7

Functions of a Complex Variable

7.1 INTRODUCTION

The theory of functions of a complex variable is of utmost importance in solving a large number of problems in the field of engineering and science. Many complicated integrals of real functions are solved with the help of functions of a complex variable.

7.2 COMPLEX VARIABLE

$x + iy$ is a complex variable and it is denoted by z.

$$z = x + iy. \qquad \text{where} \quad i = \sqrt{-1}$$

7.3 FUNCTION OF A COMPLEX VARIABLE

$f(z)$ is a function of a complex variable and is denoted by w.

$$w = f(z).$$

$$w = u + iv$$

where u and v are the real and imaginary parts of $f(z)$.

7.4 LIMIT OF A FUNCTION OF A COMPLEX VARIABLE

Let $f(z)$ be a single valued function defined at all points in some neighbourhood of a point z_0. Then the limit of $f(z)$ as z approaches z_0 is w_0.

$$\lim_{z \to z_0} f(z) = w_0$$

7.5 CONTINUITY

$f(z)$ is said to be continuous at $z = z_0$ if

$$\lim_{z \to z_0} f(z) = f(z_0)$$

7.6 DIFFERENTIABILITY

Let $f(z)$ be a single valued function of the variable z, then

$$f'(z) = \lim_{\delta z \to 0} \frac{f(z + \delta z) - f(z)}{\delta z}$$

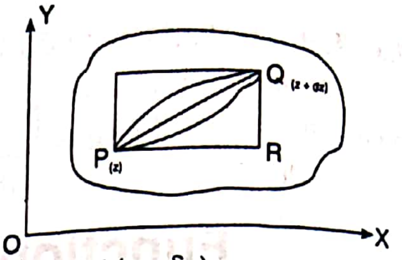
provided that the limit exists and is independent of the path along which $\delta z \to 0$.

Let P be a fixed point and Q be a neighbouring point. The point Q may approach P along any straight line or curved path.

Example 1. *Consider the function*

$$f(z) = 4x + y + i(-x + 4y)$$

and discuss $\dfrac{df}{dz}$.



Solution.

Here, $$f(z) = 4x + y + i(-x + 4y)$$
$$= u + iv$$

so $$u = 4x + y$$

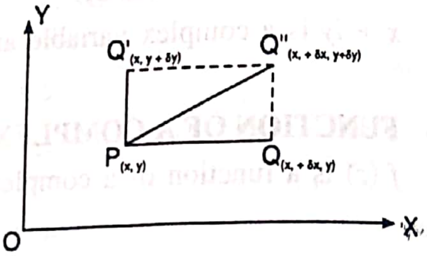
and $$v = -x + 4y$$

$$f(z + \delta z) = 4(x + \delta x) + (y + \delta y) - i(x + \delta x) + 4i(y + \delta y)$$

$$f(z + \delta z) - f(z) = 4(x + \delta x) + (y + \delta y) - i(x + \delta x) + 4i(y + \delta y) - 4x - y + ix - 4iy$$
$$= 4\delta x + \delta y - i\delta x + 4i\delta y$$

$$\frac{f(z+\delta z) - f(z)}{\delta z} = \frac{4\delta x + \delta y - i\delta x + 4i\delta y}{\delta x + i\delta y}$$

$$\Rightarrow \quad \frac{\delta f}{\delta z} = \frac{4\delta x + \delta y - i\delta x + 4i\delta y}{\delta x + i\delta y} \qquad \ldots(1)$$



(*a*) **Along real axis :** If Q is taken on the horizontal line through $P(x, y)$ and Q then approaches P along this line, we shall have $\delta y = 0$, $\delta z = \delta x$.

Putting $\delta y = 0$ and $\delta z = \delta x$ in (1) we get

$$\frac{\delta f}{\delta z} = \frac{4\delta x - i\delta x}{\delta x} = 4 - i$$

(*b*) **Along imaginary axis :** If Q is taken on the vertical line through P and then Q approaches P along this line, we have

$$z = x + iy = 0 + iy, \quad \delta z = i\delta y, \delta x = 0.$$

Putting these values in (1) we have

$$\frac{\delta f}{\delta z} = \frac{\delta y + 4i\delta y}{i\delta y} = \frac{1}{i}(1 + 4i) = 4 - i$$

(*c*) **Along a line y = x :** If Q is taken on a line $y = x$.

$$z = x + iy = x + ix = (1 + i)x$$
$$\delta z = (1 + i)\delta x, \delta y = \delta x$$

On putting these values in (1) we have

$$\frac{\delta f}{\delta z} = \frac{4\delta x + \delta x - i\delta x + 4i\delta x}{\delta x + i\delta x} = \frac{4 + 1 - i + 4i}{1 + i} = \frac{5 + 3i}{1 + i} = \frac{(5 + 3i)(1 - i)}{(1 + i)(1 - i)} = 4 - i$$

In all the three different paths approaching Q to P, we get the same values of $\frac{df}{dz} = 4 - i$. In such a case, the function is said to be differentiable at the point z in the given region.

7.7 ANALYTIC FUNCTION

A single valued function $f(z)$ which is differentiable at $z = z_0$ is said to be **Analytic** at the point $z = z_0$.

The point at which the function is not differentiable is called a **singular point** of the function.

7.8 THE NECESSARY CONDITION FOR $f(z)$ TO BE ANALYTIC

Theorem. The necessary conditions for a function $f(z) = u + iv$ to be analytic at all the points in a region R are

(i) $\dfrac{\partial u}{\partial x} = \dfrac{\partial v}{\partial y}$ (ii) $\dfrac{\partial u}{\partial y} = -\dfrac{\partial v}{\partial x}$ provided $\dfrac{\partial u}{\partial x}, \dfrac{\partial u}{\partial y}, \dfrac{\partial v}{\partial x}, \dfrac{\partial v}{\partial y}$ exist.

Proof. Let $f(z)$ be an analytic function in a region R,

$$f(z) = u + iv,$$

where u and v are the functions of x and y.

Let δu and δv be the increments of u and v respectively corresponding to increments δx and δy of x and y.

$$\therefore \quad f(z+\delta z) = (u+\delta u) + i(v+\delta v)$$

Now
$$\frac{f(z+\delta z)-f(z)}{\delta z} = \frac{(u+\delta u)+i(v+\delta v)-(u+iv)}{\delta z} = \frac{\delta u + i\delta v}{\delta z} = \frac{\delta u}{\delta z} + i\frac{\delta v}{\delta z}$$

$$\lim_{\delta z \to 0} \frac{f(z+\delta z)-f(z)}{\delta z} = \lim_{\delta z \to 0}\left(\frac{\delta u}{\delta z} + i\frac{\delta v}{\delta z}\right) \quad \text{or} \quad f'(z) = \lim_{\delta z \to 0}\left(\frac{\delta u}{\delta z} + i\frac{\delta v}{\delta z}\right) \qquad \ldots(1)$$

since δz can approach zero along any path.

(a) **Along real axis**

$$z = x + iy \qquad \text{but on } x\text{-axis}, \; y = 0$$

$$\therefore \quad z = x, \qquad \delta z = \delta x, \delta y = 0$$

Putting these values in (1) we have

$$f'(z) = \lim_{\delta x \to 0}\left(\frac{\delta u}{\delta x} + i\frac{\delta v}{\delta x}\right) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} \qquad \ldots(2)$$

(b) **Along imaginary axis (y-axis)**

$$z = x + iy \qquad \text{but on } y\text{-axis}, \; x = 0$$

$$z = 0 + iy \qquad \delta x = 0, \delta z = i\delta y.$$

Putting these values in (1) we get

$$f'(z) = \lim_{\delta y \to 0}\left(\frac{\delta u}{i\delta y} + \frac{i\delta v}{i\delta y}\right) = \lim_{\delta y \to 0}\left(-i\frac{\delta u}{\delta y} + \frac{\delta v}{\delta y}\right) = -i\frac{\partial u}{\partial y} + \frac{\partial v}{\partial y} \qquad \ldots(3)$$

If $f(z)$ is differentiable, then two values of $f'(z)$ must be the same.

Equating (2) and (3) we get

$$\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = -i\frac{\partial u}{\partial y} + \frac{\partial v}{\partial y}$$

Equating real and imaginary parts

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}$$

$$\boxed{\begin{aligned}\frac{\partial u}{\partial x} &= \frac{\partial v}{\partial y}\\ \frac{\partial u}{\partial y} &= -\frac{\partial v}{\partial x}\end{aligned}}$$ **are known as Cauchy Riemann equations.**

7.9. SUFFICIENT CONDITION FOR $f(z)$ TO BE ANALYTIC

Theorem. The sufficient condition for a function $f(z) = u + iv$ to be analytic at all the points in a region R are

(i) $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$ (ii) $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial v}{\partial x}, \frac{\partial v}{\partial y}$

are continuous functions of x and y in region R.

