1.7 THE INVERSE OF A MATRIX

In this section we restrict our attention to square matrices and formulate the notion corresponding to the reciprocal of a nonzero number.

DEFINITION An $n \times n$ matrix A is called **nonsingular** (or **invertible**) if there exists an $n \times n$ matrix B such that

$$AB = BA = I_n.$$

The matrix B is called an **inverse** of A. If there exists no such matrix B, then A is called **singular** (or **noninvertible**).

Remark From the preceding definition, it follows that if $AB = BA = I_n$, then A is also an inverse of B.

EXAMPLE 1 Let

$$A = \begin{bmatrix} 2 & 3 \\ 2 & 2 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} -1 & \frac{3}{2} \\ 1 & -1 \end{bmatrix}.$$

Since

$$AB = BA = I_2,$$

we conclude that B is an inverse of A and that A is nonsingular. ■

THEOREM 1.9 *An inverse of a matrix, if it exists, is unique.*

Proof Let B and C be inverses of A. Then $BA = AC = I_n$. Therefore,

$$B = BI_n = B(AC) = (BA)C = I_nC = C,$$

which completes the proof. ■

We shall now write the inverse of A, if it exists, as A^{-1}. Thus

$$AA^{-1} = A^{-1}A = I_n.$$

EXAMPLE 2 Let

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}.$$

To find A^{-1}, we let

$$A^{-1} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}.$$

Then we must have

$$AA^{-1} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

so that

$$\begin{bmatrix} a+2c & b+2d \\ 3a+4c & 3b+4d \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

Equating corresponding entries of these two matrices, we obtain the linear systems

$$\begin{aligned} a + 2c &= 1 \\ 3a + 4c &= 0 \end{aligned} \quad \text{and} \quad \begin{aligned} b + 2d &= 0 \\ 3b + 4d &= 1. \end{aligned}$$

The solutions are (verify) $a = -2$, $c = \frac{3}{2}$, $b = 1$, and $d = -\frac{1}{2}$. Moreover, since the matrix

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} -2 & 1 \\ \frac{3}{2} & -\frac{1}{2} \end{bmatrix}$$

also satisfies the property that

$$\begin{bmatrix} -2 & 1 \\ \frac{3}{2} & -\frac{1}{2} \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},$$

we conclude that A is nonsingular and that

$$A^{-1} = \begin{bmatrix} -2 & 1 \\ \frac{3}{2} & -\frac{1}{2} \end{bmatrix}.$$

Remark Not every matrix has an inverse. For instance, consider the following example.

EXAMPLE 3 Let

$$A = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}.$$

To find A^{-1}, we let

$$A^{-1} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}.$$

Then we must have

$$AA^{-1} = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

so that

$$\begin{bmatrix} a+2c & b+2d \\ 2a+4c & 2b+4d \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

Equating corresponding entries of these two matrices, we obtain the linear systems

$$\begin{aligned} a + 2c &= 1 \\ 2a + 4c &= 0 \end{aligned} \qquad \text{and} \qquad \begin{aligned} b + 2d &= 0 \\ 2b + 4d &= 1. \end{aligned}$$

These linear systems have no solutions, so A has no inverse. Hence A is a singular matrix.

The method used in Example 2 to find the inverse of a matrix is not a very efficient one. We shall soon modify it and thereby obtain a much faster method. We first establish several properties of nonsingular matrices.

THEOREM 1.10 **(Properties of the Inverse)**

(a) *If A is a nonsingular matrix, then A^{-1} is nonsingular and*

$$(A^{-1})^{-1} = A.$$

(b) *If A and B are nonsingular matrices, then AB is nonsingular and*

$$(AB)^{-1} = B^{-1}A^{-1}.$$

(c) *If A is a nonsingular matrix, then*

$$(A^T)^{-1} = (A^{-1})^T.$$

Proof (a) A^{-1} is nonsingular if we can find a matrix B such that

$$A^{-1}B = BA^{-1} = I_n.$$

Since A is nonsingular,

$$A^{-1}A = AA^{-1} = I_n.$$

Thus $B = A$ is an inverse of A^{-1}, and since inverses are unique, we conclude that

$$(A^{-1})^{-1} = A.$$

Thus, the inverse of the inverse of the nonsingular matrix A is A.

(b) We have

$$(AB)(B^{-1}A^{-1}) = A(BB^{-1})A^{-1} = AI_nA^{-1} = AA^{-1} = I_n$$

and

$$(B^{-1}A^{-1})(AB) = B^{-1}(A^{-1}A)B = B^{-1}I_nB = B^{-1}B = I_n.$$

Therefore, AB is nonsingular. Since the inverse of a matrix is unique, we conclude that

$$(AB)^{-1} = B^{-1}A^{-1}.$$

Thus, the inverse of a product of two nonsingular matrices is the product of their inverses in reverse order.

(c) We have

$$AA^{-1} = I_n \quad \text{and} \quad A^{-1}A = I_n.$$

Taking transposes, we obtain

$$(AA^{-1})^T = I_n^T = I_n \quad \text{and} \quad (A^{-1}A)^T = I_n^T = I_n.$$

Then

$$(A^{-1})^T A^T = I_n \quad \text{and} \quad A^T(A^{-1})^T = I_n.$$

These equations imply that

$$(A^T)^{-1} = (A^{-1})^T.$$

Thus, the inverse of the transpose of a nonsingular matrix is the transpose of its inverse. ■

EXAMPLE 4 If $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$, then from Example 2

$$A^{-1} = \begin{bmatrix} -2 & 1 \\ \frac{3}{2} & -\frac{1}{2} \end{bmatrix} \quad \text{and} \quad (A^{-1})^T = \begin{bmatrix} -2 & \frac{3}{2} \\ 1 & -\frac{1}{2} \end{bmatrix}.$$

Also (verify)

$$A^T = \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix} \quad \text{and} \quad (A^T)^{-1} = \begin{bmatrix} -2 & \frac{3}{2} \\ 1 & -\frac{1}{2} \end{bmatrix}.$$

■

COROLLARY 1.2 *If $A_1, A_2, \ldots, A_r$ are $n \times n$ nonsingular matrices, then $A_1 A_2 \cdots A_r$ is nonsingular and*

$$(A_1 A_2 \cdots A_r)^{-1} = A_r^{-1} A_{r-1}^{-1} \cdots A_1^{-1}.$$

Proof Exercise T.2. ∎

Earlier, we defined a matrix B to be the inverse of A if $AB = BA = I_n$. The following theorem, whose proof we omit, shows that one of these equations follows from the other.

THEOREM 1.11 *Suppose that A and B are $n \times n$ matrices.*

(a) *If $AB = I_n$, then $BA = I_n$.*

(b) *If $BA = I_n$, then $AB = I_n$.* ∎

A PRACTICAL METHOD FOR FINDING A^{-1}

We shall now develop a practical method for finding A^{-1}. If A is a given $n \times n$ matrix, we are looking for an $n \times n$ matrix $B = [b_{ij}]$ such that

$$AB = BA = I_n.$$

Let the columns of B be denoted by the $n \times 1$ matrices $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n$, where

$$\mathbf{x}_j = \begin{bmatrix} b_{1j} \\ b_{2j} \\ \vdots \\ b_{ij} \\ \vdots \\ b_{nj} \end{bmatrix} \qquad (1 \le j \le n).$$

Let the columns of I_n be denoted by the $n \times 1$ matrices $\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n$. Thus

$$\mathbf{e}_j = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \leftarrow j\text{th row.}$$

By Exercise T.9(a) of Section 1.3, the jth column of AB is the $n \times 1$ matrix $A\mathbf{x}_j$. Since equal matrices must agree column by column, it follows that the problem of finding an $n \times n$ matrix $B = A^{-1}$ such that

$$AB = I_n \tag{1}$$

is equivalent to the problem of finding n matrices (each $n \times 1$) $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n$ such that

$$A\mathbf{x}_j = \mathbf{e}_j \qquad (1 \le j \le n). \tag{2}$$

Thus finding B is equivalent to solving n linear systems (each is n equations in n unknowns). This is precisely what we did in Example 2. Each of these systems can be solved by the Gauss–Jordan reduction method. To solve the first linear system, we form the augmented matrix $[A \mid \mathbf{e}_1]$ and compute its reduced row echelon form. We do the same with

$$[A \mid \mathbf{e}_2], \ldots, [A \mid \mathbf{e}_n].$$

However, if we observe that the coefficient matrix of each of these n linear systems is always A, we can solve all these systems simultaneously. We form the $n \times 2n$ matrix

$$[A \mid \mathbf{e}_1 \quad \mathbf{e}_2 \quad \cdots \quad \mathbf{e}_n] = [A \mid I_n]$$

and compute its reduced row echelon form $[C \mid D]$. The $n \times n$ matrix C is the reduced row echelon form of A. Let $\mathbf{d}_1, \mathbf{d}_2, \ldots, \mathbf{d}_n$ be the n columns of D. Then the matrix $[C \mid D]$ gives rise to the n linear systems

$$C\mathbf{x}_j = \mathbf{d}_j \qquad (1 \leq j \leq n) \tag{3}$$

or to the matrix equation

$$CB = D. \tag{4}$$

There are now two possible cases.

Case 1. $C = I_n$. Then Equation (3) becomes

$$I_n\mathbf{x}_j = \mathbf{x}_j = \mathbf{d}_j,$$

and $B = D$, so we have obtained A^{-1}.

Case 2. $C \neq I_n$. It then follows from Exercise T.9 in Section 1.6 that C has a row consisting entirely of zeros. From Exercise T.3 in Section 1.3, we observe that the product CB in Equation (4) has a row of zeros. The matrix D in (4) arose from I_n by a sequence of elementary row operations, and it is intuitively clear that D cannot have a row of zeros. The statement that D cannot have a row of zeros can be rigorously established at this point, but we shall ask the reader to accept the argument now. In Section 3.2, an argument using determinants will show the validity of the result. Thus one of the equations $C\mathbf{x}_j = \mathbf{d}_j$ has no solution, so $A\mathbf{x}_j = \mathbf{e}_j$ has no solution and A is singular in this case.

