Probability Distribution (1)

Md. Ismail Hossain Riday

Variable

- A quantity or characteristic that can take different values or vary in different situations.
- For example: CGPA, Gender etc.

Random variable

- A random variable is a variable
- That can take different values (numerical)
- Determined by the outcome of an experiment

Difference

- The differences between variable and random variable are-

  a) Random variable always takes numerical values

  b) There is a probability associated with each possible values

Difference

- The differences between variable and random variable are-

a) Random variable always takes numerical values

b) There is a probability associated with each possible values

Difference

- The differences between variable and random variable are-

a) Random variable always takes numerical values

b) There is a probability associated with each possible values

Example

For example,

A coin is tossed. It has two possible outcomes-Head and Tail.

Consider a variable,

$$X = \ outcome\ of\ a\ coin\ toss = \begin{cases} 1; If\ head\ appears \\ 0; If\ tail\ appears \end{cases}$$

Here, we can write $P(X = 1) = \frac{1}{2}$, $P(X = 0) = \frac{1}{2}$

Types of Random Variable

- $X = \{1, 2, 3, 4, 5, 6\}$ Discrete
- $X \geq 0$ Continuous

Random variable

Discrete

Continuous

Discrete RV

- A random variable defined over a discrete sample space.

- For example, $X = \ Number\ of\ cars\ passing\ a\ toll\ booth\ in\ a\ day$

$$\{0, 1, 2, 3, \ldots.\}$$

$$\Downarrow$$

Discrete random variable

Continuous RV

- A random variable defined over a continuous sample space

- For example, $X = Weight\ of\ a\ person$

$$\{0 < X < finite\ number\}$$

$$\Downarrow$$

$\boxed{\text{Continuous random variable}}$

Examples

Discrete Random Variable:

1. $X = \ Number\ of\ correct\ answers\ in\ a\ 100 - MCQ\ test = \ 0, 1, 2, \dots, 100$
2. $X = \ Number\ of\ cars\ passing\ a\ toll\ both\ in\ a\ day = \ 0, 1, 2, \dots, \infty$
3. $X = \ Number\ of\ balls\ required\ to\ take\ the\ first\ wicket\ = \ 1, 2, 3, \dots, \infty$
4. $X = The\ number\ of\ telephone\ calls\ received\ in\ a\ telephone\ booth\ during\ one\ day = 1,2,..$

