1.5 Exercises

In Exercises 1 through 8, sketch **u** *and its image under the given matrix transformation* f.

1. $f: R^2 \to R^2$ defined by
$$f\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}; \quad \mathbf{u} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$$

2. $f: R^2 \to R^2$ (reflection with respect to the y-axis) defined by
$$f\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}; \quad \mathbf{u} = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$$

3. $f: R^2 \to R^2$ is a counterclockwise rotation through $30°$; $\mathbf{u} = \begin{bmatrix} -1 \\ 3 \end{bmatrix}$

H.W

4. $f: R^2 \to R^2$ is a counterclockwise rotation through $\frac{2}{3}\pi$ radians; $\mathbf{u} = \begin{bmatrix} -2 \\ -3 \end{bmatrix}$

H.W

5. $f: R^2 \to R^2$ defined by
$$f\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}; \quad \mathbf{u} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$$

H.W

6. $f: R^2 \to R^2$ defined by
$$f\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}; \quad \mathbf{u} = \begin{bmatrix} -3 \\ 3 \end{bmatrix}$$

7. $f: R^3 \to R^3$ defined by
$$f\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} 1 & 0 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix}; \quad \mathbf{u} = \begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix}$$

H.W

8. $f: R^3 \to R^3$ defined by
$$f\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} 1 & 0 & 1 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix}; \quad \mathbf{u} = \begin{bmatrix} 0 \\ -2 \\ 4 \end{bmatrix}$$

In Exercises 9 through 11, let $f: R^2 \to R^3$ *be the matrix transformation defined by* $f(\mathbf{x}) = A\mathbf{x}$, *where*
$$A = \begin{bmatrix} 1 & 3 \\ -1 & 2 \end{bmatrix}.$$

Determine whether the given vector **w** *is in the range of* f.

9. $\mathbf{w} = \begin{bmatrix} 7 \\ 3 \end{bmatrix}$ H.W 10. $\mathbf{w} = \begin{bmatrix} 4 \\ 1 \end{bmatrix}$ H.W 11. $\mathbf{w} = \begin{bmatrix} -1 \\ -9 \end{bmatrix}$

In Exercises 12 through 14, let $f: R^2 \to R^3$ *be the matrix transformation defined by* $f(\mathbf{x}) = A\mathbf{x}$, *where*
$$A = \begin{bmatrix} 1 & 2 \\ 0 & 1 \\ 1 & 1 \end{bmatrix}.$$

Determine whether the given vector **w** *is in the range of* f.

H.W 12. $\mathbf{w} = \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}$ H.W 13. $\mathbf{w} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ H.W 14. $\mathbf{w} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$.

In Exercises 15 through 17, give a geometric description of the matrix transformation $f: R^2 \to R^2$ *defined by* $f(\mathbf{u}) = A\mathbf{u}$ *for the given matrix* A.

15. (a) $A = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$ (b) $A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$

16. (a) $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ (b) $A = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$

17. (a) $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ (b) $A = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$

18. Some matrix transformations f have the property that $f(\mathbf{u}) = f(\mathbf{v})$, when $\mathbf{u} \neq \mathbf{v}$. That is, the images of different vectors can be the same. For each of the following matrix transformations $f: R^2 \to R^2$ defined by $f(\mathbf{u}) = A\mathbf{u}$, find two different vectors **u** and **v** such that $f(\mathbf{u}) = f(\mathbf{v}) = \mathbf{w}$ for the given vector **w**.

(a) $A = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & -1 \end{bmatrix}$, $\mathbf{w} = \begin{bmatrix} 0 \\ -1 \end{bmatrix}$

H.W (b) $A = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & -1 \end{bmatrix}$, $\mathbf{w} = \begin{bmatrix} 4 \\ 4 \end{bmatrix}$

19. Let $f: R^2 \to R^2$ be the linear transformation defined by $f(\mathbf{u}) = A\mathbf{u}$, where
$$A = \begin{bmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{bmatrix}.$$
For $\phi = 30°$, f defines a counterclockwise rotation by an angle of $30°$.

(a) If $T_1(\mathbf{u}) = A^2\mathbf{u}$, describe the action of T_1 on **u**.

(b) If $T_2(\mathbf{u}) = A^{-1}\mathbf{u}$, describe the action of T_2 on **u**.

(c) What is the smallest positive value of k for which $T(\mathbf{u}) = A^k\mathbf{u} = \mathbf{u}$?

Theoretical Exercises

T.1. Let $f: R^n \to R^m$ be a matrix transformation defined by $f(\mathbf{u}) = A\mathbf{u}$, where A is an $m \times n$ matrix.

(a) Show that $f(\mathbf{u} + \mathbf{v}) = f(\mathbf{u}) + f(\mathbf{v})$ for any **u** and **v** in R^n.

(b) Show that $f(c\mathbf{u}) = cf(\mathbf{u})$ for any **u** in R^n and any real number c.

(c) Show that $f(c\mathbf{u} + d\mathbf{v}) = cf(\mathbf{u}) + df(\mathbf{v})$ for any **u** and **v** in R^n and any real numbers c and d.

Ex: 1.5

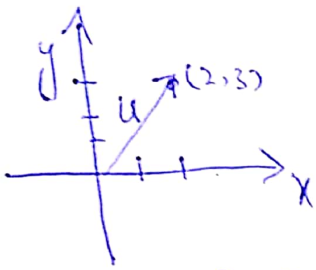
In Exercises 1 through 8, sketch u and its image under the given matrix transformation "f"

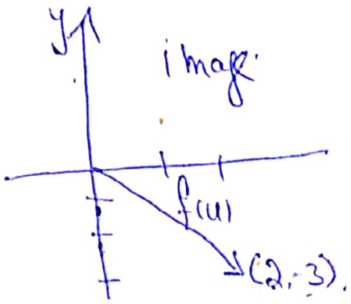
Q1 $f: R^2 \to R^2$ defined by.

$$f\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix};\quad u = \begin{bmatrix} 2 \\ 3 \end{bmatrix}.$$

Given $u = \begin{bmatrix} 2 \\ 3 \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix}$

Sketch





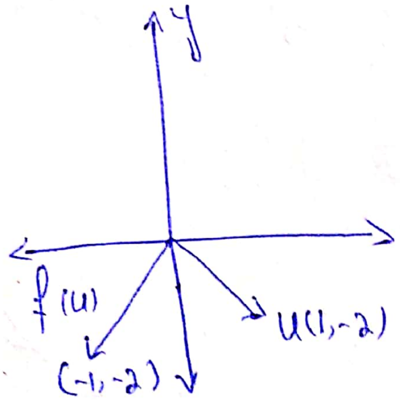
Now $f(u) = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 2 \\ -3 \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix}$

Q2 $f: R^2 \to R^2$ (reflection w.r.t y-axis) defined by

$$f\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix},\quad u = \begin{bmatrix} 1 \\ -2 \end{bmatrix}.$$

Given $u = \begin{bmatrix} 1 \\ -2 \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix}$

Sketch



Now $f(u) = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ -2 \end{bmatrix} = \begin{bmatrix} -1 \\ -2 \end{bmatrix}.$

Q3 $f: R^2 \to R^2$ is a counter-clockwise rotation through 30°, $u = \begin{bmatrix} -1 \\ 3 \end{bmatrix}$

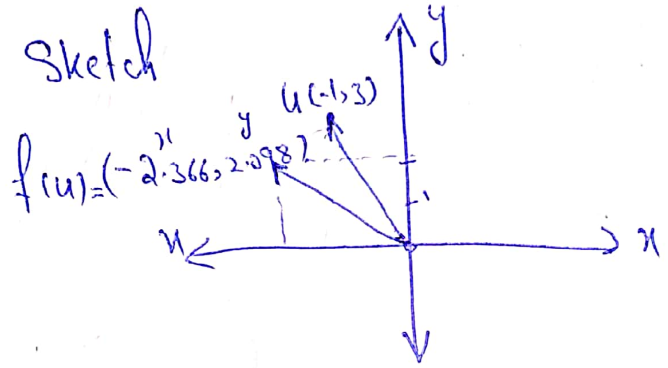
$\boxed{\text{Exp-9}}$

$$f(u) = \begin{bmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\theta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}.$$

$$= \begin{bmatrix} \cos 30^\circ & -\sin 30^\circ \\ \sin 30^\circ & \cos 30^\circ \end{bmatrix} \begin{bmatrix} -1 \\ 3 \end{bmatrix}.$$

$$= \begin{bmatrix} -\cos 30^\circ - 3\sin 30^\circ \\ -\sin 30^\circ + 3\cos 30^\circ \end{bmatrix} = \begin{bmatrix} -\frac{\sqrt{3}}{2} - 3/2 \\ -1/2 + \frac{3\sqrt{3}}{2} \end{bmatrix} = \begin{bmatrix} -2.366 \\ 2.098 \end{bmatrix}.$$

Sketch

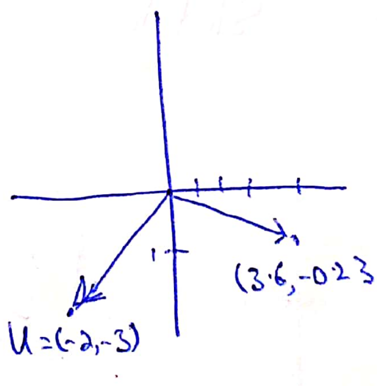


Q4 $f: R^2 \to R^2$ is a counter clock-wise rotation through $2/3\pi$ radians $u = \begin{bmatrix} -2 \\ -3 \end{bmatrix}$

$$f(u) = \begin{bmatrix} \cos 2/3\pi & -\sin 2/3\pi \\ \sin 2/3\pi & \cos 2/3\pi \end{bmatrix} \begin{bmatrix} -2 \\ -3 \end{bmatrix}.$$

$$= \begin{bmatrix} -2\cos 2/3\pi + 3\sin 2/3\pi \\ -2\sin 2/3\pi - 3\cos 2/3\pi \end{bmatrix} = \begin{bmatrix} 3.6 \\ -0.23 \end{bmatrix}$$

Sketch



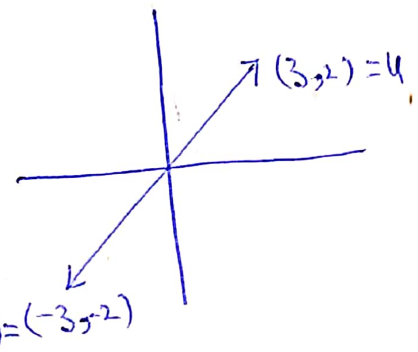
Ex: 1.5

Q5 $f: R^2 \to R^2$ defined by

$$f\begin{pmatrix} x \\ y \end{pmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}, \quad u = \begin{bmatrix} 3 \\ 2 \end{bmatrix}.$$

$$\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} 3 \\ 2 \end{bmatrix} = \begin{bmatrix} -3 \\ -2 \end{bmatrix}.$$

Sketch



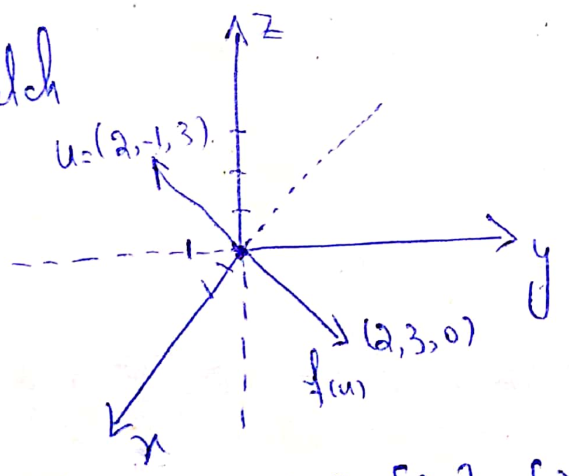
Q6 is similarly to Q2.

