12

Fourier Series

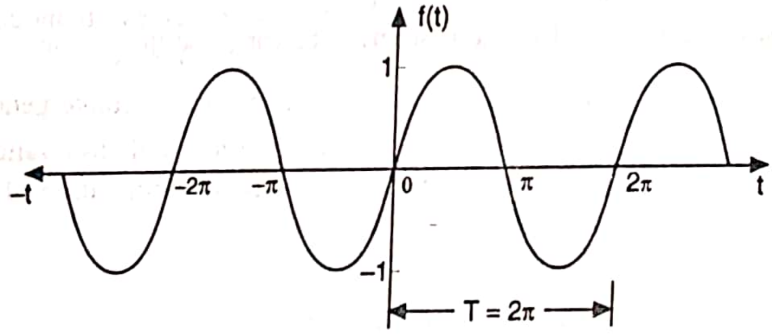
12.1 PERIODIC FUNCTIONS

If the value of each ordinate $f(t)$ repeats itself at equal intervals in the abscissa, then $f(t)$ is said to be a periodic function.

If $f(t) = f(t+T) = f(t+2T) = ...$ then T is called the period of the function $f(t)$.

For example :

$\sin x = \sin(x+2\pi) = \sin(x+4\pi) = ...$ so $\sin x$ is a periodic function with the period 2π. This is also called sinusoidal periodic function.



12.2 FOURIER SERIES

Here we will express a non-sinusoidal periodic function into a fundamental and its harmonics. A series of sines and cosines of an angle and its multiples of the form.

$$\frac{a_0}{2} + a_1 \cos x + a_2 \cos 2x + a_3 \cos 3x + \ldots + a_n \cos nx + \ldots$$

$$+ b_1 \sin x + b_2 \sin 2x + b_3 \sin 3x + \ldots + b_n \sin nx + \ldots$$

$$= \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$$

is called the *Fourier series, where* $a_0, a_1, a_2, \ldots a_n, \ldots b_1, b_2, b_3 \ldots b_n \ldots$ are constants.

A periodic function $f(x)$ can be expanded in a Fourier Series. The series consists of the following:

(*i*) A constant term a_0 (called d.c. component in electrical work).

(*ii*) A component at the fundamental frequency determined by the values of a_1, b_1.

(*iii*) Components of the harmonics (multiples of the fundamental frequency) determined by $a_2, a_3 \ldots b_2, b_3 \ldots$. And $a_0, a_1, a_2 \ldots, b_1, b_2 \ldots$ are known as *Fourier coefficients* or Fourier constants.

12.3. DIRICHLET'S CONDITIONS FOR A FOURIER SERIES

If the function $f(x)$ for the interval $(-\pi, \pi)$

(1) is single-valued (2) is bounded

(3) has at most a finite number of maxima and minima.

(4) has only a finite number of discontinuous

(5) is $f(x+2\pi) = f(x)$ for values of x outside $[-\pi, \pi]$, then

$$S_p(x) = \frac{a_0}{2} + \sum_{n=1}^{P} a_n \cos nx + \sum_{n=1}^{P} b_n \sin nx$$

converges to $f(x)$ as $P \to \infty$ at values of x for which $f(x)$ is continuous and to $\frac{1}{2}[f(x+0) + f(x-0)]$ at points of discontinuity.

12.4. ADVANTAGES OF FOURIER SERIES

1. Discontinuous function can be represented by Fourier series. Although derivatives of the discontinuous functions do not exist. (This is not true for Taylor's series).

2. The Fourier series is useful in expanding the periodic functions since outside the closed interval, there exists a periodic extension of the function.

3. Expansion of an oscillating function by Fourier series gives all modes of oscillation (fundamental and all overtones) which is extremely useful in physics.

4. Fourier series of a discontinuous function is not uniformly convergent at all points.

5. Term by term integration of a convergent Fourier series is always valid, and it may be valid if the series is not convergent. However, term by term, differentiation of a Fourier series is not valid in most cases.

12.5 USEFUL INTEGRALS

The following integrals are useful in Fourier Series.

(*i*) $\int_0^{2\pi} \sin nx \, dx = 0$ (*ii*) $\int_0^{2\pi} \cos nx \, dx = 0$

(*iii*) $\int_0^{2\pi} \sin^2 nx \, dx = \pi$ (*iv*) $\int_0^{2\pi} \cos^2 nx \, dx = \pi$

(*v*) $\int_0^{2\pi} \sin nx \cdot \sin mx \, dx = 0$ (*vi*) $\int_0^{2\pi} \cos nx \cos mx \, dx = 0$

(*vii*) $\int_0^{2\pi} \sin nx \cdot \cos mx \, dx = 0$ (*viii*) $\int_0^{2\pi} \sin nx \cdot \cos nx \, dx = 0$

(*ix*) $[uv]_1 = uv_1 - u'v_2 + u''v_3 - u'''v_4 + \dots$

where $[uv]_1 = \int uv \, dx$, $v_1 = \int v \, dx$, $v_2 = \int v_1 \, dx$ and so on. $u' = \frac{du}{dx}$, $u'' = \frac{d^2u}{dx^2}$ and so on

(*x*) $\sin n\pi = 0$, $\cos n\pi = (-1)^n$ where $n \in I$

12.6 DETERMINATION OF FOURIER COEFFICIENTS (EULER'S FORMULAE)

$$f(x) = \frac{a_0}{2} + a_1 \cos x + a_2 \cos 2x + \dots + a_n \cos nx + \dots$$
$$+ b_1 \sin x + b_2 \sin 2x + \dots + b_n \sin nx + \dots \quad \text{...(1)}$$

(*i*) **To find a_0:** Integrate both sides of (1) from $x = 0$ to $x = 2\pi$.

$$\int_0^{2\pi} f(x)\,dx = \frac{a_0}{2}\int_0^{2\pi} dx + a_1\int_0^{2\pi}\cos x\,dx + a_2\int_0^{2\pi}\cos 2x\,dx + \ldots + a_n\int_0^{2\pi}\cos nx\,dx + \ldots$$

$$+ b_1\int_0^{2\pi}\sin x\,dx + b_2\int_0^{2\pi}\sin 2x\,dx + \ldots + b_n\int_0^{2\pi}\sin nx\,dx + \ldots$$

$$= \frac{a_0}{2}\int_0^{2\pi} dx, \text{ (other integrals } = 0 \text{ by formulae } (i) \text{ and } (ii) \text{ of Art. 12.5)}$$

$$\int_0^{2\pi} f(x)\,dx = \frac{a_0}{2}2\pi, \quad a_0 = \frac{1}{\pi}\int_0^{2\pi} f(x)\,dx \qquad \ldots(2)$$

(*ii*) **To find a_n:** Multiply each side of (1) by $\cos nx$ and integrate from $x = 0$ to $x = 2\pi$.

$$\int_0^{2\pi} f(x)\cos nx\,dx = \frac{a_0}{2}\int_0^{2\pi}\cos nx\,dx + a_1\int_0^{2\pi}\cos x\cos nx\,dx + \ldots + a_n\int_0^{2\pi}\cos^2 nx\,dx \ldots$$

$$+ b_1\int_0^{2\pi}\sin x\cos nx\,dx + b_2\int_0^{2\pi}\sin 2x\cos nx\,dx + \ldots$$

$$= a_n\int_0^{2\pi}\cos^2 nx\,dx = a_n\pi \text{ (Other integrals } = 0\text{, by formulae on Page 851)}$$

$$\therefore \quad a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx\,dx \qquad \ldots(3)$$

By taking $n = 1, 2 \ldots$ we can find the values of $a_1, a_2 \ldots$.

