4. For what contours $C$ will it follow from Cauchy's theorem that

(a) $\oint_C \frac{dz}{z} = 0$, (b) $\oint_C \frac{\cos z}{z^6 - z^2}\, dz = 0$, (c) $\oint_C \frac{e^{1/z}}{z^2 + 9}\, dz = 0$?

5. The integral in Example 4 is zero. Can we conclude from this that it is zero over the contour in Prob. 1?

6. Can we conclude from Example 2 that the integral of $1/(z^2 + 4)$ taken over (a) $|z - 2| = 2$, (b) $|z - 2| = 3$ is zero? Give a reason.

Integrate $f(z)$ counterclockwise over the unit circle and indicate whether Cauchy's theorem may be applied.

7. $f(z) = |z|$ 8. $f(z) = e^{z^2}$ 9. $f(z) = \operatorname{Im} z$

10. $f(z) = 1/(2z - 5)$ 11. $f(z) = 1/\bar{z}$ 12. $f(z) = 1/(\pi z - 3)$

13. $f(z) = \tan z$ 14. $f(z) = \bar{z}$ 15. $f(z) = \bar{z}^2$

16. $f(z) = 1/|z|^3$ 17. $f(z) = 1/(z^2 + 2)$ 18. $f(z) = z^2 \sec z$

Evaluate the following integrals. (*Hint*. If necessary, represent the integrand in terms of partial fractions.)

19. $\oint_C \frac{dz}{z - i}$, $C$ the circle $|z| = 2$ (counterclockwise)

20. $\oint_C \frac{dz}{\sinh z}$, $C$ the circle $|z - \frac{1}{2}\pi i| = 1$ (clockwise)

21. $\oint_C \frac{\cos z}{z}\, dz$, $C$ consists of $|z| = 1$ (counterclockwise) and $|z| = 3$ (clockwise)

22. $\oint_C \frac{2z - 1}{z^2 - z}\, dz$, $C$ the contour in Fig. 323

23. $\oint_C \frac{dz}{z^2 - 1}$, $C$ the contour in Fig. 324

24. $\oint_C \operatorname{Re} z\, dz$, $C$ the contour in Fig. 325

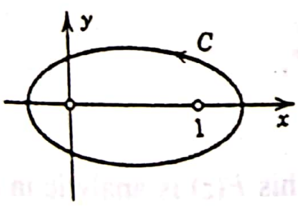



Fig. 323. Problem 22

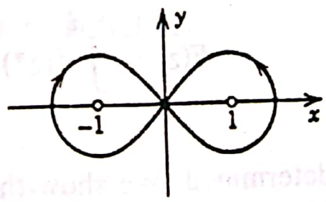



Fig. 324. Problem 23

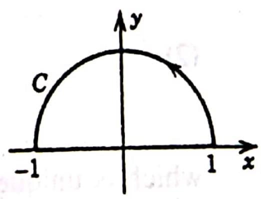



Fig. 325. Problem 24

25. $\oint_C \frac{dz}{z^2 + 1}$, $C$: (a) $|z + i| = 1$, (b) $|z - i| = 1$ (counterclockwise)

26. $\oint_C \frac{\sin z}{z + 3i}\, dz$, $C$: $|z - 2 + 3i| = 1$ (counterclockwise)

27. $\oint_C \frac{2z + 1}{z^2 + z}\, dz$, $C$: (a) $|z| = \frac{1}{4}$, (b) $|z - \frac{1}{2}| = \frac{1}{4}$, (c) $|z| = 2$ (clockwise)

Evaluate (*continued*)

28. $\oint_C \frac{dz}{1+z^3}$, $C$: $|z+1| = 1$ (counterclockwise)

29. $\oint_C \frac{3z+1}{z^3-z}\, dz$, $C$: (a) $|z| = 1/2$, (b) $|z| = 2$ (counterclockwise)

30. $\oint_C \operatorname{Re}(z^2)\, dz$, $C$ the boundary of the triangle with vertices at 0, 2, and $2+i$ (counterclockwise)

## 13.4 Existence of Indefinite Integral

In this short section we use Cauchy's integral theorem to establish the existence of an indefinite integral $F(z)$ of a given analytic function $f(z)$ and thereby justify the evaluation of line integrals by indefinite integration and substitution of the limits of integration (see Sec. 13.2):

$$\int_{z_0}^{z_1} f(z)\, dz = F(z_1) - F(z_0) \qquad [F'(z) = f(z)], \tag{1}$$

where $F(z)$ is an indefinite integral of $f(z)$, that is, $F'(z) = f(z)$, as indicated. In most applications, such an $F(z)$ can be found from differentiation formulas.

