Lecture 18

Double Integrals

Rectangular Regions

Before starting on double integrals, we want a recap the definition of definite integrals for functions of single variables.

$$\int_a^b f(x)\,dx \quad , \; a \le x \le b$$

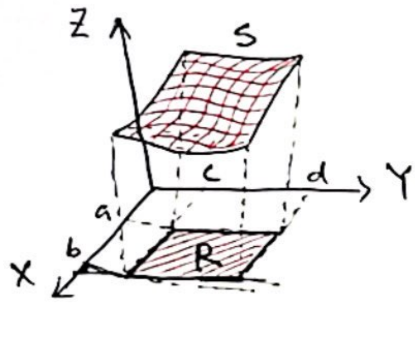
For these integrals we can say that we are integrating over the interval $a \le x \le b$. Note that this does assume that $a < b$, however, if we have $b < a$, we can use $b \le x \le a$. We broke up the interval in n subintervals of width Δx and choose a point x_i^* from each intervals. And finally we define the definite integral is

$$\int_a^b f(x)\,dx = \lim_{n \to \infty} \sum_{i=1}^{n} f(x_i^*)\,\Delta x$$

In this Lecture, we want to integrate a function of two variables $f(x,y)$. We will start out by assuming that the region in $\mathbb{R}^2$ is a rectangle which we denote

$$R = [a,b] \times [c,d]$$

i.e. the range for x and y are $a \le x \le b$ and $c \le y \le d$.



Here ~~we~~ is the definition of a double integral of a function of two variables over a rectangular region R as well as the notation that we'll use for it.

$$\text{Volume} = \iint_R f(x,y)\,dA = \lim_{n,m\to\infty} \sum_{i=1}^{n} \sum_{j=1}^{m} f(x_i^*, y_j^*)\,\Delta A$$

<u>Fubini's Theorem</u>

If $f(x,y)$ is continuous on $R = [a,b] \times [c,d]$ then,

$$\iint_R f(x,y)\,dA = \int_a^b \int_c^d f(x,y)\,dy\,dx = \int_c^d \int_a^b f(x,y)\,dx\,dy$$

These integrals are called iterated integrals.

So, Finally we get

$$\iint_R f(x,y)\,dA = \int_a^b \left[\int_c^d f(x,y)\,dy \right] dx$$

Example: Use a double integral to find the volume of the solid that is bounded above by the plane $z = 4 - x - y$ and below by the rectangle $R = [0,1] \times [0,2]$.

or,

Compute the double integrals over the indicated rectangles.

$$\iint_R (4 - x - y)\, dA, \quad R = [0,1] \times [0,2].$$

Solⁿ:

$$\text{Volume, } V = \int_0^2 \int_0^1 (4 - x - y)\, dx\, dy$$

$$= \int_0^2 \left[4x - \frac{x^2}{2} - xy \Big|_{x=0}^{1} \right] dy$$

$$= \int_0^2 \left[\left(4 - \frac{1}{2} - y\right) - (0 - 0 - 0) \right] dy$$

$$= \int_0^2 \left(\frac{7}{2} - y \right) dy$$

$$= \left(\frac{7}{2} y - \frac{y^2}{2} \right) \Big|_{y=0}^{2}$$

$$= \frac{7}{2} \cdot 2 - \frac{2^2}{2} - (0 - 0)$$

$$= 7 - 2$$

$$= 5$$

Do yourself:

1. Evaluate the iterated integrals

(i) $\int_0^{\ln 3}\int_0^{\ln 2} e^{x+2y}\, dy\, dx$

(ii) $\int_0^{2}\int_0^{1} y\cos x\, dy\, dx$

(iii) $\int_3^{5}\int_1^{2} \frac{1}{(x+y)^2}\, dy\, dx$

(iv) $\iint_R x\sqrt{1-x^2}\, dA$; $R = \{(x,y): 0 \le x \le 1,\ 2 \le y \le 3\}$.

(v) $\iint_R (x\sin y - y\sin x)\, dA$, $R = [0, \pi/2] \times [0, \pi/3]$.