The practical procedure for computing the inverse of matrix A is as follows.

Step 1. Form the $n \times 2n$ matrix $[A \mid I_n]$ obtained by adjoining the identity matrix I_n to the given matrix A.

Step 2. Compute the reduced row echelon form of the matrix obtained in Step 1 by using elementary row operations. Remember that whatever we do to a row of A we also do to the corresponding row of I_n.

Step 3. Suppose that Step 2 has produced the matrix $[C \mid D]$ in reduced row echelon form.

(a) If $C = I_n$, then $D = A^{-1}$.

(b) If $C \neq I_n$, then C has a row of zeros. In this case A is singular and A^{-1} does not exist.

EXAMPLE 5 Find the inverse of the matrix

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 3 \\ 5 & 5 & 1 \end{bmatrix}.$$

Solution *Step 1.* The 3×6 matrix $\left[A \;\vdots\; I_3\right]$ is

$$\left[A \;\vdots\; I_3\right] = \begin{array}{c} \qquad A \qquad\qquad\qquad I_3 \\ \left[\begin{array}{ccc:ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 2 & 3 & 0 & 1 & 0 \\ 5 & 5 & 1 & 0 & 0 & 1 \end{array}\right] \end{array}.$$

Step 2. We now compute the reduced row echelon form of the matrix obtained in Step 1:

$$\begin{array}{c} \qquad A \qquad\qquad\qquad I_3 \\ \left[\begin{array}{ccc:ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 2 & 3 & 0 & 1 & 0 \\ 5 & 5 & 1 & 0 & 0 & 1 \end{array}\right] \end{array}$$

$$\left[\begin{array}{ccc:ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 2 & 3 & 0 & 1 & 0 \\ 0 & 0 & -4 & -5 & 0 & 1 \end{array}\right]$$

(-5) times the first row was added to the third row.

$$\left[\begin{array}{ccc:ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & \frac{3}{2} & 0 & \frac{1}{2} & 0 \\ 0 & 0 & -4 & -5 & 0 & 1 \end{array}\right]$$

The second row was multiplied by $\frac{1}{2}$.

$$\left[\begin{array}{ccc:ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & \frac{3}{2} & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 1 & \frac{5}{4} & 0 & -\frac{1}{4} \end{array}\right]$$

The third row was multiplied by $\left(-\frac{1}{4}\right)$.

$$\left[\begin{array}{ccc:ccc} 1 & 1 & 0 & -\frac{1}{4} & 0 & \frac{1}{4} \\ 0 & 1 & 0 & -\frac{15}{8} & \frac{1}{2} & \frac{3}{8} \\ 0 & 0 & 1 & \frac{5}{4} & 0 & -\frac{1}{4} \end{array}\right]$$

$\left(-\frac{3}{2}\right)$ times the third row was added to the second row.
(-1) times the third row was added to the first row.

$$\left[\begin{array}{ccc:ccc} 1 & 0 & 0 & \frac{13}{8} & -\frac{1}{2} & -\frac{1}{8} \\ 0 & 1 & 0 & -\frac{15}{8} & \frac{1}{2} & \frac{3}{8} \\ 0 & 0 & 1 & \frac{5}{4} & 0 & -\frac{1}{4} \end{array}\right]$$

(-1) times the second row was added to the first row.

Step 3. Since $C = I_3$, we conclude that $D = A^{-1}$. Hence

$$A^{-1} = \begin{bmatrix} \frac{13}{8} & -\frac{1}{2} & -\frac{1}{8} \\ -\frac{15}{8} & \frac{1}{2} & \frac{3}{8} \\ \frac{5}{4} & 0 & -\frac{1}{4} \end{bmatrix}.$$

It is easy to verify that $AA^{-1} = A^{-1}A = I_3$. ■

If the reduced row echelon matrix under A has a row of zeros, then A is singular. Since each matrix under A is row equivalent to A, once a matrix under A has a row of zeros, every subsequent matrix that is row equivalent to A will have a row of zeros. Thus we can stop the procedure as soon as we find a matrix F that is row equivalent to A and has a row of zeros. In this case A^{-1} does not exist.

EXAMPLE 6 Find the inverse of the matrix

$$A = \begin{bmatrix} 1 & 2 & -3 \\ 1 & -2 & 1 \\ 5 & -2 & -3 \end{bmatrix}$$ if it exists.

Solution *Step 1.* The 3×6 matrix $\left[A \;\vdots\; I_3\right]$ is

$$\begin{array}{cc} & \begin{array}{ccccccc} & A & & & & I_3 & \end{array} \\ \left[A \;\vdots\; I_3\right] = & \left[\begin{array}{rrr|rrr} 1 & 2 & -3 & 1 & 0 & 0 \\ 1 & -2 & 1 & 0 & 1 & 0 \\ 5 & -2 & -3 & 0 & 0 & 1 \end{array}\right]. \end{array}$$

Step 2. We compute the reduced row echelon form of the matrix obtained in Step 1. To find A^{-1}, we proceed as follows:

$$\begin{array}{cc} \begin{array}{ccccccc} & A & & & & I_3 & \end{array} & \\ \left[\begin{array}{rrr|rrr} 1 & 2 & -3 & 1 & 0 & 0 \\ 1 & -2 & 1 & 0 & 1 & 0 \\ 5 & -2 & -3 & 0 & 0 & 1 \end{array}\right] & \\ \left[\begin{array}{rrr|rrr} 1 & 2 & -3 & 1 & 0 & 0 \\ 0 & -4 & 4 & -1 & 1 & 0 \\ 5 & -2 & -3 & 0 & 0 & 1 \end{array}\right] & \text{(−1) times the first row was added to the second row.} \\ \left[\begin{array}{rrr|rrr} 1 & 2 & -3 & 1 & 0 & 0 \\ 0 & -4 & 4 & -1 & 1 & 0 \\ 0 & -12 & 12 & -5 & 0 & 1 \end{array}\right] & \text{(−5) times the first row was added to the third row.} \\ \left[\begin{array}{rrr|rrr} 1 & 2 & -3 & 1 & 0 & 0 \\ 0 & -4 & 4 & -1 & 1 & 0 \\ 0 & 0 & 0 & -2 & -3 & 1 \end{array}\right] & \text{(−3) times the second row was added to the third row.} \end{array}$$

At this point A is row equivalent to

$$F = \begin{bmatrix} 1 & 2 & -3 \\ 0 & -4 & 4 \\ 0 & 0 & 0 \end{bmatrix}.$$

Since F has a row of zeros, we stop and conclude that A is a singular matrix. ∎

Observe that to find A^{-1} we do not have to determine, in advance, whether or not it exists. We merely start the procedure given previously and either obtain A^{-1} or find out that A is singular.

The foregoing discussion for the practical method of obtaining A^{-1} has actually established the following theorem.

THEOREM 1.12 *An $n \times n$ matrix is nonsingular if and only if it is row equivalent to I_n.* ∎

LINEAR SYSTEMS AND INVERSES

If A is an $n \times n$ matrix, then the linear system $A\mathbf{x} = \mathbf{b}$ is a system of n equations in n unknowns. Suppose that A is nonsingular. Then A^{-1} exists and we can multiply $A\mathbf{x} = \mathbf{b}$ by A^{-1} on both sides, obtaining

$$\begin{aligned} A^{-1}(A\mathbf{x}) &= A^{-1}\mathbf{b} \\ (A^{-1}A)\mathbf{x} &= A^{-1}\mathbf{b} \\ I_n\mathbf{x} &= A^{-1}\mathbf{b} \\ \mathbf{x} &= A^{-1}\mathbf{b}. \end{aligned}$$

Moreover, $\mathbf{x} = A^{-1}\mathbf{b}$ is clearly a solution to the given linear system. Thus, if A is nonsingular, we have a unique solution.

Applications This method is useful in industrial problems. Many physical models are described by linear systems. This means that if n values are used as inputs (which can be arranged as the $n \times 1$ matrix $\mathbf{x}$), then m values are obtained as outputs (which can be arranged as the $m \times 1$ matrix $\mathbf{b}$) by the rule $A\mathbf{x} = \mathbf{b}$. The matrix A is inherently tied to the process. Thus suppose that a chemical process has a certain matrix A associated with it. Any change in the process may result in a new matrix. In fact, we speak of a **black box,** meaning that the internal structure of the process does not interest us. The problem frequently encountered in systems analysis is that of determining the input to be used to obtain a desired output. That is, we want to solve the linear system $A\mathbf{x} = \mathbf{b}$ for $\mathbf{x}$ as we vary $\mathbf{b}$. If A is a nonsingular square matrix, an efficient way of handling this is as follows: Compute A^{-1} once; then whenever we change $\mathbf{b}$, we find the corresponding solution $\mathbf{x}$ by forming $A^{-1}\mathbf{b}$.

EXAMPLE 7 **(Industrial Process)** Consider an industrial process whose matrix is the matrix A of Example 5. If $\mathbf{b}$ is the output matrix

$$\begin{bmatrix} 8 \\ 24 \\ 8 \end{bmatrix},$$

then the input matrix $\mathbf{x}$ is the solution to the linear system $A\mathbf{x} = \mathbf{b}$. Then

$$\mathbf{x} = A^{-1}\mathbf{b} = \begin{bmatrix} \frac{13}{8} & -\frac{1}{2} & -\frac{1}{8} \\ -\frac{15}{8} & \frac{1}{2} & \frac{3}{8} \\ \frac{5}{4} & 0 & -\frac{1}{4} \end{bmatrix} \begin{bmatrix} 8 \\ 24 \\ 8 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 8 \end{bmatrix}.$$

On the other hand, if $\mathbf{b}$ is the output matrix

$$\begin{bmatrix} 4 \\ 7 \\ 16 \end{bmatrix},$$

then (verify)

$$\mathbf{x} = A^{-1}\begin{bmatrix} 4 \\ 7 \\ 16 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}.$$

THEOREM 1.13 *If A is an $n \times n$ matrix, the homogeneous system*

$$A\mathbf{x} = \mathbf{0} \tag{5}$$

has a nontrivial solution if and only if A is singular.

