CHAPTER 12

Complex Numbers and Functions. Conformal Mapping

Complex numbers and the complex plane are discussed in Secs. 12.1–12.2. Complex analysis is concerned with complex analytic functions, as defined in Sec. 12.3. In Sec. 12.4 we explain a check for analyticity based on the so-called Cauchy–Riemann equations. The latter are of basic importance. They are related to Laplace's equation (Sec. 12.4). In the remaining sections of Chap. 12 we study the most important elementary complex functions (exponential function, trigonometric functions, etc.), which generalize familiar real functions known from calculus. This includes discussions of geometric properties of these functions in connection with conformal mapping (defined in Sec. 12.5).

Prerequisites for this chapter: Elementary calculus.
References: Appendix 1, Part D.
Answers to problems: Appendix 2.

12.1 Complex Numbers. Complex Plane

Equations without *real* solutions, such as $x^2 = -1$ or $x^2 - 10x + 40 = 0$, were observed early in history and led to the introduction of complex numbers.[1] By definition, a **complex number** z is an ordered pair (x, y) of real numbers x and y, written

$$z = (x, y).$$

x is called the **real part** and y the **imaginary part** of z, written

$$x = \operatorname{Re} z, \qquad y = \operatorname{Im} z.$$

By definition, two complex numbers are **equal** if and only if their real parts are equal and their imaginary parts are equal.

[1]First to use complex numbers for this purpose was the Italian mathematician GIROLAMO CARDANO (1501—1576), who found the formula for solving cubic equations. The term "complex number" was introduced by the great German mathematician CARL FRIEDRICH GAUSS (see the footnote in Sec. 4.4), who also paved the way for a general use of complex numbers.

$(0, 1)$ is called the **imaginary unit** and is denoted by i,

(1) $$\boxed{i = (0, 1).}$$

Addition, Multiplication. Notation $z = x + iy$

Addition of two complex numbers $z_1 = (x_1, y_1)$ and $z_2 = (x_2, y_2)$ is defined by[2]

(2) $$z_1 + z_2 = (x_1, y_1) + (x_2, y_2) = (x_1 + x_2, \quad y_1 + y_2).$$

Multiplication is defined by

(3) $$z_1 z_2 = (x_1, y_1)(x_2, y_2) = (x_1x_2 - y_1y_2, \quad x_1y_2 + x_2y_1).$$

In particular, these two definitions imply that

$$(x_1, 0) + (x_2, 0) = (x_1 + x_2, 0) \qquad \text{and} \qquad (x_1, 0)(x_2, 0) = (x_1x_2, 0),$$

as for real numbers x_1, x_2. Hence the complex numbers **"extend"** the reals, and we can write

(4*) $$(x, 0) = x.$$

Similarly, for any real y

(4**) $$(0, y) = iy$$

because $iy = (0, 1)(y, 0)$ by (1) and (4*) (with y instead of x); and multiplication (3) gives $(0, 1)(y, 0) + (0y - 1 \cdot 0, 0 \cdot 0 + 1y) = (0, y)$, hence (4**). Together with (4) and by addition we thus have $(x, y) = (x, 0) + (0, y) = x + iy$:

In practice, complex numbers $z = (x, y)$ are written[3]

(4) $$\boxed{z = x + iy.}$$

If $x = 0$, then $z = iy$ and is called **pure imaginary.** Also, (1) and (3) give

(5) $$i^2 = -1$$

because by the definition of multiplication, $i^2 = ii = (0, 1)(0, 1) = (-1, 0) = -1$.

For **addition** the standard notation (4) gives [see (2)]

$$(x_1 + iy_1) + (x_2 + iy_2) = (x_1 + x_2) + i(y_1 + y_2).$$

---

[2]Students familiar with vectors see that this is *vector addition,* whereas this multiplication has no analog in the usual vector algebra.

[3]Electrical engineers often use j to reserve i for the current.

For **multiplication** it gives the following very simple recipe. Multiply each term by each term and use $i^2 = -1$ when it occurs [see (3)]:

$$(x_1 + iy_1)(x_2 + iy_2) = x_1x_2 + ix_1y_2 + iy_1x_2 + i^2y_1y_2$$
$$= (x_1x_2 - y_1y_2) + i(x_1y_2 + x_2y_1).$$

This agrees with (3).

