CLASS 2

(**Sections 1.3**)

Exponential and Sinusoidal Signals

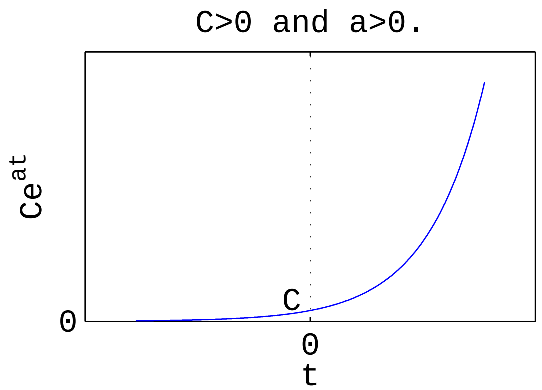
- They arise frequently in applications, and many other signals can be constructed from them.

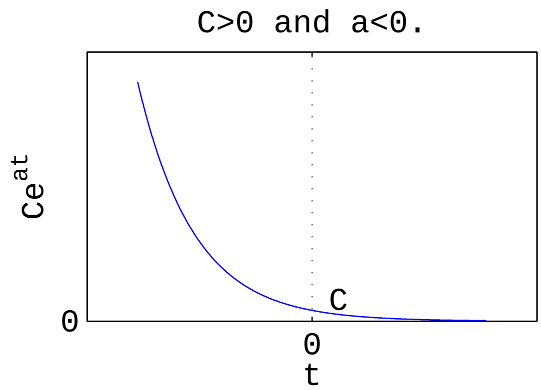
Continuous-time complex exponential and sinusoidal signals:

$$x(t) = Ce^{at}$$

where C and a are in general complex numbers.

Real exponential signals: C and a are reals.





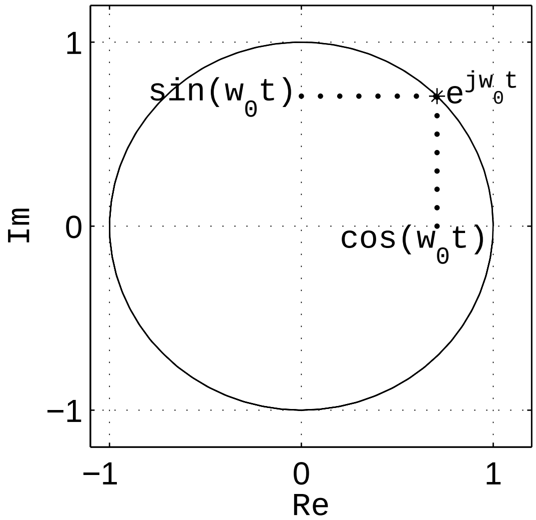
- The case $a > 0$ represents exponential growth. Some signals in unstable systems exhibit exponential growth.
- The case $a < 0$ represents exponential decay. Some signals in stable systems exhibit exponential decay.

Periodic complex exponential:

$$e^{jw_0t}$$

where $j=\sqrt{-1}$, $w_0 \neq 0$ is real, and t is the time.

Euler's formula: $e^{jw_0t} = \underbrace{\cos(w_0t)}_{\mathrm{Re}\{e^{jw_0t}\}} + j\underbrace{\sin(w_0t)}_{\mathrm{Im}\{e^{jw_0t}\}}$. Note that



- $|e^{jw_0t}| = 1$ and $\angle e^{jw_0t} = w_0t$.
- $e^{j2\pi k} = 1$, for $k = 0, \pm1, \pm2, \ldots$.

Since

$$e^{jw_0\left(t+\frac{2\pi}{|w_0|}\right)} = e^{jw_0t}e^{j2\pi\frac{w_0}{|w_0|}} = e^{jw_0t}\underbrace{e^{j2\pi\mathrm{sign}(w_0)}}_{=1} = e^{jw_0t}$$

we have

$$e^{jw_0t} \text{ is periodic with fundamental period } \frac{2\pi}{|w_0|}.$$

Note that

- e^{jw_0t} and e^{-jw_0t} have the same fundamental period.
- Energy in e^{jw_0t}: $\int_{-\infty}^{\infty}|e^{jw_0t}|dt = \int_{-\infty}^{\infty}1.dt = \infty$
- Average Power in e^{jw_0t}: $\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}|e^{jw_0t}|dt = \lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}1.dt = 1$.
- $\{e^{jkw_0t}\}_{k=0,\pm1,\ldots}$, are all periodic with period $\frac{2\pi}{|w_0|}$. They are called a harmonically related set of complex exponentials with e^{jkw_0t} being the $k-$th harmonic.

