Subject: Complex Variable

Mid-term paper

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Q No: 1:

Ans De-Moivere's Theorm:

let n be an Integer then $[(\cos\theta + i\sin\theta)]^n = (\cos n\theta + i\sin n\theta)$

$$\boxed{(\cos\theta + i\sin\theta)^n = (\cos n\theta + i\sin n\theta)}$$

(i) Proof for +ive n:

let $n=1$

$$(\cos\theta + i\sin\theta)^1 = (\cos 1\theta + i\sin 1\theta)$$

$$\boxed{(\cos\theta + i\sin\theta) = (\cos\theta + i\sin\theta)}$$

Theorm true for $n=1$

P.T.O

for $n = k+1$

$$(\cos\theta + i\sin\theta)^{k+1} = \cos(k+1)\theta + i\sin(k+1)\theta$$

↳ Required Target

As

$$(\cos\theta + i\sin\theta)^k = (\cos k\theta + i\sin k\theta) \quad -(1)$$

xing eq (1) by $(\cos\theta + i\sin\theta)$

$$(\cos\theta + i\sin\theta)^k(\cos\theta + i\sin\theta) = (\cos^k\theta + i\sin^k\theta)(\cos\theta + i\sin\theta)$$

$$(\cos\theta + i\sin\theta)^{k+1} = \cos k\theta\cos\theta - \sin k\theta\sin\theta + i(\sin k\theta\cos\theta + \cos k\theta\sin\theta)$$

$$= \cos(k\theta+\theta) + i\sin(k\theta+\theta)$$

$$\boxed{(\cos\theta + i\sin\theta)^{k+1} = \cos(k+1)\theta + i\sin(k+1)\theta}$$

Hence proved for $n = k+1$

(ii) for n is $-$ive:

let $n = -m$ then

$$(\cos\theta + i\sin\theta)^n = (\cos\theta + i\sin\theta)^{-m}$$

$$= \frac{1}{(\cos\theta + i\sin\theta)^m}$$

Now m is positive so

$$(\cos\theta + i\sin\theta)^n = \frac{1}{(\cos m\theta + i\sin m\theta)}$$

$$= \frac{1}{\cos m\theta + i\sin m\theta} \times \frac{\cos m\theta - i\sin m\theta}{\cos m\theta - i\sin m\theta}$$

$$= \frac{\cos m\theta - i\sin m\theta}{\cos^2 m\theta + \sin^2 m\theta}$$

As $\cos^2\theta + \sin^2\theta = 1$ So

$$(\cos\theta + i\sin\theta)^n = \cos m\theta - i\sin m\theta$$

(or)

$$(\cos\theta + i\sin\theta)^n = \cos(-m)\theta + i\sin(-m)\theta$$

$$\boxed{(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta}$$

Hence proved.

(iii) for $n = 0$

$m = 0$

$$(\cos\theta + i\sin\theta)^0 = (\cos 0\theta + i\sin 0\theta)$$

$$= \cos(0) + i\sin(0)$$

As $\cos 0 = 1$

$\sin 0 = 0$ So

$$(\cos\theta + i\sin\theta)^0 = 1 + i0$$

$$= 1$$

Hence proved

(iv) for n = rational no:

Sol: let $n = p/q$

Consider

$$\left(\cos\theta/q + i\sin\theta/q\right)^q = \left(\cos\theta\, q/q + i\sin\theta\, q/q\right)$$

this is according to (D.T).

$$\left(\cos\theta/q + i\sin\theta/q\right)^q = \left(\cos\theta + i\sin\theta\right)$$

Now take power $1/q$ so

$$\left(\cos\theta/q + i\sin\theta/q\right)^{q/q} = \left(\cos\theta + i\sin\theta\right)^{1/q}$$

So

$$\left(\cos\theta + i\sin\theta\right)^{1/q} = \cos\left(\theta/q + i\sin\theta/q\right)$$

Now take power p

$$\left(\cos\theta + i\sin\theta\right)^{p/q} = \cos\left(\theta/q + i\sin\theta/q\right)^p$$

According to (D.T)

$$\left(\cos\theta + i\sin\theta\right)^{p/q} = \left(\cos\tfrac{p}{q}\theta + i\sin\tfrac{p}{q}\theta\right)$$

$$\boxed{\left(\cos\theta + i\sin\theta\right)^n = \left(\cos n\theta + i\sin n\theta\right)}$$

Hence proved.

Q2:

Ans (a) Given

$$f(z) = xy^2 + ix^2y$$

$f'(z)$ exist = ?

$f(z)$ analytic = ?

