Probability Distribution (3)

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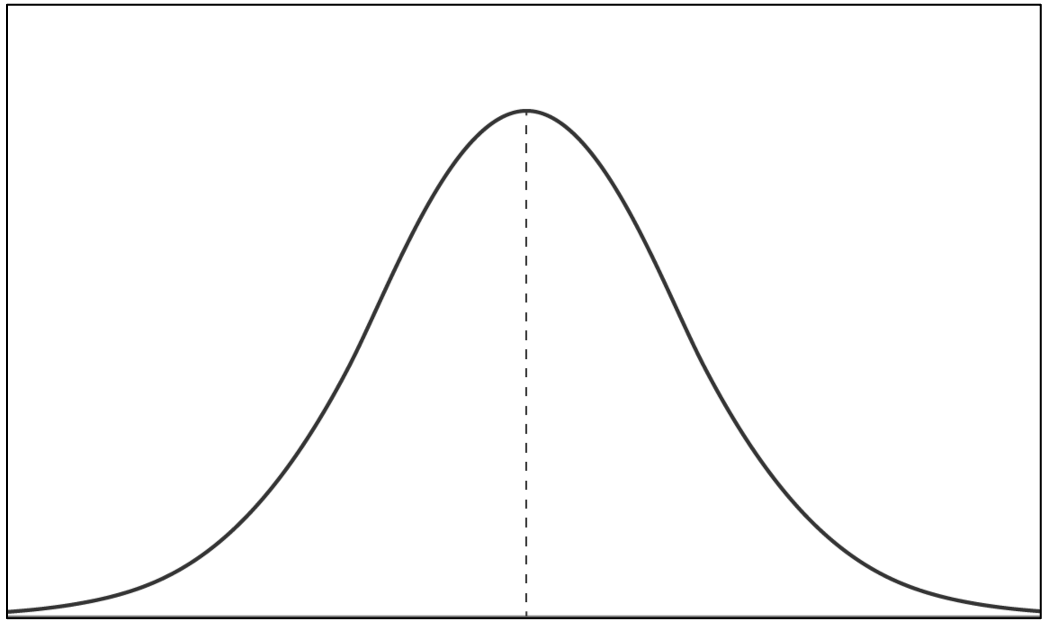
Normal Distribution

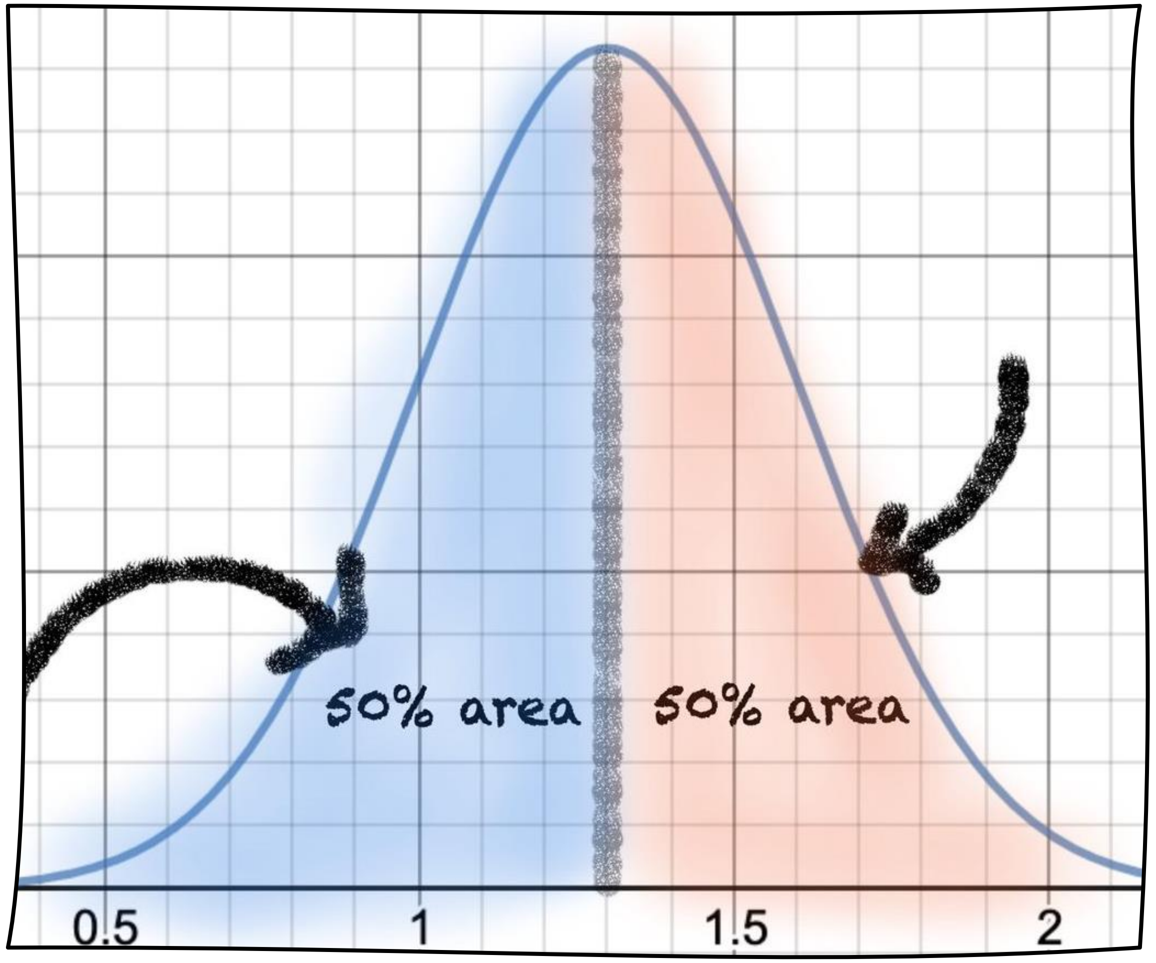
- Normal distribution is a probability distribution that is symmetric about the mean