Sinusoidal signals:

$$A\cos(w_0 t + \phi) \qquad \text{and} \qquad A\sin(w_0 t + \phi).$$

where A is real, w_0 is real, ϕ is real, and t is the time. (Graph one of the signals!)

- They arise in systems that conserve energy such as an ideal LC circuit or an ideal mass-spring system.
-
  - Periodic with the same fundamental period $T_0 = 2\pi/|w_0|$
  - $|w_0|$ is the fundamental frequency
  - $f_0 := 1/T_0 = |w_0|/(2\pi)$ is the number of cycles per unit time (large f_0 means more oscillatory)
  - $|A|$ is the amplitude
  - $|\phi|$ is the size of the phase shift.
- Since

  $$e^{j(w_0 t+\phi)} = \cos(w_0 t + \phi) + j\sin(w_0 t + \phi)$$

  we can write

  $$\begin{aligned} A\cos(w_0 t + \phi) &= A\text{Re}(e^{j(w_0 t+\phi)}) \\ A\sin(w_0 t + \phi) &= A\text{Im}(e^{j(w_0 t+\phi)}). \end{aligned}$$

- Recall, for any complex number z,

  $$z = \text{Re}(z) + j\text{Im}(z) \qquad z^* = \text{Re}(z) - j\text{Im}(z)$$

  therefore

  $$\text{Re}(z) = \frac{z+z^*}{2} \qquad \text{Im}(z) = \frac{z-z^*}{2j}.$$

  Hence, we can also write

  $$\begin{aligned} A\cos(w_0 t + \phi) &= \frac{A}{2}\left(e^{j(w_0 t+\phi)} + \left(e^{j(w_0 t+\phi)}\right)^*\right) = \frac{A}{2}\left(e^{j(w_0 t+\phi)} + e^{-j(w_0 t+\phi)}\right) \\ &= \frac{A}{2}e^{j\phi}e^{jw_0 t} + \frac{A}{2}e^{-j\phi}e^{-jw_0 t} \\ A\sin(w_0 t + \phi) &= \frac{A}{2j}\left(e^{j(w_0 t+\phi)} - \left(e^{j(w_0 t+\phi)}\right)^*\right) = \frac{A}{2}e^{-j\pi/2}\left(e^{j(w_0 t+\phi)} - e^{-j(w_0 t+\phi)}\right) \\ &= \frac{A}{2}e^{j(\phi-\pi/2)}e^{jw_0 t} - \frac{A}{2}e^{-j(\phi+\pi/2)}e^{-jw_0 t}. \end{aligned}$$

General complex exponential signals:

$$Ce^{at}$$

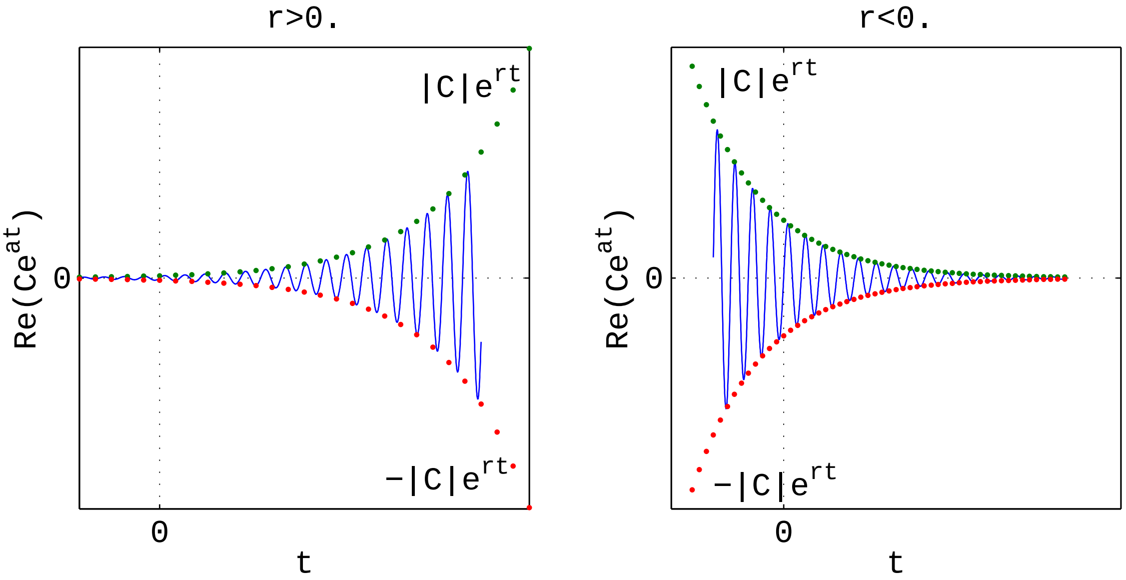
where C and a are complex numbers.

