Practice Problems - Week #5

Nonhomogeneous Linear DEs

Solutions

Note: we use the notation y_p **or** y_{NH} interchangeably for a single solution that satisfies a non-homogeneous equation. These stand for "particular" or "non-homogenous", both of which are common names for that part of the differential equation solution.

For Questions 1 to 8, find a single solution that satisfies the non-homogeneous differential equation.

1. $y'' + 16y = e^{3x}$

Given the RHS of e^{3x}, we would prefer to use $y_{NH} = Ae^{3x}$.

For y_c, we solve $r^2 + 16 = 0$ which gives $r = \pm 4i$ and $y_c = c_1 \cos(4x) + c_2 \sin(4x)$. There is no overlap between y_c then and the preferred form of y_{NH}, so we can use it as-is:

Let $y_{NH} = Ae^{3x}$. Then
$y'_{NH} = 3Ae^{3x}$ and
$y''_{NH} = 9Ae^{3x}$

Subbing in y_{NH} and its derivatives into the DE,

$$\begin{aligned} 9Ae^{3x} + 16Ae^{3x} &= e^{3x} \\ 25A &= 1 \\ A &= \frac{1}{25} \end{aligned}$$

So our particular solution is $y_{NH} = \dfrac{1}{25}e^{3x}$.

2. $y'' - y' - 2y = 3x + 4$

Given the RHS of $3x + 4$, we would prefer to use $y_{NH} = A + Bx$.

For y_c, we solve $r^2 - r - 2 = 0$ or $(r-2)(r+1) = 0$ which gives $r = 2, -1$ and $y_c = c_1 e^{2x} + c_2 e^{-x}$. There is no overlap between y_c then and the preferred form of y_{NH}, so we can use it as-is:

Let $y_{NH} = A + Bx$. Then
$y'_{NH} = B$ and
$y''_{NH} = 0$

Subbing in y_{NH} and its derivatives into the DE,

$$\begin{aligned} -(B) - 2(A + Bx) &= 3x + 4 \\ -2Bx + (-B - 2A) &= 3x + 4 \\ \text{Equating const coeffs: } -B - 2A &= 4 \\ \text{Equating } x \text{ coeffs: } -2B &= 3 \\ \text{so } B &= \frac{-3}{2} \\ \text{and } A &= \frac{-1}{2}(-4 + B) = \frac{-5}{4} \end{aligned}$$

So our particular solution is $y_{NH} = \dfrac{-5}{4} - \dfrac{3}{2}x$

3. $y'' - y' - 6y = 2\sin(3x)$

Similar to the last two problems, we would want $y_{NH} = A\sin(3x) + B\cos(3x)$.

The homogeneous aux. equation has $r = 3, -2$, so $y_c = c_1e^{3x} + c_2e^{-2x}$, so there is no overlap with our desired y_{NH} form.

$y_{NH} = A\sin(3x) + B\cos(3x)$
$y'_{NH} = 3A\cos(3x) - 3B\sin(3x)$
$y''_{NH} = -9A\sin(3x) - 9B\cos(3x)$

Subbing into the DE,

$$(-9A\sin(3x) - 9B\cos(3x)) - (3A\cos(3x) - 3B\sin(3x)) - 6(A\sin(3x) + B\cos(3x)) = 2\sin(3x)$$
$$\text{Equating sin coeffs: } -15A + 3B = 2$$
$$\text{Equating cos coeffs: } -3A - 15B = 0$$

Solving gives $A = \dfrac{-5}{39}, B = \dfrac{1}{39}$, so

$$y_{NH} = \frac{-5}{39}\sin(3x) + \frac{1}{39}\cos(3x)$$

4. $2y'' + 4y'' + 7y = x^2$

We would want $y_{NH} = A + Bx + Cx^2$.

The homogeneous aux. equation has complex roots, so y_c is made up of exponentials and sines and cosines, so there is no overlap with our desired y_{NH} form.

