4.3 Exercises

1. Which of the following are linear transformations?

(a) $L(x, y) = (x + 1, y, x + y)$

(b) $L\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} x + y \\ y \\ x - z \end{bmatrix}$

(c) $L(x, y) = (x^2 + x, y - y^2)$

2. Which of the following are linear transformations?

(a) $L(x, y, z) = (x - y, x^2, 2z)$

(b) $L\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} 2x - 3y \\ 3y - 2z \\ 2z \end{bmatrix}$

(c) $L(x, y) = (x - y, 2x + 2)$

3. Which of the following are linear transformations?

(a) $L(x, y, z) = (x + y, 0, 2x - z)$

(b) $L\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} x^2 - y^2 \\ x^2 + y^2 \end{bmatrix}$

(c) $L(x, y) = (x - y, 0, 2x + 3)$

4. Which of the following are linear transformations?

(a) $L\left(\begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{bmatrix}\right) = \begin{bmatrix} u_1 \\ u_1^2 + u_2 \\ u_1 - u_3 \end{bmatrix}$

(b) $L\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} 1 & 1 & 0 \\ 0 & -1 & 2 \\ 1 & 1 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix}$

(c) $L\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

In Exercises 5 through 12, sketch the image of the given point P or vector **u** *under the given linear transformation L.*

5. $L: R^2 \to R^2$ is defined by

$$L(x, y) = (x, -y);\ P = (2, 3).$$

6. $L: R^2 \to R^2$ is defined by

$$L\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 1 & -1 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix};\ \mathbf{u} = (1, -2).$$

7. $L: R^2 \to R^2$ is a counterclockwise rotation through 30°; $P = (-1, 3)$.

8. $L: R^2 \to R^2$ is a counterclockwise rotation through $\frac{2}{3}\pi$ radians; $\mathbf{u} = (-2, -3)$.

9. $L: R^2 \to R^2$ is defined by $L(\mathbf{u}) = -\mathbf{u}$; $\mathbf{u} = (3, 2)$.

10. $L: R^2 \to R^2$ is defined by $L(\mathbf{u}) = 2\mathbf{u}$; $\mathbf{u} = (-3, 3)$.

11. $L: R^3 \to R^2$ is defined by

$$L\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} x \\ x - y \end{bmatrix};\ \mathbf{u} = (2, -1, 3).$$

12. $L: R^3 \to R^3$ is defined by

$$L\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} 1 & 0 & 1 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix};\ \mathbf{u} = (0, -2, 4).$$

13. Let $L: R^3 \to R^3$ be the linear transformation defined by

$$L\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} x + z \\ y + z \\ x + 2y + 2z \end{bmatrix}.$$

Is $\mathbf{w}$ in range L?

(a) $\mathbf{w} = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$ (b) $\mathbf{w} = \begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix}$

14. Let $L: R^3 \to R^3$ be the linear transformation defined by

$$L\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} -1 & 2 & 0 \\ 1 & 1 & 1 \\ 2 & -1 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix}.$$

Is $\mathbf{w}$ in range L?

(a) $\mathbf{w} = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$ (b) $\mathbf{w} = \begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix}$

15. Let $L: R^3 \to R^3$ be defined by

$$L\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} 4 & 1 & 3 \\ 2 & -1 & 3 \\ 2 & 2 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix}.$$

Find an equation relating a, b, and c so that

$$\mathbf{w} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$$

will lie in range L.

16. Repeat Exercise 15 if $L: R^3 \to R^3$ is defined by

$$L\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} x + 2y + 3z \\ -3x - 2y - z \\ -2x + 2z \end{bmatrix}.$$

17. Let $L: R^2 \to R^2$ be a linear transformation such that

$$L(\mathbf{i}) = \begin{bmatrix} 2 \\ 3 \end{bmatrix} \quad \text{and} \quad L(\mathbf{j}) = \begin{bmatrix} -1 \\ 2 \end{bmatrix}.$$

Find $L\left(\begin{bmatrix} 4 \\ -3 \end{bmatrix}\right)$.

18. Let $L: R^3 \to R^3$ be a linear transformation such that

$$L(\mathbf{i}) = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix},\ L(\mathbf{j}) = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix},\ \text{and } L(\mathbf{k}) = \begin{bmatrix} 1 \\ 1 \\ 3 \end{bmatrix}.$$

Find $L\left(\begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix}\right)$.

19. Let L be the linear transformation defined in Exercise 11. Find all vectors $\mathbf{x}$ in R^3 such that $L(\mathbf{x}) = \mathbf{0}$.

20. Repeat Exercise 19, where L is the linear transformation defined in Exercise 12.

21. Describe the following linear transformations geometrically.
 (a) $L(x, y) = (-x, y)$
 (b) $L(x, y) = (-x, -y)$
 (c) $L(x, y) = (-y, x)$

22. Describe the following linear transformations geometrically.
 (a) $L(x, y) = (y, x)$
 (b) $L(x, y) = (-y, -x)$
 (c) $L(x, y) = (2x, 2y)$

In Exercises 23 and 24, determine whether L is a linear transformation.

23. $L: R^2 \to R^2$ defined by $L(x, y) = (x + y + 1, x - y)$

24. $L: R^2 \to R^1$ defined by $L(x, y) = \sin x + \sin y$

In Exercises 25 through 30, find the standard matrix representing L.

