Lecture 8

Topics:

1. Set Operations
2. Set Identities
3. De Morgan’s Law for Set Operations

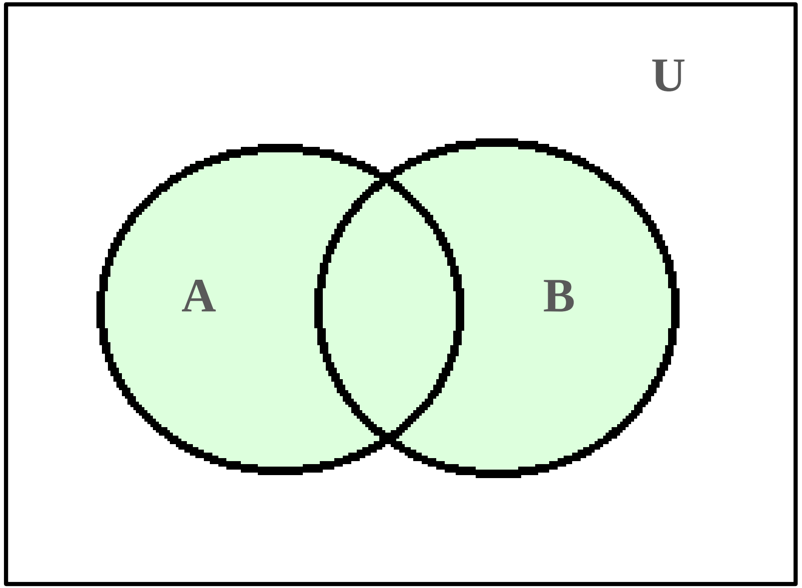
2.2 Set Operations

Definition 1: Let A and B be sets. The union of the sets A and B, denoted by $A \cup B$, is the set that contains those elements that are either in A or in B, or in both. An element x belongs to the union of the sets A and B if and only if x belongs to A or x belongs to B.
This tells us that $\mathbf{A \cup B = \{x \mid x \in A \vee x \in B\}}$.

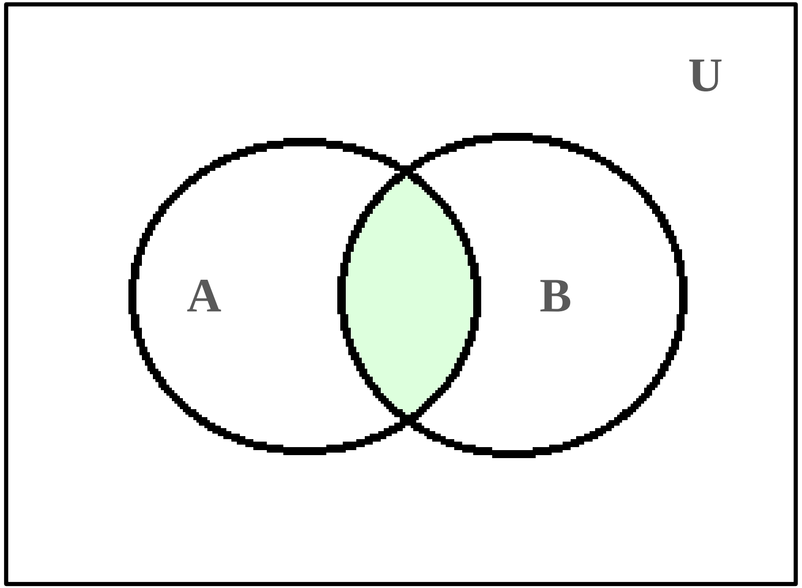
EXAMPLE 1: The union of the sets {1, 3, 5} and {1, 2, 3} is the set {1, 2, 3, 5}; that is, $\mathbf{\{1, 3, 5\} \cup \{1, 2, 3\} = \{1, 2, 3, 5\}}$

EXAMPLE 3: The intersection of the sets {1, 3, 5} and {1, 2, 3} is the set {1, 3}; that is, $\mathbf{\{1, 3, 5\} \cap \{1, 2, 3\} = \{1, 3\}}$.

Venn Diagram



A ∪ B is shaded.



A ∩ B is shaded.

Disjoint Set

Definition 3: Two sets are called disjoint if their intersection is the empty set.

EXAMPLE 5:

Let $A = \{1, 3, 5, 7, 9\}$ and $B = \{2, 4, 6, 8, 10\}$. Because $A \cap B = \emptyset$, A and B are disjoint.

Principle of inclusion-exclusion

We are often interested in finding the cardinality of a union of two finite sets A and B. Note that **|A| + |B|** counts each element that is in A but not in B or in B but not in A exactly once, and Be careful not to overcount! each element that is in both A and B exactly twice. Thus, if the number of elements that are in both A and B is subtracted from |A| + |B|, elements in A ∩ B will be counted only once.

Hence, $|A \cup B| = |A| + |B| - |A \cap B|$.

The generalization of this result to unions of an arbitrary number of sets is called the principle of **inclusion–exclusion.** The principle of inclusion–exclusion is an important technique used in enumeration. We will discuss this principle and other counting techniques in detail in Chapters 6 and 8.

Difference of two sets

Definition 4: Let A and B be sets. The difference of A and B, denoted by $A - B$, is the set containing those elements that are in A but not in B. The difference of A and B is also called the complement of B with respect to A.

Remark: The difference of sets A and B is sometimes denoted by $\mathbf{A \backslash B}$. An element x belongs to the difference of A and B if and only if $x \in A$ and $x \notin B$. This tells us that $\mathbf{A - B = \{x \mid x \in A \land x \notin B\}}$.

EXAMPLE 6: The difference of {1, 3, 5} and {1, 2, 3} is the set {5};

that is, $\{1, 3, 5\} - \{1, 2, 3\} = \{5\}$. This is different from the difference of {1, 2, 3} and {1, 3, 5}, which is the set {2}

Complement of a set

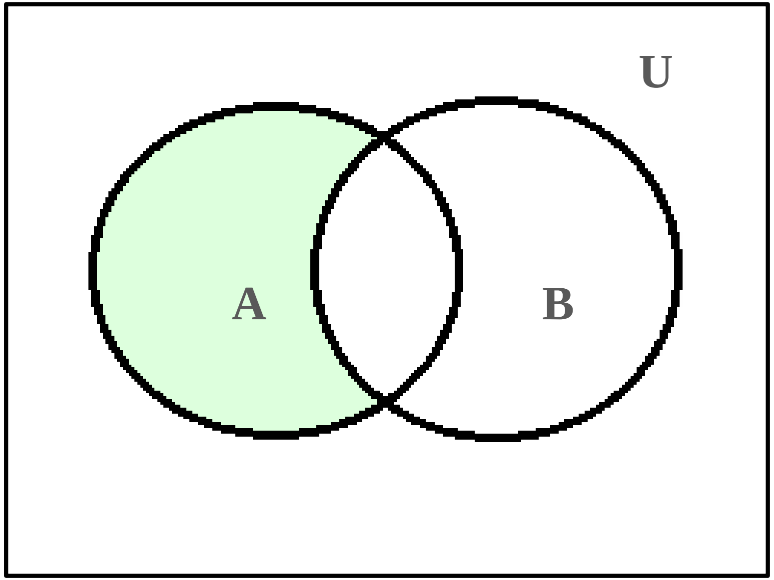
Definition 5: Let U be the universal set. The complement of the set A, denoted by A^c, is the complement of A with respect to U.

Therefore, the complement of the set A is $U - A$.

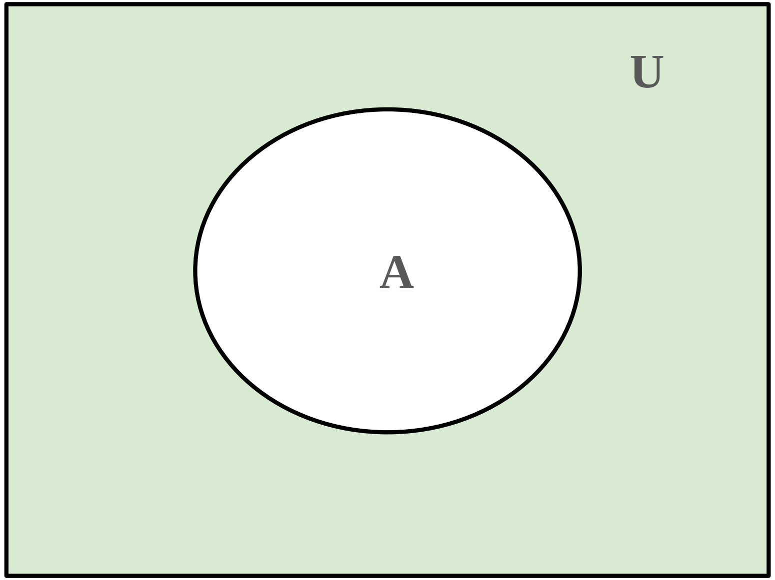
Remark: The definition of the complement of A depends on a particular universal set U. This definition makes sense for any superset U of A. If we want to identify the universal set U, we would write "the complement of A with respect to the set U." An element belongs to A^c if and only if $\mathbf{x \notin A}$.

This tells us that $A^c = \{x \in U \mid x \notin A\}$.

Venn Diagram



Venn diagram for the difference of A and B



Venn diagram for the complement of the set A.

2.2.2 Set Identities

Identity	Name
$A \cap U = A$ $A \cup \emptyset = A$	Identity laws
$A \cup U = U$ $A \cap \emptyset = \emptyset$	Domination laws
$A \cup A = A$ $A \cap A = A$	Idempotent laws
$(A^c)^c = A$	Complementation law
$A \cup B = B \cup A$ $A \cap B = B \cap A$	Commutative laws

Identity	Name
$A \cup (B \cup C) = (A \cup B) \cup C$ $A \cap (B \cap C) = (A \cap B) \cap C$	Associative laws
$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$ $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$	Distributive laws
$(A \cap B)^c = A^c \cup B^c$ $(A \cup B)^c = A^c \cap B^c$	De Morgan's laws
$A \cup (A \cap B) = A$ $A \cap (A \cup B) = A$	Absorption laws
$A \cup \bar{A} = U$ $A \cap \bar{A} = \emptyset$	Complement laws

EXAMPLE 10 Prove that $(A \cap B)^c = A^c \cup B^c$

Do it by yourself

Proof of DeMorgan's theorem

EXAMPLE 11 Use set builder notation and logical equivalences to establish the first De Morgan law $(A \cap B)^c = A^c \cup B^c$

Solution: We can prove this identity with the following steps.

$(A \cap B)^c = \{x \mid x \notin A \cap B\}$ by definition of complement

$= \{x \mid \neg(x \in (A \cap B))\}$ by definition of does not belong symbol

$= \{x \mid \neg(x \in A \land x \in B)\}$ by definition of intersection

$= \{x \mid \neg(x \in A) \lor \neg(x \in B)\}$ by the first De Morgan law for logical equivalences

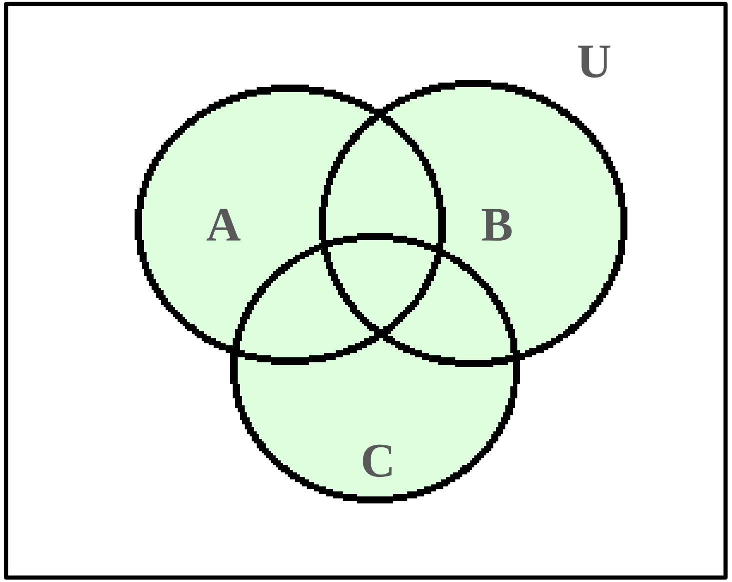
$= \{x \mid x \notin A \lor x \notin B\}$ by definition of does not belong symbol

$= \{x \mid x \in A^c \lor x \in B^c\}$ by definition of complement

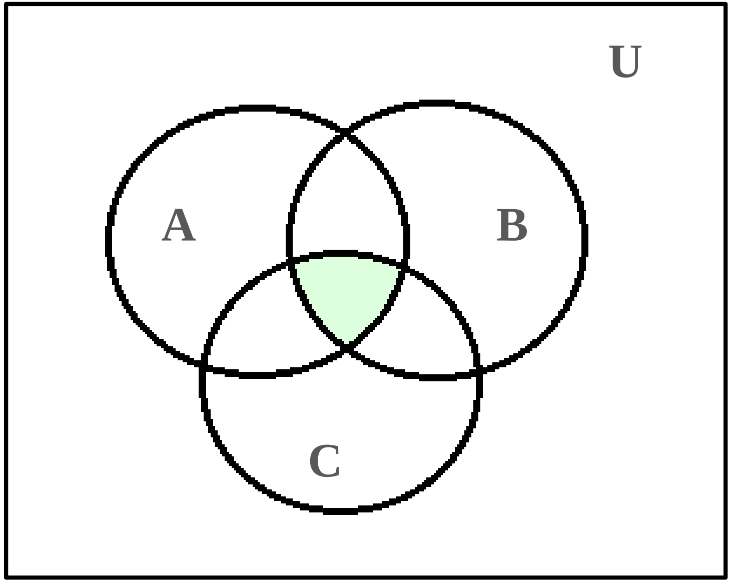
$= \{x \mid x \in A^c \cup B^c\}$ by definition of union

$= A^c \cup B^c$ by meaning of set builder notation

Union and Intersection of several sets



(a) A U B U C is shaded.



(b) A ∩ B ∩ C is shaded.

Ex 13: Use a membership table to show that $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$.

A	B	C	B U C	A ∩ (B ∪ C)	(A ∩ B)	(A ∩ C).	(A ∩ B) ∪ (A ∩ C)
1	1	1	1	1	1	1	1
1	1	0	1	1	1	0	1
1	0	1	1	1	0	1	1
1	0	0	0	0	0	0	0
0	1	1	1	0	0	0	0
0	1	0	1	0	0	0	0
0	0	1	1	0	0	0	0
0	0	0	0	0	0	0	0

EXAMPLE 14 : Let A, B, and C be sets. Let A, B, and C be sets.

Show that $(A \cup (B \cap C))^c = (C^c \cup B^c) \cap A^c$.

Solution: We have $(A \cup (B \cap C))^c$

$= A^c \cap (B \cap C)^c$ by the first De Morgan law

$= A^c \cap (B^c \cup C^c)$ by the second De Morgan law

$= (B^c \cup C^c) \cap A^c$ by the commutative law for intersections

$= (C^c \cup B^c) \cap A^c$ by the commutative law for unions.

Union and intersection of a collection of sets

Definition 6 The union of a collection of sets is the set that contains those elements that are members of at least one set in the collection.

We use the notation $A_1 \cup A_2 \cup \cdots \cup A_n$

$=\bigcup^{n}_{i=1}, A_i$ to denote the union of the sets $A_1, A_2, \ldots, A_n$

Definition 7 The intersection of a collection of sets is the set that contains those elements that are members of all the sets in the collection.

We use the notation $A_1 \cap A_2 \cap \cdots \cap A_n = \bigcap^{n}_{i=1} A_i$

The end