$y_{NH} = A + Bx + Cx^2$
$y'_{NH} = B + 2Cx$
$y''_{NH} = 2C$

Subbing into the DE,

$$2(2C) + 4(B + 2Cx) + 7(A + Bx + Cx^2) = x^2$$
$$\text{Equating } x^2 \text{ coeffs: } 7C = 1$$
$$\text{Equating } x \text{ coeffs: } 8C + 7B = 0$$
$$\text{Equating const coeffs: } 4C + 4B + 7A = 0$$

Solving gives $C = \dfrac{1}{7}, B = \dfrac{-8}{49}, A = \dfrac{4}{343}$, so

$$y_{NH} = \frac{3}{343} - \frac{8}{49}x + \frac{1}{7}x^2$$

5. $y^{(5)} + 5y^{(4)} - y = 17$

Our desired form is $y_{NH} = A$ (constant).

We can't find the roots for y_c easily for this case, *but* we can be sure that $r^5 + 5r^4 - 1 = 0$ does not have an $r = 0$ root. This means that y_c does not contain a e^{0x}/constant solution, so there is no overlap between y_c and our desired y_{NH} form.

$y_{NH} = A$
$y'_{NH} = 0$ as do all the other derivatives.

Subbing into the DE,

$$0 + 0 - A = 17$$

Solving gives $A = -17$, so

$$y_{NH} = -17$$

6. $y^{(5)} + 2y^{(3)} + 2y'' = 3x^2 - 1$

Our preferred form is $y_{NH} = A + Bx + Cx^2$.

The homogeneous aux. equation has $r^5 + 2r^3 + 2r^2 = 0$ or $r^2(r^3 + 2r + 2) = 0$.

This means $r = 0, 0$ and 3 other non-zero roots. The repeated real $r = 0$ roots mean $y_c = c_1 + c_2x +$ other terms. This **does** overlap our desired form for y_{NH}. To avoid the overlap, we try adding term one after the other, and multiplying it by x until it is neither in y_c nor the earlier terms of y_{NH}. This gives

$y_{NH} = Ax^2 + Bx^3 + Cx^4$ (three terms, as our original, and neither repeats itself nor is in y_c)
$y'_{NH} = 2Ax + 3Bx^2 + 4Cx^3$
$y''_{NH} = 2A + 6Bx + 12Cx^2$
$y^{(3)}_{NH} = 6B + 24Cx$
$y^{(4)}_{NH} = 24C$
$y^{(5)}_{NH} = 0$

Subbing into the DE,

$$0 + 2(6B + 24Cx) + 2(2A + 6Bx + 12Cx^2) = 3x^2 - 1$$
$$\text{Equating } x^2 \text{ coeffs: } 24C = 3$$
$$\text{Equating } x \text{ coeffs: } 48C + 12B = 0$$
$$\text{Equating const coeffs: } 12B + 4A = -1$$

Solving gives $A = \frac{5}{4}, B = \frac{-1}{2}, C = \frac{1}{8}$, so

$$y_{NH} = \frac{5}{4}x^2 - \frac{1}{2}x^3 + \frac{1}{8}x^4$$

7. $y^{(3)} - y = e^x + 7$

Our preferred form for y_{NH} is $y_{NH} = A + Be^x$.

The homogeneous aux. equation has $r^3 - 1 = 0$ or $(r - 1)(r^2 + r + 1) = 0$.

This means $r = 1$ is a root so $y = e^x$ is one of the solutions in y_c. This **does** overlap our desired form for y_{NH}. To avoid the overlap, we try adding term one after the other, and multiplying it by x until it is neither in y_c nor the earlier terms of y_{NH}. NOTE: only the e^x term is duplicated in y_c. The constant term is not duplicated, so it should remain unchanged.