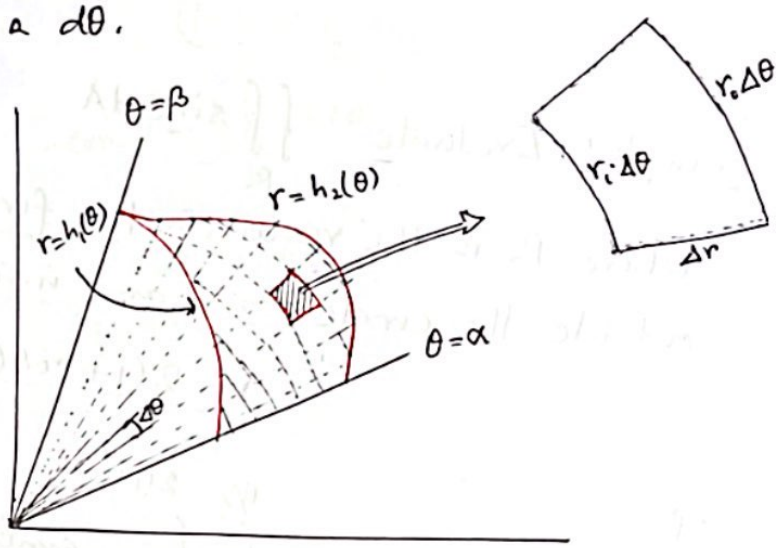
Double Integrals

Polar Coordinates

In polar coordinates, our general region will be defined by inequalities,

$$\alpha \leq \theta \leq \beta \quad \text{and} \quad h_1(\theta) \leq r \leq h_2(\theta)$$

Make sure that we can't just convert the dx and the dy into a dr and a $d\theta$.



$\Delta r = r_o - r_i$, where r_o is the radius of the outer arc and r_i is the radius of the inner arc.

$\Delta\theta$ is the angle betn. the two radial lines that form the sides of this piece.

Now, let's assume that we've taken the mesh so small that we can assume that $r_i \sim r_o = r$ and also assume that the piece is closed enough to a rectangle that we can write

$$\Delta A \approx r\Delta\theta\Delta r$$

Also assume that the mesh is small enough then

$$dA \approx \Delta A \quad d\theta \approx \Delta\theta \quad \text{and} \quad dr \approx \Delta r$$

Finally, $\boxed{dA \approx r\,dr\,d\theta}$

Theorem: If R is a simple polar region whose boundaries are that rays $\theta = \alpha$, and $\theta = \beta$ and the curves $r = r_1(\theta)$ and $r = r_2(\theta)$, and If $f(r,\theta)$ is continuous on R, then

$$\boxed{\iint_R f(r,\theta)\, dA = \int_{\alpha}^{\beta} \int_{r_1(\theta)}^{r_2(\theta)} f(r,\theta)\, r\, dr\, d\theta}$$

Example: Evaluate $\displaystyle\iint_R \sin\theta\, dA$

where R is the region in the first quadrant that is outside the circle $r = 2$ and inside the cardioid

$$r = 2(1+\cos\theta).$$

Solⁿ.

$$\iint_R \sin\theta\, dA = \int_0^{\pi/2} \int_2^{2(1+\cos\theta)} \sin\theta\, r\, dr\, d\theta$$

$$= \int_0^{\pi/2} \left[\sin\theta \int_2^{2(1+\cos\theta)} r\, dr \right] d\theta$$

$$= \int_0^{\pi/2} \sin\theta \left(\frac{r^2}{2}\right)\Bigg|_2^{2(1+\cos\theta)} d\theta$$

$$= -\int_0^{\pi/2} \sin\theta\ \frac{1}{2}\left[4 - 4(1+\cos\theta)^2\right] d\theta$$

$$= \cancel{2}\int_0^{\pi/2} \cancel{\sin\theta} \left(1 - \cancel{1} - 2\cancel{\cos\theta} \cancel{-\cos^2\theta}\right) \cancel{d}\theta$$

$$= -2\int_0^{\pi/2} \sin\theta\left[1-(1+\cos\theta)^2\right]d\theta$$

$$= -2\int_0^{\pi/2} \sin\theta\, d\theta + 2\int_0^{\pi/2} (1+\cos\theta)^2 \sin\theta\, d\theta$$

