Lecture 17

Triple Integral

There is nothing special for triple integral if you understand double integral. Last lecture we discussed about the double integral. Here we give some example of triple integral in rectangular system.

Example: Evaluate $\displaystyle\int_2^3\int_{-1}^4\int_1^0 (4x^2y - z^3)\,dz\,dy\,dx$

$$= \int_2^3\int_{-1}^4 \left[\int_1^0 (4x^2y - z^3)\,dz\right] dy\,dx$$

$$= \int_2^3\int_{-1}^4 \left(4x^2yz - \frac{1}{4}z^4\right)\Big|_1^0\, dy\,dx$$

$$= \int_2^3\int_{-1}^4 \left[(0-0) - 4x^2y + \frac{1}{4}\right] dy\,dx$$

$$= \int_2^3 \left[\int_{-1}^4 \frac{1}{4} - 4x^2y\,dy\right] dx$$

$$\frac{1}{\cancel{4}}\cdot\cancel{4} - 4.x^2\frac{\cancel{16}^{8}}{\cancel{2}}$$

$$(1 - 32x^2) - \left(\frac{-1}{4} - \right.$$

$$2\cancel{4}x^2\frac{1}{\cancel{2}}$$

$$1 - 32x^2 + \frac{1}{4} + 2x^2$$

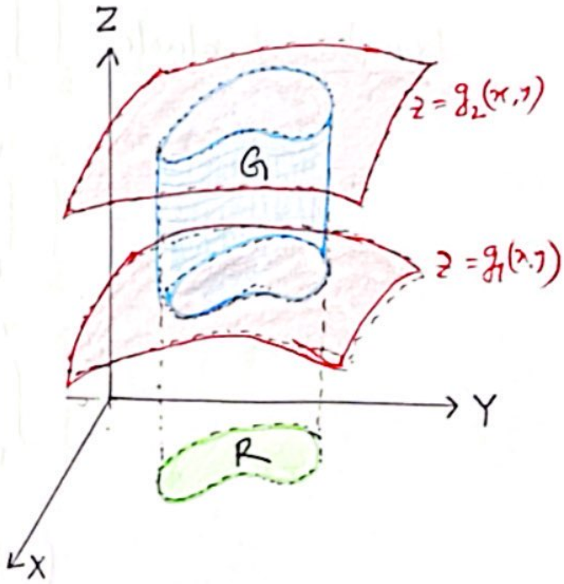
$$= \int_2^3 \left(\frac{1}{4}y - 4x^2\frac{y^2}{2}\right)\Big|_{-1}^4 dx$$

$$= \int_2^3 \left(\frac{5}{4} - 30x^2\right) dx$$

$$= \left(\frac{5}{4}x - 30\frac{x^3}{3}\right)\Big|_2^3 = -\frac{755}{4}$$

Theorem: Let G be a simple xy-solid with upper surface $z = g_2(x,y)$ and lower surface $z = g_1(x,y)$, and let R be the projection of G on the xy-plane. If $f(x,y,z)$ is continuous on G, then

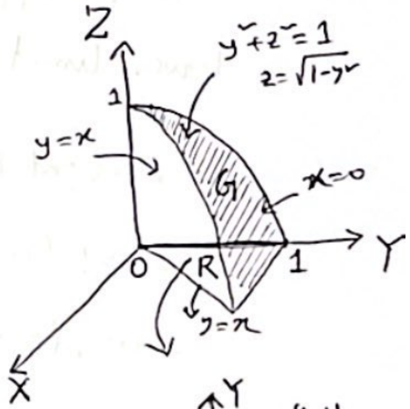
$$\iiint_G f(x,y,z)\,dV = \iint_R \left[\int_{g_1(x,y)}^{g_2(x,y)} f(x,y,z)\,dz \right] dA$$



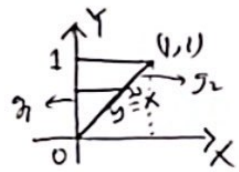
Example: Let G be the wedge in the first octant that is cut from the cylindrical solid $y^2+z^2 \le 1$ by the planes $y=x$ and $x=0$. Evaluate $\iiint_G z\,dV$.