25. $L: R^2 \to R^2$ is reflection with respect to the y-axis.

26. $L: R^2 \to R^2$ is defined by
$$L\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} x - y \\ x + y \end{bmatrix}.$$

27. $L: R^2 \to R^2$ is counterclockwise rotation through $\frac{\pi}{4}$ radians.

28. $L: R^2 \to R^2$ is counterclockwise rotation through $\frac{\pi}{3}$ radians.

29. $L: R^3 \to R^3$ is defined by
$$L\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} x - y \\ x + z \\ y - z \end{bmatrix}.$$

30. $L: R^3 \to R^3$ is defined by $L(\mathbf{u}) = -2\mathbf{u}$.

31. Use the substitution and matrix A of Example 4.
 (a) Code the message SEND HIM MONEY.
 (b) Decode the message 67 44 41 49 39 19 113 76 62 104 69 55.

32. Use the substitution scheme of Example 4 and the matrix
$$A = \begin{bmatrix} 5 & 3 \\ 2 & 1 \end{bmatrix}.$$
 (a) Code the message WORK HARD.
 (b) Decode the message

 93 36 60 21 159 60 110 43

Theoretical Exercises

T.1. Prove Theorem 4.6.

T.2. Prove Theorem 4.7.

T.3. Show that $L: R^n \to R^n$ defined by $L(\mathbf{u}) = r\mathbf{u}$, where r is a scalar, is a linear operator on R^n.

T.4. Let $\mathbf{u}_0 \neq \mathbf{0}$ be a fixed vector in R^n. Let $L: R^n \to R^n$ be defined by $L(\mathbf{u}) = \mathbf{u} + \mathbf{u}_0$. Determine whether L is a linear transformation. Justify your answer.

T.5. Let $L: R^1 \to R^1$ be defined by $L(\mathbf{u}) = a\mathbf{u} + b$, where a and b are real numbers (of course, $\mathbf{u}$ is a vector in R^1, which in this case means that $\mathbf{u}$ is also a real number). Find all values of a and b such that L is a linear transformation.

T.6. Show that the function $O: R^n \to R^m$ defined by $O(\mathbf{u}) = \mathbf{0}_{R^m}$ is a linear transformation, which is called the **zero linear transformation**.

T.7. Let $I: R^n \to R^n$ be defined by $I(\mathbf{u}) = \mathbf{u}$, for $\mathbf{u}$ in R^n. Show that I is a linear transformation, which is called the **identity operator** on R^n.

T.8. Let $L: R^n \to R^m$ be a linear transformation. Show that if $\mathbf{u}$ and $\mathbf{v}$ are vectors in R^n such that $L(\mathbf{u}) = \mathbf{0}$ and $L(\mathbf{v}) = \mathbf{0}$, then $L(a\mathbf{u} + b\mathbf{v}) = \mathbf{0}$ for any scalars a and b.

T.9. Let $L: R^2 \to R^2$ be the linear transformation defined by $L(\mathbf{u}) = A\mathbf{u}$, where
$$A = \begin{bmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{bmatrix}.$$
For $\phi = 30^\circ$, L defines a counterclockwise rotation by 30°.
 (a) If $T_1(\mathbf{u}) = A^2\mathbf{u}$, describe the action of T_1 on $\mathbf{u}$.
 (b) If $T_2(\mathbf{u}) = A^{-1}\mathbf{u}$, describe the action of T_2 on $\mathbf{u}$.
 (c) What is the smallest positive value of k for which $T(\mathbf{u}) = A^k\mathbf{u} = \mathbf{u}$?

T.10. Let $O: R^n \to R^n$ be the zero linear transformation defined by $O(\mathbf{v}) = \mathbf{0}$ for $\mathbf{v}$ in R^n (see Exercise T.6). Find the standard matrix representing O.

T.11. Let $I: R^n \to R^n$ be the identity linear transformation defined by $I(\mathbf{v}) = \mathbf{v}$ for $\mathbf{v}$ in R^n (see Exercise T.7). Find the standard matrix representing I.

Ex 4.3

Linear transformation:- A linear transformation L of R^n into R^m is a function assigning a unique vector $L(u)$ in R^m to each u in R^n such that

(i) $L(u+v) = L(u) + L(v)$, for every u and v in R^n

(ii) $L(ku) = kL(u)$, for every u in R^n and every scalar k.

Q1 Which of the following are linear transformation?

(a) $L(x,y) = (x+1, y, x+y)$.

Let $\vec{u} = (x_1, y_1)$ and $\vec{v} = (x_2, y_2)$ be the two vectors in R^2

Then $L(\vec{u}) = L(x_1, y_1) = (x_1+1, y_1, x_1+y_1)$.

& $L(\vec{v}) = L(x_2, y_2) = (x_2+1, y_2, x_2+y_2)$.

$$L(\vec{u}) + L(\vec{v}) = (x_1+1, y_1, x_1+y_1) + (x_2+1, y_2 + x_2+y_2).$$

$$= (x_1+x_2+2, y_1+y_2, x_1+y_1+x_2+y_2) \quad \text{—— ①}$$

Now $\vec{u}+\vec{v} = (x_1, y_1) + (x_2, y_2)$

$$= (x_1+x_2, y_1+y_2).$$

$L(\vec{u}+\vec{v}) = (x_1+x_2+1,\ y_1+y_2,\ x_1+x_2+y_1+y_2)$ —— ②

From eq ① & eq ② we have:

$L(\vec{u}+\vec{v}) \neq L(\vec{u})+L(\vec{v})$.

it's not a linear transformation.