Proof Suppose that A is nonsingular. Then A^{-1} exists, and multiplying both sides of (5) by A^{-1}, we have

$$\begin{aligned} A^{-1}(A\mathbf{x}) &= A^{-1}\mathbf{0} \\ (A^{-1}A)\mathbf{x} &= \mathbf{0} \\ I_n\mathbf{x} &= \mathbf{0} \\ \mathbf{x} &= \mathbf{0}. \end{aligned}$$

Hence the only solution to (5) is $\mathbf{x} = \mathbf{0}$.

We leave the proof of the converse—if A is singular, then (5) has a nontrivial solution—as an exercise (Exercise T.3). ■

EXAMPLE 8 Consider the homogeneous system $A\mathbf{x} = \mathbf{0}$, where A is the matrix of Example 5. Since A is nonsingular,

$$\mathbf{x} = A^{-1}\mathbf{0} = \mathbf{0}.$$

We could also solve the given system by Gauss–Jordan reduction. In this case we find that the matrix in reduced row echelon form that is row equivalent to the augmented matrix of the given system,

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 2 & 3 & 0 \\ 5 & 5 & 1 & 0 \end{array}\right],$$

is

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right],$$

which again shows that the solution is

$$\mathbf{x} = \mathbf{0}.$$ ■

EXAMPLE 9 Consider the homogeneous system $A\mathbf{x} = \mathbf{0}$, where A is the singular matrix of Example 6. In this case the matrix in reduced row echelon form that is row equivalent to the augmented matrix of the given system,

$$\left[\begin{array}{ccc|c} 1 & 2 & -3 & 0 \\ 1 & -2 & 1 & 0 \\ 5 & -2 & -3 & 0 \end{array}\right],$$

is (verify)

$$\left[\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right],$$

which implies that

$$\begin{aligned} x &= r \\ y &= r \\ z &= r, \end{aligned}$$

where r is any real number. Thus the given system has a nontrivial solution. ■

The proof of the following theorem is left as an exercise (Supplementary Exercise T.18).

THEOREM 1.14 *If A is an $n \times n$ matrix, then A is nonsingular if and only if the linear system $A\mathbf{x} = \mathbf{b}$ has a unique solution for every $n \times 1$ matrix $\mathbf{b}$.* ∎

We may summarize our results on homogeneous systems and nonsingular matrices in the following list of nonsingular equivalences.

List of Nonsingular Equivalences

The following statements are equivalent.

1. A is nonsingular.
2. $\mathbf{x} = \mathbf{0}$ is the only solution to $A\mathbf{x} = \mathbf{0}$.
3. A is row equivalent to I_n.
4. The linear system $A\mathbf{x} = \mathbf{b}$ has a unique solution for every $n \times 1$ matrix $\mathbf{b}$.

This means that in solving a given problem we can use any of the preceding four statements; they are interchangeable. As you will see throughout this course, a given problem can often be solved in several alternative ways, and sometimes one solution procedure is easier to apply than another. This list of nonsingular equivalences will grow as we progress through the book. By the end of Appendix B it will have expanded to 12 equivalent statements.

INVERSE OF BIT MATRICES (OPTIONAL)

The definitions and theorems developed in this section are all valid for bit matrices. Examples 10 and 11 illustrate the computational procedures developed in this section for bit matrices where, of course, we use arithmetic base 2.

EXAMPLE 10 Find the inverse of the bit matrix

$$A = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix}.$$

Solution *Step 1.* The 3×6 matrix $\left[A \mid I_3\right]$ is

$$\left[A \mid I_3\right] = \left[\begin{array}{ccc|ccc} 0 & 1 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 & 0 & 1 \end{array}\right].$$

Step 2. We now compute the reduced row echelon form of the matrix obtained in Step 1. To find A^{-1}, we proceed as follows:

$$\begin{array}{c} \begin{array}{cc} \quad A \quad & \quad I_3 \quad \end{array} \\ \left[\begin{array}{ccc|ccc} 0 & 1 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 & 0 & 1 \end{array}\right] \end{array}$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 1 \end{array}\right] \quad \text{The first and second rows were interchanged.}$$

Key Terms

Inverse
Nonsingular (or invertible) matrix
Singular (or noninvertible) matrix

1.7 Exercises

In Exercises 1 through 4, use the method of Examples 2 and 3.

1. Show that $\begin{bmatrix} 2 & 1 \\ -2 & 3 \end{bmatrix}$ is nonsingular.

2. Show that $\begin{bmatrix} 2 & 1 \\ -4 & -2 \end{bmatrix}$ is singular.

H.W **3.** Is the matrix

$$\begin{bmatrix} 1 & 1 \\ 3 & 4 \end{bmatrix}$$

singular or nonsingular? If it is nonsingular, find its inverse.

4. Is the matrix

$$\begin{bmatrix} 1 & 2 & -1 \\ 3 & 2 & 3 \\ 2 & 2 & 1 \end{bmatrix}$$

singular or nonsingular? If it is nonsingular, find its inverse.

In Exercises 5 through 10, find the inverses of the given matrices, if possible.

5. (a) $\begin{bmatrix} 1 & 3 \\ -2 & 6 \end{bmatrix}$ P.7 (b) $\begin{bmatrix} 1 & 2 & 3 \\ 1 & 1 & 2 \\ 0 & 1 & 2 \end{bmatrix}$

(c) $\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & -1 & 2 \\ 1 & -1 & 2 & 1 \\ 1 & 3 & 3 & 2 \end{bmatrix}$

H.W **6.** (a) $\begin{bmatrix} 1 & 3 \\ 2 & 6 \end{bmatrix}$ (b) $\begin{bmatrix} 1 & 2 & 3 \\ 0 & 2 & 3 \\ 1 & 2 & 4 \end{bmatrix}$

(c) $\begin{bmatrix} 1 & 1 & 2 & 1 \\ 0 & -2 & 0 & 0 \\ 0 & 3 & 2 & 1 \\ 1 & 2 & 1 & -2 \end{bmatrix}$

H.W **7.** (a) $\begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix}$ (b) $\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 3 & 1 & 2 \\ 1 & 2 & -1 & 1 \\ 5 & 9 & 1 & 6 \end{bmatrix}$

(c) $\begin{bmatrix} 1 & 2 & 1 \\ 1 & 3 & 2 \\ 1 & 0 & 1 \end{bmatrix}$

8. (a) $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 0 & 1 & 1 \end{bmatrix}$ (b) $\begin{bmatrix} 1 & 2 & 2 \\ 1 & 3 & 1 \\ 1 & 3 & 2 \end{bmatrix}$

(c) $\begin{bmatrix} 1 & 2 & 3 \\ 1 & 1 & 2 \\ 0 & 1 & 1 \end{bmatrix}$

H.W **9.** (a) $\begin{bmatrix} 1 & 2 & -3 & 1 \\ -1 & 3 & -3 & -2 \\ 2 & 0 & 1 & 5 \\ 3 & 1 & -2 & 5 \end{bmatrix}$

(b) $\begin{bmatrix} 3 & 1 & 2 \\ 2 & 1 & 2 \\ 1 & 2 & 2 \end{bmatrix}$ (c) $\begin{bmatrix} 1 & 2 & 3 \\ 1 & 1 & 2 \\ 1 & 1 & 0 \end{bmatrix}$

H.W **10.** (a) $\begin{bmatrix} 2 & 1 & 3 \\ 0 & 1 & 2 \\ 1 & 0 & 3 \end{bmatrix}$ (b) $\begin{bmatrix} 1 & -1 & 2 & 3 \\ 4 & 1 & 2 & 0 \\ 2 & -1 & 3 & 1 \\ 4 & 2 & 1 & -5 \end{bmatrix}$

(c) $\begin{bmatrix} 2 & 1 & -2 \\ 3 & 4 & 6 \\ 7 & 6 & 2 \end{bmatrix}$

P-7 B.S **11.** Which of the following linear systems have a nontrivial solution?

H.W (a) $x + 2y + 3z = 0$
$2y + 2z = 0$
$x + 2y + 3z = 0$

(b) $2x + y - z = 0$
$x - 2y - 3z = 0$
$-3x - y + 2z = 0$

H.W **12.** Which of the following linear systems have a nontrivial solution?

(a) $x + y + 2z = 0$
$2x + y + z = 0$
$3x - y + z = 0$

(b) $x - y + z = 0$
$2x + y = 0$
$2x - 2y + 2z = 0$

(c) $2x - y + 5z = 0$
$3x + 2y - 3z = 0$
$x - y + 4z = 0$

P-1 B.S **13.** If $A^{-1} = \begin{bmatrix} 2 & 3 \\ 1 & 4 \end{bmatrix}$, find A.

H.W **14.** If $A^{-1} = \begin{bmatrix} 3 & 4 \\ -1 & -1 \end{bmatrix}$, find A.

15. Show that a matrix that has a row or column consisting entirely of zeros must be singular.

P-1 B.S **16.** Find all values of a for which the inverse of

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 0 \\ 1 & 2 & a \end{bmatrix}$$

exists. What is A^{-1}?

P-2

17. Consider an industrial process whose matrix is

$$A = \begin{bmatrix} 2 & 1 & 3 \\ 3 & 2 & -1 \\ 2 & 1 & 1 \end{bmatrix}.$$

Find the input matrix for each of the following output matrices:

(a) $\begin{bmatrix} 30 \\ 20 \\ 10 \end{bmatrix}$ (b) $\begin{bmatrix} 12 \\ 8 \\ 14 \end{bmatrix}$

H.W

18. Suppose that $A = \begin{bmatrix} 1 & 3 \\ 2 & 7 \end{bmatrix}$.

(a) Find A^{-1}.

(b) Find $(A^T)^{-1}$. How do $(A^T)^{-1}$ and A^{-1} compare?