- Also known as the Gaussian distribution



Properties Normal Distribution

1. Mean is the middle value and divides the area in half
2. Mean = Median = Mode
3. Symmetric and Mesokurtic
4. Bell-shaped curve



Normal Distribution

Let, X be a continuous random variable

Then the normal distribution function can be written as

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} \quad ; -\infty < x < \infty$$

$$X \sim N(\mu, \sigma^2)$$

Normal Distribution

- Mean of normal distribution, $E(X) = \mu$
- Variance of normal distribution, $V(X) = \sigma^2$

Standard Normal Distribution

Let, $Z = \frac{X-\mu}{\sigma}$

$$f(z) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2} \quad ; -\infty < z < \infty$$

Mean, $E(Z) = E\left(\frac{X-\mu}{\sigma}\right) = \frac{E(X)-\mu}{\sigma} = \frac{\mu-\mu}{\sigma} = 0$

$\boxed{Z \sim N(0,1)}$

Variance, $V(Z) = V\left(\frac{X-\mu}{\sigma}\right) = \frac{V(X)}{\sigma^2} = \frac{\sigma^2}{\sigma^2} = 1$

Standard Normal Distribution

- $X = 3, 11, 11, 11, 15, 15$

a. Determine the standardize variable Z corresponding to X $\boxed{Z = \frac{X-\mu}{\sigma} = \frac{X-11}{4}}$

b. Find the $Z - score$ of an observed value of $X = 11$ $\boxed{Z = \frac{X-\mu}{\sigma} = \frac{11-11}{4} = 0}$

c. Compute all possible $Z - scores$

d. Find mean and standard deviation of variable Z.

Probability Calculation

- There are two methods to compute the probabilities associated with a normal variable:

  a. Using Table (Z table)

  b. Using Calculator

Steps

$$\boxed{X} \longrightarrow \boxed{Z} \longrightarrow \boxed{Z = \frac{X-\mu}{\sigma}}$$

- To find probabilities for a normal random variable X, we can transform the probability statement about X in terms of probability statement for Z

- Then calculate the probability using the standard normal distribution table/Z-table or Calculator

Example

- The number of viewers of a TV show per week has a mean of 29 million with a standard deviation of 5 million. Assume that, the number of viewers of that show follows a normal distribution.

What is the probability that, next week's show will-

a. Have between 30 and 34 million viewers?
b. Have at least 23 million viewers?
c. Exceed 40 million viewers?

Example

Transform the probability statement in terms of Z

$$Z = \left(\frac{X-\mu}{\sigma}\right)$$

Let, X = Number of viewers of the show per week (in million)

Here, $X \sim N(\mu, \sigma^2)$; i.e., $X \sim N(29, 25)$

a. Have between 30 and 34 million viewers?

$$P(30 \le X \le 34)$$

$$P\left(\frac{30-\mu}{\sigma} \le \frac{X-\mu}{\sigma} \le \frac{34-\mu}{\sigma}\right)$$

$$P\left(\frac{30-\mu}{\sigma} \le Z \le \frac{34-\mu}{\sigma}\right)$$

$$P\left(\frac{30-29}{5} \le Z \le \frac{34-29}{5}\right)$$

$$P(0.2 \le Z \le 1)$$

$$P(1) - P(0.2)$$

$$??? - ???$$

Z table

TABLE A

Standard normal probabilities (continued)

