

Lecture # 03

Linear Algebra

Outline

- Vectors
 - Operations
- Matrix
 - Operations
- Transformations
 - Scaling
 - Rotation
 - Translation
- Singular value decomposition

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Vector

- Scalar: $x \in \mathbb{R}$
- Vector: $\mathbf{x} \in \mathbb{R}^N$
 - Row Vector $\mathbf{v} \in \mathbb{R}^{1 \times n}$

$$\mathbf{x} = [\mathbf{x}_1 \quad \mathbf{x}_2 \quad \dots \quad \mathbf{x}_n]$$

- Column vector $\mathbf{v} \in \mathbb{R}^{n \times 1}$: $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = [x_1 \quad x_2 \quad \dots \quad x_n]^T$

- Transpose

Vectors – use

- Store data in memory

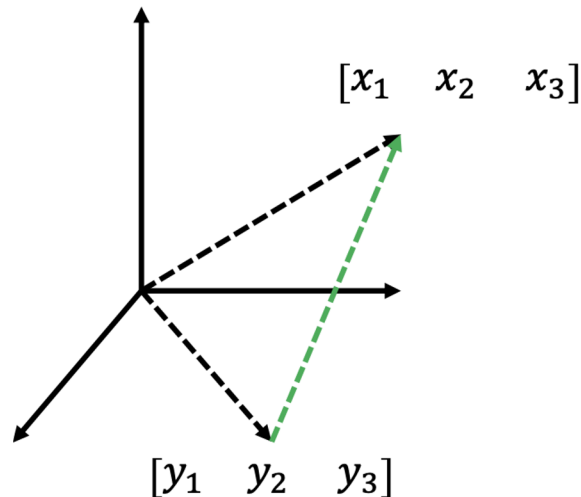
- Feature vectors
- Pixel values
- Any other data for processing

- Any point in coordinate system

- Can be n dimensional

- Difference between two points

$$[x_1 - y_1 \ x_2 - y_2 \ x_3 - y_3]$$



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Vector operations

- Norm – size of the vector

- P-norm

$$\|x\|_p = \left(\sum_i |a_i|^p \right)^{\frac{1}{p}} \quad p \geq 1$$

- Euclidean norm

$$\|x\|_2 = \left(\sum_i |a_i|^2 \right)^{1/2}$$

- L1-norm

$$\|x\|_1 = \left(\sum_i |a_i| \right)$$

Cont'd

- Inner product (dot product)
 - Scalar number
 - Multiply corresponding entries and add

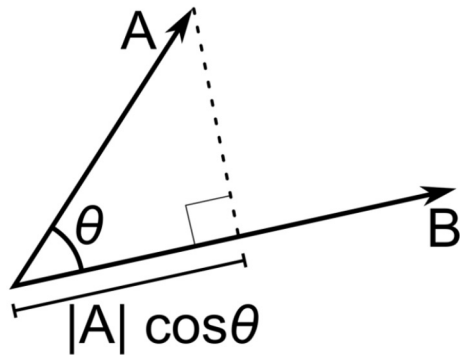
$$\mathbf{x}^T \mathbf{y} = [x_1 \quad x_2 \quad \cdots \quad x_n] \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix} = \sum_k^n x_k y_k$$

Cont'd

- Inner product (dot product)

$$\mathbf{x}_i^T \mathbf{x}_i = \sum_k^n (x_k^i)^2 = \text{squared norm of } \mathbf{x}_i$$

- $\mathbf{A} \cdot \mathbf{B}$ is also $|\mathbf{A}||\mathbf{B}|\cos$ (angle between \mathbf{A} and \mathbf{B})
- If \mathbf{B} is a unit vector, $\mathbf{A} \cdot \mathbf{B}$ gives projection of \mathbf{A} on \mathbf{B}