Proof. Let $f(z)$ be a single-valued function having

$$\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial v}{\partial x}, \frac{\partial v}{\partial y}$$

at each point in the region R. Then the $C-R$ equations are satisfied.

By Taylor's Theorem:

$$f(z+\delta z) = u(x+\delta x, y+\delta y) + iv(x+\delta x, y+\delta y)$$

$$= u(x,y) + \left(\frac{\partial u}{\partial x}\delta x + \frac{\partial u}{\partial y}\delta y\right) + \ldots + i\left[v(x,y) + \left(\frac{\partial v}{\partial x}\delta x + \frac{\partial v}{\partial y}\delta y\right) + \ldots\right]$$

$$= [u(x,y) + iv(x,y)] + \left[\frac{\partial u}{\partial x}\cdot\delta x + i\frac{\partial v}{\partial x}\cdot\delta x\right] + \left[\frac{\partial u}{\partial y}\delta y + i\frac{\partial v}{\partial y}\cdot\delta y\right] + \ldots$$

$$= f(z) + \left(\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}\right)\delta x + \left(\frac{\partial u}{\partial y} + i\frac{\partial v}{\partial y}\right)\delta y + \ldots$$

(Ignoring the terms of second power and higher powers)

or $$f(z+\delta z) - f(z) = \left(\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}\right)\cdot\delta x + \left(\frac{\partial u}{\partial y} + i\frac{\partial v}{\partial y}\right)\delta y \quad \ldots(1)$$

We know $C-R$ equations *i.e.,*

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \text{ and } \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

Replacing $\frac{\partial u}{\partial y}$ and $\frac{\partial v}{\partial y}$ by $-\frac{\partial v}{\partial x}$ and $\frac{\partial u}{\partial x}$ in (1) we get

$$f(z+\delta z) - f(z) = \left(\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}\right)\cdot\delta x + \left(-\frac{\partial v}{\partial x} + i\frac{\partial u}{\partial x}\right)\cdot\delta y \quad \text{(taking } i \text{ common)}$$

$$= \left(\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}\right)\cdot\delta x + \left(i\frac{\partial v}{\partial x} + \frac{\partial u}{\partial x}\right)\cdot i\delta y = \left(\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}\right)\cdot(\delta x + i\delta y) = \left(\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}\right)\cdot\delta z$$

or $$\frac{f(z+\delta z) - f(z)}{\delta z} = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$$

or $$\lim_{\delta z\to 0}\frac{f(z+\delta z) - f(z)}{\delta z} = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} \text{ or } f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$$

or $$f'(z) = \frac{\partial v}{\partial y} - i\frac{\partial u}{\partial y}$$ **Proved.**

Example 2. *Show that the complex variable function $f(z) = |z|^2$ is differentiable only at the origin.*

Solution. $f(z) = |z|^2$ where $z = x + iy$ or $f(z) = x^2 + y^2$

But $f(z) = u + iv \quad \therefore \quad u = x^2 + y^2, v = 0$

$$\frac{\partial u}{\partial x} = 2x,\ \frac{\partial v}{\partial y} = 0,\ \frac{\partial u}{\partial y} = 2y,\ \frac{\partial v}{\partial x} = 0$$

If $f(z)$ is differentiable then

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \text{or} \quad 2x = 0 \quad \text{or} \quad x = 0 \quad \text{and} \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \quad \text{or} \quad 2y = 0 \quad \text{or} \quad y = 0$$

$C-R$ equations are satisfied only when $x = 0, y = 0$

Thus the given function $f(z)$ is differentiable only at origin. **Proved**

Example 3. *Show that the function $e^x(\cos y + i\sin y)$ is an analytic function, find its derivative.*

Solution. Let $e^x(\cos y + i\sin y) = u + iv$

So $e^x\cos y = u$ or $e^x\sin y = v$ then $\frac{\partial u}{\partial x} = e^x\cos y,\ \frac{\partial v}{\partial x} = e^x\sin y$

$$\frac{\partial u}{\partial y} = -e^x\sin y,\ \frac{\partial v}{\partial y} = e^x\cos y$$

Here we see that $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y},\ \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$

These are $C-R$ equations and they are satisfied.

Hence $e^x(\cos y + i\sin y)$ is analytic.

$$f(z) = u + iv = e^x(\cos y + i\sin y)$$

$$f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = e^x\cos y + ie^x\sin y = e^x(\cos y + i\sin y) = e^x\cdot e^{iy} = e^{x+iy}$$

$$= e^z.$$ **Ans.**

Example 4. *Using the Cauchy-Riemann equations, show that $f(z) = z^3$ is analytic in the entire z-plane.*

Solution.
$$f(z) = z^3 = (x+iy)^3 = x^3 + 3x^2(iy) + 3x(iy)^2 + (iy)^3$$
$$= x^3 - 3xy^2 + i(3x^2y - y^3)$$

Also
$$f(z) = u + iv$$
$$u = x^3 - 3xy^2,\ v = 3x^2y - y^3$$

$$\frac{\partial u}{\partial x} = 3x^2 - 3y^2 \quad ...(1) \qquad \frac{\partial u}{\partial y} = -6xy \quad ...(2)$$

$$\frac{\partial v}{\partial x} = 6xy \quad ...(3) \qquad \frac{\partial v}{\partial y} = 3x^2 - 3y^2 \quad ...(4)$$

From (1) and (4) $$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad ...(5)$$

From (2) and (3) $$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \quad ...(6)$$

Thus $C-R$ equations are satisfied.

Hence $f(z)$ is an analytic function. **Proved**

Example 5. *Test the analyticity of the function $w = \sin z$ and hence derive that : $\frac{d}{dz}(\sin z) = \cos z$*

Solution.

$$w = \sin z = \sin(x+iy) = \sin x \cos iy + \cos x \sin iy$$

$$= \sin x \cosh y + i \cos x \sinh y \qquad \left[\begin{array}{l} \cos ix = \cosh x \\ \sin ix = i \sinh x \end{array}\right]$$

$$u = \sin x \cosh y,\ v = \cos x \sinh y$$

$$\frac{\partial u}{\partial x} = \cos x \cosh y, \quad \frac{\partial u}{\partial y} = \sin x \sinh y$$

$$\frac{\partial v}{\partial x} = -\sin x \sinh y, \quad \frac{\partial v}{\partial y} = \cos x \cosh y$$

Thus $\dfrac{\partial u}{\partial x} = \dfrac{\partial v}{\partial y}$, $\dfrac{\partial u}{\partial y} = -\dfrac{\partial v}{\partial x}$ so $C-R$ equations are satisfied.

Hence $\sin z$ is an analytical function.

$$\frac{d}{dz}(\sin z) = \frac{d}{dz}[\sin x \cosh y + i \cos x \sinh y]$$

$$= \frac{\partial}{\partial x}(\sin x \cosh y + i \cos x \sinh y)$$

$$= \cos x \cosh y - i \sin x \sinh y = \cos x \cos iy - \sin x \sin iy$$

$$= \cos(x+iy) = \cos z$$ **Ans.**

Example 6. *Show that the real and imaginary parts of the function $w = \log z$ satisfy the Cauchy-Riemann equations when z is not zero.*

Solution. $w = \log z = \log(x+iy)$ $\qquad x = r\cos\theta$

or $\quad u + iv = \log(r\cos\theta + ir\sin\theta)$ $\qquad y = r\sin\theta$

$$= \log r(\cos\theta + i\sin\theta) = \log_e r \cdot e^{i\theta}$$

$$= \log_e r + \log_e e^{i\theta} = \log r + i\theta$$

$$= \log\sqrt{x^2+y^2} + i\tan^{-1}\frac{y}{x} \qquad \left[\begin{array}{l} r = \sqrt{x^2+y^2} \\ \theta = \tan^{-1}\dfrac{y}{x} \end{array}\right.$$

so $\quad u = \log\sqrt{x^2+y^2} = \dfrac{1}{2}\log(x^2+y^2), \quad v = \tan^{-1}\dfrac{y}{x}$

On differentiating u, v we get

$$\frac{\partial u}{\partial x} = \frac{1}{2}\frac{1}{x^2+y^2}\cdot(2x) = \frac{x}{x^2+y^2} \quad ...(1)$$

$$\frac{\partial v}{\partial y} = \frac{1}{1+\frac{y^2}{x^2}}\left(\frac{1}{x}\right) = \frac{x}{x^2+y^2} \quad ...(2)$$

From (1) and (2), $\quad \dfrac{\partial u}{\partial x} = \dfrac{\partial v}{\partial y}$ ...(A)

Again differentiating u, v we have

$$\frac{\partial u}{\partial y} = \frac{1}{2}\frac{1}{x^2+y^2}(2y) = \frac{y}{x^2+y^2} \quad ...(3)$$

$$\frac{\partial v}{\partial x} = \frac{1}{1+\frac{y^2}{x^2}}\left(-\frac{y}{x^2}\right) = -\frac{y}{x^2+y^2} \quad ...(4)$$

From (3) and (4)

$$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \quad \text{...(B)}$$

Equations (A) and (B) are $C-R$ equations.