(*iii*) **To find b_n:** Multiply each side of (1) by $\sin nx$ and integrate from $x = 0$ to $x = 2\pi$.

$$\int_0^{2\pi} f(x)\sin nx\,dx = \frac{a_0}{2}\int_0^{2\pi}\sin nx\,dx + a_1\int_0^{2\pi}\cos x\sin nx\,dx + \ldots + a_n\int_0^{2\pi}\cos nx\sin nx\,dx + \ldots$$

$$+ b_1\int_0^{2\pi}\sin x\sin nx\,dx + \ldots + b_n\int_0^{2\pi}\sin^2 nx\,dx + \ldots$$

$$= b_n\int_0^{2\pi}\sin^2 nx\,dx$$

(All other integrals = 0, Article No. 12.5)

$$= b_n\pi$$

$$\therefore \quad b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx\,dx \qquad \ldots(4)$$

Note: To get similar formula of a_0, $\frac{1}{2}$ has been written with a_0 in Fourier series.

Example 1. *Find the Fourier series representing*

$$f(x) = x, \quad 0 < x < 2\pi$$

and sketch its graph from $x = -4\pi$ *to* $x = 4\pi$.

Solution. Let $f(x) = \frac{a_0}{2} + a_1\cos x + a_2\cos 2x + b_1\sin x + b_2\sin 2x + \ldots \qquad \ldots(1)$

Hence $\quad a_0 = \frac{1}{\pi}\int_0^{2\pi} f(x)\,dx = \frac{1}{\pi}\int_0^{2\pi} x\,dx = \frac{1}{\pi}\left[\frac{x^2}{2}\right]_0^{2\pi} = 2\pi$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx\,dx = \frac{1}{\pi}\int_0^{2\pi} x\cos nx\,dx$$

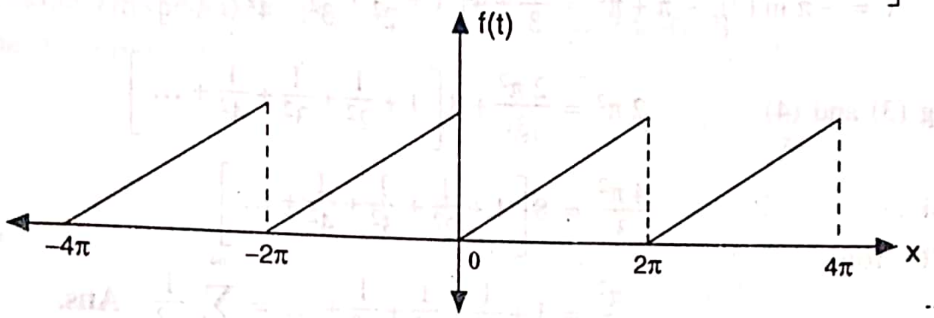
$$= \frac{1}{\pi}\left[x\frac{\sin nx}{n} - 1.\left(-\frac{\cos nx}{n^2}\right)\right]_0^{2\pi} = \frac{1}{\pi}\left[\frac{\cos 2n\pi}{n^2} - \frac{1}{n^2}\right] = \frac{1}{n^2\pi}(1-1) = 0$$

$$b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx\,dx = \frac{1}{\pi}\int_0^{2\pi} x\sin nx\,dx$$

$$= \frac{1}{\pi}\left[x\left(-\frac{\cos nx}{n}\right) - 1.\left(\frac{-\sin nx}{n^2}\right)\right]_0^{2\pi} = \frac{1}{\pi}\left[\frac{-2\pi\cos 2n\pi}{n}\right] = -\frac{2}{n}$$

Substituting the values of a_0, a_n, b_n in (1), we get

$$x = \pi - 2\left[\sin x + \frac{1}{2}\sin 2x + \frac{1}{3}\sin 3x + \ldots\right] \quad \textbf{Ans.}$$



Example 2. *Given that $f(x) = x + x^2$ for $-\pi < x < \pi$, find the Fourier expression of $f(x)$*
Deduce that $\frac{\pi^2}{6} = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \ldots$ *(U.P. II Semester, Summer 2003)*

Solution. Let $x + x^2 = \frac{a_0}{2} + a_1\cos x + a_2\cos 2x + \ldots + b_1\sin x + b_2\sin 2x + \ldots \quad \ldots(1)$

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx = \frac{1}{\pi}\int_{-\pi}^{\pi}(x+x^2)\,dx$$

$$= \frac{1}{\pi}\left[\frac{x^2}{2} + \frac{x^3}{3}\right]_{-\pi}^{\pi} = \frac{1}{\pi}\left[\frac{\pi^2}{2} + \frac{\pi^3}{3} - \frac{\pi^2}{2} + \frac{\pi^3}{3}\right] = \frac{2\pi^2}{3}$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx = \int_{-\pi}^{\pi}(x+x^2)\cos nx\,dx$$

$$= \frac{1}{\pi}\left[(x+x^2)\frac{\sin nx}{n} - (2x+1)\frac{(-\cos nx)}{n^2} + (2)\left(-\frac{\sin nx}{n^3}\right)\right]_{-\pi}^{\pi}$$

$$= \frac{1}{\pi}\left[(2\pi+1)\frac{\cos n\pi}{n^2} - (-2\pi+1)\frac{\cos(-n\pi)}{n^2}\right] = \frac{1}{\pi}\left[4\pi\frac{\cos n\pi}{n^2}\right] = \frac{4(-1)^n}{n^2}$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx = \frac{1}{\pi}\int_{-\pi}^{\pi}(x+x^2)\sin nx\,dx$$

$$= \frac{1}{\pi}\left[(x+x^2)\left(-\frac{\cos nx}{n}\right)-(2x+1)\left(\frac{-\sin nx}{n^2}\right)+2\,\frac{\cos nx}{n^3}\right]_{-\pi}^{\pi}$$

$$= \frac{1}{\pi}\left[-(\pi+\pi^2)\frac{\cos n\pi}{n}+2\frac{\cos n\pi}{n^3}+(-\pi+\pi^2)\frac{\cos n\pi}{n}-2\frac{\cos n\pi}{n^3}\right]$$

$$= \frac{1}{\pi}\left[-\frac{2\pi}{n}\cos n\pi\right] = -\frac{2}{n}(-1)^n$$

Substituting the values of a_0, a_n, b_n in (1) we get

$$x+x^2 = \frac{\pi^2}{3}+4\left[-\cos x+\frac{1}{2^2}\cos 2x-\frac{1}{3^2}\cos 3x+\ldots\right]$$

$$-2\left[-\sin x+\frac{1}{2}\sin 2x-\frac{1}{3}\sin 3x+\ldots\right] \quad \ldots(2)$$

Put $x = \pi$ in (2), $\pi+\pi^2 = \frac{\pi^2}{3}+4\left[1+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}+\ldots\right]$ ...(3)

Put $x = -\pi$ in (2), $-\pi+\pi^2 = \frac{\pi^2}{3}+4\left[1+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}+\ldots\right]$ ...(4)

Adding (3) and (4) $\quad 2\pi^2 = \frac{2\pi^2}{3}+8\left[1+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}+\ldots\right]$

$$\frac{4\pi^2}{3} = 8\left[1+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}+\ldots\right]$$

$$\frac{\pi^2}{6} = 1+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}+\ldots = \sum_{n=1}^{\infty}\frac{1}{n^2} \quad \textbf{Ans.}$$

Exercise 12.1

1. Find a Fourier series to represent, $f(x) = \pi - x$ for $0 < x < 2\pi$.

Ans. $2\left[\sin x+\frac{1}{2}\sin 2x+\frac{1}{3}\sin 3x+\ldots+\frac{1}{n}\sin nx+\ldots\right]$

2. Find a Fourier series to represent $x - x^2$ from $x = -\pi$ to π and show that

$$\frac{\pi^2}{12} = \frac{1}{1^2}-\frac{1}{2^2}+\frac{1}{3^2}-\frac{1}{4^2}+\ldots$$

Ans. $-\frac{\pi^2}{3}+4\left[\frac{\cos x}{1^2}-\frac{\cos 2x}{2^2}+\frac{\cos 3x}{3^2}-\frac{\cos 4x}{4^2}+\ldots\right]+2\left[\frac{\sin x}{1}-\frac{\sin 2x}{2}+\frac{\sin 3x}{3}-\frac{\sin 4x}{4}+\ldots\right]$

3. Find a Fourier series to represent: $f(x) = x\sin x$, for $0 < x < 2\pi$.

Ans. $-1+\pi\sin x-\frac{1}{2}\cos x+2\left[\frac{\cos 2x}{2^2-1}+\frac{\cos 3x}{3^2-1}+\frac{\cos 4x}{4^2-1}+\ldots\right]$

4. Find a Fourier series to represent the function $f(x) = e^x$, for $-\pi < x < \pi$ and hence derive a series for $\frac{\pi}{\sinh\pi}$.