**Theorem 1 (Existence of an indefinite integral)**

*If $f(z)$ is analytic in a simply connected domain $D$* (see Sec. 13.3), *then there exists an indefinite integral $F(z)$ of $f(z)$ in $D$—thus, $F'(z) = f(z)$—which is analytic in $D$, and for all paths in $D$ joining any two points $z_0$ and $z_1$ in $D$, the integral of $f(z)$ from $z_0$ to $z_1$ can be evaluated by formula* (1).

*Proof.* The conditions of Cauchy's integral theorem are satisfied. Hence the line integral of $f(z)$ from any $z_0$ in $D$ to any $z$ in $D$ is independent of path in $D$. We keep $z_0$ fixed. Then this integral becomes a function of $z$, call it $F(z)$,

$$F(z) = \int_{z_0}^{z} f(z^*)\, dz^*, \tag{2}$$

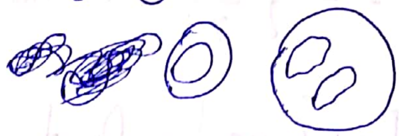
which is uniquely determined. We show that this $F(z)$ is analytic in $D$ and $F'(z) = f(z)$. The idea of doing this is as follows. We form the difference quotient

$$\begin{aligned} \frac{F(z+\Delta z) - F(z)}{\Delta z} &= \frac{1}{\Delta z}\left[\int_{z_0}^{z+\Delta z} f(z^*)\, dz^* - \int_{z_0}^{z} f(z^*)\, dz^*\right] \\ &= \frac{1}{\Delta z}\int_{z}^{z+\Delta z} f(z^*)\, dz^*, \end{aligned} \tag{3}$$

Domain:- In a complex function $f(z)$, the value of $z$ is called domain

<u>Types of Domain</u>:- There are two types of domain

① Simply connected domain. 

② Multiply connected domain. 

<u>Cauchy's Integral Theorem</u>:-

Statement:- If $f(z)$ is analytic in a simply connected Domain D, then for every simple closed path C in D.

$$I = \oint_C f(z)\,dz = 0$$

<u>Procedure</u>:- For cauchy's integral theorem, we will Prove the following conditions:

① If $f(z)$ is analytic then it will satisfy the integral theorem i.e $f(z) = u + iv$

$$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \ ; \ \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}.$$

② If all the singular points of given function lie outside the given region. then the function will satisfy Cauchy Integral theorem.

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# Ex 13.2

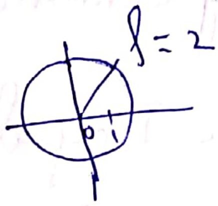
## Cauchy's Integral Theorem:-

Q17 ⟶ to ⟶ 25

Evaluate the following integral

**Q17** $I = \oint_C \frac{dz}{z-i}$ ; $C$ the circle $|z| = 2$ (c.c.w)

Sol:- For singular Point




$z - i = 0$

$z = i = (0,1)$

The singular point lies inside the given curve.

Now

$|z| = 2$

$z_0 = 0, \; \rho = 2$

$z(t) = z_0 + \rho e^{it} \quad 0 \le t \le 2\pi.$

$z(t) = 2e^{it}$

$z'(t) = 2ie^{it}$

$f[z(t)] = \frac{1}{2e^{it} - i}$

As $\oint_C f(z)dz = \int_a^b f[z(t)] \cdot z'(t) \, dt$

$$\oint_C f(z)\,dz = \int_0^{2\pi} \frac{2ie^{it}}{2e^{it}-i}\,dt\,.$$

$$= \ln[2e^{it}-i]\Big|_0^{2\pi}$$

$$= \ln[2\cos t + 2i\cancel{\sin t} - i]\Big|_0^{2\pi}$$

$$= \ln[2-i] - \ln[2-i]$$

$$\boxed{\oint_C f(z)\,dz = 0}$$

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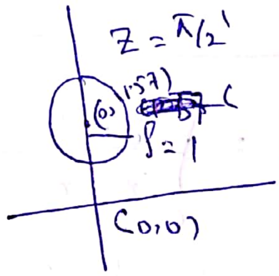
Q18 $I = \oint_C \frac{dz}{\sinh z}$ ; C the circle $|z - \pi i/2| = 1$ (C.W).

Sol: For singular Point

$\sinh z = 0$

$z = 0$

$(0,0)$



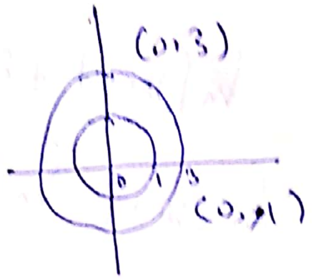

As the singular Point lies outside the given Path.

