University of Engineering and Technology, Peshawar

Engineering Economics

CSE-305

(Chapter 03d)

Principles of Money-Time Relationship

Dr. Durr-e-Nayab
Email: nayab.khan@uetpeshawar.edu.pk

Agenda

- Actual Interest VS Nominal Interest
- Cases of Actual VS Nominal Interest
- Different Compounding Periods
- Effective Interest Per Payment Period
- Auto Loan Payments
- Continuous Interest
- Interest rate that Varies over Time

Varying Interest Payments

If ***payments*** occur more frequently than annually, how do you calculate economic equivalence?

If the ***interest period*** is other than annual, how do you calculate economic equivalence?

Nominal Versus Effective Interest Rates

Nominal Interest Rate:

Interest rate quoted based on an annual period

APR = Interest rate per period * Number of interest periods

Effective Interest Rate:

Actual interest earned or paid in a year or some other time period

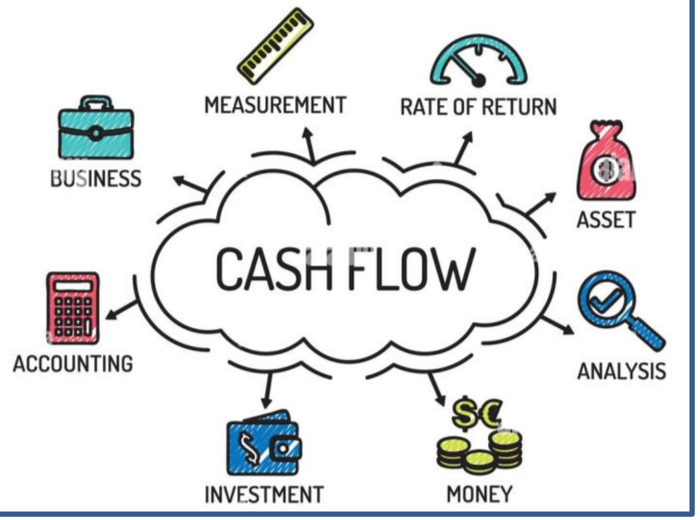
EIR = $i_a = (1 + r/M)^M - 1$

Equivalence Analysis using Effective Interest Rates

Step 1: Identify the **payment period**
(e.g., annual, quarter, month, week, etc)

Step 2: Identify the **interest period**
(e.g., annually, quarterly, monthly, etc)

Step 3: Find the **effective interest rate** that covers the **payment period.**



Case I: When Payment and Compounding Periods Coincide

Step 1: Identify the number of **compounding periods** (M) per year

Step 2: Compute the **effective interest rate per payment period** (i)

$$i = r/M$$

Step 3: Determine the total **number of payment periods** (N)

$$N = M \text{ (number of years)}$$

Step 4: Use the appropriate **interest formula using i** and ***N*** above

Nominal Versus Effective Interest Rates

- Interest period/ time between successive compounding is less than a year
- Interest rates on annual basis followed by compounding period different from a year length
- Example: If the interest rate is 6% per interest period and the interest period is 6 months, **it is customary to speak of this as 12% compounded semiannually**.
- The basic interest rate is known as **Nominal interest rate, 12%.. Represented by r.**
- The actual annual rate on principal amount is not 12% but something greater because compounding occurs twice a year.

Nominal Versus Effective Interest Rates

- **Example:** \$1000 to be invested for 3 years at a nominal interest rate of 12% compounded semiannually. The interest earned during the first year:
- **Solution:**

  *12% /2= 6%   N= 3*2   F= 1000 (1+0.06)^6=\$ 418.52*

- Now change the same example but evaluate interest compounded monthly.

  *12% /12= 1%   N= 3*12   F= 1000 (1+0.01)^12=\$ 430.77*

- **This means increasing the number of compounding periods increases the interest. The effective interest (i) in both cases is different, and the nominal interest (r) is the same.**

Nominal Versus Effective Interest Rates

18% Compounded Monthly

- **What It Really Means?**
  - Interest rate per month (i) = 18%/12 = 1.5%
  - Number of interest periods per year (N) = 12
- **In words,**
  - Bank would charge 1.5% interest each month on your unpaid balance if you borrowed money
  - You would earn 1.5% interest each month on your remaining balance if you deposited money

Nominal Versus Effective Interest Rates

18% Compounded Monthly

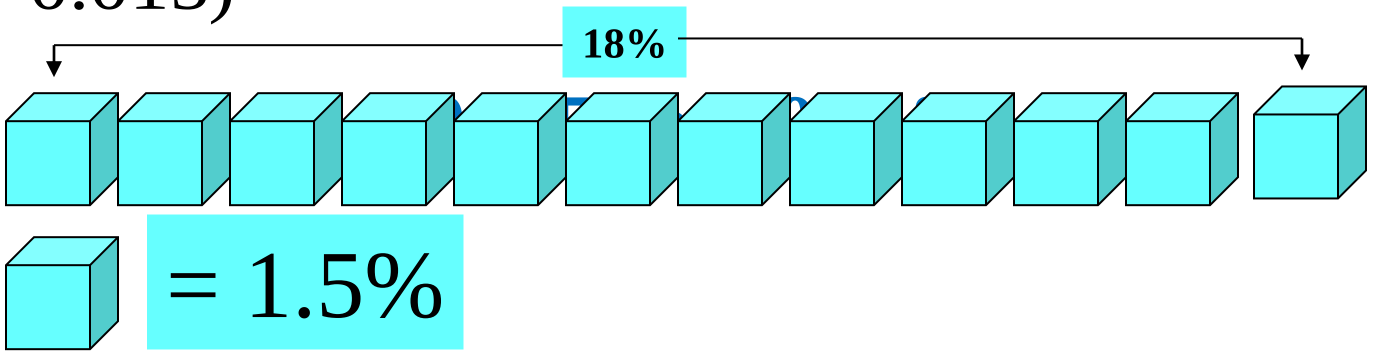
Question: Suppose that you invest $1 for 1 year at 18% compounded monthly. How much interest would you earn?