H.W

19. Is the inverse of a nonsingular symmetric matrix always symmetric? Explain.

P-5 B.S

20. (a) Is $(A + B)^{-1} = A^{-1} + B^{-1}$ for all A and B?

(b) Is $(cA)^{-1} = \frac{1}{c}A^{-1}$, for $c \neq 0$?

21. For what values of λ does the homogeneous system

$$\begin{aligned} (\lambda - 1)x + \quad\quad 2y &= 0 \\ 2x + (\lambda - 1)y &= 0 \end{aligned}$$

have a nontrivial solution?

H.W

22. If A and B are nonsingular, are $A + B$, $A - B$, and $-A$ nonsingular? Explain.

H.W

23. If $D = \begin{bmatrix} 4 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$, find D^{-1}.

P-4

24. If $A^{-1} = \begin{bmatrix} 3 & 2 \\ 1 & 3 \end{bmatrix}$ and $B^{-1} = \begin{bmatrix} 2 & 5 \\ 3 & -2 \end{bmatrix}$, find $(AB)^{-1}$.

P-4

25. Solve $A\mathbf{x} = \mathbf{b}$ for $\mathbf{x}$ if

$$A^{-1} = \begin{bmatrix} 2 & 3 \\ 4 & 1 \end{bmatrix} \quad \text{and} \quad \mathbf{b} = \begin{bmatrix} 5 \\ 3 \end{bmatrix}.$$

26. Let A be a 3×3 matrix. Suppose that $\mathbf{x} = \begin{bmatrix} 1 \\ 2 \\ -3 \end{bmatrix}$ is a solution to the homogeneous system $A\mathbf{x} = \mathbf{0}$. Is A singular or nonsingular? Justify your answer.

In Exercises 27 and 28, find the inverse of the given partitioned matrix A and express A^{-1} as a partitioned matrix.

P-7 B.S

27. $\left[\begin{array}{cc|c} 5 & 2 & 0 \\ 3 & 1 & 0 \\ \hline 0 & 0 & -4 \end{array}\right]$ **28.** $\left[\begin{array}{cc|cc} 1 & 1 & 0 & 0 \\ 2 & 3 & 0 & 0 \\ \hline 0 & 0 & 6 & 7 \\ 0 & 0 & 1 & 1 \end{array}\right]$

In Exercises 29 and 30, find the inverse of the given bit matrices, if possible.

P-6 H.W

29. (a) $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$ H.W (b) $\begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 1 \\ 0 & 1 & 0 \end{bmatrix}$

P-6

(c) $\begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix}$

30. (a) $\begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$ (b) $\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}$

(c) $\begin{bmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 \end{bmatrix}$

P-5 B.S

In Exercises 31 and 32, determine which bit linear systems have a nontrivial solution.

31. (a) $\begin{aligned} x + y + z &= 0 \\ x \quad\; + z &= 0 \\ y \quad\; &= 0 \end{aligned}$ (b) $\begin{aligned} x \quad\quad\quad &= 0 \\ x + y + z &= 0 \\ x \quad\; + z &= 0 \end{aligned}$

32. (a) $\begin{aligned} x + y \quad\; &= 0 \\ x + y + z &= 0 \\ y + z &= 0 \end{aligned}$ (b) $\begin{aligned} y + z &= 0 \\ x \quad\; + z &= 0 \\ x + y \quad\; &= 0 \end{aligned}$

Theoretical Exercises

T.1. Suppose that A and B are square matrices and $AB = O$. If B is nonsingular, find A.

T.2. Prove Corollary 1.2.

T.3. Let A be an $n \times n$ matrix. Show that if A is singular, then the homogeneous system $A\mathbf{x} = \mathbf{0}$ has a nontrivial solution. (*Hint*: Use Theorem 1.12.)

T.4. Show that the matrix

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

is nonsingular if and only if $ad - bc \neq 0$. If this condition holds, show that

$$A^{-1} = \begin{bmatrix} \dfrac{d}{ad - bc} & \dfrac{-b}{ad - bc} \\ \dfrac{-c}{ad - bc} & \dfrac{a}{ad - bc} \end{bmatrix}.$$

T.5. Show that the matrix

$$\begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$$

is nonsingular, and compute its inverse.

EX 1·7

Q1 Show that $\begin{bmatrix} 2 & 1 \\ -2 & 3 \end{bmatrix}$ is nonsigular.

Sol:- Let $A = \begin{bmatrix} 2 & 1 \\ -2 & 3 \end{bmatrix}$ To find A^{-1}, we let $A^{-1} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ — ⊛

Then we must have

$$AA^{-1} = \begin{bmatrix} 2 & 1 \\ -2 & 3 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

So that

$$\begin{bmatrix} 2a+c & 2b+d \\ -2a+3c & -2b+3d \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

$2a+c = 1$ — ① $-2a+3c=0$ — ②

$2b+d = 0$ — ③ $-2b+3d=1$ — ④

① + ②

$4c = 1 \Rightarrow \boxed{c = 1/4}$

eq ① $\Rightarrow 2a = 1 - 1/4 = 3/4 \Rightarrow \boxed{a = 3/8}$

③ + ④

$4d = 1 \Rightarrow \boxed{d = 1/4}$

eq ③ $\Rightarrow 2b = -1/4 \Rightarrow \boxed{b = -1/8}$

eq ⊛ $\Rightarrow A^{-1} = \begin{bmatrix} 3/8 & -1/8 \\ 1/4 & 1/4 \end{bmatrix}$ we conclude A^{-1} is exist

So A is nonsingular.

Q2 ~~let~~ $A = \begin{bmatrix} 2 & 1 \\ -4 & -2 \end{bmatrix}$, let $A^{-1} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$.

Now $AA^{-1} = I_2$

$$\begin{bmatrix} 2 & 1 \\ -4 & -2 \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

$$= \begin{bmatrix} 2a+c & 2b+d \\ -4a-2c & -4b-2d \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

$2a+c=1$ —(1)

$-4a-2c=0$ —(2)

$2b+d=0$ —(3)

$-4b-2d=1$ —(4)

x'ing eq (1) by (2) & adding eq (2)

$$\begin{array}{r} 4a+2c=2 \\ -4a-2c=0 \\ \hline 0=2 \end{array}$$ impossible

A^{-1} doesn't Exists So it is Singular

②

EX 1.7

Q_3 is Similarly to Q_2 & Q_1

Q4 $A = \begin{bmatrix} 1 & 2 & -1 \\ 3 & 2 & 3 \\ 2 & 2 & 1 \end{bmatrix}$ let $A^{-1} = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$

Now $AA^{-1} = I_3$

$$\begin{bmatrix} 1 & 2 & -1 \\ 3 & 2 & 3 \\ 2 & 2 & 1 \end{bmatrix}\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

$$= \begin{bmatrix} a+2d-g & b+2e-h & c+2f-i \\ 3a+2d+3g & 3b+2e+3h & 3c+2f+3i \\ 2a+2d+g & 2b+2e+h & 2c+2f+i \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$a+2d-g=1$ — ① $\quad b+2e-h=0$ — ④ $\quad c-2f-i=0$ — ⑦

$3a+2d+3g=0$ — ② $\quad 3b+2e+3h=1$ — ⑤ $\quad 3c+2f+3i=0$ — ⑧

$2a+2d+g=0$ — ③ $\quad 2b+2e+h=0$ — ⑥ $\quad 2c+2f+i=1$ — ⑨

① − ②

$a+\cancel{2d}-g=1$

$\underline{3a+\cancel{2d}+3g=0}$

$-2a-4g=1$ — ⑩

① − ③

$a+\cancel{2d}-g=1$

$\underline{2a+\cancel{2d}+g=0}$

$-a-2g=1$ — ⑪

xing 2 by eq (11) then sub: from 10

$-\cancel{2a}-\cancel{4g}=2$

$\underline{-\cancel{2a}-\cancel{4g}=1}$

$0=1$ impossible. A^{-1} doesn't exists so it is singular.

Q5 (a)

$A = \begin{bmatrix} 1 & 3 \\ -2 & 6 \end{bmatrix}$. let $A^{-1} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. — (*)

Now $AA^{-1} = \begin{bmatrix} 1 & 3 \\ -2 & 6 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.

$\begin{bmatrix} a-2b & 3a+6b \\ c-2d & 3c+6d \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.

$a - 2b = 1$ — (1) $\quad 3a + 6b = 0$ — (3)

$c - 2d = 0$ — (2) $\quad 3c + 6d = 1$ — (4)

solving the above equation we get

$\boxed{a = 1/2}$ $\boxed{b = -1/4}$ $\boxed{c = 1/6}$ $\boxed{d = 1/12}$

eq (*) $\Rightarrow A^{-1} = \begin{bmatrix} 1/2 & 1/4 \\ 1/6 & 1/12 \end{bmatrix}$.

—— 0 ——

Ex 1.7

(b) $A = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 1 & 2 \\ 0 & 1 & 2 \end{bmatrix}$.

Now $[A \vdots I_3]$ is

$$[A \vdots I_3] = \left[\begin{array}{ccc:ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 1 & 1 & 2 & 0 & 1 & 0 \\ 0 & 1 & 2 & 0 & 0 & 1 \end{array}\right]$$

We now compute the reduced row echelon form of the matrix. To find A^{-1}, we proceed as follows.

$$\begin{array}{c} \quad A \qquad\quad I_3 \\ \left[\begin{array}{ccc:ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 1 & 1 & 2 & 0 & 1 & 0 \\ 0 & 1 & 2 & 0 & 0 & 1 \end{array}\right] \end{array}$$

$$\sim \left[\begin{array}{ccc:ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & -1 & -1 & -1 & 1 & 0 \\ 0 & 1 & 2 & 0 & 0 & 1 \end{array}\right] R_2 - R_1$$

$$\sim \left[\begin{array}{ccc:ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & -1 & 0 \\ 0 & 1 & 2 & 0 & 0 & 1 \end{array}\right] -R_2$$

$$\sim \left[\begin{array}{ccc:ccc} 1 & 0 & 1 & -1 & 2 & 0 \\ 0 & 1 & 1 & 1 & -1 & 0 \\ 0 & 0 & 1 & -1 & 1 & 1 \end{array}\right] \begin{array}{l} R_3 - R_2 \\ R_1 - 2R_2 \end{array}$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 1 & -1 \\ 0 & 1 & 0 & 2 & -2 & -1 \\ 0 & 0 & 1 & -1 & 1 & 1 \end{array}\right] \begin{array}{l} R_1 - R_3 \\ R_2 - R_3 \end{array}$$

So $A^{-1} = \begin{bmatrix} 0 & 1 & -1 \\ 2 & -2 & -1 \\ -1 & 1 & 1 \end{bmatrix}$.