If

$$C = |C|e^{j\theta} \qquad \text{and} \qquad a = r + jw_0$$

then

$$Ce^{at} = |C|e^{j\theta}e^{(r+jw_0)t} = |C|e^{rt}e^{j(w_0t+\theta)} = \underbrace{|C|e^{rt}\cos(w_0t+\theta)}_{\mathrm{Re}(Ce^{at})} + j\,\underbrace{|C|e^{rt}\sin(w_0t+\theta)}_{\mathrm{Im}(Ce^{at})}.$$



- If $r = 0$, the real and imaginary part are sinusoidals.
- If $r > 0$, the real and imaginary part are sinusoidals multiplied by a growing exponential.

  Such signals arise in unstable systems.
- If $r < 0$, the real and imaginary part are sinusoidals multiplied by a decaying exponential.

  Such signals arise in stable systems, for example, in RLC circuits, or in mass-spring-friction system, where the energy is dissipated due to the resistors, friction, etc.

Discrete-time complex exponential and sinusoidal signals:

$$x[n] = Ce^{\beta n}$$

where C and β are complex numbers.

Analogous to the continuous-time case with the following differences: (w_0 is real below)

- $e^{jw_0 t} = e^{jw_1 t}$ are different signals if $w_0 \neq w_1$, whereas

  $$e^{jw_0 n} = e^{jw_1 n} \qquad \text{if} \qquad w_0 - w_1 = 2k\pi, \text{ for some } k \in \{0, \pm 1, \dots\}.$$

  (Explain this on the unit circle!)

  Therefore, it is sufficient to consider only the case $w_0 \in [0, 2\pi)$ or $w_0 \in [-\pi, \pi)$.

- As w_0 increases $e^{jw_0 n}$ oscillates at higher frequencies, whereas this is not the case for $e^{jw_0 n}$.

  In the figure below, the frequency of oscillations increases as w_0 changes from 0 to π then it decreases as w_0 changes from π to 2π.

- $e^{jw_0 t}$ is periodic with fundamental period $2\pi/|w_0|$, whereas

  $$\begin{aligned} e^{jw_0 n} \text{ is periodic} \quad &\Leftrightarrow \quad e^{jw_0 n} = e^{jw_0(n+M)} \text{ for some integer } M > 0, \text{ for all } n \\ &\Leftrightarrow \quad e^{jw_0 M} = 1 \text{ for some integer } M > 0 \\ &\Leftrightarrow \quad w_0 M = 2\pi m \text{ for some integers } m, M > 0 \\ &\Leftrightarrow \quad \frac{w_0}{2\pi} \text{ is rational.} \end{aligned}$$

- If $\frac{w_0}{2\pi} = \frac{m}{M}$ for some integers m and M which have no common factors, then the fundamental period is $M = \frac{2m\pi}{w_0}$ because

  $$e^{jw_0(n+N)} = e^{jw_0 n} e^{j\frac{2\pi m}{M} N}.$$

The same observations hold for discrete-time sinusoids.