Ans. $\frac{2\sinh\pi}{\pi}\left[\left(\frac{1}{2}-\frac{1}{1^2+1}\cos x+\frac{1}{2^2+1}\cos 2x-\frac{1}{3^2+1}\cos 3x+\ldots\right)\right.$

$\left.+\frac{1}{1^2+1}\sin x-\frac{2}{2^2+1}\sin 2x+\frac{3}{3^2+1}\sin 3x\ldots\right] \quad \frac{\pi}{\sinh\pi} = 1+2\left[-\frac{1}{2}+\frac{1}{5}-\frac{1}{10}+\ldots\right.$

5. Obtain the Fourier series for $f(x) = e^{-x}$ in the interval $0 \le x < 2\pi$.

Ans. $\frac{1-e^{-2\pi}}{\pi}\left[\frac{1}{2}+\frac{1}{2}\cos x+\frac{1}{5}\cos 2x+\frac{1}{10}\cos 3x+\frac{1}{2}\sin x+\frac{2}{5}\sin 2x+\frac{3}{10}\sin 3x+\ldots\right]$

6. If $f(x)=\left(\frac{\pi-x}{2}\right)^2$, $0<x<2\pi$, show that $f(x) = \frac{\pi^2}{12}+\sum_{n=1}^{\infty}\frac{\cos nx}{n^2}$

7. Prove that $x^2 = \frac{\pi^2}{3}+4\sum_{n=1}^{\infty}(-1)^n\frac{\cos nx}{n^2}$, $-\pi<x<\pi$.

Hence show that (*i*) $\Sigma\frac{1}{n^2} = \frac{\pi}{6}$

(*ii*) $\Sigma\frac{1}{(2n-1)^2} = \frac{\pi^2}{8}$ (*iii*) $\Sigma\frac{1}{n^4} = \frac{\pi^4}{90}$

8. If $f(x)$ is a periodic function defined over a period $(0, 2\pi)$ by $f(x) = \frac{(3x^2-6x\pi+2\pi^2)}{12}$.

Prove that $f(x) = \sum_{n=1}^{\infty}\frac{\cos nx}{n^2}$ and hence show that $\frac{\pi^2}{6} = 1+\frac{1}{2^2}+\frac{1}{3^2}+\ldots$

12.7 FUNCTION DEFINED IN TWO OR MORE SUB-RANGES

Example 3. *Find the Fourier series of the function*

$$f(x) = \begin{cases} -1 & \text{for} \quad -\pi < x < -\frac{\pi}{2} \\ 0 & \text{for} \quad -\frac{\pi}{2} < x < \frac{\pi}{2} \\ +1 & \text{for} \quad \frac{\pi}{2} < x < \pi. \end{cases}$$

Solution. Let $f(x) = \frac{a_0}{2}+a_1\cos x+a_2\cos 2x+\ldots+b_1\sin x+b_2\sin 2x+\ldots$ ...(1)

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi}f(x)\,dx = \frac{1}{\pi}\int_{-\pi}^{-\pi/2}(-1)\,dx+\frac{1}{\pi}\int_{-\pi/2}^{\pi/2}0\,dx+\frac{1}{\pi}\int_{\pi/2}^{\pi}1\,dx$$

$$= \frac{1}{\pi}\Big[-x\Big]_{-\pi}^{-\pi/2}+\frac{1}{\pi}\Big[x\Big]_{\pi/2}^{\pi} = \frac{1}{\pi}\left[\frac{\pi}{2}-\pi+\pi-\frac{\pi}{2}\right] = 0$$

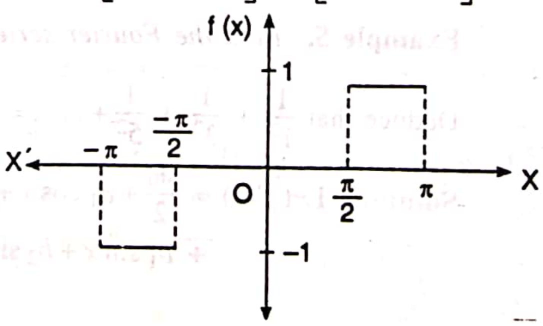
$$a_n = \frac{1}{\pi}\int_{-\pi}^{+\pi}f(x)\cos nx\,dx$$

$$= \frac{1}{\pi}\int_{-\pi}^{-\pi/2}(-1)\cos nx\,dx+\frac{1}{\pi}\int_{-\pi/2}^{\pi/2}(0)\cos nx\,dx+\frac{1}{\pi}\int_{\pi/2}^{\pi}(1)\cos nx\,dx$$

$$= -\frac{1}{\pi}\left[\frac{\sin nx}{n}\right]_{-\pi}^{-\pi/2}+\frac{1}{\pi}\left[\frac{\sin nx}{n}\right]_{\pi/2}^{\pi} = -\frac{1}{\pi}\left[-\frac{\sin\frac{n\pi}{2}}{n}\right]+\frac{1}{\pi}\left[-\frac{\sin\frac{n\pi}{2}}{n}\right] = 0$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi}f(x)\sin nx\,dx$$

$$= \frac{1}{\pi}\int_{-\pi}^{-\pi/2}(-1)\sin nx\,dx+\frac{1}{\pi}\int_{-\pi/2}^{\pi/2}(0)\sin nx\,dx$$



$$+\frac{1}{\pi}\int_{\pi/2}^{\pi}(1)\sin nx\,dx = \frac{1}{\pi}\left[\frac{\cos nx}{n}\right]_{-\pi}^{-\pi/2} - \frac{1}{\pi}\left[\frac{\cos nx}{n}\right]_{\pi/2}^{\pi}$$

$$= \frac{1}{n\pi}\left[\cos\frac{n\pi}{2} - \cos n\pi\right] - \frac{1}{n\pi}\left(\cos n\pi - \cos\frac{n\pi}{2}\right) = \frac{2}{n\pi}\left[\cos\frac{n\pi}{2} - \cos n\pi\right]$$

$$b_1 = \frac{2}{\pi},\ b_2 = -\frac{2}{\pi},\ b_3 = \frac{2}{3\pi}$$

Putting the values of a_0, a_n, b_n in (1) we get

$$f(x) = \frac{1}{\pi}\left[2\sin x - 2\sin 2x + \frac{2}{3}\sin 3x + \dots\right]$$ **Ans.**

Example 4. *Find the Fourier series for the periodic function*

$$f(x) = \begin{bmatrix} 0, & -\pi < x < 0 \\ x, & 0 < x < \pi \end{bmatrix}$$

$$f(x+2\pi) = f(x)$$

Solution. Let $f(x) = \frac{a_0}{2} + a_1\cos x + a_0\cos 2x + \dots + b_1\sin x + b_2\sin 2x + \dots$ ...(1)

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{0} 0\,.\,dx + \frac{1}{\pi}\int_0^{\pi} x\,dx = \frac{1}{\pi}\left[\frac{x^2}{2}\right]_0^{\pi} = \frac{1}{\pi}\left(\frac{\pi^2}{2}\right) = \frac{\pi}{2}$$

$$a_n = \frac{1}{\pi}\int_0^{\pi} x\cos nx\,dx = \frac{1}{\pi}\left[x\cdot\frac{\sin nx}{n} - (1)\left(-\frac{\cos nx}{n^2}\right)\right]_0^{\pi}$$

$$= \frac{1}{\pi}\left[\frac{(-1)^n}{n^2} - \frac{1}{n^2}\right] = -\frac{2}{n^2\pi} \quad \text{when } n \text{ is odd}$$

$$= 0, \text{ when } n \text{ is even.}$$

$$b_n = \frac{1}{\pi}\int_0^{\pi} x\sin nx\,dx = \frac{1}{\pi}\left[x\left(-\frac{\cos nx}{n}\right) - (1)\left(-\frac{\sin nx}{n^2}\right)\right]_0^{\pi} = \frac{1}{\pi}\left[-\pi\frac{(-1)^n}{n}\right] = -\frac{(-1)^n}{n}.$$

Substituting the values of $a_0, a_1, a_2 \dots b_1, b_2, \dots$ in (1), we get

$$f(x) = \frac{\pi}{4} - \frac{2}{\pi}\left[\frac{\cos x}{1^2} + \frac{\cos 3x}{3^2} + \frac{\cos 5x}{5^2}\dots\right] + \left[\frac{\sin x}{1} - \frac{\sin 2x}{2} + \frac{\sin 3x}{3} + \dots\right]$$ **Ans.**

DISCONTINUOUS FUNCTIONS

At a point of discontinuity, Fourier series gives the value of $f(x)$ as the arithmetic mean of left and right limits.