Q7 $f: R^3 \to R^3$ defined by

$$f\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} 1 & 0 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix}, \quad u = \begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix}.$$

Given $u = \begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$

Sketch



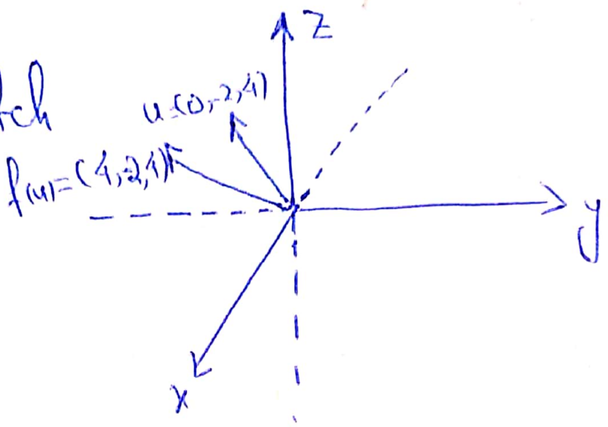
Now $f(u)\begin{bmatrix} 1 & 0 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix} = \begin{bmatrix} 2 \\ 2+1 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \\ 0 \end{bmatrix}.$

Q8 $f: R^3 \to R^3$ defined by

$$f\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} 1 & 0 & 1 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} ; \; u = \begin{bmatrix} 0 \\ -2 \\ 4 \end{bmatrix}$$

$$f(u) = \begin{bmatrix} 1 & 0 & 1 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 0 \\ -2 \\ 4 \end{bmatrix} = \begin{bmatrix} 4 \\ -2 \\ 4 \end{bmatrix}.$$

Sketch



Q9 Given that $f(x) = Ax$ & $A = \begin{bmatrix} 1 & 3 \\ -1 & 2 \end{bmatrix}$ & $W = \begin{bmatrix} 7 \\ 3 \end{bmatrix}$.

Now $Ax = f(x) = W$

$$\begin{bmatrix} 1 & 3 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 7 \\ 3 \end{bmatrix} \quad \text{—— } (*)$$

$$\begin{bmatrix} x + 3y \\ -x + 2y \end{bmatrix} = \begin{bmatrix} 7 \\ 3 \end{bmatrix}$$

$$x + 3y = 7 \quad \text{—— (i)}$$
$$-x + 2y = 3 \quad \text{—— (ii)}$$

Solving eq (ii) & (i) we get $\boxed{x = 1}$ & $\boxed{y = 2}$

Ex: 1.5

eq (*) =>
$$\begin{bmatrix}1 & 3\\ -1 & 2\end{bmatrix}\begin{bmatrix}1\\2\end{bmatrix}=\begin{bmatrix}7\\3\end{bmatrix}$$

$$\begin{bmatrix}1+6\\-1+4\end{bmatrix}=\begin{bmatrix}7\\3\end{bmatrix}$$

$$\begin{bmatrix}7\\3\end{bmatrix}=\begin{bmatrix}7\\3\end{bmatrix}$$

Hence $W=\begin{bmatrix}7\\3\end{bmatrix}$ is in range of f

Q10 & Q11 is similarly to Q9

Q12 $f: \mathbb{R}^2 \to \mathbb{R}^3$.

Given that $f(x) = Ax$ & $A=\begin{bmatrix}1 & 2\\ 0 & 1\\ 1 & 1\end{bmatrix}$ & $W=\begin{bmatrix}1\\-1\\2\end{bmatrix}$.

Now $Ax = f(x) = W$, $X=\begin{bmatrix}x\\y\end{bmatrix}$.

$$\begin{bmatrix}1\\-1\\2\end{bmatrix}=\begin{bmatrix}1 & 2\\ 0 & 1\\ 1 & 1\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix} \longrightarrow (*)$$

$1 = x+2y$ —— (i)

$-1 = \;\;+y$ —— (ii)

$2 = x+y$ —— (iii)

eq (ii) => $\boxed{y=-1}$ eq (iii) => $x = 2+1 = 3$ => $\boxed{x=3}$

eq (*) =>

$$\begin{bmatrix} 1 & 2 \\ 0 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 3 \\ -1 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}$$

$$\begin{bmatrix} 3-2 \\ 0-1 \\ 3-1 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}$$

$$\begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}$$

Hence $W = \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}$ is in the range of f.

Q13 Given $w = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ & $X = \begin{bmatrix} x \\ y \end{bmatrix}$

Now $Ax = f(x) = W$

$$\begin{bmatrix} 1 & 2 \\ 0 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \text{ ——— } (*)'$$

$$\begin{bmatrix} x+2y \\ 0+y \\ x+y \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

$x + 2y = 1$ ——— (i)

$+y = 1$ ——— (ii)

$x + y = 1$ ——— (iii)

eq (ii) => $\boxed{y = 1}$ eq (iii) => $\boxed{x = 0}$ & eq (i) => $\boxed{x = -1}$

x has two values.

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Ex: 1.5

eq (*) =>

$$\begin{bmatrix} 1 & 2 \\ 0 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} \neq \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

w is not in range of f

Again eq (*) =>

$$\begin{bmatrix} 1 & 2 \\ 0 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} -1+2 \\ 0+1 \\ -1+1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \neq \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

"w is not in range of f"

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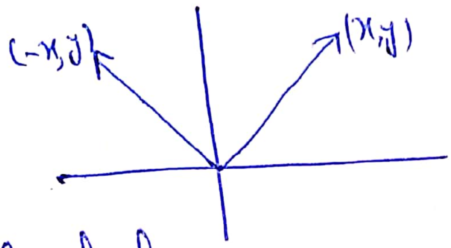
Q15 (a) $A = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$ let $u = \begin{bmatrix} x \\ y \end{bmatrix}$

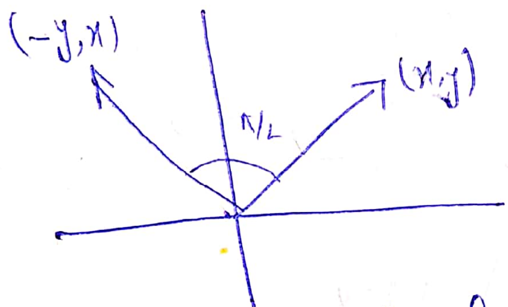
$$Au = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -x+0y \\ 0x+y \end{bmatrix} = \begin{bmatrix} -x \\ y \end{bmatrix}.$$



Reflection about y-axis.

(b) $A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$, $u = \begin{bmatrix} x \\ y \end{bmatrix}$

$$Au = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0-y \\ x+0 \end{bmatrix} = \begin{bmatrix} -y \\ x \end{bmatrix}$$



$x = -y$ = ~~$xy \Rightarrow x$~~

$y = x \Rightarrow$ ~~$y \Rightarrow x$~~

$= (x,y) = (-y,x)$

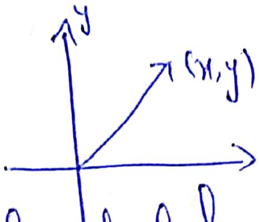
Rotate counter clockwise through $\pi/2$.

Scanned with CamScanner

Ex: 1.5

16(a) $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$. let $u = \begin{bmatrix} x \\ y \end{bmatrix}$.

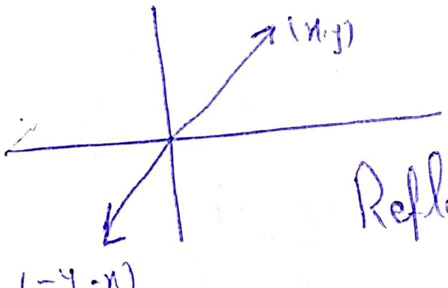
$$Au = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} y \\ x \end{bmatrix}.$$



Reflection about line $y = x$.

(b) $A = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$ let $u = \begin{bmatrix} x \\ y \end{bmatrix}$.

$$Au = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -y \\ -x \end{bmatrix}.$$



Reflection about line $y = -x$.

Q17 (a) $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$. let $u = \begin{bmatrix} x \\ y \end{bmatrix}$ $Au = \begin{bmatrix} x \\ 0 \end{bmatrix}$.

Projection onto x-axis.

(b) $A = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$, $u = \begin{bmatrix} x \\ y \end{bmatrix}$ then $Au = \begin{bmatrix} 0 \\ y \end{bmatrix}$.

Projection onto y-axis.

Q18 (a) $A = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & -1 \end{bmatrix}$, $W = \begin{bmatrix} 0 \\ -1 \end{bmatrix}$.

~~[illegible]~~ $AX = W$

$$\begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \end{bmatrix}$$

$x + 2y = 0$

$y - z = -1$

or

$x = -2y$ — (i) & $y + 1 = z$ — (2)

let

$z = r$ be real no.

let $z = 0$ then $\boxed{y = -1}$ & $\boxed{x = 2}$

let $z = 1$, $y = 0$, $x = 0$

So $u = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$, $v = \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix}$.

(b) $A = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & -1 \end{bmatrix}$, $W = \begin{bmatrix} 4 \\ 4 \end{bmatrix}$. ~~[illegible]~~

$AX = W$

$$\begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4 \\ 4 \end{bmatrix} \Rightarrow 2x + y = 4 \Rightarrow \boxed{x = 4 - y/2}$$

$$2y - z = 4 \Rightarrow \boxed{y = \frac{4+z}{2}}$$

let $z = 4$, $y = 4$, $x = 0$

let $z = 0$ $y = 2$, $x = 1$

so $u = \begin{bmatrix} 0 \\ 4 \\ 4 \end{bmatrix}$, $v = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$

T.2. Let $f: R^n \to R^m$ be a matrix transformation defined by $f(\mathbf{u}) = A\mathbf{u}$, where A is an $m \times n$ matrix. Show that if $\mathbf{u}$ and $\mathbf{v}$ are vectors in R^n such that $f(\mathbf{u}) = \mathbf{0}$ and $f(\mathbf{v}) = \mathbf{0}$, where

$$\mathbf{0} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 0 \end{bmatrix},$$

then $f(c\mathbf{u} + d\mathbf{v}) = \mathbf{0}$ for any real numbers c and d.

T.3. (a) Let $O: R^n \to R^m$ be the matrix transformation defined by $O(\mathbf{u}) = O\mathbf{u}$, where O is the $m \times n$ zero matrix. Show that $O(\mathbf{u}) = \mathbf{0}$ for all $\mathbf{u}$ in R^n.

(b) Let $I: R^n \to R^n$ be the matrix transformation defined by $I(\mathbf{u}) = I_n\mathbf{u}$, where I_n is the identity matrix (see Section 1.4). Show that $I(\mathbf{u}) = \mathbf{u}$ for all $\mathbf{u}$ in R^n.

