Lecture 10

Linear Equations

A first-order differential equation of the form

$$\boxed{a_1(x)\frac{dy}{dx} + a_0(x)y = g(x)}$$

is said to be a linear equation in the dependent variable y.

when $g(x)=0$ then we say homogeneous linear eqn.

otherwise nonhomogeneous Linear equations.

Standard Form

$$\frac{dy}{dx} + P(x)y = f(x) \qquad \text{——— (1)}$$

Solution: The above eqn. (1) has two ~~sto~~ solution and its ~~its~~ solution is the sum of the two solutions.

$$y = y_c + y_p$$

where y_c is a solution of the associated homogeneous equation $\frac{dy}{dx} + P(x)y = 0$

and y_p is a particular solution of the nonhomogeneous equation of (1).

when we write for homogeneous eqn.

$$\frac{dy}{y} + P(x)dx = 0$$

Solving the above homogeneous eqⁿ we get

$$\int \frac{dy}{y} = -\int P(x)\,dx$$

$$\ln y = -\int P(x)\,dx$$

$$\boxed{y = e^{-\int P(x)\,dx}}$$

And we called it integrating factor. $y_c = e^{-\int P(x)\,dx}$

<u>Solving Steps:</u>

Step ① Put a linear eqⁿ into the standard form ①

Step 2: Find the integrating factor, I.F. $= e^{\int P(x)\,dx}$

Step 3: Multiply integrating factor on both sides. We'll get

$$\frac{d}{dx}\left[e^{\int P(x)\,dx}\, y\right] = e^{\int P(x)\,dx} f(x).$$

Step 4: Integrating both sides of updated equation.

Solve $\frac{dy}{dx} + y = f(x)$, $y(0) = 0$ where $f(x) = \begin{cases} 1 & , 0 \le x \le 1 \\ 0 & , x > 1 \end{cases}$

For $0 \le x \le 1$ we have

$$\frac{dy}{dx} + y = 1 \quad , \text{I.F.} = e^{\int dx} = e^{x}$$

$$e^{x}\frac{dy}{dx} + e^{x}y = e^{x}$$

$$\frac{d}{dx}(e^{x}y) = e^{x} \Rightarrow$$

$$\int \frac{d}{dx}(e^{x}y)\,dx = \int e^{x}dx$$

$$e^{x}y = e^{x} + c_1 \Rightarrow y = 1 + c_1 e^{-x}$$

$x = 0,\ y = 0 \quad \Rightarrow \quad 0 = 1 + c_1 e^{0} \Rightarrow c_1 = -1$

$$\boxed{y = 1 - e^{-x}} \quad \text{——— (1)}$$

general sol:

$$y = \begin{cases} 1 - e^{-x} & 0 \le x \le 1 \\ Ae^{-x} & x > 1 \end{cases}$$

For $x > 1$, we have

$$\frac{dy}{dx} + y = 0$$

$$\frac{dy}{y} = -dx \qquad \ln y = -x + c_2$$

$$y = e^{-x + c_2} = e^{-x} \cdot e^{c_2} = Ae^{-x}$$

~~$x = 0, y = 0$, simpler~~

Extra

Now, we don't use $y(0) = 0$ because here $x > 1$ which means we use limit here.

From (1)

$$\lim_{x \to 1^{+}} y(x) = y(1)$$

$$\Rightarrow \quad Ae^{-1} = 1 - e^{-1}$$

$$A = e - 1$$

Therefore,

$$y = \begin{cases} 1 - e^{-x} & 0 \le x \le 1 \\ (e-1)e^{-x} & x > 1 \end{cases}$$

$\blacksquare$

Example: Solve $\frac{dy}{dx} + y = x$, $y(0) = 4$ [Initial-value Problem]

Sol^n:

$$\text{I.F.} = e^{\int 1 dx} = e^{x}$$

$$e^{x}\frac{dy}{dx} + e^{x}y = xe^{x}$$

$$\frac{d}{dx}(e^{x}y) = xe^{x}$$

$$\int \frac{d}{dx}(e^{x}y)\,dx = \int xe^{x}\,dx$$

$$e^{x}y = xe^{x} - e^{x} + c$$

$$y = x - 1 + ce^{-x}$$

when, $x=0$, $y=4$, then

$$4 = 0 - 1 + c\cdot e^{-0} = -1 + c$$

$$c = 5$$

Therefore, the general solution is

$$y = x - 1 + 5e^{-x}. \qquad x \in \mathbb{R}$$

Example: Solve $\frac{dy}{dx} - 3y = 0$.

Sol:

Since $\frac{dy}{dx} + P(x)\,y = 0$

Here $P(x) = -3$

so, Integrating factor, $I.F. = e^{\int -3\,dx} = e^{-\int 3\,dx} = e^{-3x}$

Now, $e^{-3x}\frac{dy}{dx} - 3e^{-3x}y = 0$

$$\frac{d}{dx}\left(e^{-3x}y\right) = 0$$

Integration on both sides,

$$\int \frac{d}{dx}\left(e^{-3x}y\right)dx = \int 0\,dx$$

$$\Rightarrow e^{-3x}y = c$$

$$\Rightarrow y = ce^{3x}$$

Example: $\frac{dy}{dx} - 3y = 6$

$$I.F. = e^{\int -3\,dx} = e^{-3x}$$

$$e^{-3x}\frac{dy}{dx} - 3e^{-3x}y = 6e^{-3x}$$

$$\frac{d}{dx}\left(e^{-3x}y\right) = 6e^{-3x}$$

$$\int \frac{d}{dx}\left(e^{-3x}y\right)dx = \int 6e^{-3x}\,dx$$

$$e^{-3x}y = 6\,\frac{e^{-3x}}{-3} + c \Rightarrow e^{-3x}y = -2e^{-3x} + c$$

$$\Rightarrow y = -2 + ce^{3x}$$

Example

Solve $x\frac{dy}{dx} - 4y = x^6 e^x$

Solⁿ:

$$\frac{dy}{dx} - \frac{4}{x}y = \frac{x^6}{x}e^x = x^5 e^x$$

$\boxed{a^{\log_a N} = N,\ N>0}$

Here $P(x) = -\frac{4}{x}$

$$\text{I.F.} = e^{\int -\frac{4}{x}dx} = e^{-4\ln x} = e^{\ln x^{-4}} = x^{-4}, \quad x>0$$

Now,

$$x^{-4}\frac{dy}{dx} - \frac{4}{x}x^{-4}y = x^{-4}.x^5 e^x$$

$$x^{-4}\frac{dy}{dx} - 4x^{-5}y = xe^x$$

$$\frac{d}{dx}(x^{-4}y) = xe^x$$

$$\int \frac{d}{dx}(x^{-4}y)dx = \int xe^x dx$$

$$x^{-4}y = xe^x - e^x + c$$

$$y = x^5e^x - x^4e^x + cx^4.$$

$x \quad e^x$

$1 \quad e^x$

$0 \quad e^x$

$xe^x - e^x$

Do yourself: 1. Find the general solution of $(x^2-9)\frac{dy}{dx} + xy = 0$.

$$\frac{dy}{dx} + \frac{x}{x^2-9}y = 0$$