(b) $L\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} x+y \\ y \\ x-z \end{bmatrix}$.

Sol:- $L: R^3 \to R^3$

Let $\vec{u} = \begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix}$ and $\vec{v} = \begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix}$ be the two vectors in R^3.

then

$$L(\vec{u}) = L\begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix} = \begin{bmatrix} x_1+y_1 \\ y_1 \\ x_1-z_1 \end{bmatrix}$$

and

$$L(\vec{v}) = L\begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix} = \begin{bmatrix} x_2+y_2 \\ y_2 \\ x_2-z_2 \end{bmatrix}.$$

Now $L(\vec{u})+L(\vec{v}) = \begin{bmatrix} x_1+y_1 \\ y_1 \\ x_1-z_1 \end{bmatrix} + \begin{bmatrix} x_2+y_2 \\ y_2 \\ x_2-z_2 \end{bmatrix}$

$$= \begin{bmatrix} (x_1+y_1)+(x_2+y_2) \\ y_1+y_1 \\ (x_1+x_2)-(z_1+z_2) \end{bmatrix} \longrightarrow ①$$

Ex 4.3

Now

$$(\vec{U}+\vec{V}) = \begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix} + \begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix} = \begin{bmatrix} x_1+x_2 \\ y_1+y_2 \\ z_1+z_2 \end{bmatrix}.$$

$$L(\vec{U}+\vec{V}) = L\begin{bmatrix} x_1+x_2 \\ y_1+y_2 \\ z_1+z_2 \end{bmatrix} = \begin{bmatrix} (x_1+x_2)+(y_1+y_2) \\ y_1+y_2 \\ (x_1+x_2)-(z_1+z_2) \end{bmatrix} \longrightarrow ②$$

From eq ① & eq ② we have.

$$L(\vec{U}+\vec{V}) = L(\vec{U})+L(\vec{V}) \text{ ——— } (A)$$

Checking 2nd Property:

$$L(k\vec{U}) = k(L\vec{U}):$$

$$k\vec{U} = k\begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix} = \begin{bmatrix} kx_1 \\ ky_1 \\ kz_1 \end{bmatrix}.$$

$$L(k\vec{U}) = L\begin{bmatrix} kx_1 \\ ky_1 \\ kz_1 \end{bmatrix} = \begin{bmatrix} kx_1+ky_1 \\ ky_1 \\ kx_1-kz_1 \end{bmatrix}.$$

$$= k\begin{bmatrix} x_1+y_1 \\ y_1 \\ x_1-z_1 \end{bmatrix}$$

$$\Rightarrow L(k\vec{U}) = k(L\vec{U}) \text{ ——— } (B)$$

From eq (A) & eq (B) it's clear that it's a Linear transformation from R^3 to R^3.

(i) $L(x,y) = (x^2+x, y-y^2)$.

Sol:- Let $\vec{u} = (x_1, y_1)$ & $\vec{V} = (x_2, y_2)$ be the two vector in R^2

Then $L(\vec{U}) = L(x_1, y_1) = (x_1^2+x_1, y_1-y_1^2)$

& $L(\vec{V}) = L(x_2, y_2) = (x_2^2+x_2, y_2-y_2^2)$.

$L(\vec{u})+L(\vec{V}) = (x_1^2+x_1, y_1-y_1^2)+(x_2^2+x_2, y_2-y_2^2)$

$= [(x_1^2+x_1)+(x_2^2+x_2), (y_1-y_1^2)+(y_2-y_2^2)]$ — (1)

Now $\vec{U}+\vec{V} = (x_1, y_1)+(x_2, y_2)$

$= (x_1+x_2, y_1+y_2)$

$L(\vec{U}+\vec{V}) = L(x_1+x_2, y_1+y_2)$

$=$

$= (x_1+x_2)^2+(x_1+x_2), (y_1+y_2)-(y_1+y_2)^2$ — (2)

From eq (1) & eq (2) we have

$L(\vec{u})+L(\vec{V}) \neq L(\vec{U}+\vec{V})$

it's not a linear transformation.

Ex 4.3

Q_2 & Q_3 is similarly to Q_1

Q4 (a)

$$L\left(\begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{bmatrix}\right) = \begin{bmatrix} u_1 \\ u_1^2+u_2 \\ u_1-u_3 \end{bmatrix}.$$

Let $\vec{U} = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{bmatrix}$ & $\vec{V} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix}$ be the two vector in R^4 then.

$$L(\vec{U}) = L\begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{bmatrix} = \begin{bmatrix} u_1 \\ u_1^2+u_2 \\ u_1-u_3 \end{bmatrix}.$$

$$L(\vec{V}) = L\begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} = \begin{bmatrix} v_1 \\ v_1^2+v_2 \\ v_1-v_3 \end{bmatrix}.$$

