Lecture : 09

Differential Equation

A differential equation is any equation which contains derivatives, either ordinary derivatives or partial derivatives.

For example: A few some examples of differential equations:

$$ay'' + by' + cy = f(t) \quad \text{——— (1)}$$

$$p\frac{d^2y}{dx^2} = (1-y)\frac{dy}{dx} + e^{-y} \quad \text{——— (2)}$$

$$y^{(3)} + 10y'' - 2y' + 2y = \sin(t) \quad \text{——— (3)}$$

$$\alpha^2 \frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t} \qquad \text{Heat eq}^n \quad \text{——— (4)}$$

$$\alpha^2 u_{xx} = u_{tt} \qquad \text{Wave eq}^n \quad \text{——— (5)}$$

$$\frac{\partial^3 u}{\partial x^2 \partial t} = 1 + \frac{\partial u}{\partial y} \quad \text{——— (6)}$$

Here eq^n 1, 2, 3, 5 are second order differential equations. Eq^n 3 and 6 are 3rd order differential equations.

Eq^n 1–3 are ordinary differential equations where as 4–6 are partial differential equations.

Solution of DEs.

When we try to solve the differential eqn. it means that a differential eqn has a solution(s) curve and we use "a" solution rather than "the" solution. The indefinite article "a" is used deliberately to suggest the possibility that other solutions may exist.

TWO fundamental questions arise in considering an initial-value problem.

Does a solution of the problem exist?

If a solution exists, is it unique?

$\boxed{\text{1. Existence}}$

$\boxed{\text{2. Uniqueness}}$

Example: Suppose $x = c_1 \cos 4t + c_2 \sin 4t$ is a two-parameter family of solutions of $x'' + 16x = 0$. Find ~~the~~ a solution of the initial-value problem. $x(\frac{\pi}{2}) = -2$, $x'(\frac{\pi}{2}) = 1$.

Solr. Here $x(\pi/2) = -2$ so,

$$-2 = c_1 \cos(4\,\pi/2) + c_2 \sin(4\,\pi/2) = c_1 \cos(2\pi) + c_2 \sin(2\pi)$$

$$\Rightarrow c_1 \cos(2\pi) = -2 \Rightarrow c_1 = -2$$

Now, $x' = -c_1 \sin 4t \cdot 4 + c_2 \cos 4t \cdot 4$

$$x'(\pi/2) = -4c_1 \sin(2t) + 4c_2 \cos(2t)$$

$$1 = 4c_2 \cdot 1 \Rightarrow c_2 = \frac{1}{4}$$

Hence, $x = -2\cos(4t) + \frac{1}{4}\sin(4t)$ is a solⁿ of $x'' + 16x = 0$.

You can solve ODEs by three way:

1. Analytical (Mathematical solⁿ)
2. Qualitative (Using computer software/graphing)
3. Numerical (Using computer program) by

Ordinary differential equations Classification

(i) Autonomous e.g. $\frac{dy}{dx} = \underbrace{1+y^2}_{f(y)}$

(ii) Nonautonomous e.g. $\frac{dy}{dx} = \underbrace{0.2\,xy}_{f(x,y)}$

First-order Differential eqⁿ Solution Method

1. Separable Variables
2. Linear Equations
3. Exact Equations

Separable Variables

A first-order differential equation of the form

$$\frac{dy}{dx} = g(x)\,h(y)$$

is said to be separable or to have separable variables.

Method of solution

Step 1: Separate the ~~vab~~ variables with their ~~pro~~ displacement.

Step 2: Integrate both sides.

Step 3: If there mention initial value, then put the initial ~~ial~~ value on the corresponding variable. then find out the ~~co~~ value of the constant.

Example: Solve: $(1+x)\,dy - y\,dx = 0$

Sol^n^: $(1+x)\,dy - y\,dx = 0$

$$(1+x)\,dy = y\,dx$$

$$\frac{1}{y}\,dy = \frac{1}{1+x}\,dx \qquad \left[\text{Dividing by } (1+x)y \text{ on both sides}\right]$$

$$\int \frac{1}{y}\,dy = \int \frac{1}{1+x}\,dx$$

$$\ln|y| = \ln|1+x| + c_1$$

$$|y| = e^{\ln|1+x| + c_1} = e^{\ln|1+x|} \cdot e^{c_1} = |1+x|\,e^{c_1}$$

$$y = \pm c\,|1+x| = \pm c\,(1+x)$$

Example: Solve the initial-value problem $\frac{dy}{dx} = -\frac{x}{y}$, $y(4) = -3$.

Sol^n:

$$\frac{dy}{dx} = -\frac{x}{y}$$

$$y \cdot dy = -x\, dx$$

$$\int y\, dy = -\int x\, dx$$

$$\frac{y^2}{2} = -\frac{x^2}{2} + c_1$$

$$y^2 = -x^2 + 2c_1 \Rightarrow y^2 = -x^2 + c$$

$$\cancel{x^2+y^2=c}$$

Now, $(-3)^2 = -4^2 + c \Rightarrow 9 + 16 = c \Rightarrow c = 25$

Hence the sol^n is $y^2 = -x^2 + 25 = 25 - x^2$

$$y = \pm\sqrt{25 - x^2}, \quad -5 < x < 5$$

~~explicit solution of the given initial-value problem.~~

Example; Find the explicit solution of the given initial-value problem.

$$\frac{dx}{dt} = 4(x^2+1) \quad , \quad x(\pi/4) = 1$$

Solⁿ:

$$\frac{dx}{dt} = 4(x^2+1)$$

$$\frac{dx}{1+x^2} = 4\,dt$$

$$\int \frac{dx}{1+x^2} = 4\int dt$$

$$\tan^{-1}(x) = 4t + c$$

$$x = \tan(4t+c)$$

$$\begin{cases} \sin(\pi+c) = -\sin c \\ \cos(\pi+c) = -\cos c \end{cases}$$

$t = \frac{\pi}{4}, x = 1$

So, $1 = \tan\left(4 \cdot \frac{\pi}{4} + c\right)$

$$\tan(\pi + c) = 1$$

$$\tan c = 1$$

$$c = \tan^{-1}(1) = \frac{\pi}{4}$$

Hence $x = \tan\left(4t + \frac{\pi}{4}\right)$.

Extra problem:

Book: Differential Equations — Dennis G. Zill

page: 50

Exercise: 2.2

Problem: 3, 6, 7, 8, 13, 18, 25, 27