Sol^n

$y^2+z^2=1$ that lies above the xy-plane, has the equation $z^2 = 1-y^2 \Rightarrow z=\sqrt{1-y^2}$, and the xy-plane has the equation $z=0$.



$$\iiint_G z\,dV = \iint_R \left[\int_0^{\sqrt{1-y^2}} z\,dz\right] dA$$



$g_1(x)=0$

$g_2(x)=y$

$$= \iint_R \left[\int_0^{\sqrt{1-y^2}} z\,dz\right] dx\,dy$$

$$= \int_0^1 \int_0^y \int_0^{\sqrt{1-y^2}} z\,dz\,dx\,dy$$

$$=$$

$$= \frac{1}{8}$$

Example: Find the volume of the solid in the first octant bounded by the coordinate planes and the plane with the equation

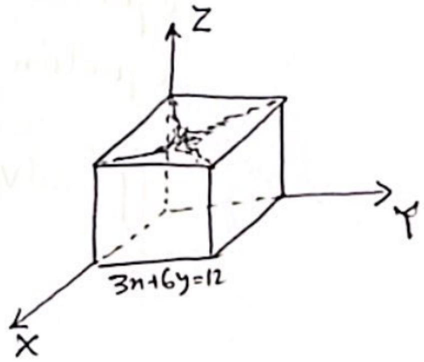
$$3x+6y+4z=12.$$

Sol^n.

$$3x+6y+4z=12$$

$$z=\frac{1}{4}(12-3x-6y)$$

So the upper limit of integration w.r. to z is $\frac{1}{4}(12-3x-6y)$ and lower limit is in the xy-plane i.e. $z=0$



x-axis: at $(4,0,0)$ $\quad 3x+0+0=12$, $\; x=4$

y-axis: $(0,2,0)$ $\quad 6y=12 \;\; y=2$

z-axis: $(0,0,3)$ $\quad 4z=12 \;\; z=3$

$$\iiint_V dV = \iint\left[\int_0^{\frac{1}{4}(12-3x-6y)} dz\right] dA$$

$dA = dy\,dx$

For dy:

$3x+6y+0=12$

$6y=12-3x$

$y=2-\frac{1}{2}x$

$$=\int_0^4\int_0^{(2-\frac{1}{2}x)}\int_0^{(3-\frac{3}{4}x-\frac{3}{2}y)} dz\,dy\,dx$$

$$=\int_0^4\int_0^{2-\frac{1}{2}x} z\Big|_0^{(3-\frac{3}{4}x-\frac{3}{2}y)} dy\,dx$$

$$=\int_0^4\int_0^{2-\frac{1}{2}x}\left(3-\frac{3}{4}x-\frac{3}{2}y\right)dy\,dx$$

$$=\int_0^4\left(3y-\frac{3}{4}xy-\frac{3}{4}y^2\right)\Big|_0^{(2-\frac{1}{2}x)} dx$$

$$=\int_0^4\left[6-\frac{3}{2}x-\frac{3}{4}x\left(2-\frac{1}{2}x\right)-\frac{3}{4}\left(2-\frac{1}{2}x\right)^2\right]dx$$

$=$ Do yourself.

Do yourself:

① Evaluate $\iiint_V (x-2y+z)\,dx\,dy\,dz$ where

$V: 0 \le x \le 1,\ 0 \le y \le x^2,\ 0 \le z \le (x+y)$.

② Evaluate: $\int_1^3 \int_x^{x^2} \int_0^{\ln z} x e^y \,dy\,dz\,dx$ in the Cartesian coordinates.

③ Evaluate: $\int_0^1 \int_0^{\sqrt{1-y^2}} \int_0^y z\,dx\,dz\,dy$.