$y_{NH} = A + Bxe^x$ (two terms, as our original, and neither repeats itself nor is in y_c)
$y'_{NH} = B(xe^x + e^x)$
$y''_{NH} = B(xe^x + 2e^x)$
$y^{(3)}_{NH} = B(xe^x + 3e^x)$

Subbing into the DE,

$$B(xe^x + 3e^x) - (A + Bxe^x) = e^x + 7$$
$$\text{Note the } Bxe^x \text{ terms cancel.}$$
$$\text{Equating } e^x \text{ coeffs: } 3B = 1$$
$$\text{Equating const coeffs: } -A = 7$$

Solving gives $A = -7, B = \frac{1}{3}$, so

$$y_{NH} = -7 + \frac{1}{3} x e^x$$

8. $4y'' + 4y' + y = 3xe^x$

We would want $y_{NH} = Axe^x + Be^x$.

The homogeneous aux. equation has $r = -0.5, -0.5$, so $y_c = c_1 e^{-x/2} + c_2 x e^{-x/2}$, so there is no overlap with our desired y_{NH} form.

$y_{NH} = Axe^x + Be^x$
$y'_{NH} = A(xe^x + e^x) + Be^x$
$y''_{NH} = A(xe^x + 2e^x) + Be^x$
Subbing into the DE,

$$4(A(xe^x + 2e^x) + Be^x) + 4(A(xe^x + e^x) + Be^x) + (Axe^x + Be^x) = 3xe^x$$
$$\text{Equating } e^x \text{ coeffs: } 12A + 9B = 0$$
$$\text{Equating } xe^x \text{ coeffs: } 9A = 3$$

Solving gives $A = \frac{1}{3}, B = \frac{-4}{9}$, so

$$y_{NH} = \frac{1}{3} x e^x - \frac{4}{9} e^x$$

For Questions 9-12, find the appropriate form for a single solution to each non-homogeneous equation. You do em not need to solve for the undetermined coefficients.

9. $y'' - 2y' + 2y = e^x \sin(x)$

Roots of auxiliary equation are $r = 1 \pm i$ so $e^x \sin(x)$ and $e^x \cos(x)$ are solutions to the homogeneous DE.

To avoid overlap with these solutions, we must select

$$y_p = Axe^x \sin(x) + Bxe^x \cos(x)$$

10. $y^{(5)} - y^{(3)} = e^x + 2x^2 - 5$

Based on the RHS, we would select 4 terms in y_{NH}: $e^x, 1, x, x^2$.

The auxiliary equation is $r^5 - r^3 = 0$, or $r^3(r^2 - 1) = 0$, or $r^3(r - 1)(r + 1) = 0$.

Roots of auxiliary equation are therefore $r = 0, 0, 0, 1,$ and -1.

Simple solutions in y_c then are $1, x, x^2, e^x,$ and e^{-x}.

To keep the same number of terms as our original form of y_{NH}, but still avoid overlap with the y_c solutions, we must select the following form for y_{NH}:

$$y_{NH} = \underbrace{Ax^3 + Bx^4 + Cx^5}_{\text{boosted } 1, x, x^2 \text{ to avoid } y_c} + \underbrace{Dxe^x}_{\text{to avoid } e^x \text{ in } y_c}$$

11. $y'' + 4y = 3x \cos(2x)$

Based on the RHS, we would select 4 terms in y_{NH}: $x\cos(2x), x\sin(2x), \cos(x), \sin(2x)$.

Roots of the auxiliary equation are $r = \pm 2i$, so $\cos(2x)$ and $\sin(2x)$ are in y_c.

To keep the same number of terms as our original form of y_{NH}, but still avoid overlap with the y_c solutions, we must select

$$y_{NH} = Ax^2 \cos(2x) + Bx^2 \sin(2x) + Cx \cos(2x) + Dx \sin(2x)$$

12. $y^{(3)} - y'' - 12y' = x - 2xe^{-3x}$

Based on the RHS, we would select 4 terms in y_{NH}: $x, 1, xe^{-3x}, e^{-3x}$.

Roots of the auxiliary equation are $r = 0, -3, 4$, so 1, e^{-3x} and e^{4x} are solutions in y_c.

