Ans. $z = 1.609 + 0.927i \pm 2n\pi i$ $(n = 0, 1, 2, \cdots)$. These are infinitely many solutions (due to the periodicity of e^z). They lie on the vertical line $x = 1.609$ at a distance 2π from their neighbors. ◀

To summarize: many properties of $e^z = \exp z$ parallel those of e^x; an exception is the periodicity of e^z with $2\pi i$, which suggested the concept of a fundamental region. Keep in mind that e^z is an *entire function*. (Do you still remember what that means?)

PROBLEM SET 12.6

Function Values. Find e^z (in the form $u + iv$) and $|e^z|$ if z equals

1. $2 + 3\pi i$ **2.** $1 + i$ **3.** $2\pi(1 + i)$ **4.** $0.95 - 1.6i$ **5.** $-\pi i/2$

Polar Form (6). Represent each of the following in the "exponential polar form" (6):

6. $1 + i$ **7.** $4 + 3i$ **8.** $\sqrt[n]{z}$ **9.** -4 **10.** $\sqrt{i}$, $\sqrt{-i}$

Equations. Find all solutions and plot some of them in the complex plane.

11. $e^z = 1$ **12.** $e^{2z} = 2$ **13.** $e^z = -3$ **14.** $e^z = 4 + 3i$ **15.** $e^z = 0$

Conformal Mapping $w = e^z$. Find and sketch or plot the image of the given region.

16. $-1 \leqq x \leqq 1, -\pi < y < \pi$ **17.** $0 < y < \frac{1}{2}\pi$ **18.** $\pi < y \leqq 3\pi$ **19.** $\ln 3 < x < \ln 5$

20. TEAM PROJECT. Further Properties of the Exponential Function. (a) Analyticity. Show that e^z is entire. What about $e^{1/z}$? $e^{\bar{z}}$? $e^x(\cos kx + i \sin kx)$? (Use the Cauchy-Riemann equations.)

(b) Special Values. Find all z such that (i) e^z is real, (ii) $|e^{-z}| < 1$, (iii) $e^{\bar{z}} = \overline{e^z}$.

(c) Harmonic Function. Show that $u = e^{xy} \cos(x^2/2 - y^2/2)$ is harmonic and find a conjugate.

(d) Uniqueness. It is interesting that $f(z) = e^z$ is uniquely determined by the two properties $f(x + i0) = e^x$ and $f'(z) = f(z)$, where f is assumed to be entire. Prove this using the Cauchy–Riemann equations.

12.7 Trigonometric Functions Hyperbolic Functions

Just as e^z extends e^x to complex, we want the *complex* trigonometric functions to extend the familiar *real* trigonometric functions. The idea of making the connection is the use of the Euler formulas (Sec. 12.6)

$$e^{ix} = \cos x + i \sin x, \qquad e^{-ix} = \cos x - i \sin x.$$

By addition and subtraction we obtain for the *real* cosine and sine

$$\cos x = \frac{1}{2}(e^{ix} + e^{-ix}), \qquad \sin x = \frac{1}{2i}(e^{ix} - e^{-ix}).$$

This suggests the following definitions for complex values $z = x + iy$:

$$\boxed{\cos z = \frac{1}{2}(e^{iz} + e^{-iz}), \qquad \sin z = \frac{1}{2i}(e^{iz} - e^{-iz}).} \tag{1}$$

(54)

Problem set 12.6

Exponential Function:- We begin with one of the most important analytic functions, the complex exponential function e^z, also written $\exp z$.

The definition of e^z in term of real function e^x, $\cos y$ + $\sin y$ is

$$\exp z = e^z = e^{x+iy} = e^x e^{iy} = e^x[\cos y + i\sin y]$$

$$\boxed{e^z = e^x[\cos y + i\sin y]}$$

① Using the CRE, show that e^z is analytic for all z.

~~compute e^z in the form $(u+iv)$ and the $|e^z|$ of z equals~~

Sol:- Since $f(z) = e^z \longrightarrow (A)$

Also $w = u+iv$

The given function is analytic if it satisfies CRE

Now $z = x+iy$

$$u+iv = e^z = e^{x+iy} = e^x[\cos y + i\sin y]$$

$$= e^x\cos y + i e^x \sin y$$

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comparing both sides

$u = e^x \cos y$ & $v = e^x \sin y$.

