Lecture 11

First-Order Linear Equation Continue...

We alread Know that the Linear (Differential) Equation of the form

$$\boxed{\frac{dy}{dx} + P(x)y = Q(x)}$$

On the right side there has no function of y.

Now, if there also has a function of y then we use another ~~method~~ form for solving ODEs. That Equation is called Bernoulli's equation.

The Bernoulli's eqn is

$$\boxed{\frac{dy}{dx} + P(x)y = Q(x)\,y^n}\,,$$ where n is any real number.

if $n=0$ then the eqn becomes linear.

Solving Method:

1. First divided y^n on both sides i.e. always right side free from y-variable.
2. Substitute $u = y^{1-n}$ and use chain rule. ie.

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}$$

3. Make the ODEs as $\frac{du}{dx}$ linear form.
4. Find Integrating Factor, I.F.
5. Use linear method for solving DEs.

<u>Example:</u> Solve $x\frac{dy}{dx}+y=x^2y^2$ Not a linear form.

<u>Sol^n.</u> We first rewrite the eq^n,

$$\frac{dy}{dx}+\frac{1}{x}y=xy^2$$

$$\frac{1}{y^2}\frac{dy}{dx}+\frac{1}{xy}=x \quad \text{———} \quad (1)$$

let $\frac{1}{y}=u \Rightarrow y^{-1}=u \Rightarrow$ ~~or~~ $y=\frac{1}{u}$

$$\cancel{-1\,dy}$$

$$-1.y^{-2}\frac{dy}{du}=1$$

$$\frac{1}{y^2}\frac{dy}{du}=-1$$

Now, $\frac{dy}{dx}=\frac{dy}{du}\frac{du}{dx}=-y^2\frac{du}{dx}=-\frac{1}{u^2}\frac{du}{dx}$ $\quad \therefore \cancel{y^{-2}\frac{dy}{dx}=}$

$$\therefore \frac{1}{y^2}\frac{dy}{dx}=-\frac{du}{dx}$$

From (1) becomes

$$-\frac{du}{dx}+\frac{1}{x}u=x$$

$$\frac{du}{dx}-\frac{1}{x}u=-x \qquad \text{which is linear form.}$$

$$\text{I.F.}=e^{-\int\frac{1}{x}dx}=e^{-\ln(x)}=e^{\ln(x^{-1})}=x^{-1} \qquad x\in(0,\infty)$$

Now $\frac{1}{x}\frac{du}{dx}-\frac{1}{x^2}u=-1$

$$\frac{d}{dx}(x^{-1}u)=-1$$

$$\int\frac{d}{dx}(x^{-1}u)\,dx=-\int dx=-x+C$$

$$x^{-1}u=-x+C$$

$$u=-x^2+Cx \Rightarrow y=\frac{1}{-x^2+Cx}. \quad , y=\frac{1}{u}$$

$\times$

Example: Solve $\frac{dy}{dx} + 2xy + xy^4 = 0$

Sol^n: We can rewrite the eq^n.

$$\frac{dy}{dx} + 2xy = -xy^4$$

$$\frac{1}{y^4}\frac{dy}{dx} + 2x\frac{1}{y^3} = -x \quad \text{---} (1)$$

Let $y^{-3} = u$

$$-3y^{-4}\frac{dy}{du} = 1$$

$$\frac{1}{y^4}\frac{dy}{du} = -\frac{1}{3}y^4$$

Now, $\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx} = -\frac{1}{3}y^4\frac{du}{dx}$

$$\frac{1}{y^4}\frac{dy}{dx} = -\frac{1}{3}\frac{du}{dx}$$

Eq^n (1) becomes.

$$-\frac{1}{3}\frac{du}{dx} + 2xu = -x$$

$$\frac{du}{dx} - 6xu = 3x \qquad \text{[Linear form]}$$

$$\text{I.F.} = e^{-\int 6x\,dx} = e^{-3x^2}$$

Therefore, $e^{-3x^2}\frac{du}{dx} - 6xe^{-3x^2}u = 3xe^{-3x^2}$

$$\frac{d}{dx}\left(e^{-3x^2}u\right) = 3xe^{-3x^2}$$

$$ue^{-3x^2} = \int 3xe^{-3x^2}dx = -\frac{1}{2}e^{-3x^2} + c$$

$$\Rightarrow \frac{1}{y^3} = -\frac{1}{2} + ce^{3x^2}$$

Extra Problem:

① Solve: (a) $\frac{dy}{dx} + \frac{1}{3}y = \frac{1}{3}(1-2x)y^4$

(b) $\frac{dy}{dx} + y = y^2(\cos x - \sin x)$

(c) $x\,dy - \{y + xy^3(1+\ln x)\}\,dx = 0$

(d) $\frac{dr}{d\theta} + r\sec\theta = \cos\theta$

(e) $x\frac{dy}{dx} + (3x+1)y = e^{-3x}$

② Solve the given initial-value problem.