Hence $w = \log z$ is an analytic function except when $x^2 + y^2 = 0$ or $x = y = 0$ or $x + iy = 0$ or $z = 0$

$$w = u + iv$$

$$\frac{dw}{dz} = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = \frac{x}{x^2+y^2} - i\frac{y}{x^2+y^2} = \frac{x-iy}{x^2+y^2}$$

$$= \frac{x-iy}{(x+iy)(x-iy)} = \frac{1}{x+iy} = \frac{1}{z} \quad \textbf{Ans.}$$

Example 7. *Find the point where the Cauchy-Riemann equations are satisfied for the function : $f(z) = xy^2 + ix^2y$ where does $f'(z)$ exist ? Where $f(z)$ is analytic ?*

Solution.

$$f(z) = xy^2 + ix^2y = f(z) = u + iv$$

$$u = xy^2, \quad v = x^2y$$

$$\frac{\partial u}{\partial x} = y^2, \quad \frac{\partial v}{\partial x} = 2xy$$

$$\frac{\partial u}{\partial y} = 2xy, \quad \frac{\partial v}{\partial y} = x^2$$

If $f(z)$ is an analytic function, then it will satisfy $C-R$ equations.

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad i.e. \quad y^2 = x^2 \quad \text{...(1)}$$

$$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \quad i.e. \quad 2xy = -2xy \text{ or. } 4xy = 0 \quad \text{...(2)}$$

Solving (1) and (2) we get $x = y = 0$

At origin $C-R$ equations are satisfied. $f'(z)$ exists at origin only and no where else. Hence $f(z)$ is analytic at origin only. **Ans.**

Example 8. *Show that the function $z|z|$ is not analytic anywhere*

Solution. Let

$$w = z|z|$$

$$w = u + iv \text{ and } z = x + iy$$

$$w = z|z| \quad \text{or} \quad u + iv = (x+iy)\sqrt{x^2+y^2}$$

$$\Rightarrow \quad u = x\sqrt{(x^2+y^2)} \quad \text{and} \quad v = y\sqrt{(x^2+y^2)}$$

$$\frac{\partial u}{\partial x} = \sqrt{x^2+y^2} + \frac{x \cdot 2x}{2\sqrt{x^2+y^2}} \qquad \frac{\partial v}{\partial y} = \sqrt{x^2+y^2} + \frac{y \cdot 2y}{2\sqrt{x^2+y^2}}$$

$$= \frac{x^2+y^2+x^2}{\sqrt{x^2+y^2}} \qquad = \frac{x^2+y^2+y^2}{\sqrt{x^2+y^2}}$$

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \text{ except when } x = y$$

Also $\dfrac{\partial u}{\partial y} = \dfrac{x \cdot 2y}{2\sqrt{x^2+y^2}}$ $\quad \dfrac{\partial v}{\partial x} = \dfrac{y \cdot 2x}{2\sqrt{x^2+y^2}}$ $= \dfrac{xy}{\sqrt{x^2+y^2}}$ $= \dfrac{xy}{\sqrt{x^2+y^2}}$

$$\frac{\partial u}{\partial y} \neq -\frac{\partial v}{\partial x}$$

Hence, $C-R$ equations are not satisfied at any point, the function $z\,|\,z\,|$ is not analytical anywhere. **Proved.**

Example 9. *Show that the function* $f(z) = u + iv$, *where*

$$f(z) = \frac{x^3(1+i) - y^3(1-i)}{x^2+y^2}, \quad z \neq 0$$

$$= 0, \qquad z = 0$$

satisfies the Cauchy-Riemann equations at $z = 0$. Is the function analytic at $z = 0$? Justify your answer.

Solution. $f(z) = \dfrac{x^3(1+i) - y^3(1-i)}{x^2+y^2} = u + iv$

$$u = \frac{x^3 - y^3}{x^2+y^2}, \quad v = \frac{x^3+y^3}{x^2+y^2}$$

[By differentiation the value of $\dfrac{\partial u}{\partial x}, \dfrac{\partial u}{\partial y}, \dfrac{\partial v}{\partial x}, \dfrac{\partial v}{\partial y}$ at (0, 0) we get $\dfrac{0}{0}$, so we apply first principle method]

At the origin

$$\frac{\partial u}{\partial x} = \lim_{h \to 0} \frac{u(0+h, 0) - u(0, 0)}{h} = \lim_{h \to 0} \frac{\frac{h^3}{h^2}}{h} = 1$$

$$\frac{\partial u}{\partial y} = \lim_{k \to 0} \frac{u(0, 0+k) - u(0, 0)}{k} = \lim_{k \to 0} \frac{\frac{-k^3}{k^2}}{k} = -1$$

$$\frac{\partial v}{\partial x} = \lim_{h \to 0} \frac{v(0+h, 0) - v(0, 0)}{h} = \lim_{h \to 0} \frac{\frac{h^3}{h^2}}{h} = 1$$

$$\frac{\partial v}{\partial y} = \lim_{k \to 0} \frac{v(0, 0+k) - v(0, 0)}{k} = \lim_{k \to 0} \frac{\frac{k^3}{k^2}}{k} = 1$$

Thus we see that

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \text{and} \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

Hence Cauchy-Riemann equations are satisfied at $z = 0$.

Again $f'(0) = \lim\limits_{z \to 0} \dfrac{f(z) - f(0)}{z}$ $\qquad = \lim\limits_{z \to 0} \left[\dfrac{x^3 - y^3 + i(x^3+y^3)}{x^2+y^2} \dfrac{1}{x+iy}\right]$

Now let $z \to 0$ along $y = x$, then

$$f'(0) = \lim_{x \to 0} \frac{x^3 - x^3 + i(x^3+x^3)}{x^2+x^2}\left(\frac{1}{x+ix}\right)$$

$$= \frac{2i}{2(1+i)} = \frac{i}{1+i} = \frac{i(1-i)}{(1+i)(1-i)} = \frac{i(1-i)}{1+1} = \frac{1}{2}(1+i)$$

Again let $z \to 0$ along $y = 0$, then $f'(0) = \lim_{x \to 0} \frac{x^3 + ix^3}{x^2} \cdot \frac{1}{x} = (1 + i)$

So we see that $f'(0)$ is not unique. Hence the function $f(z)$ is not analytic at $z = 0$. **Ans.**

7.10 C – R EQUATIONS IN POLAR FORM

$$\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta} \quad ; \qquad \frac{\partial u}{\partial \theta} = -r\frac{\partial v}{\partial r}$$

Proof. We know $x = r\cos\theta$, $y = r\sin\theta$ and u is a function x and y.

$$z = x + iy = r(\cos\theta + i\sin\theta) = re^{i\theta}$$

$$u + iv = f(z) = f(re^{i\theta}) \qquad \dots (1)$$

Differentiating (1) partially w.r.t., "r", we get $\frac{\partial u}{\partial r} + i\frac{\partial v}{\partial r} = f'(re^{i\theta}) \cdot e^{i\theta}$ ... (2)

Differentiating (1) w.r.t. "θ", we get $\frac{\partial u}{\partial \theta} + i\frac{\partial v}{\partial \theta} = f'(re^{i\theta})\, r e^{i\theta}\, i$ ... (3)

Substituting the value of $f'(re^{i\theta})\, i^{i\theta}$ from (2) in (3), we obtain

$$\frac{\partial u}{\partial \theta} + i\frac{\partial v}{\partial \theta} = r\left(\frac{\partial u}{\partial r} + i\frac{\partial v}{\partial r}\right)i \quad \text{or} \quad \frac{\partial u}{\partial \theta} + i\frac{\partial v}{\partial \theta} = ir\frac{\partial u}{\partial r} - r\frac{\partial v}{\partial r}$$

Equating real and imaginary parts, we get

$$\frac{\partial u}{\partial \theta} = -r\frac{\partial v}{\partial r} \quad \text{or} \quad \frac{\partial v}{\partial r} = \frac{-1}{r}\frac{\partial u}{\partial \theta}\ ; \qquad \frac{\partial v}{\partial \theta} = r\frac{\partial u}{\partial r} \quad \text{or} \quad \frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta}.$$

Proved.