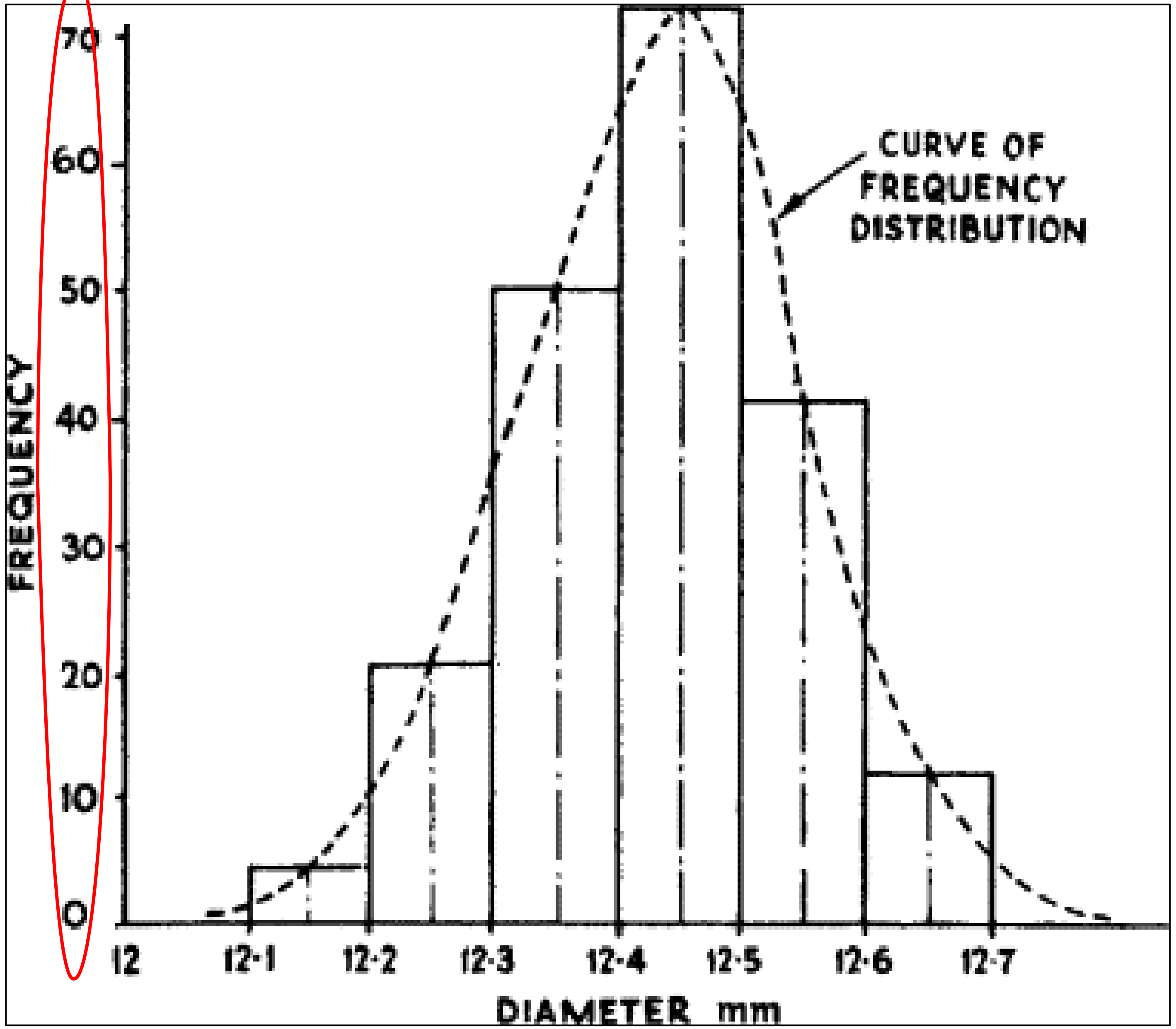
Continuous Random Variable:

