Problem Set 14.7

Expand each of the following functions in a Laurent series that converges for $0 < |z| < R$ and determine the precise region of convergence.

1. $\dfrac{e^z}{z^2}$ 2. $\dfrac{\sin 4z}{z^4}$ 3. $\dfrac{\cosh 2z}{z}$

4. $\dfrac{1}{z^3(1 - z)}$ 5. $\dfrac{1}{z(1 + z^2)}$ 6. $\dfrac{8 - 2z}{4z - z^3}$

7. $z \cos \dfrac{1}{z}$ 8. $\dfrac{e^{-1/z^2}}{z^5}$ 9. $\dfrac{1}{z^6(1 + z)^2}$

Expand each of the following functions in a Laurent series that converges for $0 < |z - z_0| < R$ and determine the precise region of convergence.

10. $\dfrac{e^z}{z - 1}$, $z_0 = 1$ 11. $\dfrac{1}{z^2 + 1}$, $z_0 = i$ 12. $z^2 \sinh \dfrac{1}{z}$, $z_0 = 0$

13. $\dfrac{\cos z}{(z - \pi)^3}$, $z_0 = \pi$ 14. $\dfrac{z^4}{(z + 2i)^2}$, $z_0 = -2i$ 15. $\dfrac{z^2 - 4}{z - 1}$, $z_0 = 1$

16. $\dfrac{\sin z}{(z - \frac{1}{4}\pi)^3}$, $z_0 = \dfrac{\pi}{4}$ 17. $\dfrac{1}{(z + i)^2 - (z + i)}$, $z_0 = -i$

18. $\dfrac{1}{1 - z^4}$, $z_0 = -1$

Find the Taylor or Laurent series of $1/(1 - z^2)$ in the region

19. $0 \leqq |z| < 1$ 20. $|z| > 1$ 21. $0 < |z - 1| < 2$

Using partial fractions, find the Laurent series of $(3z^2 - 6z + 2)/(z^3 - 3z^2 + 2z)$ in the region

22. $0 < |z| < 1$ 23. $1 < |z| < 2$ 24. $|z| > 2$

Find all Taylor and Laurent series with center $z = z_0$ and determine the precise region of convergence.

25. $\dfrac{1}{1 - z^3}$, $z_0 = 0$ 26. $\dfrac{2}{1 - z^2}$, $z_0 = 1$ 27. $\dfrac{z^2}{1 - z^4}$, $z_0 = 0$

28. $\dfrac{1}{z^2}$, $z_0 = i$ 29. $\dfrac{1}{z}$, $z_0 = 1$ 30. $\dfrac{\sinh z}{(z - 1)^2}$, $z_0 = 1$

31. $\dfrac{\sin z}{z + \frac{1}{2}\pi}$, $z_0 = -\dfrac{1}{2}\pi$ 32. $\dfrac{z^3 - 2iz^2}{(z - i)^2}$, $z_0 = i$ 33. $\dfrac{4z - 1}{z^4 - 1}$, $z_0 = 0$

34. Does $\tan(1/z)$ have a Laurent series convergent in a region $0 < |z| < R$?
35. Prove that the Laurent expansion of a given analytic function in a given annulus is unique.

<u>Laurent Series</u>:- let "Z" be any analytic function so the Laurent Series is given by:

$$f(z) = a_0 + a_1(z-z_0) + a_2(z-z_0)^2 + a_3(z-z_0)^3 + \cdots$$

$$+ \frac{b_1}{(z-z_0)} + \frac{b_2}{(z-z_0)^2} + \cdots \quad (1)$$

where $a_1, a_2, a_3 \cdots, b_1, b_2 \cdots$ are constants

We will find out their values z_0 may or may not be the Singular Point of a given function.

1^{st} Part of the Series having $a_0 + a_1(z-z_0) + a_2(z-z_0)^2 + \cdots$ is called analytic Part of Laurent Series. While the 2^{nd} Part $\frac{b_1}{(z-z_0)} + \frac{b_2}{(z-z_0)^2} + \cdots$ is called Principale Part of laurent Series.

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Exercise 14.7

Expand each of the following function in a Laurent series that converges for $0 < |z| < R$ and determine the precise region of convergence.

Q1 $f(z) = \dfrac{e^z}{z^2}$ —— (1)

for Singular Point

$$\frac{e^z}{z^2} = \infty \Rightarrow z^2 = 0 \Rightarrow z = 0$$

Here no Point is given, So we will take $z_0 = 0$.

Now let $z - z_0 = U$

$$z - 0 = U \Rightarrow z = U \quad \text{—— (2)}$$

eq(1) $\Rightarrow$ $f(U) = \dfrac{e^U}{(U^2)}$

$$= \frac{1}{U^2}\left[1 + U + \frac{U^2}{2!} + \frac{U^3}{3!} + \cdots\right]$$

$$= \frac{1}{U^2} + \frac{1}{U} + \frac{1}{2!} + \frac{U}{3!} + \frac{U^2}{4!} + \cdots$$

$$= \frac{1}{2!} + \frac{u}{3!} + \frac{u^2}{4!} + \cdots + 1/u + \frac{1}{u^2} + \cdots \quad (3)$$

Putting $u = z$ eq (3) $\Rightarrow$

$$f(z) = \frac{1}{2!} + \frac{z}{3!} + \frac{z^2}{4!} + \cdots + \frac{1}{z} + \frac{1}{z^2} + \cdots \quad (4)$$

The Laurent Series is given by

$$f(z) = a_0 + a_1(z - z_0) + a_2(z - z_0)^2 + \cdots + \frac{b_1}{z - z_0} + \frac{b_2}{(z - z_0)^2} + \cdots \quad (5)$$

Comparing eq (4) & eq (5) we get

$a_0 = 1/2!$, $a_1 = \frac{1}{3!}$; $a_2 = \frac{1}{4!}$, ...

$b_1 = 1$; $b_2 = 1$, $b_3 = b_4 = \cdots b_n = 0$ & $z_0 = 0$

Hence eq (4) gives the required Laurent Series.

Q2 $f(z) = \dfrac{\sin 4z}{z^4}$ —— (1)

The Laurent Series of $f(z)$ is given by

$$f(z) = a_0 + a_1(z-z_0) + a_2(z-z_0)^2 + \cdots + \frac{b_1}{z-z_0} + \frac{b_2}{(z-z_0)^2} + \text{—— (2)}$$

Now for Singular Point

$\therefore f(z) = \infty$

i.e $\dfrac{\sin 4z}{z^4} = \infty \Rightarrow z^4 = 0 \Rightarrow \boxed{z=0}$

$\therefore z_0 = 0$

Now Put $z - z_0 = u$

$z - 0 = u \Rightarrow \boxed{z = u}$ —→ (3)

eq (1) $\Rightarrow$

$$f(u) = \frac{\sin 4u}{u^4} = \frac{1}{u^4}\left[4u - \frac{(4u)^3}{3!} + \frac{(4u)^5}{5!} + \cdots\right]$$

$$= \frac{1}{u^4}\left[4u - \frac{4^3u^3}{3!} + \frac{4^5u^5}{5!} - \frac{4^7u^7}{7!} + \cdots\right]$$

$$= \frac{4}{u^3} - \frac{64}{3u} + \frac{4^5}{5!}u^2 - \frac{u^7}{7!}u^4 + \cdots$$

$$= \frac{4^5}{5!}u^2 - \frac{4^7}{7!}u^4 + \cdots - \frac{4^3}{3u} + \frac{4}{u^3}$$

using eq ③

$$f(z) = \frac{4^5}{5!}z^2 - \frac{4^7}{7!}z^4 + \cdots - \frac{4^3}{3z} + \frac{4}{z^3} \longrightarrow ④$$

comparing eq ② & eq ④

$a_0 = 0$; $a_1 = a_3 = a_5 = \cdots a_{n-1} = 0$

$a_2 = \frac{4^5}{5!}$; $a_4 = -\frac{4^7}{7!}$; $b_1 = -\frac{4^3}{3}$

$b = 0$, $b_3 = 4$; $b_4 = b_5 = \cdots = b_n = 0$

Eq ④ gives the required Laurent Series.

Q3 $f(z) = \frac{\cosh 2z}{z}$ —— (1)

The L. Series of $f(z)$ at z_0 is given by:

$f(z) = a_0 + (z-z_0)a_1 + a_2(z-z_0)^2 + \cdots + \frac{b_1}{z-z_0} + \frac{b_2}{z-z_0} + \cdots$ —— (2)

Now for Singular Points $f(z) = \infty$.

$\frac{\cosh 2z}{z} = \infty$

$\Rightarrow z = 0 \quad \therefore z_0 = 0$

let $z - z_0 = u \Rightarrow z - 0 = u \Rightarrow z = u \longrightarrow$ (3)

eq (1) $\Rightarrow f(u) = \frac{\cosh 2u}{u} = \frac{1}{u}\left[1 + \frac{(2u)^2}{2!} + \frac{(2u)^4}{4!} + \cdots\right]$

$= \cancel{\frac{1}{u}\{1+} \frac{1}{u} + 2u + \frac{2}{3}u^3 + \cdots$

$= 2u + \frac{2}{3}u^3 + \cdots + 1/u$

Putting $u = z$ from eq (3)

$f(z) = 2z + \frac{2}{3}z^3 + \cdots + 1/z \longrightarrow$ (4)

Comparing eq (2) and eq (4) we get

$a_0 = 0$; $a_1 = 2$; $a_2 = 0$; $a_3 = 2/3$

$b_1 = 1$, $b_2 = b_3 = \text{———} = b_n = 0$

Eq (4) gives the required Laurent Series.

Q4 $\dfrac{1}{z^3(1-z)}$

Q5 $\dfrac{1}{z(1+z^2)}$

Q6 $\dfrac{8-2z}{4z-z^3}$

Q7 $z\cos 1/z$

Q8 $\dfrac{e^{-1/z^2}}{z^5}$

Q9 $f(z) = \frac{1}{z^6(1+z)^2}$ —(1)

L. series of $f(z)$ at $z = z_0$ is

$$f(z) = a_0 + a_1(z-z_0) + a_2(z-z_0)^2 + \cdots + \frac{b_1}{z-z_0} + \frac{b_2}{(z-z_0)^2} + \cdots \quad (2)$$

for Singular Point

$\therefore f(z) = \infty$

$\Rightarrow \frac{1}{z^6(1+z)^2} = \infty \Rightarrow (1+z)^2 = 0 \Rightarrow \boxed{z = -1}$

Put $z - z_0 = U$

$z - (-1) = U \Rightarrow \boxed{z + 1 = U} \Rightarrow \boxed{z = U - 1}$

eq (1) $\Rightarrow f(z) = \frac{1}{(U-1)^6(1+(U-1))^2}$

$$= \frac{1}{(U-1)^6 (U)^2}$$

$$= \frac{1}{U^2}(1-U)^{-6}$$

$$= \frac{1}{U^2}\left[1 + 6U - 21U^2 + 56U^3 - 126U^4 + \cdots\right]$$

$$f(U-1) = \frac{1}{U^2} + \frac{6}{U} - 21 + 56U - 126U^2 + \cdots$$

$$= -21 + 56U - 126U^2 + \cdots + \frac{1}{U^2} + \frac{6}{U}$$

Since $U = 1 + z$

$$f(z) = -21 + 56(z+1) - 126(z+1)^2 + \cdots + \frac{6}{1+z} + \frac{1}{(1+z)^2}$$

comparing with eq (2)

$a_0 = -21$, $a_1 = 56$; $a_2 = -126$

$b_1 = 6$; $b_2 = 1$ & $z_0 = -1$

Hence the above eq gives the required L. Series.

<u>Zeros</u>:- $f(z)$ be an analytic function for $z=a$;
if $f(a)=0$. Then $z=a$ is called zero of $f(z)$.

e.g $f(z)=z-2$. Find its zero.

Sol:- for zero of the given function put

$$z-2=0$$
$$\Rightarrow z=2$$

$\therefore$ $z=2$ is the zero of the given function.

<u>Order of zero</u>:- For a function $f(z)$ & at $z=a$
if $f(a)=0$ & $f'(a)\neq 0$.

$\Rightarrow$ $z=a$ is called zero of order "1"

if $f(a)=0$; $f'(a)=0$ & $f''(a)\neq 0$.

$\Rightarrow$ $z=a$ is zero of order "2"

Similarly if

$$f(a)=f'(a)=f''(a)=\cdots\cdots=f^{n-1}(a)=0 \text{ \& } f^{n}(a)\neq 0$$

$\Rightarrow$ $z=a$ is zero of order "n"

Singularity: If $z=a$ is a singular Point of a function, $f(z)$ so we may say that $f(z)$ has the singularity at $z=a$

Types of Singularities :- There are two types of Singularities

1- Pole

2 - Essential Singularities

Pole :- let $z=a$ be the Singular Point of function $f(z)$ & if $\lim_{z\to a} f(z)(z-a)^m =$ constant, when $(z-a)^m$ is in the denominator of the function. Thus we may say that $z=a$ is Pole of order m. If $m=1$, So ~~we~~ $z=a$ is called Simple Pole or Pole of order "1"

Example :- $f(z) = \dfrac{3z+2}{(z-3)^2}$ — ①, Find the Pole or order of the Pole

Sol:- For Singular Point

$$(Z-3)^2 = 0$$

$\Rightarrow$ $Z = 3$ is a Pole of function ①

Now for Pole

$$\lim_{z \to 3} \frac{3z+2}{\cancel{(z-3)^2}} * \cancel{(z-3)^2}$$

$\Rightarrow$ $3(3)+2 = 11$ (Constant)

$\therefore$ $Z=3$ is Pole of order "2"

Essential Singularity:- If singular Point is not a Pole So, such singular Point is called essential singularity.

i.e $\lim_{z \to a} f(z) \cdot (z-a)^m \neq$ constant $= \infty$.

Example:- $f(z) = e^{\frac{1}{z-5}}$, for singular Point Put $e^{\frac{1}{z-5}} = \infty$

$Z-5=0 \Rightarrow Z=5 \therefore Z=5$ is singular Point of function

For E.S $\lim_{z \to 5} \frac{e^{\frac{1}{z-5}}}{1} * 1 = e^{\frac{1}{5-5}} = e^{\infty} = \infty$

$\Rightarrow$ $Z=5$ is essential Singularity.

A **meromorphic function** is an analytic function whose only singularities in the finite plane are poles.

EXAMPLE 6 Meromorphic functions

Rational functions with nonconstant denominator, $\tan z$, $\cot z$, $\sec z$, and $\csc z$ are meromorphic functions.

This is the end of Chap. 14 on power series, particularly Taylor series (which play an even greater role here than in calculus), and on Laurent series. Interestingly enough, the latter will provide us with another powerful integration method in the next chapter.

Problem Set 14.8

Singularities. Determine the location and type of the singularities of the following functions, including those at infinity. (In the case of poles also state the order.)

1. $\cot z$
2. $1/(z + a)^4$
3. $z + 1/z$
4. $\dfrac{3}{z} - \dfrac{1}{z^2} - \dfrac{2}{z^3}$
5. $\dfrac{\cos 4z}{(z^4 - 1)^3}$
6. $\dfrac{\sin^2 z}{z^4 \cos 2z}$
7. $e^{\pi z}/(z^2 - iz + 2)^2$
8. $e^{1/(z+i)} + z^2$
9. $(e^z - 1 - z)/z^3$
10. $\cosh [1/(z^2 + 1)]$
11. $\tan 1/z$
12. $(\cos z - \sin z)^{-1}$
13. $\cos z - \sin z$
14. $1/\sinh \frac{1}{2}z$
15. $e^{1/(z-1)}/(e^z - 1)$
16. Verify Theorem 1 for $f(z) = z^{-3} - z^{-1}$. Prove Theorem 1.

Zeros. Determine the location and order of the zeros of the following functions.

17. $(z^4 - 16)^2$
18. $(z - 16)^8$
19. $z \sin^2 \pi z$
20. $e^z - e^{2z}$
21. $z^{-2} \cos^3 \pi z$
22. $\cosh^2 z$
23. $(3z^2 - 1)/(z^2 - 2iz + 3)^2$
24. $(z^2 - 1)^2(e^{z^2} - 1)$
25. $(1 - \cos z)^2$
26. If $f(z)$ has a zero of order n at $z = z_0$, show that $f^2(z)$ has a zero of order $2n$, and the derivative $f'(z)$ has a zero of order $n - 1$ at $z = z_0$ (provided $n > 1$).
27. Prove Theorem 4.
28. If $f_1(z)$ and $f_2(z)$ are analytic in a domain D and equal at a sequence of points z_n in D that converges in D, show that $f_1(z) \equiv f_2(z)$ in D.
29. Show that the points at which a nonconstant analytic function $f(z)$ assumes a given value k are isolated.