Now $u_x = e^x \cos y$ & $u_y = -e^x \sin y$.

Similarly

$v_x = e^x \sin y$ & $v_y = e^x \cos y$

Now CRE's are given by

① $u_x = v_y$

$e^x \cos y = e^x \cos y$ satisfied

② $u_y = -v_x$

$-e^x \sin y = -e^x \sin y$ Satisfied

Hence CRE's are Satisfied, So the given function is analytic.

(56)

Compute e^z (in the form $u+iv$) & $|e^z|$ if z equals.

Q2 —— to ——> Q9

Q2 $3+\pi i$

Sol:- let $z = 3+\pi i$

Now $e^z = e^{3+\pi i} = e^3 . e^{\pi i} = e^3[\cos\pi + i\sin\pi]$

$= e^3\cos\pi + i e^3\sin\pi$

Also $u+iv = e^z = e^3\cos\pi + i e^3\sin\pi$

comparing

$u = e^3\cos\pi$ & $v = e^3\sin\pi$

$\boxed{u = -20.08}$ $\boxed{v = 0}$

Also $e^z = e^3\cos\pi + i e^3\sin\pi$

$|e^z| = \sqrt{(e^3\cos\pi)^2 + (e^3\sin\pi)^2}$

$= e^3\sqrt{\cos^2\pi + \sin^2\pi}$

$= e^3$

$\boxed{|e^z| = 20.08}$

5

Q3 $1+i$

let $z = 1+i$

$$e^{z} = e^{1+i} = e^{1} \cdot e^{i} = e\left[\cos(1) + i\sin(1)\right]$$

$$= e\cos(1) + i\, e\sin(1)$$

$$= 1.468 + i\, 2.287$$

As $u + iv = e^{z} = 1.468 + i\, 2.287$

comparing

$\boxed{u = 1.468}$; $\boxed{v = 2.287}$

Now $$|e^{z}| = \sqrt{(e\cos(1))^2 + (e\sin(1))^2}$$

$$= e\sqrt{\cos^2(1) + \sin^2(1)}$$

$$= e(1)$$

$$\boxed{|e^{z}| = 2.718}$$

Q4: $2+5\pi i$ Q5: $\sqrt{5} - \frac{1}{2}i$ Q6: $7\pi i/2$

Q7: $(1+i)\pi$ Q8: $-1+1.4i$ Q9: $\mp 9\pi i/2$

Find the real and imaginary Parts of

Q10 —— to ——> Q13

Q10 let $f(z) = e^{-z^2}$

Since $z = x+iy$

$$z^2 = (x+iy)^2 = (x^2-y^2) + i2xy.$$

$$f(z) = e^{-z^2} = e^{-[(x^2-y^2)+i2xy]}$$

$$= e^{-(x^2-y^2)} \cdot e^{-i2xy}$$

$$= e^{-(x^2-y^2)}[\cos 2xy - i\sin 2xy]$$

$$f(z) = e^{-(x^2-y^2)}\cos 2xy - i e^{-(x^2-y^2)}\sin 2xy$$

$$\boxed{Re[f(z)] = e^{-(x^2-y^2)}\cos 2xy}$$

&

$$\boxed{Im[f(z)] = -e^{-(x^2-y^2)}\sin 2xy}$$

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Q13 $f(z) = e^{-2z}$

Since $z = x+iy$

$f(z) = e^{-2z} = e^{-2(x+iy)} = e^{-2x-2iy} = e^{-2x} \cdot e^{-2iy}$

$= e^{-2x}[\cos 2y - i\sin 2y]$

$f(z) = e^{-2x}\cos 2y - e^{-2x} i\sin 2y$

$$\boxed{Re[f(z)] = e^{-2x}\cos 2y}$$

$$\boxed{Im[f(z)] = -e^{-2x}\sin 2y}$$

Q 1~~2~~ $f(z) = e^{z^3}$

Q12 $f(z) = e^{-\pi z}$.