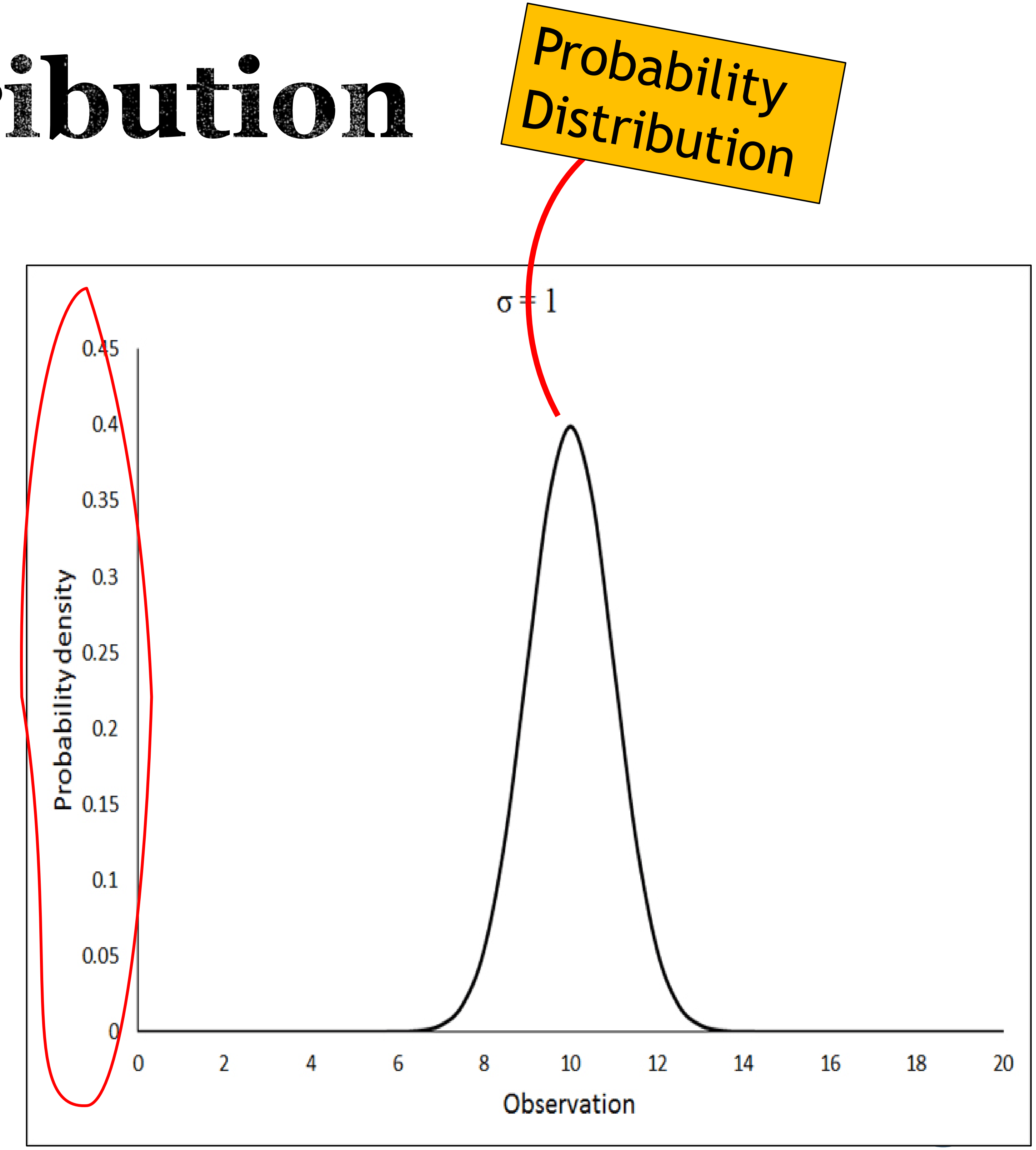
1. $X = \ Weight\ of\ a\ person.\ 0 < X < \infty$
2. $X = \ Monthly\ Profit. -\infty < X < \infty$
3. $X = Temperature\ recorded\ by\ the\ meteorological\ office.\ 0 < X < \infty$

Probability Distribution

- Distribution of the probabilities among the different values of a random variable.

Probability Distribution





Probability Distribution

- Distribution of the probabilities among the different values of a random variable.

- For example, Here's an example probability distribution that results from the rolling of a single fair die.