At the point of discontinuity, $x = c$

$$\text{At } x = c,\ f(x) = \frac{1}{2}\left[f(c-0) + f(c+0)\right]$$

Example 5. *Find the Fourier series for $f(x)$, if* $f(x) = \begin{bmatrix} -\pi & -\pi < x < 0 \\ x, & 0 < x < \pi \end{bmatrix}$.

Deduce that $\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \dots = \frac{\pi^2}{8}$

Solution. Let $f(x) = \frac{a_0}{2} + a_1\cos x + a_2\cos 2x + \dots + a_n\cos nx + \dots$

$$+ b_1\sin x + b_2\sin 2x + \dots + b_n\sin nx + \dots \quad \text{... (1)}$$

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\, dx$$

Then $a_0 = \frac{1}{\pi}\left[\int_{-\pi}^{0} (-\pi)\, dx + \int_0^{\pi} x\, dx\right] = \frac{1}{\pi}\left[-\pi\,(x)_{-\pi}^{0} + (x^2/2)_0^{\pi}\right] = \frac{1}{\pi}(-\pi^2 + \pi^2/2) = -\frac{\pi}{2}$;

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\, dx$$

$$a_n = \frac{1}{\pi}\left[\int_{-\pi}^{0} (-\pi)\cos nx\, dx + \int_0^{\pi} x\cos nx\, dx\right] = \frac{1}{\pi}\left[-\pi\left(\frac{\sin nx}{n}\right)_{-\pi}^{0} + \left(\frac{x\sin nx}{n} + \frac{\cos nx}{n^2}\right)_0^{\pi}\right]$$

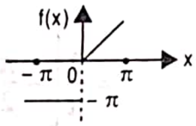
$$= \frac{1}{\pi}\left[0 + \frac{1}{n^2}\cos n\pi - \frac{1}{n^2}\right] = \frac{1}{\pi n^2}(\cos n\pi - 1), \qquad b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\, dx$$

$$b_n = \frac{1}{\pi}\left[\int_{-\pi}^{0} (-\pi)\sin nx\, dx + \int_0^{\pi} x\sin nx\, dx\right] = \frac{1}{\pi}\left[\left(\frac{\pi\cos nx}{n}\right)_{-\pi}^{0} + \left(-x\frac{\cos nx}{n} + \frac{\sin nx}{n^2}\right)_0^{\pi}\right]$$

$$= \frac{1}{\pi}\left[\frac{\pi}{n}(1 - \cos n\pi) - \frac{\pi}{n}\cos n\pi\right] = \frac{1}{n}(1 - 2\cos n\pi)$$

$$f(x) = -\frac{\pi}{4} - \frac{2}{\pi}\left(\cos x + \frac{\cos 3x}{3^2} + \frac{\cos 5x}{5^2} + \dots\right) + 3\sin x - \frac{\sin 2x}{2} + \frac{3\sin 3x}{3} - \frac{\sin 4x}{4} \quad \dots(2)$$

Putting $x = 0$ in (2), we get $f(0) = -\frac{\pi}{4} - \frac{2}{\pi}\left(1 + \frac{1}{3^2} + \frac{1}{5^2} + \dots\infty\right)$ (3)



Now $f(x)$ is discontinuous at $x = 0$.

But $f(0-0) = -\pi$ and $f(0+0) = 0$ $\therefore$ $f(0) = \frac{1}{2}[f(0-0) + f(0+0)] = -\pi/2$

From (3), $-\frac{\pi}{2} = -\frac{\pi}{4} - \frac{2}{\pi}\left[\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \dots\right]$ or $\frac{\pi^2}{8} = \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \dots$ **Proved**

Example 6. *Find the Fourier series expansion of the periodic function of period 2π, defined by*

$$f(x) = x, \quad \text{if } -\frac{\pi}{2} < x < \frac{\pi}{2}, \qquad f(x) = \pi - x, \quad \text{if } \frac{\pi}{2} < x < \frac{3\pi}{2}$$

Solution. Let $f(x) = \frac{a_0}{2} + a_1\cos x + a_2\cos 2x + \dots + b_1\sin x + b_2\sin 2x + \dots$

$$a_0 = \frac{1}{\pi}\int_{-\pi/2}^{\pi/2} x\, dx + \frac{1}{\pi}\int_{\pi/2}^{\frac{3\pi}{2}} (\pi - x)\, dx = \frac{1}{\pi}\left(\frac{x^2}{2}\right)_{-\pi/2}^{\pi/2} + \frac{1}{\pi}\left(\pi x - \frac{x^2}{2}\right)_{\pi/2}^{\frac{3\pi}{2}}$$

$$= \frac{1}{\pi}\left(\frac{\pi^2}{8} - \frac{\pi^2}{8}\right) + \frac{1}{\pi}\left(\frac{3\pi^2}{2} - \frac{9\pi^2}{8} - \frac{\pi^2}{2} + \frac{\pi^2}{8}\right) = \pi\left(\frac{3}{2} - \frac{9}{8} - \frac{1}{2} + \frac{1}{8}\right) = 0$$

$$a_n = \frac{1}{\pi}\int_{-\pi/2}^{\pi/2} x\cos nx\, dx + \frac{1}{\pi}\int_{\pi/2}^{\frac{3\pi}{2}} (\pi - x)\cos nx\, dx$$

$$= \frac{1}{\pi}\left[x\frac{\sin nx}{n} - (1)\left(-\frac{\cos nx}{n^2}\right)\right]_{-\pi/2}^{\pi/2} + \frac{1}{\pi}\left[(\pi - x)\frac{\sin nx}{n} - (-1)\left(-\frac{\cos nx}{n^2}\right)\right]_{\pi/2}^{\frac{3\pi}{2}}$$

$$= \frac{1}{\pi}\left[\frac{\pi}{2}\frac{\sin\frac{n\pi}{2}}{n} + \frac{\cos\frac{n\pi}{2}}{n^2} - \frac{\pi}{2}\frac{\sin\frac{n\pi}{2}}{n} - \frac{\cos\frac{n\pi}{2}}{n^2}\right]$$

$$+ \frac{1}{\pi}\left[-\frac{\pi}{2}\frac{\sin\frac{3n\pi}{2}}{n} - \frac{\cos\frac{3n\pi}{2}}{n^2} - \frac{\pi}{2}\frac{\sin\frac{n\pi}{2}}{n} + \cos\frac{\frac{n\pi}{2}}{n^2}\right]$$

$$= \frac{1}{\pi}\left[-\frac{\pi}{2n}\left(\sin\frac{3n\pi}{2} + \sin\frac{n\pi}{2}\right) - \frac{1}{n^2}\left(\cos\frac{3n\pi}{2} - \cos\frac{n\pi}{2}\right)\right]$$

$$= \frac{1}{\pi}\left[-\frac{\pi}{n}\sin n\pi\cos\frac{n\pi}{2} + \frac{2}{n^2}\sin\frac{n\pi}{2}\sin n\pi\right] = 0$$