Cont'd

- Outer product

$$\mathbf{x}_i \mathbf{x}_j^T = \begin{bmatrix} x_1^i x_1^j & x_1^i x_2^j & \cdots & x_1^i x_n^j \\ x_2^i x_1^j & x_2^i x_2^j & \cdots & x_2^i x_n^j \\ \vdots & \vdots & \ddots & \vdots \\ x_n^i x_1^j & x_n^i x_2^j & \cdots & x_n^i x_n^j \end{bmatrix} \Rightarrow$$

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Matrix

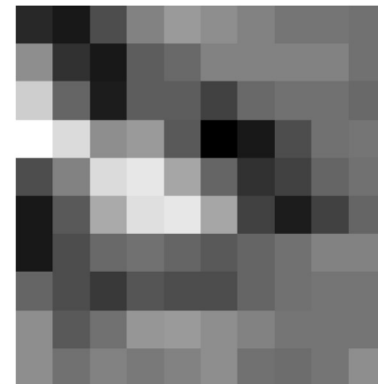
- Array $\mathbf{A} \in \mathbb{R}^{m \times n}$ of numbers with shape m by n
 - m rows and n columns

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

- A row vector is a matrix with single row
- A column vector is a matrix with single column

Matrix — Use

- Image representation - grayscale
 - One number per pixel
 - Stored as $n \times m$ matrix



0	3	2	5	4	7	6	9	8
3	0	1	2	3	4	5	6	7
2	1	0	3	2	5	4	7	6
5	2	3	0	1	2	3	4	5
4	3	2	1	0	3	2	5	4
7	4	5	2	3	0	1	2	3
6	5	4	3	2	1	0	3	2
9	6	7	4	5	2	3	0	1
8	7	6	5	4	3	2	1	0

Cont'd

- Image representation – RGB
 - 3 numbers per pixel
 - Stored as $n \times m \times 3$ matrix

0	3	2	5	4	7	6	9	8	0	3	2	5	4	7	6	9	8	0	3	2	5	4	7	6	9	8
3	0	1	2	3	4	5	6	7	3	0	1	2	3	4	5	6	7	3	0	1	2	3	4	5	6	7
2	1	0	3	2	5	4	7	6	2	1	0	3	2	5	4	7	6	2	1	0	3	2	5	4	7	6
5	2	3	0	1	2	3	4	5	5	2	3	0	1	2	3	4	5	5	2	3	0	1	2	3	4	5
4	3	2	1	0	3	2	5	4	4	3	2	1	0	3	2	5	4	4	3	2	1	0	3	2	5	4
7	4	5	2	3	0	1	2	3	7	4	5	2	3	0	1	2	3	7	4	5	2	3	0	1	2	3
6	5	4	3	2	1	0	3	2	6	5	4	3	2	1	0	3	2	6	5	4	3	2	1	0	3	2
9	6	7	4	5	2	3	0	1	9	6	7	4	5	2	3	0	1	9	6	7	4	5	2	3	0	1
8	7	6	5	4	3	2	1	0	8	7	6	5	4	3	2	1	0	8	7	6	5	4	3	2	1	0



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Matrix Operations

- Addition

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} + \begin{bmatrix} e & f \\ g & h \end{bmatrix} = \begin{bmatrix} a + e & b + f \\ c + g & d + h \end{bmatrix}$$

- Both matrices should have same shape, except with a scalar

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} + 2 = \begin{bmatrix} a + 2 & b + 2 \\ c + 2 & d + 2 \end{bmatrix}$$