$1+\cos\theta = u$

$-\sin\theta d\theta = du$

$-\int u^2 du$

$-\frac{u^3}{3}$

$-\frac{1}{3}(1+\cos\theta)^3$

$$= -2\left(-\cos\theta\right)\Big|_0^{\pi/2} + 2\left[-\frac{1}{3}(1+\cos\theta)^3\Big|_0^{\pi/2}\right]$$

$$= 2\cos\theta\Big|_0^{\pi/2} - \frac{2}{3}(1+\cos\theta)^3\Big|_0^{\pi/2}$$

$$= 2.\left(\cos\pi/2 - \cos 0\right) - \frac{2}{3}\left[\left(1+\cos\pi/2\right)^3 - (1+\cos 0)^3\right]$$

$$= 2(0-1) - \frac{2}{3}\left[(1+0)^3 - (1+1)^3\right]$$

$$= -2 - \frac{2}{3}(1-8)$$

$$= -2 + \frac{14}{3}$$

$$= \frac{8}{3}$$

*

<u>Do yourself</u>

1. page 1022: Example 2

2. page 1023 : Example 4

3. Evaluate the iterated integral

(i) $$\int_0^{\pi/4} \int_0^{\sin\theta} r\cos\theta \, dr \, d\theta$$

(ii) $$\int_{\pi/2}^{3\pi/2} \int_0^{1+\sin\theta} r \, dr \, d\theta$$

(iii) $$\int_0^{\pi} \int_0^{\cos\theta} r^3 \, dr \, d\theta$$

~~4. Express the volume of the solid described as a double integral in polar coordinates~~

4. $\iint\limits_R \frac{1}{1+x^2+y^2} \, dA$, where R is the sector in the first quadrant bounded by $y=0$, $y=x$, and $x^2+y^2=9$.

~~5, Evaluate~~

Example: Use polar coordinates to evaluate $\int_{-1}^{1}\int_{0}^{\sqrt{1-x^2}} (x^2+y^2)^{3/2}\, dy\, dx$.

Sol^n:

Here, we observe that first we ~~iterat~~ integrate the iterated integral w.r.to y i.e. x is fixed.

So, the limits ~~is~~ are $y=0$ to $y=\sqrt{1-x^2} \Rightarrow y^2 = 1-x^2$

i.e. $x^2+y^2=1$

which means it is a circle whose center at $(0,0)$ and radius 1.

For 2nd integration, $x=-1$, to $x=1$.

Thus in polar coordinates, r varies bet^n 0 ~~and~~ and 1.

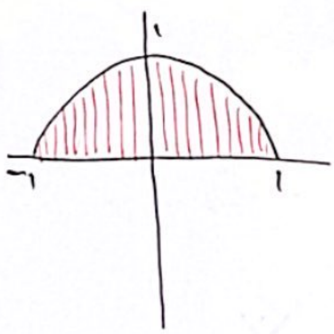
θ varies bet^n 0 and π.

~~In polar form~~, for circle eq^n: $x^2+y^2=r^2$ and

$dA = dx\,dy = r\,dr\,d\theta$

Now,
$$\int_{-1}^{1}\int_{0}^{\sqrt{1-x^2}} (x^2+y^2)^{3/2}\, dy\, dx = \int_{\theta=0}^{\pi}\int_{r=0}^{1} (r^2)^{3/2}\, r\, dr\, d\theta$$

$$= \int_{0}^{\pi}\int_{0}^{1} r^4\, dr\, d\theta$$



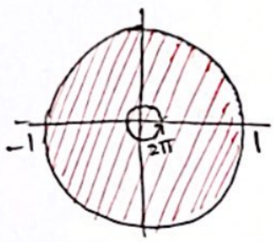
$$= \int_{0}^{\pi} \frac{r^5}{5}\Big|_0^1 d\theta$$

$$= \frac{1}{5}\int_{0}^{\pi} d\theta = \frac{\pi}{5}$$

Problem: Express the volume of the solid described as a double integral in polar coordinates.

$$\iint_R e^{-(x^2+y^2)}\, dA,$$ where R is the region enclosed by the circle $x^2+y^2=1$.