$$b_n = \frac{1}{\pi}\int_{-\pi/2}^{\pi/2} x\sin nx\,dx + \frac{1}{\pi}\int_{\pi/2}^{3\pi/2}(\pi - x)\sin nx\,dx$$

$$= \frac{2}{\pi}\int_0^{\pi/2} x\sin nx\,dx + \frac{1}{\pi}\int_{\pi/2}^{\frac{3\pi}{2}}(\pi - x)\sin nx\,dx$$

$$= \frac{2}{\pi}\left[x\left(-\frac{\cos nx}{n}\right) - (1)\left(-\frac{\sin nx}{n^2}\right)\right]_0^{\pi/2} + \frac{1}{\pi}\left[(\pi - x)\left(-\frac{\cos nx}{n}\right) - (-1)\left(-\frac{\sin nx}{n^2}\right)\right]_{\pi/2}^{\frac{3\pi}{2}}$$

$$= \frac{2}{\pi}\left[-\frac{\pi}{2}\frac{\cos\frac{n\pi}{2}}{n} + \frac{\sin\frac{n\pi}{2}}{n^2}\right] + \frac{1}{\pi}\left[\frac{\pi}{2}\frac{\cos\frac{3n\pi}{2}}{n} - \frac{\sin\frac{3n\pi}{2}}{n^2} + \frac{\pi}{2}\frac{\cos\frac{n\pi}{2}}{n} + \frac{\sin\frac{n\pi}{2}}{n^2}\right]$$

$$= \frac{1}{\pi}\left[-\frac{\pi}{2}\frac{\cos\frac{n\pi}{2}}{n} + \frac{3\sin\frac{n\pi}{2}}{n^2} + \frac{\pi}{2}\frac{\cos 3\frac{n\pi}{2}}{n} - \frac{\sin\frac{3n\pi}{2}}{n^2}\right]$$

$$= \frac{1}{\pi}\left[\frac{\pi}{2n}\left(\cos\frac{3n\pi}{2} - \cos\frac{n\pi}{2}\right) + \frac{3}{n^2}\sin\frac{n\pi}{2} - \frac{1}{n^2}\sin\frac{3n\pi}{2}\right]$$

$$= \frac{1}{\pi}\left[-\frac{\pi}{n}\sin\frac{n\pi}{2}\sin n\pi + \frac{3}{n^2}\sin\frac{n\pi}{2} - \frac{1}{n^2}\sin\frac{3n\pi}{2}\right] = \frac{1}{n^2\pi}\left[3\sin\frac{n\pi}{2} - \sin\frac{3n\pi}{2}\right]$$

Substituting the values of $a_0, a_1, a_2 \ldots b_1, b_2, \ldots$ we get

$$f(x) = \frac{4}{\pi}\left[\frac{\sin x}{1^2} - \frac{\sin 3x}{3^2} + \frac{\sin 5x}{5^2} - \ldots\right]$$ **Ans.**

Example 7. *Find the Fourier series of the function defined as*

$$f(x) = \begin{cases} x + \pi & \text{for } 0 \le x \le \pi \\ -x - \pi & -\pi \le x < 0 \end{cases} \quad \text{and} \quad f(x + 2\pi) = f(x).$$

Solution. $a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx = \frac{1}{\pi}\int_{-\pi}^{0} f(x)\,dx + \frac{1}{\pi}\int_0^{\pi} f(x)\,dx$

$$= \frac{1}{\pi}\int_{-\pi}^{0}(-x - \pi)\,dx + \frac{1}{\pi}\int_0^{\pi}(x + \pi)\,dx = \frac{1}{\pi}\left(-\frac{x^2}{2} - \pi x\right)_{-\pi}^{0} + \frac{1}{\pi}\left(\frac{x^2}{2} + \pi x\right)_0^{\pi}$$

$$= \frac{1}{\pi}\left(\frac{\pi^2}{2} - \pi^2\right) + \frac{1}{\pi}\left(\frac{\pi^2}{2} + \pi^2\right) = \pi\left(\frac{1}{2} - 1\right) + \pi\left(\frac{1}{2} + 1\right) = \pi$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx = \frac{1}{\pi}\int_{-\pi}^{0} f(x)\cos nx\,dx + \frac{1}{\pi}\int_{0}^{\pi} f(x)\cos nx\,dx$$

$$= \frac{1}{\pi}\int_{-\pi}^{0}(-x-\pi)\cos nx\,dx + \frac{1}{\pi}\int_{0}^{\pi}(x+\pi)\cos nx\,dx$$

$$= \frac{1}{\pi}\left[(-x-\pi)\frac{\sin nx}{n} - (-1)\left\{-\frac{\cos nx}{n^2}\right\}\right]_{-\pi}^{0} + \frac{1}{\pi}\left[(x+\pi)\frac{\sin nx}{n} - (1)\left\{-\frac{\cos nx}{n^2}\right\}\right]_{0}^{\pi}$$

$$= \frac{1}{\pi}\left[-\frac{1}{n^2} + \frac{(-1)^n}{n^2}\right] + \frac{1}{\pi}\left[\frac{(-1)^n}{n^2} - \frac{1}{n^2}\right] = \frac{2}{n^2\pi}[(-1)^n - 1] = \frac{-4}{n^2\pi} \quad \text{if } n \text{ is odd.}$$

$$= 0 \quad \text{if } n \text{ is even.}$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx = \frac{1}{\pi}\int_{-\pi}^{0} f(x)\sin nx\,dx + \frac{1}{\pi}\int_{0}^{\pi} f(x)\sin nx\,dx$$

$$= \frac{1}{\pi}\int_{-\pi}^{0}(-x-\pi)\sin nx\,dx + \frac{1}{\pi}\int_{0}^{\pi}(x+\pi)\sin nx\,dx$$

$$= \frac{1}{\pi}\left[(-x-\pi)\left(-\frac{\cos nx}{n}\right) - (-1)\left(-\frac{\sin nx}{n^2}\right)\right]_{-\pi}^{0}$$

$$+ \frac{1}{\pi}\left[(x+\pi)\left(-\frac{\cos nx}{n}\right) - (1)\left(-\frac{\sin nx}{n^2}\right)\right]_{0}^{\pi}$$

$$= \frac{1}{\pi}\left[\frac{\pi}{n}\right] + \frac{1}{\pi}\left[-\frac{2\pi}{n}(-1)^n + \frac{\pi}{n}\right] = \frac{1}{n}[(1) - 2(-1)^n + (1)] = \frac{2}{n}[1-(-1)^n]$$

$$= \frac{4}{n}, \qquad \text{if } n \text{ is odd.}$$

$$= 0, \qquad \text{if } n \text{ is even.}$$

Fourier series is

$$f(x) = \frac{a_0}{2} + a_1\cos x + a_2\cos 2x + \ldots + b_1\sin x + b_2\sin 2x + \ldots$$

$$f(x) = \frac{\pi}{2} - \frac{4}{\pi}\left(\frac{\cos x}{1^2} + \frac{\cos 3x}{3^2} + \ldots\right) + 4\left(\frac{\sin x}{1} + \frac{\sin 3x}{3} + \ldots\right)$$

Ans.

Exercise 12.2

1. Find the Fourier series of the function

$$f(x) = \begin{cases} -1 & \text{for} \quad -\pi < x < 0 \\ 1 & \text{for} \quad 0 < x < \pi \end{cases}$$

where $f(x+2\pi) = f(x)$.

Ans. $\frac{4}{\pi}\left[\frac{1}{1}\sin x + \frac{1}{3}\sin 3x + \frac{1}{5}\sin 5x + \frac{1}{7}\sin 7x + \ldots\right]$

2. Find the Fourier series for the function

$$f(x) = \begin{cases} -\frac{\pi}{4} & \text{for} \quad -\pi < x < 0 \\ \frac{\pi}{4} & \text{for} \quad 0 < x < \pi \end{cases}$$

and $f(-\pi) = f(0) = f(\pi) = 0$, $f(x) = f(x+2\pi)$ for all x.

