Example: Evaluate $\int \frac{dx}{x^2\sqrt{4-x^2}}$

Sol^n:

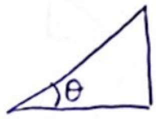
$$\int \frac{dx}{x^2\sqrt{4(1-x^2/4)}}$$

let

$\frac{x}{2} = \sin\theta$, $x = 2\sin\theta$

$dx = 2\cos\theta\, d\theta$

$$= \frac{1}{2}\int \frac{dx}{x^2\sqrt{1-\left(\frac{x}{2}\right)^2}}$$



$\sin\theta = \frac{x}{2} = \frac{\text{opp}}{\text{hyp}}$

adj $= \sqrt{4-x^2}$

$\cot\theta = \frac{\text{adj}}{\text{opp}} = \frac{\sqrt{4-x^2}}{x}$

$$= \frac{1}{2}\int \frac{2\cos\theta\, d\theta}{4\sin^2\theta\sqrt{1-\sin^2\theta}}$$

$$= \frac{1}{4}\int \frac{\cos\theta\, d\theta}{\sin^2\theta\sqrt{\cos^2\theta}}$$

$$= \frac{1}{4}\int \frac{1}{\sin^2\theta}\, d\theta$$

$$= \frac{1}{4}\int \csc^2\theta\, d\theta$$

$$= \frac{1}{4}(-\cot\theta) + C$$

$$= -\frac{1}{4}\cot\theta + C$$

$$= -\frac{1}{4}\frac{\sqrt{4-x^2}}{x} + C$$

Example : $\displaystyle\int \frac{\sqrt{x^2-9}}{x}\,dx$

$\frac{x}{3} = \sec\theta \qquad x = 3\sec\theta$

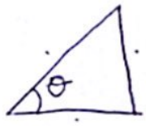
$\frac{1}{3}\,dx = \sec\theta\tan\theta\,d\theta$

$$= \int \frac{3\sqrt{\left(\frac{x}{3}\right)^2-1}}{x}\,dx$$

$$= \cancel{3}\int \cancel{\frac{\sqrt{\sec^2\theta-1}}{\cancel{x}/3}}\,dx$$

$$= 3\int \frac{\sqrt{\sec^2\theta-1}}{\cancel{3}\sec\theta}\,\cancel{3}\sec\theta\tan\theta\,d\theta$$

$$= 3\int \tan\theta \cdot \tan\theta\,d\theta$$



$$= 3\int \tan^2\theta\,d\theta$$

$\sec\theta = \frac{x}{3} = \frac{\text{hyp.}}{\text{adj}}$

$$= 3\int (\sec^2\theta - 1)\,d\theta$$

$\text{opp} = \sqrt{x^2-9}$

$$= 3\left[\tan\theta - \theta\right] + c$$

$\tan\theta = \frac{\text{opp.}}{\text{adj}} = \frac{\sqrt{x^2-9}}{3}$

$$= 3\left[\frac{\sqrt{x^2-9}}{3} - \tan^{-1}\left(\frac{\sqrt{x^2-9}}{3}\right)\right] + c$$

$\theta = \tan^{-1}\frac{\sqrt{x^2-9}}{3}$

or

$\sec^{-1}\left(\frac{x}{3}\right)$

* $\boxed{\int \tan^m x \sec^n x \, dx}$

If n even then $\sec^2 x = \tan^2 x + 1$ say, $u = \tan x$

If m odd then $\tan^2 x = \sec^2 x - 1$ say, $u = \sec x$

If $\begin{cases} m \text{ even} \\ n \text{ odd} \end{cases}$ then $\tan^2 x = \sec^2 x - 1$

<u>Example</u>: Evaluate $\int \tan^2 x \sec^4 x \, dx$

$$= \int \tan^2 x \sec^2 x . \sec^2 x \, dx$$

$$= \int \tan^2 x (\tan^2 x + 1) \underline{\sec^2 x \, dx}$$

say,
$u = \tan x$
$du = \sec^2 x \, dx$

$$= \int u^2 (u^2+1) \, du$$

$$= \frac{u^5}{5} + \frac{u^3}{3} + C$$

$$= \frac{1}{5} \tan^5 x + \frac{1}{3} \tan^3 x + C$$

□

<u>Extra</u>:

1. $\int \tan^5 x \sec^4 x \, dx$

2. $\int \tan^2 x \sec^3 x \, dx$

3. $\int \sec^5 x \, dx$

4. $\int \tan^4 x \, dx$

Lecture 5:

Partial Fractions

Form of rational function	Form of partial fraction
1. $\dfrac{px+q}{(x-a)(x-b)}$, $a \neq b$	$\dfrac{A}{x-a}+\dfrac{B}{x-b}$
2. $\dfrac{px+q}{(x-a)^2}$	$\dfrac{A}{x-a}+\dfrac{B}{(x-a)^2}$
3. $\dfrac{px^2+qx+r}{(x-a)(x-b)(x-c)}$	$\dfrac{A}{x-a}+\dfrac{B}{x-b}+\dfrac{C}{x-c}$
4. $\dfrac{px^2+qx+r}{(x-a)^2(x-b)}$	$\dfrac{A}{x-a}+\dfrac{B}{(x-a)^2}+\dfrac{C}{x-b}$
5. $\dfrac{px^2+qx+r}{(x-a)(x^2+bx+c)}$	$\dfrac{A}{x-a}+\dfrac{Bx+C}{x^2+bx+c}$

Example: Evaluate $\int \frac{2x+4}{x^3-2x^2}\,dx$

Sol^n: ~~Now~~ Here

$$\frac{2x+4}{x^3-2x^2} = \frac{2x+4}{x^2(x-2)} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x-2}$$

$$2x+4 = Ax(x-2) + B(x-2) + Cx^2$$
$$= (A+C)x^2 + (B-2A)x - 2B$$

$-2B = 4$, $B = -2$

$B - 2A = 2$
$-2 - 2A = 2$
$-2A = 4$
$A = -2$

$A + C = 0$
$C = 2$

$$\frac{2x+4}{x^3-2x^2} = \frac{-2}{x} + \frac{-2}{x^2} + \frac{2}{x-2}$$

$$\therefore \int \frac{2x+4}{x^3-2x^2}\,dx = -2\int \frac{1}{x}\,dx - 2\int \frac{1}{x^2}\,dx + 2\int \frac{1}{x-2}\,dx$$
$$= -2\ln|x| - 2\,\frac{x^{-2+1}}{-2+1} + 2.\ln|x-2| + C$$
$$= 2\ln|x-2| - 2\ln|x| + 2\frac{1}{x} + C$$
$$= 2\ln\left|\frac{x-2}{x}\right| + \frac{2}{x} + C$$

Example: Evaluate $\int \frac{x^2+x-2}{3x^3-x^2+3x-1}\,dx$

Here $\frac{x^2+x-2}{3x^3-x^2+3x-1} = \frac{x^2+x-2}{x^2(3x-1)+1(3x-1)}$

$= \frac{x^2+x-2}{(3x-1)(x^2+1)} = \frac{A}{3x-1} + \frac{Bx+C}{x^2+1}$

$\therefore\ x^2+x-2 = A(x^2+1) + (Bx+C)(3x-1)$

$= (A+3B)x^2 + (3C-B)x + A - C$

$A+3B=1$, $3C-B=1$ $A-C=-2$

$$\begin{array}{r} A - C = -2 \\ (-)\ \underline{\qquad\qquad} \\ 3B+C=3 \end{array}$$

$$\begin{array}{r} \textcircled{3}\ \underline{3C-B=1} \\ 10C = 6 \end{array}$$

$C = \frac{6}{10} = \frac{3}{5}$

$-B = 1-\frac{9}{5}$

$= -\frac{4}{5}$

$B = \frac{4}{5}$

$A = -2+\frac{3}{5}$

$= \frac{-7}{5}$

Therefore,

$$\int \frac{x^2+x-2}{3x^3-x^2+3x-1}\,dx = \int \frac{-7/5}{3x-1}\,dx + \int \frac{\frac{4}{5}x+\frac{3}{5}}{x^2+1}\,dx$$

$$= -\frac{7}{5}\int \frac{1}{3x-1}\,dx + \frac{4}{5}\int \frac{x}{x^2+1}\,dx + \frac{3}{5}\int \frac{1}{x^2+1}\,dx$$

$$= -\frac{7}{5}\cdot\frac{1}{3}\ln|3x-1| + \frac{4}{5}\cdot\frac{1}{2}\ln|x^2+1| + \frac{3}{5}\tan^{-1}x + C$$

$$= -\frac{7}{15}\ln|3x-1| + \frac{2}{5}\ln|x^2+1| + \frac{3}{5}\tan^{-1}x + C$$

$3x-1=u$

$\frac{1}{3}du = 3dx$

$\int \frac{1}{3}\frac{1}{u}\,du$

$\frac{1}{3}\ln|u|$

$u = x^2+1$

$du = 2x\,dx$

$\frac{1}{2}du = x\,dx$

Extra Problem:

1. $\displaystyle\int \frac{2x-3}{x^2-3x-10}\,dx$

3. $\displaystyle\int \frac{x^5+x^2+2}{x^3-x}\,dx$

3. $\displaystyle\int \frac{dx}{x^2-6x-7}$

4. $\displaystyle\int \frac{2x^2-10x+4}{(x+1)(x-3)^2}$

Improper Integral

We know that how to deal the definite integral. ~~But when~~ In the definite integral $\int_a^b f(x)\,dx$ where $[a, b]$ is a finite interval and that the limit that defines the integral exists, i.e., the function f is integrable.

But when we deal with infinite intervals or infinite discontinuities within the interval, then we say that type of integrals is improper integral.

e.g. $\displaystyle\int_1^{+\infty} \frac{1}{x^2}\,dx, \quad \int_1^{r} \frac{dx}{x-1}, \quad \int_0^{\pi} \tan x\,dx, \quad \int_{-\infty}^{+\infty} \frac{dx}{x^2-9}$

Example: Evaluate $\displaystyle\int_1^{+\infty} \frac{dx}{x^3}$

Sol^n

$$\int_1^{+\infty} \frac{dx}{x^3} = \lim_{a\to+\infty} \int_1^{a} \frac{dx}{x^3} = \lim_{a\to+\infty} \left. -\frac{1}{2x^2} \right|_1^a$$

$$= \lim_{a\to+\infty} \left(-\frac{1}{2a^2} + \frac{1}{2}\right)$$

$$= \lim_{a\to+\infty} \left(\frac{1}{2} - \frac{1}{2a^2}\right)$$

$$= \frac{1}{2}, \text{ converge.}$$

Example: Evaluate $\int_{-\infty}^{+\infty} \frac{dx}{1+x^2}$

Sol^n. $\int_{-\infty}^{+\infty} \frac{dx}{1+x^2} = \int_{-\infty}^{0} \frac{dx}{1+x^2} + \int_{0}^{+\infty} \frac{dx}{1+x^2}$

Now, $\int_{-\infty}^{0} \frac{dx}{1+x^2} = \lim_{b\to-\infty} \int_{b}^{0} \frac{dx}{1+x^2} = \lim_{b\to-\infty} \tan^{-1}x \Big|_{b}^{0}$

$= \lim_{b\to-\infty} [\tan^{-1}(0) - \tan^{-1} b]$

$= \lim_{b\to-\infty} (-\tan^{-1} b)$

$= \frac{\pi}{2}$

$\int_{0}^{+\infty} \frac{dx}{1+x^2} = \lim_{b\to+\infty} \int_{0}^{b} \frac{dx}{1+x^2} = \lim_{b\to+\infty} \tan^{-1}x \Big|_{0}^{b}$

$= \lim_{b\to+\infty} (\tan^{-1}b - \tan^{-1}0)$

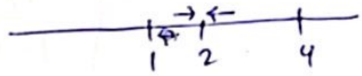
$= \lim_{b\to+\infty} (\tan^{-1} b)$

$= \frac{\pi}{2}$

Therefore, $\int_{-\infty}^{+\infty} \frac{dx}{1+x^2} = \frac{\pi}{2} + \frac{\pi}{2} = \pi$, converge.

※

Example: Evaluate $\cancel{\int_1^2 \frac{dx}{1-x}}$ $\int_1^4 \frac{dx}{(x-2)^{2/3}}$ $[1, \vec{2}]$ $[\overleftarrow{2}, 4]$



Sol^n: [since for $x=2$ the function doesn't define. So we separate the integral by two parts]

$$\int_1^4 \frac{dx}{(x-2)^{2/3}} = \int_1^2 \frac{dx}{(x-2)^{2/3}} + \int_2^4 \frac{dx}{(x-2)^{2/3}}$$

Now, $$\int_1^2 \frac{dx}{(x-2)^{2/3}} = \lim_{K\to 2^-} \int_1^K \frac{dx}{(x-2)^{2/3}} = \lim_{K\to 2^-} \left. \frac{(x-2)^{-2/3+1}}{-\frac{2}{3}+1} \right|_1^K$$

$$= \lim_{K\to 2^-} 3 \left. (x-2)^{1/3} \right|_1^K$$

$$= 3 \lim_{K\to 2^-} \left[(K-2)^{1/3} - (1-2)^{1/3}\right]$$

$$= 3 \cdot \left[0 - (-1)^{1/3}\right]$$

$$= 3 \cdot 1 = 3$$

and $$\int_2^4 \frac{dx}{(x-2)^{2/3}} = \lim_{K\to 2^+} \int_K^4 \frac{dx}{(x-2)^{2/3}} =$$

$$= 3 \lim_{K\to 2^+} \left[(4-2)^{1/3} - (K-2)^{1/3}\right]$$

$$= 3 \lim_{K\to 2^+} \left[2^{1/3} - (K-2)^{1/3}\right] = 3 \cdot 2^{1/3} - 0 = 3 \cdot 2^{1/3}$$

Therefore, $$\int_1^4 \frac{dx}{(x-2)^{2/3}} = 3 + 3 \cdot 2^{1/3}$$, converges.

Example: Evaluate $\int_1^2 \frac{dx}{1-x}$ $\quad [1, 2]$

Sol^n:

$$\int_1^2 \frac{dx}{1-x} = \lim_{K\to 1^+} \int_K^2 \frac{dx}{1-x} = \lim_{K\to 1^+} \ln|1-x|(-1)\Big|_K^2$$

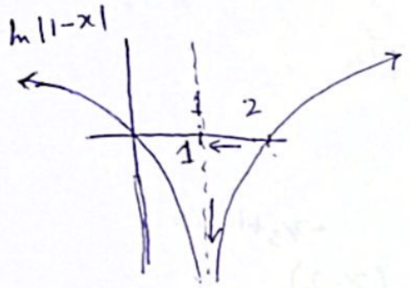
$$= \lim_{K\to 1^+} \left[-\ln|-1| + \ln|1-K|\right]$$

$\ln|-1| = \ln 1 = 0$

$$= \lim_{K\to 1^+} \ln|1-K|$$

$$= -\infty \text{ , diverges.}$$

$\ln|1-x|$



*** $\quad K \to 1^+$ means $K > 1$

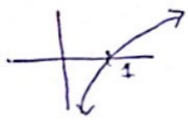
$1 - K < 0$

so, $|1-K| = -(1-K) = K-1$ $\quad |1-K| = 1-K$ $\quad 1-K \geq$

$-(1-K)$ $\quad 1-K<$

since natural log of a positive number close to zero is negative.

so, $\ln(K-1) \to -\infty$. ***



Extra Problem: Evaluate the following integral.

1. $\int_3^{+\infty} \frac{2}{x^2-1}\, dx$

2. $\int_{-\infty}^0 \frac{e^x\, dx}{3-2e^x}$

3. $\int_{-3}^1 \frac{x\, dx}{\sqrt{9-x^2}}$

4. $\int_0^1 \frac{dx}{(x-1)^{2/3}}$

5. $\int_{\pi/3}^{\pi/2} \frac{\sin x}{\sqrt{1-2\cos x}}\, dx$