Sol

$$f(z) = xy^2 + ix^2y$$

$$U = xy^2, \quad V = x^2y$$

$U_x = y^2$

$U_y = 2xy$

and

$V_x = 2xy$

$V_y = x^2$

$U_x = V_y$

$xy^2 = x^2$

$x = 0$ (or) $x = y^2$

$u_y = -v_x$

$2xy = -2xy$

$xy = 0$

$x = 0$ (or) $y = 0$

$(x = 0$ (or) $x = y^2)$

and

$(x = 0$ (or) $y = 0)$

$x = 0$

$f'(z)$ is exist on every point on line $x = 0$

$f'(z) = u_x + i v_x$.

$= y^2 + i 2xy$

"f" is not analytic at any point, since every neighbourhood of point

on $x=0$ there exists other
point which is not
derivable at that point

——— xx ——— xx ———

Q2:

(b) Given

$$\sqrt[4]{-7+24i}$$

Sol

let

$$w = \sqrt[4]{-7+24i}$$

$$z = -7+24i$$

$$|z| = 25$$

and $\arg(z) = \tan^{-1}\left(\frac{-24}{7}\right) = 1.287$

$$w_k = \sqrt[4]{25}\left[\cos\left(\frac{-1.287+2k\pi}{4}\right) + i\sin\left(\frac{-1.287+2k\pi}{4}\right)\right]$$

$i = 0, 1, 2, 3$ —

$$w_0 = \sqrt{5}\left[\cos\left(\frac{-1.287}{4}\right) + i\sin\left(\frac{-1.287}{4}\right)\right]$$

$$= 2.121 - 0.7071i$$

$$w_1 = \sqrt{5}\left[\cos\left(\frac{-1.287+2\pi}{4}\right) + i\sin\left(\frac{-1.287+2\pi}{4}\right)\right]$$

$$= 0.7071 + 2.121i$$

8 (b)

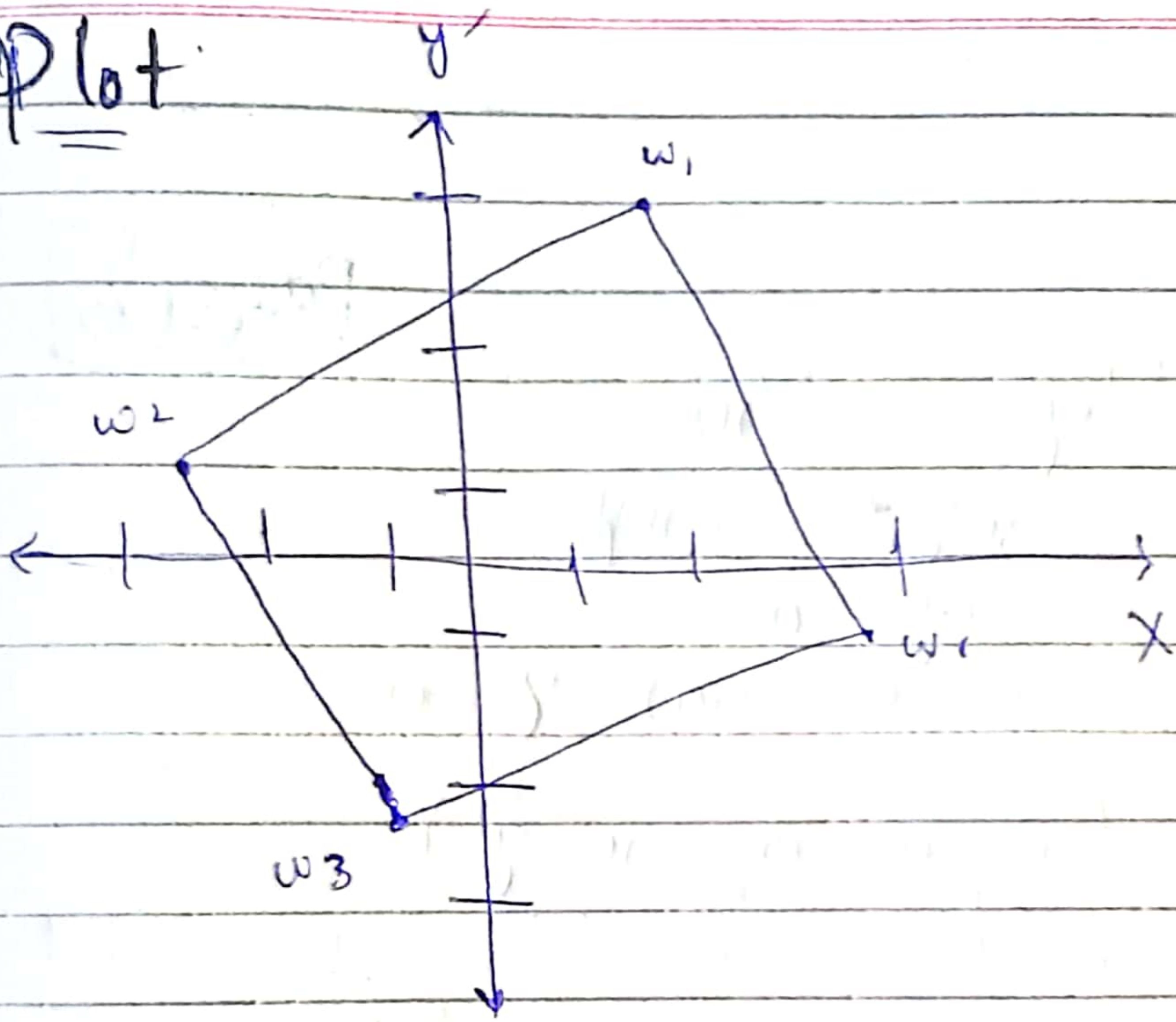
$$w_2 = \sqrt{5}\left[\cos\left(\frac{-1\times 287+4\pi}{4}\right)+i\sin\left(\frac{-1\times 28+6\pi}{4}\right)\right]$$

$$= -2\times 121 + 0\times 7071 i$$

$$w_3 = \sqrt{5}\left[\cos\left(\frac{-1\times 287+6\pi}{4}\right)+i\sin\left(\frac{-1\times 287+6\pi}{4}\right)\right]$$

$$w_3 = -0.7071 - 2\times 121 i$$

Plot



——— XY ——— XY ——— XY

(3).
Ans=
(a) Analytic function.

let $w = u+iv$ be a complex function with domain "D". This function is said to be analytic if,
(i) $f(z)$ is define at z_0
(2) $f'(z)$ is define at z_0
where z_0 is a point in domain of $f(z)$.

Example:

let $f(x,y) = u(x,y) + iv(x,y)$ be a complex function,
Since $x = (z+\bar{z})/2$ &

$y = (z-\bar{z})/2i$

Substituting for x & y

$f(z,\bar{z}) = u(x,y) + iv(x,y)$

A necessary Condition for $f(z, \bar{z})$ to be analytic is

$$\frac{\partial f}{\partial \bar{z}} = 0$$

Therefore necessary Condition for $f = u + iv$ to be analytic is that f depend only on z. In term of the real & imaginary part u, v of f Condition is equivalent to

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \text{——} (1)$$

$$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \quad \text{——} (2)$$

eq (1) & (2) are known as Cauchy-Riemann theorm eq. They are the necessary Condition for $f(z+\bar{z}) = u + iv$ to be analytic.

P.T.O

(b) $f(z) = z + \frac{1}{z}$

Sol

let $z = x + iy$

$$f(z) = z + \frac{1}{z}$$

$$= x + iy + \frac{1}{x+iy}$$

$$= \frac{(x+iy)^2 + 1}{(x+iy)}$$

$$= \frac{x^2 - y^2 + 1 + 2ixy}{x+iy}$$

$$= \frac{(x^2 - y^2 + 1) + i\,2xy}{x+iy} \times \frac{x-iy}{x-iy}$$

$$= \frac{x^3 - xy^2 + x + 2xy^2 + i(2x^2y - x^2y + y^3 - y)}{x^2+y^2}$$

$$f(z) = \frac{x^3 + x + xy^2}{x^2+y^2} + i\,\frac{x^2y + y^3 - y}{x^2+y^2}$$

Hence $u = \dfrac{x^3 + x + xy^2}{x^2+y^2}$

$$V = \frac{x^2y + y^3 - y}{x^2+y^2}$$

$$U_x = \frac{(x^2+y^2)(3x^2+1+y^2)-(x^3+x+xy^2)(2x)}{(x^2+y^2)^2}$$

$$U_x = \frac{x^4+2x^2y^2-x^2+y^4+y^2}{(x^2+y^2)^2} \quad \text{——— (1)}$$

$$\cancel{V_y = x^4+2x^2y^2-x^2+y^4+y^2}$$

$$V_y = \frac{(x^2y^2)(x^2+3y^2-1)-(x^2y+y^3-y)(2y)}{(x^2+y^2)^2}$$

$$V_y = \frac{x^4+2x^2y^2-x^2+y^4+y^2}{(x^2+y^2)^2} \quad \text{——— (2)}$$

from eq (1) & (2) we have

$$U_x = V_y$$

Also

$$U_y = \frac{(x^2+y^2)(2xy)-(x^3+x+xy^2)(2y)}{(x^2+y^2)^2}$$

$$U_y = -2xy/(x^2+y^2)^2 \quad \text{——— (3)}$$

&

$$V_x = \frac{(x^2+y^2)(2xy)-(x^2y+y^3-y)(2x)}{(x^2+y^2)^2}$$

$$V_x = 2xy/(x^2+y^2)^2$$

$$-V_x = -2xy/(x^2+y^2)^2 \quad \text{—— (4)}$$

from eq (3) & (4)

$$U_y = -V_x$$

that satisfy CRE, so the given function is Analytic

—— xx —— xx —— xx

Q4 (a)

Continuity of Complex functions:

A function $f(z)$ is Said to be continuous at $z = z_0$ if it satisfy following Condition

(1) $f(z)$ is define at z_0

(2) $\lim_{z \to z_0} f(z)$ exist

(3) $\lim_{z \to z_0} f(z) = f(z_0)$

P. T. O

Cauchy- Riemann Theorm:

Consider a Complex function given by,

$$w = f(z) = u(x,y) + iv(x,y)$$

we says that "f" is analytic if it satisfies given relation

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$$

& $$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \quad (or)$$

$$u_x = v_y \quad \& \quad u_y = -v_x$$

known as Cauchy - Riemann theorm.

Polar form:

if $f(z) = u(r,\theta) + iv(r,\theta)$

then Cauchy - Riemann theorem is.

$$\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta}$$

& $\frac{\partial v}{\partial r} = -\frac{1}{r}\frac{\partial u}{\partial \theta}$ where $r > 0$

—— xx —— xx —— xx

the END

Q 4: (b)

Given

$$f(z) = U(x,y) + iv(x,y)$$

$$U = \frac{x}{x^2+y^2}$$

Sol

$$U = \frac{x}{x^2+y^2}$$

$$U_x = \frac{1(x^2+y^2) - x(2x)}{(x^2+y^2)^2} = \frac{y^2-x^2}{(x^2+y^2)^2}$$

$$U_{xx} = \frac{-2x(x^2+y^2)^2 - (y^2-x^2)2(x^2+y^2)2x}{(x^2+y^2)^4}$$

$$U_{xx} = \frac{2x(x^2-3y^2)}{(x^2+y^2)^3}$$

$$U_y = \frac{-2x}{(x^2+y^2)^2}$$

$$U_{yy} = \frac{-2x(x^2+y^2)^2 + 2xy\,2(x^2+y^2)2y}{(x^2+y^2)^4}$$

$$U_{yy} = \frac{-2x^3-2xy^2+8xy^2}{(x^2+y^2)^3}$$

$$U_{yy} = \frac{-2x^3+6xy^2}{(x^2+y^2)^3}$$

$$\nabla^2 U = \boxed{U_{xx} + U_{yy} = 0} \Rightarrow$$

$$\nabla^2 U = U_{xx} + U_{yy} =$$

$$\frac{2x^3-6xy^2}{(x^2+y^2)^3} + \frac{-2x^3+6xy^2}{(x^2+y^2)^3}$$

$\nabla^2 U = 0 \Rightarrow$
given function u is harmonic

$U_x = V_y$ & $u_y = -V_x$

$\Rightarrow f(z) = U(x,y) + iv(x,y)$ is analytic

$$V_y = U_x = \frac{y^2-x^2}{(x^2+y^2)^2}$$

$$V = -\frac{y}{x^2+y^2} + h(x)$$

$$v_x = \frac{2xy}{(x^2+y^2)^2} + h'(x) = -4y$$

$$v_x = \frac{2xy}{(x^2+y^2)^2} + h'(x)$$

$$= \frac{2xy}{(x^2+y^2)^2}$$

$$h'(x) = 0$$

$$h(x) = c$$

$$v = -\frac{y}{x^2+y^2} + h(x)$$

$$= -\frac{y}{x^2+y^2} + c$$

$$f(z) = \frac{x}{x^2+y^2} - i\left(\frac{y}{x^2+y^2} - c\right)$$

$$f(z) = \frac{x-iy}{(x+iy)(x-iy)} + ic$$

$$f(z) = \frac{1}{x+iy} + ic = \boxed{\frac{1}{z} + ic}$$

Corresponding Analytic.

xx — xx — xx