Deduce that

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \ldots$$

Ans. $\frac{\sin x}{1} + \frac{\sin 3x}{3} + \frac{\sin 5x}{5} + \frac{\sin 7x}{7} + \ldots\ldots$

3. Find the Fourier series of the function

$$f(x) = \begin{cases} 0 & \text{for} \quad -\pi \le x \le 0 \\ 1 & \text{for} \quad 0 < x < \frac{\pi}{2} \\ 0 & \text{for} \quad \frac{\pi}{2} \le x \le \pi \end{cases}$$

4. Obtain a Fourier series to represent the following periodic function

$$f(x) = 0 \quad \text{when} \quad 0 < x < \pi$$
$$f(x) = 1 \quad \text{when} \quad \pi < x < 2\pi$$

Ans. $\frac{1}{2} - \frac{2}{\pi}\left(\sin x + \frac{1}{3}\sin 3x + \frac{1}{5}\sin 5x + \ldots\right)$

5. Find the Fourier expansion of the function defined in a single period by the relations.

$$f(x) = \begin{cases} 1 & \text{for} \quad 0 < x < \pi \\ 2 & \text{for} \quad \pi < x < 2\pi \end{cases}$$

and from it deduce that $\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \ldots$

Ans. $\frac{3}{2} - \frac{2}{\pi}\left(\sin x + \frac{1}{3}\sin 3x + \frac{1}{5}\sin 5x + \ldots\right)$

6. Find a Fourier series to represent the function

$$f(x) = \begin{cases} 0 & \text{for} \quad -\pi < x \le 0 \\ \frac{1}{4}\pi x & \text{for} \quad 0 < x < \pi \end{cases}$$

and hence deduce that $\frac{\pi^2}{8} = \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \ldots$

Ans. $\frac{\pi^2}{16} + \sum_{n=1}^{\infty}\left(\frac{[(-1)^n - 1]}{4n^2}\cos nx - \frac{(-1)^n \pi}{4n}\sin nx + \ldots\right)$

7. Find the Fourier series for $f(x)$, if

$$f(x) = -\pi \quad \text{for} \quad -\pi < x \le 0$$
$$= x \quad \text{for} \quad 0 < x < \pi$$
$$= \frac{-\pi}{2} \quad \text{for} \quad x = 0$$

Deduce that $\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \ldots = \frac{\pi^2}{8}$

Ans. $-\frac{\pi}{4} - \frac{2}{\pi}\left(\cos x + \frac{\cos 3x}{3^2} + \frac{\cos 5x}{5^2} + \ldots\right) + 3\sin x - \frac{1}{2}\sin 2x + \frac{3}{3}\sin 3x - \frac{1}{4}\sin 4x + \ldots$

8. Obtain a Fourier series to represent the function

$$f(x) = |x| \quad \text{for} \quad -\pi < x < \pi$$

and hence deduce $\frac{\pi^2}{8} = \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \ldots$

Ans. $\frac{\pi}{2} - \frac{4}{\pi}\left[\cos x + \frac{1}{3^2}\cos 3x + \frac{1}{5^2}\cos 5x + \ldots\right]$

9. Expand as a Fourier series, the function $f(x)$ defined as

$$f(x) = \pi + x \quad \text{for} \quad -\pi < x < -\frac{\pi}{2}$$
$$= \frac{\pi}{2} \quad \text{for} \quad -\frac{\pi}{2} < x < \frac{\pi}{2}$$
$$= \pi - x \quad \text{for} \quad \frac{\pi}{2} < x < \pi$$

Ans. $\frac{3\pi}{8} + \frac{2}{\pi}\left[\frac{1}{1^2}\cos x - \frac{2}{2^2}\cos 2x + \frac{1}{3^2}\cos 3x + \ldots\right]$

10. Obtain a Fourier series to represent the function

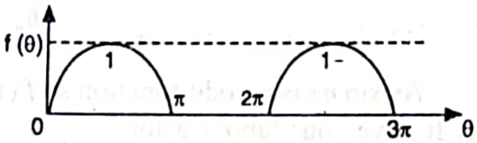
$$f(x) = |\sin x| \quad \text{for} \quad -\pi < x < \pi \qquad \left\{ \textbf{Hint} \quad \begin{array}{lll} f(x) = -\sin x & \text{for} & -\pi < x < 0 \\ \quad\;\; = \sin x & \text{for} & 0 < x < \pi \end{array} \right\}$$

Ans. $\frac{2}{\pi} - \frac{4}{\pi}\left[\frac{1}{3}\cos 2x + \frac{1}{15}\cos 4x + \frac{1}{35}\cos 6x + \ldots\right]$

11. An alternating current after passing through a rectifier has the form

$$i = I \sin\theta \quad \text{for} \quad 0 < \theta < \pi$$
$$= 0 \quad \text{for} \quad \pi < \theta < 2\pi$$

Find the Fourier series of the function.



Ans. $\frac{I}{\pi} - \frac{2I}{\pi}\left(\frac{\cos 2\theta}{3} + \frac{\cos 4\theta}{15} + \ldots\right) + \frac{I}{2}\sin\theta$

12. If $f(x) = 0$ for $-\pi < x < 0$

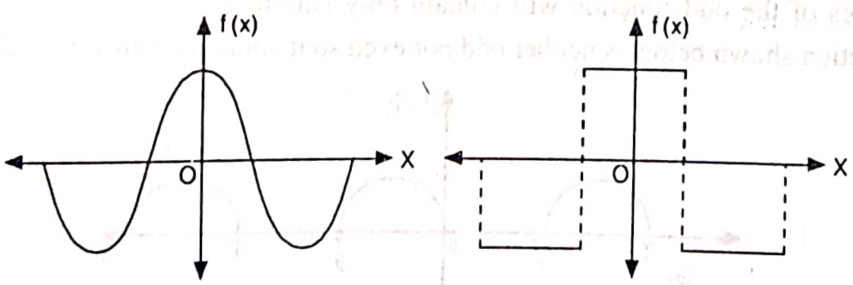
$= \sin x$ for $0 < x < \pi$

Prove that $f(x) = \frac{1}{\pi} + \frac{\sin x}{2} - \frac{2}{\pi}\sum_{m=1}^{\infty}\frac{\cos 2mx}{4m^2 - 1}$. Hence show that $\frac{1}{1.3} - \frac{1}{3.5} + \frac{1}{5.7} \ldots \infty = \frac{1}{4}(\pi - 2)$

12.8 (a) EVEN FUNCTION

A function $f(x)$ is said to be even (or symmetric) function if, $f(-x) = f(x)$

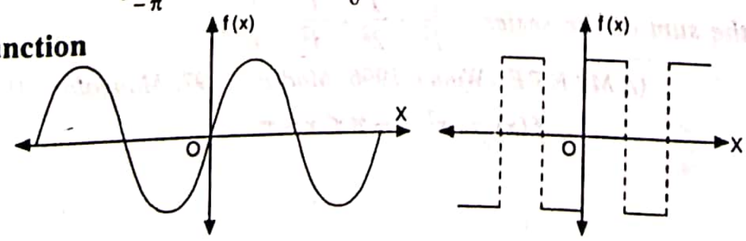
The graph of such a function is symmetrical with respect to y-axis [$f(x)$ axis]. Here y-axis is a mirror for the reflection of the curve.



The area under such a curve from $-\pi$ to π is double the area from 0 to π.

$$\therefore \quad \int_{-\pi}^{\pi} f(x)\,dx = 2\int_{0}^{\pi} f(x)\,dx$$

(b) Odd Function



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A function $f(x)$ is called odd (or skew symmetric) function if

$$f(-x) = -f(x)$$

Here the area under the curve from $-\pi$ to π is zero.

$$\int_{-\pi}^{\pi} f(x)\, dx = 0$$

Expansion of an even function:

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\, dx = \frac{2}{\pi}\int_0^{\pi} f(x)\, dx$$

$$a_n = \frac{1}{\pi}\int_{\pi}^{\pi} f(x)\cos nx\, dx = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\, dx$$

As $f(x)$ and $\cos nx$ are both even functions .