To keep the same number of terms as our original form of y_{NH}, but still avoid overlap with the y_c solutions, we must select

$$y_{NH} = Ax^2 + Bx + Cxe^{-3x} + Dx^2e^{-3x}$$

13. Solve the initial value problem where $y'' - 2y' - 3y = 3xe^{2x}$, $y(0) = 1$, and $y'(0) = 0$.

The characteristic equation is $r^2 - 2r - 3 = (r-3)(r+1)$, so the general solution to the corresponding homogeneous equation is $C_1e^{3x} + C_2e^{-x}$. Using the method of undetermined coefficients, we consider a solution of the form $y_g(t) := A_1xe^{2x} + A_2e^{2x}$ for some constants A_1 and A_2. It follows that

$$\begin{aligned} 3xe^{2x} &= y_g'' - 2y_g' - 3y_g \\ &= \left(4A_1xe^{2x} + 4A_1e^{2x} + 4A_2e^{2x}\right) - 2\left(2A_1xe^{2x} + A_1e^{2x} + 2A_2e^{2x}\right) - 3\left(A_1xe^{2x} + A_2e^{2x}\right) \\ &= -3A_1xe^{2x} + (2A_1 - 3A_2)e^{2x} \end{aligned}$$

Hence, we have $A_1 = -1$ and $A_2 = -\frac{2}{3}$, so the general solution is $C_1e^{3x} + C_2e^{-x} - xe^{2x} - \frac{2}{3}e^{2x}$. Since $\frac{d}{dx}\left(C_1e^{3x} + C_2e^{-x} - xe^{2x} - \frac{2}{3}e^{2x}\right) = 3C_1e^{3x} - C_2e^{-x} - 2xe^{2x} - e^{2x} - \frac{4}{3}e^{2x}$, setting $x = 0$ yields $C_1 + C_2 - \frac{2}{3} = 1$ and $3C_1 - C_2 - \frac{7}{3} = 0$. Therefore, we have $C_1 = 1$, $C_2 = \frac{2}{3}$, and the solution is $e^{3x} + \frac{2}{3}e^{-x} - xe^{2x} - \frac{2}{3}e^{2x}$.

14. Find the particular solution to the IVP $y'' + 4y = 2x$, $y(0) = 1$, $y'(0) = 2$.

$y_c = c_1\cos(2x) + c_2\sin(2x)$

$y_{NH} = A + Bx$
$y'_{NH} = B$
$y''_{NH} = 0$

Substituting into the DE gives $A = \frac{1}{2}$ and $B = 0$, so the general solution is

$$y = y_c + y_{NH} = c_1\cos(2x) + c_2\sin(2x) + \frac{x}{2}$$

To find c_1 and c_2, use the initial conditions:

$$\begin{aligned} y(0) = 1: &\qquad 1 = c_1 + 0 + 0 \\ y'(0) = 2: &\qquad 2 = 0 + 2c_2 + \frac{1}{2} \end{aligned}$$

Solving gives

$$y = 1 \cdot \cos(2x) + \frac{3}{4}\sin(2x) + \frac{x}{2}$$

15. Find the particular solution to the IVP $y'' + 9y = \sin(2x)$, $y(0) = 1$, $y'(0) = 0$.

$y_c = c_1\cos(3x)c_2\sin(3x)$

$y_{NH} = A\cos(2x) + B\sin(2x)$. Solving for A and B gives $A = 0$ and $B = \frac{1}{5}$. The general solution is

$$y = c_1\cos(3x) + c_2\sin(3x) + \frac{1}{5}\sin(2x)$$

Using the initial conditions, we can solve for $c_1 = 1$ and $c_2 = -\frac{2}{15}$

$$y = \cos(3x) - \frac{2}{15}\sin(3x) + \frac{1}{5}\sin(2x)$$

Applications

16. For a spring/mass system with $m = 1$ kg, $c = 6$ N/(m/s), and $k = 45$, approximately what frequency of external forcing would produce the largest amplitude steady-state vibration?

The natural frequency is found using the auxiliary equation $r^2+6r+45 = 0$, giving $r = -3\pm 6i$, so $x_c = c_1e^{-3t}\cos(6t)+c_2e^{-3t}\sin(6t)$

Since this system's natural or intrinsic oscillations are at 6 rad/s, the spring/mass will show the largest amplitude response to an outside force if that outside force also has a frequency close to 6 rad/s.

17. For a spring/mass system with $m = 1$ kg, $c = 10$ N/(m/s), and $k = 650$, approximately what frequency of external forcing would produce the largest amplitude steady-state vibration?

$m = 1, c = 10, k = 650, F_0 = 100$, so

$$x'' + 10x' + 650x = \cos(\omega t)$$

The natural frequency is found using the auxiliary equation $r^2 + 10r + 650 = 0$, giving $r = -5 \pm 25i$, so $x_c = c_1e^{-5t}\cos(25t) + c_2e^{-5t}\sin(25t)$

It will show its largest amplitude response to stimuli with frequencies close to 25 rad/s.

18. Consider the undamped spring model $y'' + \omega_0^2 y = \cos(\omega t)$.
    (a) Solve $y'' + \omega_0^2 y = \cos(\omega t)$ where $\omega^2 \neq \omega_0^2$.
    (b) What does this predict will happen to the amplitude of the oscillations as the stimulus frequence ω is brought closer and closer to the natural frequency ω_0?

(a) The characteristic equation is $r^2 + \omega_0^2 = \left(r - \omega_0\sqrt{-1}\right)\left(r + \omega_0\sqrt{-1}\right)$, so the general solution to the corresponding homogeneous equation is $C_1\cos(\omega_0 t) + C_2\sin(\omega_0 t)$. Using the method of undetermined coefficients, we consider a solution of the form $y_g(t) := A_1\cos(\omega t) + A_2\sin(\omega t)$ for some constants A_1 and A_2. It follows that

$$\begin{aligned}\cos(\omega t) = y_g'' + \omega_0^2 y_g &= \left(-A_1\omega^2\cos(\omega t) - A_2\omega^2\sin(\omega t)\right) + \omega_0^2\left(A_1\cos(\omega t) + A_2\sin(\omega t)\right)\\ &= (\omega_0^2 - \omega^2)\left(A_1\cos(\omega t) + A_2\sin(\omega t)\right)\end{aligned}$$

Since $A_1 = \frac{1}{\omega_0^2-\omega^2}$ and $A_2 = 0$, the general solution is

$$C_1\cos(\omega_0 t) + C_2\sin(\omega_0 t) + \frac{1}{(\omega_0^2 - \omega^2)}\cos(\omega t)\,.$$

(b) As the stimulus frequency ω approaches the natural frequency ω_0, the amplitude given by $\frac{1}{(\omega_0^2-\omega^2)} \to \infty$. This corresponds to the result of resonance, when $\omega = \omega_0$, and the solutions change to $y_c = At\cos(\omega t)$, which has an linearly growing (unbounded) amplitude.

19. Consider the equation for the spring/mass system with $m = 1$ kg, $c = 4$ N/(m/s) and $k = 4$ N/m, and which is being forced by an external periodic force of $10\cos(3t)$ N:

$$x'' + 4x' + 4x = 10\cos(3t)$$

Find the formula for the steady-state oscillations, and find their amplitude.

The system is damped, so the x_c solution will have negative exponentials in it. i.e. that part of the solution will be **transient** because its contribution $\to 0$ over time. That is why x_c is referred to as $x_{\text{transient}}$ or x_{tr} throughout the problems in this section.

Since the RHS is pure cos/sine, its differential family will not overlap with the exp sin/cos in x_c. This means x_{NH} will be a simple linear combination of $\cos(3t)$ and $\sin(3t)$.

Let $x_{NH} = A\cos(3t) + B\sin(3t)$.
$x'_{NH} = -3A\sin(3t) + 3B\cos(3t)$

$x''_{NH} = -9A\cos(3t) - 9B\sin(3t)$

Substituting into the DE,

$$(-9A\cos(3t) - 9B\sin(3t)) + 4(-3A\sin(3t) + 3B\cos(3t)) + 4(A\cos(3t) + B\sin(3t)) = 10\cos(3t)$$

$$\begin{array}{rlr} \text{sine coeffs:} & -9B - 12A + 4B = 0 & -12A - 5B = 0 \\ \text{cos coeffs:} & -9A + 12B + 4A = 10 & -5A + 12B = 10 \end{array}$$

$$\begin{aligned} -60A - 25B &= 0 \\ -60A + 144B &= 120 \end{aligned}$$

$$-169B = -120$$

Solving gives $A = \dfrac{-50}{169}, B = \dfrac{120}{169}$, so

$$x_{NH} = x_{sp} = \frac{-50}{169}\cos(3t) + \frac{120}{169}\sin(3t)$$

The amplitude of the sum $A\cos(bt)+B\sin(bt)$ is given by $C = \sqrt{A^2+B^2}$, so the amplitude here is $\sqrt{\left(\dfrac{-50}{169}\right)^2 + \left(\dfrac{120}{169}\right)^2} \approx$ 0.77 meters or 77 cm.

20. Consider the equation for the spring/mass system

$$x'' + 3x' + 5x = 4\cos(5t)$$

Find the formula for the steady-state oscillations, and find their amplitude.

Again, because there is damping, x_c will have negative exponentials, and will fade away with time (transient part of the solution). Also, there is no overlap between the functions in the transient x_c and steady-state x_{NH}, so we can set the form of x_{NH} as a linear combination of $\cos(5t)$ and $\sin(5t)$.

Let $x_{NH} = A\cos(5t) + B\sin(5t)$ so
$x'_{NH} = -5A\sin(5t) + 5B\cos(5t)$
$x''_{NH} = -25A\cos(5t) - 25B\sin(5t)$

Substituting into the DE,

$$(-25A\cos(5t) - 25B\sin(5t)) + 3(-5A\sin(5t) + 5B\cos(5t)) + 5(A\cos(5t) + B\sin(5t)) = 4\cos(5t)$$

$$\begin{array}{rlr} \text{sine coeffs:} & -25B - 15A + 5B = 0 & -15A - 20B = 0 \\ \text{cos coeffs:} & -25A + 15B + 5A = 4 & -20A + 15B = 4 \end{array}$$

$$\begin{aligned} -60A - 80B &= 0 \\ -60A + 45B &= 12 \end{aligned}$$

$$-125B = 12$$

Solving gives $A = \dfrac{16}{125}, B = \dfrac{-12}{125}$ so

$$x_{NH} = x_{sp} = \frac{16}{125}\cos(5t) - \frac{12}{125}\sin(5t)$$

The amplitude of the sum $A\cos(bt)+B\sin(bt)$ is given by $C=\sqrt{A^2+B^2}$, so the amplitude here is $\sqrt{\left(\frac{16}{125}\right)^2+\left(\frac{-12}{125}\right)^2}=$ 0.16 m, or 16 cm.

21. The charge at a point in an electrical circuit is called $Q(t)$. For a particular series circuit with a resistor, a conductor, and a capacitor, $Q(t)$ satistifies the differential equation

$$2Q''+60Q'+1/0.0025Q=100e^{-t}$$

(a) Given the initial conditions that $Q(0)=0$, $Q'(0)=0$, find the particular solution for $Q(t)$.

(b) Knowing that $I=Q'$ is the current in the circuit, find an expression for $I(t)$.