Q6 — to — Q10 is similarly to Q5

Q11 Which of the following linear system have a nontrivial solution?

(b) $2x + y - z = 0$

$x - 2y - 3z = 0$

$-3x - y + 2z = 0$

Method 1 A.M

$$\left[\begin{array}{ccc|c} 2 & 1 & -1 & 0 \\ 1 & -2 & -3 & 0 \\ -3 & -1 & 2 & 0 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|c} 1 & -2 & -3 & 0 \\ 2 & 1 & -1 & 0 \\ -3 & -1 & 2 & 0 \end{array}\right] R_{12}$$

EX 1.7

$$\sim \left[\begin{array}{ccc:c} 1 & -2 & -3 & 0 \\ 0 & 5 & 5 & 0 \\ 0 & -7 & -7 & 0 \end{array}\right] \begin{array}{l} R_2-2R_1 \\ R_3+3R_1 \end{array}$$

$$\sim \left[\begin{array}{ccc:c} 1 & -2 & -3 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & -7 & -7 & 0 \end{array}\right] 1/5 R_2$$

$$\sim \left[\begin{array}{ccc:c} 1 & -2 & -3 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] R_3+7R_2$$

As infinite solution exists so it is non-trivial.

Method 2 $[A \,\vdots\, I_3]$

$$A \vdots I_3 \Rightarrow \begin{array}{c} \quad A \qquad\qquad I_3 \\ \left[\begin{array}{ccc:ccc} 2 & 1 & -1 & 1 & 0 & 0 \\ 1 & -2 & -3 & 0 & 1 & 0 \\ -3 & -1 & 2 & 0 & 0 & 1 \end{array}\right] \end{array}$$

$$\sim \left[\begin{array}{ccc:ccc} 1 & 1/2 & -1/2 & 1/2 & 0 & 0 \\ 1 & -2 & -3 & 0 & 1 & 0 \\ -3 & -1 & 2 & 0 & 0 & 1 \end{array}\right] \frac{1}{2}R_1$$

$$\sim \left[\begin{array}{ccc:ccc} 1 & 0.5 & -0.5 & 0.5 & 0 & 0 \\ 0 & -2.5 & -2.5 & -0.5 & 1 & 0 \\ 0 & 0.5 & 0.5 & 1.5 & 0 & 1 \end{array}\right] \begin{array}{l} R_2-R_1 \\ R_3+3R_1 \end{array}$$

$$\sim \left[\begin{array}{ccc:ccc} 1 & 0.5 & -0.5 & 0.5 & 0 & 0 \\ 1 & 1 & 1 & 0.2 & 0.4 & 0 \\ 0 & 0.5 & 0.5 & 1.5 & 0 & 1 \end{array}\right] \frac{-R_2}{2.5}$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 0.5 & -0.5 & 0.5 & 0 & 0 \\ 1 & 1 & 1 & 0.2 & 0.4 & 0 \\ 0 & 0 & 0 & 1.5 & 0 & 1 \end{array}\right] R_3 - 0.5R_2$$

So no inverse exist hence it's non-trivial.

Q12 is similarly to Q11

Q13 $A^{-1} = \begin{bmatrix} 2 & 3 \\ 1 & 4 \end{bmatrix}$ find A let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ —— (*)

Since $A^{-1}A = I_2$

$$\begin{bmatrix} 2 & 3 \\ 1 & 4 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

$$\begin{bmatrix} 2a+3c & 2b+3d \\ a+4c & b+4d \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

$2a+3c=1$ —— (1) $\quad 2b+3d=0$ —— (3)

$a+4c=0$ —— (2) $\quad b+4d=1$ —— (4)

Solving the above equation we get

$\boxed{a = 4/5}$, $\boxed{b = -3/5}$ $\boxed{c = -1/5}$ $\boxed{d = 2/5}$

eq (*) $\Rightarrow A = \begin{bmatrix} 4/5 & -3/5 \\ -1/5 & 2/5 \end{bmatrix}.$

Q14 is similarly to Q13.

Ex 1.7

Q16 Given $A = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 0 \\ 1 & 2 & a \end{bmatrix}$ $A^{-1} = ?$ let $A^{-1} = \begin{bmatrix} b & c & d \\ e & f & g \\ h & i & j \end{bmatrix}$ —— $*$

Since $AA^{-1} = I_3$

$$\begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 0 \\ 1 & 2 & a \end{bmatrix} \begin{bmatrix} b & c & d \\ e & f & g \\ h & i & j \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

$$= \begin{bmatrix} b+e & c+f & d+g \\ b & c & d \\ b+2e+ha & c+2f+ai & d+2g+ja \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

$b+e=1$ —— (1) $\quad b=0$ —— (4) $\quad b+2e+ha=0$ —— (7)

$c+f=0$ —— (2) $\quad c=1$ —— (5) $\quad c+2f+ai=0$ —— (8)

$d+g=0$ —— (3) $\quad d=0$ —— (6) $\quad d+2g+ja=1$ —— (9)

eq (1) $\Rightarrow 0+e=1 \Rightarrow \boxed{e=1}$

eq (2) $\Rightarrow 1+f=0 \Rightarrow \boxed{f=-1}$

eq (3) $\Rightarrow 0+g=0 \Rightarrow \boxed{g=0}$

eq (7) $\Rightarrow 0+2+ha=0 \Rightarrow \boxed{h=-2/a}$

eq (8) $\Rightarrow 1+2(-1)+ai \Rightarrow \boxed{i=1/a}$

eq (9) $\Rightarrow 0+2(0)+ja=1 \Rightarrow \boxed{j=1/a}$

eq $(*)$ $\Rightarrow A^{-1} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & -1 & 0 \\ -2/a & 1/a & 1/a \end{bmatrix}$ when $a \neq 0$.

Q17 Let x be input matrix i.e $x = A^{-1}b$ given output matrix.

first we find A^{-1} let $A^{-1} = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$ (*) & $A = \begin{bmatrix} 2 & 1 & 3 \\ 3 & 2 & -1 \\ 2 & 1 & 1 \end{bmatrix}$.

Now $AA^{-1} = I_3$

$$\begin{bmatrix} 2 & 1 & 3 \\ 3 & 2 & -1 \\ 2 & 1 & 1 \end{bmatrix}\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

$$\begin{bmatrix} 2a+d+3g & 2b+1e+3h & 2c+f+3i \\ 3a+2d-1g & 3b+2e-h & 3c+2f-i \\ 2a+d+g & 2b+e+h & 2c+1f+i \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$2a+d+3g=1$ — (1) $\quad 2b+e+3h=0$ — (4) $\quad 2c+h+3i=0$ — (7)

$3a+2d-g=0$ — (2) $\quad 3b+2e-h=1$ — (5) $\quad 3c+2f-i=0$ — (8)

$2a+d+g=0$ — (3) $\quad 2b+e+h=0$ — (6) $\quad 2c+f+i=1$ — (9)

(3) + (2) are added

$$\begin{aligned} 3a+2d-g &= 0 \\ 2a+d+g &= 0 \\ \hline 5a+3d &= 0 \quad \text{—— } (2)' \end{aligned}$$

add 3 times (2) to (1)

$$\begin{aligned} 2a+d+3g &= 1 \\ 9a+6d-3g &= 0 \\ \hline 11a+7d &= 1 \quad \text{—— } (3)' \end{aligned}$$

(6)

Ex 1.7

Solving (2)' & (3)'

$5a = -3d \Rightarrow a = -3/5 d$

$11(-3/5 d) + 7d = 1$

$-33d + 35d = 5 \Rightarrow \boxed{d = 5/2}$

$\boxed{a = -3/2}$

Put values in (1)

$2a + d + 3g = 1$

$2(-3/2) + (5/2) + 3g = 1$

$-3 + 5/2 + 3g = 1 \Rightarrow \boxed{g = 1/2}$

(5) + (6)

$3b + 2e - \cancel{h} = 1$

$2b + e + \cancel{h} = 0$

$5b + 3e = 1$ —— (5)'

Adding 3 times (5) to (4)

$2b + e + 3\cancel{h} = 0$

$9b + 6e - \cancel{3h} = 3$

$11b + 7e = 3$ —— (6)'

Solving (5)' & (6)'

$5b = 1 - 3e \Rightarrow b = \frac{1-3e}{5}$

$11\left(\frac{1-3e}{5}\right) + 7e = 3$

$\Rightarrow \boxed{e = 2}$

$b = \frac{1-3(2)}{5} = -5/5 = -1$

$\boxed{b = -1}$

eq (4) $\Rightarrow$ $2b + e + 3h = 0$

$2(-1) + 2 + 3h = 0$

$\Rightarrow \boxed{h = 0}$

(8) + (9)

$3c + 2f - \cancel{i} = 0$

$2c + f + \cancel{i} = 1$

$5c + 3f = 1$ —— (8)'

Add (3) times (8) to (7)

$2c + f + \cancel{3i} = 0$

$9c + 6f - \cancel{3i} = 0$

$11c + 7f = 0$ —— (9)'

Solving (8)' & (9)'