EXERCISE 7.1

Determine which of the following functions are analytic :

1. $x^2 + iy^2$ **Ans.** Analytic at all points $y = x$
2. $2xy + i(x^2 - y^2)$ **Ans.** Not analytic
3. $\frac{x - iy}{x^2 + y^2}$ **Ans.** Not analytic
4. $\frac{1}{(z-1)(z+1)}$ **Ans.** Analytic at all points, except $z = \pm 1$
5. $\frac{x - iy}{x - iy + a}$ **Ans.** Not analytic
6. $\sin x \cosh y + i \cos x \sinh y$ **Ans.** Yes, analytic
7. $xy + iy^2$ **Ans.** Yes, analytic at origin only
8. Show that the function $I_m(z - i)$ is analytic everywhere except on the half line $y = 1,\ x \le 0$ *(A.M.I.E.T.E., Dec. 2006)*
9. Choose the correct answer :

   (*a*) In order that the function $f(z) = \frac{|z|^2}{z}$, $z \ne 0$ be continuous at $z = 0$, we should define $f(0)$ equal to

   (*i*) 2 (*ii*) – 1 (*iii*) 0 (*iv*) 1 **Ans.** (*iii*)

   (*b*) If $f(z)$ is analytic and equals $u(x, y) + iv(x, y)$ then $f'(z)$ equals

   (*i*) $\frac{\partial u}{\partial x} - i\frac{\partial u}{\partial y}$ (*ii*) $\frac{\partial u}{\partial x} - i\frac{\partial v}{\partial x}$ (*iii*) $\frac{\partial v}{\partial y} - i\frac{\partial v}{\partial x}$ (*iv*) none of these **Ans.** (*iii*)

   (*c*) The only function among the following, that is analytic, is :

   (*i*) $f(z) = Riz$ (*ii*) $f(z) = Imz$ (*iii*) $f(z) = \bar{z}$, (*iv*) $f(z) = \sin z$. **Ans.** (*iv*)
10. Discuss the analyticity of the function $f(z) = z\bar{z} + \bar{z}^2$ in the complex plane, where $\bar{z}$ is the complex conjugate of z. Also find the points where it is differentiable but not analytic.

    Ans. Differentiable only $z = 0$, No where analytic.
11. Show the functions $\bar{z}$ is not analytic any where. *(A.M.I.E.T.E., Dec. 2006)*
12. Show that the function $f(z) = e^{-z^{-4}}$ and $f(0) = 0$ is not analytic at $z = 0$, although Cauchy-Riemann equations are satisfied at this point.

13. For what values of z do the function w defined by the following equations, ceases to be analytic?

$$z = \sin u \cosh v + i \cos u \sinh v$$ **Ans.** $z = \pm 1$

14. Show that the function $w = \dfrac{x}{x^2+y^2} - \dfrac{iy}{x^2+y^2}$ is an analytic function. Find $\dfrac{dw}{dz}$. **Ans.** $-\dfrac{1}{z^2}$

15. Show that the function $f(z) = \sqrt{|xy|}$ is not regular at the origin. although C-R equations are satisfied at this point. *(A.M.I.E.T.E. Dec. 2005)*

16. Determine the analytic function $f(z) = u + iv$, given that $3u + 2v = y^2 - x^2 + 16xy$. *(A.M.I.E.T.E., Dec. 2004)*

(Madras 2006)

17. Show that an analytic function with constant real part is constant.

18. If $w = u(x, y) + iv(x, y)$ is an analytic function of $z = x + iy$, then $\dfrac{dw}{dz}$ equals

(*a*) $i\dfrac{\partial w}{\partial x}$ (*b*) $-i\dfrac{\partial w}{\partial x}$ (*c*) $i\dfrac{\partial w}{\partial y}$ (*d*) $-i\dfrac{\partial w}{\partial y}$ *(A.M.I.E.T.E., Dec. 2004)*

19. Let (i) and (ii) denote the facts

(i) : f is continous at $z = 0$ ' (ii) : f is differentiable at $z = 0$

Then for function $f(z) = \bar{z}$. which is correct statement ?

(*a*) Both (i) & (ii) are true (*b*) (i) is true, (ii) is false.

(*c*) (i) is false, (ii) is true. (*d*) Both (i) & (ii) are false

20. Which of the following is an entire function

(*a*) $\dfrac{z}{1+z^2}$ (*b*) $z\bar{z}$ (*c*) e^{-z^2} (*d*) $e^{z^{-2}}$ *(A.M.I.E.T.E., June 2006)* **Ans.** (*b*)

21. Let $f(z) = |z|^3$. Then which of the following statements is not correct.

(*a*) f is differentiable at $z = 0$ (*b*) f is differentiable at $z \neq 0$

(*c*) f is not analytic at $z = 0$ (*d*) f is not analytic at $z \neq 0$. *(A.M.I.E.T.E., June 2006)* **Ans.** (*d*)

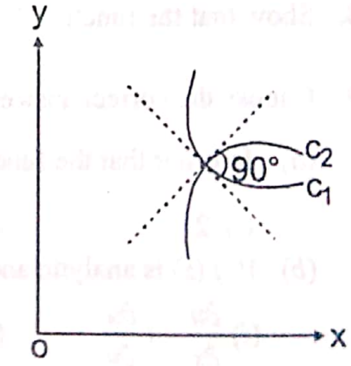
7.11 ORTHOGONAL CURVES

Two curves are said to be orthogonal to each other, when they intersect at right angle at each of their points of intersection.

The analytic function $f(z) = u + iv$ consists of two families of curves $u(x, y) = c_1$ and $v(x, y) = c_2$ which form an orthogonal system.

$$u(x, y) = c_1 \quad \ldots(1) \qquad v(x, y) = c_2 \quad \ldots(2)$$

Differentiating (1), $\dfrac{\partial u}{\partial x}dx + \dfrac{\partial u}{\partial y}dy = 0$ or $\dfrac{dy}{dx} = -\dfrac{\frac{\partial u}{\partial x}}{\frac{\partial u}{\partial y}} = m_1$ (say)



Similarly from (2), $\dfrac{dy}{dx} = -\dfrac{\frac{\partial v}{\partial x}}{\frac{\partial v}{\partial y}} = m_2$ (say)

The product of two slopes

$$m_1 m_2 = \left(-\frac{\frac{\partial u}{\partial x}}{\frac{\partial u}{\partial y}}\right)\left(-\frac{\frac{\partial v}{\partial x}}{\frac{\partial v}{\partial y}}\right) = \left(-\frac{\frac{\partial u}{\partial x}}{\frac{\partial u}{\partial y}}\right)\left(-\frac{\frac{-\partial u}{\partial y}}{\frac{\partial u}{\partial x}}\right) \quad C-R \text{ equations}$$

$$= -1$$

Since $m_1 m_2 = -1$, two curves $u = c_1$, and $v = c_2$ are orthogonal, and c_1, c_2 are parameters, $u = c_1$ and $v = c_2$ form an orthogonal system.

7.12 HARMONIC FUNCTION

Any function which satisfies the Laplace's equation is known as a harmonic function.

Theorem. If $f(z) = u + iv$ is an analytic function, then u and v are both harmonic functions.

Proof. Let $f(z) = u + iv$, be an analytic function, then we have

$$\left.\begin{aligned} \frac{\partial u}{\partial x} &= \frac{\partial v}{\partial y} \qquad \ldots(1) \\ \frac{\partial u}{\partial y} &= -\frac{\partial v}{\partial x} \qquad \ldots(2) \end{aligned}\right] C-R \text{ equations.}$$

Differentiating (1) with respect to x, we get $\dfrac{\partial^2 u}{\partial x^2} = \dfrac{\partial^2 v}{\partial y\, \partial x}$ ...(3)

Differentiating (2) w.r.t. 'y' we have $\dfrac{\partial^2 u}{\partial y^2} = -\dfrac{\partial^2 v}{\partial x\, \partial y}$ ..(4)

Adding (3) and (4) we have $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = \dfrac{\partial^2 v}{\partial x\, \partial y} - \dfrac{\partial^2 v}{\partial y\, \partial x}$

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \qquad \left(\frac{\partial^2 v}{\partial x\, \partial y} = \frac{\partial^2 v}{\partial y\, \partial x}\right)$$

Similarly $$\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial^2 y} = 0$$

Therefore both u and v are harmonic functions.

Such functions u, v are called **Conjugate harmonic functions as $u + iv$ is also analytic function.**

Q. If $w = u + i\,v$ is an analytic function, then show that the family of curves $u(x, y) = a$, cut the family of curves $v(x, y) = b$ orthogonally, a, b being parameters.