EXAMPLE 1 Real part, imaginary part, sum and product of complex numbers

Let $z_1 = 8 + 3i$ and $z_2 = 9 - 2i$. Then $\text{Re } z_1 = 8$, $\text{Im } z_1 = 3$, $\text{Re } z_2 = 9$, $\text{Im } z_2 = -2$ and

$$z_1 + z_2 = (8 + 3i) + (9 - 2i) = 17 + i,$$

$$z_1z_2 = (8 + 3i)(9 - 2i) = 72 + 6 + i(-16 + 27) = 78 + 11i.$$ ◀

Subtraction, Division

Subtraction and **division** are defined as the inverse operations of addition and multiplication. Thus the **difference** $z = z_1 - z_2$ is the complex number z for which $z_1 = z + z_2$. Hence by (2),

$$z_1 - z_2 = (x_1 - x_2) + i(y_1 - y_2). \tag{6}$$

The **quotient** $z = z_1/z_2$ $(z_2 \neq 0)$ is the complex number z for which $z_1 = zz_2$. If we equate the real and the imaginary parts on both sides of this equation, setting $z = x + iy$, we obtain $x_1 = x_2x - y_2y$, $y_1 = y_2x + x_2y$. The solution is

$$z = \frac{z_1}{z_2} = x + iy, \qquad x = \frac{x_1x_2 + y_1y_2}{x_2^2 + y_2^2}, \qquad y = \frac{x_2y_1 - x_1y_2}{x_2^2 + y_2^2}. \tag{7*}$$

The ***practical rule*** used to get this is by multiplying numerator and denominator of z_1/z_2 by $x_2 - iy_2$ and simplifying:

$$z = \frac{x_1 + iy_1}{x_2 + iy_2} = \frac{(x_1 + iy_1)(x_2 - iy_2)}{(x_2 + iy_2)(x_2 - iy_2)} = \frac{x_1x_2 + y_1y_2}{x_2^2 + y_2^2} + i\,\frac{x_2y_1 - x_1y_2}{x_2^2 + y_2^2}. \tag{7}$$

EXAMPLE 2 Difference and quotient of complex numbers

For $z_1 = 8 + 3i$ and $z_2 = 9 - 2i$ we get $z_1 - z_2 = (8 + 3i) - (9 - 2i) = -1 + 5i$ and

$$\frac{z_1}{z_2} = \frac{8 + 3i}{9 - 2i} = \frac{(8 + 3i)(9 + 2i)}{(9 - 2i)(9 + 2i)} = \frac{66 + 43i}{81 + 4} = \frac{66}{85} + \frac{43}{85}i.$$

Check the division by multiplication to get $8 + 3i$. ◀

Complex numbers satisfy the same commutative, associative, and distributive laws as real numbers (see the problem set).

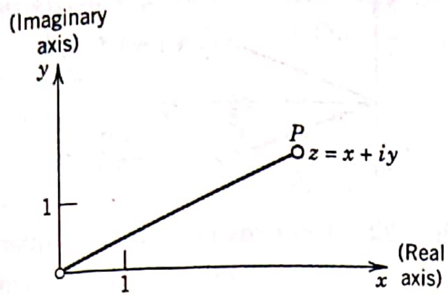


Fig. 288. The complex plane

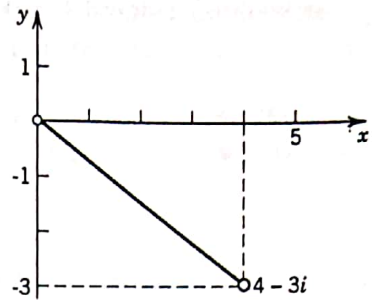


Fig. 289. The number 4 − 3i in the complex plane

Complex Plane

This was algebra. Now comes geometry: the geometrical representation of complex numbers as points in the plane. This is of great practical importance. The idea is quite simple and natural. We choose two perpendicular coordinate axes, the horizontal x-axis, called the **real axis,** and the vertical y-axis, called the **imaginary axis.** On both axes we choose the same unit of length (Fig. 288). This is called a **Cartesian coordinate system.** We now plot a given complex number $z = (x, y) = x + iy$ as the point P with coordinates x, y. The xy-plane in which the complex numbers are represented in this way is called the **complex plane.**[4] Figure 289 shows an example.