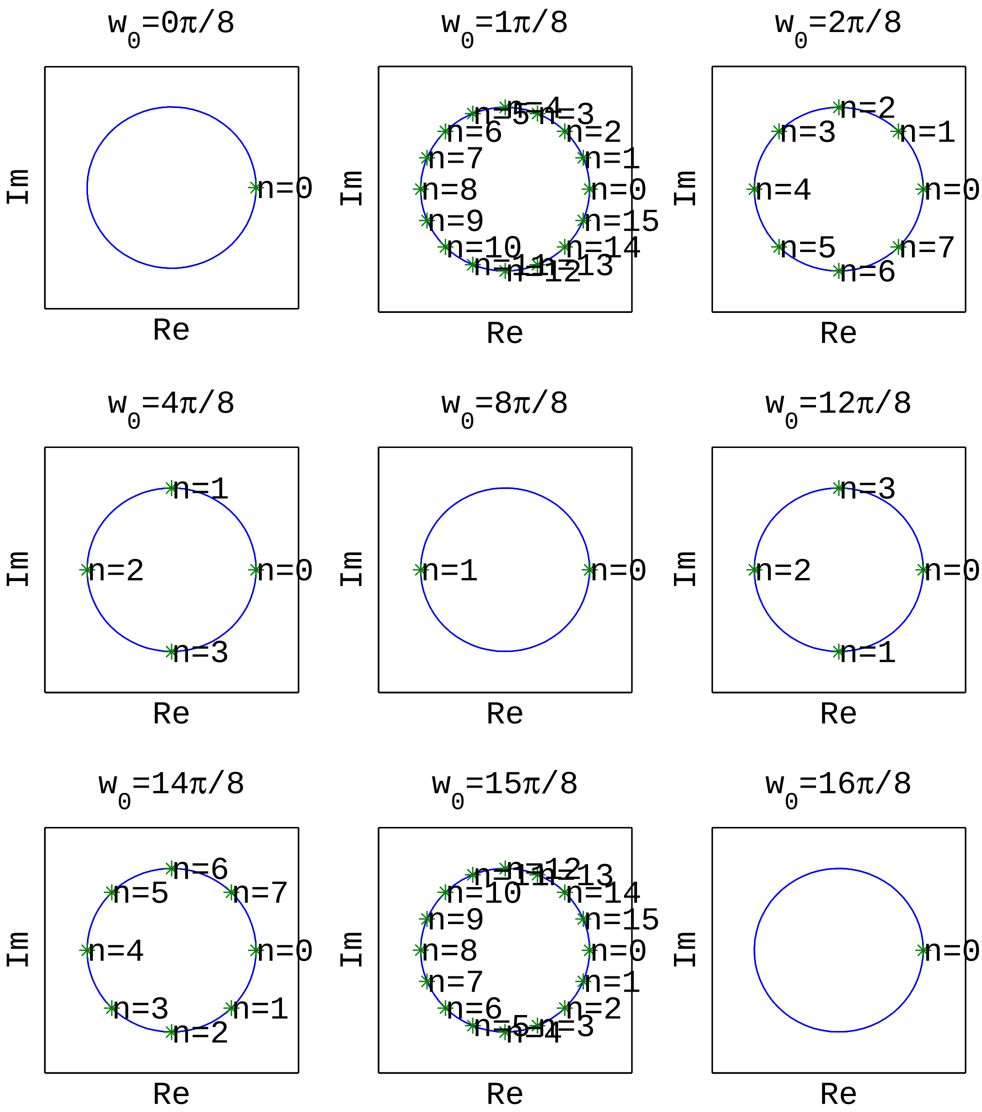


Fig. 1. To determine the fundamental period, count the number of steps to get back to 1!

Examples:

1) Is $x[n] = e^{jn2\pi/3} + e^{jn3\pi/4}$ periodic? If it is periodic, what's its fundamental period?

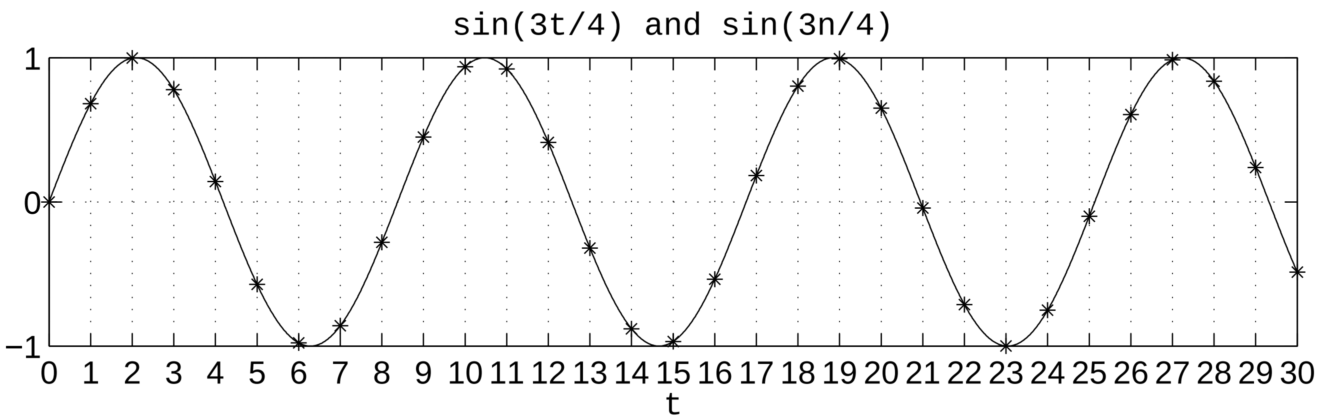
   For $e^{jn2\pi/3}$, $w_0/(2\pi) = 1/3$, so $e^{jn2\pi/3}$ is periodic with fundamental period 3.