Solution:

$$F = \$1(1+i)^{12} = \$1(1+0.015)^{12}$$

$$= \mathbf{\$1.1956}$$

$$i_a = \mathbf{0.1956 \text{ or } 19.56\%}$$

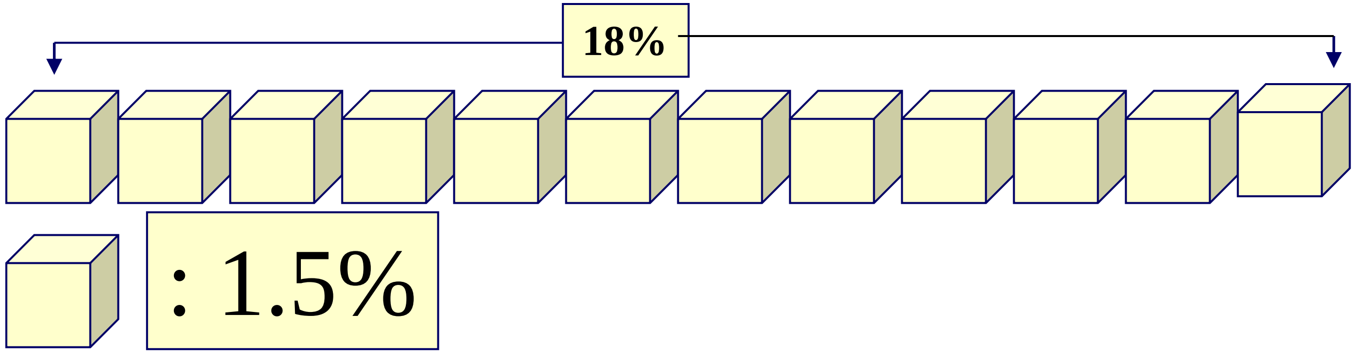


Effective Annual Interest Rate: Yield

$$i_a = (1 + r/M)^M - 1$$

r = nominal interest rate per year
i_a = effective annual interest rate
M = number of interest periods per year

Effective Annual Interest Rate: Yield

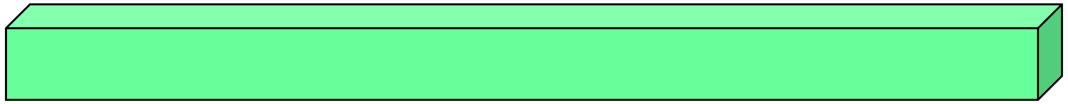


18% compounded **monthly**

or

1.5% per month for **12 months**

=



19.56 % compounded **annually**

Practice Problem: Effective Interest

Suppose your savings account pays 9% interest compounded quarterly. If you deposit $10,000 for one year, how much would you have?

(a) Interest rate per quarter:

$$i = \frac{9\%}{4} = 2.25\%$$

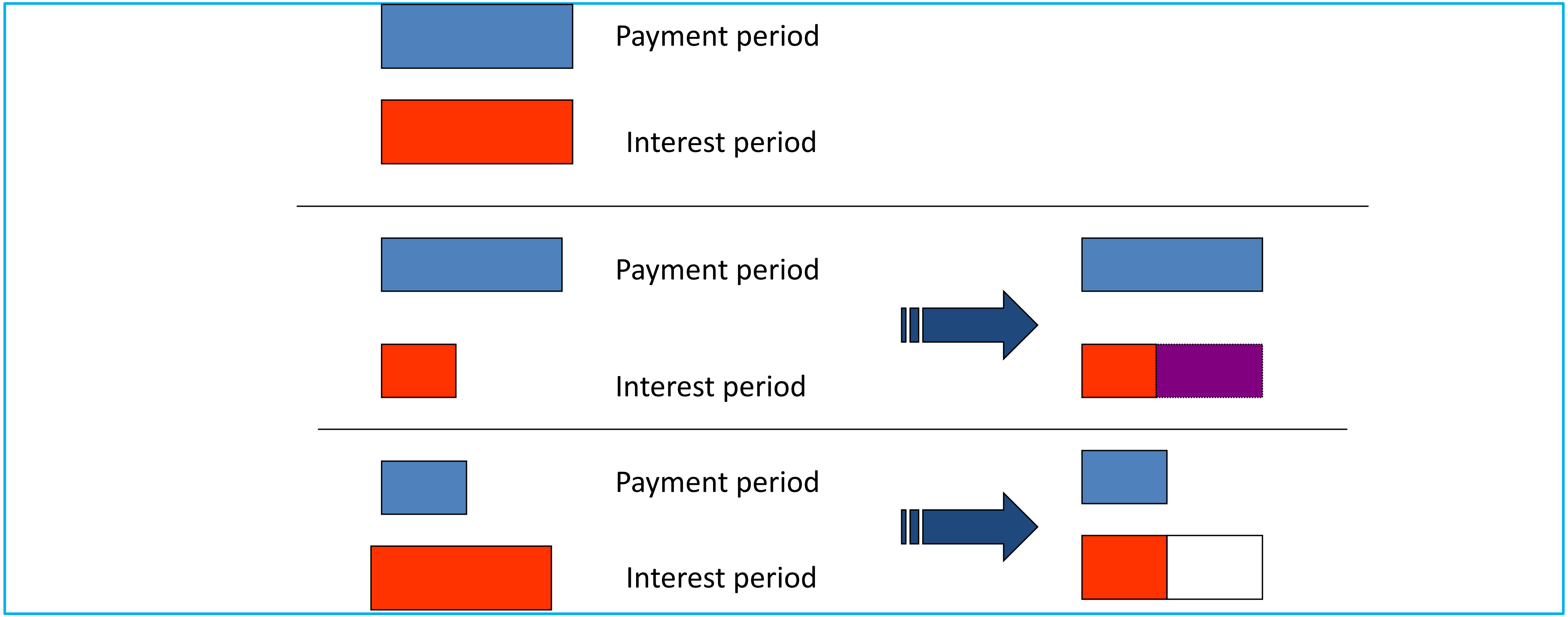
(b) Annual effective interest rate:

$$i_a = (1 + 0.0225)^4 - 1 = 9.31\%$$

(c) Balance at the end of one year (after 4 quarters)

$$\begin{aligned} F &= \$10,000(F/P, 2.25\%, 4) \\ &= \$10,000(F/P, 9.31\%, 1) \\ &= \$10,931 \end{aligned}$$

Why Do We Need an Effective Interest Rate per Payment Period?



Nominal and Effective Interest Rates with Different Compounding Periods

	Effective Rates				
Nominal Rate	Compounding Annually	Compounding Semi-annually	Compounding Quarterly	Compounding Monthly	Compounding Daily
4%	4.00%	4.04%	4.06%	4.07%	4.08%
5	5.00	5.06	5.09	5.12	5.13
6	6.00	6.09	6.14	6.17	6.18
7	7.00	7.12	7.19	7.23	7.25
8	8.00	8.16	8.24	8.30	8.33
9	9.00	9.20	9.31	9.38	9.42
10	10.00	10.25	10.38	10.47	10.52
11	11.00	11.30	11.46	11.57	11.62
12	12.00	12.36	12.55	12.68	12.74

Effective Annual Interest Rates (9% compounded quarterly)

First quarter	Base amount + Interest (2.25%)	$10,000 + $225
Second quarter	= New base amount + Interest (2.25%)	= $10,225 +$230.06
Third quarter	= New base amount + Interest (2.25%)	= $10,455.06 +$235.24
Fourth quarter	= New base amount + Interest (2.25 %) = Value after one year	= $10,690.30 + $240.53 = $10,930.83

Example: Calculating Auto Loan Payments

Given:

Invoice Price = \$21,599

Sales tax at 4% = \$21,599 (0.04) = \$863.96

Dealer's freight = \$21,599 (0.01) = \$215.99

Total purchase price = \$22,678.95

Down payment = \$2,678.95

Dealer's interest rate = 8.5% APR

Length of financing = 48 months

Find: the monthly payment

Payment Period = Interest Period

Given: P = \$20,000, r = 8.5% per year

K = 12 payments per year

N = 48 payment periods

Find A

- Step 1: $M = 12$
- Step 2: $i = r/M$ = 8.5%/12 = 0.7083% per month
- Step 3: $N = (12)(4) = 48$ months
- Step 4: A = \$20,000(A/P, 0.7083%,48) = \$492.97

Example: Dollars Up in Smoke

What three levels of smokers who bought cigarettes every day for 50 years at $1.75 a pack would have if they had instead banked that money each week:

Level of smoker	Would have had
1 pack a day	$169,325
2 packs a day	$339,650
3 packs a day	$507,976

Note: Assume constant price per pack, the money banked weekly and an annual interest rate of 5.5%

Calculation: One Pack per Day

Step 1: Determine the effective interest rate per payment period.

Payment period = weekly

"5.5% interest compounded weekly"

$i = 5.5\%/52 = 0.10577\%$ per week

Step 2: Compute the equivalence value.

Weekly deposit amount

$A = \$1.75 \text{ x } 7 = \$12.25$ per week

Total number of deposit periods

$N = (52 \text{ weeks/yr.})(50 \text{ years})$

$= 2600$ weeks

$F = \$12.25\ (F/A, 0.10577\%, 2600)$

$= \$169{,}325$

Finding Equivalence: Practice Problem

You have a habit of drinking a cup of Starbuck coffee ($2.00 a cup) on the way to work every morning for 30 years. If you put the money in the bank for the same period, how much would you have, assuming your accounts earns 5% interest compounded daily.

NOTE: Assume you drink a cup of coffee every day including weekends

$$i = \frac{5\%}{365} = 0.0137\% \text{ per day}$$

$$N = 30 \times 365 = 10,950 \text{ days}$$

$$F = \$2(F/A, 0.0137\%, 10950)$$

$$= \$50,831$$

Case II: When Payment Periods Differ from Compounding Periods

Step 1: Identify the following parameters

M = No. of compounding periods

K = No. of payment periods

C = No. of interest periods per payment period

Step 2: Compute the effective interest rate per payment period

For discrete compounding

$$i = [1 + r / CK]^{C} - 1$$

For continuous compounding

$$i = e^{r/K} - 1$$

Step 3: Find the total no. of payment periods

$$N = K \text{ (no. of years)}$$

Step 4: Use i and N in the appropriate equivalence formula

Effective Interest Rate per Payment Period with Continuous Compounding

$$\boxed{i = [1 + r / CK]^C - 1}$$

where CK = number of compounding periods per year

K = 4 payments per year

C = 1 interest period per quarter

continuous compounding =>

$$i = \lim[(1 + r / CK)^C - 1]$$
$$= (e^r)^{1/K} - 1$$

Case 0: 8% compounded quarterly

Payment Period = Quarter

Interest Period = Quarterly

1st Q	2nd Q	3rd Q	4th Q

1 interest period

Given $r = 8\%$,

$K = 4$ payments per year

$C = 1$ interest period per quarter

$M = 4$ interest periods per year

$$i = [1 + r / CK]^{C} - 1$$

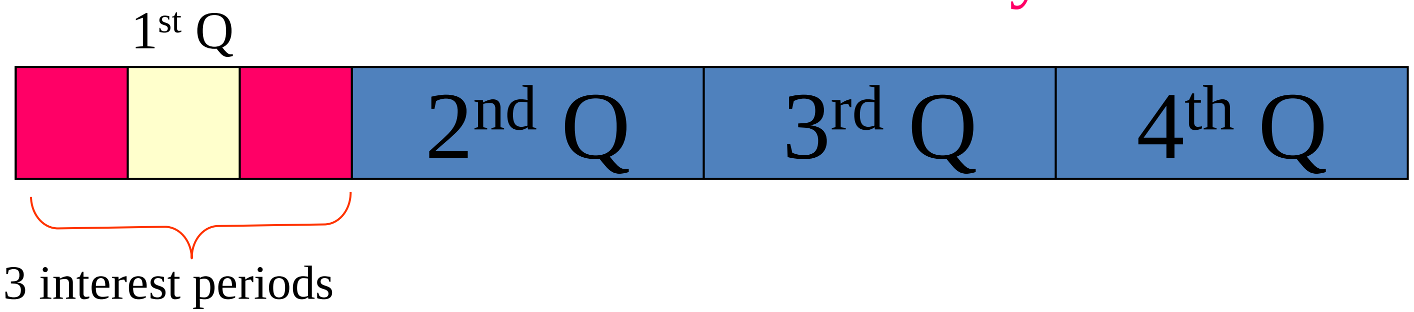
$$= [1 + 0.08 / (1)(4)]^{1} - 1$$

$$= 2.000\% \text{ per quarter}$$

Case 1: 8% compounded Monthly

Payment Period = Quarter

Interest Period = Monthly



Given r = 8%,

K = 4 payments per year

C = 3 interest periods per quarter

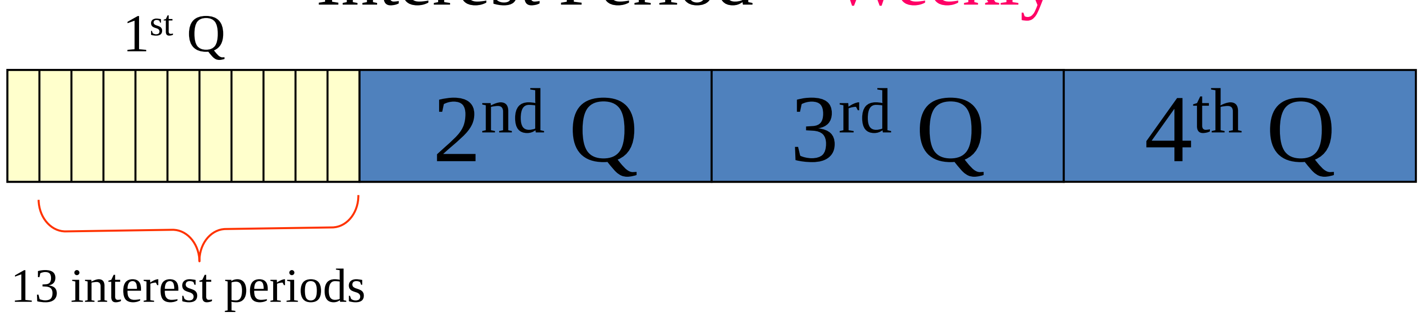
M = 12 interest periods per year

$$i = [1 + r / CK]^C - 1$$
$$= [1 + 0.08 / (3)(4)]^3 - 1$$
$$= 2.013\% \text{ per quarter}$$

Case 2: 8% compounded weekly

Payment Period = Quarter

Interest Period = Weekly



Given r = 8%,

K = 4 payments per year

C = 13 interest periods per quarter

M = 52 interest periods per year

$$i = [1 + r / CK]^{C} - 1$$

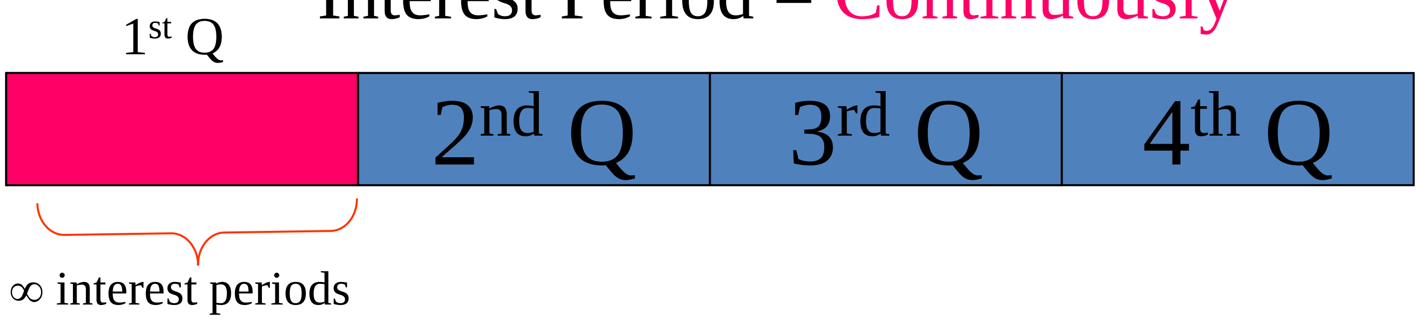
$$= [1 + 0.08 / (13)(4)]^{13} - 1$$

$$= 2.0186\% \text{ per quarter}$$

Case 3: 8% compounded continously

Payment Period = Quarter

Interest Period = Continuously



Given $r = 8\%$,

$K = 4$ payments per year

$$i = e^{r/K} - 1$$

$$= e^{0.02} - 1$$

$$= 2.0201\% \text{ per quarter}$$

Summary: Effective Interest Rate Per Quarter

Case 0	Case 1	Case 2	Case 3
8% compounded quarterly	8% compounded monthly	8% compounded weekly	8% compounded continuously
Payments occur quarterly	Payments occur quarterly	Payments occur quarterly	Payments occur quarterly
2.000% per quarter	2.013% per quarter	2.0186% per quarter	2.0201% per quarter

Effective Interest Rate per Payment Period

$$i = [1 + r / CK]^{C} - 1$$

$C =$ number of interest periods per payment period

$K =$ number of payment periods per year

$CK =$ total number of interest periods per year, or M

$r/K =$ nominal interest rate per payment period

Effective Interest Rate per Payment Period

12% compounded monthly

Payment Period = Quarter

Compounding Period = Month

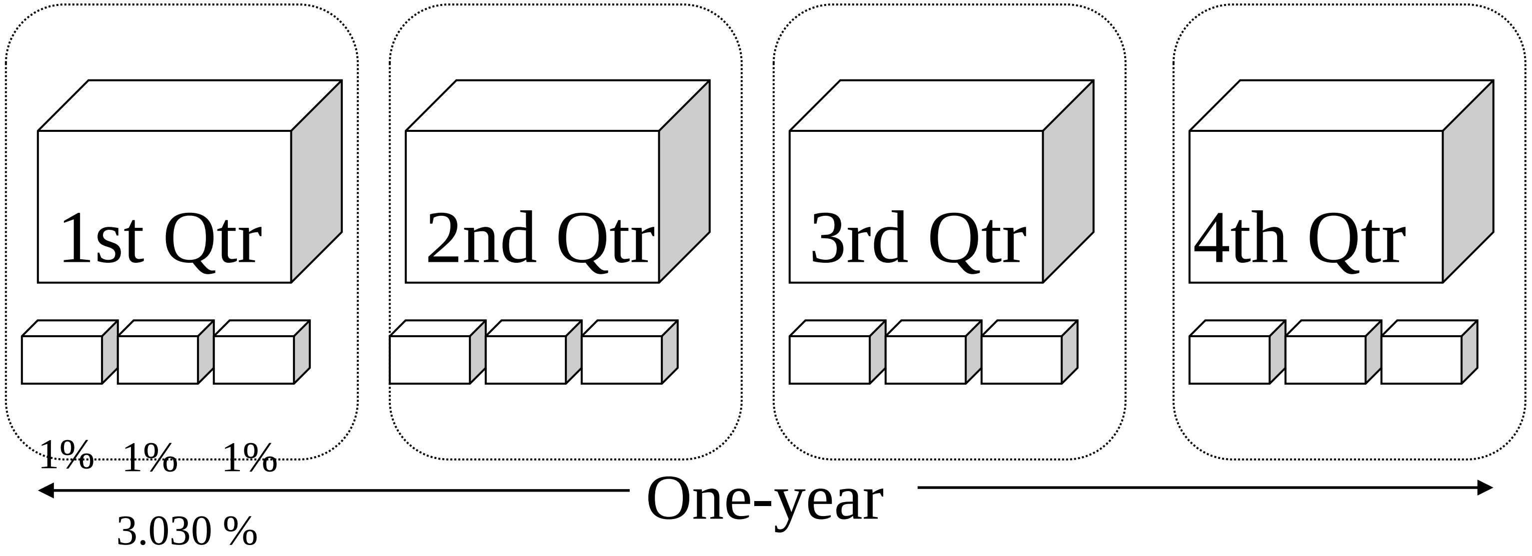
Effective interest rate per quarter

$$i = (1 + 0.01)^3 - 1 = 3.030\%$$

Effective annual interest rate:

$$i_a = (1 + 0.01)^{12} - 1 = 12.68\%$$

$$i_a = (1 + 0.03030)^4 - 1 = 12.68\%$$



Continuous Interest Rate

Continuously Compounded Interest is a great thing when you are earning it! Continuously compounded interest means that your principal is constantly earning interest and the interest keeps earning on the interest earned.

$$i = e^{r} - 1$$

Continuous Interest Rate: Practice Problem

An Investor requires an effective return of 15% per year. What is the minimum annual nominal rate that is acceptable if the interest on his investment is compounded continuously?

$$e^r - 1 = 0.15$$

$$e^r - 1 = 0.15$$

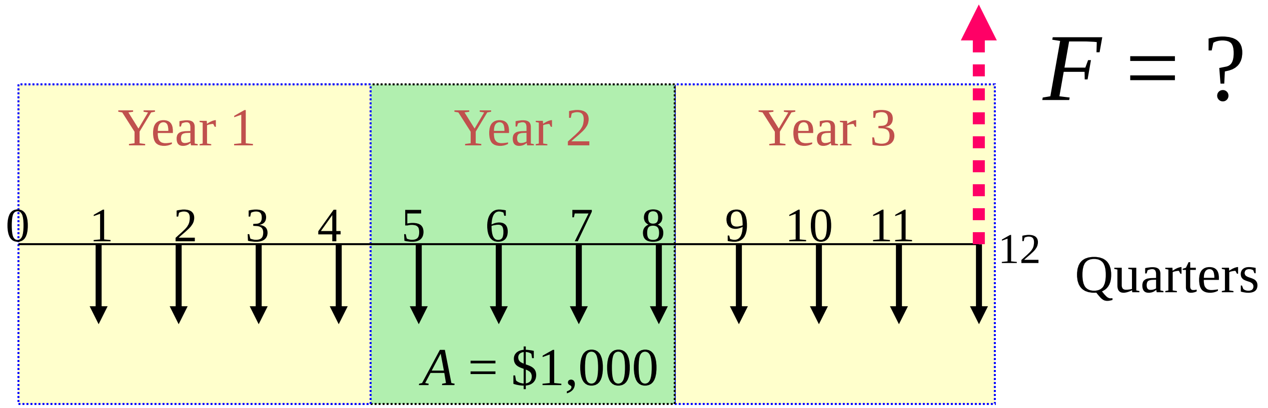
$$e^r = 1.15$$

$$\ln(e^r) = \ln(1.15)$$

$$r = \ln(1.15) = 0.1398 = \underline{13.98\%}$$

A rate of 13.98% per year, cc. generates the same as 15% true effective annual rate.

Continuous Case: Quarterly Deposits with Continuous Compounding



Step 1: K = 4 payment periods/year
$C = \infty$ interest periods per quarter

Step 2:

Step 3: $N = 4(3) = 12$

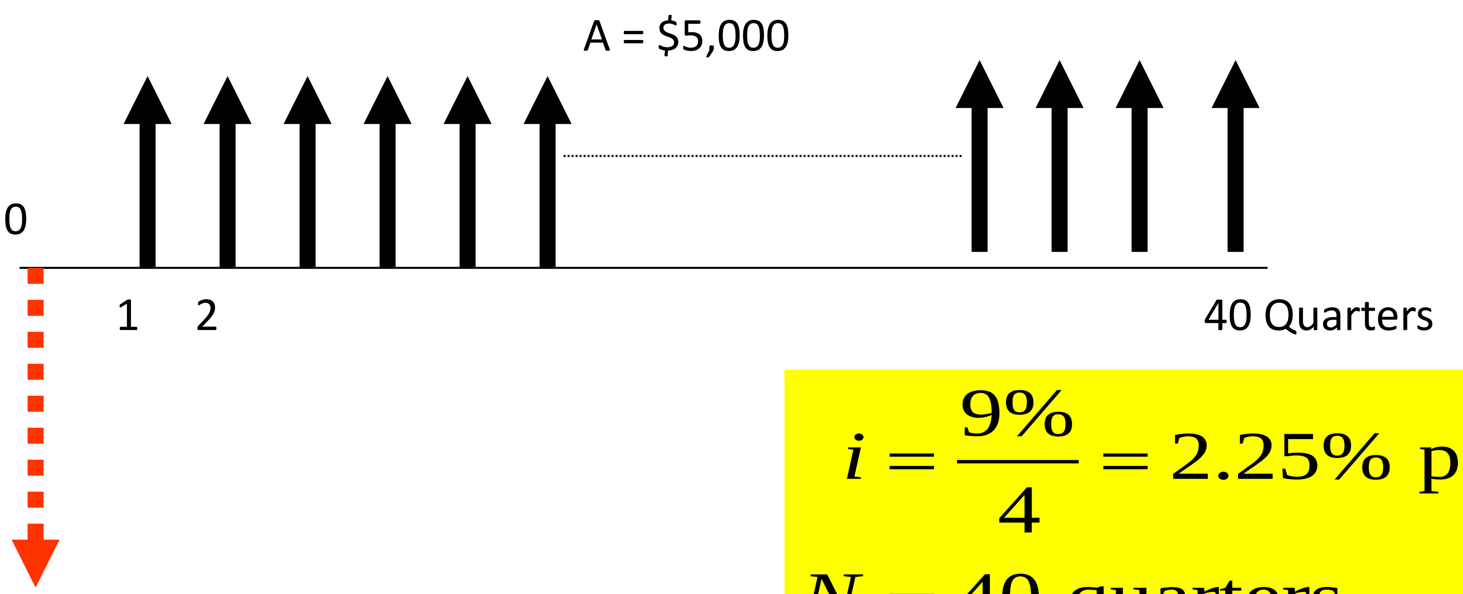
Step 4: F = \$1,000 (F/A, 3.045%, 12)
= **\$14,228.37**

Effective Interest: Practice Problem

A series of equal quarterly payments of $5,000 for 10 years is equivalent to what present amount at an interest rate of 9% compounded:

(a) quarterly
(b) monthly
(c) continuously

Effective Interest: Practice Problem



$$i = \frac{9\%}{4} = 2.25\% \text{ per quarter}$$

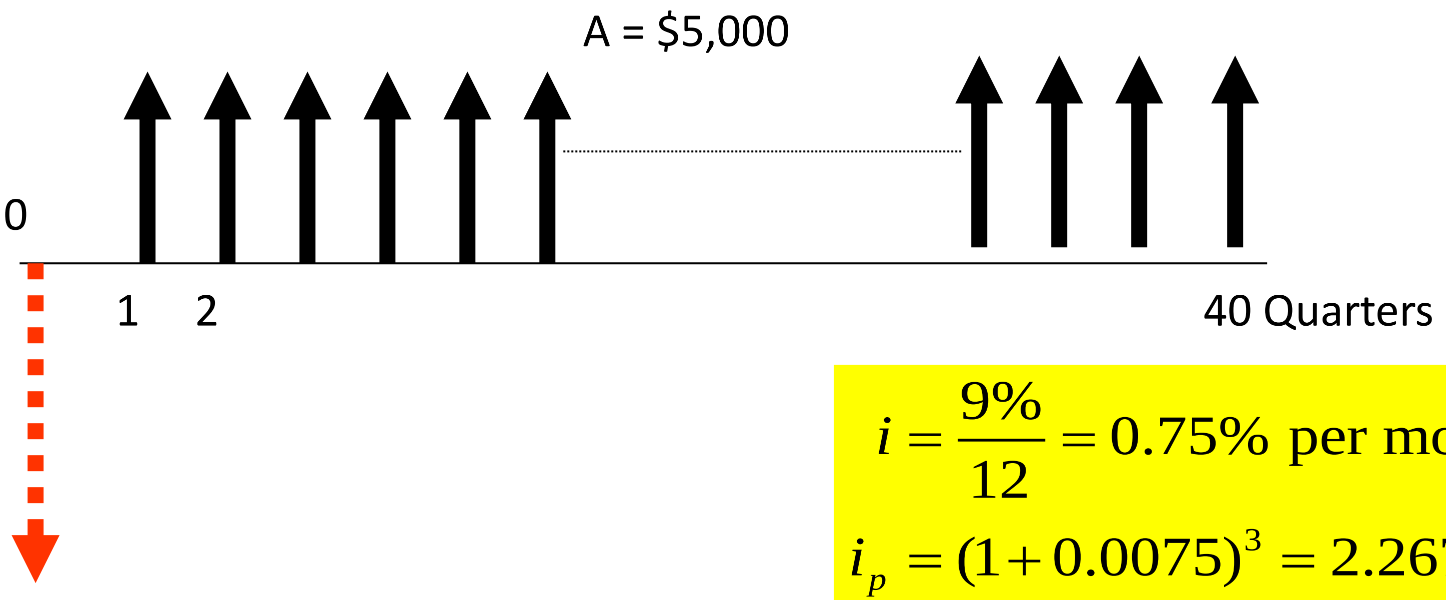
$$N = 40 \text{ quarters}$$

$$P = \$5,000(P/A, 2.25\%, 40)$$
$$= \$130,968$$

Payment period: Quarterly

Interest Period: Quarterly

Effective Interest: Practice Problem



$$i = \frac{9\%}{12} = 0.75\% \text{ per month}$$

$$i_p = (1+0.0075)^3 = 2.267\% \text{ per quarter}$$

$$N = 40 \text{ quarters}$$

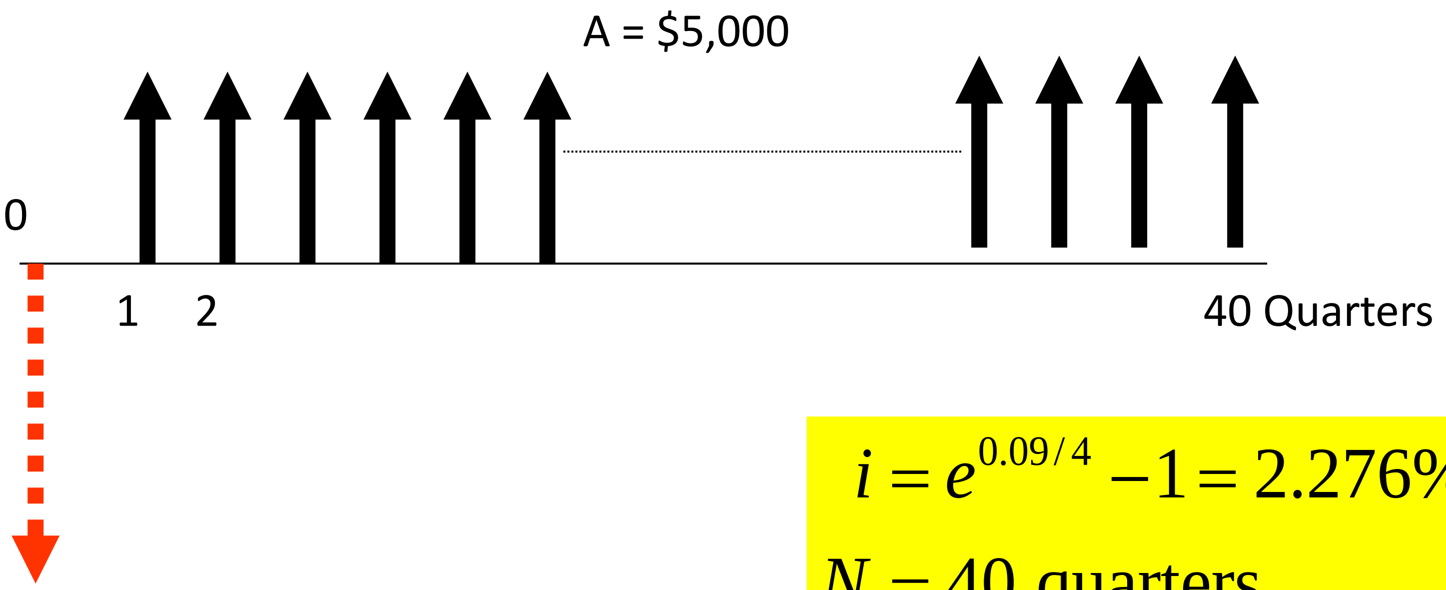
$$P = \$5,000(P/A, 2.267\%, 40)$$

$$= \$130,586$$

Payment period: Quarterly

Interest Period: Monthly

Effective Interest: Practice Problem



Payment period: Quarterly

Interest Period: Continuously

$$i = e^{0.09/4} - 1 = 2.276\% \text{ per quarter}$$

$$N = 40 \text{ quarters}$$

$$P = \$5,000(P/A, 2.276\%, 40)$$

$$= \$130,384$$

Effective Interest: Practice Problem

Three different interest charging plans. Payments are made on a loan every 6 months. Three interest plans are presented:

a) 1. 9% (compounded quarterly)
b) 2. 3% (compounded quarterly)
c) 3. 8.8% (compounded monthly)

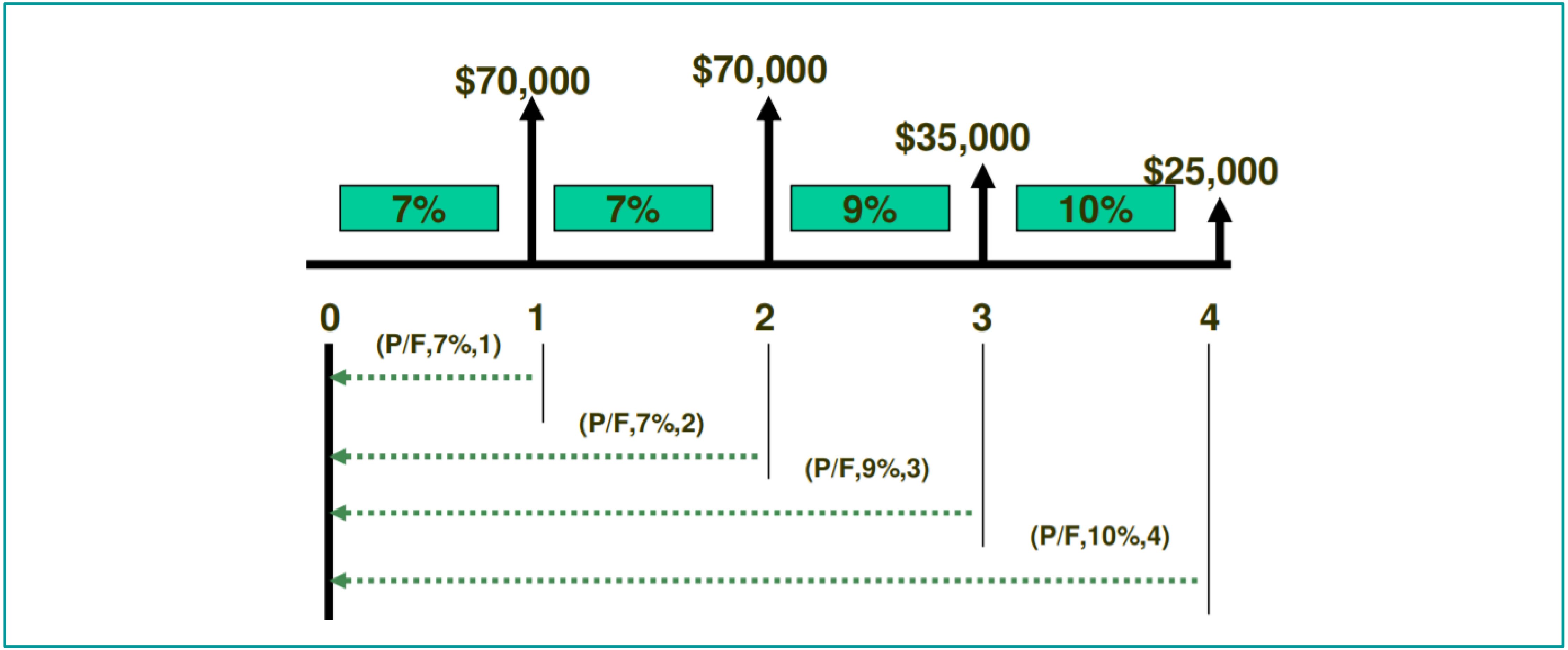
Interest Rate that Vary Over Time

In practice – interest rates do not stay the same over time unless by contractual obligation.

There can exist "variation" of interest rates over time quite normal!

If required, how do you handle that situation?

Interest Rate that Vary Over Time



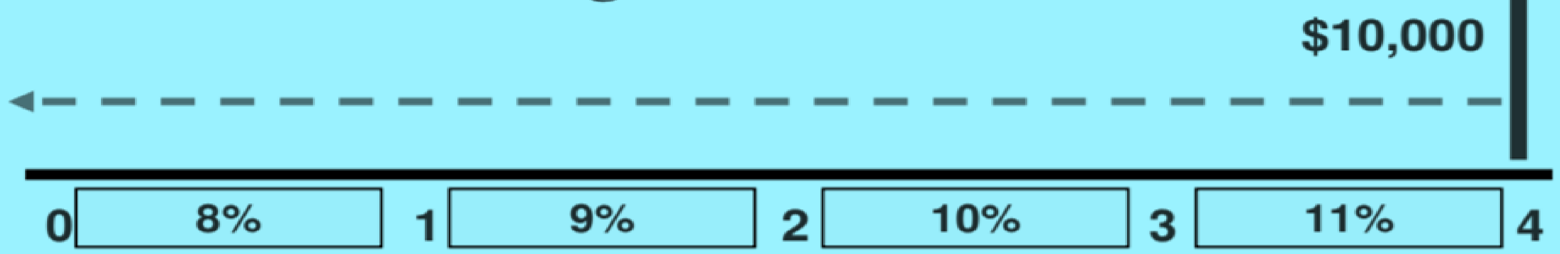
Interest Rate that Vary Over Time

1. $\$7000(P/F,7\%,1)$
2. $\$7000(P/F,7\%,1)(P/F,7\%,1)$
3. $\$35000(P/F,9\%,1)(P/F,7\%,1)^2$
4. $\$25000(P/F,10\%,1)(P/F,9\%,1)(P/F,7\%,1)^2$

Equals $1,72,816 at 0

Interest Rate that Vary Over Time

- **Assume the following Cash Flow:**



Objective: Find P_0 at the varying rates

$$P_0 = \$10{,}000(P/F,8\%,1)(P/F,9\%,1)(P/F,10\%,1)(P/F,11\%,1)$$

$$= \$10{,}000(0.9259)(0.9174)(0.9091)(0.9009)$$

$$= \$10{,}000(0.6957) = \underline{\$6{,}957}$$

Summary

- Actual Interest VS Nominal Interest
- Cases of Actual VS Nominal Interest
- Different Compounding Periods
- Effective Interest Per Payment Period
- Auto Loan Payments
- Continuous Interest
- Interest rate that Varies over Time