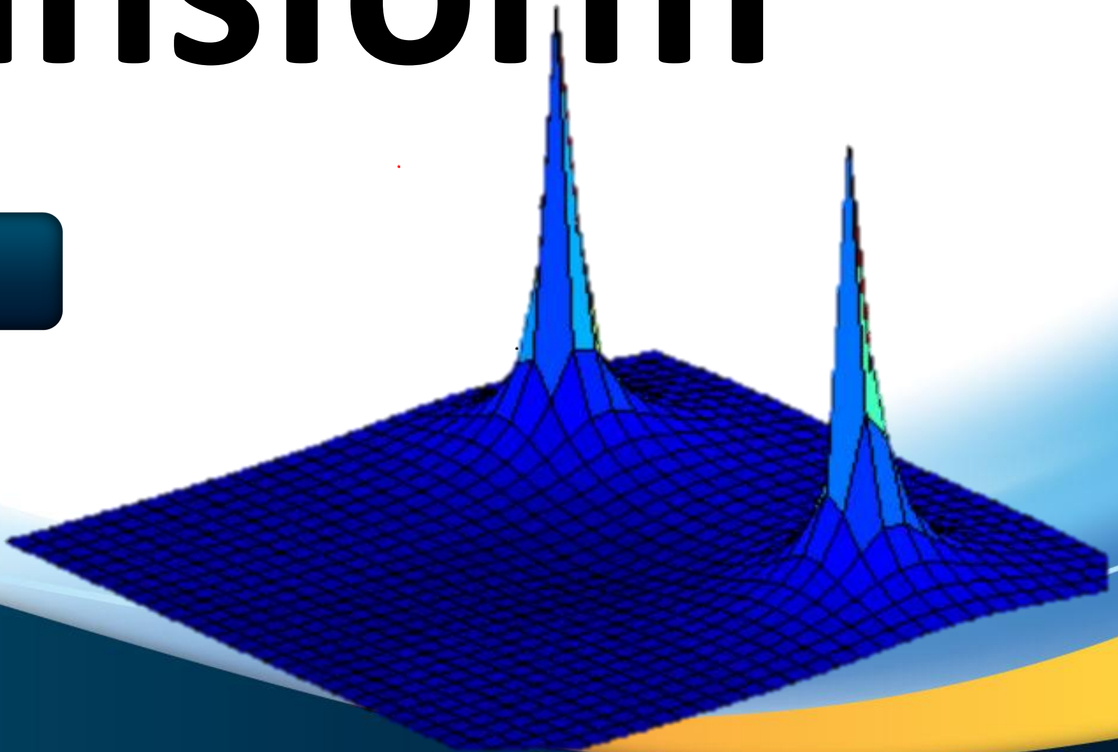
Undergraduate Course in Mathematics

Laplace Transform

Basic Laplace Transformations



Conducted By

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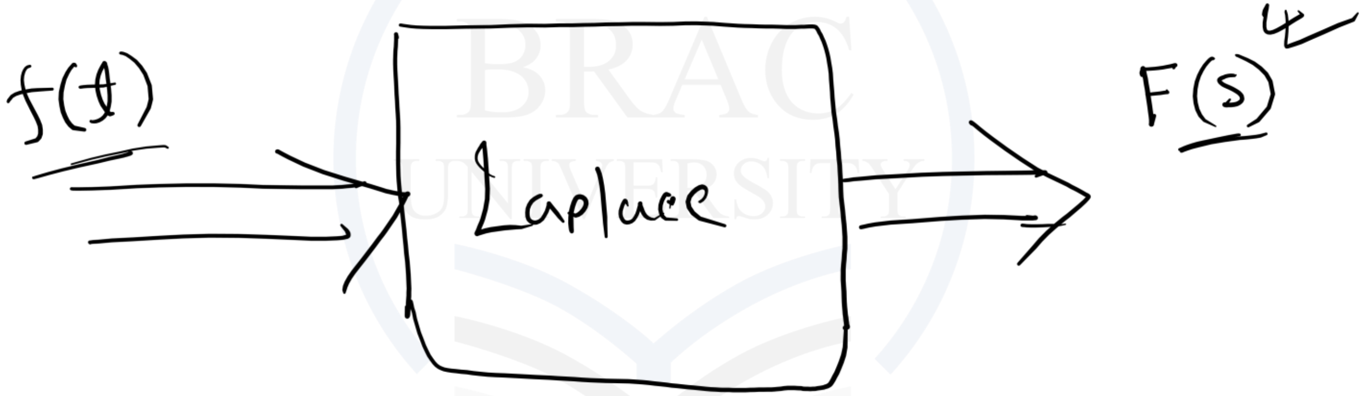

$$F(s) = \int_0^\infty f(t)\, e^{-st}\, dt$$

Formal Definition of the Laplace Transform

Let $f(t)$ be a function defined for $t \geq 0$. Then the integral

$$F(s) = \mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} f(t)\,dt$$

is said to be the Laplace transform of f, provided that the integral converges. Where $\boldsymbol{s}$ is a complex number.