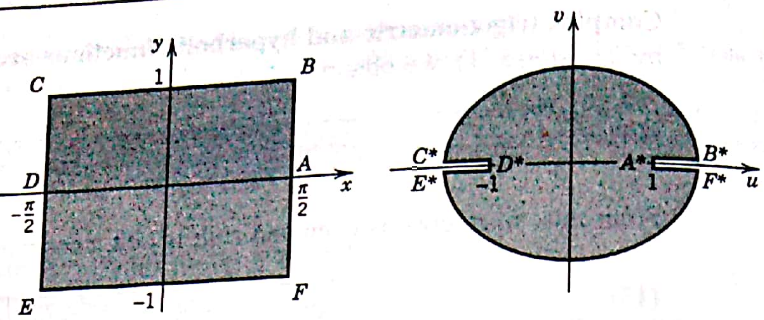

Fig. 317. Mapping by $w = \sin z$

Cosine. The mapping $w = \cos z$ could be discussed independently, but since

$$w = \cos z = \sin\left(z + \tfrac{1}{2}\pi\right),$$

we see at once that this is the same mapping as $\sin z$ preceded by a translation to the right through $\frac{1}{2}\pi$ units.

Hyperbolic sine. Since

$$w = \sinh z = -i \sin(iz),$$

the mapping is a counterclockwise rotation $Z = iz$ through $\frac{1}{2}\pi$ (i.e., 90°), followed by the sine mapping $Z^* = \sin Z$, followed by a clockwise 90°-rotation $w = -iZ^*$.

Hyperbolic cosine. This function

$$w = \cosh z = \cos(iz)$$

defines a mapping that is a rotation $Z = iz$ followed by the mapping $w = \cos Z$.

The mapping by $w = \tan z$ is more tricky. We shall discuss it in Sec. 12.9.

PROBLEM SET 12.7

Formulas for Hyperbolic Functions. Show that

1. $$\boxed{\begin{aligned}\cosh z &= \cosh x \cos y + i \sinh x \sin y,\\ \sinh z &= \sinh x \cos y + i \cosh x \sin y.\end{aligned}}$$

2. $$\boxed{\begin{aligned}\cosh(z_1 + z_2) &= \cosh z_1 \cosh z_2 + \sinh z_1 \sinh z_2,\\ \sinh(z_1 + z_2) &= \sinh z_1 \cosh z_2 + \cosh z_1 \sinh z_2.\end{aligned}}$$

3. $\cosh^2 z - \sinh^2 z = 1, \qquad \cosh^2 z + \sinh^2 z = \cosh 2z$

Function Values. Find (in the form $u + iv$)

4. $\cos(1 + i)$ **5.** $\sin \pi i$ **6.** $\cos\left(\frac{1}{2}\pi - \pi i\right)$ **7.** $\cosh(-3 - 6i)$ **8.** $\sinh(4 + 5i)$

Equations. Find all solutions of the following equations.

9. $\cos z = 3i$ **10.** $\cosh z = 0$ **11.** $\cosh z = \frac{1}{2}$ **12.** $\sin z = 1000$ **13.** $\sin z = \cosh 3$

Conformal Mapping $w = \sin z$. Find and sketch or plot the image of the region

14. $0 < x < \pi/6$, y arbitrary **15.** $0 < x < \pi/2, 0 < y < 2$ **16.** $-\pi/4 < x < \pi/4, 0 < y < 3$

Conformal Mapping $w = \cos z$. Find and sketch or plot the image of the region

17. $0 < x < \pi, 0 < y < 1$ **18.** $0 < x < 2\pi, \frac{1}{2} < y < 1$ **19.** $0 < x < \pi, y > 0$

20. CAS PROJECT. Orthogonal Nets of Level Curves. Let $w = u + iv = f(z)$. Plot the orthogonal net of the two families of level curves $u = \text{Re } f(z) = const$ and $v = \text{Im } f(z) = const$ where $f(z)$ equals (a) $\cos z$, (b) $\sin z$, (c) $\cosh z$, (d) $\sinh z$. Then try to find the relations between the four functions experimentally by looking at the four plots. Can you discover these relations completely from the plots?

