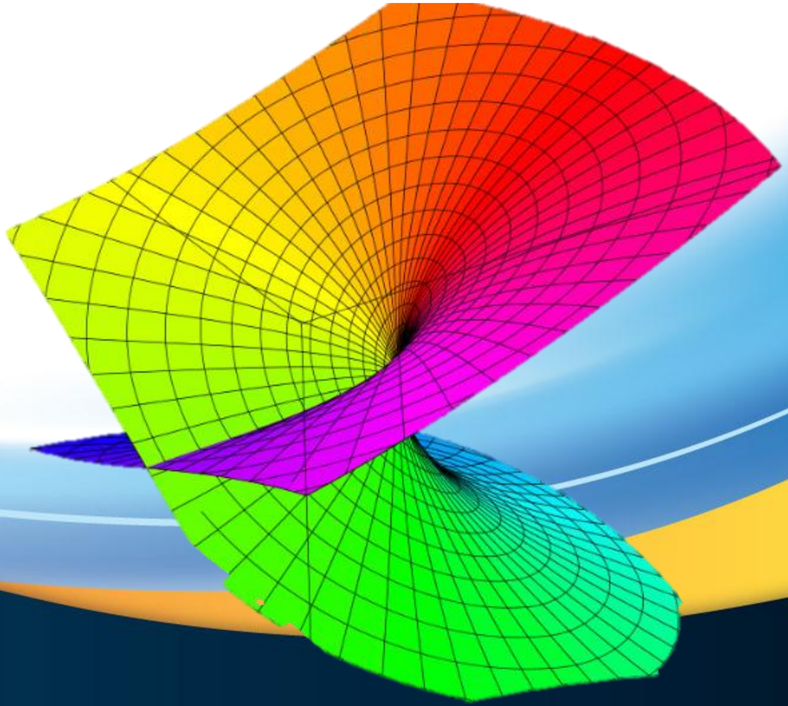
Undergraduate Course in Mathematics

Complex Variables

Topic: n^{th} root Finding and Graphing



Conducted By

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Modulus

$$z = a + ib$$

$$|z| = \sqrt{a^2 + b^2}$$

Arguments Formulae

$z = a + bi$, n any integer.

$$\theta = \pi - \tan^{-1}\left|\frac{b}{a}\right| + 2n\pi$$

$$\theta = \tan^{-1}\left|\frac{b}{a}\right| + 2n\pi$$

$$\theta = -\pi + \tan^{-1}\left|\frac{b}{a}\right| + 2n\pi$$

$$\theta = -\tan^{-1}\left|\frac{b}{a}\right| + 2n\pi$$

Euler’s Formula and Polar Form

$$\Rightarrow \quad e^{i\theta} = \cos\theta + i\sin\theta$$

$$a + bi = re^{i\theta} = r(\cos\theta + i\sin\theta)$$

De Moivre's Theorem

$$(\cos\theta + i\sin\theta)^n = \cos(n\theta) + i\sin(n\theta)$$

$$n = 1, 2, 3, 4, \ldots$$

General De Moivre's Theorem

$$(\cos\theta + i\sin\theta)^n = \cos(n\theta) + i\sin(n\theta)$$

Calculating Large Power using De Moivre's Theorem

$$(a+ib)^n = \left(r(\cos\theta + i\sin\theta)\right)^n$$

$$(1+\sqrt{3}\,i)^{10}$$

$$\text{mod} = \sqrt{1^2+(\sqrt{3})^2} = 2$$

$$\text{Arg} = \tan^{-1}\left|\frac{\sqrt{3}}{1}\right| + 2n\pi = \frac{\pi}{3} + 2n\pi$$

$$= \left(2\cdot\left[\cos\left(\frac{\pi}{3}+2n\pi\right) + i\,\sin\left(\frac{\pi}{3}+2n\pi\right)\right]\right)^{10}$$

$$= 2^{10}\cdot\left[\cos\left(\frac{\pi}{3}+2n\pi\right) + i\,\sin\left(\frac{\pi}{3}+2n\pi\right)\right]^{10}$$

$$= 1024\left[\cos\left(\frac{10\pi}{3}+20n\pi\right) + i\,\sin\left(\frac{10\pi}{3}+20n\pi\right)\right]$$

$$= 1024\left(-\frac{1}{2} - i\frac{\sqrt{3}}{2}\right)$$

B

Finding n^{th} Roots using De Moivre's Theorem

5th root

$$(a+bi)^{\frac{1}{5}} \rightarrow$$

Using De Moivre's theorem, find all the 4^{th} roots of the complex number (-16)

$$-16 = 16\left(\cos(\pi+2n\pi)+i\sin(\pi+2n\pi)\right)$$

$$\Rightarrow (-16)^{\frac{1}{4}} = \left[2^4\left(\cos(\pi+2n\pi)+i\sin(\pi+2n\pi)\right)\right]^{\frac{1}{4}}$$

$$= 2\left(\cos\left(\frac{\pi+2n\pi}{4}\right)+i\sin\left(\frac{\pi+2n\pi}{4}\right)\right)$$

n is any integer.