$$\mathcal{L}\{f(t)\} =$$

Transforms of some Algebraic and Exponential functions

- $\mathcal{L}\{1\} = \frac{1}{s}$ for $Re(s) > 0$
- $\mathcal{L}\{t\} = \frac{1}{s^2}$ for $Re(s) > 0$
- $\mathcal{L}\{t^n\} = \frac{n!}{s^{n+1}}$, where n is a non-negative integer, for $Re(s) > 0$
- $\mathcal{L}\{t^n\} = \frac{\Gamma(n+1)}{s^{n+1}}$, when n is not an integer, for $Re(s) > 0$
- $\mathcal{L}\{e^{at}\} = \frac{1}{s-a}$ for $Re(s) > a$

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$$\int uv\,dt = u\int v\,dt - \int\left(\frac{du}{dt}\int v\,dt\right)dt$$

$$e^{-50} = \frac{1}{e^{50}} \approx 0$$

$$e^{-\text{big}} = \text{small}$$

$$e^{+\infty} = \infty$$

$$e^{-\infty} \to 0$$

Using definition show that

$$\mathcal{L}\{t\} = \frac{1}{s^2} \text{ for } Re(s) > 0$$

$$\mathcal{L}\{f(t)\} = \int_0^\infty f(t)\, e^{-st}\, dt$$

$$\mathcal{L}\{t\} = \int_0^\infty t \cdot e^{-st}\, dt$$

$$= \left[t \cdot \frac{e^{-st}}{-s} \right]_0^\infty - \int_0^\infty 1 \cdot \frac{e^{-st}}{-s}\, dt$$

$$= 0 - 0 + \frac{1}{s} \int_0^\infty e^{-st}\, dt$$

$$= \frac{1}{s} \cdot \left[\frac{e^{-st}}{-s} \right]_0^\infty = \frac{1}{s}\left[0 - \frac{e^{-s \cdot 0}}{-s} \right] = \frac{1}{s^2}.$$

Using definition show that

$$\mathcal{L}\{e^{at}\} = \frac{1}{s-a} \text{ for } Re(s) > a$$

$$\mathcal{L}\{f(t)\} = \int_0^\infty f(t)\, e^{-st}\, dt$$

$$\mathcal{L}\{e^{at}\} = \int_0^\infty e^{at} \cdot e^{-st}\, dt$$

$$= \int_0^\infty e^{-(s-a)t}\, dt$$

$$= \left[\frac{e^{-(s-a)t}}{-(s-a)} \right]_0^\infty$$

$$= 0 - \frac{e^{-(s-a)\cdot 0}}{-(s-a)} = \frac{1}{s-a}.$$

Transforms of some Trigonometric and Hyperbolic functions

- $\mathcal{L}\{\sin at\} = \dfrac{a}{s^2+a^2}$ for $Re(s) > |a|$
- $\mathcal{L}\{\cos at\} = \dfrac{s}{s^2+a^2}$ for $Re(s) > |a|$
- $\mathcal{L}\{\sinh at\} = \dfrac{a}{s^2-a^2}$ for $Re(s) > |a|$
- $\mathcal{L}\{\cosh at\} = \dfrac{s}{s^2-a^2}$ for $Re(s) > |a|$

Using definition show that

$$\mathcal{L}\{\cos at\} = \frac{s}{s^2+a^2} \quad \text{for } Re(s) > |a|$$

$$\mathcal{L}\{\cos at\} = \int_0^\infty \cos at \cdot e^{-st}\, dt$$

$$I = \int_0^\infty \cos at \cdot e^{-st}\, dt$$

$$= \left[\cos at \cdot \frac{e^{-st}}{-s}\right]_0^\infty - \int_0^\infty (-a)\sin at \, \frac{e^{-st}}{-s}\, dt$$

$$= 0 - \cos(0)\frac{e^{-0}}{-s} - \frac{a}{s}\int_0^{\infty} \sin at \; e^{-st}\, dt$$

$$= \frac{1}{s} - \frac{a}{s}\left\{ \left[\sin at \cdot \frac{e^{-st}}{-s}\right]_0^{\infty} - \int_0^{\infty} a \cos at \cdot \frac{e^{-st}}{-s}\, dt \right\}$$