12.8 Logarithm General Power

As the last of the functions we introduce the *complex logarithm*, which is more complicated than the real logarithm (which it includes as a special case) and historically puzzled mathematicians for some time (so if you first get puzzled—which need not happen!—be patient and work this section with extra care).

The **natural logarithm** of $z = x + iy$ is denoted by $\ln z$ (sometimes also by $\log z$) and is defined as the inverse of the exponential function; that is, $w = \ln z$ is defined for $z \neq 0$ by the relation

$$e^w = z.$$

(Note that $z = 0$ is impossible, since $e^w \neq 0$ for all w; see Sec. 12.6.) If we set $w = u + iv$ and $z = re^{i\theta}$, this becomes

$$e^w = e^{u+iv} = re^{i\theta}.$$

Now from Sec. 12.6 we know that e^{u+iv} has the absolute value e^u and the argument v. These must be equal to the absolute value and argument on the right:

$$e^u = r, \qquad v = \theta.$$

$e^u = r$ gives $u = \ln r$, where $\ln r$ is the familiar *real* natural logarithm of the positive number $r = |z|$. Hence $w = u + iv = \ln z$ is given by

$$\text{(1)} \qquad \boxed{\ln z = \ln r + i\theta} \qquad (r = |z| > 0, \quad \theta = \arg z).$$

Now comes an important point (without analog in real calculus). Since the argument of z is determined only up to integer multiples of 2π, ***the complex natural logarithm*** $\ln z$ ***($z \neq 0$) is infinitely many-valued.***

The value of $\ln z$ corresponding to the principal value $\text{Arg } z$ (see Sec. 12.2) is denoted by $\text{Ln } z$ (Ln with capital L) and is called the **principal value** of $\ln z$. Thus

$$\text{(2)} \qquad \boxed{\text{Ln } z = \ln |z| + i \text{ Arg } z} \qquad (z \neq 0).$$

Trigonometric functions, Hyperbolic function

Since $e^{ix} = \cos x + i\sin x$ —①

& $e^{-ix} = \cos x - i\sin x \longrightarrow$ ②

adding eq ① & eq ② we get

$e^{ix} + e^{-ix} = \cos x + \cancel{i\sin x} + \cos x - \cancel{i\sin x}$

$2\cos x = e^{ix} + e^{-ix}$

$\Rightarrow \boxed{\cos x = \frac{1}{2}(e^{ix} + e^{-ix})}$

Subtraction we get eq ① & eq ②

$e^{ix} - e^{-ix} = \cancel{\cos x} + i\sin x - \cancel{\cos x} + i\sin x$

$2i\sin x = e^{ix} - e^{-ix}$

$\boxed{\sin x = \frac{1}{2i}(e^{ix} - e^{-ix})}$

for complex no. Replacing x by z

$\boxed{\cos z = \frac{1}{2}(e^{iz} + e^{-iz})}$, $\boxed{\sin z = \frac{1}{2i}(e^{iz} - e^{-iz})}$

$\boxed{\tan z = \frac{\sin z}{\cos z}}$, $\boxed{\cot z = \frac{\cos z}{\sin z}}$

$\boxed{\sec z = \frac{1}{\cos z}}$; $\boxed{\operatorname{cosec} z = \frac{1}{\sin z}}$

$$\boxed{e^{iz} = \cos z + i \sin z}$$

$$\boxed{\cos z = \cos x \cosh y - i \sin x \sinh y}$$

$$\boxed{\sin z = \sin x \cosh y + i \cos x \sinh y}$$

Complex Hyperbolic Functions:

Def:- The hyperbolic functions having complex numbers z are called complex hyperbolic functions.

The complex hyperbolic functions are given by:

① $\cosh z = \frac{1}{2}(e^{z} + e^{-z})$

② $\sinh z = 1/2\,(e^{z} - e^{-z})$

③ $\tanh z = \frac{\sinh z}{\cosh z} = \frac{e^{z} - e^{-z}}{e^{z} + e^{-z}}$.

