Control Systems - 7^{th} Semester - Week 2

Mathematical Modeling of Systems

Dr. Salman Ahmed

Stability Analysis

Example: Compute the poles, zeros and analyze stability of the following seven transfer functions:

$$\boldsymbol{G_1(s) = \frac{(s-3)}{(s+5)}}$$

$$\boldsymbol{G_2(s) = \frac{(s-3)}{(s-5)}}$$

Stability Analysis

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$$G_1(s) = \frac{(s-3)}{(s+5)}$$

$$G_2(s) = \frac{(s-3)}{(s-5)}$$

$$G_3(s) = \frac{(s-3)(s+2)}{(s+5)(s-10)}$$

$$G_4(s) = \frac{s(s+2)}{(s+5)(s-10)}$$

$$G_5(s) = \frac{3s}{(s+5)(s-10)}$$

Stability Analysis

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$$G_4(s) = \frac{s(s+2)}{(s+5)(s-10)}$$
$$G_5(s) = \frac{3s}{(s+5)(s-10)}$$
$$G_6(s) = \frac{3s}{2s(s+5)(s-10)}$$
$$G_7(s) = (s+3)(s+5)$$

Model

A model is representation or abstraction of reality/system.

Who invent model? We, human beings, invent model based on our knowledge.

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What is mathematical model?

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A model is representation or abstraction of reality/system.

Who invent model? We, human beings, invent model based on our knowledge.

This means the more knowledge a person has, the better he/she can write a model.

What is mathematical model? A set of equations (linear or differential) that describes the relationship between input and output of a system.

Types of Model

There are three types of mathematical models

- Black Box
- Grey Box
- White Box

Black Box Model

It is used when only input and output data are known

The internal dynamics are either too complex or totally unknown

Figure: Black Box Model of a System

For modeling purpose, it is an easy model but for analysis purpose it is very hard to analyze or conclude something based on I/O data

Grey Box Model

It is used when input and output data is known, plus some information about internal dynamics of the system are known.

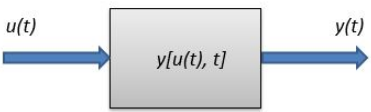

Figure: Grey Box Model of a System

In complex systems, we use grey box modeling to identify or estimate the system model

White Box Model

It is used when the input, output and internal dynamics of the system are known.

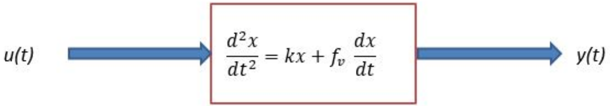

Figure: White Box Model of a System

White Box Model

It is used when the input, output and internal dynamics of the system are known.

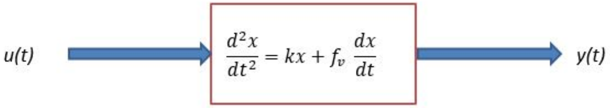

Figure: White Box Model of a System

For modeling purpose, it is an most difficult model but for analysis purpose it is very easy to predict any future values

Obtaining white box models requires us to know exact mathematical formulas and equations

Equations

Formulas used in electrical systems are as follows:

$$\boldsymbol{V_R = I_R R}$$

$$\boldsymbol{i_c = C \frac{dv_c}{dt}}$$

$$\boldsymbol{v_L = L \frac{di_L}{dt}}$$

Equations

Formulas used in electrical systems are as follows:

$$V_R = I_R R$$

$$i_c = C\frac{dv_c}{dt}$$

$$v_L = L\frac{di_L}{dt}$$

Formulas used in mechanical systems are as follows:

$$F = f_v\frac{dx}{dt}$$

$$F = Kx$$

$$F = Ma = M\frac{d^2x}{dt^2}$$

Mechanical Systems

In electrical circuits, we apply KCL and KVL (sum of voltages in a loop is zero or sum of currents at node is zero) .

Mechanical systems obey Newton law; the sum of forces equal to zero (or sum of applied forces equals sum of transmitted/reactive forces).

There are three basic elements of mechanical systems

- Mass
- Spring
- Damper

Mechanical Element - Mass

Mass: Inertial element usually denoted by M.