∴ According to cauchy's integral formula.

$$I = \oint_C \frac{dz}{\sinh z} = 0.$$

Q19 $\oint_c \frac{\cos z}{z}\,dz$, c consists of $|z|=1$ (C.C.W) & $z=|3|$ (C.W)

Sol:- For Singular Point




$z=0 \Rightarrow (0,0)$

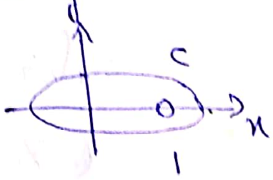
which clearly shows that the Singular Point lies outside the given ~~Point~~ region

According to cauchy's Integral theorem

$$I = \oint_c \frac{\cos z}{z}\,dz = 0.$$

---

Q20 $I = \oint_c \frac{2z-1}{z^2-z}\,dz$, c is contour.




Sol:- For Singular Point

$z^2 - z = 0$

$z(z-1) = 0$

$z=0,\quad z-1=0 \Rightarrow z=1$

$z=0,1$

$(0,0)\ ,\ (1,0)$

both the Singular Points lies inside the given Path

Using Partial fractions.

$$\frac{2z-1}{z^2-z} = \frac{2z-1}{z(z-1)}.$$

$$= \frac{A}{z} + \frac{B}{z-1}$$

$$2z-1 = A(z-1)+Bz$$

Put $z=0$, $\boxed{A=1}$

Put $z=1$, $\boxed{B=1}$

$$\frac{2z-1}{z^2-z} = \frac{1}{z} + \frac{1}{z-1}$$

$$I = \oint_c \left(\frac{1}{z} + \frac{1}{z-1}\right) dz$$

$$= \oint_c \frac{1}{z} dz + \oint_c \frac{1}{z-1} dz$$

We know that

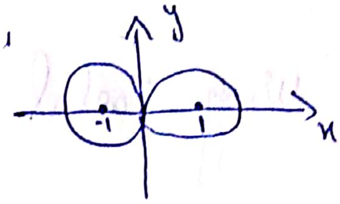
$$\oint_c (z-z_0)^{-m} dz = \begin{cases} 2\pi i & m=-1 \\ 0 & m \neq -1 \end{cases}$$

Now $\oint_c \frac{1}{z} dz = \oint_c z^{-1} dz = 2\pi i$ for $m=-1$

& $\oint_c \frac{1}{z-1} dz = \oint_c (z-1)^{-1} dz = 2\pi i$ for $m=-1$

$$I = 2\pi i + 2\pi i = \boxed{4\pi i}$$

21. $\oint_C \frac{dz}{z^2-1}$ ; c the contour




Sol:- For Singular Point

$z^2-1=0$

$z=\pm 1 \Rightarrow (1,0),(-1,0)$

∵ The singular points lie inside the given path

Now

$$\frac{1}{(z+1)(z-1)} = \frac{A}{z+1} + \frac{B}{z-1}$$

$$1 = A(z-1)+B(z+1)$$

Put $z=1$

$\boxed{B=1/2}$

Put $z=-1 \Rightarrow \boxed{A=-1/2}$

∴ The given integral becomes

$$I = \oint_C \left(\frac{-1/2}{1+z} + \frac{1/2}{z-1}\right) dz$$

$$= -1/2 \oint_C \frac{1}{z+1}\, dz + 1/2 \oint_C \frac{1}{z-1}\, dz$$

$$= -1/2 \oint_C (z+1)^{-1} dz + 1/2 \oint_C (z-1)^{-1} dz$$

$$I = -\tfrac{1}{2} \oint_C [z-(-1)]^{-1} dz + \tfrac{1}{2} \oint_C (z-1)^{-1} dz.$$

C.W $(z=-1)$ C.C.W $(z=1)$

As we know

$$\oint_C (z-z_0)^{-m} = \begin{cases} 2\pi i & m=-1 \\ 0 & m \neq -1 \end{cases}$$

$$I = -\tfrac{1}{2}(-2\pi i) + \tfrac{1}{2}(2\pi i)$$

$$I = \pi i + \pi i$$

$$\boxed{I = 2\pi i}$$

~ o ~ o ~

**Q22** $\oint_C \dfrac{2z+1}{z^2+z} dz$ C:

(a) $|z| = 1/4$ (C.W)

(b) $|z - 1/2| = 1/4$ (C.W)

(c) $|z| = 2$ (C.W).