$11c = -7f \Rightarrow c = -7/11\, f$

$5(-7/11)f + 3f = 1 \Rightarrow \boxed{f = -11/2}$

$$C = -\frac{7}{\cancel{11}} \times \frac{-\cancel{11}}{2} = \frac{7}{2}$$

$$\boxed{C = 7/2}$$

$$3c + 2f - i = 0$$

$$3(7/2) + 2(-11/2) - i = 0$$

$$\Rightarrow \boxed{i = -1/2}$$

eq (*) $\Rightarrow$ $A = \begin{bmatrix} -3/2 & -1 & 7/2 \\ 5/2 & 2 & -11/2 \\ 1/2 & 0 & -1/2 \end{bmatrix}$

$X = A^{-1}b$, $b = \begin{bmatrix} 30 \\ 20 \\ 10 \end{bmatrix}$

$$X = \begin{bmatrix} -3/2 & -1 & 7/2 \\ 5/2 & 2 & -11/2 \\ 1/2 & 0 & -1/2 \end{bmatrix} \begin{bmatrix} 30 \\ 20 \\ 10 \end{bmatrix}$$

$$X = \begin{bmatrix} -30 \\ 60 \\ 10 \end{bmatrix} \text{ A}$$

17(b) $x = A^{-1}b$, $b = \begin{bmatrix} 12 \\ 8 \\ 14 \end{bmatrix}$

$$X = \begin{bmatrix} -3/2 & -1 & 7/2 \\ 5/2 & 2 & -11/2 \\ 1/2 & 0 & -1/2 \end{bmatrix} \begin{bmatrix} 12 \\ 8 \\ 14 \end{bmatrix} = \begin{bmatrix} 33 \\ -31 \\ -1 \end{bmatrix} \text{ A}$$

Q18 $A = \begin{bmatrix} 1 & 3 \\ 2 & 7 \end{bmatrix}$

(a) A^{-1}

Now $AA^{-1} = I_2$ let $A^{-1} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. — (*)

$$\Rightarrow \begin{bmatrix} 1 & 3 \\ 2 & 7 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} a+3c & b+3d \\ 2a+7c & 2b+7d \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$a+3c = 1$ — (1) $\quad b+3d = 0$ — (3)

$2a+7c = 0$ — (2) $\quad 2b+7d = 1$ — (4)

solving the above equation we get

$\boxed{a=7}$ $\boxed{b=-3}$ $\boxed{c=-2}$ $\boxed{d=1}$

eq (*) $\Rightarrow A^{-1} = \begin{bmatrix} 7 & -3 \\ -2 & 1 \end{bmatrix}$ — (a)

(b) $(A^T)^{-1} = ?$

$A^T = \begin{bmatrix} 1 & 2 \\ 3 & 7 \end{bmatrix}$. let $(A^T)^{-1} = \begin{bmatrix} x & y \\ z & w \end{bmatrix}$ — (**)

$(A^T)(AT)^{-1} = I_2$

Scanned with CamScanner

$$= \begin{bmatrix} 1 & 2 \\ 3 & 7 \end{bmatrix} \begin{bmatrix} x & y \\ z & w \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

$$= \begin{bmatrix} x+2z & y+2w \\ 3x+7z & 3y+7w \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

$x+2z=1$ — ① $y+2w=0$ — ③

$3x+7z=0$ — ② $3y+7w=1$ — ④

Solving the above equations we get

$\boxed{x=7}$ $\boxed{y=-2}$ $\boxed{z=-3}$ $\boxed{w=1}$

$eq(*) \Rightarrow (A^T)^{-1} = \begin{bmatrix} 7 & -2 \\ -3 & 1 \end{bmatrix}$. Ans.

$eq(a) \Rightarrow (A^{-1})^T = \begin{bmatrix} 7 & -2 \\ -3 & 1 \end{bmatrix}$

So $(A^{-1})^T = (A^T)^{-1}$.

—— o ——

Q2 (a) Let $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ & $B = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$

Then $(A+B)^{-1}$ exists but A^{-1} & B^{-1} don't.

(b) Yes for A non-singular & $c \neq 0$

$(cA)^{-1} = \frac{1}{c} A^{-1}$

$I_n = cA(\frac{1}{c}A)^{-1}$

$I_n = \cancel{c}(1/\cancel{c}) \cdot A A^{-1}$

$I_n = I_n.$

Q21 $(\lambda-1)x + 2y = 0$ —(1)

$2x + (\lambda-1)y = 0$ —(2)

from (2) $x = \frac{(1-\lambda)y}{2}$

eq (1) $\Rightarrow \frac{(\lambda-1)(1-\lambda)y}{2} + 2y = 0$

$\Rightarrow -\frac{(1-\lambda)(1-\lambda)y}{2} + 2y = 0$

$= \frac{-(1-\lambda)^2 y + 4y}{2} = 0$

$= (-(1-\lambda)^2 + 4)y = 0$

$= -(1+\lambda^2-2\lambda) + 4 = 0$

$= +\lambda^2 - 2\lambda - 3 = 0 \quad \lambda = \frac{2 \pm \sqrt{4+12}}{2} = \frac{2 \pm 4}{2}$

$\boxed{\lambda = 3, -1}$

Ex 1.7

Q23 Similarly to Q5

Q24 $A^{-1}=\begin{bmatrix}3&2\\1&3\end{bmatrix}$ + $B^{-1}=\begin{bmatrix}2&5\\3&-2\end{bmatrix}$

find $(AB)^{-1}$

Since $(AB)^{-1}=(B^{-1})(A^{-1})=\begin{bmatrix}2&5\\3&-2\end{bmatrix}\begin{bmatrix}3&2\\1&3\end{bmatrix}$

$=\begin{bmatrix}6+5&4+15\\9-2&6-6\end{bmatrix}=\begin{bmatrix}11&19\\7&0\end{bmatrix}$.

Q25 $X=A^{-1}b$

$X=\begin{bmatrix}2&3\\4&1\end{bmatrix}\begin{bmatrix}5\\3\end{bmatrix}=\begin{bmatrix}10+9\\20+3\end{bmatrix}=\begin{bmatrix}19\\23\end{bmatrix}$.

Q27 $\begin{array}{c} \qquad A \qquad\qquad I_3 \\ \left[\begin{array}{ccc:ccc}5&2&0&1&0&0\\3&1&0&0&1&0\\0&0&-4&0&0&1\end{array}\right]\end{array}$

$\sim\left[\begin{array}{ccc:ccc}1&0.4&0&0.2&0&0\\3&1&0&0&1&0\\0&0&-4&0&0&1\end{array}\right] R_1/5$

$$\sim \left[\begin{array}{ccc:ccc} 1 & 0.4 & 0 & 0.2 & 0 & 0 \\ 0 & -0.2 & 0 & -0.6 & 1 & 0 \\ 0 & 0 & -4 & 0 & 0 & 1 \end{array}\right] R_2 - 3R_1$$

$$\sim \left[\begin{array}{ccc:ccc} 1 & 0.4 & 0 & 0.2 & 0 & 0 \\ 0 & 1 & 0 & 3 & -5 & 0 \\ 0 & 0 & -4 & 0 & 0 & 1 \end{array}\right] -\frac{R_2}{0.2}$$

$$\sim \left[\begin{array}{ccc:ccc} 1 & 0 & 0 & -1 & 2 & 0 \\ 0 & 1 & 0 & 3 & -5 & 0 \\ 0 & 0 & -4 & 0 & 0 & 1 \end{array}\right] R_1 - 0.4R_2$$

$$\sim \left[\begin{array}{ccc:ccc} 1 & 0 & 0 & -1 & 2 & 0 \\ 0 & 1 & 0 & 3 & -5 & 0 \\ 0 & 0 & 1 & 0 & 0 & -1/4 \end{array}\right] -\frac{R_3}{4}$$

So $$A^{-1} = \begin{bmatrix} -1 & 2 & 0 \\ 3 & -5 & 0 \\ 0 & 0 & -1/4 \end{bmatrix}.$$

Mr. Jamal Nasir

Lecturer U.E.T Peshawar

Ex 1.7

Q29(c)

$$\left[\begin{array}{cccc:cccc} 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 1 \end{array}\right]$$

$$\sim \left[\begin{array}{cccc:cccc} 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 1 \end{array}\right] R_2+R_1$$

$$\sim \left[\begin{array}{cccc:cccc} 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 1 \end{array}\right] R_{32}$$

$$\sim \left[\begin{array}{cccc:cccc} 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 0 & 1 \end{array}\right] R_4+R_3$$

$$\sim \left[\begin{array}{cccc:cccc} 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 0 & 1 \end{array}\right] R_2+R_4$$

$$\sim \left[\begin{array}{cccc:cccc} 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 0 & 1 \end{array}\right] R_2+R_3$$

$$\sim \left[\begin{array}{cccc:cccc} 1 & 0 & 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 0 & 1 \end{array}\right] R_1+R_2$$

So $A^{-1}=\begin{bmatrix}1&0&1&1\\0&0&1&1\\1&1&0&0\\1&1&0&1\end{bmatrix}$.

Q30 is similarly to Q29.

Q32 is similarly to Q31

Q31 (a)

$$\left[\begin{array}{ccc:ccc}1&1&1&1&0&0\\1&0&1&0&1&0\\0&1&0&0&0&1\end{array}\right]$$

$$\sim\left[\begin{array}{ccc:ccc}1&1&1&1&0&0\\0&1&0&1&1&0\\0&1&0&0&0&1\end{array}\right] R_2+R_3$$

$$\sim\left[\begin{array}{ccc:ccc}1&1&1&1&0&0\\0&1&0&1&1&0\\0&0&0&1&1&1\end{array}\right] R_3+R_2$$

it's nontrivial so matrix is singular.

(b)

$$\left[\begin{array}{ccc:ccc}1&0&0&1&0&0\\1&1&1&0&1&0\\1&0&1&0&0&1\end{array}\right],$$

$$\sim\left[\begin{array}{ccc:ccc}1&0&0&1&0&0\\0&1&1&1&1&0\\0&0&1&1&0&1\end{array}\right] \begin{matrix}R_2+R_1\\R_3+R_1\end{matrix}$$

$$\sim\left[\begin{array}{ccc:ccc}1&0&0&1&0&0\\0&1&0&0&1&1\\0&0&1&1&0&1\end{array}\right] R_2+R_3$$ As inverse exist so it's trivial.