(a) Solving for Q_c, we solve $2r^2+60r+400=0$, for $r=-20,-10$

$$Q_c=c_1e^{-20t}+c_2e^{-10t}$$

Solving for Q_{NH}, we assume $Q_{NH}=Ae^{-t}$. Subbing into the DE gives $2A-60A+400A=100$ or $A=100/342=50/171$.

$$Q_{NH}=\frac{50}{171}e^{-t}$$

This gives the general solution

$$Q=Q_c+Q_{NH}=c_1e^{-20t}+c_2e^{-10t}+\frac{50}{171}e^{-t}$$

Solving for c_1 and c_2 given $Q(0)=0$ and $Q'(0)=0$, gives

$$0=c_1+c_2+\frac{50}{171}$$
$$0=-20c_1-10c_2-\frac{50}{171}$$

Solving gives $c_1=45/171=5/19$, $c_2=-95/171=-5/9$, so

$$Q=\frac{5}{19}e^{-20t}+\frac{-5}{9}e^{-10t}+\frac{50}{171}e^{-t}$$

(b) Differentiating $Q(t)$ to find I we obtain

$$I=\frac{-100}{19}e^{-20t}+\frac{50}{9}e^{-10t}-\frac{50}{171}e^{-t}$$

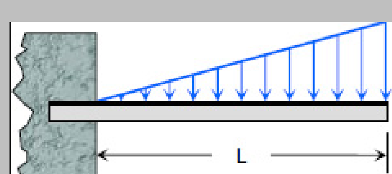
22. A cantilevered beam which is $L=2$ m long, made out of a pine "2 by 4" has $I=2.23\times10^{-6}$ m^4, $E=9.1\times10^9$ N/m^2, and its deformation satisfies the differential equation

$$EIy^{(4)}=p(x)$$

where $p(x)$ is the loading on the beam in N/m, and boundary conditions $y(0)=0$, $y'(0)=0$, $y''(2)=0$, and $y'''(2)=0$.

(a) Find the formula for $y(x)$ if the load applied is *triangular* (increases linearly), given by $p(x)=20x$ N/m. I.e. as x increases, moving away from the mounting point of the beam, the load increases, until at the tip of the beam ($x=2$), the load is 40 N/m.

(b) Find the amount of deflection of the beam at the tip under this load.



(a) Since the corresponding homogeneous DE is simply $EIy^{(4)} = 0$, with roots to the characteristic equation of zero (repeated four times), $y_c = c_1 + c_2x + c_3x^2 + c_4x^3$.

Since the RHS $= 20x$, y_p would usually have been $A + Bx$. However, this is not linearly independent with the functions in y_c, so we boost by powers of x until it is, so we choose $y_{NH} = Ax^4 + Bx^5$.

Subbing y_{NH} into the DE and solving for A and B,

$$EI\left(\underbrace{24A + 120Bx}_{y^{(4)}}\right) = 20x$$

$$\text{const coeff's:} \quad A = 0 \text{ and}$$

$$x \text{ coeff's:} \quad 120EIB = 20, \text{ or } B = \frac{1}{6EI}$$

$$\text{so } y_P = \frac{1}{6EI}x^5$$

This means the general solution is

$$y = \underbrace{c_1 + c_2x + c_3x^2 + c_4x^3}_{y_c} + \underbrace{\frac{1}{6EI}x}_{y_{NH}}$$

To solve for $c_1, \ldots, c_4$, we use the boundary conditions.

$$\begin{aligned}
y(0) = 0 &\implies 0 = c_1 \\
y'(0) = 0 &\implies 0 = c_2 \\
y''(2) = 0 &\implies 0 = 2c_3 + 6c_4(2) + \frac{1}{6EI}20(2)^3 \\
y'''(2) = 0 &\implies 0 = 6c_4 + \frac{1}{6EI}60(2)^2
\end{aligned}$$

Solving gives $c_4 = \dfrac{-20}{3EI}$ and $c_3 = \dfrac{80}{3EI}$, so the particular solution, or the formula for the beam deflection at x is

$$y = \frac{1}{EI}\left(\frac{80}{3}x^2 - \frac{20}{3}x^3 + \frac{1}{6}x^5\right)$$

(b) The deflection at the tip, $x = 2$, is simply the value of $y(2)$:

$$y(2) = \frac{1}{EI}\left(\frac{80}{3}2^2 - \frac{20}{3}2^3 + \frac{1}{6}2^5\right) \approx 0.00289$$

This means the tip of the beam deflects by 0.00289 m or roughly 3 mm under this load.