X	**1**	**2**	**3**	**4**	**5**	**6**
$P(x)$	1/6	1/6	1/6	1/6	1/6	1/6

Types of PD

Probability Distribution

- Discrete
- Continuous

Discrete PD

- Probability distribution of a discrete random variable

- For example, $X = \ Number\ of\ cars\ passing\ a\ toll\ booth\ in\ a\ day$

$$\{0, 1, 2, 3, \ldots.\}$$

$$\Downarrow$$

Discrete random variable $\Rightarrow$ Discrete PD

Continuous PD

- Probability distribution of a continuous random variable

- For example, $X = Weight\ of\ a\ person$

$$\{0 < X < finite\ number\}$$

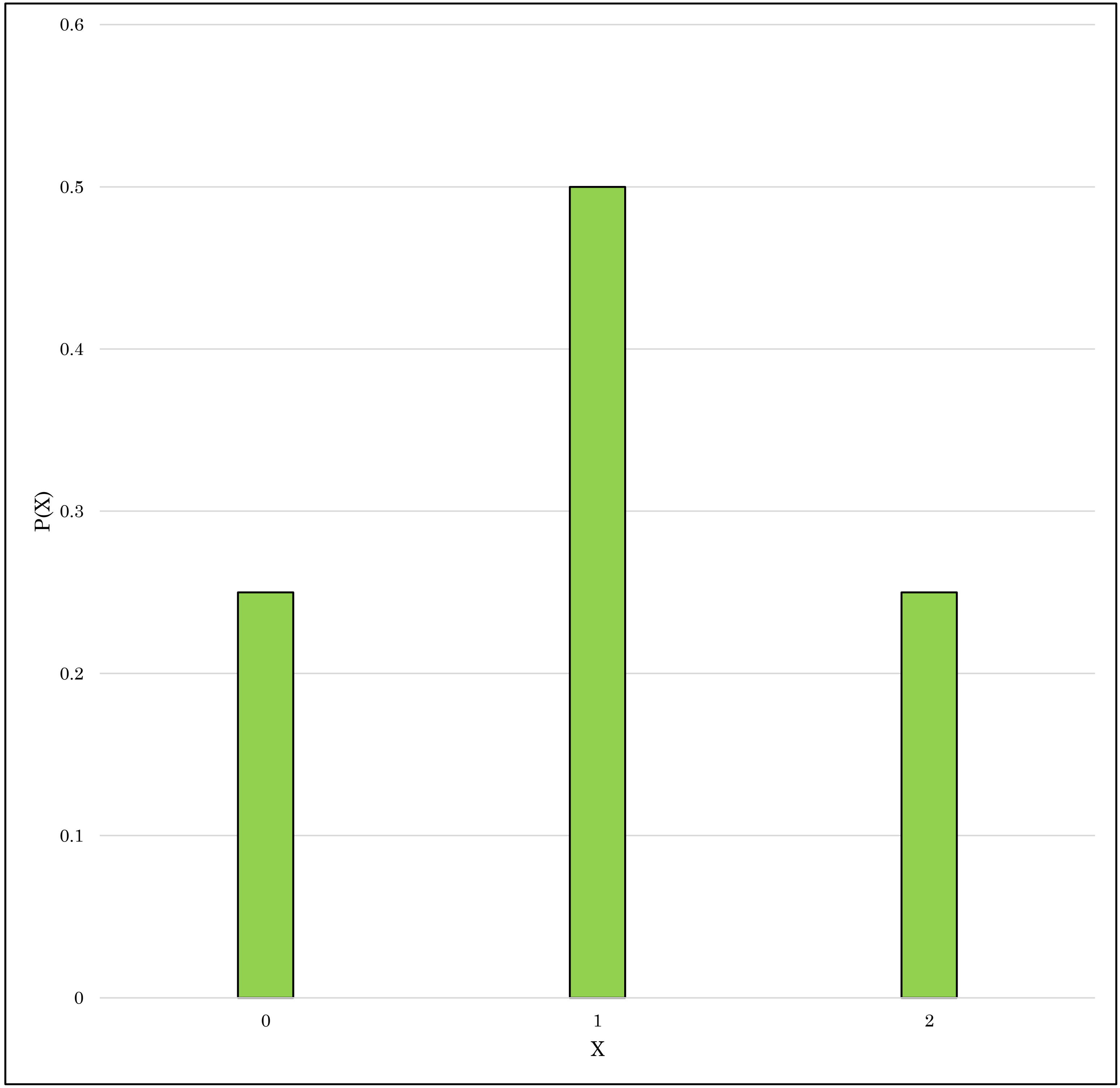
$\Downarrow$

Continuous random variable $\Rightarrow$ Continuous PD

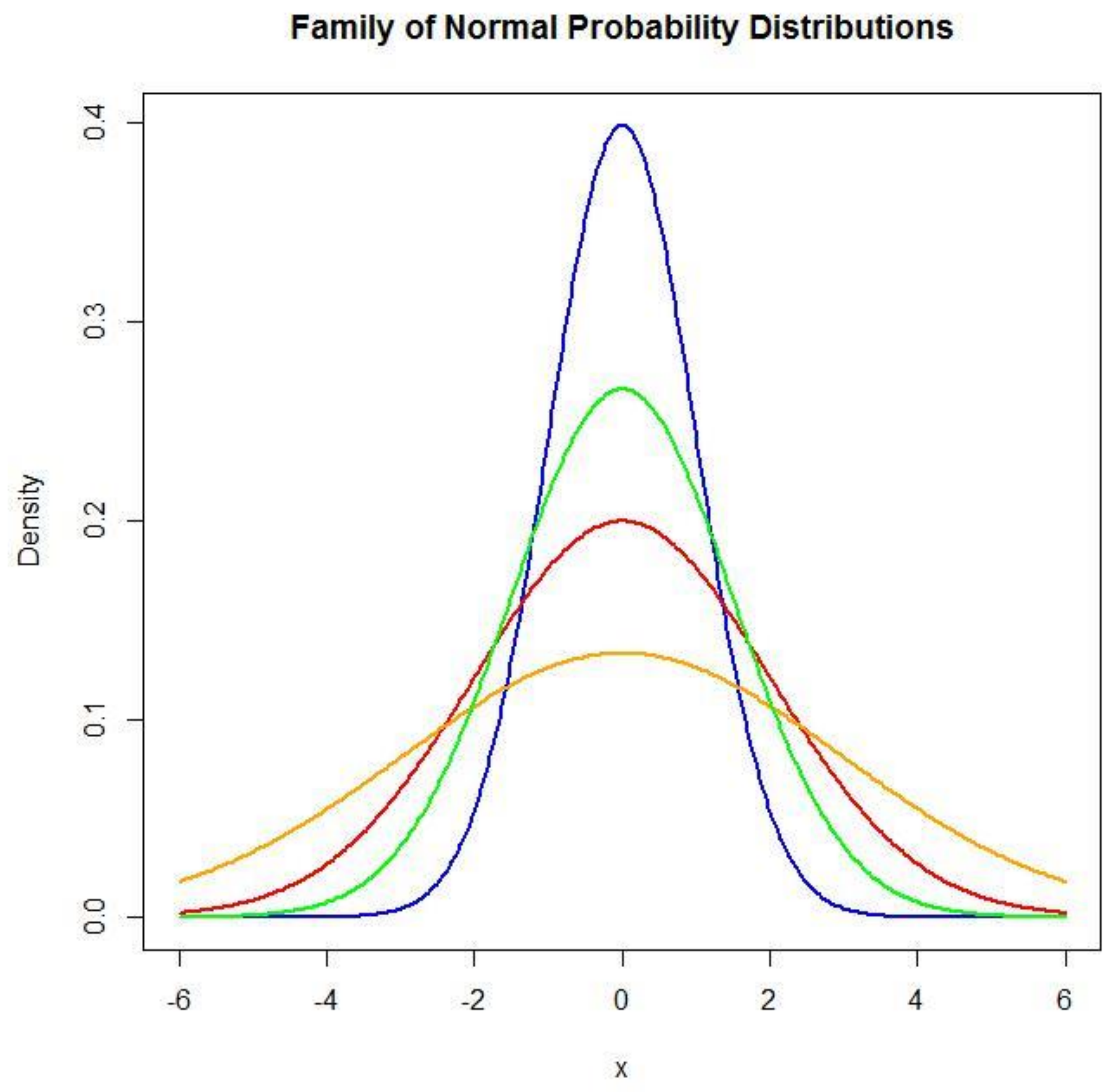
Example

- Tossing a coin 2 times.
- $X = Number\ of\ head\ appeared$
- $S = \{HH, HT, TH, TT\}$

X	0	1	2
$P(x)$	1/4	2/4	1/4



Example



Family of Normal Probability Distributions
Density
0.0
0.1
0.2
0.3
0.4
-6
-4
-2
0
2
4
6
x

Probability Mass Function (PMF)

- The probability distribution function of a discrete random variable X is called a **PMF** and is denoted by $\boldsymbol{P(x)}$

- $\boldsymbol{P(x)}$ satisfy three properties

$\boldsymbol{X}$	**0**	**1**	**2**
$P(x)$	1/4	2/4	1/4

a) $0 \leq P(x) \leq 1$
b) $\sum P(x_i) = 1$
c) $P(x) = P(X = x)$

Probability Mass Function (PMF)

- The probability distribution function of a discrete random variable X is called a **PMF** and is denoted by $\boldsymbol{P(x)}$

- $\boldsymbol{P(x)}$ satisfy three properties

Random Variable (X)

X	0	1	2
$P(x)$	1/4	2/4	1/4

a) $0 \leq P(x) \leq 1$
b) $\sum P(x_i) = 1$
c) $P(x) = P(X = x)$

Probability Mass Function (PMF)

- The probability distribution function of a discrete random variable X is called a **PMF** and is denoted by $\boldsymbol{P(x)}$

Outcomes (x)

X	0	1	2
$P(x)$	1/4	2/4	1/4

- $\boldsymbol{P(x)}$ satisfy three properties

a) $0 \leq P(x) \leq 1$
b) $\sum P(x_i) = 1$
c) $P(x) = P(X = x)$

$P(2)$ ➡ $P(X = 2)$

Probability Mass Function (PMF)

- The probability distribution function of a discrete random variable X is called a **PMF** and is denoted by $\boldsymbol{P(x)}$

- $\boldsymbol{P(x)}$ satisfy three properties

X	0	1	2
$P(x)$	1/4	2/4	1/4

a) $0 \le P(x) \le 1$

b) $\sum P(x_i) = 1$

c) $P(x) = P(X = x)$

Probability Density Function (PDF)

- The probability distribution function of a continuous random variable X is called a **PDF** and is denoted by $\boldsymbol{f(x)}$

- $\boldsymbol{f(x)}$ satisfy three properties

a) $f(x) \geq 0$
b) $\int f(x)\, dx = 1$
c) $P(a \leq X \leq b) = \int_a^b f(x)\, dx$

Example

X	0	1	2	3
$P(x)$	1/27	6/27	12/27	8/27

$$P(X \geq 1) \Rightarrow P(X = 1) + P(X = 2) + P(X = 3) \Rightarrow \frac{6}{27} + \frac{12}{27} + \frac{8}{27} = 0.96$$

$$P(X \leq 2) \Rightarrow P(X = 0) + P(X = 1) + P(X = 2) \Rightarrow \frac{1}{27} + \frac{6}{27} + \frac{12}{27} = 0.73$$

Example

- Suppose you roll a fair six-sided die twice. Let, X denote the number of times you roll a three.

a) Summarize the probability function of X.

b) Determine,

   a) $P(X > 0)$ Ans:11/36
   b) $P(X \leq 1)$ Ans:35/36
   c) $P(X = 2)$ Ans:1/36

X	0	1	2
$P(x)$	25/36	10/36	1/36

Example

- Suppose that 2 batteries are randomly chosen from a box containing 10 batteries of which 7 are good and 3 are defective. Let $\boldsymbol{X}$ denote the number of defective batteries chose.

a) Summarize the probability function of $\boldsymbol{X}$.

b) Determine the probability,
   a) $P(X > 0)$
   b) $P(X \leq 1)$
   c) $P(X = 1)$

$\boldsymbol{X}$	0	1	2
$P(x)$	21/45	21/45	3/45

Example

- Let X be a discrete random variable whose only possible values are -1, 2, and 5. Suppose that the probability function of X is,

$$P(x) = \begin{cases} \frac{1}{4}; for\ x = -1 \\ \frac{1}{2}; for\ x = 2 \\ \frac{1}{4}; for\ x = 5 \end{cases}$$

a) Find the cumulative distribution function of X.