1.6 SOLUTIONS OF LINEAR SYSTEMS OF EQUATIONS

In this section we shall systematize the familiar method of elimination of unknowns (discussed in Section 1.1) for the solution of linear systems and thus obtain a useful method for solving such systems. This method starts with the augmented matrix of the given linear system and obtains a matrix of a certain form. This new matrix represents a linear system that has exactly the same solutions as the given system but is easier to solve. For example, if

$$\left[\begin{array}{cccc|c} 1 & 0 & 0 & 2 & 4 \\ 0 & 1 & 0 & -1 & -5 \\ 0 & 0 & 1 & 3 & 6 \end{array}\right]$$

represents the augmented matrix of a linear system, then the solution is easily found from the corresponding equations

$$\begin{aligned} x_1 \quad\quad\; + 2x_4 &= 4 \\ x_2 \quad - x_4 &= -5 \\ x_3 + 3x_4 &= 6. \end{aligned}$$

The task of this section is to manipulate the augmented matrix representing a given linear system into a form from which the solution can easily be found.

DEFINITION An $m \times n$ matrix A is said to be in **reduced row echelon form** if it satisfies the following properties:

(a) All zero rows, if there are any, appear at the bottom of the matrix.

(b) The first nonzero entry from the left of a nonzero row is a 1. This entry is called a **leading one** of its row.

(c) For each nonzero row, the leading one appears to the right and below any leading one's in preceding rows.

(d) If a column contains a leading one, then all other entries in that column are zero.

A matrix in reduced row echelon form appears as a staircase ("echelon") pattern of leading ones descending from the upper left corner of the matrix.

An $m \times n$ matrix satisfying properties (a), (b), and (c) is said to be in **row echelon form**.

EXAMPLE 1 The following are matrices in reduced row echelon form since they satisfy

Key Terms

Reduced row echelon form
Leading one
Row echelon form
Elementary row operation
Row equivalent
Reduced row echelon form of a matrix
Row echelon form of a matrix
Gauss–Jordan reduction
Gaussian elimination
Back substitution
Consistent linear system
Inconsistent linear system
Homogeneous system
Trivial solution
Nontrivial solution
Bit linear systems

1.6 Exercises

In Exercises 1 through 8, determine whether the given matrix is in reduced row echelon form, row echelon form, or neither.

√ **1.** $\begin{bmatrix} 1 & 0 & 0 & 0 & -3 \\ 0 & 0 & 1 & 0 & 4 \\ 0 & 0 & 0 & 1 & 2 \end{bmatrix}$

2. $\begin{bmatrix} 0 & 1 & 0 & 0 & 5 \\ 0 & 0 & 1 & 0 & -4 \\ 0 & 0 & 0 & -1 & 3 \end{bmatrix}$

3. $\begin{bmatrix} 1 & 0 & 0 & 0 & 2 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$

4. $\begin{bmatrix} 0 & 1 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$

5. $\begin{bmatrix} 1 & 2 & 3 & 1 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 1 & -4 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

6. $\begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

7. $\begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 2 & -3 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$

8. $\begin{bmatrix} 0 & 1 & 0 & 0 & 5 \\ 0 & 0 & 1 & 0 & 4 \\ 0 & 1 & 0 & -2 & 3 \end{bmatrix}$

9. Let

$$A = \begin{bmatrix} 1 & 0 & 3 \\ -3 & 1 & 4 \\ 4 & 2 & 2 \\ 5 & -1 & 5 \end{bmatrix}.$$

Find the matrices obtained by performing the following elementary row operations on A.

(a) Interchanging the second and fourth rows

(b) Multiplying the third row by 3

(c) Adding (-3) times the first row to the fourth row

10. Let

$$A = \begin{bmatrix} 2 & 0 & 4 & 2 \\ 3 & -2 & 5 & 6 \\ -1 & 3 & 1 & 1 \end{bmatrix}.$$

Find the matrices obtained by performing the following elementary row operations on A.

(a) Interchanging the second and third rows

(b) Multiplying the second row by (-4)

(c) Adding 2 times the third row to the first row

P.H **11.** Find three matrices that are row equivalent to

$$A = \begin{bmatrix} 2 & -1 & 3 & 4 \\ 0 & 1 & 2 & -1 \\ 5 & 2 & -3 & 4 \end{bmatrix}.$$

H.W **12.** Find three matrices that are row equivalent to

$$\begin{bmatrix} 4 & 3 & 7 & 5 \\ -1 & 2 & -1 & 3 \\ 2 & 0 & 1 & 4 \end{bmatrix}.$$

In Exercises 13 through 16, find a row echelon form of the given matrix.

H.W **13.** $\begin{bmatrix} 0 & -1 & 2 & 3 \\ 2 & 3 & 4 & 5 \\ 1 & 3 & -1 & 2 \\ 3 & 2 & 4 & 1 \end{bmatrix}$

P.9 **14.** $\begin{bmatrix} 1 & -2 & 0 & 2 \\ 2 & -3 & -1 & 5 \\ 1 & 3 & 2 & 5 \\ 1 & 1 & 0 & 2 \\ 2 & -6 & -2 & 1 \end{bmatrix}$

H.W **15.** $\begin{bmatrix} 1 & 2 & -3 & 1 \\ -1 & 0 & 3 & 4 \\ 0 & 1 & 2 & -1 \\ 2 & 3 & 0 & -3 \end{bmatrix}$

H.W **16.** $\begin{bmatrix} 2 & -1 & 0 & 1 & 4 \\ 1 & -2 & 1 & 4 & -3 \\ 5 & -4 & 1 & 6 & 5 \\ -7 & 8 & -3 & -14 & 1 \end{bmatrix}$

H.W H.W H.W

17. For each of the matrices in Exercises 13 through 16, find the reduced row echelon form of the given matrix.

P-11

18. Let

$$A = \begin{bmatrix} 1 & 2 & 1 \\ -1 & 1 & 2 \\ 2 & 1 & -2 \end{bmatrix}.$$

In each part, determine whether $\mathbf{x}$ is a solution to the linear system $A\mathbf{x} = \mathbf{b}$.

(a) $\mathbf{x} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$; $\mathbf{b} = \mathbf{0}$ (b) $\mathbf{x} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$; $\mathbf{b} = \mathbf{0}$

(c) $\mathbf{x} = \begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix}$; $\mathbf{b} = \begin{bmatrix} 3 \\ 6 \\ -5 \end{bmatrix}$

H.W

(d) $\mathbf{x} = \begin{bmatrix} 1 \\ 2 \\ -3 \end{bmatrix}$; $\mathbf{b} = \begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix}$

19. Let

$$A = \begin{bmatrix} 1 & 2 & -1 & 3 \\ 1 & 3 & 0 & 2 \\ -1 & 2 & 1 & 3 \end{bmatrix}.$$

In each part, determine whether $\mathbf{x}$ is a solution to the homogeneous system $A\mathbf{x} = \mathbf{0}$.

(a) $\mathbf{x} = \begin{bmatrix} 5 \\ -3 \\ 5 \\ 2 \end{bmatrix}$ (b) $\mathbf{x} = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}$

(c) $\mathbf{x} = \begin{bmatrix} 1 \\ -\frac{3}{5} \\ 1 \\ \frac{2}{5} \end{bmatrix}$ (d) $\mathbf{x} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ -1 \end{bmatrix}$

In Exercises 20 through 22, find all solutions to the given linear system.

P-1

20. (a)
$$\begin{aligned} x + y + 2z &= -1 \\ x - 2y + z &= -5 \\ 3x + y + z &= 3 \end{aligned}$$

H.W

(b)
$$\begin{aligned} x + y + 3z + 2w &= 7 \\ 2x - y \quad\quad + 4w &= 8 \\ 3y + 6z \quad\quad &= 8 \end{aligned}$$

H.W

(c)
$$\begin{aligned} x + 2y - 4z &= 3 \\ x - 2y + 3z &= -1 \\ 2x + 3y - z &= 5 \\ 4x + 3y - 2z &= 7 \\ 5x + 2y - 6z &= 7 \end{aligned}$$

H.W

(d)
$$\begin{aligned} x + y + z &= 0 \\ x \quad\quad + z &= 0 \\ 2x + y - 2z &= 0 \\ x + 5y + 5z &= 0 \end{aligned}$$

H.W

21. (a)
$$\begin{aligned} x + y + 2z + 3w &= 13 \\ x - 2y + z + w &= 8 \\ 3x + y + z - w &= 1 \end{aligned}$$

H.W

(b)
$$\begin{aligned} x + y + z &= 1 \\ x + y - 2z &= 3 \\ 2x + y + z &= 2 \end{aligned}$$

(c)
$$\begin{aligned} 2x + y + z - 2w &= 1 \\ 3x - 2y + z - 6w &= -2 \\ x + y - z - w &= -1 \\ 6x \quad\quad + z - 9w &= -2 \\ 5x - y + 2z - 8w &= 3 \end{aligned}$$

H.W

(d)
$$\begin{aligned} x + 2y + 3z - w &= 0 \\ 2x + y - z + w &= 3 \\ x - y \quad\quad + w &= -2 \end{aligned}$$

H.W

22. (a)
$$\begin{aligned} 2x - y + z &= 3 \\ x - 3y + z &= 4 \\ -5x \quad\quad - 2z &= -5 \end{aligned}$$

(b)
$$\begin{aligned} x + y + z + w &= 6 \\ 2x + y - z \quad\quad &= 3 \\ 3x + y \quad\quad + 2w &= 6 \end{aligned}$$

(c)
$$\begin{aligned} 2x - y + z &= 3 \\ 3x + y - 2z &= -2 \\ x - y + z &= 7 \\ x + 5y + 7z &= 13 \\ x - 7y - 5z &= 12 \end{aligned}$$

(d)
$$\begin{aligned} x + 2y - z &= 0 \\ 2x + y + z &= 0 \\ 5x + 7y + z &= 0 \end{aligned}$$

In Exercises 23 through 26, find all values of a for which the resulting linear system has (a) no solution, (b) a unique solution, and (c) infinitely many solutions.

P-8

23.
$$\begin{aligned} x + y - \quad\quad\quad z &= 2 \\ x + 2y + \quad\quad\quad z &= 3 \\ x + y + (a^2 - 5)z &= a \end{aligned}$$

24.
$$\begin{aligned} x + y + \quad\quad\quad z &= 2 \\ 2x + 3y + \quad\quad\quad 2z &= 5 \\ 2x + 3y + (a^2 - 1)z &= a + 1 \end{aligned}$$

25.
$$\begin{aligned} x + y + \quad\quad\quad z &= 2 \\ x + 2y + \quad\quad\quad z &= 3 \\ x + y + (a^2 - 5)z &= a \end{aligned}$$

H.W

26.
$$\begin{aligned} x + \quad\quad\quad y &= 3 \\ x + (a^2 - 8)y &= a \end{aligned}$$

H.W

In Exercises 27 through 30, solve the linear system with the given augmented matrix.