z	.00	.01	.02	.03	.04	.05	.06	.07	.08	.09
0.0	.5000	.5040	.5080	.5120	.5160	.5199	.5239	.5279	.5319	.5359
0.1	.5398	.5438	.5478	.5517	.5557	.5596	.5636	.5675	.5714	.5753
0.2	.5793	.5832	.5871	.5910	.5948	.5987	.6026	.6064	.6103	.6141
0.3	.6179	.6217	.6255	.6293	.6331	.6368	.6406	.6443	.6480	.6517
0.4	.6554	.6591	.6628	.6664	.6700	.6736	.6772	.6808	.6844	.6879
0.5	.6915	.6950	.6985	.7019	.7054	.7088	.7123	.7157	.7190	.7224
0.6	.7257	.7291	.7324	.7357	.7389	.7422	.7454	.7486	.7517	.7549
0.7	.7580	.7611	.7642	.7673	.7704	.7734	.7764	.7794	.7823	.7852
0.8	.7881	.7910	.7939	.7967	.7995	.8023	.8051	.8078	.8106	.8133
0.9	.8159	.8186	.8212	.8238	.8264	.8289	.8315	.8340	.8365	.8389
1.0	.8413	.8438	.8461	.8485	.8508	.8531	.8554	.8577	.8599	.8621
1.1	.8643	.8665	.8686	.8708	.8729	.8749	.8770	.8790	.8810	.8830
1.2	.8849	.8869	.8888	.8907	.8925	.8944	.8962	.8980	.8997	.9015
1.3	.9032	.9049	.9066	.9082	.9099	.9115	.9131	.9147	.9162	.9177
1.4	.9192	.9207	.9222	.9236	.9251	.9265	.9279	.9292	.9306	.9319

Example

Transform the probability statement in terms of Z

$$Z = \left(\frac{X-\mu}{\sigma}\right)$$

Let, X = Number of viewers of the show per week (in million)

Here, $X \sim N(\mu, \sigma^2)$

a. Have between 30 and 34 million viewers?

$$P(30 \leq X \leq 34)$$

$$P\left(\frac{30-\mu}{\sigma} \leq \frac{X-\mu}{\sigma} \leq \frac{34-\mu}{\sigma}\right)$$

$$P\left(\frac{30-\mu}{\sigma} \leq Z \leq \frac{34-\mu}{\sigma}\right)$$

$$P\left(\frac{30-29}{5} \leq Z \leq \frac{34-29}{5}\right)$$

$$P\left(\frac{30-29}{5} \leq Z \leq \frac{34-29}{5}\right)$$

$$P(0.2 \leq Z \leq 1)$$

$$P(1) - P(0.2)$$

$$??? - ???$$

Example

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$$P(1) - P(0.2)$$

$$0.8431 - ???$$

Example

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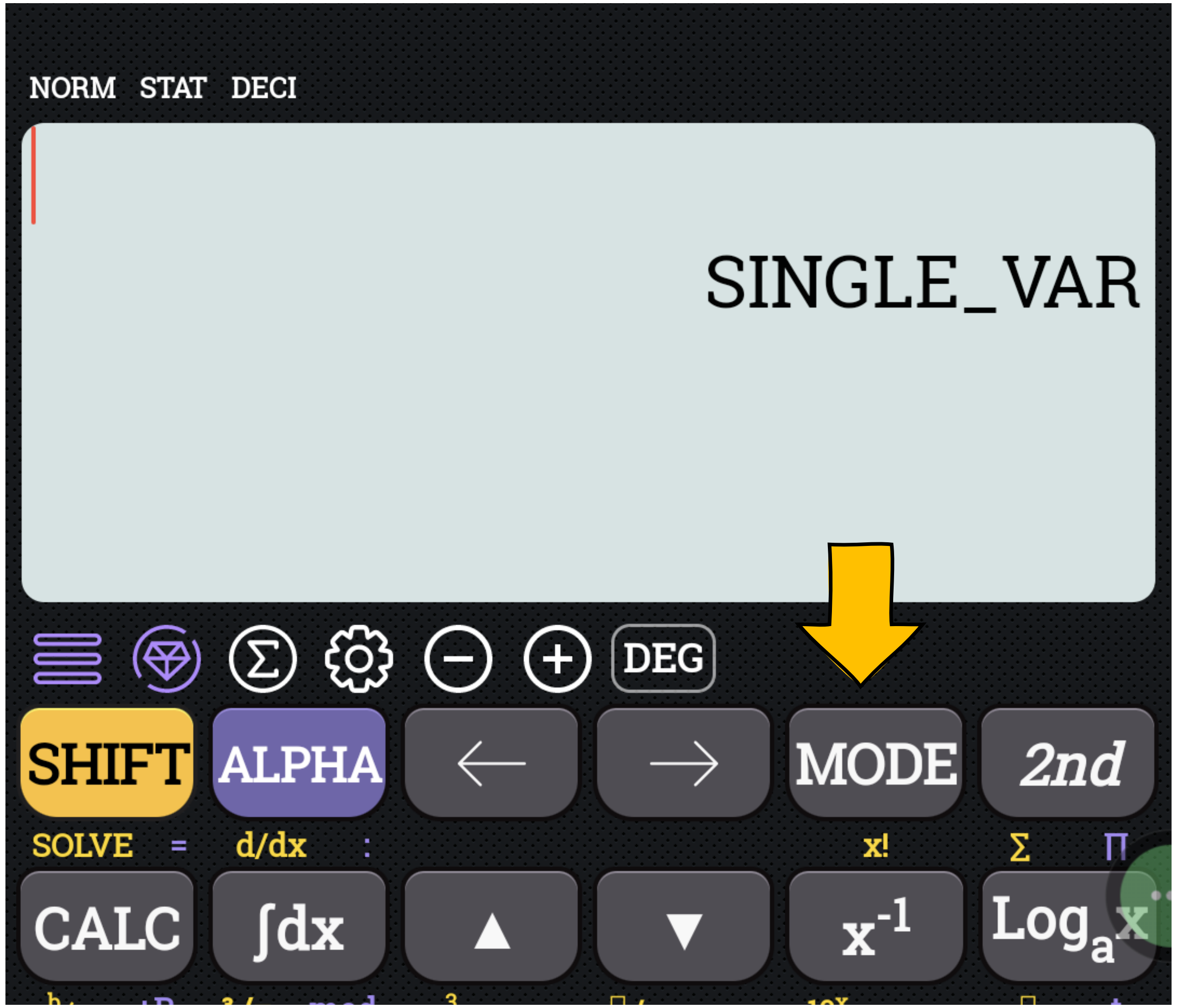
$$P(0.2 \le Z \le 1)$$