Example

$P(X > 1.2) = ???$

$P(1.2 \leq X \leq 1.6) = ???$

$P(X = 1.5) = ???$

$P(X > 3.5) = ???$

- Let, X be a random variable with probability function,

$$f(x) = \begin{cases} \frac{2x}{3}; & if\ 1 \leq x \leq 2 \\ 0, & Otherwise \end{cases}$$

$$P(X \leq 1.2) \Rightarrow \int_{1}^{1.2} \frac{2x}{3}\, dx \Rightarrow \frac{2}{3}\left[\frac{x^2}{2}\right]_1^{1.2} \Rightarrow \frac{1}{3}[1.44 - 1] = 0.1467$$

Example

- The pressure measured in pounds per cm^2 at a certain valve is a random variable X whose probability function is,

$$f(x) = \begin{cases} \frac{2}{9}(3x - x^2); if\ 0 < x < 3 \\ 0; Otherwise \end{cases}$$

Find the probability that the pressure at this valve is,

a) Not more than 2 pounds per cm^2

b) Greater than 2 pounds per cm^2

c) Between 1.5 and 2.5 pounds per cm^2

Example

- The cumulative distribution function of a continuous random variable $\boldsymbol{X}$ can be written as,

$$F(x) = \begin{cases} 0; for\ X \leq 0 \\ x^2; for\ 0 \leq x \leq 1 \\ 1; for\ x \geq 1 \end{cases}$$

Find the probability density function and calculate,

a) $P(X \leq 0.5)$

b) $P(0.2 \leq X \leq 0.5)$

Mathematical Expectation → $E(X)$

- The probability of X contains all of the probabilistic information about X.
- The whole distribution usually too cumbersome for presentation
- Summary measure $\boxed{\text{(Mean, Variance, Median)}}$ may be useful

X	0	1	2
$P(x)$	1/4	2/4	1/4

Mathematical Expectation

- For a discrete random variable X with PMF $P(x)$, the mathematical expectation of X is-

$$E(X) = \sum x\,P(x)$$

- For a continuous random variable X with PDF $f(x)$, the mathematical expectation of X is-

$$E(X) = \int x\,f(x)\,dx$$

Properties

- $E(X) = Mean\ (X) = \mu(X)$
- $E(X^2) - \left(E(X)\right)^2 = Variance\ (X) = Var(X)$
- $E(c) = c; c\ is\ constant$
- $E(cX) = c\ E(X); c\ is\ constant$
- $E(X + c) = E(X) + c; c\ is\ constant$
- $E(X \pm Y) = E(X) \pm E(Y)$
- $E(XY) = E(X) \times E(Y)$

Properties

- $Var(c) = 0; c \text{ is constant}$
- $Var(cX) = c^2\, Var(X)$
- $Var(X + c) = Var(X) + Var(c) = Var(X)$
- $Var(X \pm Y) = Var(X) \pm Var(Y)$

Example

- Find the mean and variance of

X	0	1	2
$P(x)$	1/4	2/4	1/4

$$Mean, E(X) = \sum x\, P(x) = \left(0 \times \frac{1}{4}\right) + \left(1 \times \frac{2}{4}\right) + \left(2 \times \frac{1}{4}\right) = 1$$

$$Var(X) = E(X^2) - \left(E(X)\right)^2 = ???$$

Example

- Individuals applying for a driving license are allowed up to four attempts to pass the license exam. An applicant is randomly selected and let X denote the number of attempts made by the applicants. The probability function of X is as follows,

$$P(x) = \begin{cases} \frac{x}{10}; for\ x = 1,2,3,4 \\ 0; Otherwise \end{cases}$$

Find mean of X.

$$\boxed{E(X) = \sum x \times P(x) = 3}$$

The mean number of attempts required to pass the license exam is 3

Example

- Let, X be a continuous random variable with pdf,

$$f(x) = \begin{cases} \dfrac{2x}{3}; & if\ 1 \le x \le 2 \\ 0, & Otherwise \end{cases}$$

??????

Mean

$$E(X) = \int_1^2 x\, f(x) dx$$

$$E(X) = \int_1^2 x \times \frac{2x}{3} dx$$

$$E(X) = \int_1^2 \frac{2x^2}{3} dx$$

Mathematical exercise

To access additional mathematical problems,

please refer to the PDF lecture notes.

Thank You