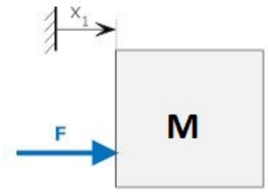

Figure: Schematic and Symbol of Mass

$$\boldsymbol{F = Ma = M\frac{dv}{dt} = M\frac{d^2x_1}{dt^2}}$$

If we reverse the direction of $\boldsymbol{x_1}$, then we write the following:

$$\boldsymbol{F = -Ma = -M\frac{d^2x_1}{dt^2}}$$

Mechanical Element - Spring

Spring: Element that can store or release energy depending upon force applied (compress/expand). Usually denoted by K (Actually K denotes stiffness of spring).

Figure: Schematic and Symbol of Spring

A spring obey Hooke's law, as expressed below:

$$\boldsymbol{F = Kx_1}$$

Mechanical Element - Damper

Damper: An element that dissipates or absorb energy. Usually denoted by D (but book denotes it by f_v).

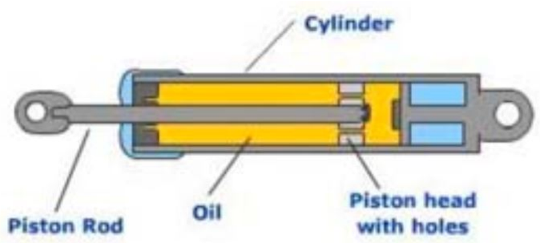

Figure: Schematic of Damper

If the fluid cannot move easily, we drill a hole inside the piston head.

Normally, fluids which are less compressible are chosen (based on application of damper)

Mechanical Element - Damper

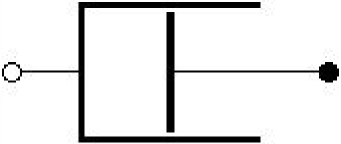

Figure: Symbol of Damper

$$F = f_v \frac{dx}{dt}$$

Mechanical Element - Damper

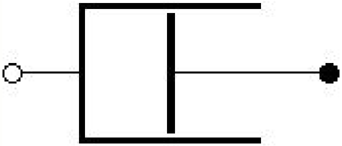

Figure: Symbol of Damper

$$\boldsymbol{F = f_v \frac{dx}{dt}}$$

System Parameters

To recap, in control systems literature, a system must have input and output.

In a system, we have constants and variables.

State-space variables: Those variables which completely describe the behavior of a system.

State-space variables are abbreviated as *ss* variables (or sometimes state variable).

State-space Model

State-space variables are used to obtain mathematical model a system.

The standard state-space model is expressed as follows:

$$\frac{dx}{dt} = Ax + Bu$$
$$y = Cx + Du$$

where x denotes the vector containing state-space variables, $\frac{dx}{dt}$ represent the derivative of state-space variables, u denotes input and y denotes the output.

Electrical Circuit Model

Obtain the state-space model of the following circuit. Choose current across the resistor as output variable.

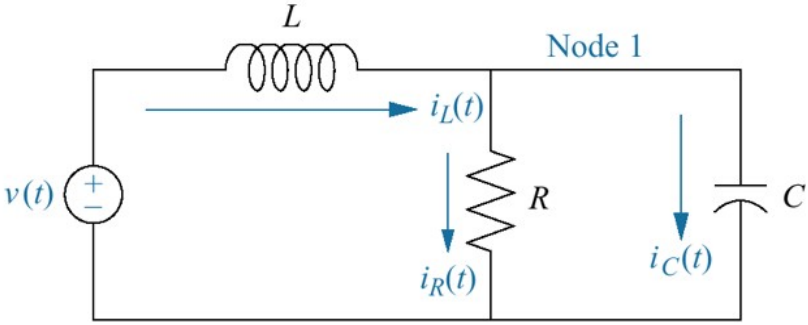

Figure: RLC example used to obtain state-space model

Solution: Step 1. Identify input, output, variables and constants in this circuit.

Terminologies

Input: $v(t)$

Terminologies

Input: $v(t)$

Output: $i_R(t)$

Terminologies

Input: $v(t)$

Output: $i_R(t)$

Variables: (Total 6)

$$\begin{array}{cc} v_R & i_R \\ v_C & i_C \\ v_L & i_L \end{array}$$

Terminologies

Input: $v(t)$

Output: $i_R(t)$

Variables: (Total 6)

$$\begin{matrix} v_R & i_R \\ v_C & i_C \\ v_L & i_L \end{matrix}$$

Constants: (Total 3)

$$R \quad L \quad C$$

Next question: Identify state-space variables. Let us write formulas for all 6 variables

Electrical Circuit Formulas

$$\boldsymbol{v_R = i_R R}$$

$$\boldsymbol{i_R = \frac{v_R}{R}}$$

Electrical Circuit Formulas

$$v_R = i_R R$$

$$i_R = \frac{v_R}{R}$$

$$i_C = C\frac{dv_c}{dt}$$

$$v_C = \frac{1}{C}\int_0^t i_C dt$$

Electrical Circuit Formulas

$$v_R = i_R R$$
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$$i_L = \frac{1}{L}\int_0^t v_L dt$$

Now, which variables derivatives are used in formulas:

Electrical Circuit Formulas

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$$v_C \qquad \text{and} \qquad i_L$$

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Now, which variables derivatives are used in formulas:

$$v_C \qquad \text{and} \qquad i_L$$

So, these 2 variables are state-space variables.

First state-space equation

Now the objective is to write the equation for state-space variable $\boldsymbol{x}$ as follows:

$$\frac{\boldsymbol{dx}}{\boldsymbol{dt}} = \boldsymbol{f}(\boldsymbol{x}, \text{inputs, constants})$$

In other words, we can state the following:

$$\frac{\boldsymbol{dx}}{\boldsymbol{dt}} = \boldsymbol{f}(\text{state-space variables, inputs, constants})$$

First state-space equation

Now the objective is to write the equation for state-space variable x as follows:

$$\frac{dx}{dt} = f(x, \text{inputs, constants})$$

In other words, we can state the following:

$$\frac{dx}{dt} = f(\text{state-space variables, inputs, constants})$$

What are the equations in our examples?

$$\frac{dv_C}{dt} = \frac{1}{C} i_C$$

$$\frac{di_L}{dt} = \frac{1}{L} v_L$$

Let us analyze each term in these equations. i_C is a problematic term and let us eliminate it.

First state-space equation

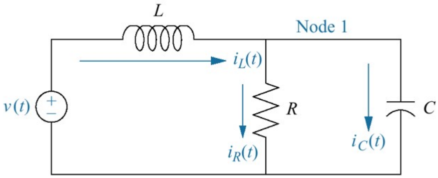

Figure: RLC example used to obtain state-space model

We can write the following:

$$i_L = i_R + i_C$$
$$i_C = i_L - i_R$$

Let us analyze again (by substituting it in the equation)

$$\frac{dv_C}{dt} = \frac{1}{C} i_C$$
$$\frac{dv_C}{dt} = \frac{1}{C} [i_L - i_R]$$

First state-space equation

Let us analyze each term in this equation:

$$\frac{dv_C}{dt} = \frac{1}{C}[i_L - i_R]$$

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It seems i_R is problematic term, as $\frac{1}{C}$ is constant and i_L is state-space variable. Let us further solve i_R.

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It seems i_R is problematic term, as $\frac{1}{C}$ is constant and i_L is state-space variable. Let us further solve i_R. We can write the following:

$$i_R = \frac{v_R}{R} = \frac{v_C}{R}$$

We obtain the first state-space equation as follows:

$$\frac{dv_C}{dt} = \frac{1}{C}\left\{i_L - \frac{v_C}{R}\right\}$$

$$\frac{dv_C}{dt} = \frac{i_L}{C} - \frac{v_C}{RC}$$

First state-space equation

Let us analyze each term in this equation:

$$\frac{dv_C}{dt} = \frac{1}{C}[i_L - i_R]$$

It seems i_R is problematic term, as $\frac{1}{C}$ is constant and i_L is state-space variable. Let us further solve i_R. We can write the following:

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Second state-space equation

The second ss variable was i_L. Let us obtain the second state-space equation.

$$\frac{di_L}{dt} = f(\text{state-space variables, inputs, constants})$$

$$\frac{di_L}{dt} = f(i_L, v_C, \text{ inputs, R, L, C})$$

Second state-space equation

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$$\frac{di_L}{dt} = f(\text{state-space variables, inputs, constants})$$

$$\frac{di_L}{dt} = f(i_L, v_C, \text{ inputs, R, L, C})$$

We have the following equation:

$$\frac{di_L}{dt} = \frac{1}{L} v_L$$

Let us analyze each term, v_L is problematic term and needs to be eliminated.

Second state-space equation

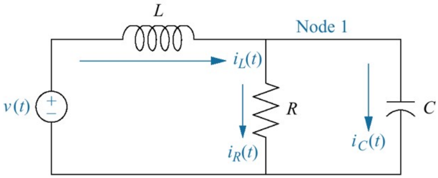

Figure: RLC example used to obtain state-space model

We can write the following:

$$v(t) = v_L + v_C$$
$$v_L = v(t) - v_C$$

$$\frac{di_L}{dt} = \frac{1}{L} v_L = \frac{1}{L}[v(t) - v_C]$$

Output equation

The two state-space equations in standard form are written as follows:

$$\frac{dv_C}{dt} = \frac{i_L}{C} - \frac{v_C}{RC} \tag{1}$$

$$\frac{di_L}{dt} = \frac{1}{L}[v(t) - v_C] \tag{2}$$

Now, let us write the equation for output in standard form. The output is current across resistor i_R.

$$i_R = f(\text{state-space variables, inputs, constants})$$

$$i_R = \frac{v_R}{R} = \frac{v_C}{R}$$

State-space Model

The state-space model is as follows:

$$\frac{dv_C}{dt} = \frac{i_L}{C} - \frac{v_C}{RC} \tag{3}$$

$$\frac{di_L}{dt} = \frac{1}{L}[v(t) - v_C] \tag{4}$$

$$i_R = \frac{v_C}{R} \tag{5}$$

Now, let us convert it to matrix form.

State-space Model

$$\begin{bmatrix} \frac{dv_c}{dt} \\ \\ \frac{di_L}{dt} \end{bmatrix} = \begin{bmatrix} & \\ & \end{bmatrix} \begin{bmatrix} v_C \\ \\ i_L \end{bmatrix} + \begin{bmatrix} . \\ \\ . \end{bmatrix} v(t)$$

$$y = \begin{bmatrix} . & & . \end{bmatrix} \begin{bmatrix} v_C \\ \\ i_L \end{bmatrix} + \begin{bmatrix} . \end{bmatrix} v(t)$$

State-space Model

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$$y = \begin{bmatrix} . & & . \end{bmatrix} \begin{bmatrix} v_C \\ \\ i_L \end{bmatrix} + \begin{bmatrix} . \end{bmatrix} v(t)$$

Substituting values from equations, we obtain the following:

$$\begin{bmatrix} \frac{dv_c}{dt} \\ \\ \frac{di_L}{dt} \end{bmatrix} = \begin{bmatrix} -\frac{1}{RC} & \frac{1}{C} \\ \\ -\frac{1}{L} & 0 \end{bmatrix} \begin{bmatrix} v_C \\ \\ i_L \end{bmatrix} + \begin{bmatrix} 0 \\ \\ \frac{1}{L} \end{bmatrix} v(t)$$

$$y = \begin{bmatrix} \frac{1}{R} & & 0 \end{bmatrix} \begin{bmatrix} v_C \\ \\ i_L \end{bmatrix} + \begin{bmatrix} 0 \end{bmatrix} v(t)$$

Example of Mechanical System 1

Write equation of the following mechanical system (assuming frictionless surface).

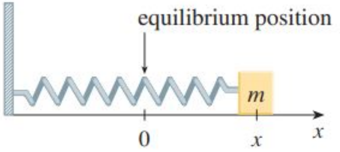

Figure: Mechanical System Example 1

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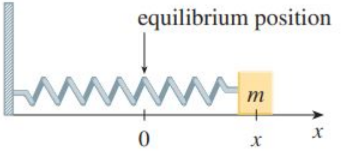

Figure: Mechanical System Example 1

$$\text{Force on spring} + \text{Force on mass} = 0$$

$$kx + m\frac{d^2x}{dt^2} = 0$$

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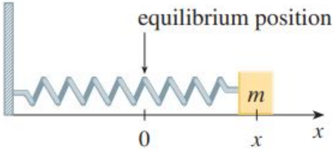

Figure: Mechanical System Example 1

$$\text{Force on spring} + \text{Force on mass} = 0$$

$$kx + m\frac{d^2x}{dt^2} = 0$$

Example of Mechanical System 2

Write equation of the following mechanical system (assuming frictionless surface).

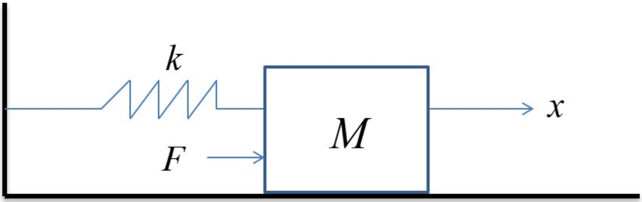

Figure: Mechanical System Example 2

Example of Mechanical System 2

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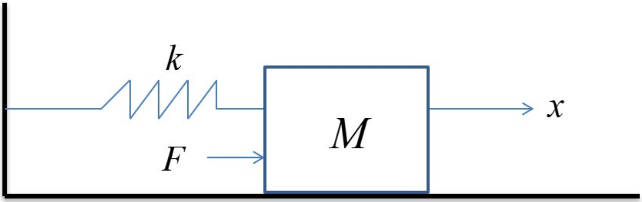
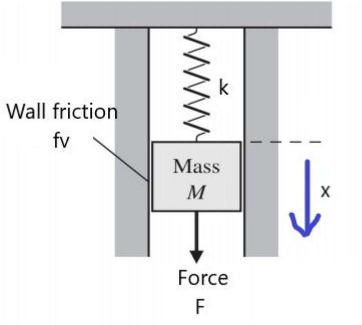


Figure: Mechanical System Example 2

$$kx + M\frac{d^2x}{dt^2} = F$$

Example of Mechanical System 3

Write equation of the automobile shock absorber system shown below



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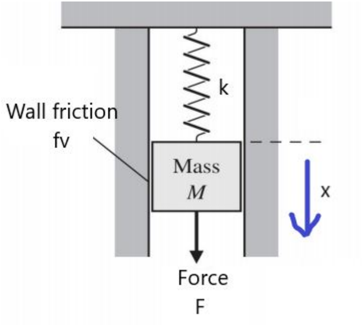

Figure: Mechanical System Example 3

Force on spring + Force on mass + Wall Friction = Force, F

$$kx + M\frac{d^2x}{dt^2} + f_v\frac{dx}{dt} = F$$

Example of Mechanical System 4

Write equation of the system shown below (here f_v is represented by b)

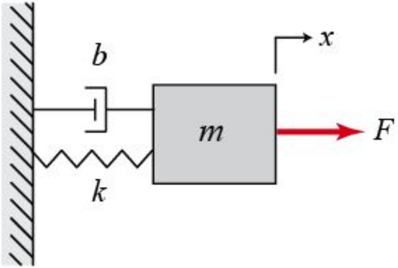

Figure: Mechanical System Example 4

Example of Mechanical System 5

Write equation of the system shown below

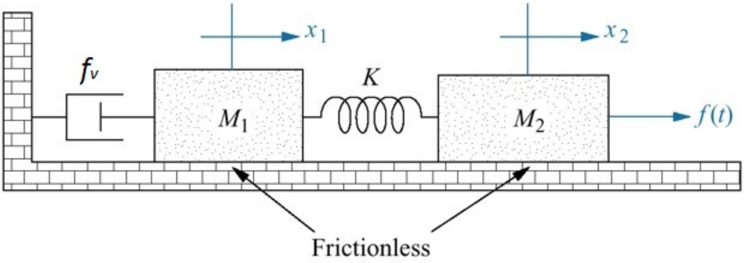

Figure: Mechanical System Example 5

Now, we have 2 masses namely $\boldsymbol{M_1}$ and $\boldsymbol{M_2}$. So, we will write equations for $\boldsymbol{M_1}$ due to itself and due to $\boldsymbol{M_2}$. Similarly, we will write equations for $\boldsymbol{M_2}$ due to $\boldsymbol{M_2}$ as well as $\boldsymbol{M_1}$.

Example of Mechanical System 5

Forces on M_1 due to M_1 only:

$$f_v \frac{dx_1}{dt} + M_1 \frac{d^2 x_1}{dt^2} + K x_1$$

Forces on M_1 due to M_2 only:

$$K x_2$$

Total forces on M_1

$$f_v \frac{dx_1}{dt} + M_1 \frac{d^2 x_1}{dt^2} + K x_1 = K x_2$$

Example of Mechanical System 5

Forces on M_2 due to M_2 only:

$$M_2 \frac{d^2 x_2}{dt^2} + K x_2 = f(t)$$

Forces on M_2 due to M_1 only:

$$K x_1$$

Total forces on M_2

$$M_2 \frac{d^2 x_2}{dt^2} + K x_2 = K x_1 + f(t)$$

or

$$M_2 \frac{d^2 x_2}{dt^2} + K x_2 - K x_1 = f(t)$$

Next week

Next week, we will be studying the following topics:

- Write state-space for mechanical systems
- Convert state-space to transfer function
- Convert transfer function to state-space