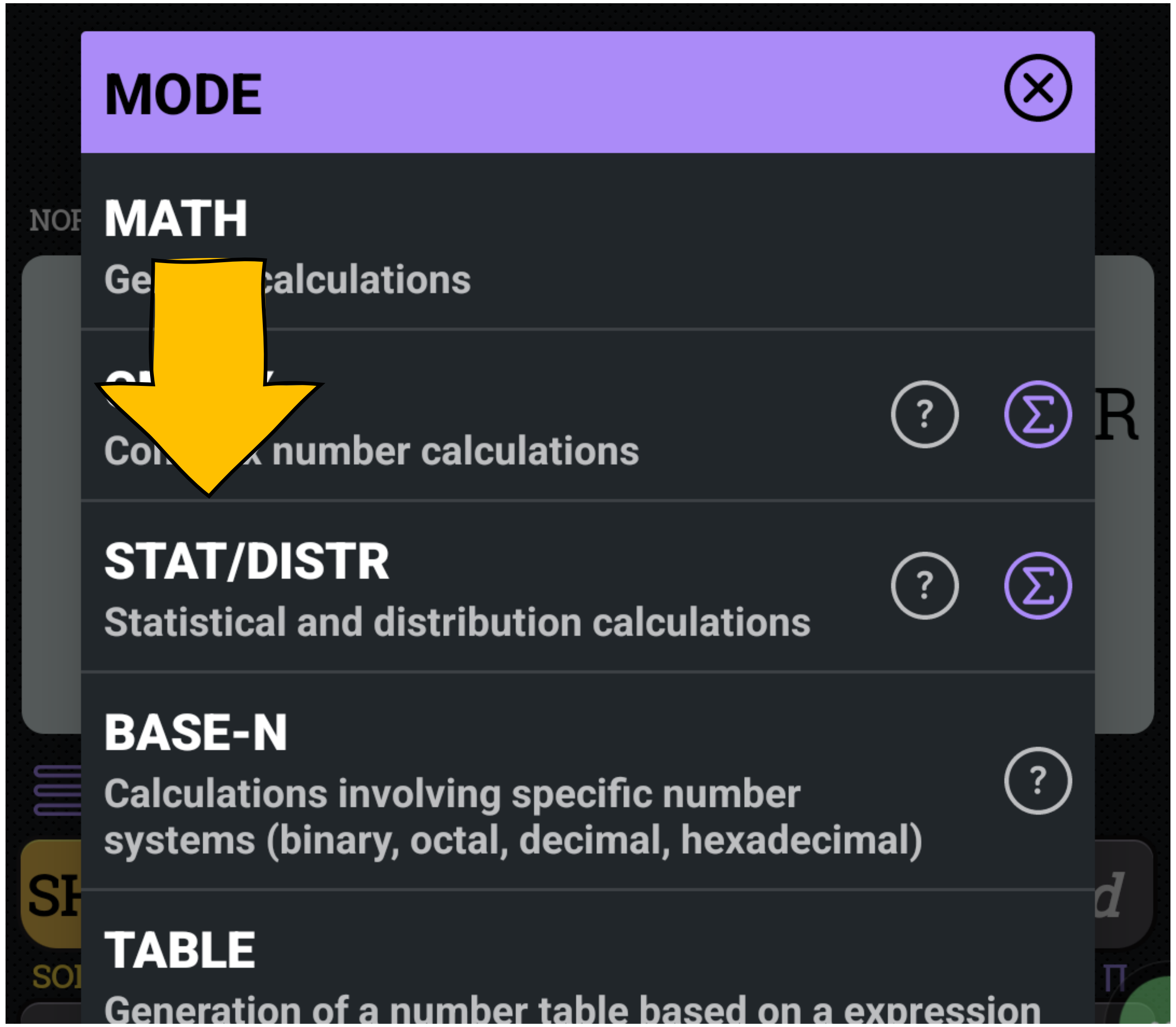
$$P(1) - P(0.2)$$

$$0.8431 - 0.5793$$

$$0.262$$

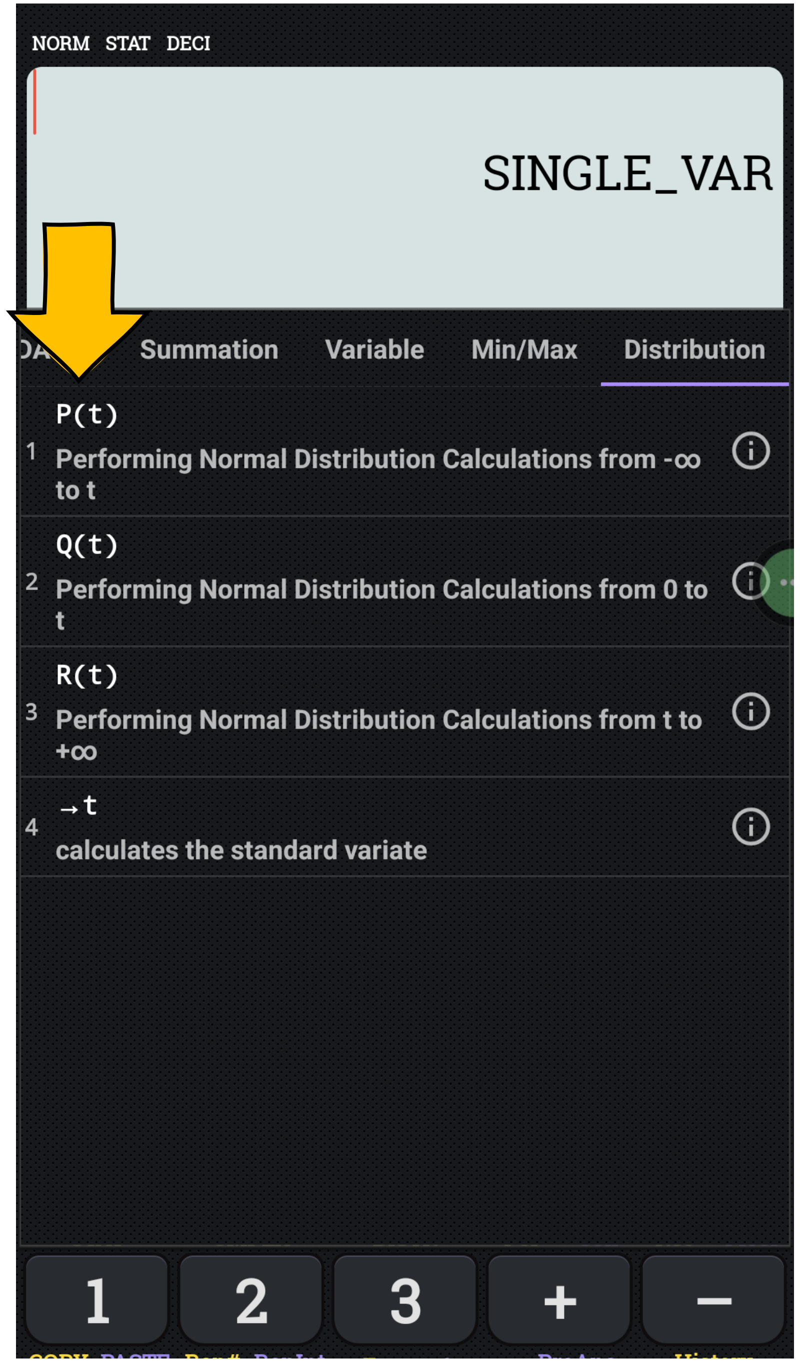
Calculator


NORM STAT DECI
SINGLE_VAR
DEG
SHIFT
ALPHA
MODE
2nd
SOLVE
=
d/dx
:
x!
Σ
Π
CALC
∫dx
x^{-1}
$Log_a x$