Sol:- Now

$$\frac{2z+1}{z(z+1)} = \frac{A}{z} + \frac{B}{z+1}$$

$$2z+1 = A(z+1) + Bz$$

$$\Rightarrow \boxed{A=1} ; \boxed{B=1}$$

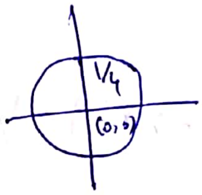
$$\frac{2z+1}{z(z+1)} = \frac{1}{z} + \frac{1}{z+1}$$

The given integral becomes

$$\oint_c f(z)\,dz = \oint_c \frac{1}{z}\,dz + \oint_c \frac{1}{z+1}\,dz \quad \text{—(1)}$$

(a) $|z| = 1/4$

$z_0 = 0, \quad \rho = 1/4$




As

$$\oint_c f(z)\,dz = \underbrace{\oint_c \frac{1}{z}\,dz}_{I_1} + \underbrace{\oint_c \frac{1}{z+1}\,dz}_{I_2}$$

Now for $I_1$

Put $z = 0$

$\because$ $(0,0)$ lies inside the given Path

Similarly for $I_2$

Put $z+1 = 0$

$z = -1$

$(-1,0)$ it is out side the given Path

According to cauchy Integral theorem

$$\oint_c \frac{1}{z+1}\,dz = 0$$

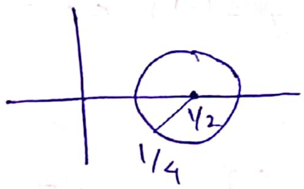
So $\oint_C f(z)\,dz = \oint_C \frac{1}{z}\,dz$

As $\oint_C \frac{1}{z}\,dz = -2\pi i$ (C.W)

$\oint_C f(z) = -2\pi i$

(b) $|z - 1/2| = 1/4$

$z_0 = 1/2$ ; $\rho = 1/4$



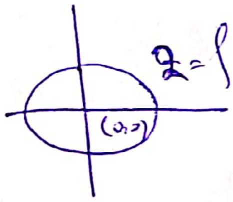

Here Both singular points $(0,0)$, $(-1,0)$ are outside the given path.

∴ According to cauchy integral formula

$\oint_C 1/z\,dz = 0$ ; $\oint_C \frac{1}{z+1}\,dz = 0$

(c) $|z| = 2$

Here $z_0 = 0$ ; $\rho = 2$




Now:

Both singular points $(0,0)$ + $(-1,0)$ are inside the given curve.

$$\oint_C f(z)\,dz = \oint_C \frac{1}{z}\,dz + \oint_C \frac{1}{z+1}\,dz.$$

Now $\oint_C \frac{1}{z}\,dz = -2\pi i$ (C.W)

& $\oint_C (z-z_0)^{-m} = \begin{cases} -2\pi i & m=-1 \quad \text{C.W} \\ 0 & m \neq -1 \end{cases}$

$$\oint_C (z-(-1))^{-1}\,dz = -2\pi i$$

$$\oint_C f(z)\,dz = -2\pi i - 2\pi i$$

$$\boxed{\oint_C f(z)\,dz = -4\pi i}$$

### Contour Integrals

**Evaluate (showing the details and using a partial fraction representation of the integrand if necessary)**

**17.** $\oint_C \frac{dz}{z - 3i}$, $C$ the circle $|z| = \pi$, counterclockwise

**18.** $\oint_C \text{Ln}\,(1 - z)\,dz$, $C$ the boundary of the parallelogram with vertices $\pm i$, $\pm(1 + i)$

**19.** $\oint_C \frac{e^z}{z}\,dz$, $C$ consists of $|z| = 2$ (counterclockwise) and $|z| = 1$ (clockwise)

**20.** $\oint_C \text{Re}\, z\,dz$, $C$ the contour in Fig. 339

**21.** $\oint_C \frac{dz}{z^2 - 1}$, $C$ the contour in Fig. 340

**22.** $\oint_C \frac{2z - 1}{z^2 - z}\,dz$, $C$ the contour in Fig. 341

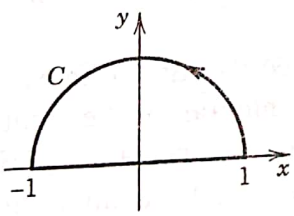



**Fig. 339.** Problem 20

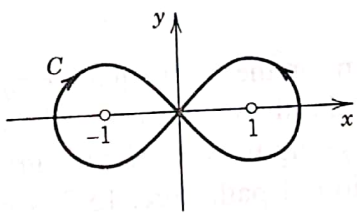



**Fig. 340.** Problem 21

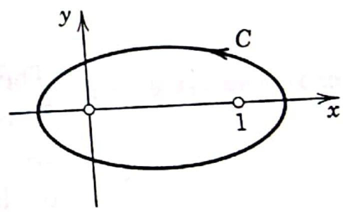



**Fig. 341.** Problem 22

**23.** $\oint_C \frac{dz}{z^2 + 1}$, $C$: (a) $|z + i| = 1$, (b) $|z - i| = 1$, counterclockwise

**24.** $\oint_C \coth \tfrac{1}{2}z\,dz$, $C$ the circle $|z - \tfrac{1}{2}\pi i| = 1$, clockwise

**25.** $\oint_C \frac{2z^3 + z^2 + 4}{z^4 + 4z^2}\,dz$, $C$ the circle $|z - 2| = 4$, clockwise