T.6. Show that the inverse of a nonsingular upper (lower) triangular matrix is upper (lower) triangular.

T.7. Show that if A is singular and $A\mathbf{x} = \mathbf{b}$, $\mathbf{b} \neq \mathbf{0}$ has one solution, then it has infinitely many. (*Hint:* Use Exercise T.13 in Section 1.6.)

T.8. Show that if A is a nonsingular symmetric matrix, then A^{-1} is symmetric.

T.9. Let A be a diagonal matrix with nonzero diagonal entries $a_{11}, a_{22}, \ldots, a_{nn}$. Show that A^{-1} is nonsingular and that A^{-1} is a diagonal matrix with diagonal entries $1/a_{11}, 1/a_{22}, \ldots, 1/a_{nn}$.

T.10. If $B = PAP^{-1}$, express $B^2, B^3, \ldots, B^k$, where k is a positive integer, in terms of A, P, and P^{-1}.

T.11. Make a list of all possible 2×2 bit matrices and then determine which are nonsingular. (See Exercise T.13 in Section 1.2.)

T.12. If A and B are nonsingular 3×3 bit matrices, is it possible that $AB = O$? Explain.

T.13. Determine which 2×2 bit matrices A have the property that $A^2 = O$. (See Exercise T.13 in Section 1.2.)

MATLAB Exercises

In order to use MATLAB *in this section, you should first have read Chapter 12 through Section 12.5.*

ML.1. Using MATLAB, determine which of the following matrices are nonsingular. Use command **rref**.

(a) $A = \begin{bmatrix} 1 & 2 \\ -2 & 1 \end{bmatrix}$

(b) $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$

(c) $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 0 \end{bmatrix}$

ML.2. Using MATLAB, determine which of the following matrices are nonsingular. Use command **rref**.

(a) $A = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$ (b) $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}$

(c) $A = \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 2 \\ 1 & 0 & 0 \end{bmatrix}$

ML.3. Using MATLAB, determine the inverse of each of the following matrices. Use command **rref([A eye(size(A))])**.

(a) $A = \begin{bmatrix} 1 & 3 \\ 1 & 2 \end{bmatrix}$ (b) $A = \begin{bmatrix} 1 & 1 & 2 \\ 2 & 1 & 1 \\ 1 & 2 & 1 \end{bmatrix}$

ML.4. Using MATLAB, determine the inverse of each of the following matrices. Use command

rref([A eye(size(A))]).

(a) $A = \begin{bmatrix} 2 & 1 \\ 2 & 3 \end{bmatrix}$ (b) $A = \begin{bmatrix} 1 & -1 & 2 \\ 0 & 2 & 1 \\ 1 & 0 & 0 \end{bmatrix}$

ML.5. Using MATLAB, determine a positive integer t so that $(tI - A)$ is singular.

(a) $A = \begin{bmatrix} 1 & 3 \\ 3 & 1 \end{bmatrix}$ (b) $A = \begin{bmatrix} 4 & 1 & 2 \\ 1 & 4 & 1 \\ 0 & 0 & -4 \end{bmatrix}$

Exercises ML.6 through ML.9 use bit matrices and the supplemental instructional commands described in Section 12.9.

ML.6. Determine which of the bit matrices in Exercises 29 and 30 have an inverse using **binreduce**.

ML.7. Determine which of the bit linear systems in Exercises 31 and 32 have a nontrivial solution using **binreduce**.

ML.8. Determine which of the following matrices has an inverse using **binreduce**.

(a) $\begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 \end{bmatrix}$ (b) $\begin{bmatrix} 1 & 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 & 1 \end{bmatrix}$

ML.9. Let $B = \mathbf{bingen}(1, 7, 3)$; that is, the matrix whose columns are the binary representations of the integers 1 through 7 using three bits. Determine two 3×3 submatrices that have an inverse and two that do not.

1.8 LU-FACTORIZATION (OPTIONAL)

In this section we discuss a variant of Gaussian elimination (presented in Section 1.6) that decomposes a matrix as a product of a lower triangular matrix and an upper triangular matrix. This decomposition leads to an algorithm for solving a linear system $A\mathbf{x} = \mathbf{b}$ that is the most widely used method on computers for solving a linear system. A main reason for the popularity of this method is that it provides the cheapest way of solving a linear system

Thus the solution to the given linear system is

$$\mathbf{x} = \begin{bmatrix} 4.5 \\ 6.9 \\ -1.2 \\ -4 \end{bmatrix}.$$

■

Next, we show how to obtain an LU-factorization of a matrix by modifying the Gaussian elimination procedure from Section 1.6. No row interchanges will be permitted and we do not require that the diagonal entries have value 1. At the end of this section we provide a reference that indicates how to enhance the LU-factorization scheme presented to deal with matrices where row interchanges are necessary. We observe that the only elementary row operation permitted is the one that adds a multiple of one row to a different row.

To describe the LU-factorization, we present a step-by-step procedure in the next example.

EXAMPLE 3 Let A be the coefficient matrix of the linear system of Example 2.

$$A = \begin{bmatrix} 6 & -2 & -4 & 4 \\ 3 & -3 & -6 & 1 \\ -12 & 8 & 21 & -8 \\ -6 & 0 & -10 & 7 \end{bmatrix}.$$

We proceed to "zero out" entries below the diagonal entries using only the row operation that adds a multiple of one row to a different row.

Procedure | **Matrices Used**

Step 1. "Zero out" below the first diagonal entry of A. Add $\left(-\frac{1}{2}\right)$ times the first row of A to the second row of A. Add 2 times the first row of A to the third row of A. Add 1 times the first row of A to the fourth row of A. Call the new resulting matrix U_1.

$$U_1 = \begin{bmatrix} 6 & -2 & -4 & 4 \\ 0 & -2 & -4 & -1 \\ 0 & 4 & 13 & 0 \\ 0 & -2 & -14 & 11 \end{bmatrix}$$

We begin building a lower triangular matrix, L_1, with 1s on the main diagonal, to record the row operations. Enter the *negatives of the multipliers* used in the row operations in the first column of L_1, below the first diagonal entry of L_1.

$$L_1 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ \frac{1}{2} & 1 & 0 & 0 \\ -2 & * & 1 & 0 \\ -1 & * & * & 1 \end{bmatrix}$$

Step 2. "Zero out" below the second diagonal entry of U_1. Add 2 times the second row of U_1 to the third row of U_1. Add (-1) times the second row of U_1 to the fourth row of U_1. Call the new resulting matrix U_2.

$$U_2 = \begin{bmatrix} 6 & -2 & -4 & 4 \\ 0 & -2 & -4 & -1 \\ 0 & 0 & 5 & -2 \\ 0 & 0 & -10 & 12 \end{bmatrix}$$

Enter the negatives of the multipliers from the row operations below the second diagonal entry of L_1. Call the new matrix L_2.

$$L_2 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ \frac{1}{2} & 1 & 0 & 0 \\ -2 & -2 & 1 & 0 \\ -1 & 1 & * & 1 \end{bmatrix}$$

Step 3. "Zero out" below the third diagonal entry of U_2. Add 2 times the third row of U_2 to the fourth row of U_2. Call the new resulting matrix U_3.

$$U_3 = \begin{bmatrix} 6 & -2 & -4 & 4 \\ 0 & -2 & -4 & -1 \\ 0 & 0 & 5 & -2 \\ 0 & 0 & 0 & 8 \end{bmatrix}$$

Enter the negative of the multiplier below the third diagonal entry of L_2. Call the new matrix L_3.

$$L_3 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ \frac{1}{2} & 1 & 0 & 0 \\ -2 & -2 & 1 & 0 \\ -1 & 1 & -2 & 1 \end{bmatrix}$$

Let $L = L_3$ and $U = U_3$. Then the product LU gives the original matrix A (verify). This linear system of equations was solved in Example 2 using the LU-factorization just obtained. ■

Remark In general, a given matrix may have more than one LU-factorization. For example, if A is the coefficient matrix considered in Example 2, then another LU-factorization is LU, where

$$L = \begin{bmatrix} 2 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ -4 & 2 & 1 & 0 \\ -2 & -1 & -2 & 2 \end{bmatrix} \quad \text{and} \quad U = \begin{bmatrix} 3 & -1 & -2 & 2 \\ 0 & 2 & 4 & 1 \\ 0 & 0 & 5 & -2 \\ 0 & 0 & 0 & 4 \end{bmatrix}.$$

There are many methods for obtaining an LU-factorization of a matrix, besides the scheme for **storage of multipliers** described in Example 3. It is important to note that if $a_{11} = 0$, then the procedure used in Example 3 fails. Moreover, if the second diagonal entry of U_1 is zero or if the third diagonal entry of U_2 is zero, then the procedure also fails. In such cases we can try rearranging the equations of the system and beginning again or using one of the other methods for LU-factorization. Most computer programs for LU-factorization incorporate row interchanges into the storage of multipliers scheme and use additional strategies to help control roundoff error. If row interchanges are required, then the product of L and U is not necessarily A—it is a matrix that is a permutation of the rows of A. For example, if row interchanges occur when using the **lu** command in MATLAB in the form **[L,U] = lu(A)**, then MATLAB responds as follows: The matrix that it yields as L is not lower triangular, U is upper triangular, and LU is A. The book *Experiments in Computational Matrix Algebra*, by David R. Hill (New York: Random House, 1988, distributed by McGraw-Hill) explores such a modification of the procedure for LU-factorization.

erms

ostitution

substitution

rization (or LU-decomposition)

1.8 Exercises

In Exercises 1 through 4, solve the linear system $A\mathbf{x} = \mathbf{b}$ with the given LU-factorization of the coefficient matrix A. Solve the linear system using a forward substitution followed by a back substitution.