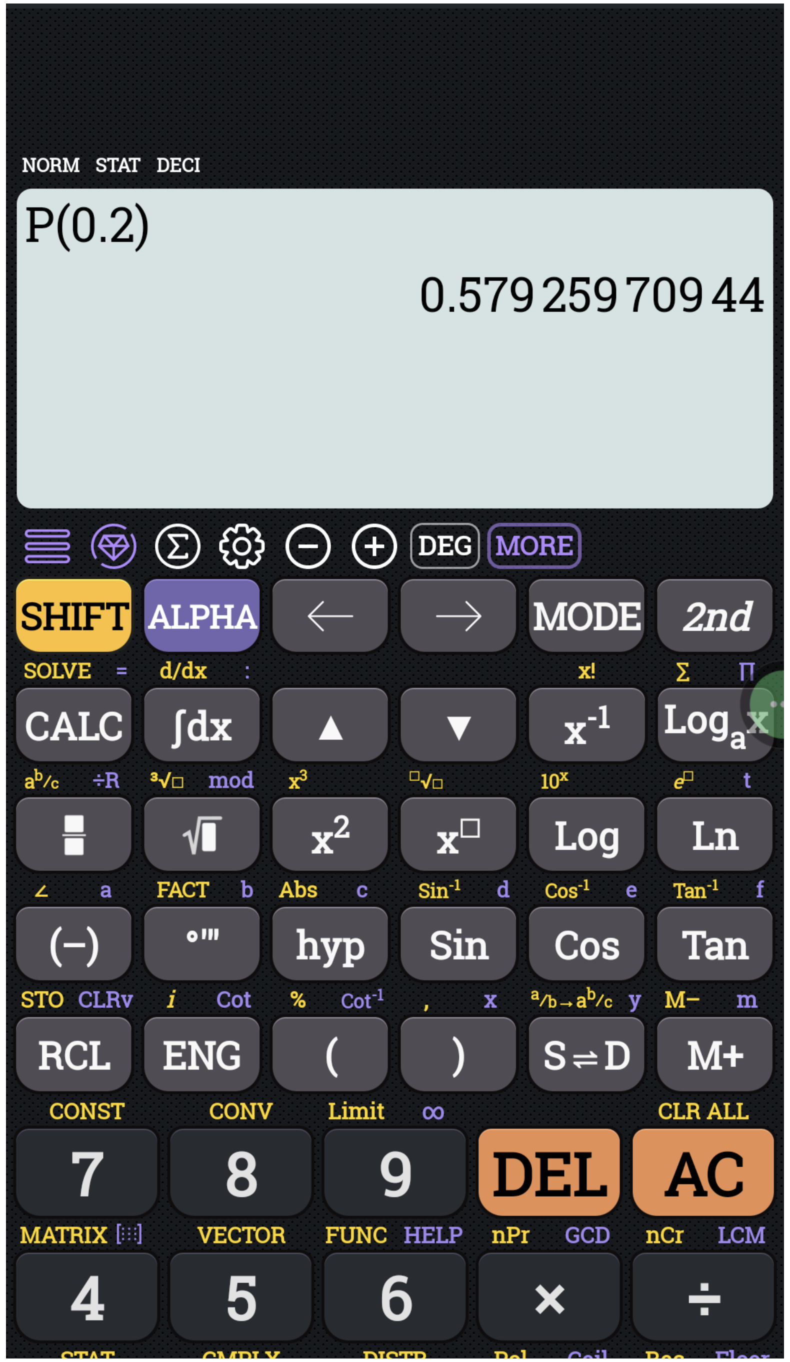

MODE
MATH
calculations
number calculations
STAT/DISTR
Statistical and distribution calculations
BASE-N
Calculations involving specific number systems (binary, octal, decimal, hexadecimal)
TABLE
Generation of a number table based on a expression

Calculator

NORM STAT DECI
SINGLE_VAR
SHIFT ALPHA MODE 2nd
CALC ∫dx Log
STAT



NORM STAT DECI
SINGLE_VAR
Summation Variable Min/Max Distribution
P(t)
Performing Normal Distribution Calculations from -∞ to t
Q(t)
Performing Normal Distribution Calculations from 0 to t
R(t)
Performing Normal Distribution Calculations from t to +∞
→t
calculates the standard variate



NORM STAT DECI
P(0.2)
0.579 259 709 44

Example

Transform the probability statement in terms of Z

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$$P\left(\frac{30-\mu}{\sigma} \leq Z \leq \frac{34-\mu}{\sigma}\right)$$

$$P\left(\frac{30-29}{5} \leq Z \leq \frac{34-29}{5}\right)$$

$$P\left(\frac{30-29}{5} \leq Z \leq \frac{34-29}{5}\right)$$

$$P(0.2 \leq Z \leq 1)$$

$$P(1) - P(0.2)$$

$$0.8431 - 0.5793$$

$$0.262$$

Example

Transform the probability statement in terms of Z

$$Z = \left(\frac{X-\mu}{\sigma}\right)$$

b. Have at least 23 million viewers?