Sol^n: The enclosed region by the circle and so, ~~radius is~~ r varies bet^n 0 and 1 and θ varies bet^n 0 and 2π.



$$\iint_R e^{-(x^2+y^2)}\, dA = \int_0^{2\pi}\int_0^1 e^{-r^2}\, r\, dr\, d\theta$$

$r^2 = u$

$2r\,dr = du$

$r\,dr = \frac{1}{2}du$

r	0	1
u	0	1

$$= \int_0^{2\pi}\int_0^1 e^{-u}\, du\, d\theta$$

$$= \int_0^{2\pi} -\cdot e^{-u}\Big|_0^1\, d\theta$$

$$= \int_0^{2\pi} -(e^{-1} - e^{0})\, d\theta$$

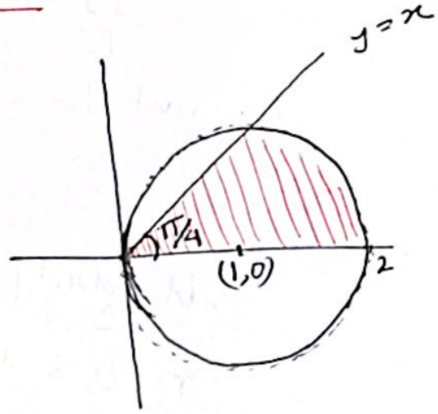
$$= \int_0^{2\pi} \left(1 - \frac{1}{e}\right) d\theta$$

$$= \left(\theta - \frac{\theta}{e}\right)\Big|_0^{2\pi}$$

$$= 2\pi - \frac{2\pi}{e}$$

$$= 2\pi\left(1 - \frac{1}{e}\right)$$

Problem; Use polar coordinate to evaluate $\iint\limits_R 2y\,dA$, where R is the region in the first quadrant bounded above by the circle $(x-1)^2+y^2=1$ and below by the line $y=x$.



Sol^n: $x = r\cos\theta$, $y = r\sin\theta$

$$(x-1)^2+y^2=1$$

$$(r\cos\theta-1)^2+r^2\sin^2\theta=1$$

$$r^2\cos^2\theta-2r\cos\theta+1+r^2\sin^2\theta=1$$

$$r^2-2r\cos\theta=0$$

$$r(r-2\cos\theta)=0$$

$$r=0,\quad r=2\cos\theta$$

Now, For line $y=x$

$$\tan\theta = \pi/4 \quad \text{[see previous example]}$$

$$\theta = 0 \text{ to } \pi/4$$

Now, $dA = r\,dr\,d\theta$

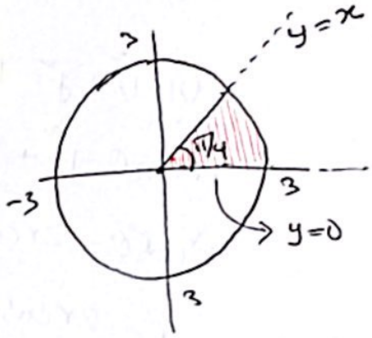
$$\therefore \iint\limits_R 2y\,dA = \int_0^{\pi/4}\int_0^{2\cos\theta} 2r\sin\theta\; r\,dr\,d\theta$$

$$= 2\int_0^{\pi/4}\int_0^{2\cos\theta} r^2 dr \sin\theta\, d\theta$$

Do yourself !!!

Problem: Express the volume of the solid described as a double integral in polar coordinates:

$$\iint_R \frac{1}{1+x^2+y^2}\, dA$$, where R is the sector in the first quadrant bounded by $y=0, y=x$, and $x^2+y^2=9$



Sol^n:

$dA = dxdy = r dr d\theta$

$r = 0$ and $r = 3$ from $x^2+y^2 = 3^2$

$\theta = 0$ and $\theta = \frac{\pi}{4}$ from $y=0, y=x$

Now,

$$\int_0^{\pi/4}\int_0^3 \frac{1}{1+r^2}\, r dr d\theta$$

$x = r\cos\theta$

$y = r\sin\theta$

Since, $y = x$

$r\sin\theta = r\cos\theta$

$\frac{\sin\theta}{\cos\theta} = 1 \quad r \neq 1$

$\tan\theta = 1$

$\theta = \frac{\pi}{4}$

$\boxed{\text{Do yourself}} \Rightarrow$