(*A.M.I.E.T.E., Dec. 2004*)

Example 10. *Prove that $u = x^2 - y^2$ and $v = \dfrac{y}{x^2 + y^2}$ are harmonic functions of (x, y), but are not harmonic conjugates.*

Solution. $$u = x^2 - y^2$$

$$\frac{\partial u}{\partial x} = 2x, \quad \frac{\partial^2 u}{\partial x^2} = 2, \quad \frac{\partial u}{\partial y} = -2y, \quad \frac{\partial^2 u}{\partial y^2} = -2$$

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 2 - 2 = 0$$

$u(x, y)$ satisfies Laplace equation, hence $u(x, y)$ is harmonic

$$v = \frac{y}{x^2 + y^2}, \quad \frac{\partial v}{\partial x} = -\frac{2xy}{(x^2 + y^2)^2}$$

$$\frac{\partial^2 v}{\partial x^2} = \frac{(x^2 + y^2)^2(-2y) - (-2xy)\,2(x^2 + y^2)\,2x}{(x^2 + y^2)^4}$$

$$= \frac{(x^2 + y^2)(-2y) - (-2xy)\,4x}{(x^2 + y^2)^3} = \frac{6x^2y - 2y^3}{(x^2 + y^2)^3} \qquad \ldots(1)$$

$$\frac{\partial v}{\partial y} = \frac{(x^2 + y^2)\cdot 1 - y(2y)}{(x^2 + y^2)^2} = \frac{x^2 - y^2}{(x^2 + y^2)^2}$$

$$\frac{\partial^2 v}{\partial y^2} = \frac{(x^2 + y^2)^2(-2y) - (x^2 - y^2)\,2(x^2 + y^2)(2y)}{(x^2 + y^2)^4}$$

$$= \frac{(x^2+y^2)(-2y)-(x^2-y^2)(4y)}{(x^2+y^2)^3}$$

$$= \frac{-2x^2y-2y^3-4x^2y+4y^3}{(x^2+y^2)^3} = \frac{-6x^2y+2y^3}{(x^2+y^2)^3} \qquad ...(2)$$

On adding (1) and (2), we get $\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2} = 0$

$v(x, y)$ also satisfies Laplace equations, hence $v(x, y)$ is also harmonic function.

But $\qquad \frac{\partial u}{\partial x} \neq \frac{\partial v}{\partial y}$ and $\frac{\partial u}{\partial y} \neq -\frac{\partial v}{\partial x}$ **Proved**

Therefore u and v are not harmonic conjugates.

Example 11. *Show that the function x^2-y^2+2y which is harmonic remains harmonic under the transformation $z = w^3$.*

Solution. $\qquad u = x^2-y^2+2y$

$$\frac{\partial u}{\partial x} = 2x, \qquad \frac{\partial^2 u}{\partial x^2} = 2 \qquad ... (1)$$

$$\Rightarrow \qquad \frac{\partial u}{\partial y} = -2y+2 \qquad \frac{\partial^2 u}{\partial y^2} = -2 \qquad ... (2)$$

Adding the equation (1) with equation (2)

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2} = 2-2 = 0 \quad \Rightarrow \quad \frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2} = 0$$

Hence function is harmonic.

Transformation:

$$z = w^3, \ z = r e^{i\theta} \text{ and } w = R e^{i\phi}$$

$$\Rightarrow \qquad r e^{i\theta} = \left(R e^{i\phi}\right)^3$$

$$\Rightarrow \qquad r e^{i\theta} = R^3 e^{3i\phi}$$

By comparing both side

$$r = R^3, \ \theta = 3\phi$$

Given function,

$$f(x, y) = x^2+y^2+2y \qquad \text{where } x = r\cos\theta \text{ and } y = r\sin\theta$$

$$f(r\cos\theta, r\sin\theta) = (r\cos\theta)^2-(r\sin\theta)^2+2\times r\sin\theta$$

$$= r^2\cos^2\theta - r^2\sin^2\theta + 2r\sin\theta$$

$$= r^2(\cos^2\theta-\sin^2\theta)+2r\sin\theta = r^2\cos 2\theta + 2r\sin\theta$$

$$f(R^3\cos 3\phi, R^3\sin 3\phi) = R^6\cos 6\phi + 2R^3\sin 3\phi$$

This is function in cos and sin. Hence it will be harmonic function.

7.13 METHOD TO FIND THE CONJUGATE FUNCTION

Given. If $f(z) = u+iv$, and u is known.

To find. v, conjugate function.

Method. We know $\qquad dv = \frac{\partial v}{\partial x}\cdot dx + \frac{\partial v}{\partial y}\cdot dy \qquad ...(1)$

Replacing $\frac{\partial v}{\partial x}$ by $-\frac{\partial u}{\partial y}$ and $\frac{\partial v}{\partial y}$ by $\frac{\partial u}{\partial x}$, we get

$$dv = -\frac{\partial u}{\partial y}\cdot dx + \frac{\partial u}{\partial x}\cdot dy$$

$$v = -\int \frac{\partial u}{\partial y} dx + \int \frac{\partial u}{\partial x}\cdot dy$$

or

$$v = \int M\,dx + \int N\,dy \qquad ...(2)$$

where $M = -\frac{\partial u}{\partial y}$ and $N = \frac{\partial u}{\partial x}$

so that $\frac{\partial M}{\partial y} = \frac{\partial}{\partial y}\left(-\frac{\partial u}{\partial y}\right) = -\frac{\partial^2 u}{\partial y^2}$ and $\frac{\partial N}{\partial x} = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right) = \frac{\partial^2 u}{\partial x^2}$

since u is a conjugate function, so $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$

or

$$-\frac{\partial^2 u}{\partial y^2} = \frac{\partial^2 u}{\partial x^2} \quad \text{or} \quad \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} \qquad ...(3)$$

Equation (3) satisfies the condition of an exact differential equation.

So equation (2) can be integrated and thus v is determined.

Example 12. *Let $f(z) = u(x, y) + iv(x, y)$ be an analytic function. If $u = 3x - 2xy$, then find v and express $f(z)$ in terms of z.*

Solution.

$$u = 3x - 2xy$$

$$\frac{\partial u}{\partial x} = 3 - 2y, \quad \frac{\partial u}{\partial y} = -2x$$

$$dv = \frac{\partial v}{\partial x} dx + \frac{\partial v}{\partial y} dy \qquad \text{(Total differentiation)}$$

$$= \left(-\frac{\partial u}{\partial y}\right) dx + \left(\frac{\partial u}{\partial x}\right) dy \qquad \left(\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y},\ \frac{\partial v}{\partial y} = \frac{\partial u}{\partial x}\right)$$

$$= 2x\,dx + (3 - 2y)\,dy$$

$$v = \int 2x\,dx + \int (3 - 2y)\,dy = x^2 + 3y - y^2 + c$$

$$f(z) = u(x, y) + iv(x, y) = (3x - 2xy) + i(x^2 + 3y - y^2 + c)$$

$$= (ix^2 - iy^2 - 2xy) + (3x + 3yi) + ic = i(x^2 - y^2 + 2ixy) + 3(x + iy) + ic$$

$$= i(x + iy)^2 + 3(x + iy) + ic = iz^2 + 3z + ic$$

Ans.

Example 13. *Show that the function $u(x, y) = 4xy - 3x + 2$ is harmonic. Construct the corresponding analytic function*

$$f(z) = u(x, y) + iv(x, y).$$

Express $f(z)$ in terms of complex variable z.

Solution.

$$u = 4xy - 3x + 2$$

$$\frac{\partial u}{\partial x} = 4y - 3, \quad \frac{\partial^2 u}{\partial x^2} = 0, \quad \frac{\partial u}{\partial y} = 4x, \quad \frac{\partial^2 u}{\partial y^2} = 0$$

On adding (2) and (3) we get $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$

$u(x, y)$ satisfies Laplace equation, hence $u(x, y)$ is harmonic.

(Total differentiation)

$$dv = \frac{\partial v}{\partial x} dx + \frac{\partial v}{\partial y} dy$$

$$= -\frac{\partial u}{\partial y} dx + \frac{\partial u}{\partial x} dy \qquad \left[\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}, \frac{\partial v}{\partial y} = \frac{\partial u}{\partial x} \quad C-R \text{ equations}\right]$$

$$= -4\,x dx + (4\,y - 3)\,dy$$

(Exact differential equation)

$$v = \int -4\,x dx + \int (4\,y - 3)\,dy$$

$$= -2\,x^2 + 2\,y^2 - 3\,y + c$$

$$f(z) = u(x, y) + iv(x, y)$$

$$= 4\,xy - 3\,x + 2 + i(-2\,x^2 + 2\,y^2 - 3\,y) + ic = -2\,ix^2 + 4\,xy + 2\,iy^2 - 3\,x - 3\,iy + 2 + ic$$

$$= -2\,i(x^2 + 2\,ixy - y^2) - 3\,(x + iy) + 2 + ic = -2\,i\,(x + iy)^2 - 3\,(x + iy) + 2 + ic$$

$$= -2\,iz^2 - 3\,z + 2 + ic$$

Ans.

Example 14. *If $w = \phi + i\,\psi$ represents the complex potential for an electric field and*

$$\psi = x^2 - y^2 + \frac{x}{x^2 + y^2},$$

determine the function ϕ.