1. $A = \begin{bmatrix} 2 & 8 & 0 \\ 2 & 2 & -3 \\ 1 & 2 & 7 \end{bmatrix}$, $\mathbf{b} = \begin{bmatrix} 18 \\ 3 \\ 12 \end{bmatrix}$,

$L = \begin{bmatrix} 2 & 0 & 0 \\ 2 & -3 & 0 \\ 1 & -1 & 4 \end{bmatrix}$, $U = \begin{bmatrix} 1 & 4 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{bmatrix}$

2. $A = \begin{bmatrix} 8 & 12 & -4 \\ 6 & 5 & 7 \\ 2 & 1 & 6 \end{bmatrix}$, $\mathbf{b} = \begin{bmatrix} -36 \\ 11 \\ 16 \end{bmatrix}$,

$L = \begin{bmatrix} 4 & 0 & 0 \\ 3 & 2 & 0 \\ 1 & 1 & 1 \end{bmatrix}$, $U = \begin{bmatrix} 2 & 3 & -1 \\ 0 & -2 & 5 \\ 0 & 0 & 2 \end{bmatrix}$

3. $A = \begin{bmatrix} 2 & 3 & 0 & 1 \\ 4 & 5 & 3 & 3 \\ -2 & -6 & 7 & 7 \\ 8 & 9 & 5 & 21 \end{bmatrix}$, $\mathbf{b} = \begin{bmatrix} -2 \\ -2 \\ -16 \\ -66 \end{bmatrix}$,

$L = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ -1 & 3 & 1 & 0 \\ 4 & 3 & 2 & 1 \end{bmatrix}$,

$U = \begin{bmatrix} 2 & 3 & 0 & 1 \\ 0 & -1 & 3 & 1 \\ 0 & 0 & -2 & 5 \\ 0 & 0 & 0 & 4 \end{bmatrix}$

4. $A = \begin{bmatrix} 4 & 2 & 1 & 0 \\ -4 & -6 & 1 & 3 \\ 8 & 16 & -3 & -4 \\ 20 & 10 & 4 & -3 \end{bmatrix}$, $\mathbf{b} = \begin{bmatrix} 6 \\ 13 \\ -20 \\ 15 \end{bmatrix}$,

$L = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 2 & -3 & 1 & 0 \\ 5 & 0 & -1 & 1 \end{bmatrix}$,

$U = \begin{bmatrix} 4 & 2 & 1 & 0 \\ 0 & -4 & 2 & 3 \\ 0 & 0 & 1 & 5 \\ 0 & 0 & 0 & 2 \end{bmatrix}$.

In Exercises 5 through 10, find an LU-factorization of the coefficient matrix of the given linear system $A\mathbf{x} = \mathbf{b}$. Solve the linear system using a forward substitution followed by a back substitution.

P-1 BS

5. $A = \begin{bmatrix} 2 & 3 & 4 \\ 4 & 5 & 10 \\ 4 & 8 & 2 \end{bmatrix}$, $\mathbf{b} = \begin{bmatrix} 6 \\ 16 \\ 2 \end{bmatrix}$.

6. $A = \begin{bmatrix} -3 & 1 & -2 \\ -12 & 10 & -6 \\ 15 & 13 & 12 \end{bmatrix}$, $\mathbf{b} = \begin{bmatrix} 15 \\ 82 \\ -5 \end{bmatrix}$ Q3(a)

7. $A = \begin{bmatrix} 4 & 2 & 3 \\ 2 & 0 & 5 \\ 1 & 2 & 1 \end{bmatrix}$, $\mathbf{b} = \begin{bmatrix} 1 \\ -1 \\ -3 \end{bmatrix}$

8. $A = \begin{bmatrix} -5 & 4 & 0 & 1 \\ -30 & 27 & 2 & 7 \\ 5 & 2 & 0 & 2 \\ 10 & 1 & -2 & 1 \end{bmatrix}$, $\mathbf{b} = \begin{bmatrix} -17 \\ -102 \\ -7 \\ -6 \end{bmatrix}$

9. $A = \begin{bmatrix} 2 & 1 & 0 & -4 \\ 1 & 0 & 0.25 & -1 \\ -2 & -1.1 & 0.25 & 6.2 \\ 4 & 2.2 & 0.3 & -2.4 \end{bmatrix}$,

$\mathbf{b} = \begin{bmatrix} -3 \\ -1.5 \\ 5.6 \\ 2.2 \end{bmatrix}$

10. $A = \begin{bmatrix} 4 & 1 & 0.25 & -0.5 \\ 0.8 & 0.6 & 1.25 & -2.6 \\ -1.6 & -0.08 & 0.01 & 0.2 \\ 8 & 1.52 & -0.6 & -1.3 \end{bmatrix}$,

$\mathbf{b} = \begin{bmatrix} -0.15 \\ 9.77 \\ 1.69 \\ -4.576 \end{bmatrix}$

MATLAB Exercises

Routine **lupr** *provides a step-by-step procedure in* MATLAB *for obtaining the LU-factorization discussed in this section. Once we have the LU-factorization, routines* **forsub** *and* **bksub** *can be used to perform the forward and back substitution, respectively. Use* **help** *for further information on these routines.*

ML.1. Use **lupr** in MATLAB to find an LU-factorization of

$$A = \begin{bmatrix} 2 & 8 & 0 \\ 2 & 2 & -3 \\ 1 & 2 & 7 \end{bmatrix}.$$

ML.2. Use **lupr** in MATLAB to find an LU-factorization of

$$A = \begin{bmatrix} 8 & -1 & 2 \\ 3 & 7 & 2 \\ 1 & 1 & 5 \end{bmatrix}.$$

ML.3. Solve the linear system in Example 2 using **lupr**, **forsub**, and **bksub** in MATLAB. Check your LU-factorization using Example 3.

ML.4. Solve Exercises 7 and 8 using **lupr**, **forsub**, and **bksub** in MATLAB.

Ex 1.8

Q1 $A=\begin{bmatrix} 2 & 8 & 0 \\ 2 & 2 & -3 \\ 1 & 2 & 7 \end{bmatrix}$; $b=\begin{bmatrix} 18 \\ 3 \\ 12 \end{bmatrix}$, $L=\begin{bmatrix} 2 & 0 & 0 \\ 2 & -3 & 0 \\ 1 & -1 & 4 \end{bmatrix}$, $U=\begin{bmatrix} 1 & 4 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{bmatrix}$.

Now $UX=Z$ — (i) $LZ=b$ — (2)

eq (ii) => ~~Lx~~ $LZ=b$

$$\begin{bmatrix} 2 & 0 & 0 \\ 2 & -3 & 0 \\ 1 & -1 & 4 \end{bmatrix}\begin{bmatrix} Z_1 \\ Z_2 \\ Z_3 \end{bmatrix}=\begin{bmatrix} 18 \\ 3 \\ 12 \end{bmatrix}$$

$2Z_1=18$ — (i) $2Z_1-3Z_2=3$ — (2)

$Z_1-Z_2+4Z_3=12$ — (3)

solving the above equation we get

$\boxed{Z_1=9,}$ $\boxed{Z_2=5}$ $\boxed{Z_3=2}$

eq (1) => $UX=Z$

$$\begin{bmatrix} 1 & 4 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}=\begin{bmatrix} 9 \\ 5 \\ 2 \end{bmatrix}=\begin{bmatrix} Z_1 \\ Z_2 \\ Z_3 \end{bmatrix}$$

$x_1+4x_2=9$ — (1)

$2x_2+x_3=5$ — (2)

$2x_3=2$ — (5)

From above equation we get

$\boxed{x_1 = 1}$ $\boxed{x_2 = 2}$ $\boxed{x_3 = 1}$

& $X = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$ Ans.

Q2 & Q4 is similarly to Q1

Q5 $A = \begin{bmatrix} 2 & 3 & 4 \\ 4 & 5 & 10 \\ 4 & 8 & 2 \end{bmatrix}$; $b = \begin{bmatrix} 6 \\ 16 \\ 2 \end{bmatrix}$.

For matrix U (upper triangular)

$\sim \begin{bmatrix} 2 & 3 & 4 \\ 0 & -1 & 2 \\ 0 & 2 & -6 \end{bmatrix}$ $R_2 - 2R_1$, $R_3 - 2R_1$

$\sim \begin{bmatrix} 2 & 3 & 4 \\ 0 & -1 & 2 \\ 0 & 0 & -2 \end{bmatrix}$ $R_3 + 2R_2$

Ex 1.8

For matrix L (lower triangular)

$$L = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 2 & -2 & 1 \end{bmatrix}$$

Now $LZ = b$

$$\begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 2 & -2 & 1 \end{bmatrix} \begin{bmatrix} Z_1 \\ Z_2 \\ Z_3 \end{bmatrix} = \begin{bmatrix} 6 \\ 16 \\ 2 \end{bmatrix}$$

$Z_1 = 6$ — (1)
$2Z_1 + Z_2 = 16$ — (2)
$2Z_1 - 2Z_2 + Z_3 = 2$ — (3)

Solving we get

$\boxed{Z_1 = 6}$ $\boxed{Z_2 = 4}$ $\boxed{Z_3 = -2}$

Now solving

$UX = Z$

$$\begin{bmatrix} 2 & 3 & 4 \\ 0 & -1 & 2 \\ 0 & 0 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 6 \\ 4 \\ -2 \end{bmatrix}$$

$2x_1 + 3x_2 + 4x_3 = 6$ — (1)
$-x_2 + 2x_3 = 4$ — (2) $-2x_3 = -2$ — (3)

Now we get

$\boxed{x_1 = 4}$ $\boxed{x_2 = -2}$ $\boxed{x_3 = 1}$

$$X = \begin{bmatrix} 4 \\ -2 \\ 1 \end{bmatrix}$$

Q6 — to Q10 similarly to Q5

Mr. Jamal Nasir
Lecturer U.E.T Peshawar
Spring Semester - 2012
Electrical Engg: