Undergraduate Course in Mathematics

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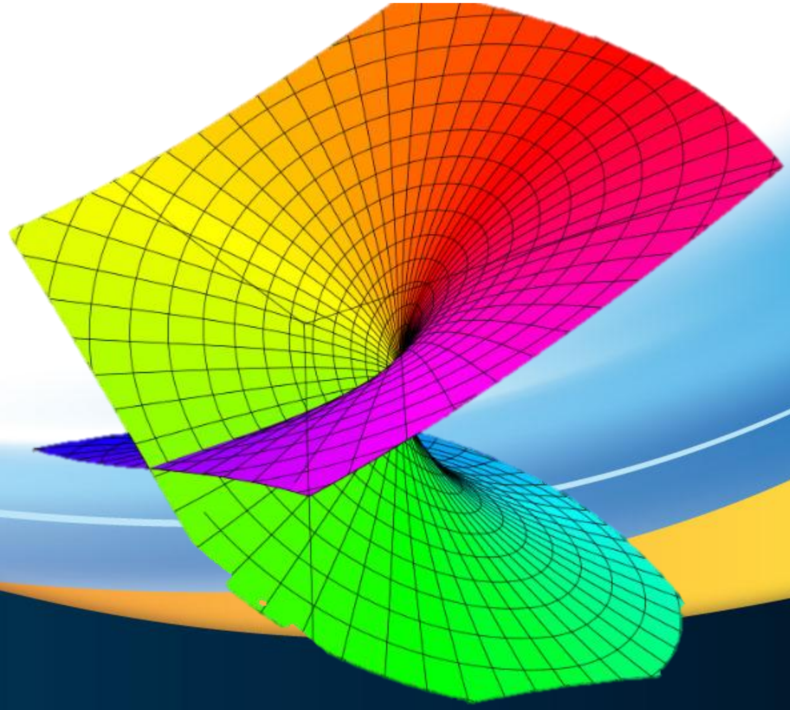
Complex Variables

Topic: Complex Valued Functions



Conducted By

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Real Valued and Complex Valued Functions

$$f(x) = \sqrt{x-2}$$

$$f(6) = \sqrt{6-2} = \sqrt{4} = 2$$

$$f(-2) = \sqrt{-4} = \text{undefined}$$

$$f(z) = \sqrt{z-2}$$

$$f(-2) = \sqrt{-2-2} = \sqrt{-4} = 2i$$

$$f(2i) = \sqrt{2i} = 1+i$$

Graph of Complex Valued Functions

Real

$f(x) = x^2$

$y = x^2$

2D

$f(z) = z^2$

$f(x+iy) = (x+iy)^2$

$= (x^2-y^2) + i(2xy)$

4D

2-Variable function

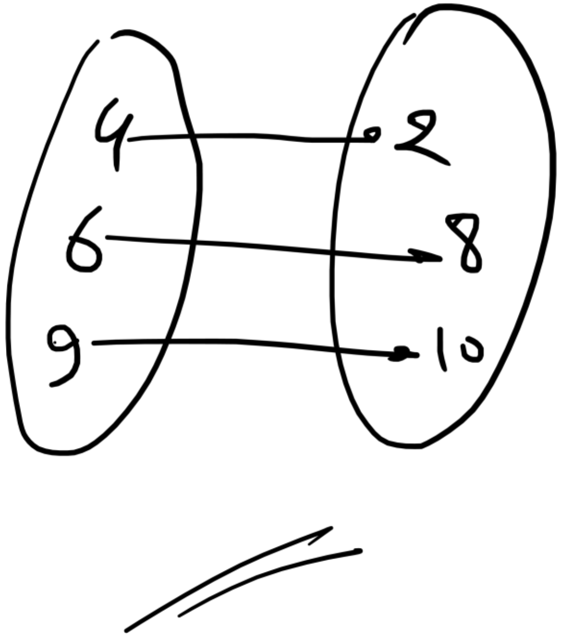
$f(x,y) = x^2 + y^2$

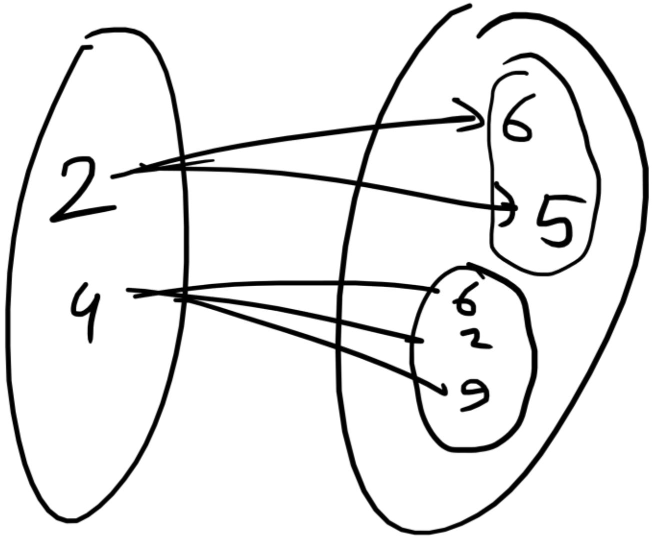
Re (Im → color)

$f(z) = z^2$

$f(x+iy) = (x^2-y^2) + i\,(2xy)$

Single Valued and Multi Valued Functions





Examples of some multi-valued Functions

- $f(z) = z^{\frac{1}{2}}$
- $f(z) = z^{\frac{1}{n}}$
- $f(z) = Arg(z)$
- $f(z) = \ln(z)$

$f(4) = 4^{\frac{1}{2}} = 2, -2$

$g(z) = z^{\frac{1}{3}}$

$g(1) = 1^{\frac{1}{3}} = 1, -\frac{1}{2}+\frac{\sqrt{3}i}{2}, -\frac{1}{2}-\frac{\sqrt{3}i}{2}$

Why some functions are multi-valued?

$z = re^{i\theta}$

single valued

$f(z) = z^2$

$4 = 4 \cdot e^{i(0+2n\pi)}$

$f(4) = \left[4 \cdot e^{i(2n\pi)}\right]^2$

$= 16 \cdot e^{i\,4n\pi}$

$= 16\left[\cos(4n\pi) + i\sin(4n\pi)\right]$

$n = 0, 1$

$f(z) = z^{\frac{1}{2}}$

$f(4) = \left[4e^{i\,2n\pi}\right]^{\frac{1}{2}}$

$= 2 \cdot e^{i\,n\pi}$

$= 2\left[\cos n\pi + i \sin n\pi\right]$

$n = 0, 1$

2 -2

$f(z) = \ln(z)$

$f(-1) = \ln\left(1 \cdot e^{i(\pi+2n\pi)}\right)$

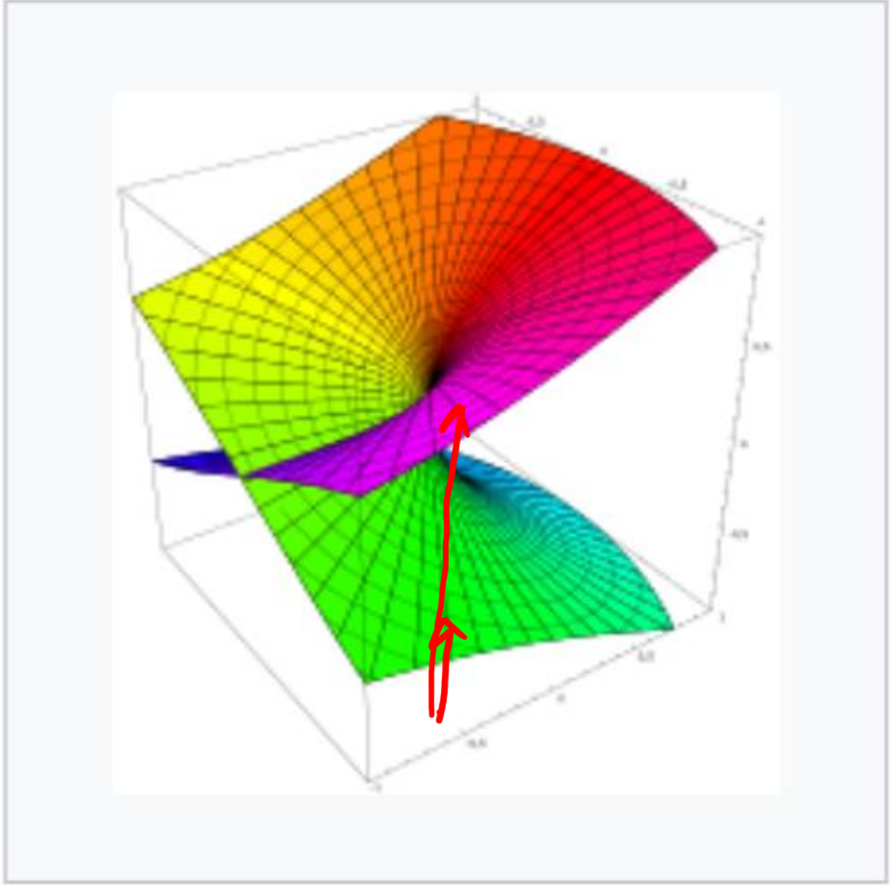
$= i(\pi+2n\pi) \cdot \ln(e)$

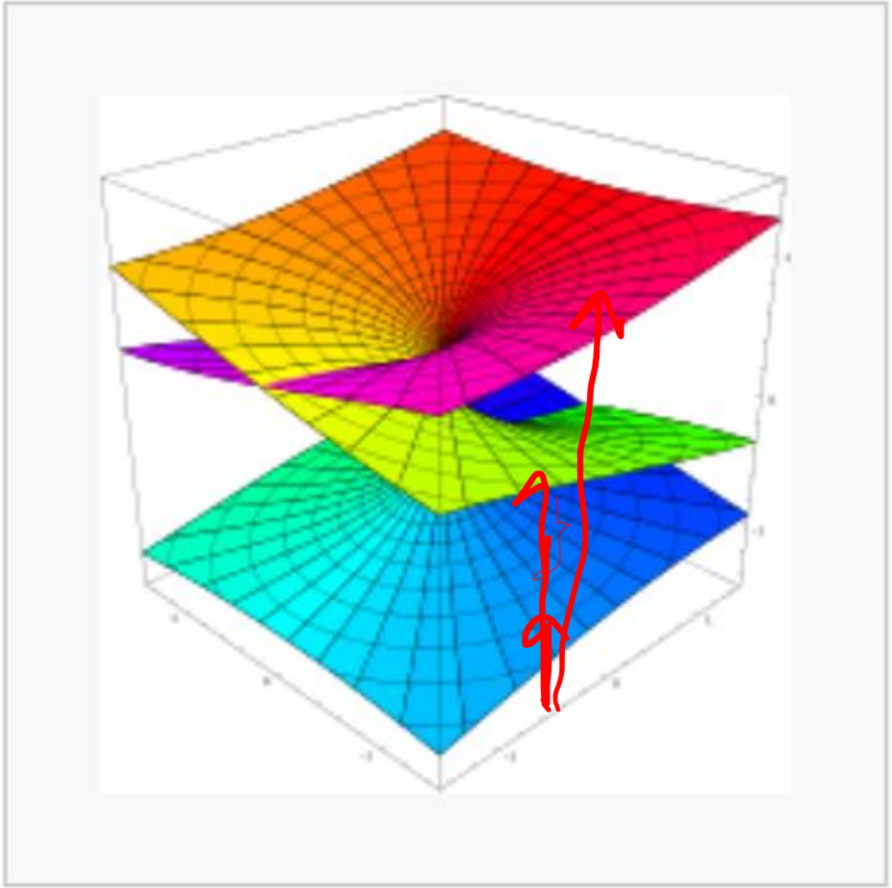
$= i(\pi+2n\pi)$

$n=0$	$n=1$	$n=2$	
$i\pi$	$3i\pi$	$5i\pi$	, - - -

Branch

Riemann Surface





$f(z) = z^{1/2}$ $\quad$ $f(z) = z^{1/3}$

Some Basic Complex Valued Functions

Algebraic $\longrightarrow z^{\frac{1}{2}}, z^2, z^3, \ldots$

Trigonometric (circular) $\longrightarrow \sin z, \cos z, \tan z$

Inverse circular $\longrightarrow \sin^{-1} z$

hyperbolic functions $\longrightarrow \sinh z$

log function

The Log Function

$$\log_2 16$$

$$= 4$$

Mul	Add
1	0
2	1
4	2
8	3
16	4
32	5
64	6
128	7

$$\ln(z) = \log_e z$$

Scientific calculator

$\ln \longrightarrow \log_e$

$\log \longrightarrow \log_{10}$, $\log_e$, $\log_2$

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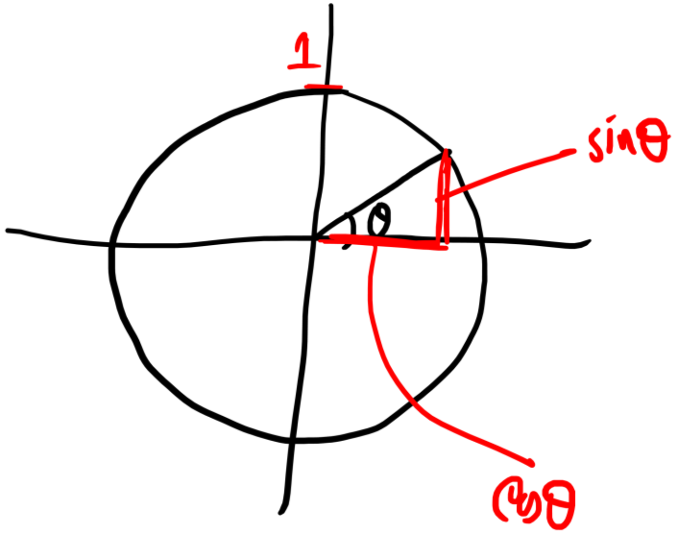
Circular Function

$$e^{i\theta} = \cos\theta + i\sin\theta$$

$$e^{-i\theta} = \cos\theta - i\sin\theta$$



Adding $2\cos\theta = e^{i\theta} + e^{-i\theta}$

$$\therefore \boxed{\cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2}}$$

Subtraction

$$2i\sin\theta = e^{i\theta} - e^{-i\theta}$$

$$\therefore \boxed{\sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i}}$$

$$\tan\theta = \frac{\sin\theta}{\cos\theta}$$

$$= \frac{\frac{e^{i\theta} - e^{-i\theta}}{\cancel{2}i}}{\frac{e^{i\theta} + e^{-i\theta}}{\cancel{2}}}$$

$$= \frac{1}{i} \frac{e^{i\theta} - e^{-i\theta}}{e^{i\theta} + e^{-i\theta}} \checkmark$$

$$x^2 + y^2 = 1$$

$$\left.\begin{aligned} x &= \cos\theta \\ y &= \sin\theta \end{aligned}\right\}$$

$$\underline{\underline{\cos^2\theta + \sin^2\theta = 1}}$$

Hyperbolic Function

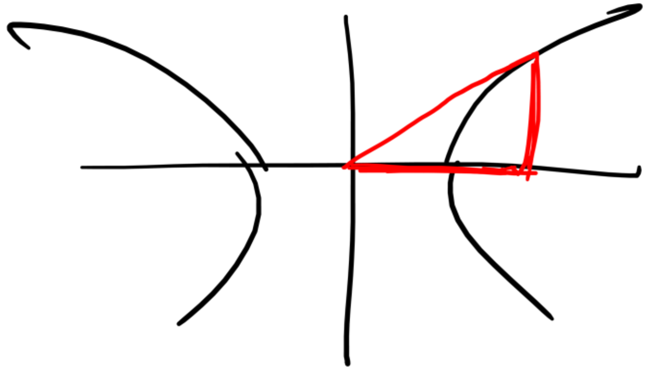
$$\cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2}$$

$$\sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i}$$

$$\boxed{\cosh\theta = \frac{e^{\theta} + e^{-\theta}}{2}}$$

$$\boxed{\sinh\theta = \frac{e^{\theta} - e^{-\theta}}{2}}$$

$$\int \frac{dx}{\sqrt{x^2-a^2}}$$



$$x^2 - y^2 = 1$$

$$\left.\begin{aligned} x &= \cosh\theta \\ y &= \sinh\theta \end{aligned}\right\}$$

$$\cosh^2\theta - \sinh^2\theta = 1$$

$\cosh^2\theta - \sinh^2\theta = 1$

$$\int \frac{dx}{\sqrt{x^2-a^2}}$$

$x = a\cosh\theta$

$dx = a\sinh\theta\, d\theta$

$$= \int \frac{\cancel{a\sinh\theta}\, d\theta}{\cancel{a\sinh\theta}}$$

$$= \int d\theta = \theta + c$$

$$= \cosh^{-1}\frac{x}{a} + c.$$

$x^2 - a^2$

$= (a\cosh\theta)^2 - a^2$

$= a^2(\cosh^2\theta - 1)$

$= a^2\sinh^2\theta$

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Inverse Circular Function

$$\cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2}$$

$$\cos^{-1} z = \log\left(\quad \right)$$

Show that:

(a) $\sin^{-1}(z) = -i \ln\left(iz \pm \sqrt{1-z^2}\right)$

(b) ~~$\cot^{-1}(z) = \frac{1}{2i} \ln\left(\frac{z+i}{z-i}\right)$~~

let $\sin^{-1} z = \theta$

$\Rightarrow z = \sin\theta$

$\Rightarrow z = \dfrac{e^{i\theta} - e^{-i\theta}}{2i}$

$\Rightarrow e^{i\theta} - e^{-i\theta} = 2iz$ ---- (1)

$$\left(e^{i\theta} + e^{-i\theta}\right)^2 = \left(e^{i\theta} - e^{-i\theta}\right)^2 + 4 \cdot e^{i\theta} \cdot e^{-i\theta}$$

$$= (2iz)^2 + 4$$

$$= 4 - 4z^2$$

$\therefore e^{i\theta} + e^{-i\theta} = \pm\sqrt{4-4z^2}$

$\therefore e^{i\theta} + e^{-i\theta} = \pm 2\sqrt{1-z^2}$ ---- (2)

Adding ① and ②.

$$2e^{i\theta} = 2iz \pm 2\sqrt{1-z^2}$$

$$\Rightarrow e^{i\theta} = iz \pm \sqrt{1-z^2}$$

$$\Rightarrow \ln\left(e^{i\theta}\right) = \ln\left(iz \pm \sqrt{1-z^2}\right)$$

$$\Rightarrow i\theta = \ln\left(iz \pm \sqrt{1-z^2}\right)$$

$$\theta = \frac{1}{i}\ln\left(iz \pm \sqrt{1-z^2}\right)$$

$$= -i\ln\left(iz \pm \sqrt{1-z^2}\right)$$

$$\therefore \sin^{-1} z = -i\ln\left(iz \pm \sqrt{1-z^2}\right)$$

$\sin^{-1} z, \cos^{-1} z, \sec^{-1} z, \operatorname{cosec}^{-1} z$

HW

Show that:

(a) ~~$\sin^{-1}(z) = -i\ln\left(iz + \sqrt{1-z^2}\right)$~~

(b) $\cot^{-1}(z) = \frac{1}{2i}\ln\left(\frac{z+i}{z-i}\right)$

let $\cot^{-1} z = \theta$

$\Rightarrow z = \cot\theta$

$\Rightarrow z = \dfrac{\cos\theta}{\sin\theta}$

$= \dfrac{\frac{e^{i\theta}+e^{-i\theta}}{2}}{\frac{e^{i\theta}-e^{-i\theta}}{2i}}$

$\therefore z = i\,\dfrac{e^{i\theta}+e^{-i\theta}}{e^{i\theta}-e^{-i\theta}}$

$\Rightarrow \dfrac{z}{i} = \dfrac{e^{i\theta}+e^{-i\theta}}{e^{i\theta}-e^{-i\theta}}$

$\Rightarrow \dfrac{z+i}{z-i} = \dfrac{e^{i\theta}+\cancel{e^{-i\theta}}+e^{i\theta}-\cancel{e^{-i\theta}}}{\cancel{e^{i\theta}}+e^{-i\theta}-\cancel{e^{i\theta}}+e^{i\theta}}$

$$\Rightarrow \frac{z+i}{z-i} = \frac{\cancel{2}e^{i\theta}}{\cancel{2}e^{-i\theta}}$$

$$\Rightarrow e^{2i\theta} = \frac{z+i}{z-i}$$

$$\Rightarrow \ln\left(e^{2i\theta}\right) = \ln\left(\frac{z+i}{z-i}\right)$$

$$\Rightarrow 2i\theta = \ln\left(\frac{z+i}{z-i}\right)$$

$$\theta = \frac{1}{2i}\ln\left(\frac{z+i}{z-i}\right)$$

$$\therefore \cot^{-1} z = \frac{1}{2i}\ln\left(\frac{z+i}{z-i}\right)$$

find an expression for $\operatorname{sech}^{-1} z$

let $\operatorname{sech}^{-1} z = \theta$

$\Rightarrow z = \operatorname{sech}\theta$

$\Rightarrow z = \dfrac{1}{\cosh\theta}$

$z = \dfrac{1}{\frac{e^{\theta}+e^{-\theta}}{2}}$

$\Rightarrow \dfrac{e^{\theta}+e^{-\theta}}{2} = \dfrac{1}{z}$

$\Rightarrow e^{\theta}+e^{-\theta} = \dfrac{2}{z}$ ——— (1)

$$\left(e^{\theta}-e^{-\theta}\right)^2 = \left(e^{\theta}+e^{-\theta}\right)^2 - 4\cdot e^{\theta}\cdot e^{-\theta}$$

$$= \left(\frac{2}{z}\right)^2 - 4$$

$$= \frac{4}{z^2} - 4$$

$$= \frac{4-4z^2}{z^2}$$

$$e^{\theta}-e^{-\theta} = \pm\sqrt{\frac{4-4z^2}{z^2}}$$

$$e^{\theta}-e^{-\theta} = \pm\frac{2\cdot\sqrt{1-z^2}}{z}$$

(2)

Adding ① and ②,

$$2e^{i\theta} = \frac{2}{z} \pm \frac{2\sqrt{1-z^2}}{z}$$

$$\Rightarrow e^{i\theta} = \frac{1 \pm \sqrt{1-z^2}}{z}$$

$$\Rightarrow i\theta = \ln\left(\frac{1 \pm \sqrt{1-z^2}}{z}\right)$$

$$\theta = \frac{1}{i} \ln\left(\frac{1 \pm \sqrt{1-z^2}}{z}\right)$$

$$\operatorname{sech}^{-1} z = -i \cdot \ln\left(\frac{1 \pm \sqrt{1-z^2}}{z}\right)$$

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