$$= \frac{1}{s} - \frac{a}{s}\left\{ 0 - \cancel{\sin(0)}\frac{e^{-0}}{-s} + \frac{a}{s}\int_0^{\infty} \cos at \; e^{-st}\, dt \right\}$$

$$I = \frac{1}{s} - \frac{a^2}{s^2}\int_0^{\infty} \cos at \cdot e^{-st}\, dt \quad \Rightarrow \quad I = \frac{1}{s} - \frac{a^2}{s^2} I$$

$$\Rightarrow \quad s^2 I = s - a^2 I$$

$$\Rightarrow (s^2+a^2) I = s$$

$$\Rightarrow I = \frac{s}{s^2+a^2}$$

$$\therefore \mathcal{L}\{\cos at\} = \frac{s}{s^2+a^2}$$

Using definition show that

$$\mathcal{L}\{\sin at\} = \frac{a}{s^2+a^2} \quad \text{for } Re(s) > |a|$$

$$\mathcal{L}\{\sin at\} = \int_0^\infty \sin at \; e^{-st}\, dt$$

$$I = \int_0^\infty \sin at \; e^{-st}\, dt$$

$$= \left[\sin at \cdot \frac{e^{-st}}{-s}\right]_0^\infty - \int_0^\infty a \cos at \cdot \frac{e^{-st}}{-s}\, dt$$

$$= 0 - 0 - \frac{a}{s}\int_0^\infty \cos at \; e^{-st}\, dt$$

$$= \frac{a}{s}\left\{ \left[\cos at \cdot \frac{e^{-st}}{-s}\right]_0^\infty - \int_0^\infty (-a)\sin at \, \frac{e^{-st}}{-s}\, dt \right\}$$

$$= \frac{a}{s}\left\{ 0 - 1 \cdot \frac{1}{-s} - \frac{a}{s}\int_0^\infty \sin at \; e^{-st}\, dt \right\}$$

$$I = \frac{a}{s^2} - \frac{a^2}{s^2}\int_0^\infty \sin at \cdot e^{-st}\, dt \quad \Rightarrow \quad I = \frac{a}{s^2} - \frac{a^2}{s^2} I$$

$$\Rightarrow s^2 I = a - a^2 I$$

$$\Rightarrow (s^2 + a^2) I = a$$

$$\Rightarrow I = \frac{a}{s^2 + a^2}$$

$$\mathcal{L}\{\sin at\} = \frac{a}{s^2 + a^2}.$$

Piecewise functions

$$f(t) = \begin{cases} \text{—} & a < t \leq b \\ \text{—} & \text{—} \\ \text{—} & \text{—} \end{cases}$$

Using definition find the Laplace Transform of $f(t)$

$$f(t) = \begin{cases} 0, & 0 < t < \frac{\pi}{2} \\ 2\cos t, & t \geq \frac{\pi}{2} \end{cases}$$

$$\mathcal{L}\{f(t)\} = \int_0^\infty f(t)\, e^{-st}\, dt$$

$$= \int_0^{\frac{\pi}{2}} \cancel{0 \cdot e^{-st}}\, dt + \int_{\frac{\pi}{2}}^\infty 2\cos t\, e^{-st}\, dt = 2\int_{\frac{\pi}{2}}^\infty \cos t\, e^{-st}\, dt$$

Let $I = \int_{\frac{\pi}{2}}^{\infty} \cos t \cdot e^{-st}\, dt$

$$= \left[\cos t \cdot \frac{e^{-st}}{-s}\right]_{\frac{\pi}{2}}^{\infty} - \int_{\frac{\pi}{2}}^{\infty} (-\sin t)\, \frac{e^{-st}}{-s}\, dt$$

$$= 0 - 0 - \frac{1}{s}\int_{\frac{\pi}{2}}^{\infty} \sin t \cdot e^{-st}\, dt$$

$$= -\frac{1}{s}\left\{ \left[\sin t \cdot \frac{e^{-st}}{-s}\right]_{\frac{\pi}{2}}^{\infty} - \int_{\frac{\pi}{2}}^{\infty} \cos t \cdot \frac{e^{-st}}{-s}\, dt \right\}$$

$$= \frac{-1}{s}\left\{ 0 - \sin\frac{\pi}{2} \cdot \frac{e^{-s\cdot\frac{\pi}{2}}}{-s} + \frac{1}{s}\int_{\frac{\pi}{2}}^{\infty} \cos t \cdot e^{-st}\, dt \right\}$$

$$I = \frac{-1}{s^2} \cdot e^{-\frac{\pi}{2}s} - \frac{1}{s^2} \cdot \int_{\frac{\pi}{2}}^{\infty} \cos t\, e^{-st}\, dt$$