$P(X \geq 23)$

$1 - P(X < 23)$

$$1 - P\left(\frac{X-\mu}{\sigma} \leq \frac{23-\mu}{\sigma}\right)$$

$$1 - P\left(Z \leq \frac{23-29}{5}\right)$$

$1 - P(Z \leq -1.2)$

$1 - P(-1.2)$

$1 - 0.1151$

0.8849

Normal Distribution

Transform the probability statement in terms of Z

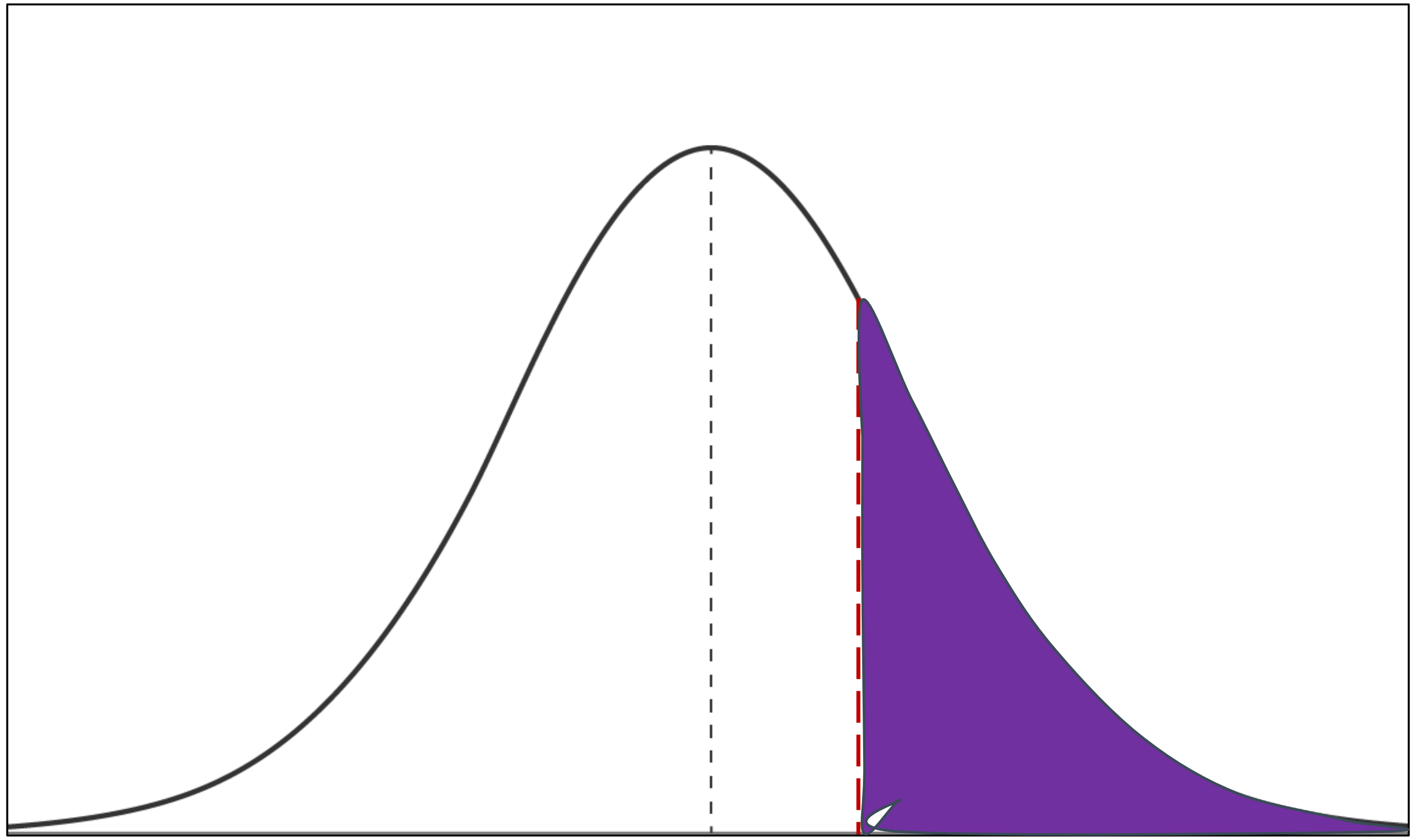
$$Z = \left(\frac{X-\mu}{\sigma}\right)$$

c. Exceed 40 million viewers?

$$P(X > 40)$$

$$1 - P(X \leq 40)$$

$$1 - P\left(\frac{X-\mu}{\sigma} \leq \frac{40-29}{5}\right)$$



$$1 - P(Z \leq 2.2)$$

$$1 - P(2.2)$$

$$0.0139$$

Example

- Suppose that the average temperature in July in a certain region is a normal random variable with parameters $\mu = 90^{\circ}F$ and $\sigma = 5^{\circ}F$. Find the probability that in a given year the average temperature in July in that region will be,

a. Above $100°F$ (Ans: 0.00228)
b. Below $95°F$ (Ans: 0.8413)
c. Between $85°F$ and $95°F$ (Ans: 0.6826)

