{bmatrix} + c_4\begin{bmatrix}0\\1\\1\\1\end{bmatrix} = 0$$

$$\begin{bmatrix}c_1\\0\\0\\c_1\end{bmatrix} + \begin{bmatrix}0\\c_2\\0\\0\end{bmatrix} + \begin{bmatrix}c_3\\c_3\\c_3\\c_3\end{bmatrix} + \begin{bmatrix}0\\c_4\\c_4\\c_4\end{bmatrix} = 0$$

$c_1 + c_3 = 0 \rightarrow$ (i) , $c_2 + c_3 + c_4 = 0 \rightarrow$ (ii)

$c_3 + c_4 = 0 \rightarrow$ (iii) , $c_3 + c_4 + c_1 = 0 \rightarrow$ (iv)

from (i) $c_1 + c_3 = 0$

Put in eq (iv)

$$c_1 + c_3 + c_4 = 0$$
$$0 + c_4 = 0 \Rightarrow c_4 = 0$$

Put c_4 in (iii)

$$c_4 + c_3 = 0$$
$$0 + c_3 = 0$$
$$\Rightarrow c_3 = 0$$

Put c_3 in (i)

$$c_1 + c_3 = 0$$
$$c_1 + 0 = 0 \Rightarrow c_1 = 0$$

Put c_3 & c_4 in (ii)

$$c_2 + c_3 + c_4 = 0$$
$$c_2 + 0 + 0 = 0 \Rightarrow c_2 = 0$$

As $c_1 = c_2 = c_3 = c_4 = 0$, so it is linear independent.

(b) <u>Span:-</u>

$$c_1V_1 + c_2V_2 + c_3V_3 + c_4V_4 = V$$

let $V = (a, b, c, d)$

$$c_1\begin{bmatrix}1\\0\\0\\1\end{bmatrix} + c_2\begin{bmatrix}0\\1\\0\\0\end{bmatrix} + c_3\begin{bmatrix}1\\1\\1\\1\end{bmatrix} + c_4\begin{bmatrix}0\\1\\1\\1\end{bmatrix} = \begin{bmatrix}a\\b\\c\\d\end{bmatrix}$$

$$\begin{bmatrix}c_1\\0\\0\\c_1\end{bmatrix} + \begin{bmatrix}0\\c_2\\0\\0\end{bmatrix} + \begin{bmatrix}c_3\\c_3\\c_3\\c_3\end{bmatrix} + \begin{bmatrix}0\\c_4\\c_4\\c_4\end{bmatrix} = \begin{bmatrix}a\\b\\c\\d\end{bmatrix}$$

$c_1 + c_3 = a \rightarrow$ (i)

$c_2 + c_3 + c_4 = b \rightarrow$ (ii)

$c_3 + c_4 = c \rightarrow$ (iii)

$c_3 + c_1 + c_4 = d \rightarrow$ (iv)

Put (i) in (iv)

$a + c_4 = d$

$c_4 = d - a$

Put c_4 in (iii)

$c_3 + c_4 = c$

$c_3 + (d - a) = c$

$c_3 = c - d + a$

Put c_3 and c_4 in (ii)

$c_2 + c_3 + c_4 = b$

$c_2 + (c - d + a) + (d - a) = b$

$c_2 + c - d + a + \cancel{d} - \cancel{a} = b$

$c_2 = b - c$

Put c_3 in (i)

$c_1 + c_3 = a$

$c_1 + (c - d + a) = a$

$c_1 + c - d + \cancel{a} = \cancel{a}$

$c_1 = d - c$

As we can find c_1, c_2, c_3 for any values of a, b, c so, it

is span S.
As it satisfied both conditions, so it means, it forms basis for $\mathbb{R}^4$.

Question No. 4:-

$\{-t^2+t+2, 2t^2+2t+3, 4t^2-1\}$. Is this basis for P_2?

Solution:-

For ~~vector~~ basis vector must be
(a) linear independent &
(b) form span.

(a) Linear Independence:-

$$c_1V_1 + c_2V_2 + c_3V_3 = 0$$

$$c_1\begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix} + c_2\begin{bmatrix} 2 \\ 2 \\ 3 \end{bmatrix} + c_3\begin{bmatrix} 4 \\ 0 \\ -1 \end{bmatrix} = 0$$

$$\begin{bmatrix} -c_1 \\ c_1 \\ 2c_1 \end{bmatrix} + \begin{bmatrix} 2c_2 \\ 2c_2 \\ 3c_2 \end{bmatrix} + \begin{bmatrix} 4c_3 \\ 0 \\ -c_3 \end{bmatrix} = 0$$

$$\left[\begin{array}{ccc|c} -1 & 2 & 4 & 0 \\ 1 & 2 & 0 & 0 \\ 2 & 3 & -1 & 0 \end{array}\right] \underset{R}{\sim} \left[\begin{array}{ccc|c} 1 & -2 & -4 & 0 \\ 1 & 2 & 0 & 0 \\ 2 & 3 & -1 & 0 \end{array}\right] -R_1$$

$$\underset{R}{\sim} \left[\begin{array}{ccc|c} 1 & -2 & -4 & 0 \\ 0 & 4 & 4 & 0 \\ 0 & 7 & 7 & 0 \end{array}\right] \begin{matrix} +R_2 - R_1 \\ R_3 - 2R_1 \end{matrix}$$

$$\underset{R}{\sim} \left[\begin{array}{ccc|c} 1 & -2 & -4 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \end{array}\right] \begin{matrix} 1/4\, R_2 \\ 1/7\, R_3 \end{matrix}$$

$$\underset{R}{\sim} \left[\begin{array}{ccc|c} 1 & -2 & -4 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] R_3 - R_2$$

$c_2 + c_3 = 0 \longrightarrow$ (i)
$c_1 - 2c_2 - 4c_3 = 0 \longrightarrow$ (ii)

from (i) $c_2 + c_3 = 0 \Rightarrow c_2 = -c_3$

Put c_2 in (ii)

$c_1 - 2c_2 - 4c_3 = 0$

$c_1 - 2(-c_3) - 4c_3 = 0$

$c_1 + 2c_3 - 4c_3 = 0$

$c_1 - 2c_3 = 0$

$c_1 = 2c_3$ & $c_2 = -c_3$

Put $c_3 = 1$.

so $c_1 = 2$, $c_2 = -1$

So, as $c_1 \neq c_2 \neq c_3 \neq 0$

So, not linear independent

So vector donot for basis in P_2.

Question No. 5:-

$\{t^3+t^2+1,\ t^3-1,\ t^3+t^2+t\}$. Is this basis for P_3?

Solution:-

For basis vector must be

(a) linear independent &

(b) form span.

(a) Linear independent:-

$c_1v_1 + c_2v_2 + c_3v_3 = 0$

$$c_1\begin{bmatrix}1\\1\\0\\1\end{bmatrix} + c_2\begin{bmatrix}1\\0\\0\\-1\end{bmatrix} + c_3\begin{bmatrix}1\\1\\1\\0\end{bmatrix} = \begin{bmatrix}0\\0\\0\\0\end{bmatrix}$$

$$\begin{bmatrix}c_1\\c_1\\0\\c_1\end{bmatrix} + \begin{bmatrix}c_2\\0\\0\\-c_2\end{bmatrix} + \begin{bmatrix}c_3\\c_3\\c_3\\0\end{bmatrix} = \begin{bmatrix}0\\0\\0\\0\end{bmatrix}$$

$c_1 + c_2 + c_3 = 0 \rightarrow$ (i)

$c_1 + c_3 = 0 \rightarrow$ (ii)

$c_3 = 0 \rightarrow$ (iii)

$c_1 - c_2 = 0 \rightarrow$ (iv)

from (iii) put c_3 in (ii)

$c_1 + c_3 = 0 \Rightarrow c_1 + 0 = 0 \Rightarrow c_1 = 0$

Put c_1 and c_3 in (i)

$$c_1+c_2+c_3=0$$
$$0+c_2+0=0$$
$$c_2=0$$

As $c_1=c_2=c_3=0$

so they are linearly independent.

(b) Span:-

$$c_1V_1+c_2V_2+c_3V_3=V$$

let $V=(a,b,c,d)$

$$c_1\begin{bmatrix}1\\1\\0\\1\end{bmatrix}+c_2\begin{bmatrix}1\\0\\0\\-1\end{bmatrix}+c_3\begin{bmatrix}1\\1\\1\\0\end{bmatrix}=\begin{bmatrix}a\\b\\c\\d\end{bmatrix}$$

$$\begin{bmatrix}c_1\\c_1\\0\\c_1\end{bmatrix}+\begin{bmatrix}c_2\\0\\0\\-c_2\end{bmatrix}+\begin{bmatrix}c_3\\c_3\\c_3\\0\end{bmatrix}=\begin{bmatrix}a\\b\\c\\d\end{bmatrix}$$

$$c_1+c_2+c_3=a \rightarrow (i)$$
$$c_1+c_3=b \rightarrow (ii)$$
$$c_3=\cancel{b}\,c \rightarrow (iii)$$
$$c_1-c_2=d \rightarrow (iv)$$

Put $c_3=c$ in (ii)

$$c_1+c_3=b$$
$$c_1+c=b$$
$$c_1=b-c$$

Put c_1 and c_3 in (i)

$$c_1+c_2+c_3=a$$
$$(b-c)+c_2+c=a$$
$$b-\cancel{c}+c_2+\cancel{c}=a$$
$$c_2=a-b$$

Put c_1 and c_2 in (iv)

$$c_1-c_2=d$$
$$(b-c)-(a-b)=d \Rightarrow b-c-a+b=d$$

$2b - c - a \neq d$

so this doesn't exist. So we can't find values of c_1, c_2 & c_3 for any values of a, b & c, it doesn't form span.