- Same with subtraction

Cont'd

- Scaling

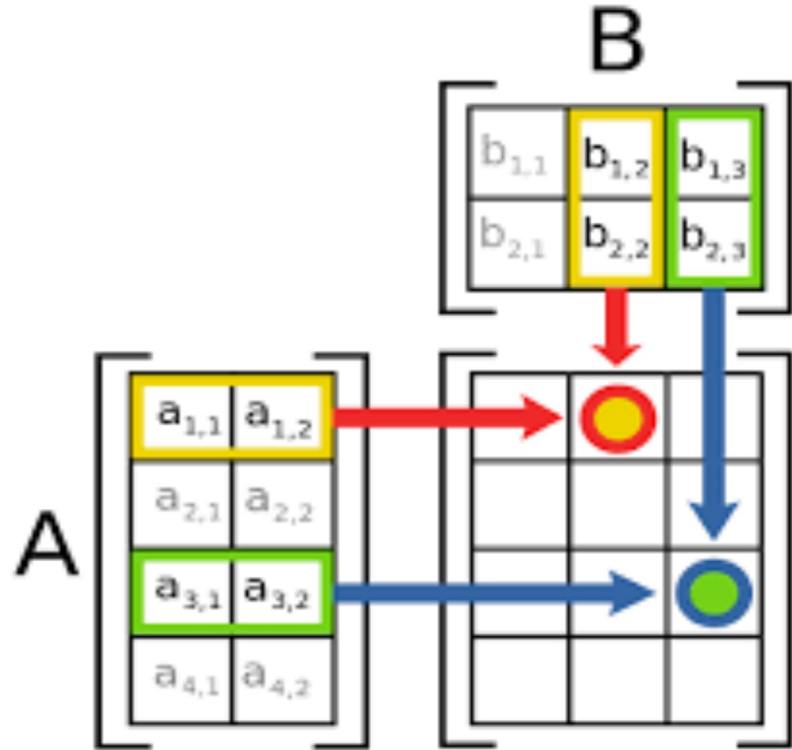
$$s \times \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} s \times a & s \times b \\ s \times c & s \times d \end{bmatrix}$$

- Hadamard product

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \odot \begin{bmatrix} e & f \\ g & h \end{bmatrix} = \begin{bmatrix} a \times e & b \times f \\ c \times g & d \times h \end{bmatrix}$$

Matrix Operations

- Matrix Multiplication
 - Complexity?
 - $m \times n$ and $n \times p$
 - Results in $m \times p$ matrix



Cont'd

- Matrix multiplication
- Let \mathbf{a}_i denote the i -th column of the matrix \mathbf{A} , and
- \mathbf{b}_j denote the j -th column of the matrix \mathbf{B}

$$\mathbf{A} = [\mathbf{a}_1 \quad \mathbf{a}_2 \quad \cdots \quad \mathbf{a}_n], \text{ and } \mathbf{B} = [\mathbf{b}_1 \quad \mathbf{b}_2 \quad \cdots \quad \mathbf{b}_m]$$

- The product of the two matrices is defined as

$$\mathbf{A}^T \mathbf{B} = \begin{bmatrix} \mathbf{a}_1^T \mathbf{b}_1 & \mathbf{a}_1^T \mathbf{b}_2 & \cdots & \mathbf{a}_1^T \mathbf{b}_m \\ \mathbf{a}_2^T \mathbf{b}_1 & \mathbf{a}_2^T \mathbf{b}_2 & \cdots & \mathbf{a}_2^T \mathbf{b}_m \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{a}_n^T \mathbf{b}_1 & \mathbf{a}_n^T \mathbf{b}_2 & \cdots & \mathbf{a}_n^T \mathbf{b}_m \end{bmatrix}$$

Matrix Operations

- Matrix Multiplication - another interpretation (very intuitively)
- The first of column of AB
 - Linear combination of all the columns in A

$$[a_1b_{11} + a_2b_{12} + \dots + a_nb_{1n}]$$

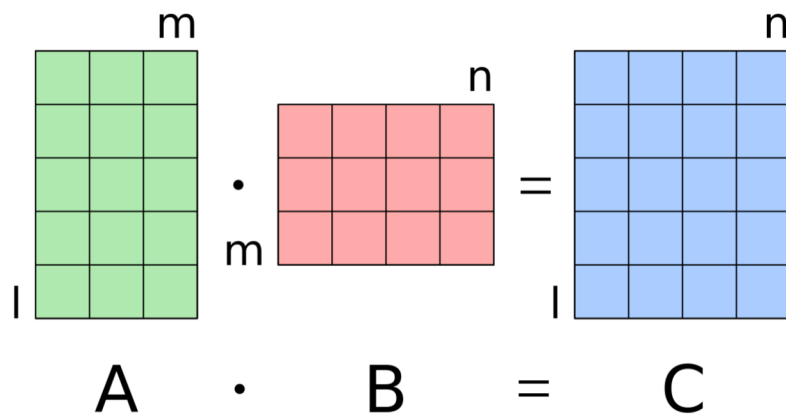
- Similarly we can get other columns...

$$A \cdot B = C$$

$$[a_1b_{11} + a_2b_{12} + \dots + a_nb_{1n}]$$

Cont'd

- How about linear combination of all rows of **B**?
 - • Each row of $\mathbf{C} = \mathbf{AB}$ is a linear combination of rows of **B**



$$[\mathbf{b}_1 a_{11} + \mathbf{b}_2 a_{21} + \dots + \mathbf{b}_n a_{n1}]$$

Matrix Operations

- Transpose

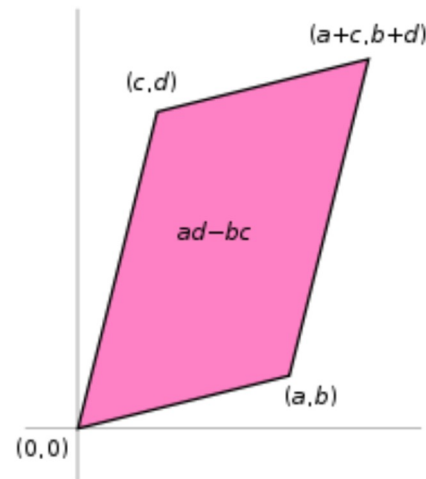
$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

$$\mathbf{A}^T = \begin{bmatrix} a_{11} & a_{21} & \cdots & a_{m1} \\ a_{12} & a_{22} & \cdots & a_{m2} \\ \vdots & \vdots & \ddots & \vdots \\ a_{1n} & a_{2n} & \cdots & a_{mn} \end{bmatrix}$$

Cont'd

- Determinant
 - A scalar

- For $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, $\det(A) = ad - bc$



- Represents area of the parallelogram described by the vectors in the rows of the matrix

Cont'd

- Determinant

$$|\mathbf{A}| = \sum_{i=1}^K a_{ij} C_{ij}$$

where C_{ij} is the cofactor of a_{ij} defined by

$$C_{ij} = (-1)^{i+j} |M_{ij}|, \text{ and}$$

M_{ij} is the minor of matrix \mathbf{A} formed by eliminating row i and column j of \mathbf{A}

- Some Properties

- $|\mathbf{AB}| = |\mathbf{A}||\mathbf{B}|$
- $|\mathbf{AB}| = |\mathbf{BA}|$
- $|\mathbf{A}^T| = |\mathbf{A}|$

$$|A| = \begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = a \begin{vmatrix} e & f \\ h & i \end{vmatrix} - b \begin{vmatrix} d & f \\ g & i \end{vmatrix} + c \begin{vmatrix} d & e \\ g & h \end{vmatrix}$$

Matrix Operations

- Trace
 - $\text{Tr}(A)$ = Sum of diagonal elements

$$\text{Tr} \begin{bmatrix} a_{00} & a_{01} & a_{02} \\ a_{10} & a_{11} & a_{12} \\ a_{20} & a_{21} & a_{22} \end{bmatrix} = a_{00} + a_{11} + a_{22}$$

- Properties
 - $\text{tr}(AB) = \text{tr}(BA)$
 - $\text{tr}(A+B) = \text{tr}(A) + \text{tr}(B)$

Cont'd

- Inverse

- Given a matrix \mathbf{A} , its inverse \mathbf{A}^{-1} is a matrix such that

$$\mathbf{A}\mathbf{A}^{-1} = \mathbf{A}^{-1}\mathbf{A} = \mathbf{I}$$

- Inverse does not always exist

- Singular vs non-singular

- Properties

- $(\mathbf{A}^{-1})^{-1} = \mathbf{A}$
- $(\mathbf{AB})^{-1} = \mathbf{B}^{-1}\mathbf{A}^{-1}$

Special matrices

- Symmetric matrix

$$A^T = A$$

$$\begin{bmatrix} 1 & 1 & -1 \\ 1 & 2 & 0 \\ -1 & 0 & 5 \end{bmatrix}$$

- Skew- symmetric matrix

$$A^T = -A$$

$$\begin{bmatrix} 0 & 1 & -2 \\ -1 & 0 & 3 \\ 2 & -3 & 0 \end{bmatrix}$$

Cont'd

- Diagonal matrix
 - Used for row scaling
- Identity matrix
 - Special diagonal matrix
 - 1 along diagonals

$$I.A = A$$

$$A = \begin{bmatrix} A_1 & 0 & \cdots & 0 \\ 0 & A_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & A_n \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

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- Vectors
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Transformation - scaling

- Matrices are useful for vector transformations
- Matrix multiplication

$$X' = AX$$

$$\begin{bmatrix} s_x & 0 \\ 0 & s_y \end{bmatrix} \times \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} s_x x \\ s_y y \end{bmatrix}$$

- Linear combination of columns

Transformation

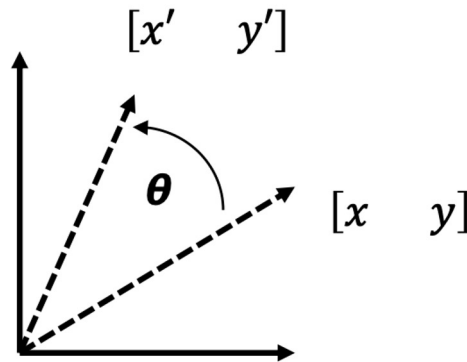
- Rotation
 - Matrix multiplication to rotate a vector

- Rotation matrix

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

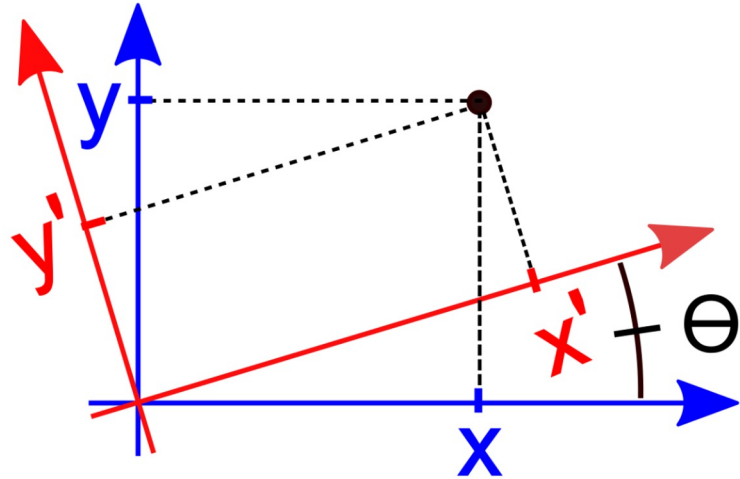
$$x' = \cos\theta x - \sin\theta y$$

$$y' = \sin\theta x + \cos\theta y$$



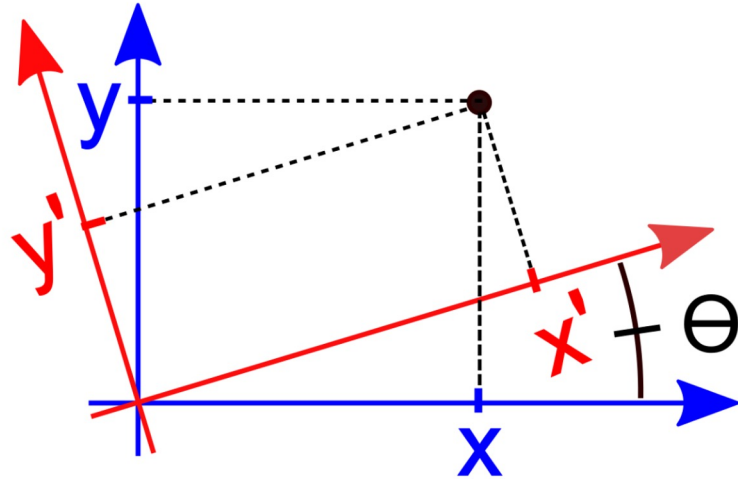
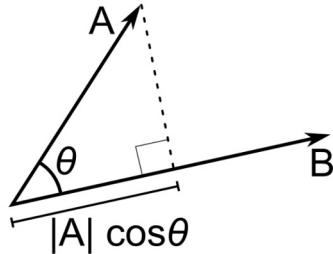
Transformation - rotation

- Rotating axis first
- Vector $v = [x \ y]$
 - x - projection of v on x axis
 - y - projection of v on y axis



Transformation - rotation

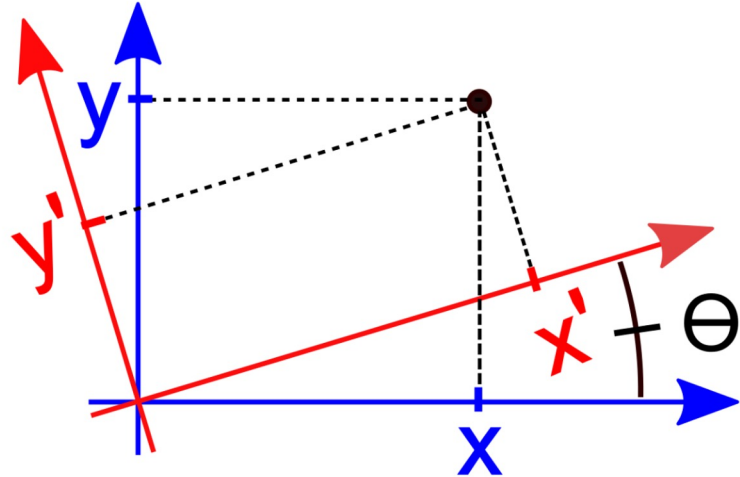
- Rotating axis first
- Vector $v = [x \ y]$
 - x - projection of v on x axis
 - y - projection of v on y axis
- Remember vector dot product?



Transformation - rotation

- Now we need new x and y axis
 - $x' = v \cdot \text{new x-axis}$
 - $y' = v \cdot \text{new y-axis}$
- For rotation of θ
 - New x-axis = $[\cos\theta, \sin\theta]$
 - New y-axis = $[-\sin\theta, \cos\theta]$
- We can form a matrix using new axis

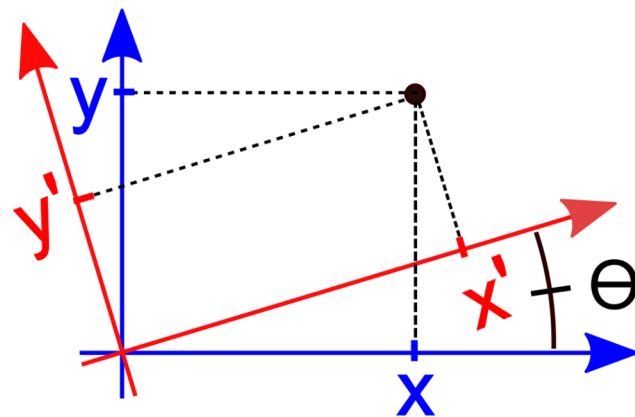
$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$



Cont'd

- When we rotate the axis to left
- We are rotating the vector to right
- We can use rotation matrix to rotate the vector
- We need new x y axis coordinates
 - When we rotate the axis right
- Updated rotation matrix

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$



Transformation

- Linear combination

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \times \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} ax + by \\ cx + dy \end{bmatrix}$$

- Sufficient for
 - Scaling
 - Rotating and
 - Skew transformations
- But no shifting

Cont'd

- Linear combination

$$\begin{bmatrix} a & b & 0 \\ c & d & 0 \\ 0 & 0 & 1 \end{bmatrix} \times \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \begin{bmatrix} ax + by \\ cx + dy \\ 1 \end{bmatrix}$$

- Still be able to:
 - Scaling
 - Rotating and
 - Skew transformations

Transformation

- Homogeneous System

$$\begin{bmatrix} a & b & c \\ d & e & f \\ 0 & 0 & 1 \end{bmatrix} \times \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \begin{bmatrix} ax + by + c \\ dx + ey + f \\ 1 \end{bmatrix}$$

- We can also perform shifting now
- This is called homogeneous coordinates

Scaling + rotation + translation

- Careful about the order
 - $V' = (TRS)V$

$$\begin{bmatrix} 1 & 0 & t_y \\ 0 & 1 & t_x \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} s_x & 0 & 0 \\ 0 & s_y & 0 \\ 0 & 0 & 1 \end{bmatrix} \times \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

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Linear independence

- Linear Independence

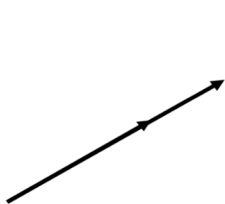
- $\{\mathbf{x}_1, \mathbf{x}_2 \dots, \mathbf{x}_M\}$ is a set of linearly independent vectors provided

$$a_1\mathbf{x}_1 + a_2\mathbf{x}_2 + \dots + a_M\mathbf{x}_M = \mathbf{0} \Rightarrow a_1 = 0 = a_2 = \dots = a_M$$

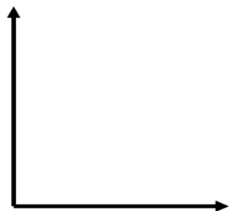
- In other words, none of the vectors can be expressed as a linear combination of the other vectors
 - Each vector is perpendicular to every other vector
 - For example axis in cartesian coordinate system

Intuition

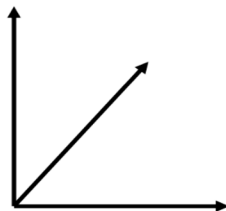
- In terms of features
 - Person recognition - [height, hair color, weight, specs, eye color, etc.]



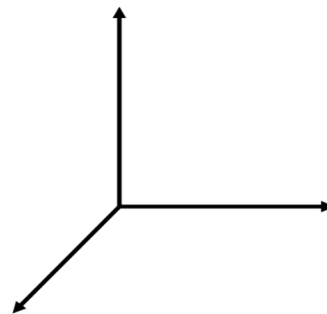
Linearly dependent



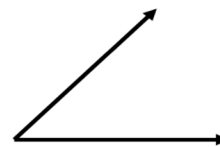
Linearly independent



Linearly dependent



Linearly independent



Linearly independent

Matrix factorization

- Singular value decomposition (SVD)

$$A = U\Sigma V^T$$

- If A is $m \times n$ matrix, then
 - U will be $m \times m$,
 - Σ will be $m \times n$, and
 - V^T will be $n \times n$
- U and V are unitary matrices
 - Each Column is a unit Vector
- Σ is a diagonal matrix

Singular value decomposition

- Interpretation

$$\mathbf{A} = \mathbf{U}\mathbf{\Sigma}\mathbf{V}^T$$

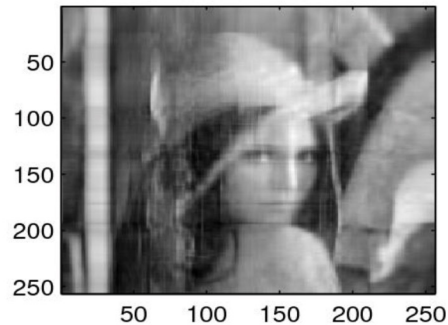
- Columns of \mathbf{U} are scaled by values in $\mathbf{\Sigma}$
- The resultant columns are linearly combined by \mathbf{V}
- \mathbf{A} is formed as a linear combination of columns of \mathbf{U}
- If we use all column, we will get original \mathbf{A}
- We can just use few columns of \mathbf{U} and we get an approximation
 - We call these columns principal components

Application

Original image 256 singular values



retaining 20 singular values



retaining 50 singular values



retaining 85 singular values