27. (a) $\left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 3 \\ 0 & 1 & 1 & 1 \end{array}\right]$

(b) $\left[\begin{array}{ccc|c} 1 & 2 & 3 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 2 & 0 \\ 1 & 3 & 3 & 0 \end{array}\right]$

28. (a) $\left[\begin{array}{ccc|c} 1 & 2 & 3 & 0 \\ 1 & 1 & 1 & 0 \\ 5 & 7 & 9 & 0 \end{array}\right]$

(b) $\left[\begin{array}{ccc|c} 1 & 2 & 1 & 7 \\ 2 & 0 & 1 & 4 \\ 1 & 0 & 2 & 5 \\ 1 & 2 & 3 & 11 \\ 2 & 1 & 4 & 12 \end{array}\right]$

29. (a) $\left[\begin{array}{cccc|c} 1 & 2 & 3 & 1 & 8 \\ 1 & 3 & 0 & 1 & 7 \\ 1 & 0 & 2 & 1 & 3 \end{array}\right]$

(b) $\left[\begin{array}{ccc|c} 1 & -2 & 3 & 4 \\ 2 & -1 & -3 & 5 \\ 3 & 0 & 1 & 2 \\ 3 & -3 & 0 & 7 \end{array}\right]$

30. (a) $\left[\begin{array}{ccc|c} 4 & 2 & -1 & 5 \\ 3 & 3 & 6 & 1 \\ 5 & 1 & -8 & 8 \end{array}\right]$

(b) $\left[\begin{array}{cccc|c} 1 & 1 & 3 & -3 & 0 \\ 0 & 2 & 1 & -3 & 3 \\ 1 & 0 & 2 & -1 & -1 \end{array}\right]$

Q2 (a)

31. Let $f: R^3 \to R^3$ be the matrix transformation defined by

$$f\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} 4 & 1 & 3 \\ 2 & -1 & 3 \\ 2 & 2 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}.$$

H.W

Find x, y, z so that $f\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} 4 \\ 5 \\ -1 \end{bmatrix}$.

32. Let $f: R^3 \to R^3$ be the matrix transformation defined by

$$f\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} 1 & 2 & 3 \\ -3 & -2 & -1 \\ -2 & 0 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}.$$

H.W

Find x, y, z so that $f\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} 2 \\ 2 \\ 4 \end{bmatrix}$.

33. Let $f: R^3 \to R^3$ be the matrix transformation defined by

$$f\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} 4 & 1 & 3 \\ 2 & -1 & 3 \\ 2 & 2 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}.$$

Find an equation relating a, b, and c so that we can always compute values of x, y, and z for which

$$f\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} a \\ b \\ c \end{bmatrix}.$$

H.W

34. Let $f: R^3 \to R^3$ be the matrix transformation defined by

$$f\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} 1 & 2 & 3 \\ -3 & -2 & -1 \\ -2 & 0 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}.$$

Find an equation relating a, b, and c so that we can always compute values of x, y, and z for which

$$f\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} a \\ b \\ c \end{bmatrix}.$$

In Exercises 35 and 36, solve the linear systems $A\mathbf{x} = \mathbf{b}_1$ and $A\mathbf{x} = \mathbf{b}_2$ separately and then by obtaining the reduced row echelon form of the augmented matrix $\left[A \mid \mathbf{b}_1 \;\; \mathbf{b}_2\right]$. Compare your answers.

35. $A = \begin{bmatrix} 1 & -1 \\ 2 & 3 \end{bmatrix}$, $\mathbf{b}_1 = \begin{bmatrix} 1 \\ -8 \end{bmatrix}$, $\mathbf{b}_2 = \begin{bmatrix} 5 \\ -5 \end{bmatrix}$

36. $A = \begin{bmatrix} 1 & -2 & 0 \\ -3 & 2 & -1 \\ 4 & -2 & 3 \end{bmatrix}$, $\mathbf{b}_1 = \begin{bmatrix} 3 \\ -7 \\ 12 \end{bmatrix}$, $\mathbf{b}_2 = \begin{bmatrix} -4 \\ 6 \\ -10 \end{bmatrix}$

In Exercises 37 and 38, let

$$A = \begin{bmatrix} 1 & 0 & 5 \\ 1 & 1 & 1 \\ 0 & 1 & -4 \end{bmatrix}.$$

37. Find a nontrivial solution to the homogeneous system $(-4I_3 - A)\mathbf{x} = \mathbf{0}$.*

38. Find a nontrivial solution to the homogeneous system $(2I_3 - A)\mathbf{x} = \mathbf{0}$.*

39. Find an equation relating a, b, and c so that the linear system

$$\begin{aligned} x + 2y - 3z &= a \\ 2x + 3y + 3z &= b \\ 5x + 9y - 6z &= c \end{aligned}$$

is consistent for any values of a, b, and c that satisfy that equation.

40. Find an equation relating a, b, and c so that the linear system

$$\begin{aligned} 2x + 2y + 3z &= a \\ 3x - \;\; y + 5z &= b \\ x - 3y + 2z &= c \end{aligned}$$

is consistent for any values of a, b, and c that satisfy that equation.

*This type of problem will play a key role in Chapter 8.

H.W

41. Find a 2×1 matrix $\mathbf{x}$ with entries not all zero such that $A\mathbf{x} = 4\mathbf{x}$, where

$$A = \begin{bmatrix} 4 & 1 \\ 0 & 2 \end{bmatrix}.$$

[*Hint*: Rewrite the matrix equation $A\mathbf{x} = 4\mathbf{x}$ as $4\mathbf{x} - A\mathbf{x} = (4I_2 - A)\mathbf{x} = \mathbf{0}$ and solve the homogeneous system.].

H.W

42. Find a 2×1 matrix $\mathbf{x}$ with entries not all zero such that $A\mathbf{x} = 3\mathbf{x}$, where

$$A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}.$$

H.W

43. Find a 3×1 matrix with entries not all zero such that $A\mathbf{x} = 3\mathbf{x}$, where

$$A = \begin{bmatrix} 1 & 2 & -1 \\ 1 & 0 & 1 \\ 4 & -4 & 5 \end{bmatrix}.$$

H.W

44. Find a 3×1 matrix $\mathbf{x}$ with entries not all zero such that $A\mathbf{x} = 1\mathbf{x}$, where

$$A = \begin{bmatrix} 1 & 2 & -1 \\ 1 & 0 & 1 \\ 4 & -4 & 5 \end{bmatrix}.$$

Q2 (b)

P.5. B. side

In Exercises 45 and 46, solve the given linear system and write the solution $\mathbf{x}$ *as* $\mathbf{x} = \mathbf{x}_p + \mathbf{x}_h$, *where* $\mathbf{x}_p$ *is a particular solution to the given system and* $\mathbf{x}_h$ *is a solution to the associated homogeneous system.*

45.
$$\begin{aligned} x + 2y - z - 2w &= 2 \\ 2x + y - 2z + 3w &= 2 \\ x + 2y + 3z + 4w &= 5 \\ 4x + 5y - 4z - w &= 6 \end{aligned}$$

46.
$$\begin{aligned} x - y - 2z + 3w &= 4 \\ 3x + 2y - z + 2w &= 5 \\ - y - 7z + 9w &= -2 \end{aligned}$$

In Exercises 47 and 48, find the quadratic polynomial that interpolates the given points.

H.W P-6

47. (1, 2), (3, 3), (5, 8)

48. (1, 5), (2, 12), (3, 44)

In Exercises 49 and 50, find the cubic polynomial that interpolates the given points.

H.W

49. (−1, −6), (1, 0), (2, 8), (3, 34)

H.W

50. (−2, 2), (−1, 2), (1, 2), (2, 10)

P6.B.S

51. A furniture manufacturer makes chairs, coffee tables, and dining-room tables. Each chair requires 10 minutes of sanding, 6 minutes of staining, and 12 minutes of varnishing. Each coffee table requires 12 minutes of sanding, 8 minutes of staining, and 12 minutes of varnishing. Each dining-room table requires 15 minutes of sanding, 12 minutes of staining, and 18 minutes of varnishing. The sanding bench is available 16 hours per week, the staining bench 11 hours per week, and the varnishing bench 18 hours per week. How many (per week) of each type of furniture should be made so that the benches are fully utilized?

P-6.B.S

52. A book publisher publishes a potential best seller in three different bindings: paperback, book club, and deluxe. Each paperback book requires 1 minute for sewing and 2 minutes for gluing. Each book club book requires 2 minutes for sewing and 4 minutes for gluing. Each deluxe book requires 3 minutes for sewing and 5 minutes for gluing. If the sewing plant is available 6 hours per day and the gluing plant is available 11 hours per day, how many books of each type can be produced per day so that the plants are fully utilized?

P-X

53. (*Calculus Required*) Construct a linear system of equations to determine a quadratic polynomial

$$p(x) = ax^2 + bx + c$$

that satisfies the conditions $p(0) = f(0)$, $p'(0) = f'(0)$, and $p''(0) = f''(0)$, where $f(x) = e^{2x}$.

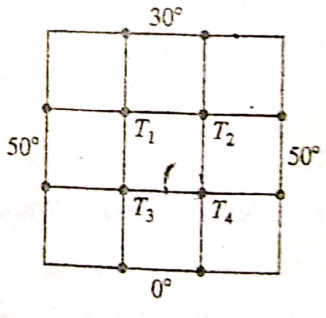
P-X

54. (*Calculus Required*) Construct a linear system of equations to determine a quadratic polynomial

$$p(x) = ax^2 + bx + c$$

that satisfies the conditions $p(1) = f(1)$, $p'(1) = f'(1)$, and $p''(1) = f''(1)$, where $f(x) = xe^{x-1}$.

55. Determine the temperatures at the interior points T_i, $i = 1, 2, 3, 4$ for the plate shown in the figure. (See Example 17.)



P7.B.S

In Exercises 56 through 59, solve the bit linear systems.

56. (a)
$$\begin{aligned} x + y + z &= 0 \\ y + z &= 1 \\ x + y \quad\;\; &= 1 \end{aligned}$$

(b)
$$\begin{aligned} x + y + z &= 1 \\ x \quad\;\; + z &= 0 \\ y + z &= 1 \end{aligned}$$

H.W

57. (a)
$$\begin{aligned} x + y \quad\;\; + w &= 0 \\ x \quad\;\; + z + w &= 1 \\ y + z + w &= 1 \end{aligned}$$

(b)
$$\begin{aligned} x + y \quad\;\;\quad\;\; &= 0 \\ x + y + z \quad\;\; &= 1 \\ x + y + z + w &= 0 \end{aligned}$$

H.W

58. (a)
$$\begin{aligned} x + y + z \quad\;\; &= 1 \\ y + z + w &= 1 \\ x \quad\;\;\quad\;\; + w &= 1 \end{aligned}$$

(b)
$$\begin{aligned} x + y + z \quad\;\; &= 0 \\ y + z + w &= 0 \\ x \quad\;\;\quad\;\; + w &= 0 \end{aligned}$$

*This type of problem will play a key role in Chapter 8.

29 ✓

59. Solve the bit linear system $A\mathbf{x} = \mathbf{c}$, where

(a) $A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}, \mathbf{c} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$

H.W

(b) $A = \begin{bmatrix} 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix}, \mathbf{c} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$

Theoretical Exercises

T.1. Show that properties (a), (b), and (c) alone [excluding (d)] of the definition of the reduced row echelon form of a matrix A imply that if a column of A contains a leading entry of some row, then all other entries in that column *below the leading entry* are zero.

T.2. Show that

(a) Every matrix is row equivalent to itself.

(b) If A is row equivalent to B, then B is row equivalent to A.

(c) If A is row equivalent to B and B is row equivalent to C, then A is row equivalent to C.

T.3. Prove Corollary 1.1.

T.4. Show that the linear system $A\mathbf{x} = \mathbf{b}$, where A is $n \times n$, has no solution if and only if the reduced row echelon form of the augmented matrix has a row whose first n elements are zero and whose $(n + 1)$st element is 1.

T.5. Let

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}.$$

Show that A is row equivalent to I_2 if and only if $ad - bc \neq 0$.