(a) $xy' + y = e^x, \quad y(1) = 2$

(b) $(x+1)\frac{dy}{dx} + y = \ln x, \quad y(1) = 10$

(c) $\frac{dT}{dt} = k(T - T_m)\,; \quad T(0) = T_0,$

where k, T_m and T_0 constants.

Exact Equations

Definition:

A differential expression $M(x,y)\,dx + N(x,y)\,dy$ is an exact differential in a region $\mathbb{R}$ of the xy-plane if it corresponds to the differential of some function $f(x,y)$ defined in $\mathbb{R}$.

A first-order differential equation, called exact equation, form

$$M(x,y)\,dx + N(x,y)\,dy = 0$$

Criterion for an Exact Differential

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Say, $M(x,y) = \dfrac{\partial f}{\partial x}$ and $N(x,y) = \dfrac{\partial f}{\partial y}$

so, $\dfrac{\partial M}{\partial y} = \dfrac{\partial f}{\partial x \partial y}$, $\dfrac{\partial N}{\partial x} = \dfrac{\partial f}{\partial y \partial x}$

Method for Solution:

1. Make the eqn. as this form $M(x,y)\,dx + N(x,y)\,dy = 0$
2. Introducing $\dfrac{\partial f}{\partial x} = M(x,y)$ $\left[\dfrac{\partial f}{\partial y} = N(x,y)\right]$
3. Integrating step 2

$$f(x,y) = \int M(x,y)\,dx + g(y) \quad \text{——— (1)}$$

here $g(y)$ is the arbitrary function we treat it as a "constant" of integration.

4. Differentiate preceding step 3 w.r.to. y

$$\frac{\partial f}{\partial y} = \frac{\partial}{\partial y}\int M(x,y)\,dx + g'(y) = N(x,y)$$

⑤ Now equation on both side, we will get

$$g'(y) = u(y) \quad \text{———} \text{②}$$

⑥ Then integrating eqn ② we will get $g(y)$.

⑦ The solution of the eqn in implicit form is

$$f(x,y) = c.$$

<u>Example:</u> Solve $2xy\,dx + (x^2-1)\,dy = 0$

<u>Soln.</u> $M(x,y) = 2xy \quad , N(x,y) = x^2-1$

$$\frac{\partial M}{\partial y} = 2x \qquad \frac{\partial N}{\partial x} = 2x$$

$$\therefore \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} \quad \text{exact.}$$

Now, $\frac{\partial f}{\partial x} = 2xy$ and $\frac{\partial f}{\partial y} = x^2-1$

$$f(x,y) = 2\frac{x^2}{2}y + g(y)$$

$$\frac{\partial}{\partial y}f(x,y) = x^2\frac{\partial}{\partial y}y + g'(y) = x^2 + g'(y) = x^2-1$$

Equating both sides,

$$g'(y) = -1$$

$$g(y) = -y$$

Therefore, $f(x,y) = x^2y - y$

So the Soln. of the exact eqn is

$$x^2y - y = c$$

$$y = \frac{c}{x^2-1}$$

Example: Solve $(2x^3+3y)\,dx + (3x+y-1)\,dy = 0$.

Soln. Say $M(x,y) = 2x^3+3y$ and $N(x,y) = 3x+y-1$

$$\frac{\partial M}{\partial y} = 3 \qquad \frac{\partial N}{\partial x} = 3$$

$$\therefore \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} \text{ which is exact.}$$

Now

$$\frac{\partial f}{\partial x} = M(x,y) = 2x^3+3y^2 \qquad \frac{\partial f}{\partial y} = N(x,y) = 3x+y-1$$

Integrating w.r.to x

$$f(x,y) = 2.\frac{x^4}{4} + 3y^2x + g(y)$$

$$= \frac{1}{2}x^4 + 3xy^2 + g(y) \quad \text{———} (1)$$

$$\frac{\partial f}{\partial y} = 6xy + g'(y) = 3x+y-1$$

Equating both sides. $g'(y) = y-1$

$$g(y) = \frac{y^2}{2} - y$$

Eqn (1) becomes,

$$f(x,y) = \frac{1}{2}x^4 + 3xy^2 + \frac{1}{2}y^2 - y = c_1$$

$$x^4 + 6xy^2 + y^2 - 2y = c$$