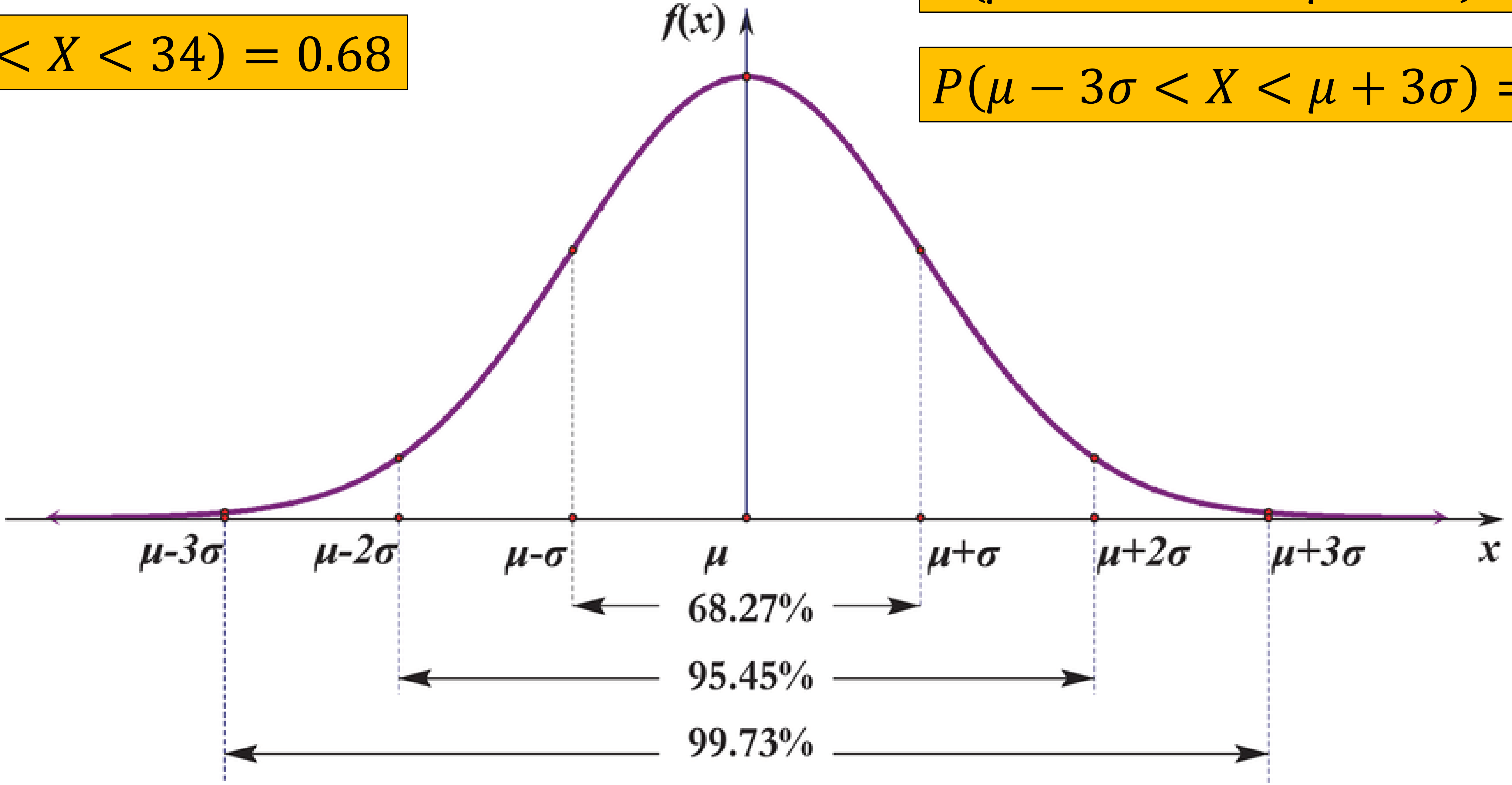
3σ Rules

$P(\mu - \sigma < X < \mu + \sigma) = 0.68$

$P(\mu - 2\sigma < X < \mu + 2\sigma) = 0.95$

$P(\mu - 3\sigma < X < \mu + 3\sigma) = 0.99$

$P(30 < X < 34) = 0.68$



Example

- The IQ score of students follows normal distribution with mean 100 and standard deviation 16. In what interval would you expect the central 95% of IQ scores to be found?

It is expected that 95% of students have an IQ between 68 and 132

Here, $\mu = 100,\ \sigma = 16$

We know that, $P(\mu - 2\sigma < X < \mu + 2\sigma) = 0.95$

$\mu - 2\sigma = 100 - (2 \times 16) = 68$

$\mu + 2\sigma = 100 + (2 \times 16) = 132$

Example

- For what value of "c", $P(X > c) = 0.01$
- $P\left(\frac{X-\mu}{\sigma} > \frac{c-100}{16}\right) = 0.01$
- $P\left(Z > \frac{c-100}{16}\right) = 0.01$
- $1 - P\left(Z \leq \frac{c-100}{16}\right) = 0.01$
- $P\left(Z \leq \frac{c-100}{16}\right) = 0.99$
- $P(Z \leq 2.33) = 0.99$

$$\boxed{\frac{c-100}{16} = 2.33}$$

$$\boxed{c = ???}$$

TABLE A

Standard normal probabilities (continued)

z	.00	.01	.02	.03
0.0	.5000	.5040	.5080	.5120
0.1	.5398	.5438	.5478	.5517
0.2	.5793	.5832	.5871	.5910
0.3	.6179	.6217	.6255	.6293
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1.3	.9032	.9049	.9066	.9082
1.4	.9192	.9207	.9222	.9236
1.5	.9332	.9345	.9357	.9370
1.6	.9452	.9463	.9474	.9484
1.7	.9554	.9564	.9573	.9582
1.8	.9641	.9649	.9656	.9664
1.9	.9713	.9719	.9726	.9732
2.0	.9772	.9778	.9783	.9788
2.1	.9821	.9826	.9830	.9834
2.2	.9861	.9864	.9868	.9871
2.3	.9893	.9896	.9898	.9901
2.4	.9918	.9920	.9922	.9925

Mathematical exercise

To access additional mathematical problems, please refer to the PDF lecture notes.

Thank You