④ $\coth z = \frac{1}{\tanh z}$

(5) $\operatorname{sech} z = \frac{1}{\cosh z}$.

(6) $\operatorname{cosech} z = \frac{1}{\sinh z}$.

Relation between complex Trigonometric & Hyperbolic functions:-

(1) $\cosh iz = \cos z$

(2) $\sinh iz = i \sin z$

(3) $\cos iz = \cosh z$

(4) $\sin iz = i \sinh z$.

(1) Prove $\cosh iz = \cos z$

Proof $\cosh iz = \frac{1}{2}(e^{iz} + e^{-iz})$ — (1)

& $\cos z = \frac{1}{2}(e^{iz} + e^{-iz})$ — (2)

comparing eq (1) & eq (2) we get

$$\boxed{\cosh iz = \cos z}$$

(2) $\sinh iz = i \sin z$

Proof:- $\sinh iz = \frac{1}{2}\left[e^{iz} - e^{-iz}\right]$ —— (1)

& $i \sin z = i\left[\frac{1}{2i}\left[e^{iz} - e^{-iz}\right]\right]$

$= \frac{1}{2}\left[e^{iz} - e^{-iz}\right]$ —— (2)

comparing eq (1) & eq (2) we get

$$\boxed{\sinh iz = i \sin z}$$

(3) $\cos iz = \cosh z$

Proof:- $\cos iz$

$= \frac{1}{2}\left(e^{i(iz)} + e^{-i(iz)}\right]$

$= \frac{1}{2}\left[e^{-z} + e^{z}\right]$

$= \frac{1}{2}\left[e^{z} + e^{-z}\right]$

$$\boxed{\cos iz = \cosh z}$$

(4) $\sin iz = i\sinh z$

Proof:- $\sin iz = \frac{1}{2i}\left(e^{i(iz)} - e^{-i(iz)}\right)$

$= \frac{1}{2i} \times \frac{i}{i}\left[e^{-z} - e^{z}\right]$

$= \frac{-i}{2}\left[e^{-z} - e^{z}\right]$

$= \frac{i}{2}\left[e^{z} - e^{-z}\right]$

$= i\left[\frac{1}{2}\left(e^{z} - e^{-z}\right)\right]$

$$\boxed{\sin iz = i\sinh z}$$

Prove

1) $\cos z = \cos x\cosh y - i\sin x\sinh y$.

Proof:- Let $z = x+iy$

$\cos z = \cos(x+iy)$

$= \cos x \cdot \cos(iy) - \sin x \cdot (\sin(iy))$

Since $\cos iy = \cosh y$

& $\sin iy = i\sinh y$.

$\cos z = \cos x \cosh y - \sin x \, (i \sinh y)$

$$\boxed{\cos z = \cos x \cosh y - i \sin x \sinh y}$$

0

② $\sin z = \sin x \cosh y + i \cos x \sinh y$.

Proof $z = x+iy$

$\sin z = \sin(x+iy)$

$= \sin x \cos(iy) + \cos x \sin(iy)$.

$= \sin x \cosh y + \cos x \, (i \sinh y)$

$$\boxed{\sin z = \sin x \cosh y + i \cos x \sinh y.}$$

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Exercise – 12.7

Show that

① (i) $\cosh z = \cosh x \cos y + i \sinh x \sin y$

Proof:- Let $z = x+iy$

$\cosh z = \cosh(x+iy)$

$= \cosh x \cosh(iy) + \sinh x \sinh(iy)$

As $\cosh iy = \cos y$ & $\sinh(iy) = i\sin y$.