Now $L(\vec{U})+L(\vec{V}) = \begin{bmatrix} u_1 \\ u_1^2+u_2 \\ u_1-u_3 \end{bmatrix} + \begin{bmatrix} v_1 \\ v_1^2+v_2 \\ v_1-v_3 \end{bmatrix}$

$$= \begin{bmatrix} u_1+v_1 \\ (u_1^2+u_2)+(v_1^2+v_2) \\ (u_1-u_3)+(v_1-v_3) \end{bmatrix} \longrightarrow ①$$

Again Now

$$\vec{U}+\vec{V} = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{bmatrix} + \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} = \begin{bmatrix} u_1+v_1 \\ u_2+v_2 \\ u_3+v_3 \\ u_4+v_3 \end{bmatrix}.$$

$$L(\vec{U}+\vec{V}) = L\left(\begin{bmatrix} U_1+V_1 \\ U_2+V_2 \\ U_3+V_3 \\ U_4+V_4 \end{bmatrix}\right) = \begin{bmatrix} U_1+V_1 \\ (U_1+V_1)^2+(U_2+V_2) \\ (U_1+V_1)-(U_3+V_3) \end{bmatrix} \rightarrow (2)$$

From eq (1) & eq (2) we have.

$$L(\vec{U}+\vec{V}) \neq L(\vec{U})+L(\vec{V}).$$

it's not a linear transformation.

(b) $$L\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} 1 & 1 & 0 \\ 0 & -1 & 2 \\ 1 & 1 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

$$= \begin{bmatrix} x+y+0z \\ 0x-y+2z \\ x+y-z \end{bmatrix} = \begin{bmatrix} x+y \\ -y+2z \\ x+y-z \end{bmatrix}.$$

Let $\vec{U} = \begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix}$ and $\vec{V} = \begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix}$ be the two vector in R^3, then.

$$L(\vec{U}) = L\left(\begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix}\right) = \begin{bmatrix} x_1+y_1 \\ -y_1+2z_1 \\ x_1+y_1-z_1 \end{bmatrix}.$$

$$\& \; L(\vec{V}) = L\begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix} = \begin{bmatrix} x_2+y_2 \\ -y_2+2z_2 \\ x_2+y_2-z_2 \end{bmatrix}.$$

Scanned with CamScanner

Ex 4.3

$$L(\vec{U})+L(\vec{V}) = \begin{bmatrix} x_1+y_1 \\ -y_1+2z_1 \\ x_1+y_1-z_1 \end{bmatrix} + \begin{bmatrix} x_2+y_2 \\ -y_2+2z_2 \\ x_2+y_2-z_2 \end{bmatrix}$$

$$\begin{bmatrix} (x_1+x_2)+(y_1+y_2) \\ -(y_1+y_2)+2(z_1+z_2) \\ (x_1+x_2)+(y_1+y_2)-(z_1+z_2) \end{bmatrix} \longrightarrow 1.$$

Now $\vec{U}+\vec{V} = \begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix} + \begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix} = \begin{bmatrix} x_1+x_2 \\ y_1+y_2 \\ z_1+z_2 \end{bmatrix}.$

$$L(\vec{U}+\vec{V}) = L\left(\begin{bmatrix} x_1+x_2 \\ y_1+y_2 \\ z_1+z_2 \end{bmatrix}\right)$$

$$= \begin{bmatrix} (x_1+x_2)+(y_1+y_2) \\ -(y_1+y_2)+2(z_1+z_2). \\ (x_1+x_2)+(y_1+y_2)-(z_1+z_2) \end{bmatrix} \longrightarrow 2$$

From eq 1 & eq 2 we have.

$$L(\vec{U})+L(\vec{V}) = L(\vec{U}+\vec{V}). \text{———} A$$

Checking 2nd Property.

$L(k\vec{U}) = k(L\vec{U})$ where $k \in R$ then.

$$k\vec{U} = k\begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix} = \begin{bmatrix} kx_1 \\ ky_1 \\ kz_1 \end{bmatrix}$$

$$L(k\vec{u}) = L\begin{bmatrix} kx_1 \\ ky_1 \\ kz_1 \end{bmatrix}$$

$$= \begin{bmatrix} kx_1 + ky_1 \\ -ky_1 + 2z_1 \\ kx_1 + ky_1 + kz_1 \end{bmatrix} = k\begin{bmatrix} x_1 + y_1 \\ -y_1 + 2z_1 \\ x_1 + y_1 - z_1 \end{bmatrix}.$$

$$\Rightarrow L(k\vec{u}) = k(L\vec{u}) \longrightarrow (B)$$

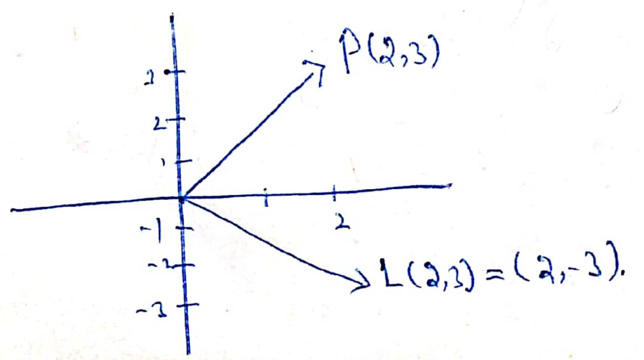
From eq (A) + eq (B) it's clear that it's a linear transformation from R^3 to R^3.

(C) Same as

In Exercises 5-12, sketch the image of the given Point P or vector u under the given linear transformation L.

Q5 $L(x,y) = (x-y)$: $P(2,3)$

$L(2,3) = (2,-3)$.



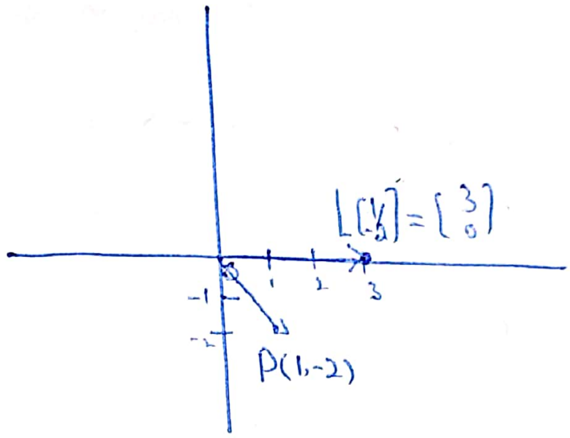
Ex 4.3

Q6 $L\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 1 & -1 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$; $u = (1,-2)$

$L\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} x-y \\ 2x+y \end{bmatrix}$

$L\left(\begin{bmatrix} 1 \\ -2 \end{bmatrix}\right) = \begin{bmatrix} 1-(-2) \\ 2(1)+(-2) \end{bmatrix} = \begin{bmatrix} 1+2 \\ 2-2 \end{bmatrix} = \begin{bmatrix} 3 \\ 0 \end{bmatrix}.$

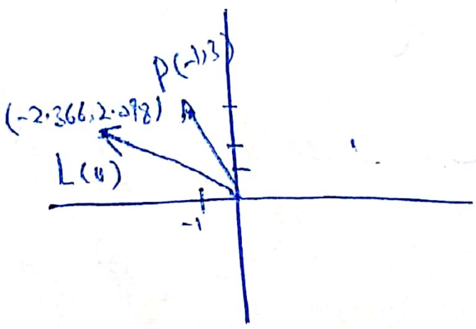
$L\left(\begin{bmatrix} 1 \\ -2 \end{bmatrix}\right) = \begin{bmatrix} 3 \\ 0 \end{bmatrix}$



Q7 $L = \begin{bmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{bmatrix}$ $\phi = 30^\circ$ $P = (-1, 3) = u$

$LU = \begin{bmatrix} \cos 30^\circ & -\sin 30^\circ \\ \sin 30^\circ & \cos 30^\circ \end{bmatrix}\begin{bmatrix} -1 \\ 3 \end{bmatrix}.$

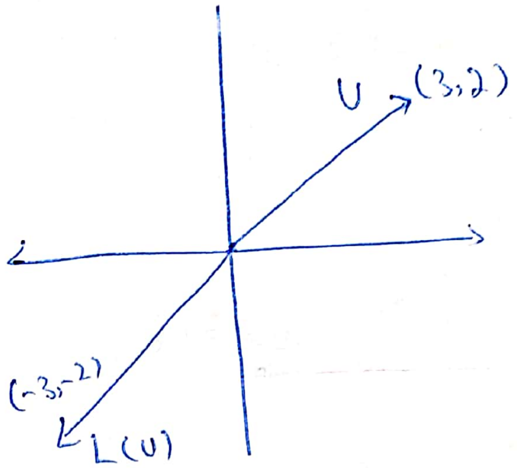
$= \begin{bmatrix} -\cos 30^\circ - 3\sin 30^\circ \\ -\sin 30 + 3\cos 30^\circ \end{bmatrix} = \begin{bmatrix} -2.366, 2.098 \end{bmatrix}.$



Q8 is similarly to Q7.

Q9 $L(U) = -U$, $U = (3,2)$

$L(3,2) = -(3,2) = (-3,-2)$



Q10 is similarly to Q9.

Q11 $L\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} x \\ x-y \end{bmatrix}$ $\quad u = \overset{x\ \ y\ \ z}{(2,-1,3)}$

$L(u) = L\left(\begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix}\right) = \begin{bmatrix} 2 \\ 2-(-1) \end{bmatrix} = \begin{bmatrix} 2 \\ 2+1 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}.$

Q12 is similarly to Q11

~ o ~

Ex 4.3

Q13 (a) $L\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} x+z \\ y+z \\ x+2y+2z \end{bmatrix}$ $W = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$.

Since $L(\vec{U}) = \vec{W}$

$$L\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} x+z \\ y+z \\ x+2y+2z \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$$

$\Rightarrow$ $x+z=1$ — (i) $y+z=-1$ — (ii), $x+2y+2z=0$ — (iii)

$x = 1-z$ — (iv)

& $y = -1-z$ — (v)

eq (iii) $\Rightarrow$ $(1-z)+2(-1-z)+2z=0$

$1-z-2-\cancel{2z}+\cancel{2z}=0$

$-z-1=0 \Rightarrow \boxed{z=-1}$

eq (iv) $\Rightarrow$ $x = 1-(-1) = \boxed{x=2}$

eq (v) $\Rightarrow$ $y = -1-(-1) = -1+1 = 0 = \boxed{y=0}$.

eq (iii) $\Rightarrow$ $2+2(0)+2(-1)=0$

$\cancel{2}+0-\cancel{2}=0$

$0=0$.

$\therefore \vec{U} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ -1 \end{bmatrix}$ This shows the image and so it's the range L.

(b) $L(\vec{v}) = w$

$$\begin{bmatrix} x+z \\ y+z \\ x+2y+2z \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix}.$$

$x+z = 2$ —— (i)

$y+z = -1$ —— (ii)

$x+2y+2z = 3$ —— (iii)

eq (i) $\Rightarrow x = 2-z$ —— (iv)

eq (ii) $\Rightarrow y = -1-z$ —— (v)

eq (iii) $\Rightarrow (2-z)+2(-1-z)+2z = 3$

$\cancel{2}-z+(-\cancel{2}-\cancel{2z})+\cancel{2z} = 3$

$-z = 3 \Rightarrow \boxed{z=-3}$

eq (iv) $\Rightarrow x = 2+3 = 5 \Rightarrow \boxed{x=5}$

eq (v) $\Rightarrow y = -1+3 \Rightarrow \boxed{y=2}$

eq (iii) $\Rightarrow 5+2(2)+2(-3) = 3$

$5+4-6 = 3$

$9-6 = 3$

$3 = 3$

$\therefore \vec{v} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 5 \\ 2 \\ -3 \end{bmatrix}$ This shows the image so it's the range L.

Ex 4.3

Q14 is similarly to Q13.

Q15 Let $\vec{U} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ ~~the~~ such that

$L\vec{U} = \vec{W}$

$L\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}.$

$\begin{bmatrix} 4 & 1 & 3 \\ 2 & -1 & 3 \\ 2 & 2 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}.$

$\begin{bmatrix} 4x+y+3z \\ 2x-y+3z \\ 2x+2y \end{bmatrix} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}.$

$\Rightarrow$ $4x+y+3z = a$ —— (i)

$2x-y+3z = b$ —— (ii)

$2x+2y = c$ —— (iii).

x eq (ii) by (2) then subtr: from (i)

$4x+y+3z = a$

$\cancel{4x}-2y+6z = 2b$

$3y-3z = a-2b$ —— (4).

(i) (iii)

~~$4x+y+3z=a$~~

eq ② − eq ③

$$\cancel{2x} - y + 3z = b$$
$$\underline{\cancel{2x} \pm 2y \quad = c}$$
$$-3y + 3z = b - c \quad \text{——— ⑤}$$

eq ④ + eq ⑤

$$\cancel{3y} - \cancel{3z} = a - 2b$$
$$\underline{-\cancel{3y} + \cancel{3z} = b - c}$$
$$0 = a - 2b + b - c$$

$$a - b - c = 0.$$

$$\Rightarrow c - a + b = 0 \quad \text{Ans.}$$

—o—o—.

Q16 is similarly to Q15.

Ex 4.3

Q17 As $\begin{bmatrix} 4 \\ -3 \end{bmatrix} = 4i - 3j$ where $L(i) = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$ & $L(j) = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$.

$$\Rightarrow L\begin{bmatrix} 4 \\ -3 \end{bmatrix} = L(4i - 3j)$$

$$= 4L(i) - 3L(j)$$

$$= 4\begin{bmatrix} 2 \\ 3 \end{bmatrix} - 3\begin{bmatrix} -1 \\ 2 \end{bmatrix}$$

$$= \begin{bmatrix} 8 \\ 12 \end{bmatrix} + \begin{bmatrix} 3 \\ -6 \end{bmatrix}.$$

$$= \begin{bmatrix} 8+3 \\ 12-6 \end{bmatrix} = \begin{bmatrix} 11 \\ 6 \end{bmatrix}.$$

$$\Rightarrow L\left(\begin{bmatrix} 4 \\ -3 \end{bmatrix}\right) = \begin{bmatrix} 11 \\ 6 \end{bmatrix}.$$

Q18 As $\begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix} = 2i + j + 3k$

$$\Rightarrow L\begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix} = L(2i - j + 3k) = 2L(i) - L(j) + 3L(k)$$

$$= 2\begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} - \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} + 3\begin{bmatrix} 1 \\ 1 \\ 3 \end{bmatrix}$$

$$= \begin{bmatrix} 2 \\ 4 \\ -2 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} + \begin{bmatrix} 3 \\ 3 \\ 9 \end{bmatrix} = \begin{bmatrix} 2-1+3 \\ 4-0+3 \\ -2-2+9 \end{bmatrix}$$

$$L\begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix} = \begin{bmatrix} 4 \\ 7 \\ 5 \end{bmatrix} \text{ Ans.}$$

Ex 4.3

Q2: let $\vec{u}=(x_1,y_1)$ & $\vec{v}=(x_2,y_2)$. be the two vector in R^2, then

$L(\vec{u})=L(x_1,y_1)=(x_1+y_1+1,\ x_1-y_1)$

$L(\vec{v})=L(x_2,y_2)=(x_2+y_2+1,\ x_2-y_2)$.

$L(\vec{u})+L(\vec{v})=(x_1+y_1+1,\ x_1-y_1)+(x_2+y_2+1,\ x_2-y_2)$

$=((x_1+y_1)+(x_2+y_2)+2\ ,(x_1+x_2)-(y_1+y_2)) \rightarrow (1)$

Now $\vec{u}+\vec{v}=(x_1,y_1)+(x_2+y_2)$

$=(x_1+x_2,\ y_1+y_2)$

$L(\vec{u}+\vec{v})=L(x_1+x_2,\ y_1+y_2)$

$=(x_1+x_2+y_1+y_2+1,\ x_1+x_2-(y_1-y_2)) \text{ — } (2)$

From eq (1) & eq (2) it's clearly that

$L(\vec{u}+\vec{v})\neq L(\vec{u})+L(\vec{v})$.

$\therefore$ it's not a linear transformation.

Q24 let $\vec{U} = (x_1, y_1)$ and $\vec{V} = (x_2, y_2)$ be the two vector in R_2, then

$$\vec{U} + \vec{V} = (x_1 + y_1) + (x_2 + y_2) = (x_1 + x_2, y_1 + y_2).$$

$$L(\vec{U} + \vec{V}) = L(x_1 + x_2, y_1 + y_2).$$

$$L(\vec{U} + \vec{V}) = (\sin(x_1 + x_2), \sin(y_1 + y_2)) \text{ —— } (1)$$

Now $L(\vec{U}) = L(x_1, y_1)$

$$= \sin x_1 + \sin y_1$$

& $L(\vec{V}) = L(x_2, y_2)$

$$= \sin x_2 + \sin y_2.$$

$$\Rightarrow L(\vec{U}) + L(\vec{V}) = (\sin x_1 + \sin y_1) + (\sin x_2 + \sin y_2)$$

$$= (\sin(x_1 + x_2), \sin(y_1 + y_2)) \text{ —— } (2).$$

From eq (1) & eq (2) we have

$$L(\vec{U} + \vec{V}) = L(\vec{U}) + L(\vec{V}). \text{ —— } (A)$$

Now let $k \in R$ then $k\vec{U} = k(x_1, y_1) = (kx_1, ky_1)$

$$L(k\vec{U}) = L(kx_1, ky_1) = (k\sin x_1 + k\sin y_1) = k(\sin x_1 + \sin y_1)$$

$$\Rightarrow L(k\vec{U}) = k(L\vec{U}) \text{ —— } (B)$$

From eq (A) & eq (B), we can say that the given function is a linear transformation.

Scanned with CamScanner

Ex 4.3

Theorem If $L: R^n \to R^m$ is a L.T. then

$L(c_1\vec{u_1} + c_2\vec{u_2} + \dots + c_k\vec{u_k})$

$= c_1 L(\vec{u_1}) + c_2 L(\vec{u_2}) + \dots + c_k L(\vec{u_k})$.

for any vector $\vec{u_1}, \vec{u_2}, \dots \vec{u_k}$ in R^n and any scalar

$c_1, c_2 \dots c_k$.

For R^2
$\vec{i} = \vec{e_1} = (1, 0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$.

$\vec{j} = \vec{e_2} = (0, 1) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$.

For R^3
$\vec{i} = \vec{e_1} = (1, 0, 0) = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$.

$\vec{j} = \vec{e_2} = (0, 1, 0) = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$

$\vec{k} = \vec{e_3} = (0, 0, 1) = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$.

For R^n
$\vec{i} = \vec{e_1} = (1, 0, 0, \dots 0)$

$\vdots$

$\vec{e_n} = (0, 0, 0 \dots 1)$.

Q26 $L: R^2 \to R^2$ $L\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} x-y \\ x+y \end{bmatrix}$

For R^2 $e_1 = (1,0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$. & $e_2 = (0,1) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$.

$L(e_1) = L\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1-0 \\ 1+0 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

& $L(e_2) = L\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0-1 \\ 0+1 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$.

then $A = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$. Ans

Q28 $\theta = 60°$

$L(x) = \begin{bmatrix} \cos 60 & -\sin 60 \\ \sin 60 & \cos 60 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \frac{x}{2} & -\frac{y\sqrt{3}}{2} \\ \frac{x\sqrt{3}}{2} & y/2 \end{bmatrix}$.

For R^2 $e_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ & $e_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

$L(e_1) = L\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1/2 & 0 \\ \sqrt{3}/2 & 0 \end{bmatrix} = \begin{bmatrix} 1/2 \\ \sqrt{3}/2 \end{bmatrix}$.

$L(e_2) = L\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 & -\sqrt{3}/2 \\ 0 & 1/2 \end{bmatrix} = \begin{bmatrix} -\sqrt{3}/2 \\ 1/2 \end{bmatrix}$.

$A = \begin{bmatrix} 1/2 & -\sqrt{3}/2 \\ \sqrt{3}/2 & 1/2 \end{bmatrix}$.

Q27 is same as Q28

EX 4.3

Q29 $L\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} x-y \\ x+z \\ y-z \end{bmatrix}.$

For R^3 $e_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, $e_2 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ & $e_3 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$.

$$L(e_1) = L\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1-0 \\ 1+0 \\ 0-0 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}.$$

$$L(e_2) = L\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0-1 \\ 0+0 \\ 1-0 \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}.$$

$$L(e_3) = L\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0-0 \\ 0+1 \\ 0-1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}.$$

Thus $A = \begin{bmatrix} 1 & -1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & -1 \end{bmatrix}.$

Q30 $L(u) = -2u = -2\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -2x \\ -2y \\ -2z \end{bmatrix}.$

$$L(e_1) = L\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} -2 \\ 0 \\ 0 \end{bmatrix}$$

$$L(e_2) = L\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ -2 \\ 0 \end{bmatrix}$$

$$L(e_3) = L\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ -2 \end{bmatrix}.$$

Thus $A = \begin{bmatrix} -2 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -2 \end{bmatrix}.$

Q31 (a)

S E N D H I M M O N E Y

19 5 14 4 8 9 13 13 15 14 5 25.

if $A = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 1 & 2 \\ 0 & 1 & 2 \end{bmatrix}$.

Breaking message into 4 vector

$\begin{bmatrix} 19 \\ 5 \\ 14 \end{bmatrix} \begin{bmatrix} 4 \\ 8 \\ 9 \end{bmatrix} \begin{bmatrix} 13 \\ 13 \\ 15 \end{bmatrix} \begin{bmatrix} 14 \\ 5 \\ 25 \end{bmatrix}$.

Now $L(x) = AX$.

$$Ax = A\begin{bmatrix} 19 \\ 5 \\ 14 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 1 & 2 \\ 0 & 1 & 2 \end{bmatrix}\begin{bmatrix} 19 \\ 5 \\ 14 \end{bmatrix} = \begin{bmatrix} 71 \\ 52 \\ 33 \end{bmatrix}$$

$$A\begin{bmatrix} 19 \\ 5 \\ 14 \end{bmatrix} = \begin{bmatrix} 71 \\ 52 \\ 33 \end{bmatrix}.$$

$$Ax = A\begin{bmatrix} 4 \\ 8 \\ 9 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 1 & 2 \\ 0 & 1 & 2 \end{bmatrix}\begin{bmatrix} 4 \\ 8 \\ 9 \end{bmatrix} = \begin{bmatrix} 47 \\ 30 \\ 26 \end{bmatrix}$$

$$A\begin{bmatrix} 4 \\ 8 \\ 9 \end{bmatrix} = \begin{bmatrix} 47 \\ 30 \\ 26 \end{bmatrix}.$$

$$Ax = A\begin{bmatrix} 13 \\ 13 \\ 15 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 1 & 2 \\ 0 & 1 & 2 \end{bmatrix}\begin{bmatrix} 13 \\ 13 \\ 15 \end{bmatrix} = \begin{bmatrix} 84 \\ 56 \\ 43 \end{bmatrix}$$

$$A\begin{bmatrix} 13 \\ 13 \\ 15 \end{bmatrix} = \begin{bmatrix} 84 \\ 56 \\ 43 \end{bmatrix}.$$

$$Ax = A\begin{bmatrix} 14 \\ 5 \\ 25 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 1 & 2 \\ 0 & 1 & 2 \end{bmatrix}\begin{bmatrix} 14 \\ 5 \\ 25 \end{bmatrix} = \begin{bmatrix} 99 \\ 69 \\ 56 \end{bmatrix}.$$

Ex 4.3

So the massage is 71 52 33 47 30 26 84 56 43 99 69 56.

(b) 67 44 41 49 39 19 113 76 62 104 69 55

Breaking into vector in R^3

$$\begin{array}{cccc} L(x_1) & L(x_2) & L(x_3) & L(x_4) \\ \begin{bmatrix}67\\44\\41\end{bmatrix} & \begin{bmatrix}49\\39\\19\end{bmatrix} & \begin{bmatrix}113\\76\\62\end{bmatrix} & \begin{bmatrix}104\\69\\55\end{bmatrix} \end{array}.$$

find $A^{-1} = ?$ $\qquad L(x) = Ax. \Rightarrow x = A^{-1}L(x)$

$$A^{-1} = \begin{bmatrix} 0 & 1 & -1 \\ 2 & -2 & -1 \\ -1 & 1 & 1 \end{bmatrix}.$$

$$x_1 = A^{-1}L(x_1) = \begin{bmatrix} 0 & 1 & -1 \\ 2 & -2 & -1 \\ -1 & 1 & 1 \end{bmatrix}\begin{bmatrix}67\\44\\41\end{bmatrix} = \begin{bmatrix}3\\5\\18\end{bmatrix}.$$

$$x_2 = A^{-1}L(x_2) = \begin{bmatrix} 0 & 1 & -1 \\ 2 & -2 & -1 \\ -1 & 1 & 1 \end{bmatrix}\begin{bmatrix}49\\39\\19\end{bmatrix} = \begin{bmatrix}20\\1\\9\end{bmatrix}$$

$$x_3 = A^{-1}L(x_3) = \begin{bmatrix} 0 & 1 & -1 \\ 2 & -2 & -1 \\ -1 & 1 & 1 \end{bmatrix}\begin{bmatrix}113\\76\\62\end{bmatrix} = \begin{bmatrix}14\\12\\25\end{bmatrix}.$$

$$x_4 = A^{-1}L(x_4) = \begin{bmatrix} 0 & 1 & -1 \\ 2 & -2 & -1 \\ -1 & 1 & 1 \end{bmatrix}\begin{bmatrix}104\\69\\55\end{bmatrix} = \begin{bmatrix}14\\15\\20\end{bmatrix}$$

So the massage is

3 5 18 20 1 9 14 12 25 14 15 20

C E R T A I N L Y N O T

certainly not.

Q32 is same as Q31.

Jamal Nasir

Lecturer U.E.T Peshawar. ~~main campus~~.

Scanned with CamScanner