$\therefore$ The product of $f(x)$. $\cos nx$ is also an even function. page 846

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\, dx = 0$$

As $\sin nx$ is an odd function so $f(x)$. $\sin nx$ is also an odd function. We need not to calculate b_n. It saves our labour a lot.

The series of the even function will contain only cosine terms.

Expansion of an odd function :

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\, dx = 0$$

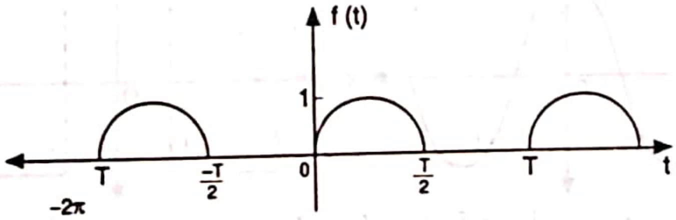
$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\, dx = 0 \quad [f(x).\ \cos nx \text{ is odd function.}]$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\, dx = \frac{2}{\pi}\int_0^{\pi} f(x)\sin nx\, dx$$

[$f(x)$. $\sin nx$ is even function.]

The series of the odd function will contain only sine terms.

The function shown below is neither odd nor even so it contains both sine and cosine terms



Example 8. *Find the Fourier series expansion of the periodic function of period 2π*

$$f(x) = x^2, \quad -\pi \le x \le \pi$$

Hence, find the sum of the series $\frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \dots$

(A.M.I.E.T.E., Winter 1996, Madras 1997, Mangalore 1997, Warangal 1996)

Solution. $\quad f(x) = x^2, \quad -\pi \le x \le \pi$

Solution.

$$f(x) = x^2, \quad -\pi \le x \le \pi$$

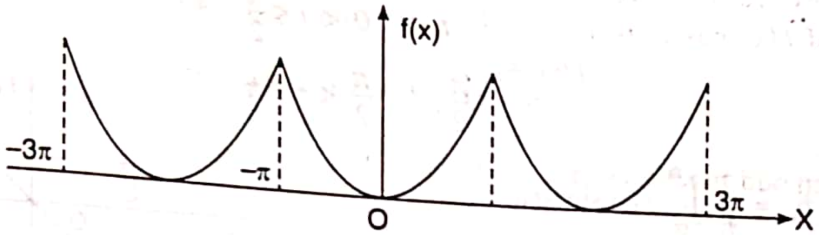
This is an even function. $\therefore\ b_n = 0$

$$a_0 = \frac{2}{\pi}\int_0^{\pi} f(x)\,dx = \frac{2}{\pi}\int_0^{\pi} x^2\,dx = \frac{2}{\pi}\left[\frac{x^3}{3}\right]_0^{\pi} = \frac{2\pi^2}{3}$$

$$a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\,dx = \frac{2}{\pi}\int_0^{\pi} x^2\cos nx\,dx$$

$$= \frac{2}{\pi}\left[x^2\left(\frac{\sin nx}{n}\right) - (2x)\left(-\frac{\cos nx}{n^2}\right) + (2)\left(-\frac{\sin nx}{n^3}\right)\right]_0^{\pi}$$

$$= \frac{2}{\pi}\left[\frac{\pi^2\sin n\pi}{n} + \frac{2\pi\cos n\pi}{n^2} - \frac{2\sin n\pi}{n^3}\right] = \frac{4(-1)^n}{n^2}$$



Fourier series is

$$f(x) = \frac{a_0}{2} + a_1\cos x + a_2\cos 2x + a_3\cos 3x + \ldots + a_n\cos nx + \ldots$$

$$x^2 = \frac{\pi^2}{3} - 4\left[\frac{\cos x}{1^2} - \frac{\cos 2x}{2^2} + \frac{\cos 3x}{3^2} - \frac{\cos 4x}{4^2} + \ldots\right]$$

On putting $x = 0$, we have

$$0 = \frac{\pi^2}{3} - 4\left[\frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2}\ldots\right]$$

or

$$\frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2}\ldots = \frac{\pi^2}{12}$$

Ans

Example 9. *Obtain a Fourier expression for*

$$f(x) = x^3 \quad \textit{for} \quad -\pi < x < \pi.$$

Solution. $f(x) = x^3$ is an odd function.

$\therefore$ $\quad a_0 = 0$ and $a_n = 0$

$$b_n = \frac{2}{\pi}\int_0^{\pi} f(x)\sin nx\,dx = \frac{2}{\pi}\int_0^{\pi} x^3\sin nx\,dx$$

$$\left[\int uv = uv_1 - u'v_2 + u''v_3 - u'''v_4 + \ldots\right]$$

$$= \frac{2}{\pi}\left[x^3\left(\frac{\cos nx}{n}\right) - 3x^2\left(-\frac{\sin nx}{n^2}\right) + 6x\left(\frac{\cos nx}{n^3}\right) - 6\left(\frac{\sin nx}{n^4}\right)\right]_0^{\pi}$$

$$= \frac{2}{\pi}\left[-\frac{\pi^3\cos n\pi}{n} + \frac{6\pi\cos n\pi}{n^3}\right] = 2.(-1)^n\left[-\frac{\pi^2}{n} + \frac{6}{n^3}\right]$$

$$\therefore \quad x^3 = 2\left[-\left(-\frac{\pi^2}{1} + \frac{6}{1^3}\right)\sin x + \left(-\frac{\pi^2}{2} + \frac{6}{2^3}\right)\sin 2x - \left(-\frac{\pi^2}{3} + \frac{6}{3^3}\right)\sin 3x \ldots\right]$$

Ans.

12.9 HALF-RANGE SERIES, PERIOD 0 TO π

The given function is defined in the interval $(0, \pi)$ and it is immaterial whatever the function may be outside the interval $(0, \pi)$. To get the series of cosines only we assume that $f(x)$ is an even function in the interval $(-\pi, \pi)$.

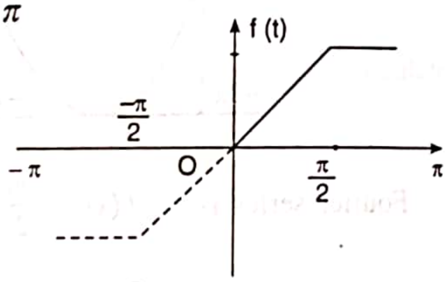
$$a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\,dx \quad \text{and} \quad b_n = 0$$

To expand $f(x)$ as a sine series we extend the function in the interval $(-\pi, \pi)$ as an odd function.

$$b_n = \frac{2}{\pi}\int_0^{\pi} f(x)\sin nx\,dx \quad \text{and} \quad a_n = 0$$

Example 10. *Represent the following function by a Fourier sine series :*

$$f(t) = \begin{cases} t, & 0 < t \le \dfrac{\pi}{2} \\ \dfrac{\pi}{2}, & \dfrac{\pi}{2} < t \le \pi \end{cases}$$