Solution. $w = \phi + i\,\psi$ and $\psi = x^2 - y^2 + \dfrac{x}{x^2 + y^2}$

$$\frac{\partial \psi}{\partial x} = 2\,x + \frac{(x^2 + y^2)\cdot 1 - x \cdot 2\,x}{(x^2 + y^2)^2} = 2\,x + \frac{y^2 - x^2}{(x^2 + y^2)^2}$$

$$\frac{\partial \psi}{\partial y} = -2\,y - \frac{x\,(2\,y)}{(x^2 + y^2)^2} = -2\,y - \frac{2\,xy}{(x^2 + y^2)^2}$$

$$d\phi = \frac{\partial \phi}{\partial x} dx + \frac{\partial \phi}{\partial y} dy = \frac{\partial \psi}{\partial y} dx - \frac{\partial \psi}{\partial x} dy$$

$$= \left(-2\,y - \frac{2\,xy}{(x^2 + y^2)^2}\right) dx - \left(2\,x + \frac{y^2 - x^2}{(x^2 + y^2)^2}\right) dy$$

This is an exact differential equation.

$$\phi = \int \left[-2\,y - \frac{2\,xy}{(x^2 + y^2)^2}\right] dx + c$$

$$\phi = -2\,xy + \frac{y}{x^2 + y^2} + c$$

Ans.

Example 15. *Construct the analytic function $f(z)$ of which the real part is $e^x \cos y$.*

Solution. $f(z) = u(x, y) + iv(x, y), \quad u(x, y) = e^x \cos y$

$$\frac{\partial u}{\partial x} = e^x \cos y, \quad \frac{\partial u}{\partial y} = -e^x \sin y$$

$$dv = \frac{\partial v}{\partial x} dx + \frac{\partial v}{\partial y} dy$$

(Total differential)

$$= -\frac{\partial u}{\partial y}dx + \frac{\partial u}{\partial x}dy \qquad \left[\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}, \frac{\partial v}{\partial y} = \frac{\partial u}{\partial x} \quad C-R \text{ equations}\right]$$

$$= e^x \sin y\, dx + e^x \cos y\, dy$$

This is an exact differential equation.

$$v = \underset{y \text{ as constant}}{\int e^x \sin y \cdot dx} + \underset{\text{Ignoring the term containing } x}{\int e^x \cos y dy}$$

$$v = e^x \sin y$$

$$f(z) = u + iv = e^x \cos y + ie^x \sin y = e^x (\cos y + i \sin y)$$

$$= e^x \cdot e^{iy} = e^{x+iy} = e^z.$$ **Ans.**

Example 16. *Find an analytic function $w = u + iv$ given that*

$$v = \frac{x}{x^2 + y^2} + \cosh x \cos y.$$

Solution.

$$w = u + iv$$

given that

$$v = \frac{x}{x^2+y^2} + \cosh x \cos y$$

$$du = \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy \qquad \text{(Total differentiation)}$$

$$= \frac{\partial v}{\partial y}dx - \frac{\partial v}{\partial x}dy \qquad \left(\begin{aligned}\frac{\partial u}{\partial x} &= \frac{\partial v}{\partial y}\\ \frac{\partial u}{\partial y} &= -\frac{\partial v}{\partial x}\end{aligned}\right)$$

On substituting the values of $\frac{\partial v}{\partial x}$ and $\frac{\partial v}{\partial y}$ we have

$$du = \left(\frac{-2xy}{(x^2+y^2)^2} - \cosh x \sin y\right)dx - \left[\frac{(x^2+y^2) - x \cdot 2x}{(x^2+y^2)^2} + \sinh x \cos y\right]dy$$

$$= \left[\frac{-2xy}{(x^2+y^2)^2} - \cosh x \sin y\right]dx - \left[\frac{y^2-x^2}{(x^2+y^2)^2} + \sinh x \cos y\right]dy$$

This is an exact differential equation

$$\int du = \int\left(\frac{-2xy}{(x^2+y^2)^2} - \cosh x \sin y\right)dx - \int\left(\frac{x^2-y^2}{(x^2+y^2)^2} + \sinh x \cos y\right)dy$$

$$u = \frac{y}{x^2+y^2} - \sinh x \sin y$$

$$w = u + iv = \frac{y}{x^2+y^2} - \sinh x \sin y + i\left[\frac{x}{x^2+y^2} + \cosh x \cos y\right]$$

$$= \frac{y + ix}{x^2+y^2} - \sinh x \sin y + i \cosh x \cos y$$

$$= \frac{i(x - iy)}{x^2+y^2} + i \sin ix \sin y + i \cos ix \cos y$$

$$= \frac{i\bar{z}}{|z|^2} + i\cos(ix - y) = \frac{i\bar{z}}{|z|^2} + i\cos i(x+iy)$$

$$= \frac{i\bar{z}}{|z|^2} + i\cosh z \qquad \textbf{Ans.}$$

Example 17. *If $u - v = (x-y)(x^2 + 4xy + y^2)$ and $f(z) = u + iv$ is an analytic function of $z = x + iy$, find $f(z)$ in terms of z.*

Solution. $u + iv = f(z), \quad iu - v = if(z)$

Adding these, $(u - v) + i(u + v) = (1 + i) f(z)$

or $(1+i)(u + iv) = (1+i) f(z)$ or $u + iv = f(z)$

where $U = u - v,\ V = u + v$

$$U = u - v = (x-y)(x^2 + 4xy + y^2)$$

$$= x^3 + 3x^2y - 3xy^2 - y^3 \qquad ...(1)$$

$$\frac{\partial U}{\partial x} = 3x^2 + 6xy - 3y^2, \quad \frac{\partial U}{\partial y} = 3x^2 - 6xy - 3y^2$$

$$dV = \frac{\partial V}{\partial x}\cdot dx + \frac{\partial V}{\partial y}dy = -\frac{\partial U}{\partial y}\cdot dx + \frac{\partial U}{\partial x}\cdot dy$$

$$= (-3x^2 + 6xy + 3y^2)\,dx + (3x^2 + 6xy - 3y^2)\cdot dy$$

$$V = \underset{(y \text{ as constant})}{\int(-3x^2 + 6xy + 3y^2)\,dx} + \underset{(\text{Ignoring terms of } x)}{\int(-3y^2)\,dy}$$

$$= -x^3 + 3x^2y + 3xy^2 - y^3 + C$$

$$F(z) = U + iV$$

$$= (x^3 + 3x^2y - 3xy^2 - y^3) + i(-x^3 + 3x^2y + 3xy^2 - y^3) + ic$$

$$= (1-i)x^3 + (1+i)3x^2y - (1-i)3xy^2 - (1+i)y^3 + ic$$

$$= (1-i)x^3 + i(1-i)3x^2y - (1-i)3xy^2 - i(1-i)y^3 + ic$$

$$= (1-i)[x^3 + 3ix^2y - 3xy^2 - iy^3] + ic$$

$$= (1-i)(x+iy)^3 + iC = (1-i)z^3 + ic$$

$$(1+i)f(z) = (1-i)z^3 + ic \qquad F(z) = (1+i)f(z)$$

$$f(z) = \frac{1-i}{1+i}z^3 + \frac{ic}{1+i} = -iz^3 + \frac{1+i}{2}c \qquad \textbf{Ans.}$$

Example 18. *If $f(z) = u + iv$, is any analytic function of the complex variable z and $u - v = e^x(\cos y - \sin y)$, find $f(z)$ in terms of z.*

Solution. $u + iv = f(z) \qquad \therefore\ iu - v = if(z)$

Adding we have

$$u + iv + iu - v = f(z) + if(z)$$

$$(u - v) + i(u + v) = (1+i)f(z) = F(z) \text{ say}$$

Put $u - v = U$ and $u + v = V$, then $F(z) = U + iV$ is an analytic function.