$$\Rightarrow I = \frac{-1}{s^2} e^{-\frac{\pi}{2}s} - \frac{1}{s^2} I$$

$$\Rightarrow s^2 I = -e^{-\frac{\pi}{2}s} - 1$$

$$\Rightarrow I = \frac{-e^{-\frac{\pi}{2}s}}{s^2+1}$$

$$\therefore \mathcal{L}\{f(t)\} = 2I = \frac{-2e^{-\frac{\pi}{2}s}}{s^2+1}$$

Using definition find the Laplace Transform of $f(t)$

$$f(t) = \begin{cases} \cos 3t\,, & 0 < t < \pi \\ 0, & t \geq \pi \end{cases}$$

$$\mathcal{L}\{f(t)\} = \int_0^\infty f(t)\, e^{-st}\, dt$$

$$I = \int_0^\pi \cos 3t\, e^{-st}\, dt + \int_\pi^\infty 0 \cdot e^{-st}\, dt = \int_0^\pi \cos 3t \cdot e^{-st}\, dt$$

$$I = \int_0^{\pi} \cos 3t \, e^{-st} \, dt$$

$$= \left[\cos 3t \cdot \frac{e^{-st}}{-s} \right]_0^{\pi} - \int_0^{\pi} (-3)\sin 3t \, \frac{e^{-st}}{-s} \, dt$$

$$= \cos(3\pi) \frac{e^{-\pi s}}{-s} - \cos(0) \frac{e^{-0}}{-s} - \frac{3}{s} \int_0^{\pi} \sin 3t \, e^{-st} \, dt$$

$$= \frac{e^{-\pi s}}{s} + \frac{1}{s} - \frac{3}{s} \left\{ \left[\sin 3t \cdot \frac{e^{-st}}{-s} \right]_0^{\pi} - \int_0^{\pi} 3 \cdot \cos 3t \cdot \frac{e^{-st}}{-s} \, dt \right\}$$

$$= \frac{e^{-\pi s}+1}{s} - \frac{3}{s}\left\{ \sin 3\pi \frac{e^{-\pi s}}{-s} - \sin(0)\frac{e^{-0}}{-s} + \frac{3}{s}\int_0^\infty \cos 3t\, e^{-st}\, dt \right\}$$

$$= \frac{e^{-\pi s}+1}{s} - \frac{3}{s}\left\{ 0 - 0 + \frac{3}{s}\int_0^\infty \cos 3t\, e^{-st}\, dt \right\}$$

$$I = \frac{e^{-\pi s}+1}{s} - \frac{9}{s^2}\int_0^\infty \cos 3t \cdot e^{-st}\, dt$$

$$\Rightarrow I = \frac{e^{-\pi s}+1}{s} - \frac{9}{s^2} I$$

$$s^2 I = s \cdot e^{-\pi s} + s - 9I$$

$$\Rightarrow I = \frac{s \cdot e^{-\pi s} + s}{s^2+9}$$

$$\therefore \mathcal{L}\{f(t)\} = \frac{s \cdot e^{-\pi s} + s}{s^2+9}$$

Using definition find the Laplace Transform of $f(t)$

$$f(t) = \begin{cases} 0, & 0 \le t < \pi \\ \sin(t), & \pi \le t < 2\pi \\ 0, & t \ge 2\pi \end{cases}$$

$$\mathcal{L}\{f(t)\} = \int_0^\infty f(t)\, e^{-st}\, dt$$

$$= \int_0^\pi 0 \cdot \cancel{e^{-st}}\, dt + \int_\pi^{2\pi} \sin t\, e^{-st}\, dt + \int_{2\pi}^\infty 0 \cdot \cancel{e^{-st}}\, dt$$

$$I = \int_\pi^{2\pi} \sin t\, e^{-st}\, dt \implies ??$$

Laplace Transform using Formula Sheet/Table

Linearity of Laplace Transformation

- $\mathcal{L}\{\, f(t) \pm g(t) \,\} = \mathcal{L}\{\, f(t)\} \pm \mathcal{L}\{\, g(t)\}$
- $\mathcal{L}\{k \cdot f(t)\} = k \cdot \mathcal{L}\{\, f(t)\}$