Instead of saying "the point represented by z in the complex plane" we say briefly and simply ***"the point z in the complex plane."*** This will cause no misunderstandings.

Addition and subtraction can now be visualized as illustrated in Figs. 290 and 291.

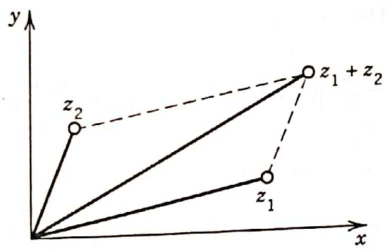


Fig. 290. Addition of complex numbers

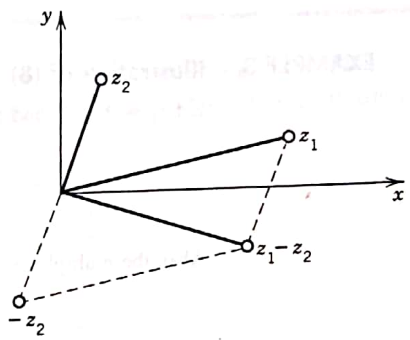


Fig. 291. Subtraction of complex numbers

Complex Conjugate Numbers

The **complex conjugate** $\bar{z}$ of a complex number $z = x + iy$ is defined by

$$\boxed{\bar{z} = x - iy.}$$

[4]Sometimes called the **Argand diagram,** after the French mathematician JEAN ROBERT ARGAND (1768—1822), born in Geneva and later librarian in Paris. His paper on the complex plane appeared in 1806, nine years after a similar memoir by the Norwegian mathematician CASPAR WESSEL (1745—1818), a surveyor of the Danish Academy of Science.

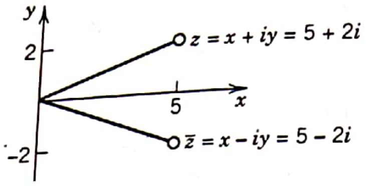


Fig. 292. Complex conjugate numbers

It is obtained geometrically by reflecting the point z in the real axis. Figure 292 shows this for $z = 5 + 2i$ and its conjugate $\bar{z} = 5 - 2i$.

It is important because it permits us to switch from complex to real. Indeed, by multiplication, $z\bar{z} = x^2 + y^2$ (verify!). By addition and subtraction, $z + \bar{z} = 2x$, $z - \bar{z} = 2iy$. We thus obtain for the real part x and the imaginary part y (not iy!) of $z = x + iy$ the important formulas

$$\boxed{\operatorname{Re} z = x = \frac{1}{2}(z + \bar{z}), \qquad \operatorname{Im} z = y = \frac{1}{2i}(z - \bar{z}).} \tag{8}$$

If z is real, $z = x$, then $\bar{z} = z$ by the definition of $\bar{z}$, and conversely. Working with conjugates is easy, since we have

$$\begin{aligned} \overline{(z_1 + z_2)} &= \bar{z}_1 + \bar{z}_2, & \overline{(z_1 - z_2)} &= \bar{z}_1 - \bar{z}_2, \\ \overline{(z_1 z_2)} &= \bar{z}_1 \bar{z}_2, & \overline{\left(\frac{z_1}{z_2}\right)} &= \frac{\bar{z}_1}{\bar{z}_2}. \end{aligned} \tag{9}$$

EXAMPLE 3 Illustration of (8) and (9)

Let $z_1 = 4 + 3i$ and $z_2 = 2 + 5i$. Then by (8),

$$\operatorname{Im} z_1 = \frac{1}{2i}[(4 + 3i) - (4 - 3i)] = \frac{3i + 3i}{2i} = 3.$$

Also, the multiplication formula in (9) is verified by

$$\overline{(z_1 z_2)} = \overline{(4 + 3i)(2 + 5i)} = \overline{-7 + 26i} = -7 - 26i,$$

$$\bar{z}_1 \bar{z}_2 = (4 - 3i)(2 - 5i) = -7 - 26i. \quad \blacktriangleleft$$

PROBLEM SET 12.1

1. **(Powers of the imaginary unit)** Show that

$$\begin{aligned} &i^2 = -1, \quad i^3 = -i, \quad i^4 = 1, \quad i^5 = i, \cdots \\ &\frac{1}{i} = -i, \quad \frac{1}{i^2} = -1, \quad \frac{1}{i^3} = i, \cdots, \end{aligned} \tag{10}$$

2. **Multiplication by i** is geometrically a counterclockwise rotation through $\pi/2$ (90°). Verify this by plotting z and iz and the angle of rotation for $z = 4 + 2i$, $z = -1 + i$, $z = 5 - 2i$.