Now $U = e^x(\cos y - \sin y)$

$$\therefore \quad \frac{\partial U}{\partial x} = e^x(\cos y - \sin y) \quad \text{or} \quad \frac{\partial U}{\partial y} = e^x(-\sin y - \cos y)$$

$$dV = \frac{\partial V}{\partial x} dx + \frac{\partial V}{\partial y} dy = -\frac{\partial U}{\partial y} dx + \frac{\partial U}{\partial x} dy$$

$$= e^x (\sin y + \cos y)\, dx + e^x (\cos y - \sin y)\, dy.$$

Integrating, we have,

$$V = e^x (\sin y + \cos y) + c$$

$$F(z) = U + iV$$

$$= e^x (\cos y - \sin y) + ie^x (\sin y + \cos y) + ic$$

$$= e^x (\cos y + i \sin y) + ie^x (\cos y + i \sin y) + ic$$

$$= e^x \cdot e^{iy} + ie^x e^{iy} + ic = e^{x+iy} + ie^{x+iy} + ic = e^z + ie^z + ic$$

$$(1+i) f(z) = (1+i)\, e^z + ic$$

$$f(z) = e^z + \frac{ic}{1+i}, \quad f(z) = e^z + c_1.$$ **Ans.**

Example 19. *Let $f(z) = u(r, \theta) + iv(r, \theta)$ be an analytic function. If $u = -r^3 \sin 3\theta$, then construct the corresponding analytic function $f(z)$ in terms of z.*

Solution.

$$u = -r^3 \sin 3\theta$$

$$\frac{\partial u}{\partial r} = -3r^2 \sin 3\theta, \quad \frac{\partial u}{\partial \theta} = -3r^3 \cos 3\theta$$

$$dv = \frac{\partial v}{\partial r} dr + \frac{\partial v}{\partial \theta} d\theta$$

$$= \left(-\frac{1}{r} \frac{\partial u}{\partial \theta}\right) dr + \left(r \frac{\partial u}{\partial r}\right) d\theta \qquad \left(\begin{array}{l} C\text{–}R \text{ equations} \\ \dfrac{\partial u}{\partial r} = \dfrac{1}{r}\dfrac{\partial v}{\partial \theta}. \\ \dfrac{\partial u}{\partial \theta} = -r\dfrac{\partial v}{\partial r} \end{array}\right)$$

$$= -\frac{1}{r}(-3r^3 \cos 3\theta)\, dr + r(-3r^2 \sin 3\theta)\, d\theta$$

$$= 3r^2 \cos 3\theta \cdot dr - 3r^3 \sin 3\theta\, d\theta$$

$$v = \int (3r^2 \cos 3\theta)\, dr + c = r^3 \cos 3\theta + c$$

$$f(z) = u + iv = -r^3 \sin 3\theta + ir^3 \cos 3\theta + ic = ir^3 (\cos 3\theta + i \sin 3\theta) + ic$$

$$= i\, r^3 e^{i3\theta} + ic = i(r e^{i\theta})^3 + ic = i z^3 + ic$$ **Ans.**

Example 20. *Find analytic function $f(z) = u(r, \theta) + iv(r, \theta)$ such that*

$$v(r, \theta) = r^2 \cos 2\theta - r \cos \theta + 2.$$

Solution.

$$v = r^2 \cos 2\theta - r \cos \theta + 2 \qquad \ldots (1)$$

Differentiating (1), we get

$$\frac{\partial v}{\partial \theta} = -2r^2 \sin 2\theta + r \sin \theta \qquad \ldots (2)$$

$$\frac{\partial v}{\partial r} = 2r \cos 2\theta - \cos \theta \qquad \ldots (3)$$

Using $C - R$ equations in polar coordinates, we get

... [From (2)]

$$r\frac{\partial u}{\partial r} = \frac{\partial v}{\partial \theta} = -2r^2 \sin 2\theta + r\sin\theta$$

... (4)

or
$$\frac{\partial u}{\partial r} = -2r\sin 2\theta + \sin\theta$$

... [From (3)]

$$-\frac{1}{r}\frac{\partial u}{\partial \theta} = \frac{\partial v}{\partial r} = 2r\cos 2\theta - \cos\theta$$

... (5)

$$\frac{\partial u}{\partial \theta} = -2r^2\cos 2\theta + r\cos\theta$$

By total differentiation formula

$$du = \frac{\partial u}{\partial r}dr + \frac{\partial u}{\partial \theta}d\theta = (-2r\sin 2\theta + \sin\theta)\,dr + (-2r^2\cos 2\theta + r\cos\theta)\,d\theta$$

$$= -[(2r\,dr)\sin 2\theta + r^2(2\cos 2\theta\,d\theta)] + [\sin\theta\cdot dr + r(\cos\theta\,d\theta)]$$

$$= -d(r^2\sin 2\theta) + d(r\sin\theta)$$

Integrating, we get

$$u = -r^2\sin 2\theta + r\sin\theta + c$$

Hence,
$$f(z) = u + iv$$
$$= (-r^2\sin 2\theta + r\sin\theta + c) + i(r^2\cos 2\theta - r\cos\theta + 2)$$
$$= ir^2(\cos 2\theta + i\sin 2\theta) - ir(\cos\theta + i\sin\theta) + 2i + c$$
$$= ir^2 e^{2i\theta} - ir\,e^{i\theta} + 2i + c$$
$$= i(r^2 e^{2i\theta} - r\,e^{i\theta}) + 2i + c.$$ **Ans.**

Example 21. *If $f(z)$ is a regular function of z, prove that*

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)|f(z)|^2 = 4|f'(z)|^2.$$

(A.M.I.E.T.E., Summer 2005, 2002, 2001)

Solution.
$$f(z) = u + iv$$
$$|f(z)|^2 = u^2 + v^2$$

Let
$$\phi = u^2 + v^2 \quad \text{... (1)}$$

Differentiating (1) w.r.t. 'x', we get $\frac{\partial \phi}{\partial x} = 2u\frac{\partial u}{\partial x} + 2v\frac{\partial v}{\partial x}$

$$\frac{\partial^2 \phi}{\partial x^2} = 2\left[u\frac{\partial^2 u}{\partial x^2} + \left(\frac{\partial u}{\partial x}\right)^2 + v\frac{\partial^2 v}{\partial x^2} + \left(\frac{\partial v}{\partial x}\right)^2\right] \quad \text{... (2)}$$

Similarly,
$$\frac{\partial^2 \phi}{\partial y^2} = 2\left[u\frac{\partial^2 u}{\partial y^2} + \left(\frac{\partial u}{\partial y}\right)^2 + v\frac{\partial^2 v}{\partial y^2} + \left(\frac{\partial v}{\partial y}\right)^2\right] \quad \text{... (3)}$$

Adding (2) and (3), we get

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 2\left\{u\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) + v\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right)\right\}$$
$$+ 2\left\{\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2 + \left(\frac{\partial v}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial y}\right)^2\right\} \quad \text{... (4)}$$

Substituting the following in (4)

$$\left(\frac{\partial u}{\partial x}\right)^2 = \left(\frac{\partial v}{\partial y}\right)^2 \qquad C-R \text{ equation}$$

$$\left(\frac{\partial u}{\partial y}\right)^2 = \left(-\frac{\partial v}{\partial x}\right)^2 \qquad C-R \text{ equation}$$

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2} = 0 \qquad \text{Laplace equation}$$

$$\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2} = 0 \qquad \text{Laplace equation}$$

Equation (4) reduces to

$$\frac{\partial^2 \phi}{\partial x^2}+\frac{\partial^2 \phi}{\partial y^2} = 4\left\{\left(\frac{\partial u}{\partial x}\right)^2+\left(\frac{\partial v}{\partial x}\right)^2\right\}$$

$$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)\phi = 4\left|\frac{\partial u}{\partial x}+i\frac{\partial v}{\partial x}\right|^2$$

$$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)|f(z)|^2 = 4\,|f'(z)|^2 \qquad \text{[From (1)]}$$

Proved.

7.14 MILNE THOMSON METHOD

By this method $f(z)$ is directly constructed without finding v and the method is given below:

Since $\quad z = x+iy$ and $\bar{z} = x-iy$

$\therefore \quad x = \dfrac{z+\bar{z}}{2},\; y = \dfrac{z-\bar{z}}{2i}$

$$f(z) \equiv u(x,y)+iv(x,y) \qquad ...(1)$$

$$f(z) \equiv u\left(\frac{z+\bar{z}}{2},\frac{z-\bar{z}}{2i}\right)+iv\left(\frac{z+\bar{z}}{2},\frac{z-\bar{z}}{2i}\right)$$

This relation can be regarded as a formal identity in two independent variables z and $\bar{z}$. Replacing $\bar{z}$ by z, we get

$$f(z) \equiv u(z,0)+iv(z,0)$$

which can be obtained by replacing x by z and y by 0 in (1).

We have $\quad f(z) = u+iv$

$\therefore \quad f'(z) = \dfrac{\partial u}{\partial x}+i\dfrac{\partial v}{\partial x},\; f'(z) = \dfrac{\partial u}{\partial x}-i\dfrac{\partial u}{\partial y} \qquad (C-R \text{ equations})$

If we write $\quad \dfrac{\partial u}{\partial x} = \phi_1(x,y),\; \dfrac{\partial u}{\partial y} = \phi_2(x,y)$

$$f'(z) = \phi_1(x,y)-i\,\phi_2(x,y) \text{ or } f'(z) = \phi_1(z,0)-i\,\phi_2(z,0)$$

On integrating $\quad f(z) = \int \phi_1(z,0)\,dz - i\int \phi_2(z,0)\,dz + c \qquad$ [when u is given]

$$f(z) = \int \psi_1(z,0)\,dz + i\int \psi_2(z,0)\,dz + c \qquad \text{[when } v \text{ is given]}$$

when $\quad \psi_1(x,y) = \dfrac{\partial v}{\partial y},\; \psi_2(x,y) = \dfrac{\partial v}{\partial x}.$

Example 22. *If $u = x^2 - y^2$, find a corresponding analytic function.*

Solution. $\dfrac{\partial u}{\partial x} = 2x = \phi_1(x, y)$, $\dfrac{\partial u}{\partial y} = -2y = \phi_2(x, y)$

$$f(z) = \int [\phi_1(z, 0) - i\,\phi_2(z, 0)]\,dz + c$$

$$= \int [2z - i(0)]\,dz + c = \int 2z\,dz + c = z^2 + c$$

Ans.