Solution. $b_n = \dfrac{2}{\pi}\displaystyle\int_0^{\pi} f(t)\sin nt\,dt$

$$= \frac{2}{\pi}\int_0^{\pi/2} t\sin nt\,dt + \frac{2}{\pi}\int_{\pi/2}^{\pi} \frac{\pi}{2}\sin nt\,dt$$

$$= \frac{2}{\pi}\left[t\left(-\frac{\cos nt}{n}\right) - (1)\left(-\frac{\sin nt}{n^2}\right)\right]_0^{\pi/2} + \frac{2}{\pi}\frac{\pi}{2}\left[-\frac{\cos nt}{n}\right]_{\pi/2}^{\pi}$$

$$= \frac{2}{\pi}\left[-\frac{\pi}{2}\frac{\cos\frac{n\pi}{2}}{n} + \frac{\sin\frac{n\pi}{2}}{n^2}\right] + \left[-\frac{\cos n\pi}{n} + \frac{\cos\frac{n\pi}{2}}{n}\right]$$

$$b_1 = \frac{2}{\pi}\left[-\frac{\pi}{2}\cos\frac{\pi}{2} + \sin\frac{\pi}{2}\right] + \left[-\cos\pi + \cos\frac{\pi}{2}\right] = \frac{2}{\pi}[0+1] + [1] = \frac{2}{\pi} + 1$$

$$b_2 = \frac{2}{\pi}\left[-\frac{\pi}{2}\frac{\cos\pi}{2} + \frac{\sin\pi}{2^2}\right] + \left[-\frac{\cos 2\pi}{2} + \frac{\cos\pi}{2}\right] = \frac{2}{\pi}\left[-\frac{\pi}{2}\frac{(-1)}{2} + 0\right] + \left[-\frac{1}{2} - \frac{1}{2}\right]$$

$$= \frac{2}{\pi}\left[\frac{\pi}{4}\right] - 1 = \frac{1}{2} - 1 = -\frac{1}{2}$$

$$b_3 = \frac{2}{\pi}\left[-\frac{\pi}{2}\frac{\cos\frac{3\pi}{2}}{3} + \frac{\sin\frac{3\pi}{2}}{3^2}\right] + \left[-\frac{\cos 3\pi}{3} + \frac{\cos\frac{3\pi}{2}}{3}\right]$$

$$= \frac{2}{\pi}\left[-\frac{\pi}{2}(0) - \frac{1}{9}\right] + \left[\frac{1}{3} + 0\right] = -\frac{2}{9\pi} + \frac{1}{3}$$

$$f(t) = \left(\frac{2}{\pi} + 1\right)\sin t - \frac{1}{2}\sin 2t + \left(-\frac{2}{9\pi} + \frac{1}{3}\right)\sin 3t + \ldots \quad \textbf{Ans.}$$

Example 11. *Find the Fourier sine series for the function*

$$f(x) = e^{ax} \quad \text{for} \quad 0 < x < \pi$$

where a is constant.

Solution.

$$\int e^{ax} \sin bx \, dx = \frac{e^{ax}}{a^2+b^2}[a \sin bx - b \cos bx]$$

$$b_n = \frac{2}{\pi}\int_0^{\pi} e^{ax} \sin nx \, dx$$

$$= \frac{2}{\pi}\left[\frac{e^{ax}}{a^2+n^2}(a \sin nx - n \cos nx)\right]_0^{\pi}$$

$$= \frac{2}{\pi}\left[\frac{e^{a\pi}}{a^2+n^2}(a \sin n\pi - n \cos n\pi) + \frac{n}{a^2+n^2}\right]$$

$$= \frac{2}{\pi}\frac{n}{a^2+n^2}\left[-(-1)^n e^{a\pi} + 1\right] = \frac{2n}{(a^2+n^2)\pi}\left[1 - (-1)^n e^{a\pi}\right]$$

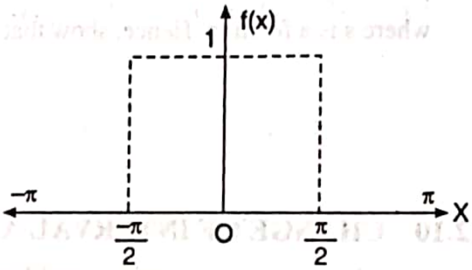
$$b_1 = \frac{2(1+e^{a\pi})}{(a^2+1^2)\pi}, \quad b_2 = \frac{2 \,.\, 2 \,.\, (1-e^{a\pi})}{.\,(a^2+2^2)\pi}$$

$$e^{ax} = \frac{2}{\pi}\left[\frac{1+e^{a\pi}}{a^2+1^2}\sin x + \frac{2(1-e^{a\pi})}{a^2+2^2}\sin 2x + \dots\right] \qquad \textbf{Ans.}$$

Exercise 12.3

1. Find the Fourier cosine series for the function

$$f(x) = \begin{cases} 1 & \text{for} \quad 0 < x < \dfrac{\pi}{2} \\ 0 & \text{for} \quad \dfrac{\pi}{2} < x < \pi. \end{cases}$$



Ans. $\frac{1}{2} + \frac{2}{\pi}\left[\cos x - \frac{1}{3}\cos 3x + \frac{1}{5}\cos 5x - \dots\right]$

2. Find a series of cosine of multiples of x which will represent $f(x)$ in $(0, \pi)$ where

$$f(x) = 0 \quad \text{for} \quad 0 < x < \frac{\pi}{2}$$

$$f(x) = \frac{\pi}{2} \quad \text{for} \quad \frac{\pi}{2} < x < \pi$$

Deduce that $1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \dots \infty = \frac{\pi}{4}$ **Ans.** $\frac{\pi}{4} - \cos x + \frac{1}{3}\cos 3x - \frac{1}{5}\cos 5x + \dots$

3. Express $f(x) = x$ as a sine series in $0 < x < \pi$. **Ans.** $2\left[\sin x - \frac{1}{2}\sin 2x + \frac{1}{3}\sin 3x - \dots\right]$

4. Find the cosine series for $f(x) = \pi - x$ in the interval $0 < x < \pi$.

Ans. $\frac{\pi}{2} + \frac{4}{\pi}\left(\cos x + \frac{\cos 3x}{3^2} + \frac{\cos 5x}{5^2} + \dots\right)$

5. If $f(x) = x, \quad \text{for} \quad 0 < x < \frac{\pi}{2}$

$= \pi - x, \quad \text{for} \quad \frac{\pi}{2} < x < \pi$

Show that:

(i) $f(x) = \frac{4}{\pi}\left(\sin x - \frac{1}{3^2}\sin 3x + \frac{1}{5^2}\sin 5x - \ldots\right)$

(ii) $f(x) = \frac{\pi}{4} - \frac{2}{\pi}\left(\frac{1}{1^2}\cos 2x + \frac{1}{3^2}\cos 6x + \frac{1}{5^2}\cos 10x + \ldots\right)$

6. Obtain the half-range cosine series for $f(x) = x^2$ in $0 < x < \pi$.

Ans. $\frac{\pi^2}{3} - \frac{4}{\pi}\left(\cos x - \frac{1}{2^2}\cos 2x + \frac{1}{3^2}\cos 3x - \ldots\right)$

7. Find (i) sine series and (ii) cosine series for the function

$$f(x) = e^x \quad \text{for} \quad 0 < x < \pi.$$

Ans. (i) $\frac{2}{\pi}\sum_1^\infty n\left[\frac{1-(-1)^n e^\pi}{n^2+1}\right]\sin nx$ (ii) $\frac{e^\pi - 1}{\pi} - \frac{2}{\pi}\sum_1^\infty \frac{1-(-1)^n e^\pi}{n^2+1}\cos nx$

8. If $f(x) = x + 1$, for $0 < x < \pi$, find its Fourier (i) sine series (ii) cosine series. Hence deduce that

(i) $1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \ldots = \frac{\pi}{4}$ (ii) $1 + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \ldots = \frac{\pi^2}{8}$

Ans. (i) $\frac{2}{\pi}\left[(\pi+2)\sin x - \frac{\pi}{2}\sin 2x + \frac{1}{3}(\pi+2)\sin 3x - \frac{\pi}{4}\sin 4x + \ldots\right]$

(ii) $\frac{\pi}{2} + 1 - 4\left(\cos x + \frac{\cos 3x}{3^2} + \frac{\cos 5x}{5^2} + \ldots\right)$

9. Find the Fourier series expansion of the function

$$f(x) = \cos(sx), \quad -\pi \le x \le \pi$$

where s is a fraction. Hence, show that $\cot\theta = \frac{1}{\theta} + \frac{2\theta}{\theta^2 - \pi^2} + \frac{2\theta}{\theta^2 - 4\pi^2} + \ldots$

Ans. $\frac{\sin \pi x}{\pi s} + \frac{1}{\pi}\Sigma\left(\frac{\sin(s\pi + n\pi)}{s+n} + \frac{\sin(s\pi - n\pi)}{s-n}\right]\cos nx$