T.6. (a) Let

$$A = \begin{bmatrix} a & b \\ ka & kb \end{bmatrix}.$$

Use Exercise T.5 to determine if A is row equivalent to I_2.

(b) Let A be a 2×2 matrix with a row consisting entirely of zeros. Use Exercise T.5 to determine if A is row equivalent to I_2.

T.7. Determine the reduced row echelon form of the matrix

$$\begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}.$$

T.8. Let

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}.$$

Show that the homogeneous system $A\mathbf{x} = \mathbf{0}$ has only the trivial solution if and only if $ad - bc \neq 0$.

T.9. Let A be an $n \times n$ matrix in reduced row echelon form. Show that if A is not equal to I_n, then A has a row consisting entirely of zeros.

T.10. Show that the values of λ for which the homogeneous system

$$\begin{aligned} (a - \lambda)x + \quad\quad by &= 0 \\ cx + (d - \lambda)y &= 0 \end{aligned}$$

has a nontrivial solution satisfy the equation $(a - \lambda)(d - \lambda) - bc = 0$. (*Hint*: See Exercise T.8.)

T.11. Let $\mathbf{u}$ and $\mathbf{v}$ be solutions to the homogeneous linear system $A\mathbf{x} = \mathbf{0}$.

(a) Show that $\mathbf{u} + \mathbf{v}$ is a solution.

(b) Show that $\mathbf{u} - \mathbf{v}$ is a solution.

(c) For any scalar r, show that $r\mathbf{u}$ is a solution.

(d) For any scalars r and s, show that $r\mathbf{u} + s\mathbf{v}$ is a solution.

T.12. Show that if $\mathbf{u}$ and $\mathbf{v}$ are solutions to the linear system $A\mathbf{x} = \mathbf{b}$, then $\mathbf{u} - \mathbf{v}$ is a solution to the associated homogeneous system $A\mathbf{x} = \mathbf{0}$.

T.13. Let $A\mathbf{x} = \mathbf{b}$, $\mathbf{b} \neq \mathbf{0}$, be a consistent linear system.

(a) Show that if $\mathbf{x}_p$ is a particular solution to the given nonhomogeneous system and $\mathbf{x}_h$ is a solution to the associated homogeneous system $A\mathbf{x} = \mathbf{0}$, then $\mathbf{x}_p + \mathbf{x}_h$ is a solution to the given system $A\mathbf{x} = \mathbf{b}$.

(b) Show that every solution $\mathbf{x}$ to the nonhomogeneous linear system $A\mathbf{x} = \mathbf{b}$ can be written as $\mathbf{x}_p + \mathbf{x}_h$, where $\mathbf{x}_p$ is a particular solution to the given nonhomogeneous system and $\mathbf{x}_h$ is a solution to the associated homogeneous system $A\mathbf{x} = \mathbf{0}$. [*Hint*: Let $\mathbf{x} = \mathbf{x}_p + (\mathbf{x} - \mathbf{x}_p)$.]

T.14. Justify the second remark following Example 12.

Ex: 1.6

Q1 Reduced row echelon form,

Q2 Neither Q3 Reduced row echelon form.

Q4 Neither Q5 Row echelon form Q6 Neither.

Q7 Neither Q8. Neither.

Q9 $A = \begin{bmatrix} 1 & 0 & 3 \\ -3 & 1 & 4 \\ 4 & 2 & 2 \\ 5 & -1 & 5 \end{bmatrix}$.

a) Interchanging the second and fourth rows.

$A = \begin{bmatrix} 1 & 0 & 3 \\ 5 & -1 & 5 \\ 4 & 2 & 2 \\ -3 & 1 & 4 \end{bmatrix} \underset{\sim}{R_{24}}$ or $R_2 \leftrightarrow R_4$

b) Multiplying the third row by 3.

$A = \begin{bmatrix} 1 & 0 & 3 \\ -3 & 1 & 4 \\ 12 & 6 & 6 \\ 5 & -1 & 5 \end{bmatrix} \underset{\sim}{3R_3}$

(c) Adding (-3) times the 1st row to the fourth row

$$A = \begin{bmatrix} 1 & 0 & 3 \\ -3 & 1 & 4 \\ 4 & 2 & 2 \\ 2 & -1 & -4 \end{bmatrix} \quad -3R_1 + R_4$$

R.W

$-3 \quad 0 \quad -9 \quad (-3R_1)$

$+$with R_4

$$\begin{array}{ccc} -3 & 0 & -9 \\ 5 & -1 & 5 \\ \hline \end{array}$$
$$\boxed{2 \quad -1 \quad -4}$$

Q10 is similarly to Q9.

Q11 Find three matrices that are row equivalent to

$$A = \begin{bmatrix} 2 & -1 & 3 & 4 \\ 0 & 1 & 2 & -1 \\ 5 & 2 & -3 & 4 \end{bmatrix}.$$

1st Possibility

(a) $\begin{bmatrix} 2 & -1 & 3 & 4 \\ 5 & 2 & -3 & 4 \\ 0 & 1 & 2 & -1 \end{bmatrix} \underset{\sim}{R_{23}}$

2nd Possibility

(b) $\begin{bmatrix} 4 & -2 & 6 & 8 \\ 0 & 1 & 2 & -1 \\ 5 & 2 & -3 & 4 \end{bmatrix} \underset{\sim}{2R_1}$

3rd Possibility

(c) $\begin{bmatrix} 2 & -1 & 3 & 4 \\ 0 & 1 & 2 & -1 \\ 7 & 1 & 0 & 8 \end{bmatrix} \underset{\sim}{R_1 + R_3}.$

Ex: 1.6

Q12 In Exercise 13 through 16, find a row echelon form of the given matrix.

Q13

$$\begin{bmatrix} 0 & -1 & 2 & 3 \\ 2 & 3 & 4 & 5 \\ 1 & 3 & -1 & 2 \\ 3 & 2 & 4 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 3 & -1 & 2 \\ 2 & 3 & 4 & 5 \\ 0 & -1 & 2 & 3 \\ 3 & 2 & 4 & 1 \end{bmatrix} \underset{\sim}{R_{13}} \text{ low operation}$$

$$\begin{bmatrix} 1 & 3 & -1 & 2 \\ 0 & -3 & 6 & 1 \\ 0 & -1 & 2 & 3 \\ 0 & -7 & 7 & -5 \end{bmatrix} \frac{R_2 - 2R_1}{R_3 - 3R_1}$$

$$\begin{bmatrix} 1 & 3 & -1 & 2 \\ 0 & -1 & 2 & 3 \\ 0 & -3 & 6 & 1 \\ 0 & -7 & 7 & -5 \end{bmatrix} \underset{\sim}{R_{23}}$$

$$\begin{bmatrix} 1 & 3 & -1 & 2 \\ 0 & 1 & -2 & -3 \\ 0 & -3 & 6 & 1 \\ 0 & -7 & 7 & -5 \end{bmatrix} (-1)R_2$$

$$= \begin{bmatrix} 1 & 3 & -1 & 2 \\ 0 & 1 & -2 & -3 \\ 0 & 0 & 0 & -8 \\ 0 & 0 & -7 & -26 \end{bmatrix} \begin{matrix} R_3+3R_2 \\ \sim \\ R_4+7R_2 \\ \sim \end{matrix}$$

$$= \begin{bmatrix} 1 & 3 & -1 & 2 \\ 0 & 1 & -2 & -3 \\ 0 & 0 & -7 & -26 \\ 0 & 0 & 0 & -8 \end{bmatrix} \begin{matrix} R_{34} \\ \sim \end{matrix}$$

$$= \begin{bmatrix} 1 & 3 & -1 & 2 \\ 0 & 1 & -2 & -3 \\ 0 & 0 & 1 & 26/7 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{matrix} R_3/-7 \\ \sim \\ R_4/-8 \\ \sim \end{matrix}$$

A.

Q4

$$\begin{bmatrix} 1 & -2 & 0 & 2 \\ 2 & -3 & -1 & 5 \\ 1 & 3 & 2 & 5 \\ 1 & 1 & 0 & 2 \\ 2 & -6 & -2 & 1 \end{bmatrix}.$$

$$\begin{bmatrix} 1 & -2 & 0 & 2 \\ 0 & 1 & -1 & 1 \\ 0 & 5 & 2 & 3 \\ 0 & 3 & 0 & 0 \\ 0 & -2 & -2 & -3 \end{bmatrix} \begin{matrix} R_2-2R_1 \\ \sim \\ R_3-R_1 \\ \sim \\ R_4-R_1 \\ \sim \\ R_5-R_1 \\ \sim \end{matrix}.$$

③

Ex: 1.6

$$= \begin{bmatrix} 1 & -2 & 0 & 2 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 7 & -2 \\ 0 & 0 & 3 & -3 \\ 0 & 0 & 4 & 1 \end{bmatrix} \begin{matrix} \underline{R_3 - 5R_2} \\ \underline{R_4 - 3R_2} \\ \underline{R_5 + 2R_2} \end{matrix}.$$

$$= \begin{bmatrix} 1 & -2 & 0 & 2 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 1 & -2/7 \\ 0 & 0 & 4 & 1 \end{bmatrix} \underline{1/7 R_3}.$$

$$= \begin{bmatrix} 1 & -2 & 0 & 2 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 1 & -2/7 \\ 0 & 0 & 0 & -15/7 \\ 0 & 0 & 0 & 15/7 \end{bmatrix} \begin{matrix} \underline{R_4 - 3R_3} \\ \underline{R_5 - 4R_3} \end{matrix}.$$

$$= \begin{bmatrix} 1 & -2 & 0 & 2 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 1 & -2/7 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 15/7 \end{bmatrix} \left(-\frac{7}{15}\right) R_4.$$

$$= \begin{bmatrix} 1 & -2 & 0 & 2 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 1 & -2/7 \\ 0 & 0 & 0 & 0 \end{bmatrix} R_5 - \frac{15}{7} R_4.$$

Ex, 1.6

Q18 Let $A=\begin{bmatrix}1&2&1\\-1&1&2\\2&1&-2\end{bmatrix}$. & $Ax=b$.

(a) $X=\begin{bmatrix}1\\2\\3\end{bmatrix}$; $b=0$

Now $Ax=b$

$$\begin{bmatrix}1&2&1\\-1&1&2\\2&1&-2\end{bmatrix}\begin{bmatrix}1\\2\\3\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$$

$$\Rightarrow \begin{bmatrix}1+4+3\\-1+2+6\\2\;2-6\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$$

$$\Rightarrow \begin{bmatrix}8\\7\\-2\end{bmatrix}\neq\begin{bmatrix}0\\0\\0\end{bmatrix}$$ No Solution.

(b) $X=\begin{bmatrix}0\\0\\0\end{bmatrix}$; $b=0$

$Ax=b$

$$\begin{bmatrix}1&2&1\\-1&1&2\\2&1&-2\end{bmatrix}\begin{bmatrix}0\\0\\0\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$$

$$\begin{bmatrix}0+0+0\\0+0+0\\0+0+0\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}\Rightarrow\begin{bmatrix}0\\0\\0\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$$

Solution Exist.

Qc & Qd similarly.

In Exercises 20 through 22, find all solution to the given linear system.