So, it doesn't form basis for P_3.

Question No. 6:-

Show that the matrices

$$\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}$$ form basis

for vector space M_{22}.

Solution:-

(a) Linear independence:-

$$c_1v_1 + c_2v_2 + c_3v_3 + c_4v_4 = 0$$

$$c_1\begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + c_2\begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix} + c_3\begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix} + c_4\begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} c_1 \\ c_1 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ c_2 \\ c_2 \end{bmatrix} + \begin{bmatrix} c_3 \\ 0 \\ 0 \\ c_3 \end{bmatrix} + \begin{bmatrix} 0 \\ c_4 \\ c_4 \\ c_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$c_1 + c_3 = 0 \rightarrow$ (i)

$c_1 + c_4 = 0 \rightarrow$ (ii)

$c_2 + c_4 = 0 \rightarrow$ (iii)

$c_2 + c_3 + c_4 = 0 \rightarrow$ (iv)

from (i) $c_1 = -c_3 \Rightarrow c_3 = -c_1$

from (iii) $c_2 = -c_4$ &

from (ii) $c_1 = -c_4$ so it means

$c_1 = c_2$

Put in (iv)

$c_2 + c_3 + c_4 = 0$

Put $c_2 = -c_4$.

$-c_4 + c_3 + c_4 = 0$

$c_3 = 0$

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(5)

Put $c_3 = 0$ in (i)

$c_1 = -c_3$

$c_1 = 0$

Put $c_1 = 0$ in (ii)

$c_1 + c_4 = 0$

$0 + c_4 = 0$

$c_4 = 0$

Put $c_4 = 0$ in (iii)

$c_2 + c_4 = 0$

$c_2 = 0$

As $c_1 = c_2 = c_3 = c_4 = 0$

so they are linear independent.

(b) <u>Span:-</u>

$$c_1V_1 + c_2V_2 + c_3V_3 + c_4V_4 = V$$

let $V = (a, b, c, d)$

$$c_1\begin{bmatrix}1\\1\\0\\0\end{bmatrix} + c_2\begin{bmatrix}0\\0\\1\\1\end{bmatrix} + c_3\begin{bmatrix}1\\0\\0\\1\end{bmatrix} + c_4\begin{bmatrix}0\\1\\1\\1\end{bmatrix} = \begin{bmatrix}a\\b\\c\\d\end{bmatrix}$$

$$\begin{bmatrix}c_1\\c_1\\0\\0\end{bmatrix} + \begin{bmatrix}0\\0\\c_2\\c_2\end{bmatrix} + \begin{bmatrix}c_3\\0\\0\\c_3\end{bmatrix} + \begin{bmatrix}0\\c_4\\c_4\\c_4\end{bmatrix} = \begin{bmatrix}a\\b\\c\\d\end{bmatrix}$$

$c_1 + c_3 = a \rightarrow$ (i)

$c_1 + c_4 = b \rightarrow$ (ii)

$c_2 + c_4 = c \rightarrow$ (iii)

$c_2 + c_3 + c_4 = d \rightarrow$ (iv)

$$\left[\begin{array}{cccc|c}1&0&1&0&a\\1&0&0&1&b\\0&1&0&1&c\\0&1&1&1&d\end{array}\right] \underset{R}{\sim} \left[\begin{array}{cccc|c}1&0&1&0&a\\0&0&-1&1&b-a\\0&1&0&1&c\\0&1&1&1&d\end{array}\right] R_2 - R_1$$

$$R_3 \sim \left[\begin{array}{cccc|c} 1 & 0 & 1 & 0 & a \\ 0 & 0 & -1 & 1 & b-a \\ 0 & 1 & 0 & 1 & c \\ 0 & 0 & 1 & 0 & d-c \end{array}\right] \quad R_4 - R_3$$

$c_3 = d-c \rightarrow (v)$

$c_2 + c_4 = c \rightarrow (vi)$

$c_4 - c_3 = b-a \rightarrow (vii)$

$c_1 + c_3 = a \rightarrow (viii)$

Put $c_3 = d-c$ in (vii)

$c_4 - c_3 = b-a$

$c_4 - (d-c) = b-a$

$c_4 - d + c = b-a$

$c_4 = b-a-c+d$

Put c_3 in (viii)

$c_1 + c_3 = a$

$c_1 + d - c = a$

$c_1 = a-d+c$

Put c_4 in (vi)

$c_2 + c_4 = c$

$c_2 + (b-a-c+d) = c$

$c_2 + b - a - c + d = c$

$c_2 = a-b+2c-d$

check:-

$c_2 + c_4 = c$

$\cancel{-d} + \cancel{a} - \cancel{b} + 2c + \cancel{b} - \cancel{a} - c + \cancel{d} = c$

$2c - c = c$

$c = c \rightarrow$ eq (vi) satisfied.

Now.

$c_4 - c_3 = b-a$

$b-a-\cancel{c}+\cancel{d}-\cancel{d}+\cancel{c} = b-a \Rightarrow b-a = b-a \rightarrow$ eq(vii) satisfied.

Now eq (viii)

$$c_1 + c_3 = a$$

$$a - d + e + d - e = a$$

$$a = a \quad \text{satisfied.}$$

So soln exist in any values of a, b and c so, vector span S and it form basis for M_{22}.

Question No. 7 :-

$\{(1,1,1), (1,2,3), (0,1,0)\}$. Is these basis for R^3. Express vector $(2,1,3)$ as a linear combination of vectors in subset that is basis.

Solution:-

(a) Linear Independence:-

$$c_1v_1 + c_2v_2 + c_3v_3 = 0$$

$$c_1\begin{bmatrix}1\\1\\1\end{bmatrix} + c_2\begin{bmatrix}1\\2\\3\end{bmatrix} + c_3\begin{bmatrix}0\\1\\0\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix}$$

$$\begin{bmatrix}c_1\\c_1\\c_1\end{bmatrix} + \begin{bmatrix}c_2\\2c_2\\3c_2\end{bmatrix} + \begin{bmatrix}0\\c_3\\0\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix}$$

$$c_1 + c_2 = 0 \rightarrow (i)$$

$$c_1 + 2c_2 + c_3 = 0 \rightarrow (ii)$$

$$c_1 + 3c_2 = 0 \rightarrow (iii)$$

from (i) $c_1 = -c_2$

Put c_1 in (iii)

$$c_1 + 3c_2 = 0$$

$$-c_2 + 3c_2 = 0$$

$$2c_2 = 0$$

$$c_2 = 0$$

Put it in (iii)

$$c_1 + 0 = 0 \Rightarrow c_1 = 0$$

Put it in (ii)

$c_1 + 2c_2 + c_3 = 0$

$0 + 0 + c_3 = 0$

$c_3 = 0$

As $c_1 = c_2 = c_3 = 0$ so it forms linear independence.

(b) <u>Span:-</u>

$c_1 v_1 + c_2 v_2 + c_3 v_3 = v$

let $v = (a, b, c)$

$$c_1 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + c_3 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$$

$$\begin{bmatrix} c_1 \\ c_1 \\ c_1 \end{bmatrix} + \begin{bmatrix} c_2 \\ 2c_2 \\ 3c_2 \end{bmatrix} + \begin{bmatrix} 0 \\ c_3 \\ 0 \end{bmatrix} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$$

$c_1 + c_2 = a \rightarrow$ (i)

$c_1 + 2c_2 + c_3 = b \rightarrow$ (ii)

$c_1 + 3c_2 = c \rightarrow$ (iii)

from (i) $c_1 = a - c_2$

Put it in (ii) $c_1 + 2c_2 + c_3 = b$

$(a - c_2) + 2c_2 + c_3 = b$

$a - c_2 + 2c_2 + c_3 = b$

$a + c_2 + c_3 = b$

Put c_1 in (iii)

$c_1 + 3c_2 = c$

$(a - c_2) + 3c_2 = c$

$a + 2c_2 = c$

$2c_2 = c - a$

$c_2 = \dfrac{c-a}{2}$

Put c_2 in (i)

$c_1 + c_2 = a \Rightarrow c_1 + (c-a)/2 = a$

$2c_1 + c - a = 2a$

$2c_1 = 2a + a - c$

$c_1 = \frac{3a-c}{2}$

Put c_1 and c_2 in (ii)

$c_1 + 2c_2 + c_3 = b$

$\frac{3a-c}{2} + 2\left(\frac{c-a}{2}\right) + c_3 = b$

$\frac{3a-c+2(c-a)+2c_3}{2} = b$

$3a - c + 2c - 2a + 2c_3 = 2b$

$a + c + 2c_3 = 2b$

$2c_3 = 2b - a - c$

$c_3 = \frac{2b-a-c}{2}$

As we can find c_1, c_2, c_3 for any value of a, b & c, so its span. and thus forms basis for R^3

Given vector is $(2,1,3)$

so $V = (a, b, c)$

$V = (2, 1, 3)$

so, $c_1V_1 + c_2V_2 + c_3V_3 = V$ (by solving)

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 3/2 \\ 0 & 1 & 0 & 1/2 \\ 0 & 0 & 1 & -3/2 \end{array}\right]$$

$(2,1,3) = 3/2(1,1,1) + 1/2(1,2,3) - 3/2(0,1,0)$

$a = 2, \quad b = 1, \quad c = 3$

$c_1 = \frac{3a-c}{2}$

$c_1 = \frac{3(2)-3}{2}$

$c_1 = 3/2$

$c_2 = \frac{c-a}{2}$

$c_2 = \frac{3-2}{2}$

$c_2 = 1/2$

$c_3 = \frac{-a+2b-c}{2}$

$c_3 = \frac{-2+2(1)-3}{2}$

$c_3 = -3/2$

$V = c_1V_1 + c_2V_2 + c_3V_3$

so $(2,1,3) = 3/2\,(1,1,1) + 1/2\,(+1,2,3) - 3/2\,(0,1,0)$.

Question No. 8:-

$\{(1,1,2),(2,2,0),(3,4,-1)\}$. Is this basis for R^3

Express vector $(2,1,3)$ as a linear combination of vectors in subset that is basis.

Solution:-

(a) Linear independence:-

$c_1V_1 + c_2V_2 + c_3V_3 = 0$

$$c_1\begin{bmatrix}1\\1\\2\end{bmatrix}+c_2\begin{bmatrix}2\\2\\0\end{bmatrix}+c_3\begin{bmatrix}3\\4\\-1\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$$

$$\begin{bmatrix}c_1\\c_1\\2c_1\end{bmatrix}+\begin{bmatrix}2c_2\\2c_2\\0\end{bmatrix}+\begin{bmatrix}3c_3\\4c_3\\-c_3\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$$

$c_1+2c_2+3c_3=0 \rightarrow (i)$

$c_1+2c_2+4c_3=0 \rightarrow (ii)$

$2c_1-c_3=0 \rightarrow (iii)$

from (iii)

$c_3=2c_1$

Put it in (i)

$c_1+2c_2+3c_3=0$

$c_1+2c_2+3(2c_1)=0$

$c_1+2c_2+6c_1=0$

$7c_1+2c_2=0 \rightarrow (iv)$

Put c_3 in (ii)

$c_1+2c_2+4c_3=0$

$c_1+2c_2+4(2c_1)=0$

$c_1+2c_2+8c_1=0$

$9c_1+2c_2=0 \rightarrow (v)$

subtract (v) from (iv)

$$\begin{array}{r} 7c_1+\cancel{2c_2}=0 \\ \underline{\ominus 9c_1 \ominus \cancel{2c_2}=0} \\ -2c_1=0 \end{array}$$

$c_1=0$

Put in (iii)

$c_3=2(0)$

$c_3=0$

Put in (ii)

$c_1 + 2c_2 + 4c_3 = 0$

$0 + 2c_2 + 4(0) = 0$

$2c_2 = 0$

$c_2 = 0$

As $c_1 = c_2 = c_3 = 0$ so it is linear independent

Span:-

$c_1v_1 + c_2v_2 + c_3v_3 = V$

let $V = (a, b, c)$

$$c_1\begin{bmatrix}1\\1\\2\end{bmatrix} + c_2\begin{bmatrix}2\\2\\0\end{bmatrix} + c_3\begin{bmatrix}3\\4\\-1\end{bmatrix} = \begin{bmatrix}a\\b\\c\end{bmatrix}$$

$$\begin{bmatrix}c_1\\c_1\\2c_1\end{bmatrix} + \begin{bmatrix}2c_2\\2c_2\\0\end{bmatrix} + \begin{bmatrix}3c_3\\4c_3\\-c_3\end{bmatrix} = \begin{bmatrix}a\\b\\c\end{bmatrix}$$

$c_1 + 2c_2 + 3c_3 = a \rightarrow (i)$

$c_1 + 2c_2 + 4c_3 = b \rightarrow (ii)$

$2c_1 - c_3 = c \rightarrow (iii)$

from (iii) $c_3 = 2c_1 - c$

Put it in (i)

$c_1 + 2c_2 + 3c_3 = a$

$c_1 + 2c_2 + 3(2c_1 - c) = a$

$c_1 + 2c_2 + 6c_1 - 3c = a$

$7c_1 + 2c_2 = a + 3c \rightarrow (iv)$

Put c_1 in (ii)

$c_1 + 2c_2 + 4c_3 = b$

$c_1 + 2c_2 + 4(2c_1 - c) = b$

$c_1 + 2c_2 + 8c_1 - 4c = b$

$9c_1 + 2c_2 = b + 4c \rightarrow (v)$

subtract (v) from (iv)

$$7c_1 + \cancel{7c_2} = a + 3c$$
$$\underset{-}{+} 9c_1 \underset{-}{+} \cancel{7c_2} = \underset{-}{+} b \underset{-}{+} 4c$$

$$-2c_1 = a - b + 3c - 4c$$
$$-2c_1 = a - b - c$$
$$c_1 = \frac{a - b - c}{-2}$$

$$c_1 = \frac{c + b - a}{2}$$

Put c_1 in (iii)

$$2c_1 - c_3 = c$$
$$\cancel{2}\left(\frac{c + b - a}{\cancel{2}}\right) - c_3 = c$$

$$c + b - a - c_3 = c$$

$$c_3 = \cancel{c} + b - a - \cancel{c}$$
$$c_3 = b - a$$

Put c_1 & c_3 in (i)

$$c_1 + 2c_2 + 4c_3 = b$$
$$\frac{c + b - a}{2} + 2c_2 + 4(b - a) = b$$

$$\frac{c + b - a + 4c_2 + 8(b - a)}{2} = b$$

$$c + b - a + 4c_2 + 8b - 8a = 2b$$

$$c + 9b - 9a + 4c_2 = 2b$$
$$4c_2 = 2b + 9a - 9b - c$$
$$4c_2 = 9a - 7b - c$$
$$c_2 = \frac{9a - 7b - c}{4}$$

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As we can find c_1, c_2, c_3 for any value of a, b & c so it is span and thus forms basis for R^3.

Given vector is $(2,1,3)$

so $v = (a, b, c)$

$v = (2,1,3)$

so $c_1 v_1 + c_2 v_2 + c_3 v_3 = v$

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & 2 \\ 1 & 2 & 4 & 1 \\ 2 & 0 & -1 & 3 \end{array}\right] \underset{\sim}{R} \left[\begin{array}{ccc|c} 1 & 2 & 3 & 2 \\ 0 & 0 & 1 & -1 \\ 0 & -4 & -7 & -1 \end{array}\right] \begin{array}{l} R_2 - R_1 \\ R_3 - 2R_1 \end{array}$$

$$\underset{\sim}{R} \left[\begin{array}{ccc|c} 1 & 2 & 3 & 2 \\ 0 & -4 & -6 & -2 \\ 0 & -4 & -7 & -1 \end{array}\right] \cdot R_2 + 2R_3 \quad \underset{\sim}{R} \left[\begin{array}{ccc|c} 1 & 2 & 3 & 2 \\ 0 & 1 & 3/2 & 1/2 \\ 0 & -4 & -7 & -1 \end{array}\right] \cdot -1/4 R_2$$

$$\underset{\sim}{R} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 3/2 & 1/2 \\ 0 & 0 & -1 & 1 \end{array}\right] \begin{array}{l} R_1 - 2R_2 \\ R_3 + 4R_2 \end{array} \quad \underset{\sim}{R} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 3/2 & 1/2 \\ 0 & 0 & 1 & -1 \end{array}\right] -R_3$$

$$\underset{\sim}{R} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & -1 \end{array}\right] R_2 - 3/2 R_3$$

so,

$(2,1,3) = 1(1,1,2) + 2(2,2,0) - 1(3,4,-1)$

$a = 2, \; b = 1, \; c = 3$

$$c_1 = \frac{c + b - a}{2}$$

$$c_1 = \frac{3 + 1 - 2}{2}$$

$$c_1 = \frac{2}{2}$$

$c_1 = 1$

$$c_2 = \frac{9a-7b-c}{4}$$

$$c_2 = \frac{9(2)-7(1)-3}{4}$$

$$c_2 = \frac{18-7-3}{4}$$

$$c_2 = \frac{18-10}{4}$$

$$c_2 = \frac{8}{4}$$

$$c_2 = 2$$

$$c_3 = b-a$$

$$c_3 = 1-2$$

$$c_3 = -1$$

$$V = c_1V_1 + c_2V_2 + c_3V_3$$

so $(2,1,3) = 1(1,1,2) + 2(2,2,0) - 1(3,4,-1)$.

Question No. 9:-

$\{t^2+t, t-1, t+1\}$. Is this basis for P_2. Express $5t^2-3t+8$ as a linear combination of vectors in each subset that is basis.

Solution:-

(a) Linear Independence:-

$$c_1V_1 + c_2V_2 + c_3V_3 = 0$$

$$c_1\begin{bmatrix}1\\1\\0\end{bmatrix} + c_2\begin{bmatrix}0\\1\\-1\end{bmatrix} + c_3\begin{bmatrix}0\\1\\1\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix}$$

$$\begin{bmatrix} c_1 \\ c_1 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ c_2 \\ -c_2 \end{bmatrix} + \begin{bmatrix} 0 \\ c_3 \\ c_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$c_1 = 0 \rightarrow (i)$

$c_1 + c_2 + c_3 = 0 \rightarrow (ii)$

$c_3 - c_2 = 0 \rightarrow (iii)$

Put $c_1 = 0$ in (ii)

$c_1 + c_2 + c_3 = 0$

$0 + c_2 + c_3 = 0$

$c_2 + c_3 = 0 \rightarrow (iv)$

Subtract (iv) from (iii)

$$\begin{array}{r} c_3 - c_2 = 0 \\ \underline{\oplus c_3 \oplus c_2 = 0} \\ -2c_2 = 0 \end{array}$$

$c_2 = 0$

Put $c_2 = 0$ in (iv)

$c_2 + c_3 = 0$

$0 + c_3 = 0$

$c_3 = 0$

As $c_1 = c_2 = c_3 = 0$ so vectors are linearly independent.

(b) Span:-

$c_1 v_1 + c_2 v_2 + c_3 v_3 = v$

let $v = (a, b, c)$

$$c_1 \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} + c_3 \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$$

$$\begin{bmatrix} c_1 \\ c_1 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ c_2 \\ -c_2 \end{bmatrix} + \begin{bmatrix} 0 \\ c_3 \\ c_3 \end{bmatrix} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$$

$c_1 = a \rightarrow$ (i)

$c_1 + c_2 + c_3 = b \rightarrow$ (ii)

$c_3 - c_2 = c \rightarrow$ (iii)

Put $c_1 = a$ in (ii)

$c_1 + c_2 + c_3 = b$

$a + c_2 + c_3 = b$

$c_2 + c_3 = b - a \rightarrow$ (iv)

subtract it from (iii)

$$\begin{aligned} c_3 - c_2 &= c \\ \pm c_3 \pm c_2 &= \pm b \mp a \\ \hline -2c_2 &= a - b + c \end{aligned}$$

$$c_2 = \frac{a-b+c}{-2}$$

$$c_2 = \frac{b-a-c}{2}$$

Put c_2 in (iii)

$$c_3 - c_2 = c$$

$$c_3 - \left(\frac{b-a-c}{2}\right) = c$$

$$c_3 = c + \left(\frac{b-a-c}{2}\right)$$

$$c_3 = \frac{2c+b-a-c}{2}$$

$$c_3 = \frac{b-a+c}{2}$$

As we can find c_1, c_2, c_3 for any value of a, b & c

So polynomial span P_2

Now given polynomial $5t^2 - 3t + 8$

$a = 5, \; b = -3, \; c = 8$

solving $c_1v_1 + c_2v_2 + c_3v_3 = v$

~~Rearranging~~ we get $c_1 = 5, \; c_2 = -8, \; c_3 = 0$

So $c_1v_1 + c_2v_2 + c_3v_3 = v$

As:

$c_1 = a$

$c_1 = 5$

$$c_2 = \frac{b-a-c}{2}$$

$$c_2 = \frac{-3-5-8}{2}$$

$$= \frac{-16}{2} = -8$$

$$c_3 = \frac{b-a+c}{2} = \frac{-3-5+8}{2}$$

$$= \frac{0}{2} = 0$$

So, $c_1v_1 + c_2v_2 + c_3v_3 = v$

$$5(t^2+t) - 8(t-1) = v$$

$$5t^2 - 3t + 8 = 5(t^2+t) - 8(t-1)$$

Question No. 10:-

(iii) $\{t^2+t, t^2, t^2+1\}$. Is this basis for P_2. Express $5t^2-3t+8$ as a linear combination of vectors.

Solution:-

(a) Linear Independence:-

$$c_1v_1 + c_2v_2 + c_3v_3 = 0$$

$$c_1\begin{bmatrix}1\\1\\0\end{bmatrix} + c_2\begin{bmatrix}1\\0\\0\end{bmatrix} + c_3\begin{bmatrix}1\\0\\1\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix}$$

$$\begin{bmatrix}c_1\\c_1\\0\end{bmatrix} + \begin{bmatrix}c_2\\0\\0\end{bmatrix} + \begin{bmatrix}c_3\\0\\c_3\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix}$$

$c_1 + c_2 + c_3 = 0 \rightarrow (i)$

$c_1 = 0 \rightarrow (ii)$

$c_3 = 0 \rightarrow (iii)$

Put c_1 & c_3 in eq(i)

$0 + c_2 + 0 = 0$

$c_2 = 0$

As $c_1 = c_2 = c_3 = 0$, so vectors are linearly independent.

(b) Span:-

$$c_1v_1 + c_2v_2 + c_3v_3 = v$$

let $v = (a, b, c)$

$$c_1\begin{bmatrix}1\\1\\0\end{bmatrix} + c_2\begin{bmatrix}1\\0\\0\end{bmatrix} + c_3\begin{bmatrix}1\\0\\1\end{bmatrix} = \begin{bmatrix}a\\b\\c\end{bmatrix}$$

$$\begin{bmatrix}c_1\\c_1\\0\end{bmatrix} + \begin{bmatrix}c_2\\0\\0\end{bmatrix} + \begin{bmatrix}c_3\\0\\c_3\end{bmatrix} = \begin{bmatrix}a\\b\\c\end{bmatrix}$$

$c_1 + c_2 + c_3 = a \longrightarrow (i)$

$c_1 = b \longrightarrow (ii)$

$c_3 = c \longrightarrow (iii)$

Put c_1 & c_3 in (1)

$c_1 + c_2 + c_3 = a$

$b + c_2 + c = a$

$c_2 = a - b - c$

As we can find c_1, c_2, c_3 for any values of a, b & c so polynomial span P_2.

Now given polynomial is $5t^2 - 3t + 8$

$a = 5, \; b = -3, \; c = 8$

so,

$c_1 = b$

$c_1 = -3$

$c_2 = a - b - c$

$c_2 = 5 + 3 - 8$

$= 0$

$c_3 = c$

$c_3 = 8$

so

$c_1 = -3, \; c_2 = 0, \; c_3 = 8.$

so,

$c_1 v_1 + c_2 v_2 + c_3 v_3 = v$

$-3(t^2 + t) + 8(t^2 + 1) = v$

$-3(t^2 + t) + 8(t^2 + 1) = 5t^2 - 3t + 8.$

Question No. 11.

Let $S = \{v_1, v_2, v_3, v_4\}$, where

$v_1 = \{1,2,2\}$, $v_2 = \{3,2,1\}$

$v_3 = \{11,10,7)$ & $v_4 = (7,6,4)$.

Find a basis for subspace $W =$ span S of R^3. What is dim W?

Solution:-

$$\left[\begin{array}{cccc|c} 1 & 3 & 11 & 7 & 0 \\ 2 & 2 & 10 & 6 & 0 \\ 2 & 1 & 7 & 4 & 0 \end{array}\right]$$

$$\left[\begin{array}{cccc|c} 1 & 3 & 11 & 7 & 0 \\ 2 & 2 & 10 & 6 & 0 \\ 0 & -1 & -3 & -2 & 0 \end{array}\right] \quad R_3 - R_2$$

$$\left[\begin{array}{cccc|c} 1 & 3 & 11 & 7 & 0 \\ 0 & -4 & -12 & -8 & 0 \\ 0 & -1 & -3 & -2 & 0 \end{array}\right] \quad R_2 - 2R_1$$

$$\left[\begin{array}{cccc|c} 1 & 3 & 11 & 7 & 0 \\ 0 & 1 & 3 & 2 & 0 \\ 0 & -1 & -3 & -2 & 0 \end{array}\right] \quad -\frac{1}{4}R_2$$

$$\left[\begin{array}{cccc|c} 1 & 3 & 11 & 7 & 0 \\ 0 & 1 & 3 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \quad R_3 + R_2$$

so v_1 & v_2 form basis so

$S = \{v_1, v_2\}$. The dim $W = 2$

Question No. 12:-

Let $S = \{v_1, v_2, v_3, v_4, v_5\}$, where

$v_1 = (1,1,0,-1)$, $v_2 = (0,1,2,1)$, $v_3 = (1,0,1,-1)$,

$v_4 = (1,1,-6,-3)$ & $v_5 = (-1,-5,1,0)$. Find a basis for subspace $W = \text{span } S$ of R^4. What is dim?

Solution:-

$$\left[\begin{array}{ccccc|c} 1 & 0 & 1 & 1 & -1 & 0 \\ 1 & 1 & 0 & 1 & -5 & 0 \\ 0 & 2 & 1 & -6 & 1 & 0 \\ -1 & 1 & -1 & -3 & 0 & 0 \end{array}\right]$$

$$\left[\begin{array}{ccccc|c} 1 & 0 & 1 & 1 & -1 & 0 \\ 0 & 1 & -1 & 0 & -4 & 0 \\ 0 & 2 & 1 & -6 & 1 & 0 \\ 0 & 1 & 0 & -2 & -1 & 0 \end{array}\right] \quad \begin{array}{l} R_2 - R_1 \\ R_4 + R_1 \end{array}$$

$$\left[\begin{array}{ccccc|c} 1 & 0 & 1 & 1 & -1 & 0 \\ 0 & 1 & -1 & 0 & -4 & 0 \\ 0 & 2 & 1 & -6 & 1 & 0 \\ 0 & 0 & 1 & -2 & 3 & 0 \end{array}\right] \quad R_4 - R_2$$

$$\left[\begin{array}{ccccc|c} 1 & 0 & 1 & 1 & -1 & 0 \\ 0 & 1 & -1 & 0 & -4 & 0 \\ 0 & 0 & 3 & -6 & 9 & 0 \\ 0 & 0 & 1 & -2 & 3 & 0 \end{array}\right] \quad R_3 - 2R_2$$

$$\left[\begin{array}{ccccc|c} 1 & 0 & 1 & 1 & -1 & 0 \\ 0 & 1 & -1 & 0 & -4 & 0 \\ 0 & 0 & 1 & -2 & 3 & 0 \\ 0 & 0 & 1 & -2 & 3 & 0 \end{array}\right] \quad \tfrac{1}{3} R_3$$

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$$\left[\begin{array}{ccccc|c} 1 & 0 & 1 & 1 & -1 & 0 \\ 0 & 1 & -1 & 0 & -4 & 0 \\ 0 & 0 & 1 & -2 & 3 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right] \quad R_4 - R_3$$

So v_1, v_2 and v_3 form bases so

$S = \{v_1, v_2, v_3\}$ The $\dim W = 3$

Question No. 13

Consider the following subset of P_3.

$$S = \{t^3 + t^2 - 2t + 1,\ t^2 + 1,\ t^3 - 2t,\ 2t^3 + 3t^2 - 4t + 3\}$$

Find basis for subspace $W = \text{span}\, S$. What is $\dim W$?

Solution:-

$$c_1 v_1 + c_2 v_2 + c_3 v_3 + c_4 v_4 = 0$$

$$c_1 \begin{bmatrix} 1 \\ 1 \\ -2 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix} + c_3 \begin{bmatrix} 1 \\ 0 \\ -2 \\ 0 \end{bmatrix} + c_4 \begin{bmatrix} 2 \\ 3 \\ -4 \\ 3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\left(\begin{array}{cccc|c} 1 & 0 & 1 & 2 & 0 \\ 1 & 1 & 0 & 3 & 0 \\ -2 & 0 & -2 & -4 & 0 \\ 1 & 1 & 0 & 3 & 0 \end{array}\right)$$

$$\left[\begin{array}{cccc|c} 1 & 0 & 1 & 2 & 0 \\ 0 & 1 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 1 & 0 \end{array}\right] \quad \begin{array}{l} R_2 - R_1 \\ R_3 + 2R_1 \\ R_4 - R_1 \end{array}$$

$$\left[\begin{array}{cccc|c} 1 & 0 & 1 & 2 & 0 \\ 0 & 1 & -1 & 1 & 0 \\ 0 & 1 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \quad R_3 \longleftrightarrow R_4$$

So V_1, V_2, V_3 form basis for W & The dim of W is 3

Question No. 14.

Let

$$S = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} \right\}$$

Find basis for subspace $W =$ span S. What is dim W?

Solution:-

(a) Linear Independence:-

$$c_1V_1 + c_2V_2 + c_3V_3 + c_4V_4 = 0$$

$$c_1 \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix} + c_3 \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} + c_4 \begin{bmatrix} -1 \\ 1 \\ 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} c_1 \\ 0 \\ 0 \\ c_1 \end{bmatrix} + \begin{bmatrix} 0 \\ c_2 \\ c_2 \\ 0 \end{bmatrix} + \begin{bmatrix} c_3 \\ c_3 \\ c_3 \\ c_3 \end{bmatrix} + \begin{bmatrix} -c_4 \\ c_4 \\ c_4 \\ -c_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$c_1 + c_3 - c_4 = 0 \rightarrow (i)$$
$$c_2 + c_3 + c_4 = 0 \rightarrow (ii)$$
$$c_2 + c_3 + c_4 = 0 \rightarrow (iii)$$
$$c_1 + c_3 - c_4 = 0 \rightarrow (iv)$$

Add ~~subtract~~ (i) ~~from~~ to (ii)

$\cancel{c_2+c_3+c_4=0}$
$\cancel{c_1+c_3-c_4=0}$
$\cancel{c_1+c_2+2c_3=0}$

$$\left[\begin{array}{cccc|c} 1 & 0 & 1 & -1 & 0 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 \\ 1 & 0 & 1 & -1 & 0 \end{array}\right]$$

$$\left[\begin{array}{cccc|c} 1 & 0 & 1 & -1 & 0 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

so this is basis for span S and

$\dim W = 3$.

Question No. 15:-

Find a basis for M_{23}. What is dimension of M_{23}? Generalize to M_{mn}.

Solution:-

$$S = \left\{ \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right\}$$

$$c_1V_1 + c_2V_2 + c_3V_3 + c_4V_4 + c_5V_5 + c_6V_6 = 0$$

$$c_1\begin{bmatrix}1\\0\\0\\0\\0\\0\end{bmatrix} + c_2\begin{bmatrix}0\\1\\0\\0\\0\\0\end{bmatrix} + c_3\begin{bmatrix}0\\0\\1\\0\\0\\0\end{bmatrix} + c_4\begin{bmatrix}0\\0\\0\\1\\0\\0\end{bmatrix} + c_5\begin{bmatrix}0\\0\\0\\0\\1\\0\end{bmatrix} + c_6\begin{bmatrix}0\\0\\0\\0\\0\\1\end{bmatrix} = \begin{bmatrix}0\\0\\0\\0\\0\\0\end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

columns consisting leading 1 are vectors that form bases for W. so $v_1, v_2, v_3, v_4, v_5, v_6$ form basis for W. so $\dim W = 6$.

Question No. 16:-

$S = \{\cos^2 t, \sin^2 t, \cos 2t\}$. Find basis for subspace W = spans. What is dim W?

Solution:-

$$S = \{\cos^2 t, \sin^2 t, \cos 2t\}$$

$$= \{\cos^2 t, 1 - \cos^2 t, 2\cos^2 t - 1\}$$

$$= \{\cos^2 t, -\cos^2 t + 1, 2\cos^2 t - 1\}$$

so $\quad c_1 v_1 + c_2 v_2 + c_3 v_3 = 0$

$$c_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + c_1 \begin{bmatrix} -1 \\ 1 \end{bmatrix} + c_3 \begin{bmatrix} 2 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\left[\begin{array}{ccc|c} 1 & -1 & 2 & 0 \\ 0 & 1 & -1 & 0 \end{array}\right] \quad \cancel{R_2 \leftrightarrow R_1}$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & -1 & 0 \end{array}\right] \quad R_1 + R_2$$

So v_1 & v_2 form bases for W. Dim $W = 2$.

Question No. 17:-

Find basis for subspaces

All vectors of the form (a,b,c) where $b = a+c$.

Solution:-

let $v_1 = (1,2,1)$, $v_2 = (2,5,3)$, $v_3 = (-1,0,1)$

$v_4 = (1,4,3)$

$$c_1v_1 + c_2v_2 + c_3v_3 + c_4v_4 = 0$$

$$c_1\begin{bmatrix}1\\2\\1\end{bmatrix} + c_2\begin{bmatrix}2\\5\\3\end{bmatrix} + c_3\begin{bmatrix}-1\\0\\1\end{bmatrix} + c_4\begin{bmatrix}1\\4\\3\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix}$$

$$\cancel{\left[\begin{array}{ccc|c}1 & 2 & -1 & 0\\2 & 5 & 0 & 0\\1 & 3 & 1 & 0\end{array}\right]}$$

$$\left[\begin{array}{cccc|c}1 & 2 & -1 & 1 & 0\\2 & 5 & 0 & 4 & 0\\1 & 3 & 1 & 3 & 0\end{array}\right]$$

$$\left[\begin{array}{cccc|c}1 & 2 & -1 & 1 & 0\\0 & 1 & 2 & 2 & 0\\0 & 1 & 2 & 2 & 0\end{array}\right] \begin{array}{l}R_2-2R_1\\R_3-R_1\end{array}$$

$$\left[\begin{array}{cccc|c}1 & 2 & -1 & 1 & 0\\0 & 1 & 2 & 2 & 0\\0 & 0 & 0 & 0 & 0\end{array}\right] \quad R_3-R_2$$

So v_1 & v_2 form bases for W. so, $\dim W = 2$.

Question No. 18:-

Find bases for given subspaces.

All vectors of form (a,b,c) where $a=0$

Solution:-

Let
$v_1 = (0,1,3)$, $v_2 = (0,4,6)$, $v_3 = (0,-1,1)$, for $a=0$

$$c_1v_1 + c_2v_2 + c_3v_3 = 0$$

$$c_1\begin{bmatrix}0\\1\\3\end{bmatrix} + c_2\begin{bmatrix}0\\4\\6\end{bmatrix} + c_3\begin{bmatrix}0\\-1\\1\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix}$$

$$\left[\begin{array}{ccc|c}0 & 0 & 0 & 0\\1 & 4 & -1 & 0\\3 & 6 & 1 & 0\end{array}\right]$$

$$\left[\begin{array}{ccc|c}0 & 0 & 0 & 0\\1 & 4 & -1 & 0\\0 & -6 & 4 & 0\end{array}\right] \quad R_3-3R_2$$

$$\left[\begin{array}{ccc|c}1 & 4 & -1 & 0\\0 & 0 & 0 & 0\\0 & -6 & 4 & 0\end{array}\right] \quad R_2 \longleftrightarrow R_1$$

$$\left[\begin{array}{ccc|c}1 & 4 & -1 & 0\\0 & -6 & 4 & 0\\0 & 0 & 0 & 0\end{array}\right] \quad R_2 \longleftrightarrow R_3$$

$$\left[\begin{array}{ccc|c}1 & 4 & -1 & 0\\0 & +1 & -2/3 & 0\\0 & 0 & 0 & 0\end{array}\right] \quad R_2/6$$

$$\left[\begin{array}{ccc|c}1 & 0 & 5/3 & 0\\0 & 1 & -2/3 & 0\\0 & 0 & 0 & 0\end{array}\right] \quad R_1-4R_2$$

so v_1 & v_2 form bases.

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Question No. 19:-

Find dimension of

All vectors of the form (a,b,c,d) where $d = a+b$.

Solution:-

Let $V_1 = (1,2,4,3)$, $V_2 = (3,0,1,3)$, $V_3 = (5,0,2,5)$,

$V_4 = (5,1,0,6)$

$$c_1V_1 + c_2V_2 + c_3V_3 + c_4V_4 = 0$$

$$c_1\begin{bmatrix}1\\2\\4\\3\end{bmatrix} + c_2\begin{bmatrix}3\\0\\1\\3\end{bmatrix} + c_3\begin{bmatrix}5\\0\\2\\5\end{bmatrix} + c_4\begin{bmatrix}5\\1\\0\\6\end{bmatrix} = \begin{bmatrix}0\\0\\0\\0\end{bmatrix}$$

$$\left[\begin{array}{cccc|c}1 & 3 & 5 & 5 & 0\\2 & 0 & 0 & 1 & 0\\4 & 1 & 2 & 0 & 0\\3 & 3 & 5 & 6 & 0\end{array}\right]$$

$$\left[\begin{array}{cccc|c}1 & 3 & 5 & 5 & 0\\0 & -6 & -10 & -9 & 0\\0 & -11 & -18 & -20 & 0\\0 & -6 & -10 & -9 & 0\end{array}\right] \begin{matrix}R_2-2R_1\\R_3-4R_1\\R_4-3R_1\end{matrix}$$

$$\left[\begin{array}{cccc|c}1 & 3 & 5 & 5 & 0\\0 & -6 & -10 & -9 & 0\\0 & -11 & -18 & -20 & 0\\0 & 0 & 0 & 0 & 0\end{array}\right] R_4-R_2$$

So V_1 & V_2 & V_3 form bases & $\dim W = 3$

Question No. 20 :-

Find dimension of

All vectors of the form (a,b,c,d) where $a=b$

Solution:-

Let $V_1 = (1,1,2,3)$, $V_2 = (0,0,1,2)$, $V_3 = (3,3,0,0)$

$V_4 = (0,0,0,9)$

$C_1V_1 + C_2V_2 + C_3V_3 + C_4V_4 = 0$

$$C_1\begin{bmatrix}1\\1\\2\\3\end{bmatrix} + C_2\begin{bmatrix}0\\0\\1\\2\end{bmatrix} + C_3\begin{bmatrix}3\\3\\0\\0\end{bmatrix} + C_4\begin{bmatrix}0\\0\\0\\2\end{bmatrix} = \begin{bmatrix}0\\0\\0\\0\end{bmatrix}$$

$$\left[\begin{array}{cccc|c}1 & 0 & 3 & 0 & 0\\ 1 & 0 & 3 & 0 & 0\\ 2 & 1 & 0 & 0 & 0\\ 3 & 2 & 0 & 2 & 0\end{array}\right]$$

$$\left[\begin{array}{cccc|c}1 & 0 & 3 & 0 & 0\\ 0 & 0 & 0 & 0 & 0\\ 2 & 1 & 0 & 0 & 0\\ 3 & 2 & 0 & 2 & 0\end{array}\right] \quad R_2 - R_1$$

$$\left[\begin{array}{cccc|c}1 & 0 & 3 & 0 & 0\\ 3 & 2 & 0 & 2 & 0\\ 2 & 1 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0\end{array}\right] \quad R_2 \longleftrightarrow R_4$$

$$\left[\begin{array}{cccc|c}1 & 0 & 3 & 0 & 0\\ 0 & 2 & -9 & 2 & 0\\ 0 & 1 & -6 & 0 & 0\\ 0 & 0 & 0 & 0 & 0\end{array}\right] \quad \begin{array}{l}R_2 - 3R_1\\ R_3 - 2R_1\end{array}$$

$$\left[\begin{array}{cccc|c}1 & 0 & 3 & 0 & 0\\ 0 & 1 & -9/2 & 1 & 0\\ 0 & 1 & -6 & 0 & 0\\ 0 & 0 & 0 & 0 & 0\end{array}\right] \quad \tfrac{1}{2}R_2$$

Hence $\dim W = 3$.

Question No. 21:-

Find basis for subspace of P_2 consisting of all vectors of the form $at^2 + bt + c$, where $c = 2a - 3b$.

Solutions

$V_1 = t^2 + t + 1$ when $a = b = 1$

$V_2 = t - 3$ when $a = 0$, $b = 1$

$V_3 = t^2 + 2$ when $a = 1$, $b = 0$

$$c_1v_1 + c_2v_2 + c_3v_3 = 0$$

$$c_1(t^2+t+1) + c_2(t-3) + c_3(t^2+2) = (0,0,0)$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & -3 & 2 & 0 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & -3 & 1 & 0 \end{array}\right] \begin{array}{l} R_2 - R_1 \\ R_3 - R_1 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & -2 & 0 \end{array}\right] R_3 + 3R_2$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right] -R_3/2$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right] R_2 + R_3$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right] R_1 - R_3$$

Question No. 22:-

Find basis for subspace of P_3 consisting of all vectors of the form $at^3 + bt^2 + ct + d$, where $a = b$ & $c = d$.

Solution:-

$v_1 = t^3 + t^2 + t + 1.$ when $a = b = 1$, $c = d = 1$

$v_2 = t^3 + t^2$ when $a = b = 1$, $c = d = 0$

$v_3 = t + 1$ when $a = b = 0$, $c = d = 1$

$$c_1 v_1 + c_2 v_2 + c_3 v_3 = 0$$

$$c_1 \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + c_3 \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & -1 & 1 & 0 \end{array}\right] \begin{array}{l} R_2 - R_1 \\ R_3 - R_1 \\ R_4 - R_1 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \quad R_2 \leftrightarrow R_4$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \quad -R_2$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \begin{array}{l} R_1 - R_2 \\ R_3 + R_2 \end{array}$$

so $\operatorname{Dim} W = 2$

Question No. 23.

Find dimension of subspaces of R^2 spanned by the vectors in $\{(1,3),(1,-1)\}$

Solution:-

$$\{(1,3),(1,-1)\}$$

$$c_1(1,3)+c_2(1,-1)=(0,0)$$

$$\left[\begin{array}{cc|c} 1 & 1 & 0 \\ 3 & -1 & 0 \end{array}\right]$$

$$\left[\begin{array}{cc|c} 1 & 1 & 0 \\ 0 & -4 & 0 \end{array}\right] \quad R_2-3R_1$$

$$\left[\begin{array}{cc|c} 1 & 1 & 0 \\ 0 & 1 & 0 \end{array}\right] \quad -\tfrac{1}{4}R_2$$

$$\left[\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 0 \end{array}\right] \quad R_1-R_2$$

so Dim $=2$

Question No. 24:-

Find dim of subspaces of R^3 spanned by vectors in Ex. 2?

$$\{(1,2,0),(0,1,-1)\}$$

Solution:-

$$\{(1,2,0),(0,1,-1)\}$$

$c_1(1,2,0) + c_2(0,1,-1) = (0,0,0)$

$$\left[\begin{array}{cc|c} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & -1 & 0 \end{array}\right]$$

$$\left[\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 0 \end{array}\right] \quad R_2 - 2R_1$$

$$\left[\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right] \quad R_3 + R_2$$

So Dim = 2.

Question No. 25

Find dim. of subspaces of R^4 spanned by the vectors in $\{(1,0,0,1), (0,1,0,0), (1,1,1,1), (0,1,1,1)\}$

Solution:-

$$c_1(1,0,0,1) + c_2(0,1,0,0) + c_3(1,1,1,1) + c_4(0,1,1,1) = (0,0,0,0)$$

$$\left[\begin{array}{cccc|c} 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 & 0 \end{array}\right]$$

$$\left[\begin{array}{cccc|c} 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{array}\right] \quad R_4 - R_1$$

$$\left[\begin{array}{cccc|c} 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{array}\right] R_2 - R_3$$

$$\left[\begin{array}{cccc|c} 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{array}\right] R_1 - R_3$$

$$\left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{array}\right] \begin{array}{l} R_1 - R_4 \\ R_3 - R_4 \end{array}$$

So Dim $= 4$.

Question No. 26:-

Find the dim. of the subspace of P_2 consisting of all vectors of the form $at^2 + bt + c$ where $c = b - 2a$.

Solution:-

$v_1 = t^2 + t - 1$ $\quad a = b = 1$

$v_2 = t^2 - 2$ $\quad a = 1, b = 0$

$v_3 = t + 1$ $\quad a = 0, b = 1$

$$c_1 v_1 + c_2 v_2 + c_3 v_3 = 0$$

$$c_1(t^2 + t - 1) + c_2(t^2 - 2) + c_3(t + 1) = (0, 0, 0)$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ -1 & -2 & 1 & 0 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & -1 & 1 & 0 \end{array}\right] \begin{array}{l} R_2-R_1 \\ R_3+R_1 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] R_3-R_2$$

so $\quad \text{Dim} = 2.$

Question No. 27:-

Find dim of subspace of P_3 consisting of all vectors of the form at^3+bt^2+ct+d, where $b=3a-5d$ & $c=d+4a$.

Solution:-

$V_1 = (1,-2,5,1)$, $V_2 = (0,0,0,0)$

$V_3 = (0,-5,1,-1)$, $V_4 = (1,3,4,0)$

$$c_1V_1 + c_2V_2 + c_3V_3 + c_4V_4 = 0$$

$$c_1(1,-2,5,1) + c_2(0,0,0,0) + c_3(0,-5,1,1)$$
$$+c_4(1,3,4,0) = (0,0,0,0)$$

$$\left[\begin{array}{cccc|c} 1 & 0 & 0 & 1 & 0 \\ -2 & 0 & -5 & 3 & 0 \\ 5 & 0 & 1 & 4 & 0 \\ 1 & 0 & 1 & 0 & 0 \end{array}\right]$$

$$\left[\begin{array}{cccc|c} 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & -5 & 5 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & -1 & 0 \end{array}\right] \begin{array}{l} R_2+2R_1 \\ R_3-5R_1 \\ R_4-R_1 \end{array}$$

$$\left[\begin{array}{cccc|c} 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 \end{array}\right] \begin{array}{l} R_2+5R_4 \\ R_3-R_4 \end{array}$$

$$\left[\begin{array}{cccc|c} 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] R_4 \leftrightarrow R_2$$

$$\left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] R_2+R_1$$

Dim = 2 (only 2 columns contain leading 1)

Question No. 28:-

Find basis for R^3 that includes the vectors $(1,0,2)$

Solution:-

let $\{e_1, e_2, e_3\}$ be natural basis for R^3

$e_1 = (1,0,0)$, $e_2 = (0,1,0)$, $e_3 = (0,0,1)$

form set $S = \{v_1, e_1, e_2, e_3\}$

Thus,

$$c_1v_1 + c_2e_1 + c_3e_2 + c_4e_3 = 0$$

$$c_1\begin{bmatrix}1\\0\\2\end{bmatrix} + c_2\begin{bmatrix}1\\0\\0\end{bmatrix} + c_3\begin{bmatrix}0\\1\\0\end{bmatrix} + c_4\begin{bmatrix}0\\0\\1\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix}$$

$$\begin{bmatrix} c_1 \\ 0 \\ 2c_1 \end{bmatrix} + \begin{bmatrix} c_2 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ c_3 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ c_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$c_1 + c_2 = 0$

$c_3 = 0$

$2c_1 + c_4 = 0$

$$\left[\begin{array}{cccc|c} 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 2 & 0 & 0 & 1 & 0 \end{array}\right]$$

$$\left[\begin{array}{cccc|c} 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & -2 & 0 & 1 & 0 \end{array}\right] \quad R_3 - 2R_1$$

$$\left[\begin{array}{cccc|c} 1 & 1 & 0 & 0 & 0 \\ 0 & -2 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{array}\right] \quad R_3 \leftrightarrow R_2$$

$$\left[\begin{array}{cccc|c} 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1/2 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{array}\right] \quad -\tfrac{1}{2}R_2$$

$$\left[\begin{array}{cccc|c} 1 & 0 & 0 & 1/2 & 0 \\ 0 & 1 & 0 & -1/2 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{array}\right] \quad R_1 - R_2$$

Since leading 1s appear in column 1, 2, & 3
so $\{v_1, e_1, e_2\}$ is basis containing v_1

Question No. 29:-

Find basis for R_4 that includes vector $(1,0,1,0)$ & $(0,1,-1,0)$

Solution:-

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Let $\{e_1, e_2, e_3, e_4\}$ be natural basis for R^4

$e_1 = (1,0,0,0)$, $e_2 = (0,1,0,0)$, $e_3 = (0,0,1,0)$, $e_4 = (0,0,0,1)$

form set $S = \{v_1, v_2, e_1, e_2, e_3, e_4\}$

As linear combination

$$c_1 v_1 + c_2 v_2 + c_3 e_1 + c_4 e_2 + c_5 e_3 + c_6 e_4$$

$$c_1(1,0,1,0) + c_2(0,1,-1,0) + c_3(1,0,0,0) + c_4(0,1,0,0)$$

$$+ c_5(0,0,1,0) + c_6(0,0,0,1)$$

$$\Rightarrow [A \mid 0] = \left[\begin{array}{cccccc|c} 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{array}\right]$$

$$\underset{\sim}{R} \left[\begin{array}{cccccc|c} 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & -1 & \cancel{0}+1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{array}\right] R_3 - R_1$$

$$\underset{\sim}{R} \left[\begin{array}{cccccc|c} 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{array}\right] R_3 + R_2$$

$$\underset{\sim}{R} \left[\begin{array}{cccccc|c} 1 & 0 & 0 & -1 & -1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{array}\right] R_1 - R_3$$

As leading 1 is appearing in columns 1, 2, 3 & 6 so $\{v_1, v_2, e_1, e_4\}$ is basis containing v_1 & v_2.

Question No. 30:-

Find all values of a for which $\{(a^2,0,1),(0,a,2),(1,0,1)$ is a basis for R^3.

Solution:-

$$c_1v_1+c_2v_2+c_3v_3=0$$

$$c_1(a^2,0,1)+c_2(0,a,2)+c_3(1,0,1)=0$$

$$\begin{bmatrix} c_1a^2 \\ 0 \\ c_1 \end{bmatrix}+\begin{bmatrix} 0 \\ ac_2 \\ 2c_2 \end{bmatrix}+\begin{bmatrix} c_3 \\ 0 \\ c_3 \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$a^2c_1+c_3=0 \rightarrow (i)$$
$$ac_2=0 \rightarrow (ii)$$
$$c_1+2c_2+c_3=0 \rightarrow (iii)$$

$$[A \vdots 0]=\left[\begin{array}{ccc|c} a^2 & 0 & 1 & 0 \\ 0 & a & 0 & 0 \\ 1 & 2 & 1 & 0 \end{array}\right]$$

$$R \sim \left[\begin{array}{ccc|c} 1 & 0 & 1/a^2 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 2 & 1 & 0 \end{array}\right] \quad \begin{array}{l} \frac{1}{a^2}R_1 \\ \frac{1}{a}R_2 \end{array}$$

$$R \sim \left[\begin{array}{ccc|c} 1 & 0 & 1/a^2 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 2 & \frac{a^2-1}{a^2} & 0 \end{array}\right] \quad R_3-R_1$$

$$R \sim \left[\begin{array}{ccc|c} 1 & 0 & 1/a^2 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \frac{a^2-1}{a^2} & 0 \end{array}\right] \quad R_3-2R_2$$

$$\cancel{\left[\begin{array}{ccc|c} 1 & 0 & 1/a^2 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right] \frac{a^2}{a^2-1} R_3}$$

$$\cancel{\left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right] R_1 - \frac{1}{a^2} R_3}$$

so $\dfrac{a^2-1}{a^2} = 0$

$a^2 - 1 = 0$

$a^2 = 1$

$a = \pm 1.$

Question No. 31:-

Find basis for subspace W of M_{33} consisting of all symmetric matrices.

Solution:-

let $A = \begin{bmatrix} a & b & c \\ b & d & e \\ c & e & f \end{bmatrix}$

Now we will write it as sum of matrices.

$$\begin{bmatrix} a & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & b & 0 \\ b & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & c \\ 0 & 0 & 0 \\ c & 0 & 0 \end{bmatrix} +$$

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & d & 0 \\ 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & d & e \\ 0 & e & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & f \end{bmatrix}$$

$$+ a\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + b\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + f$$

$$c\begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} + d\begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} + e$$

$$= e\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} + f\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\Rightarrow \left\{ \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right\}$$

Question No. 32:-

Find basis for subspace of M_{33} consisting of all diagonal matrices.

Solution:-

Diagonal matrix is

$$\begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{bmatrix}$$

we write it as sum of matrices

$$\begin{bmatrix} a & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & c \end{bmatrix}$$

$$a\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + b\begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} + c\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Hence

$$= \left\{ \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right\}$$

Question No. 33.

Give an e.g of two dim subspace of $\mathbb{R}^4$.

Solution:-

Question No. 35:-

Find basis for given plane.

$$2x-3y+4z=0$$

Solution:-

$$2x-3y+4z=0$$

$$x=\frac{3y-4z}{2}$$

As dim of plane is 3 i.e

at $y=1$, $z=0$, $x=3/2$

$$V_1=(3/2, 1, 0)$$

at $y=0$, $z=1$, $x=-2$

so

$$V_2=(-2,0,1)$$

at $y=2$, $z=1$, $x=1$

$$V_3=(1,2,1)$$

so

$$S=\{(3/2,1,0),(-2,0,1),(1,2,1)\}$$

for basis

$$c_1V_1+c_2V_2+c_3V_3=0$$

$$[A\,|\,0]=\left[\begin{array}{ccc|c} 3/2 & -2 & 1 & 0 \\ 1 & 0 & 2 & 0 \\ 0 & 1 & 1 & 0 \end{array}\right]$$

$$=\left[\begin{array}{ccc|c} 1 & 0 & 2 & 0 \\ 3/2 & -2 & 1 & 0 \\ 0 & 1 & 1 & 0 \end{array}\right] \quad R_2 \leftrightarrow R_1$$

$$= \left[\begin{array}{ccc|c} 1 & 0 & 2 & 0 \\ 0 & -2 & -2 & 0 \\ 0 & 1 & 1 & 0 \end{array}\right] \quad R_2 - 3/2\, R_1$$

$$= \left[\begin{array}{ccc|c} 1 & 0 & 2 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \end{array}\right] \quad -1/2\, R_2$$

$$= \left[\begin{array}{ccc|c} 1 & 0 & 2 & \\ 0 & 1 & 1 & \\ 0 & 0 & 0 & \end{array}\right] \quad R_3 - R_2$$

So this is the basis for given plane.

Question No.36:-

Find basis for given plane

$$x + y - 3z = 0$$

Solution:-

$$x + y - 3z = 0$$

$$x = 3z - y$$

At $y = 1$, $z = 1$, $x = 2$

$$V_1 = (2, 1, 1)$$

At $y = 0$, $z = 1$, $x = 3$

$$V_2 = (3, 0, 1)$$

At $y = 1$, $z = 0$, $x = -1$

$$V_3 = (-1, 1, 0)$$

So

$$S = \{(2,1,1), (3,0,1), (-1,1,0)\}$$

For basis

$c_1v_1 + c_2v_2 + c_3v_3 = 0$

$$[A \mid 0] = \left[\begin{array}{ccc|c} 2 & 3 & -1 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 \end{array}\right]$$

$$= \left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 2 & 3 & -1 & 0 \\ 1 & 1 & 0 & 0 \end{array}\right] \quad R_2 \leftrightarrow R_1$$

$$= \left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 3 & -3 & 0 \\ 0 & 1 & -1 & 0 \end{array}\right] \quad \begin{array}{l} R_2 - 2R_1 \\ R_3 - R_1 \end{array}$$

$$= \left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 3 & -3 & 0 \end{array}\right] \quad \begin{array}{l} R_2 \leftrightarrow R_3 \\ 3R_2 \end{array}$$

$$= \left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \quad R_3 - 3R_2$$

So this is the basis for given plane

Question No. 37:-

Determine if vectors

$$\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \quad \text{are basis for } \mathbb{R}^3$$

Solution:-

(a) Span:-

$$c_1V_1 + c_2V_2 + c_3V_3 = V$$

$$c_1\begin{bmatrix}1\\1\\1\end{bmatrix} + c_2\begin{bmatrix}1\\1\\0\end{bmatrix} + c_3\begin{bmatrix}1\\0\\1\end{bmatrix} = \begin{bmatrix}a\\b\\c\end{bmatrix}$$

$$[A \vdots b] = \left[\begin{array}{ccc|c}1 & 1 & 1 & a\\1 & 1 & 0 & b\\1 & 0 & 1 & c\end{array}\right]$$

$$\left[\begin{array}{ccc|c}1 & 1 & 1 & a\\0 & 0 & +1 & b+a\\0 & +1 & 0 & c+a\end{array}\right] \quad \begin{array}{l}R_2+R_1\\R_3+R_1\end{array}$$

$$\left[\begin{array}{ccc|c}1 & 1 & 1 & a\\0 & +1 & 0 & c+a\\0 & 0 & +1 & b+a\end{array}\right] \quad R_2 \leftrightarrow R_3$$

$$\left[\begin{array}{ccc|c}1 & 1 & 1 & a\\0 & 1 & 0 & a+c\\0 & 0 & 1 & a+b\end{array}\right]$$

$$\left[\begin{array}{ccc|c}1 & 0 & 0 & c+3a+b\\0 & 1 & 0 & a+c\\0 & 0 & 1 & a+b\end{array}\right] \quad \begin{array}{l}R_1+R_2\\R_1+R_3\end{array}$$

$$c_1 = c+3a+b$$
$$c_2 = a+c$$
$$c_3 = a+b$$

So it is span.

(b) Linear Independent

$$[A \vdots 0] = \left[\begin{array}{ccc|c}1 & 1 & 1 & 0\\1 & 1 & 0 & 0\\1 & 0 & 1 & 0\end{array}\right]$$

$$\sim\left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 \end{array}\right] \begin{array}{l} R_2+R_1 \\ R_3+R_1 \end{array}$$

$$\sim\left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \end{array}\right] \quad R_2 \longleftrightarrow R_3$$

$$\sim\left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right] \begin{array}{l} -R_2 \\ -R_3 \end{array}$$

$$\sim\left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right] \quad R_1+R_2$$

$$\sim\left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right] \quad R_1+R_3$$

As $c_1 = c_2 = c_3 = 0$ so it is linear independent & thus is basis.

Question No. 38:-

Determine if vectors

$\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$ are basis for R^3.

Solution:-

(a) Linear Independence:-

$$[A \vdots 0] = \left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right]$$

$$R \sim \left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right] R_2 + R_1$$

$$R \sim \left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right] R_1 + R_2$$

$$R \sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right] \begin{array}{l} R_1 + R_3 \\ R_2 + R_3 \end{array}$$

As $c_1 = c_2 = c_3 = 0$. so its linear independent.

(b) Span:-

$$c_1 v_1 + c_2 v_2 + c_3 v_3 = v$$

$$c_1 \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + c_3 \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$$

$$[A:b] = \left[\begin{array}{ccc|c} 1 & 1 & 0 & a \\ 1 & 0 & 1 & b \\ 0 & 0 & 1 & c \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|c} 1 & 1 & 0 & a \\ 0 & 1 & 1 & a+b \\ 0 & 0 & 1 & c \end{array}\right] \quad R_2+R_1$$

$$\sim \left[\begin{array}{ccc|c} 1 & 0 & 1 & 2a+b \\ 0 & 1 & 1 & a+b \\ 0 & 0 & 1 & c \end{array}\right] \quad R_1+R_2$$

$$\sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & 2a+b+c \\ 0 & 1 & 0 & a+b+c \\ 0 & 0 & 1 & c \end{array}\right] \quad \begin{array}{l} R_1+R_3 \\ R_2+R_3 \end{array}$$

$c_1 = 2a+b+c$

$c_2 = a+b+c$

$c_3 = c$

so it forms span & hence is basis for B^3.

Question No. 39:-

Determine if vectors

$$\begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

are basis for B^4.

Solution:-

(a) Linear Independence:

$$[A|0] = \left[\begin{array}{cccc|c} 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \end{array}\right]$$

$$R \left[\begin{array}{cccc|c} 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \end{array}\right] R_4+R_1$$

$$R \left[\begin{array}{cccc|c} 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 \end{array}\right] R_2 \leftrightarrow R_3$$

$$R \left[\begin{array}{cccc|c} 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 \end{array}\right] \begin{array}{l} R_4+R_2 \\ R_2+R_3 \end{array}$$

$$R \left[\begin{array}{cccc|c} 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] R_4+R_3$$

$$R \left[\begin{array}{cccc|c} 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] R.$$

So it is not linear independent eq

hence not basis.

Question No. 40:-

Determine if vectors

$$\begin{bmatrix}1\\1\\0\\1\end{bmatrix}, \begin{bmatrix}1\\0\\1\\1\end{bmatrix}, \begin{bmatrix}0\\1\\1\\0\end{bmatrix}, \begin{bmatrix}1\\1\\1\\1\end{bmatrix}$$ are basis for B^4.

Solution:-

(a) Linear Independence:-

$$[A\,\vdots\,0] = \left[\begin{array}{cccc|c}1 & 1 & 0 & 1 & 0\\1 & 0 & 1 & 1 & 0\\0 & 1 & 1 & 1 & 0\\1 & 1 & 0 & 1 & 0\end{array}\right]$$

$$\approx \left[\begin{array}{cccc|c}1 & 1 & 0 & 1 & 0\\0 & 1 & 1 & 0 & 0\\0 & 1 & 1 & 1 & 0\\0 & 0 & 0 & 0 & 0\end{array}\right] \begin{array}{l}R_2+R_1\\R_4+R_1\\ \\ \\ \end{array}$$

As 4 ~~eqs~~ unknowns and 3 eqs so forms so its not linear independent & Hence it is not basis for B^4.

FINISHED