Arithmetical Operations. Real and Imaginary Parts. Complex Conjugates
Let $z_1 = 4 + 3i$ and $z_2 = 2 - 5i$. Find each of the following in the form $x + iy$, showing the details of your work:

3. $z_1 z_2$ **4.** $(3z_1 - z_2)^2$ **5.** $1/z_1$ **6.** $25z_2/z_1$
7. $\text{Re}\,(z_1^3)$, $(\text{Re}\,z_1)^3$ **8.** $(z_1 - z_2)/(z_1 + z_2)$ **9.** $z_1\bar{z}_2$, $\bar{z}_1 z_2$ **10.** $1/z_1^2$, $1/\bar{z}_1^2$
11. $\bar{z}_1/\bar{z}_2$, $\overline{(z_1/z_2)}$ **12.** $z_2\bar{z}_2/(z_1\bar{z}_1)$

Let $z = x + iy$. Find (showing the details of your work)
13. $\text{Im}\,(1/z)$ **14.** $\text{Im}\,z^4$, $(\text{Im}\,z^2)^2$ **15.** $(1 + i)^{16}$ **16.** $\text{Re}\,(z/\bar{z})$ **17.** $\text{Re}\,(z^2/\bar{z})$

18. **(Laws of addition and multiplication)** Derive the following laws for complex numbers from the corresponding laws for real numbers.

$$z_1 + z_2 = z_2 + z_1, \qquad z_1 z_2 = z_2 z_1 \qquad \textit{(Commutative laws)}$$

$$(z_1 + z_2) + z_3 = z_1 + (z_2 + z_3), \qquad (z_1 z_2)z_3 = z_1(z_2 z_3) \qquad \textit{(Associative laws)}$$

$$z_1(z_2 + z_3) = z_1 z_2 + z_1 z_3 \qquad \textit{(Distributive law)}$$

$$0 + z = z + 0 = z, \qquad z + (-z) = (-z) + z = 0, \qquad z \cdot 1 = z.$$

19. **(Laws for conjugates)** Verify (9) for $z_1 = 38 + 18i$, $z_2 = 3 + 5i$.
20. **(Multiplication)** If the product of two complex numbers is zero, show that at least one factor must be zero.

12.2 Polar Form of Complex Numbers Powers and Roots

We can substantially increase the usefulness of the complex plane and gain further insight into the nature of complex numbers if besides the xy-coordinates we also employ the usual polar coordinates r, θ defined by

$$x = r\cos\theta, \qquad y = r\sin\theta. \tag{1}$$

Then $z = x + iy$ takes the so-called **polar form**

$$z = r(\cos\theta + i\sin\theta). \tag{2}$$

r is called the **absolute value** or **modulus** of z and is denoted by $|z|$. Hence

$$|z| = r = \sqrt{x^2 + y^2} = \sqrt{z\bar{z}}. \tag{3}$$

Geometrically, $|z|$ is the distance of the point z from the origin (Fig. 293). Similarly, $|z_1 - z_2|$ is the distance between z_1 and z_2 (Fig. 294).

θ is called the **argument** of z and is denoted by $\arg z$. Thus (Fig. 293)

$$\theta = \arg z = \arctan\frac{y}{x} \qquad (z \neq 0). \tag{4}$$

Geometrically, θ is the directed angle from the positive x-axis to OP in Fig. 293. Here,

Properties of complex conjugates

Let z_1 & z_2 be two complex numbers. Then it satisfied the following properties.

① $\overline{z_1+z_2} = \bar{z}_1+\bar{z}_2$

② $\overline{z_1 \cdot z_2} = \bar{z}_1 \cdot \bar{z}_2$

③ $\overline{\left(\frac{z_1}{z_2}\right)} = \frac{\bar{z}_1}{\bar{z}_2}$

④ $\overline{z_1-z_2} = \bar{z}_1-\bar{z}_2$.

~o~o~o~o

Exers # 12.1

Let $z_1 = 4-5i$ and $z_2 = 2+3i$. Find (in form $x+iy$).

Q2 $z_1 z_2$

Sol:- $z_1 z_2 = (4-5i)(2+3i)$

$= \boxed{23+2i}$

Q3 $(z_1+z_2)^2$

Sol:- $z_1+z_2 = (4-5i)+(2+3i) = 6-2i$

Sq: Both sides.

$(z_1+z_2)^2 = (6-2i)^2 = \boxed{32-24i}$

Q5

$$\frac{Z_2}{Z_1} = \frac{(2+3i)}{(4-5i)} = \frac{2+3i}{4-5i} \times \frac{4+5i}{4+5i} =$$

$$= \frac{8+15i^2+12i+10i}{(4)^2-(5i)^2} = \frac{-7+22i}{16+25} = \frac{-7+22i}{41}$$

$$= \boxed{\frac{-7}{41} + \frac{22}{41}i}$$

Q10 $\operatorname{Re} \frac{1}{1+i}$

$$\frac{1}{1+i} = \frac{1}{1+i} \times \frac{1-i}{1-i} = \frac{1-i}{2} = \frac{1}{2} - \frac{i}{2}$$

$$\boxed{\operatorname{Re}\left(\frac{1}{1+i}\right) = \frac{1}{2}}$$

Q11 $\operatorname{Im} \frac{3+4i}{7-i}$

$$\frac{3+4i}{7-i} = \frac{3+4i}{7-i} \times \frac{7+i}{7+i} = \frac{17}{50} + \frac{31}{50}i$$

$$\boxed{\operatorname{Im}\left(\frac{3+4i}{7-i}\right) = \frac{31}{50}}$$

Q13 $\text{Im}\,\dfrac{z}{\bar{z}}$

Sol. let $z = x+iy$ & $\bar{z} = x-iy$

$$\frac{z}{\bar{z}} = \frac{x+iy}{x-iy} \Rightarrow \frac{x+iy}{x-iy} \times \frac{x+iy}{x+iy} = \frac{(x+iy)^2}{(x)^2-(iy)^2}$$

$$= \frac{x^2-y^2+2ixy}{x^2+y^2}$$

$$\frac{z}{\bar{z}} = \frac{x^2-y^2}{x^2+y^2} + i\,\frac{2xy}{x^2+y^2}$$

$$\boxed{\text{Im}\left(\frac{z}{\bar{z}}\right) = \frac{2xy}{x^2+y^2}}$$

~ o ~ o ~ o ~ o ~

Q16 $\text{Im}\,z^3, (\text{Im}\,z)^3$

Sol. let $z = x+iy$

$$(z)^3 = (x+iy)^3 = x^3 + i^3y^3 + 3x^2(iy) + 3x(iy)^2$$

$$= x^3 - iy^3 + 3ix^2y - 3xy^2$$

$$z^3 = x^3 - 3xy^2 + i(3x^2y - y^3)$$

$$\boxed{\text{Im}(z)^3 = 3x^2y - y^3}$$

Also $z = x + iy$

$Im(z) = y$

$\boxed{(Im z)^3 = y^3}$

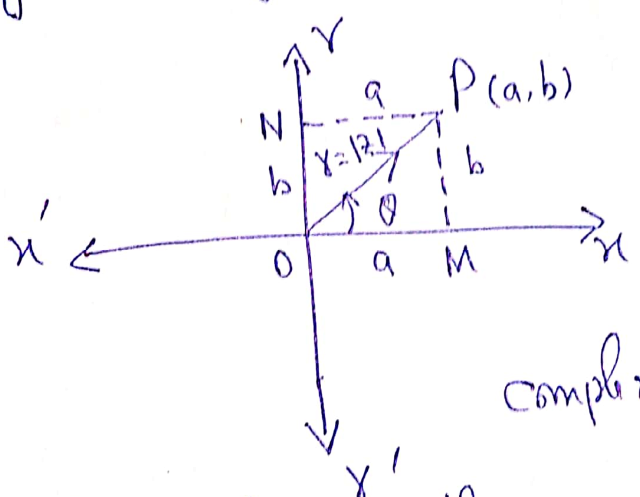
———x———x———x

Ex: 12.1 8^{th} Edition $\boxed{P/656}$

Q3 —— to Q 17

———x———x———x

Argand Diagram:-



complex plane

Applying Pythagorous theorem

$|\overrightarrow{OP}|^2 = |\overrightarrow{OM}|^2 + |\overrightarrow{MP}|^2$

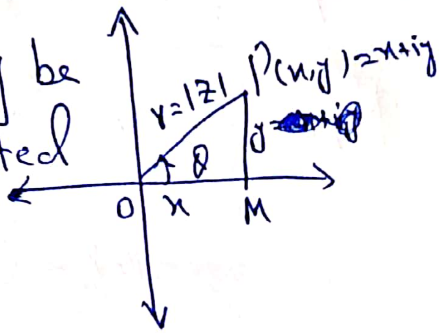
$|\overrightarrow{OP}|^2 = a^2 + b^2$

$|z| = |\overrightarrow{OP}| = \sqrt{a^2+b^2} = r$

$\boxed{|z| = r}$ & $\theta = \tan^{-1}(b/a)$

Polar Form of a complex No :-

Let us suppose that $Z = x+iy$ be a complex number represented by the point $P(x,y)$ in the Argand diagram.



Modulus "or" Absolute value :-

The distance of the point $P(x,y)$ from the origin is called magnitude or absolute value of the complex no:- It is denoted by $|Z|$ or by r Form the fig.

Let $|Z| = \overrightarrow{OP}$ makes an angle "θ" with the +ve direction of x-axis as shown. We draw $\overline{PM}$ $\perp$ to $\overrightarrow{\cancel{OX}}$ $\overrightarrow{OM}$ from the point "P" upon X-axis clearly:

$$\overline{OM} = x$$

$$\& \ \overline{MP} = y$$

$$\text{let } |\overline{OP}| = r$$

By Pythagorous theorm:

$$|OP|^2 = |\overline{OM}|^2 + |\overline{PM}|^2$$

$\Rightarrow r^2 = x^2 + y^2$

$r = |z| = \sqrt{x^2+y^2} \longrightarrow (1)$

Also, by the def: of trigonometric ratios.

$\tan\theta = y/x$

$\boxed{\theta = \tan^{-1} y/x} \longrightarrow (2)$

θ is the angle and is called argument (amplitude) of z & is denoted by $\arg(z)$.

The Pair $(r, \theta) = (\sqrt{x^2+y^2}, \tan^{-1}(y/x))$ is called Polar form of the complex no:

let $z = x+iy \longrightarrow (A)$ be a complex no: Then from the fig.

$\cos\theta = x/r \Rightarrow x = r\cos\theta$.

& $\sin\theta = y/r \Rightarrow y = r\sin\theta$

eq (A) $\Rightarrow z = r\cos\theta + ir\sin\theta$

$z = r(\cos\theta + i\sin\theta) \longrightarrow (B)$

eq (B) represents the Polar form of a complex no:

$$Z = |Z|(\cos\theta + i\sin\theta)$$

Where $\arg(z) = \theta = \tan^{-1}(y/x)$.

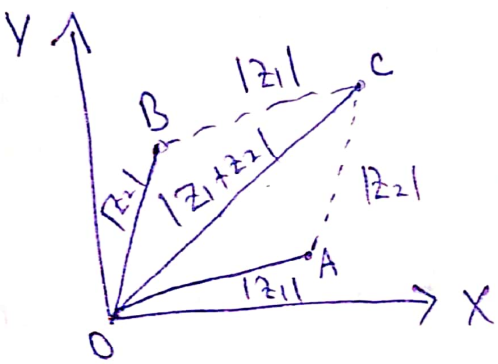
— X — X —

Triangular Inequality :-

For any two complex number we have

$$|z_1 + z_2| \leq |z_1| + |z_2|$$

This is known as triangular inequality.



Complex plane.

The generalized form of triangular inequality is

$$|z_1 + z_2 + z_3 + \cdots z_n| \leq |z_1| + |z_2| + \cdots |z_n|$$

— X —

De Moivre's Theorem

Statement:- let n be an integer then

$$(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta.$$

Proof Case No 1:- Suppose n is a +ve integer

We use Mathematical induction

let $n=1$

$$(\cos\theta + i\sin\theta)^1 = \cos 1\theta + i\sin 1\theta$$

$\cos\theta + i\sin\theta = \cos\theta + i\sin\theta$ which is true

$\therefore$ the theorem is true for $n=1$

Suppose it is true for $n=k$ i.e

$$(\cos\theta + i\sin\theta)^k = \cos k\theta + i\sin k\theta \text{ ——① (hypothesis)}.$$

To Prove for $n=k+1$ i.e

$$(\cos\theta + i\sin\theta)^{k+1} = \cos(k+1)\theta + i\sin(k+1)\theta \rightarrow \text{Target}.$$

Xing eq ① by $\cos\theta + i\sin\theta$

$$(\cos\theta + i\sin\theta)^k(\cos\theta + i\sin\theta) = (\cos k\theta + i\sin k\theta)(\cos\theta + i\sin\theta)$$

$$= \cos k\theta\cos\theta - \sin k\theta\sin\theta + i(\sin k\theta\cos\theta + \cos k\theta\sin\theta)$$

$$= \cos(k\theta + \theta) + i\sin(k\theta + \theta)$$

Scanned with CamScanner

$(\cos\theta + i\sin\theta)^{k+1} = \cos(k+1)\theta + i\sin(k+1)\theta$

$\therefore$ true for $n = k+1$

Hence by Mathematical Induction it is true for all +ve integer

$\boxed{\text{Case No 2}}$ Suppose n is a -ve integer then we can write $n = -m$ where m is +ve integer

$$(\cos\theta + i\sin\theta)^n = (\cos\theta + i\sin\theta)^{-m}$$

$$= \frac{1}{(\cos\theta + i\sin\theta)^m} \quad (\because m \text{ is +ve integer})$$

$$= \frac{1}{\cos m\theta + i\sin m\theta}$$

$$= \frac{1}{\cos m\theta + i\sin m\theta} \times \frac{\cos m\theta - i\sin m\theta}{\cos m\theta - i\sin m\theta}$$

$$= \frac{\cos m\theta - i\sin m\theta}{\cos^2 m\theta + \sin^2 m\theta}$$

$$= \cos m\theta - i\sin m\theta$$

$$= \cos(-m)\theta + i\sin(-m)\theta$$

$$(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$$

$\therefore$ it is true for -ve integer

Case No 3 :- $n=0$

$$(\cos\theta + i\sin\theta)^0 = \cos 0\theta + i\sin 0\theta$$
$$= \cos 0 + i\sin 0$$
$$= 1 + i0$$
$$= 1$$

$\therefore$ true for $n=0$.

From above three cases the theorem is true for all integers

Rational numbers

$$Q = \{ p/q \mid p, q \in Z, q \neq 0 \}$$

Prove De Moivre's Theorem for rational numbers.

Statement:- If n is a rational number then

$$(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$$

Proof:- let $n = p/q$, $p, q \in Z$ & $q \neq 0$.

consider

$$(\cos\theta/q + i\sin\theta/q)^q = \cos \frac{\cancel{q}\theta}{\cancel{q}} + i\sin \frac{\cancel{q}}{\cancel{q}}\theta \quad \text{(D.Th: for integer}$$

Raising Both the side to Power $1/q$

$$\left[\left(\cos\theta/q + i\sin\theta/q\right)^{q}\right]^{1/q} = (\cos\theta + i\sin\theta)^{1/q}$$

or $\cos\theta/q + i\sin\theta/q = (\cos\theta + i\sin\theta)^{1/q}$

or $(\cos\theta + i\sin\theta)^{1/q} = \cos\theta/q + i\sin\theta/q$

raising both the sides to the Power P

$$\left((\cos\theta + i\sin\theta)^{1/q}\right)^{p} = \left(\cos\theta/q + i\sin\theta/q\right)^{p}$$

$(\cos\theta + i\sin\theta)^{p/q} = \cos p/q\,\theta + i\sin p/q\,\theta$ (D.Th: for integer)

$$\boxed{(\cos\theta + i\sin\theta)^{n} = \cos n\theta + i\sin n\theta}$$

∴ true for rational no: