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Final-Term Paper

Complex Veriable

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Section: B

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Answer No (1)

Part (a)

Cauchy-Riemann equation:

Consider a complex function given by.

$$w = f(z) = u(x, y) + i\,v(x, y)$$

we say that "f" is analytic if it satisfy given relation.

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$$

&

$$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

(or) $u_x = v_y$ & $u_y = -v_x$

Known as Cauchy Riemann theorem.

Polar form:

if $f(z) = u(r, \theta) + i\,v(r, \theta)$

then Cauchy - Riemann

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Theorem is,

$$\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta}$$

& $$\frac{\partial v}{\partial r} = -\frac{1}{r}\frac{\partial u}{\partial \theta} \qquad r > 0$$

Example:

$$f(z) = e^{x}\,[\cos y + i \sin y]$$

Hence $u = e^{x}\cos y$ & $v = e^{x}\sin y$

Now $u_x = e^{x}\cos y$ — (A)

$u_y = -e^{x}\sin y$ — (B)

$v_x = e^{x}\sin y$ — (C)

$v_y = e^{x}\cos y$ — (d)

Compare (A) & (d)

$u_x = v_y$

Compare (B) & (C)

$u_y = -v_x$

thus given function is analytic

it was a general example.

by using CRE we check analyticity of function.

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Q No 1 ∃ Part (b)

Given.

$$f(z) = \frac{\text{Re}\, z - \text{Im}\, z}{|z|^2}$$

Continus =?? at $z=0$

Sol

$$f(z) = \frac{\text{Re}\, z - \text{Im}\, z}{|z|^2} \rightarrow (1)$$

for a function to be Continus at $z = z_0$, it must Satisfy the Condition,

1) $f(z)$ = is define at z

2) $\lim_{z \to z_0} f(z)$ exist.

3) $\lim_{z \to z_0} f(z) = f(z_0)$

let $z = x + iy$

Re of $z = x$

Im of $z = y$

So

$$|z| = \sqrt{x^2 + y^2}$$

eq (1) become

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$$f(z) = \frac{x-y}{\left(\sqrt{x^2+y^2}\right)^2}$$

$$\boxed{f(z) = \frac{x-y}{x^2+y^2}} \quad \text{at } z=0$$

$$f(0) = \frac{0-0}{0+0}$$

$$f(0) = 0 \longrightarrow \text{Condition (1)}$$

Now

$$\lim_{z\to 0} f(z) = \lim_{(x,y)\to(0,0)} \left(\frac{x-y}{x^2+y^2}\right)$$

$$= \lim_{x\to 0}\left[\lim_{y\to 0} \frac{x-y}{x^2+y^2}\right]$$

$$= \lim_{x\to 0}\left[\frac{\cancel{x}}{x^{\cancel{2}}}\right]$$

$$= \lim_{x\to 0}\left[\frac{1}{x}\right]$$

$$= \frac{1}{0}$$

$$\lim_{z\to 0} f(z) = \infty$$

Now

$$\lim_{z\to 0} f(z) = \lim_{y\to 0}\left[\lim_{x\to 0} \frac{x-y}{x^2+y^2}\right]$$

$$= \lim_{y\to 0}\left[\frac{-\cancel{y}}{y^{\cancel{2}}}\right]$$

$$= \lim_{y\to 0}\left[-\frac{1}{y}\right]$$

$$= -\frac{1}{0}$$

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$$\lim_{z \to 0} = -\infty$$

As

$$\lim_{y \to 0}\left[\lim_{x \to 0} f(z)\right] \neq \lim_{x \to 0}\left[\lim_{y \to 0} f(z)\right]$$

So

$\lim_{z \to 0} f(z)$ doesnot exist

(2nd Condition)

Also

$\lim_{z \to 0} f(z) \neq f(0)$ (3rd Condition

Thus given function is not Continus at $z = 0$

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Q2 Part - A

Given:

$$\int_c \frac{z+4}{z^2+2z+5}\,dz = ?$$

$|z+1| = 1$

Sol

$$\int \frac{z+4}{z^2+2z+5}\,dz \quad \text{—— (1)}$$

Roots of z^2+2z+5 are

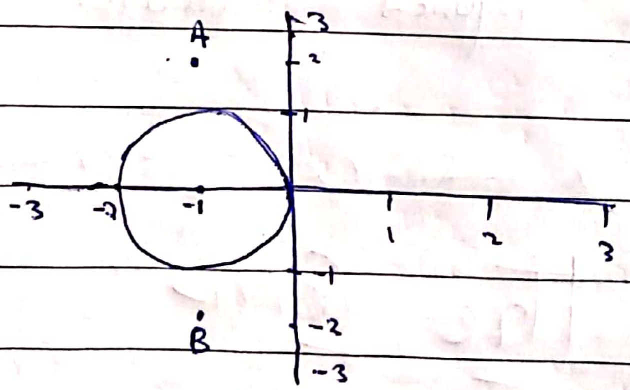
$z = (-1, 2i)$, $z = (-1, -2i)$

$|z+1| = 1$

Center $= (-1, 0)$

Redius $= 1$

Point $A = (-1, 2)$ Point $B = (-1, -2)$



Since the points A & B both are lies

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Outside of the circle So the Integral of function.

$$\boxed{\int_C \frac{z+4}{z^2+2z+5} = 0}$$

This is according to Cauchy Integral theorm.

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Q No (2) Part (b)

Given:

$$f(x) = \begin{cases} \sin x & \text{if } 0 < x < \pi \\ 0 & \text{if } x > \pi \end{cases}$$

Sol

the Fourier Sin Integral of $f(x)$ is

$$f(x) = \frac{2}{\pi} \int_0^\infty \sin \lambda x \int_0^\infty f(u) \sin \lambda t \, dt \cdot$$

$$= \frac{2}{\lambda} \int_0^\pi \sin \lambda x \; \frac{1}{2} \int_0^\pi 2 \sin \lambda t \, dt \, d\lambda$$

$$= \frac{1}{\pi} \int_0^\infty \sin \lambda x \int_0^\pi \cos(\lambda - 1)t - \cos(\lambda + 1)t \Big].$$

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$$f(x) = \frac{1}{\pi}\int_0^{\alpha} \sin \lambda x \left| \frac{\sin(\lambda-1)}{\lambda-1} - \frac{\sin(\lambda+1)t}{(\lambda+1)} \right|_0^{\pi} dt$$

$$= \frac{1}{\pi}\int_0^{\alpha} \sin \lambda x \left[\frac{\sin(\lambda x-\pi)}{\lambda-1} - \frac{\sin(\lambda\pi - x)}{\lambda+1} \right] d\lambda$$

$$= \frac{1}{\pi}\int_0^{\alpha} \sin \lambda x \left[-\frac{\sin\lambda\pi}{\lambda-1} + \frac{\sin\lambda\pi}{\lambda+1} \right] d\lambda$$

$$= \frac{1}{\pi}\int_0^{\alpha} \sin x \left(-\frac{\sin\lambda\pi}{\lambda-1} - \frac{\sin\lambda\pi}{\lambda+1} \right] d\lambda$$

$$= \frac{1}{\pi}\int_0^{\alpha} \sin \lambda x \sin \lambda x \frac{f(2)}{\lambda^2-1} d\lambda$$

$$= \int_0^{\alpha} \frac{\sin \lambda x \cdot \sin \pi x \, d\lambda}{1-\lambda^2}$$

$$= \frac{\pi}{2} f(x)$$

$$f(x) = \int_0^{\alpha} \begin{cases} \pi/2 \sin x & 0 \le \pi \le \pi \\ 0 & x > \pi \end{cases}$$

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Q No : 3

Part – (a)

Fourier Series:

Fourier Series is the approximation of function in term of fundamental trignometric function i.e Cosx and Sin x. let us assume that $f(x)$ is a periodic function of period 2π that can be represented by a trignometric Series.

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$$

That is we assume that this Series Converges and has $f(x)$ as its Sum. Given Such a function $f(x)$ we want to determine the Cofficient a_n & b_n of Corresponding Series of equation (1)

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where a_n to b_n are called the fourier co-efficient and are given by.

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx$$

&

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} \cancel{f(x)\sin nx\,dx}$$

where $n = 0, 1, 2, 3 \ldots$

$f(x) = x + |x| \quad (-\pi < x < \pi)$

Sol

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left[a_n\cos nx + b_n\sin nx\right] \quad \text{--- (i)}$$

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi}(x+|x|)\,dx$$

$$= \frac{1}{\pi}\frac{x^2}{2}\Big|_{-\pi}^{\pi} + \frac{1}{\pi}\frac{x^2}{2}\Big|_{-\pi}^{\pi}$$

$$= \frac{1}{2\pi}(\cancel{\pi^2} - \cancel{\pi^2}) + \frac{1}{2\pi}(\cancel{\pi^2} - \cancel{\pi^2})$$

$$\boxed{a_0 = 0}$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi}(x+|x|)\cos nx\,dx$$

$$a_n = \frac{1}{\pi}\left[x\frac{\sin nx}{n}\Big|_{-\pi}^{\pi} + \frac{1}{\pi}\frac{\cos nx}{n^2}\Big|_{-\pi}^{\pi} + \frac{1}{\pi}|x|\cdot\frac{\sin nx}{n}\Big|_{-\pi}^{\pi} + \frac{1}{\pi}\frac{\cos nx}{n^2}\Big|_{-\pi}^{\pi}\right.$$

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$$\boxed{a_n = \frac{2}{n}\sin n\pi}$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi}(x+|x|)\sin nx\,dx$$

$$= \frac{1}{\pi}\left[-x\frac{\cos nx}{n}\Big|_{-\pi}^{\pi} + \frac{1}{n^2}\sin nx\Big|_{-\pi}^{\pi} - |x|\frac{\cos nx}{n}\Big|_{-\pi}^{\pi} + 1/n^2 \sin n\pi\Big|_{-\pi}^{\pi}\right]$$

$$\boxed{b_n = \frac{1}{\pi}\left(-\frac{2\pi}{n}\cos n\pi + \frac{2}{n^2}\sin n\pi + \frac{2}{n^2}\sin n\pi\right.}$$

eq (1) become.

$$f(x) = \frac{0}{2} + \sum_{n=1}^{\infty}\left[\left(\frac{2}{n}\sin n\pi\right)\cos nx + \frac{1}{\pi}\left(-\frac{2\pi}{n}\cos n\pi + \frac{2}{n^2}\sin n\pi + \frac{2}{n^2}\sin n\pi\right]\sin nx\right]$$

$$f(x) = 2\sin x - \sin 2x + \frac{2}{3}\sin 3x - \frac{2}{4}\sin 4x + \dots\dots .$$

$$f(x) = 2\left(\sin x - \frac{1}{2}\sin 2x + \frac{1}{3}\sin 3x - \frac{1}{4}\sin 4x + \dots\dots\right).$$

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Q3 Part (b):

Given

$$f(x) = \begin{cases} K & \text{if } -\pi/2 < x < \pi/2 \\ 0 & \text{if } \pi/2 < x < 3\pi/2 \end{cases}$$

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$f(x) = K \Rightarrow f(-x) = x = f(x)$

& $f(x) = 0 \Rightarrow f(-x) = 0 = f(x)$

Hence the function is even

The Fourier series for an even function is,

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{L} \quad \text{---} (1)$$

$$a_0 = \frac{2}{L}\int_0^L f(x)\,dx \qquad \cancel{2}L = \cancel{2}\pi$$

$$\therefore L = \pi$$

$$= \frac{2}{\pi}\left[\int_0^{\pi/2} x\,dx + \int_{\pi/2}^{\pi} 0\,dx\right]$$

$$\boxed{a_0 = K}$$

$$a_n = \frac{2}{L}\int_0^L f(x)\cos\frac{n x\pi}{L}\,dx$$

$$= \frac{2}{\pi}\int_0^{\pi} f(x)\cos\frac{n x\cancel{\pi}}{\cancel{\pi}}\,dx$$

$$= \frac{2}{\pi}\left[\int_0^{\pi/2} K\cos nx\,dx + \int_{\pi/2}^{\pi} 0\,dx\right]$$

$$a_n = \frac{2K}{n\pi}\sin n\pi/L$$

eq (1) becomes,

$$f(x) = K/2 + \sum_{n=1}^{\infty}\left[\frac{2K}{n\pi}\sin\frac{n\pi}{2}\cos\frac{n\cancel{\pi}x}{\cancel{\pi}}\right]$$

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$$f(x) = K/2 + \sum_{n=1}^{\infty} \left[\frac{2K}{n\pi} \sin \frac{n\pi}{2} \cos nx \right]$$

$$f(x) = K/2 + \frac{2K}{\pi} \left[\cos x + \frac{1}{3} \cos 3x + \cdots\cdots \right]$$

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Q No. 4
Part (A)

Given

Compute $\int_c f(z)\,dz$

$$f(z) = 2z^4 - z^{-4} \quad (C.C.W)$$

Sol

$$\int_c \cancel{f(z)\,dz = ?}$$

$$f(z) = 2z^4 - z^{-4}$$

Since standard form of a circle,

$|z - z_0| = \rho$ —— ①

Given unit circle

$|z| = 1$ —— ②

Compare eq ① & ②

$z_0 = 0 \quad \rho = 1$

The parametric form of a circle is given by

$z(t) = z_0 + \rho e^{it} \quad 0 \leq t \leq 2\pi$ (full ci

$z(t) = 0 + e^{it}$

Diff w.r.t t on B.S

$$z'(t) = i e^{it}$$

& $$f(z(t)) = 2(e^{it})^4 - (e^{it})^{-4}$$

$$f(z(t)) = 2e^{4it} - e^{-4it}$$

According to theorem

$$\int_C f(z)\,dz = \int_0^{2\pi} f(z(t))\,z'(t)\,dt$$

$$= \int_0^{2\pi} (2e^{4it} - e^{-4it})(ie^{it})\,dt$$

$$= 2i\int_0^{2\pi} e^{5it}\,dt - i\int_0^{2\pi} e^{-3it}\,dt$$

$$= \frac{2i}{5i} e^{5it}\Big|_0^{2\pi} - \frac{i}{-3i} e^{-3it}\Big|_0^{2\pi}$$

$$= \frac{2}{5}\left[e^{10\pi i} - 1\right] + \frac{1}{3}\left[e^{-6\pi i} - 1\right]$$

$$= \frac{2}{5}\left[\cos 10\pi + i\sin 10\pi\right]$$
$$+ \frac{1}{3}\left[\cos(-6\pi) + i\sin(-6\pi) - 1\right]$$

$$= \frac{2}{5}[1-1] + \frac{1}{3}[1-1]$$

$$\int_C f(z)\,dz = \frac{2}{5}(0) + \frac{1}{3}(0)$$

$$\boxed{\int_C f(z)\,dz = 0}$$

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Q 4 ⇒ Part (b)

Given.

$$f(z) = e^z / z^2 \qquad 0 < |z| < R$$

Laurent Series = ???

Sol

$$f(z) = e^z / z^2 \quad \text{——} (1)$$

For Singular point,

$$\frac{e^z}{z^2} = \infty \Rightarrow z^2 = 0 \Rightarrow z = 0$$

here no point is given, take $z_0 = 0$,

let $z - z_0 = u$

$$z - 0 = u \Rightarrow z = u \quad \text{——} (2)$$

eq (1) ⇒

$$f(u) = e^u / u^2$$

$$= \frac{1}{u^2}\left[1 + u + \frac{u^2}{2!} + \frac{u^3}{3!} + \cdots\right]$$

$$f(u) = \frac{1}{u^2} + \frac{u}{u^2} + \frac{u^2}{2u^2} + \frac{u^3}{2u^2} + \cdots$$

$$f(u) = \frac{1}{u^2} + \frac{1}{u} + \frac{1}{2} + \frac{u}{6} + \cdots$$

$$f(u) = \frac{1}{2} + \frac{u}{6} + \cdots + \frac{1}{u} + \frac{1}{u^2} + \cdots \quad (3)$$

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Putting $u=2$ in eq (3),

$\Rightarrow f(z) = \frac{1}{2} + \frac{z}{6} + \cdots + \frac{1}{z} + \frac{1}{z^2} + \cdots$

(or)

$f(z) = \frac{1}{2!} + \frac{z}{3!} + \cdots \frac{1}{z} + \frac{1}{z^2} + \cdots$ (4)

the lauren Series is given by

$f(z) = a_0 + a_1(z-z_0) + a_2(z-z_0)^2 + \cdots$

$+ \frac{b_1}{(z-z_0)} + \frac{b_2}{(z-z_0)^2} + \cdots \rightarrow (5)$

Compare eq (4) & (5)

$a_0 = \frac{1}{2!}$, $a_1 = \frac{1}{3!}$, ...

$b_1 = 1$, $b = 1$, $b_3 = b_4 = \cdots b_n = 0$

and $z_0 = 0$

Hence equation (4) gives the required laurent series.

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The END

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