$= \cosh x \cos y + \sinh x \cdot i \sin y$.

$$\boxed{\cosh z = \cosh x \cos y + i \sinh x \sin y}$$

(ii) $\sinh z = \sinh x \cos y + i \cosh x \sin y$

Proof:- Let $z = x+iy$

$\sinh z = \sinh(x+iy)$

$= \sinh x \cosh(iy) + \cosh x \sinh(iy)$

$= \sinh x \cos y + \cosh x \cdot i \sin y$

$$\boxed{\sinh z = \sinh x \cos y + i \cosh x \sin y}$$

Q2 (i) $\cosh(z_1+z_2) = \cosh z_1 \cosh z_2 + \sinh z_1 \sinh z_2$

Proof:- Since

$$\cos iz = \cosh z \quad \& \quad \sin iz = i\sinh z.$$

Now $\cos(iz_1)\cdot\cos(iz_2) = \cosh z_1 \cosh z_2$ —— (1)

Similarly

$$\sin(iz_1)\cdot\sin(iz_2) = i\sinh z_1 \cdot i\sinh z_2$$
$$= i^2 \sinh z_1 \sinh z_2$$
$$= -\sinh z_1 \sinh z_2 \quad \text{—— (2)}$$

Subtracting eq (2) from eq (1)

$$\cos(iz_1)\cdot\cos(iz_2) - \sin(iz_1)\sin(iz_2) = \cosh z_1 \cosh z_2 + \sinh z_1 \sinh z_2$$

$$\cos i(z_1+z_2) = \cosh z_1 \cosh z_2 + \sinh z_1 \sinh z_2$$

$$\boxed{\cosh(z_1+z_2) = \cosh z_1 \cosh z_2 + \sinh z_1 \sinh z_2}$$

Q2 (ii) $\sinh(z_1+z_2) = \sinh z_1 \cosh z_2 + \cosh z_1 \sinh z_2$

Proof: Now

$\sin(iz_1)\cos(iz_2) = i\sinh z_1 \cosh z_2$ —— (1)

Similarly

$$\cos(iz_1)\sin(iz_2) = \cosh z_1 \cdot i\sinh z_2$$
$$= i\cosh z_1 \sinh z_2 \quad \text{—— (2)}$$

Adding eq ① & eq ② we get

$$\sin(iz_1)\cos(iz_2) + \cos(iz_1)\sin(iz_2) = i\sinh z_1 \cosh z_2 + i\cosh z_1 \sinh z_2.$$

$$\sin i(z_1+z_2) = i[\sinh z_1 \cosh z_2 + \cosh z_1 \sinh z_2].$$

$$\cancel{i}\sinh(z_1+z_2) = \cancel{i}[\sinh z_1 \cosh z_2 + \cosh z_1 \sinh z_2]$$

$$\boxed{\sinh(z_1+z_2) = \sinh z_1 \cosh z_2 + \cosh z_1 \sinh z_2}$$

Q3 $\cosh^2 z - \sinh^2 z = 1$

Proof: $\cosh^2 z - \sinh^2 z$

$$= \left(\frac{e^z + e^{-z}}{2}\right)^2 - \left(\frac{e^z - e^{-z}}{2}\right)^2$$

$$= \tfrac{1}{4}\left[e^{2z} + e^{-2z} + 2\right] - \tfrac{1}{4}\left[e^{2z} + e^{-2z} - 2\right]$$

$$= \tfrac{1}{4}\left[\cancel{e^{2z}} + \cancel{e^{-2z}} + 2 - \cancel{e^{2z}} - \cancel{e^{-2z}} + 2\right]$$

$$= \tfrac{1}{\cancel{4}}[\cancel{4}]$$

$$= 1$$

$$\boxed{\cosh^2 z - \sinh^2 z = 1}$$ Proved.

Q3 $\cosh^2 z + \sinh^2 z = \cosh 2z$

S.Y.S

~ o ~ o ~ o ~ o

Find (in the form $u+iv$).

Q4 — to — Q8

Q4 $\cos(1.7+1.5i)$

Sol:- $f(z) = \cos(1.7+1.5i)$

$= \cos(1.7)\cos(1.5i) - \sin(1.7)\sin(1.5i)$

As $\cos i\theta = \cosh\theta$

& $\sin i\theta = i\sinh\theta$.

$f(z) = \cos(1.7)\cosh(1.5) - \sin(1.7)(i\sinh(1.5))$

$= \cos(1.7)\cosh(1.5) - i\sin(1.7)\sinh(1.5)$

$= (-0.1288)(2.352) - i(0.991)(2.129)$

$$\boxed{f(z) = -0.3029 - 2.11i}$$