Find the Laplace Transform of each of the following functions:

i) $3e^{-2t}$ ii) $4t^3 - e^{-t}$ iii) $(t^2 + 1)^2$

iv) $7\sin 2t - 3\cos 2t$ v) $(4e^{2t} - 2)^3$

Evaluate

$$\mathcal{L}\{3e^{-2t}\}$$

$$= 3 \cdot \underline{\underline{\mathcal{L}\{e^{-2t}\}}}$$

$$= 3 \cdot \frac{1}{s-(-2)}$$

$$= \frac{3}{s+2}.$$

Evaluate

$$\mathcal{L}\{4t^3 - e^{-t}\}$$

$$= 4 \cdot \mathcal{L}\{t^3\} - \mathcal{L}\{e^{-t}\}$$

$$= 4 \cdot \frac{3!}{s^4} - \frac{1}{s-(-1)}$$

$$= \frac{24}{s^4} - \frac{1}{s+1}$$

Evaluate

$$\mathcal{L}\{(t^2+1)^2\}$$

$$= \mathcal{L}\{t^4 + 2t^2 + 1\}$$

$$= \mathcal{L}\{t^4\} + 2\cdot\mathcal{L}\{t^2\} + \mathcal{L}\{1\}$$

$$= \frac{4!}{s^5} + 2\cdot\frac{2!}{s^3} + \frac{1}{s}$$

$$= \frac{24}{s^5} + \frac{4}{s^3} + \frac{1}{s}$$

Evaluate

$$\mathcal{L}\{7\sin 2t - 3\cos 2t\}$$

$$= 7 \cdot \mathcal{L}\{\sin 2t\} - 3 \cdot \mathcal{L}\{\cos 2t\}$$

$$= 7 \cdot \frac{2}{s^2+2^2} - 3 \cdot \frac{s}{s^2+2^2}$$

$$= \frac{14 - 3s}{s^2+4}$$

Evaluate

$$\mathcal{L}\{7 \sinh 5t - 3 \cosh 2t\}$$

$$= 7 \cdot \mathcal{L}\{\sinh 5t\} - 3 \cdot \mathcal{L}\{\cosh 2t\}$$

$$= 7 \cdot \frac{5}{s^2 - 5^2} - 3 \cdot \frac{s}{s^2 - 2^2}$$

$$= \frac{35}{s^2 - 25} - \frac{3s}{s^2 - 4}$$

Ans

Evaluate

$$\mathcal{L}\{3\sin 7t\cos 4t\}$$

$$2\sin A\cos B = \sin(A+B) + \sin(A-B)$$

$$= \frac{3}{2}\cdot\mathcal{L}\{2\cdot\sin 7t\cdot\cos 4t\}$$

$$= \frac{3}{2}\mathcal{L}\{\sin 11t + \sin 3t\}$$

$$= \frac{3}{2}\cdot\mathcal{L}\{\sin(11t)\} + \frac{3}{2}\mathcal{L}\{\sin 3t\} = \frac{3}{2}\cdot\frac{11}{s^2+121} + \frac{3}{2}\cdot\frac{3}{s^2+9}$$

Evaluate

$$\mathcal{L}\{3\cos 3t \cos 5t\}$$

$$2\cos A\cos B = \cos(A+B) + \cos(A-B)$$

$$= \frac{3}{2}\cdot \mathcal{L}\left\{2\cos 3t\cdot\cos 5t\right\}$$

$$= \frac{3}{2}\mathcal{L}\left\{\cos(8t) + \cos(-2t)\right\}$$

$$= \frac{3}{2}\cdot\mathcal{L}\left\{\cos 8t\right\} + \frac{3}{2}\mathcal{L}\left\{\cos 2t\right\} = \frac{3}{2}\cdot\frac{s}{s^2+64} + \frac{3}{2}\cdot\frac{s}{s^2+4}$$

Evaluate

$$\mathcal{L}\{3\sin 5t \sin 3t\}$$

$$2\sin A \sin B = \cos(A-B) - \cos(A+B)$$

$$= \frac{3}{2}\mathcal{L}\{2\sin 5t \cdot \sin 3t\}$$

$$= \frac{3}{2}\mathcal{L}\{\cos(2t) - \cos 8t\}$$

$$= \frac{3}{2}\frac{s}{s^2+4} + \frac{3}{2}\cdot\frac{s}{s^2+64}$$

Trigonometric Identity

$$2\ \sin A\ \sin B = \cos(A - B) - \cos(A + B)$$

$$2\ \cos A\ \cos B = \cos(A + B) + \cos(A - B)$$

$$* \quad 2\ \sin A\ \cos B = \sin(A + B) + \sin(A - B)$$

$$\left.\begin{aligned} \sin(-\theta) &= -\sin(\theta) \\ \cos(-\theta) &= \cos(\theta) \end{aligned}\right\}$$

মুখস্থবিদ্যা প্রতিভাকে ধ্বংস করে কিন্তু সফলতাকে ত্বরান্বিত করে।


Thank you!