Q20(a) $x+y+2z=-1$

$x-2y+z=-5$

$3x+y+z=3$.

$$\left[\begin{array}{ccc|c} 1 & 1 & 2 & -1 \\ 1 & -2 & 1 & -5 \\ 3 & 1 & 1 & 3 \end{array}\right].$$

$$\sim \left[\begin{array}{ccc|c} 1 & 1 & 2 & -1 \\ 0 & -3 & -1 & -4 \\ 0 & -2 & -5 & 6 \end{array}\right] \begin{array}{l} R_2-R_1 \\ R_3-3R_1 \end{array}$$

$$\sim \left[\begin{array}{ccc|c} 1 & 1 & 2 & -1 \\ 0 & 3 & 1 & 4 \\ 0 & -2 & -5 & 6 \end{array}\right] (-1)R_2$$

$$\sim \left[\begin{array}{ccc|c} 1 & 1 & 2 & -1 \\ 0 & 1 & -4 & 10 \\ 0 & -2 & -5 & 6 \end{array}\right] R_2+R_3$$

$$\sim \left[\begin{array}{ccc|c} 1 & 1 & 2 & -1 \\ 0 & 1 & -4 & 10 \\ 0 & 0 & -13 & 26 \end{array}\right] R_3+2R_2.$$

Ex. 1.6

$x + y + 2z = -1$ —— (i)

$y - 4z = 10$ —— (ii)

$-13z = 26$ —— (iii)

$eq\ (iii) \Rightarrow z = \dfrac{26}{-13} \Rightarrow \boxed{z = -2}$

$eq\ (ii) \Rightarrow y - 4(-2) = 10$

$y = 10 - 8$

$\boxed{y = 2}$

$eq\ (i) \Rightarrow x + (2) + 2(-2) = -1$

$x + 2 - 4 = -1 \Rightarrow \boxed{x = 1}$

Qb, Qc, Qd Similarly

Q2 (a) $x + y + 2z + 3w = 13$

$x - 2y + z + w = 8$

$3x + y + z - w = 1$

$$\left[\begin{array}{cccc|c} 1 & 1 & 2 & 3 & 13 \\ 1 & -2 & 1 & 1 & 8 \\ 3 & 1 & 1 & -1 & 1 \end{array}\right]$$

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$$\left[\begin{array}{cccc|c} 1 & 1 & 2 & 3 & 13 \\ 0 & -3 & -1 & -2 & -5 \\ 0 & -2 & -5 & -10 & -38 \end{array}\right] \underset{\sim}{R_2 - R_3} \quad \underset{\sim}{R_3 - 3R_1}$$

$$\sim \left[\begin{array}{cccc|c} 1 & 1 & 2 & 3 & 13 \\ 0 & 3 & 1 & 2 & 5 \\ 0 & -2 & -5 & -10 & -38 \end{array}\right] (-1)R_2$$

$$\sim \left[\begin{array}{cccc|c} 1 & 1 & 2 & 3 & 13 \\ 0 & 1 & -4 & -8 & -33 \\ 0 & -2 & -5 & -10 & -38 \end{array}\right] R_2 + R_3$$

$$\sim \left[\begin{array}{cccc|c} 1 & 1 & 2 & 3 & 13 \\ 0 & 1 & -4 & -8 & -33 \\ 0 & 0 & -13 & -26 & -104 \end{array}\right] R_3 + 2R_2$$

$x + y + 2z + 3w = 13$ — (i)

$y - 4z - 8w = -33$ — (ii)

$-13z - 26w = -104$ — (iii)

Let w $\boxed{w = r}$

By (iii) $\Rightarrow -13z - 26r = -104$.

$-13(z + 2r) = -104$

$z + 2r = 8$

$\boxed{z = 8 - 2r}$

EX 1.6

eq (ii) $\Rightarrow$

$$y-4(8-2r)-8r=-33$$

$$y-32+\cancel{8r}-\cancel{8r}=-33$$

$$y=-33+32=-1 \Rightarrow \boxed{y=-1}$$

eq (i) $\Rightarrow$ $x+(-1)+2(8-2r)+3r=13$

$$x-1+16-4r+3r=13$$

$$x+15-r=13$$

$$x=13-15+r$$

$$\boxed{x=r-2}$$

$\boxed{\text{Qb, Qd, Similarly}}$

Q21(c) $\left[\begin{array}{cccc|c} 2 & 1 & 1 & -2 & 1 \\ 3 & -2 & 1 & -6 & -2 \\ 1 & 1 & -1 & -1 & -1 \\ 6 & 0 & 1 & -9 & -2 \\ 5 & -1 & 2 & -8 & 3 \end{array}\right]$

$\sim \left[\begin{array}{cccc|c} 1 & 1 & -1 & -1 & -1 \\ 3 & -2 & 1 & -6 & -2 \\ 2 & 1 & 1 & -2 & 1 \\ 6 & 0 & 1 & -9 & -2 \\ 5 & -1 & 2 & -8 & 3 \end{array}\right]$ R_{13}

$$\sim \left[\begin{array}{cccc|c} 1 & 1 & -1 & -1 & -1 \\ 0 & -5 & 4 & -3 & 1 \\ 0 & -1 & 3 & 0 & 3 \\ 0 & -6 & 7 & -3 & 4 \\ 0 & -6 & 7 & -3 & 8 \end{array}\right] \begin{array}{l} R_2-3R_1 \\ R_3-2R_1 \\ R_4-6R_1 \\ R_5-5R_1 \end{array}$$

$$\sim \left[\begin{array}{cccc|c} 1 & 1 & -1 & -1 & -1 \\ 0 & 1 & -3 & 0 & -3 \\ 0 & -5 & 4 & -3 & 1 \\ 0 & -6 & 7 & -3 & 4 \\ 0 & -6 & 7 & -3 & 8 \end{array}\right] \begin{array}{l} -R_3 \\ R_{32}. \end{array}$$

$$\sim \left[\begin{array}{cccc|c} 1 & 1 & -1 & -1 & -1 \\ 0 & 1 & -3 & 0 & -3 \\ 0 & 0 & -11 & -3 & 1 \\ 0 & 0 & -11 & -3 & -14 \\ 0 & 0 & -11 & -3 & -10 \end{array}\right] \begin{array}{l} R_3+5R_2 \\ R_4+6R_2 \\ R_5+6R_2 \end{array}$$

$$\sim \left[\begin{array}{cccc|c} 1 & 1 & -1 & -1 & -1 \\ 0 & 1 & -3 & 0 & -3 \\ 0 & 0 & 1 & 3/11 & -1/11 \\ 0 & 0 & -11 & -3 & -14 \\ 0 & 0 & -11 & -3 & -10 \end{array}\right] -\frac{1}{11}(R_3)$$

$$\sim \left[\begin{array}{cccc|c} 1 & 1 & -1 & -1 & -1 \\ 0 & 1 & -3 & 0 & -3 \\ 0 & 0 & 1 & 3/11 & -1/11 \\ 0 & 0 & 0 & 0 & -15 \\ 0 & 0 & 0 & 0 & -11 \end{array}\right] \begin{array}{l} R_4+11R_3 \\ R_5+11R_3 \end{array}$$

No Solution.

Ex: 1.6

Q23 $x + y - z = 2$
$x + 2y + z = 3$
$x + y + (a^2 - 5)z = a$

$$\left[\begin{array}{ccc|c} 1 & 1 & -1 & 2 \\ 1 & 2 & 1 & 3 \\ 1 & 1 & a^2-5 & a \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & a^2-4 & a-2 \end{array}\right] \begin{array}{l} \\ R_2 - R_1 \\ R_3 - R_1 \end{array}$$

a) If $a = -2$ no solution
(b) If $a \neq \pm 2$ then unique solution
(c) If $a = 2$ Infinite many solution

Next $a^2 - 4 = 0 \Rightarrow a^2 = 4 \Rightarrow a = \pm 2$

If $\overset{a=-2}{a^2 - 4 = a - 2}$
$\cancel{4} - \cancel{4} = -2 - 2 \Rightarrow 0 = -4$ not soln

If $a = 2$ $4 - 4 = 2 - 2 \Rightarrow 0 = 0$ Infinite Sol:

If $a \neq \pm 2$ unique soln.

Q24

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 2 & 3 & 2 & 5 \\ 2 & 3 & (a^2-1) & a+1 \end{array}\right].$$

$$\sim \left[\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & a^2-3 & a-3 \end{array}\right] \begin{array}{l} R_2-2R_1 \\ R_3-2R_1 \end{array}$$

$$\sim \left[\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & a^2-3 & a-4 \end{array}\right]$$

a) $a=\pm\sqrt{3}$ (b) $a\neq\pm\sqrt{3}$, (c) None.

Q25

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 1 & 2 & 1 & 3 \\ 1 & 1 & a^2-5 & a \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & a^2-6 & a-2 \end{array}\right] \begin{array}{l} R_2-R_1 \\ R_3-R_1 \end{array}$$

a) $a=\pm\sqrt{6}$, (b) $a\neq\pm\sqrt{6}$ (c) None.

Q26 $\left[\begin{array}{cc|c} 1 & 1 & 3 \\ 1 & a^2-8 & a \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & 1 & 3 \\ 0 & a^2-9 & a-3 \end{array}\right] R_2-R_1$

(a) $a=-3$, (b) $a\neq\pm 3$, (c) $a=3$.

Ex: 1.6

Q27 to Q30 similarly to Q20 ...

Q31 Sol: $f\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} 4 & 1 & 3 \\ 2 & -1 & 3 \\ 2 & 2 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ —— (i)

& $f\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} 4 \\ 5 \\ -1 \end{bmatrix}$ —— (ii)

comparing eq (i) & (ii) we get

$$\begin{bmatrix} 4 & 1 & 3 \\ 2 & -1 & 3 \\ 2 & 2 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4 \\ 5 \\ -1 \end{bmatrix}.$$

$$\begin{bmatrix} 4x + y + 3z \\ 2x - y + 3z \\ 2x + 2y \end{bmatrix} = \begin{bmatrix} 4 \\ 5 \\ -1 \end{bmatrix}.$$

By augmented matrix

$$\left[\begin{array}{ccc|c} 4 & 1 & 3 & 4 \\ 2 & -1 & 3 & 5 \\ 2 & 2 & 0 & -1 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|c} 4 & 1 & 3 & 4 \\ 2 & -1 & 3 & 5 \\ 1 & 1 & 0 & -1/2 \end{array}\right] \frac{R_3}{2}$$

$$\sim \left[\begin{array}{ccc|c} 1 & 1 & 0 & -1/2 \\ 2 & -1 & 3 & 5 \\ 4 & 1 & 3 & 4 \end{array}\right] R_{13}$$

$$\sim \left[\begin{array}{ccc|c} 1 & 1 & 0 & -\frac{1}{2} \\ 0 & -3 & 3 & 6 \\ 0 & -3 & 3 & 6 \end{array}\right] \begin{array}{l} R_2 - 3R_1 \\ R_3 - 4R_1 \end{array}$$

$$\sim \left[\begin{array}{ccc|c} 1 & 1 & 0 & -\frac{1}{2} \\ 0 & 1 & -1 & -2 \\ 0 & -3 & 3 & 6 \end{array}\right] -R_2/3$$

$$\sim \left[\begin{array}{ccc|c} 1 & 1 & 0 & -\frac{1}{2} \\ 0 & 1 & -1 & -2 \\ 0 & 0 & 0 & 0 \end{array}\right] R_3 + 3R_2.$$

Now $x + y = -\frac{1}{2}$ —— (i)

$y - z = -2$ —— (ii)

Let $z = t$

eq (ii) $\Rightarrow \boxed{y = -2 + t}$

eq (i) $\Rightarrow x + (-2 + t) = -\frac{1}{2}$

$x = -\frac{1}{2} + 2 - t$

$\boxed{x = \frac{3}{2} - t}$

~ . o ~

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Ex 1.6

Q33 Sol:-

$$\begin{bmatrix} 4 & 1 & 3 \\ 2 & -1 & 3 \\ 2 & 2 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$$

A.M

$$\begin{bmatrix} 4x+y+3z \\ 2x-y+3z \\ 2x+2y \end{bmatrix} = \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \left[\begin{array}{ccc|c} 4 & 1 & 3 & a \\ 2 & -1 & 3 & b \\ 2 & 2 & 0 & c \end{array}\right]$$

~~Augmented matrix~~

$$\sim \left[\begin{array}{ccc|c} 2 & 2 & 0 & c \\ 2 & -1 & 3 & b \\ 4 & 1 & 3 & a \end{array}\right] R_{13}$$

$$\sim \left[\begin{array}{ccc|c} 1 & 1 & 0 & c/2 \\ 2 & -1 & 3 & b \\ 4 & 1 & 3 & a \end{array}\right] (1/2)R_1$$

$$\sim \left[\begin{array}{ccc|c} 1 & 1 & 0 & c/2 \\ 0 & -3 & 3 & b-c \\ 0 & -3 & 3 & a-2c \end{array}\right] \begin{matrix} R_2-2R_1 \\ R_3-4R_1 \end{matrix}$$

$$\sim \left[\begin{array}{ccc|c} 1 & 1 & 0 & c/2 \\ 0 & 1 & -1 & (\frac{-b+c}{3}) \\ 0 & -3 & 3 & a-2c \end{array}\right] -R_2/3$$

$$\sim \left[\begin{array}{ccc|c} 1 & 1 & 0 & c/2 \\ 0 & 1 & -1 & -b+c/3 \\ 0 & 0 & 0 & a-c-b \end{array}\right] R_3+3R_2$$

solution is possible if

$a-c-b=0$

$\boxed{-a+c+b=0}$

Q34 is similarly to Q33

Q35 $A = \begin{bmatrix} 1 & -1 \\ 2 & 3 \end{bmatrix}$, $b_1 = \begin{bmatrix} 1 \\ -8 \end{bmatrix}$, $b_2 = \begin{bmatrix} 5 \\ -5 \end{bmatrix}$.

$Ax = b_1$

$\begin{bmatrix} 1 & -1 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ -8 \end{bmatrix}$.

$\begin{bmatrix} x-y \\ 2x+3y \end{bmatrix} = \begin{bmatrix} 1 \\ -8 \end{bmatrix}$.

New Augmented matrix

$\left[\begin{array}{cc:c} 1 & -1 & 1 \\ 2 & 3 & -8 \end{array}\right] \Rightarrow \sim \left[\begin{array}{cc:c} 1 & -1 & 1 \\ 0 & 5 & -10 \end{array}\right] R_2 - 2R_1$

$\Rightarrow 5y = -10 \Rightarrow \boxed{y = -2}$

$x - y = 1 \Rightarrow x = -2 + 1 \Rightarrow \boxed{x = -1}$

$\boxed{x = -1, \ y = -2}$

2nd Part $Ax = b_2$

$\begin{bmatrix} 1 & -1 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 5 \\ -5 \end{bmatrix}$. $\Rightarrow \begin{bmatrix} x-y \\ 2x+3y \end{bmatrix} = \begin{bmatrix} 5 \\ -5 \end{bmatrix}$.

A.M

$\left[\begin{array}{cc:c} 1 & -1 & 5 \\ 2 & 3 & -5 \end{array}\right]$

Ex 1.6

$$\sim\left[\begin{array}{cc:c} 1 & -1 & 5 \\ 0 & 5 & -15 \end{array}\right] R_2 - 2R_1$$

$5y = -15 \Rightarrow \boxed{y = -3}$

$x - y = 5 \Rightarrow x = 5 - 3 \Rightarrow \boxed{x = 2}$

$$\boxed{x = 2,\ y = -3}$$

3rd Part $[A \,\vdots\, b_1\ b_2]$

$$\left[\begin{array}{cc:cc} 1 & -1 & 1 & 5 \\ 2 & 3 & -8 & -5 \end{array}\right]$$

$$\sim\left[\begin{array}{cc:cc} 1 & -1 & 1 & 5 \\ 0 & 5 & -10 & -15 \end{array}\right] R_2 - 2R_1$$

Same results

Q36 is similarly to Q35.

Q37

$A = \begin{bmatrix} 1 & 0 & 5 \\ 1 & 1 & 1 \\ 0 & 1 & -4 \end{bmatrix}$ & $(-4I_3 - A)X = 0$

$I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$-4I_3 = \begin{bmatrix} -4 & 0 & 0 \\ 0 & -4 & 0 \\ 0 & 0 & -4 \end{bmatrix}.$

$-4I_3 - A = \begin{bmatrix} -4 & 0 & 0 \\ 0 & -4 & 0 \\ 0 & 0 & -4 \end{bmatrix} - \begin{bmatrix} 1 & 0 & 5 \\ 1 & 1 & 1 \\ 0 & 1 & -4 \end{bmatrix} = \begin{bmatrix} -5 & 0 & -5 \\ -1 & -5 & -1 \\ 0 & -1 & 0 \end{bmatrix}.$

Now $(-4I_3 - A)X = 0$

$\Rightarrow \begin{bmatrix} -5 & 0 & -5 \\ -1 & -5 & -1 \\ 0 & -1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$\begin{bmatrix} -5x + 0y - 5z \\ -1x - 5y - 1z \\ 0x - 1y + 0z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

A.M.

$\left[\begin{array}{ccc|c} -5 & 0 & -5 & 0 \\ -1 & -5 & -1 & 0 \\ 0 & -1 & 0 & 0 \end{array}\right]$

$\sim \left[\begin{array}{ccc|c} -1 & -5 & -1 & 0 \\ -5 & 0 & -5 & 0 \\ 0 & -1 & 0 & 0 \end{array}\right] R_{21}$

$\sim \left[\begin{array}{ccc|c} 1 & 5 & 1 & 0 \\ -5 & 0 & -5 & 0 \\ 0 & -1 & 0 & 0 \end{array}\right] -R_1$

$$\sim \left[\begin{array}{ccc|c} 1 & 5 & 1 & 0 \\ 0 & 25 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{array}\right] R_2+5R_1$$

$$\sim \left[\begin{array}{ccc|c} 1 & 5 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{array}\right] R_2/25$$

$$\sim \left[\begin{array}{ccc|c} 1 & 5 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] R_3+R_2$$

$x+5y+z=0$ — (i)

$y=0$ — (2)

Let $z=r$ (real no:)

eq (i) $\Rightarrow x+0+r=0 \Rightarrow \boxed{x=-r}$

$$\boxed{x=-r,\ y=0,\ z=r}$$

Q_{38} is similarly to $Q\,37$

Q39

Augmented matrix

$$\left[\begin{array}{ccc|c} 1 & 2 & -3 & a \\ 2 & 3 & 3 & b \\ 5 & 9 & -6 & c \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|c} 1 & 2 & -3 & a \\ 0 & -1 & 9 & b-2a \\ 0 & -1 & 9 & c-5a \end{array}\right] \begin{array}{l} R_2-2R_1 \\ R_3-5R_1 \end{array}$$

$$\sim \left[\begin{array}{ccc|c} 1 & 2 & -3 & a \\ 0 & 1 & -9 & 2a-b \\ 0 & 0 & 0 & c-3a-b \end{array}\right] \quad R_2+R_3, \; -R_2$$

$$c-3a-b=0$$

$$\Rightarrow \boxed{-3a-b+c=0} \text{ Ans.}$$

Q40 is similarly to Q39

Ex. 1.6

Q4) $A = \begin{bmatrix} 4 & 1 \\ 0 & 2 \end{bmatrix}$, $X = \begin{bmatrix} x \\ y \end{bmatrix}$

$$AX = \begin{bmatrix} 4 & 1 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = 4\begin{bmatrix} x \\ y \end{bmatrix} = 4X.$$

Now $4X - AX = 0$

$$(4I_2 - A)X = 0$$

$$\left(\begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix} - \begin{bmatrix} 4 & 1 \\ 0 & 2 \end{bmatrix}\right)\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

$$\begin{bmatrix} 4-4 & 0-1 \\ 0-0 & 4-2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & -1 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 0x - y \\ 0x + 2y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

A.M

$$\left[\begin{array}{cc|c} 0 & -1 & 0 \\ 0 & 2 & 0 \end{array}\right].$$

$$\sim \left[\begin{array}{cc|c} 0 & 1 & 0 \\ 0 & 2 & 0 \end{array}\right] -R_1$$

$$\sim \left[\begin{array}{cc|c} 0 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right] R_2 - 2R_1$$

$y = 0$, let $x = r$ (real no.)

$$X = \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} r \\ 0 \end{bmatrix}.$$

Q42 $A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$ & $(3I_2 - A)X = 0$

Now $3X - AX = 0$

$(3I_2 - A)X = 0$

$$\left(3\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}\right)\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\left(\begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix} - \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}\right)\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 3-2 & 0-1 \\ 0-1 & 3-2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow \begin{bmatrix} x-y \\ -x+y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

A·M

$$\left[\begin{array}{cc|c} 1 & -1 & 0 \\ -1 & 1 & 0 \end{array}\right].$$

$$\sim \left[\begin{array}{cc|c} 1 & -1 & 0 \\ 0 & 0 & 0 \end{array}\right] R_2+R_1$$

~~let~~ $x - y = 0$ let $x = \gamma$.

$y = \gamma$

$X = \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \gamma \\ \gamma \end{bmatrix}$ Ans

Ex 1.6

Q43 $(3I_3 - A)X = 0$

$$\left(\begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix} - \begin{bmatrix} 1 & 2 & -1 \\ 1 & 0 & 1 \\ 4 & -4 & 5 \end{bmatrix} \right) \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

$$\begin{bmatrix} 2 & -2 & 1 \\ -1 & 3 & -1 \\ -4 & 4 & -2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 2x - 2y + z \\ -1x + 3y - z \\ -4x + 4y - 2z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

A. M

$$\left[\begin{array}{ccc|c} 2 & -2 & 1 & 0 \\ -1 & 3 & -1 & 0 \\ -4 & 4 & -2 & 0 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|c} 1 & -3 & 1 & 0 \\ 2 & -2 & 1 & 0 \\ -4 & 4 & -2 & 0 \end{array}\right] \begin{array}{l} -R_2 \\ R_{21} \\ \end{array}$$

$$\sim \left[\begin{array}{ccc|c} 1 & -3 & 1 & 0 \\ 0 & 4 & -1 & 0 \\ 0 & -8 & 2 & 0 \end{array}\right] \begin{array}{l} \\ R_2 - 2R_1 \\ R_4 + 4R_3 \end{array}$$

$$\sim \left[\begin{array}{ccc|c} 1 & -3 & 1 & 0 \\ 0 & 1 & -1/4 & 0 \\ 0 & -8 & 2 & 0 \end{array}\right] \begin{array}{l} \\ R_4/4 \\ \end{array}$$

$$\sim \left[\begin{array}{ccc|c} 1 & -3 & 1 & 0 \\ 0 & 1 & -1/4 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] R_3+8R_2$$

$x-3y+z=0$ — (i)

$y-\frac{1}{4}z=0$ — (2)

let $\boxed{z=r}$ (real no any)

eq (2) $\Rightarrow$ $\boxed{y=1/4\, r}$

eq (i) $\Rightarrow$ $x-\frac{3}{4}r+r=0$

$x - \frac{-3r+4r}{4}=0 \Rightarrow x+r/4=0 \Rightarrow \boxed{x=-r/4}$

Hence $\boxed{x=-r/4,\ y=1/4\, r,\ z=r}$

Q44 is similarly to Q43

—— 0 ——

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EX 1.6

Q4) Given $X = X_p + X_h$,

when X_p is Particular solution

& X_h is a solution to the associated homg: system.

A.M

$$\left[\begin{array}{cccc|c} 1 & 2 & -1 & -2 & 2 \\ 2 & 1 & -2 & 3 & 2 \\ 1 & 2 & 3 & 4 & 5 \\ 4 & 5 & -4 & -1 & 6 \end{array}\right]$$

$$\sim \left[\begin{array}{cccc|c} 1 & +2 & -1 & -2 & 2 \\ 0 & -3 & 0 & 7 & -2 \\ 0 & 0 & 4 & 6 & 3 \\ 0 & -3 & 0 & 7 & -2 \end{array}\right] \begin{array}{l} \\ R_2-2R_1 \\ R_3-R_1 \\ R_4-4R_1 \end{array}$$

$$\sim \left[\begin{array}{cccc|c} 1 & 2 & -1 & -2 & 2 \\ 0 & 1 & 0 & -7/3 & 2/3 \\ 0 & 0 & 4 & 6 & 3 \\ 0 & -3 & 0 & 7 & -2 \end{array}\right] -R_2/3$$

$$\sim \left[\begin{array}{cccc|c} 1 & 2 & -1 & -2 & 2 \\ 0 & 1 & 0 & -7/3 & 2/3 \\ 0 & 0 & 4 & 6 & 3 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] R_4+3R_2$$

$$x+2y-z-2w = 2 \quad \text{—— (i)}$$

$$y - \tfrac{7}{3}w = \tfrac{2}{3} \quad \text{—— (ii)}$$

$$4z+6w = 3 \quad \text{—— (iii)}$$

Let $w = r$ (any real no.)

eq(iii) $\Rightarrow 4z + 6r = 3$

$$4z = 3 - 6r$$

$$z = 3/4 - \frac{6}{4}r \Rightarrow \boxed{z = 3/4 - 3/2\, r}$$

eq(ii) $\Rightarrow \boxed{y = 3/2 + 7/3\, r}$

eq(i) $\Rightarrow x + 2(3/2 + 7/3\, r) - (3/4 - 3/2\, r) - 2r = 2$

$$x = 17/12 - \frac{50}{12}r$$

$$x = 17/12 - 25/6\, r$$

$X = X_p + X_h$

$$\begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = \begin{bmatrix} 17/12 \\ 3/2 \\ 3/4 \\ 0 \end{bmatrix} + \begin{bmatrix} -25/6\, r \\ 7/3\, r \\ -3/2\, r \\ r \end{bmatrix}$$ Ans.

Q46 is similarly to Q45

Ex 1.6

Q47 (1,2) (3,3) (5,8).

Since quadratic polynomials $a_2x^2+a_1x+a_0=y=\cancel{f(x)}$ *

(1,2) Point

eq * => $2=a_2+a_1+a_0$ — (i)

(3,3)

eq * => $3=9a_2+3a_2+a_0$ — (ii)

For Point (5,8)

eq * => $8=25a_2+5a_1+a_0$ — (iii)

in ~~xx~~ A.M

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 9 & 3 & 1 & 3 \\ 25 & 5 & 1 & 8 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 0 & -6 & -8 & -15 \\ 0 & -20 & -24 & -42 \end{array}\right].$$

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$$\sim \left[\begin{array}{ccc:c} 1 & 1 & 1 & 2 \\ 0 & 1 & 4/3 & 5/2 \\ 0 & -20 & -24 & -42 \end{array}\right] -\frac{1}{6}R_2$$

$$\sim \left[\begin{array}{ccc:c} 1 & 1 & 1 & 2 \\ 0 & 1 & 4/3 & 5/2 \\ 0 & 0 & 8/3 & 8 \end{array}\right] R_3+20R_2$$

$a_2 + a_1 + a_0 = 2$ —— (a)

$a_1 + \frac{4}{3}a_0 = \frac{5}{2}$ —— (b)

$\frac{8}{3}a_0 = 8$ —— (c)

$q(c) \Rightarrow a_0 = \cancel{8} \times \frac{3}{\cancel{8}} \Rightarrow \boxed{a_0 = 3}$

$q(b) \Rightarrow a_1 + \frac{4}{\cancel{3}}(\cancel{3}) = \frac{5}{2}$

$a_1 = \frac{5}{2} - 4 = \frac{5-8}{2} = -\frac{3}{2}$

$\boxed{a_1 = -\frac{3}{2}}$

$q(a) \Rightarrow a_2 + (-\frac{3}{2}) + (3) = 2$

$a_2 = 2 - 3 + \frac{3}{2} = \frac{4-6+3}{2} = \frac{1}{2}$

$\boxed{a_2 = \frac{1}{2}}$

$q(*) \Rightarrow \boxed{y = \frac{1}{2}x^2 - \frac{3}{2}x + 3}$ Ans.

Ex. 1.6

Q48, Q49, Q50 similarly to Q47

Q51 $10x_1 + 12x_2 + 15x_3 = 16 \times 60 = 960$

$6x_1 + 8x_2 + 12x_3 = 11 \times 60 = 660$

$12x_1 + 12x_2 + 18x_3 = 18 \times 60 = 1080$

Aug: M

$$\left[\begin{array}{ccc:c} 10 & 12 & 15 & 960 \\ 6 & 8 & 12 & 660 \\ 12 & 12 & 18 & 1080 \end{array}\right]$$

S.Y.S.

Q52

$x_1 + 2x_2 + 3x_3 = 6 \times 60 = 360$

$2x_1 + 4x_2 + 5x_3 = 11 \times 60 = 660$

Aug: Matrix

$$\left[\begin{array}{ccc:c} 1 & 2 & 3 & 360 \\ 2 & 4 & 5 & 660 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc:c} 1 & 2 & 3 & 360 \\ 0 & 0 & -1 & -60 \end{array}\right] \quad R_2 - 2R_1$$

$\boxed{x_3 = 60}$ (deluxe binding)

$x_1 + 2x_2 + 3x_3 = 360$ — (i)

let $\boxed{x_2 = r}$ (any real no)

eq (i) $\Rightarrow x_1 + 2r + 3(60) = 360$

$x_1 = 360 - 180 - 2r$

$\boxed{x_1 = 180 - 2r}$

$\boxed{x_1 = 180 - 2r, \quad x_2 = r, \quad x_3 = 60}$

EX 1.6

Q53: Given $P(x) = ax^2 + bx + c$ — ⊛

$P(0) = f(0)$, $P'(0) = f'(0)$, & $P''(0) = f''(0)$ & $f(x) = e^{2x}$.

$f(x) = e^{2x}$ — (i)

Put $x = 0$

$f(0) = e^0 = 1 \Rightarrow \boxed{f(0) = 1}$

eq ⊛ $\Rightarrow P(0) = a0 + b0 + c$

$\boxed{P(0) = c}$

But Given $P(0) = f(0)$

$\boxed{c = 1.}$

eq ⊛ $\Rightarrow P'(x) = 2ax + b$

$P'(0) = 2a(0) + b \Rightarrow P'(0) = b.$

eq ① $\Rightarrow f'(x) = 2e^{2x}$

$f'(0) = 2e^0 = 2$

But Given $f'(0) = P'(0)$

$\boxed{2 = b}$

Similarly $\boxed{a = 2}$

eq (x) $\Rightarrow$ $\boxed{P(x) = 2x^2 + 2x + 1}$

Q54 is similarly to Q53

Ex17

Q55 $T_1 = \dfrac{30^\circ + 50^\circ + T_2 + T_3}{4}$

$4T_1 = 80^\circ + T_2 + T_3$

$4T_1 - T_2 - T_3 = 80^\circ$ —— (i)

$T_2 = \dfrac{30 + 50^\circ + T_1 + T_4}{4}$

$4T_2 - T_1 - T_4 = 80^\circ$ —— (ii)

$T_3 = \dfrac{T_1 + T_4 + 50^\circ + 0^\circ}{4}$

$4T_3 - T_1 - T_4 = 50^\circ$ —— (iii)

$T_4 = \dfrac{T_3 + T_2 + 50^\circ + 0}{4}$

$4T_4 - T_3 - T_2 = 50^\circ$ —— (iv)

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Ex 1.6

New Aug: Matrix

$$\left[\begin{array}{cccc:c} 4 & -1 & -1 & 0 & 80 \\ -1 & 4 & 0 & -1 & 80 \\ -1 & 0 & 4 & -1 & 50 \\ 0 & -1 & -1 & 4 & 50 \end{array}\right] \quad S.Y.S$$

BIT Matrices (optional) P-17

Table ①

+	0	1
0	0	1
1	1	0

Table

×	0	1
0	0	0
1	0	1

Exp) $A = \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 0 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 1 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}$

$$A+B = \begin{bmatrix} 1+1 & 0+1 \\ 1+0 & 1+1 \\ 0+1 & 1+0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \\ 1 & 1 \end{bmatrix}.$$

Note:- We have 0+0=0 and 1+1=0 Thus the additive inverse of 0 is 0 and the additive inverse of 1 is 1. Hence to compute the difference of bit matrices A & B we Proceed as follows:

$$A-B = A + (\text{inverse of } 1)\,B = A + 1B = A+B$$

$$A-B = A+B.$$

Note:- A bit is a binary digit i.e either 0 or 1

A bit matrix is also called a Boolean matrix.

Ex. 16

Q57(a) Aug: M

$$\left[\begin{array}{cccc:c} 1 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 \end{array}\right].$$

$$\sim \left[\begin{array}{cccc:c} 1 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 1 & 1 \end{array}\right] R_2+R_1$$

$$\sim \left[\begin{array}{cccc:c} 1 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{array}\right] R_3+R_2$$

$x+y+w=0$ —— (i)

$y+z=1$ —— (ii)

$\boxed{w=0}$

eq (ii) $\Rightarrow y=1-z$

eq (i) $x=-y.$

$x=-(1-z)$

let $z=0$

$y=1-0=1$

$y=1.$

$x=-(1-0)$

$x=1$

let $z=1$

$y=1-1=0$

$x=-(1-1)$

$x=0$

$$\begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} t.$$

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(b) $\left[\begin{array}{cccc:c} 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 & 0 \end{array}\right]$

$\sim \left[\begin{array}{cccc:c} 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 \end{array}\right]$ R_2+R_1 R_3+R_1

$\sim \left[\begin{array}{cccc:c} 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 \end{array}\right]$ R_3+R_2

$x+y=0 \Rightarrow x=-y$

$z=1$ let $y=0$

$w=1$ $x=0$

let $y=1$ $x=1$

$\begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}$ & $\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$ An

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Ex 1.6

Q59 (a) $Ax = C$

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} x+y+0z \\ 0x+y+0z \\ x+y+z \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

Aug: M

$$\left[\begin{array}{ccc:c} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc:c} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{array}\right] \quad R_3 + R_1$$

$x+y=0 \Rightarrow x=-y \Rightarrow x=1$

$y=1$

$z=0$

$$x = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$