Riemann number sphere. Assuming that we let the image of the x-axis be the meridians 0° and 180°, describe and sketch the images of the following regions on the Riemann number sphere.

30. $|z| \leqq 1$
31. First quadrant
32. Second quadrant
33. $|z| > 100$
34. Lower half-plane
35. $\frac{1}{2} \leqq |z| \leqq 2$

Exercise 14.8

Determine the location & type of the singularity of the following function including those at infinity.

Q1 $f(z) = \cot z$ — (1)

For singularity

$f(z) = \infty$

$\cot z = \infty \Rightarrow \dfrac{\cos z}{\sin z} = \infty \Rightarrow \sin z = 0$

$\Rightarrow z = n\pi$ where $n = 0, \pm 1, \pm 2, \text{------}$

$\therefore$ These are the singular points of the given function.

for $n = 0$

$z = 0$

Now $\lim\limits_{z \to 0} \dfrac{\cos z}{\cancel{\sin z}} * \cancel{\sin z} = \cos(0) = 1 = \text{constant}$

$\therefore z = 0$ is pole of order "1"

Now for $n = 1$

$z = \pi$

$\lim\limits_{z \to 0} \dfrac{\cos z}{\cancel{\sin z}} * \cancel{\sin z} = \cos \pi = -1 = \text{constant}$

$Z=1$ is Pole of order "1"

Similarly,

For $n=-1, \pm 2, \ldots\ldots$

$Z=n\pi$ are Poles of order "1"

Now, for $n=\infty$

$$\lim_{z\to\infty} \frac{\cos z}{\cancel{\sin z}} * \cancel{\sin z} = \cos\infty = \infty$$

$\therefore Z=\infty$ is essential singularity.

Q_2 $f(z) = 1/(z+a)^4$ —①

for Singularity

$(z+a)^4 = 0 \Rightarrow z=-a$

$Z=-a$ is the singular point of function ①

$$\lim_{z\to -a} \frac{1}{\cancel{(z+a)^4}} * (\cancel{z+a})^4 = 1 = \text{constant}.$$

$\therefore Z=-a$ is a Pole of order "4"

Similarly at $z=\infty$

$$\lim_{z\to\infty} \frac{1}{\cancel{(z+a)^4}} * \cancel{(z+a)^4} = 1 = \text{constant.}$$

$\Rightarrow z = \infty$ is also a Pole of order 4.

o o

Q4 $f(z) = \frac{3}{z} - \frac{1}{z^2} - \frac{2}{z^3}$

$$f(z) = \frac{3z^2 - z - 2}{z^3} \quad \text{—— (1)}$$

for Singular Point

$z^3 = 0 \Rightarrow z = 0$ is a Singular Point of $f(z)$.

Now

$$\lim_{z\to 0} \frac{3z^2 - z - 2}{\cancel{z^3}} * \cancel{z^3} = 0 - 0 - 2 = -2 = \text{constant}$$

$\therefore z = 0$ is a Pole of order "3"

At $z = \infty$

$$\lim_{z\to\infty} \frac{3z^2 - z - 2}{\cancel{z^3}} * \cancel{z^3} = \infty$$

$\therefore z = \infty$ is essential singularity of "1"

Q5 $f(z) = \dfrac{\cos 4z}{(z^4-1)^3}$ —— (1)

Sol:- for Singular Point

$(z^4-1)^3 = 0 \Rightarrow z^4-1=0 \Rightarrow z^4 = 1$

$z = \pm 1$

At $z = 1$

$\lim_{z\to 1} \dfrac{\cos 4z}{\cancel{(z^4-1)^3}} * \cancel{(z^4-1)^3} = \cos 4(1) = \text{Constant}$

$z=1$ is a Pole of order "3"

Similarly $z=-1$ is also the Pole of order "3"

At $z=\infty$.

$\lim_{z\to\infty} \dfrac{\cos 4z}{\cancel{(z^4-1)^3}} * \cancel{(z^4-1)^3} = \cos\infty = \infty$

$z=\infty$ is essential singularity of $f(z)$.

Q6 $\dfrac{\sin^2 z}{z^4 \cos 2z}$ Q7 $(e^z-1-z)/z^3$, Q8 $\tan 1/z$

Q9 $\cos z - \sin z$