   For $e^{jn3\pi/4}$, $w_0/(2\pi) = 3/8$, so $e^{jn3\pi/4}$ is periodic with fundamental period 8.

   $x[n]$ is periodic with fundamental period $24 = lcm(3, 8)$.

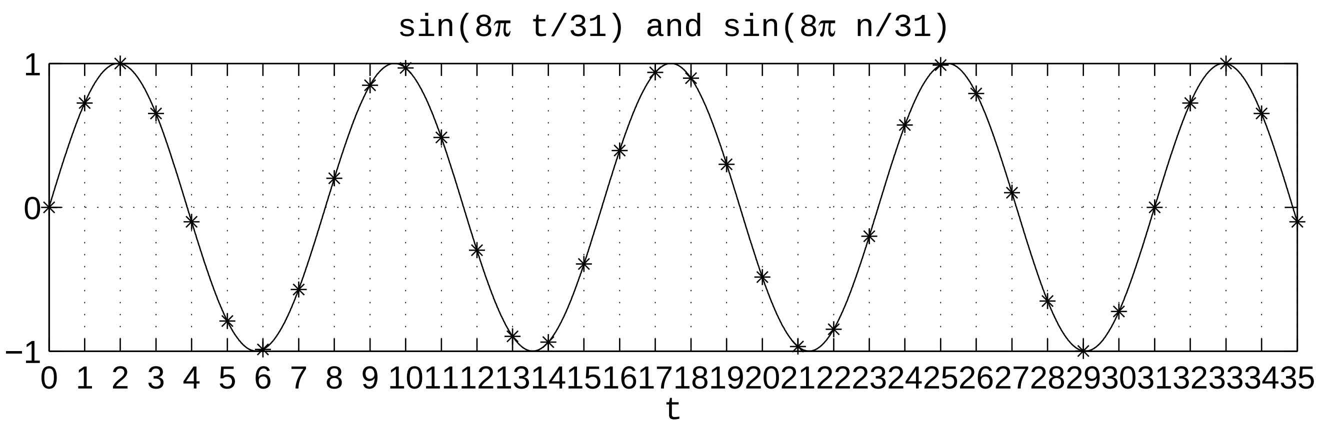
2) Is $x[n] = \sin(3n/4)$ periodic? If it is periodic, what's its fundamental period?

   Since $\frac{w_0}{2\pi} = \frac{3}{8\pi}$ is irrational, $x[n]$ is not periodic; see the figure where $x[n] = 0$ only at $n = 0$.



3) Is $x[n] = \sin(8\pi n/31)$ periodic? If it is periodic, what's its fundamental period?

   Since $w_0/(2\pi) = 4/31$, $x[n]$ is periodic with fundamental period 31; see the figure where $x[0] = x[31] = 0$. Note that the continuous-time signal $\sin(8\pi t/31)$ has fundamental period $31/4$, hence it is 0 at $t = 31/4$. But $x[n]$ has no $31/4-$th sample and it misses 0 between $x[7]$ and $x[8]$.



Harmonically related discrete-time periodic exponentials:

$\phi_k[n] = \{e^{jk(2\pi/N)n}\}_{k=0,\pm1,\ldots}$, are all periodic with period N.

However, unlike the continuous-time signals, these signals are not all distinct because

$$\phi_{k+N}[n] = e^{j(k+N)(2\pi/N)n} = e^{jk(2\pi/N)n}e^{j2\pi n} = \phi_k[n].$$

This implies that there are only N distinct signals in this set, for example,

$$\begin{aligned}
\phi_0[n] &= 1 \\
\phi_1[n] &= e^{j2\pi n/N} \\
\phi_2[n] &= e^{j4\pi n/N} \\
&\vdots \\
\phi_1[n] &= e^{j2(N-1)\pi/N}.
\end{aligned}$$