Indefinite Integral

Lecture 2

$\boxed{\text{Recap}}$ We already know ~~how~~ the antiderivative of a function f.

The process of finding antiderivatives is called antidifferentiation or integration.

If $F(x)$ be the antiderivative of $f(x)$ then,

$$\int f(x)\,dx = F(x) + C$$

Example: Evaluate the following indefinite integral.

$$\int x^4 + 3x - 9 \; dx$$

Solⁿ:

$$\int (x^4 + 3x - 9)\,dx = \frac{x^5}{5} + \frac{3}{2}x^2 - 9x + C \qquad \square$$

here C is integral constant.

Integration Formulas

page 324 : Table 5.2.1 Book. Calculus: Early Transcendentals — H. Anton

e.g.

$$\int dx = x + C$$

$$\int x^r\,dx = \frac{x^{r+1}}{r+1} + C \qquad r \neq -1$$

$$\int \cos x\,dx = \sin x + C$$

$$\int \sin x\,dx = -\cos x + C$$

$$\int \sec^2 x\,dx = \tan x + C$$

$$\int \csc^2 x\,dx = -\cot x + C$$

and so on.

Properties

Suppose that $F(x)$ and $G(x)$ are antiderivatives of $f(x)$ and $g(x)$ respectively, and a is a constant. Then:

ⓐ $\int a f(x)\,dx = aF(x) + C$

ⓑ $\int [f(x) \pm g(x)]\,dx = F(x) \pm G(x) + C$

Example: $\int (3x^6 - 2x^2 + 7x)\,dx = 3\int x^6\,dx - 2\int x^2\,dx + 7\int x\,dx$

$$= 3\,\frac{x^{6+1}}{6+1} - 2\,\frac{x^{2+1}}{2+1} + 7\,\frac{x^2}{2} + C$$

$$= \frac{3}{7}x^7 - \frac{2}{3}x^3 + \frac{7}{2}x^2 + C$$

Problem: Evaluate: 1. $\int \frac{\cos x}{\sin^2 x}\,dx$

Sol^n.

1. $\int \frac{\cos x}{\sin^2 x}\,dx = \int \frac{1}{\sin x}\,\frac{\cos x}{\sin x}\,dx$

$$= \int \csc x \cot x\,dx = -\csc x + C$$

2. $\int \frac{x^2}{x^2+1}\,dx = \int \frac{x^2+1-1}{x^2+1}\,dx = \int \left(\frac{x^2+1}{x^2+1} - \frac{1}{x^2+1}\right)dx$

$$= \int \left(1 - \frac{1}{1+x^2}\right)dx$$

$$= x - \tan^{-1}x + C$$

Exercise: TRY YOURSELF !!!

Evaluate the following integral:

1. $\displaystyle\int \frac{x^5+2x^2-1}{x^4}\,dx$

2. $\displaystyle\int (3\sin x - 2\sec^2 x)\,dx$

3. $\displaystyle\int \frac{\sin x}{\cos^2 x}\,dx$

4. $\displaystyle\int (1+\sin^2\theta \csc\theta)\,d\theta$

5. $\displaystyle\int \left[\frac{1}{2\sqrt{1-x^2}} - \frac{3}{1+x^2}\right] dx$

Not Now 6. $\displaystyle\int \sqrt{16-x^2}\,dx$

7. If $f'(x) = x^4+3x-9$. what was $f(x)$?

8. $\displaystyle\int \sin\left(\frac{t}{2}\right)\cos\left(\frac{t}{2}\right) dt$

Integration by Substitution

Some times we don't integrate straight forward. Then we use substitute to other variable, and we call sometimes u-substitution method. When we have a composite function and we need to integrate it then we use substitution method.

Suppose $F(g(x))$ is a composite function.

$$\frac{d}{dx}\left[F(g(x))\right] = F'(g(x))\, g'(x)$$

$$\int F'(g(x))\, g'(x)\, dx = F(g(x)) + C$$

Since F is an antiderivative of f so, we write

$$\int f(g(x))\, g'(x)\, dx = F(g(x)) + C$$

Here we say. $u = g(x)$

$$\frac{du}{dx} = g'(x)$$

$$du = g'(x)\, dx$$

which implies

$$\int f(u)\, du = F(u) + C.$$

Problem: 1. Evaluate $\int 3(8y-1)e^{4y^2-y}dy$

Say $u = 4y^2 - y$ $\qquad \frac{du}{dx} = g'(x)$

$\frac{du}{dx} = \cancel{8y\,dy} \cdot 8y-1$

$du = (8y-1)dy$

Now, $\int 3e^{4y^2-y}(8y-1)dy$

$= \int 3e^{u}du = 3\int e^{u}du = 3e^{u}+C$

$= 3e^{4y^2-y} + C$

2. Evaluate: $\int x^2(3-10x^3)^4 dx$

Say, $u = 3-10x^3$

$\frac{du}{dx} = -30x^2 \cancel{dx}$

$du = -30x^2 dx$

$-\frac{1}{30}du = x^2 dx$

$\int x^2(3-10x^3)^4 dx = \int u^4\left(-\frac{1}{30}\right)du$

$= -\frac{1}{30}\int u^4 du$

$= -\frac{1}{30}\frac{u^5}{5} + C$

$= -\frac{1}{150}(3-10x^3)^5 + C$

3. Evaluate: $\int \cos 5x\,dx$

Why we substitute $u = 5x$ (?) because ~~their integrand~~ we integrate this integrand w.r.to x NOT $5x$.

So $u = 5x$

$\frac{du}{dx} = 5$

$du = 5dx \Rightarrow \frac{1}{5}du = dx$

$$\int \cos 5x\,dx = \int \cos u \frac{1}{5}du = \frac{1}{5}\int \cos u\,du$$

$$= \frac{1}{5}\sin u + C$$

$$= \frac{1}{5}\sin 5x + C$$

4. Evaluate: $\int \sin^2 x \underline{\cos x}\,dx$

let $u = \sin x$

$\frac{du}{dx} = \cos x$

$du = \cos x\,dx$

$$\int \sin^2 x \cos x\,dx = \int u^2 du = \frac{u^3}{3} + C = \frac{1}{3}\sin^3 x + C$$

Exercise: Evaluate: the following integral.

1. $\int t^4 \sqrt{3-5t^5}\, dt$

2. $\int \frac{e^x}{\sqrt{1-e^{2x}}}\, dx$

3. $\int t \sqrt{7t^2+12}\, dt$

4. $\int \frac{x^2+1}{\sqrt{x^3+3x}}\, dx$

5. $\int e^{\sin x} \cos x\, dx$

6. $\int \frac{e^x+e^{-x}}{e^x-e^{-x}}\, dx$

7. $\int \frac{\sec^2 x}{\sqrt{1-\tan^2 x}}\, dx$

8. $\int \frac{3^x}{(3^x+1)^2}\, dx$