Example 23. *If $u = \dfrac{\sin 2x}{\cosh 2y + \cos 2x}$, find $f(z)$.* (*A.M.I.E.T.E., Summer 2003*)

Solution. $$\frac{\partial u}{\partial x} = \frac{(\cosh 2y + \cos 2x)\,2\cos 2x - \sin 2x(-2\sin 2x)}{(\cosh 2y + \cos 2x)^2}$$

$$= \frac{2\cosh 2y\cos 2x + 2(\cos^2 2x + \sin^2 2x)}{(\cosh 2y + \cos 2x)^2} = \frac{2\cosh 2y\cos 2x + 2}{(\cosh 2y + \cos 2x)^2} = \phi_1(x, y)$$

$$\frac{\partial u}{\partial y} = \frac{-\sin 2x\,(2\sinh 2y)}{(\cosh 2y + \cos 2x)^2} = \frac{-2\sin 2x\sinh 2y}{(\cosh 2y + \cos 2x)^2} = \phi_2(x, y)$$

$$f(z) = \int [\phi_1(z, 0) - i\,\phi_2(z, 0)]\,dz + c = \int \frac{(2\cos 2z + 2)}{(1 + \cos 2z)^2}\,dz + c = 2\int \frac{1}{1 + \cos 2z}\,dz + c$$

$$= 2\int \frac{1}{2\cos^2 z}\,dz + c = \int \sec^2 z\,dz + c = \tan z + c$$

Ans.

Example 24. *Find the analytic function $f(z) = u + iv$, given that*

$$v = e^x(x\sin y + y\cos y)$$

Solution. $\dfrac{\partial v}{\partial x} = e^x(x\sin y + y\cos y) + e^x\sin y = \psi_2(x, y)$

$$\frac{\partial v}{\partial y} = e^x(x\cos y + \cos y - y\sin y) = \psi_1(x, y)$$

$$f(z) = \int [\psi_1(z, 0) + i\,\psi_2(z, 0)]\,dz + c$$

$$= \int [e^z(z + 1) + i(0)]\,dz + c = (z + 1)e^z - \int e^z\,dz + c$$

$$= (z + 1)e^z - e^z + c = ze^z + c$$

Ans.

Exercise 7.2

Show that the following functions are harmonic and determine the conjugate functions.

1. $u = 2x(1 - y)$ **Ans.** $v = x^2 - y^2 + 2y + C$
2. $u = 2x - x^3 + 3xy^2$ **Ans.** $v = 2y - 3x^2y + y^3 + C$
3. $u = \frac{1}{2}\log(x^2 + y^2)$ **Ans.** $v = -\tan^{-1}\frac{x}{y} + C$
4. (a) If u is a harmonic function, then show that $w = u^2$ is not a harmonic function, unless u is a constant. (*A.M.I.E.T.E., Dec. 2006*)

Determine the analytic function, whose real part is

4. $2x(1 - y)$ **Ans.** $iz^2 + 2z + C$
5. $\log\sqrt{x^2 + y^2}$ **Ans.** $\log z + C$
6. $x^2 - y^2 + 5x + y - \dfrac{y}{x^2 + y^2}$ **Ans.** $z^2 + (5 - i)z - \dfrac{i}{z} + C$
7. $\cos x\cosh y$ **Ans.** $\cos z + c$
8. $3x^2y + 2x^2 - y^3 - 2y^2$ **Ans.** $2z^2 - iz^3 + C$
9. $x^3 - 3xy^2 + 3x^2 - 3y^2 + 2x + 1$ **Ans.** $z^3 + 3z^2 + 2z + C$
10. $e^{-x}(x\sin y - y\cos y)$ **Ans.** $i(ze^{-z} + C)$
11. (a) $e^{2x}(x\cos 2y - y\sin 2y)$ **Ans.** $ze^{2z} + iC$

(*b*) $e^{-x}(x\cos y + y\sin y)$ and $f(0) = i$.

Ans. $z e^{-z} + i$

Determine the analytic function, whose imaginary part is

12. $v = \log(x^2 + y^2) + x - 2y$

Ans. $2i\log z - (2 - i)z + C$

13. $v = \sinh x \cos y$ **Ans.** $\sin iz + C$

14. $v = \dfrac{x - y}{x^2 + y^2}$ **Ans.** $(1+i)\frac{1}{z} + c$

15. $v = -\dfrac{y}{x^2 + y^2}$

Ans. $\dfrac{1}{z} + C$

16. $u - v = \dfrac{\cos x + \sin x - e^{-y}}{2\cos x - 2\cosh y}$ and $f\left(\dfrac{\pi}{2}\right) = 0$ (*A.M.I.E.T.E., Dec. 2005, Winter 2000*) **Ans.** $\dfrac{1}{2}\left[1 - \cot\dfrac{z}{2}\right]$

17. $v = \left(r - \dfrac{1}{r}\right)\sin\theta$

Ans. $z + \dfrac{1}{z} + c$

18. If $f(z) = u + iv$ is an analytic function of $z = x + iy$ and $u - v = \dfrac{e^y - \cos x + \sin x}{\cosh y - \cos x}$, find $f(z)$ subject to the condition that $f(\frac{\pi}{2}) = \dfrac{3 - i}{2}$.

Ans. $f(z) = \cot\frac{z}{2} + \dfrac{1 - i}{2}$

19. Find an analytic function $f(z) = u(r, \theta) + iv(r, \theta)$ such that

$$V(r, \theta) = r^2\cos 2\theta - r\cos\theta + 2.$$

Ans. $i[z^2 - z + 2]$

20. Show that the function $u = x^2 - y^2 - 2xy - 2x - y - 1$ is harmonic. Find the conjugate harmonic function v and express $u + iv$ as a function of z where $z = x + iy$. **Ans.** $(1 + i)z^2 + (-2 + i)z - 1$

21. Construct an analytic function of the form $f(z) = u + iv$, where v is $v = \tan^{-1}(y/x)$, $x \neq 0$, $y \neq 0$.

Ans. $\log cz$

22. Choose the correct answer: The value of m so that $2x - x^2 + my^2$ may be harmonic is

(*a*) 0 (*b*) 1 (*c*) 2 (*d*) 3 **Ans.** (*b*)

23. If $f(z) = u + iv$ is an analytic function of z in any domain, show that

(*a*) $\left(\dfrac{\partial^2}{\partial x^2} + \dfrac{\partial^2}{\partial y^2}\right)|f(z)|^p = p^2\,|f(z)|^{p-2}\,|f'(z)|^2$ (*A.M.I.E.T.E., Summer 2000*)

24. If $u + v = \dfrac{2\sin 2x}{e^{2y} + e^{-2y} - 2\cos 2x}$, and $f(z) = u + iv$ is an analytic function of $z = x + iy$, find $f(z)$ in terms of z. (*A.M.I.E.T.E., Summer 2004 Winter 2002*) **Ans.** $\frac{1}{2}(1 + i)\cot z + u$

7.15 GEOMETRICAL REPRESENTATION

To draw curve of complex variable (x, iy) we take two axes *i.e.*, one real axis and the other imaginary axis. A number of points (x, y) are plotted on z-plane, by taking different value of z (different values of x and y). The curve C is drawn by joining the plotted points. The diagram obtained is called **Argand diagram.**

Let $\qquad w = f(z) = f(x + iy) = u + iv$

To draw a curve of w, we take u-axis and v-axis. By plotting different points (u, v) on w plane and joining them, we get a curve C_1 on w plane.

7.16 TRANSFORMATION

For every point (x, y) in the z plane, the relation $w = f(z)$ defines a corresponding point (u, v) in the w-plane. We call this "transformation or mapping of z plane into w plane". If a point z_0 maps into the point w_0, w_0 is also known as the image of z_0.

If the point $P(x, y)$ moves along a curve C in z-plane, the point $P'(u, v)$ will move along a corresponding curve C_1 in w plane. We, then, say that a curve C in the z plane is mapped into the corresponding curve C_1 in the w plane by the relation $w = f(z)$.