$\underline{\underline{n=0}}$

$$(-16)^{\frac{1}{4}} = 2\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right) = \boxed{\sqrt{2} + i\sqrt{2}}$$

$\underline{\underline{n=1}}$

$$(-16)^{\frac{1}{4}} = 2\left(\cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4}\right) = \boxed{-\sqrt{2} + i\sqrt{2}}$$

$\underline{\underline{n=2}}$

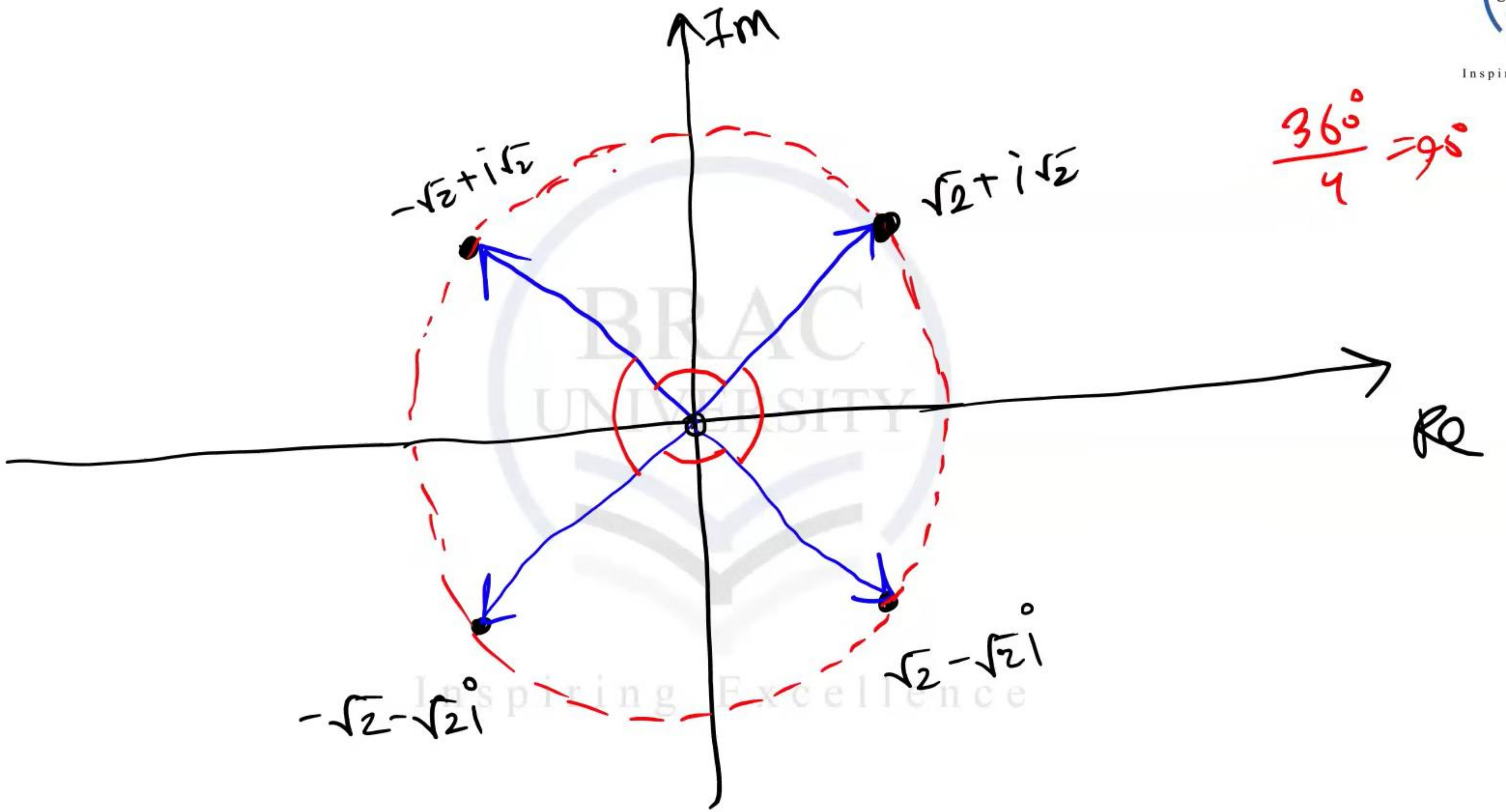
$$(-16)^{\frac{1}{4}} = 2\left(\cos\frac{5\pi}{4} + i\sin\frac{5\pi}{4}\right) = \boxed{-\sqrt{2} - i\sqrt{2}}$$

$\underline{n=3}$

$$(-16)^{\frac{1}{4}} = 2\left(\cos\frac{7\pi}{4} + i\sin\frac{7\pi}{4}\right) = \boxed{\sqrt{2} - i\sqrt{2}}$$

$\underline{\underline{n=4}}$

$$(-16)^{\frac{1}{4}} = 2\left(\cos\frac{9\pi}{4} + i\sin\frac{7\pi}{4}\right) = \boxed{\sqrt{2} + i\sqrt{2}}$$



$\frac{360^\circ}{4} = 90^\circ$

Find each of the indicated roots and locate them graphically.

(a) $(-1+i)^{1/3}$; (b) $(-2\sqrt{3}-2i)^{1/4}$

(a) $$(-1+i)^{\frac{1}{3}} = \left[\sqrt{2}\left(\cos\left(\frac{3\pi}{4}+2n\pi\right)+i\sin\left(\frac{3\pi}{4}+2n\pi\right)\right)\right]^{\frac{1}{3}}$$

$$= 2^{\frac{1}{6}}\left[\cos\left(\frac{3\pi+8n\pi}{4}\right)+i\sin\left(\frac{3\pi+8n\pi}{4}\right)\right]^{\frac{1}{3}}$$

$$= 2^{\frac{1}{6}}\left(\cos\left(\frac{3\pi+8n\pi}{12}\right)+i\sin\left(\frac{3\pi+8n\pi}{12}\right)\right)$$

when $n=0,1,2$

$\underline{n=0}$

$$(-1+i)^{\frac{1}{3}} = 2^{\frac{1}{6}}\cdot\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right) = 2^{\frac{1}{6}}\left(\sqrt{2} + i\sqrt{2}\right)$$

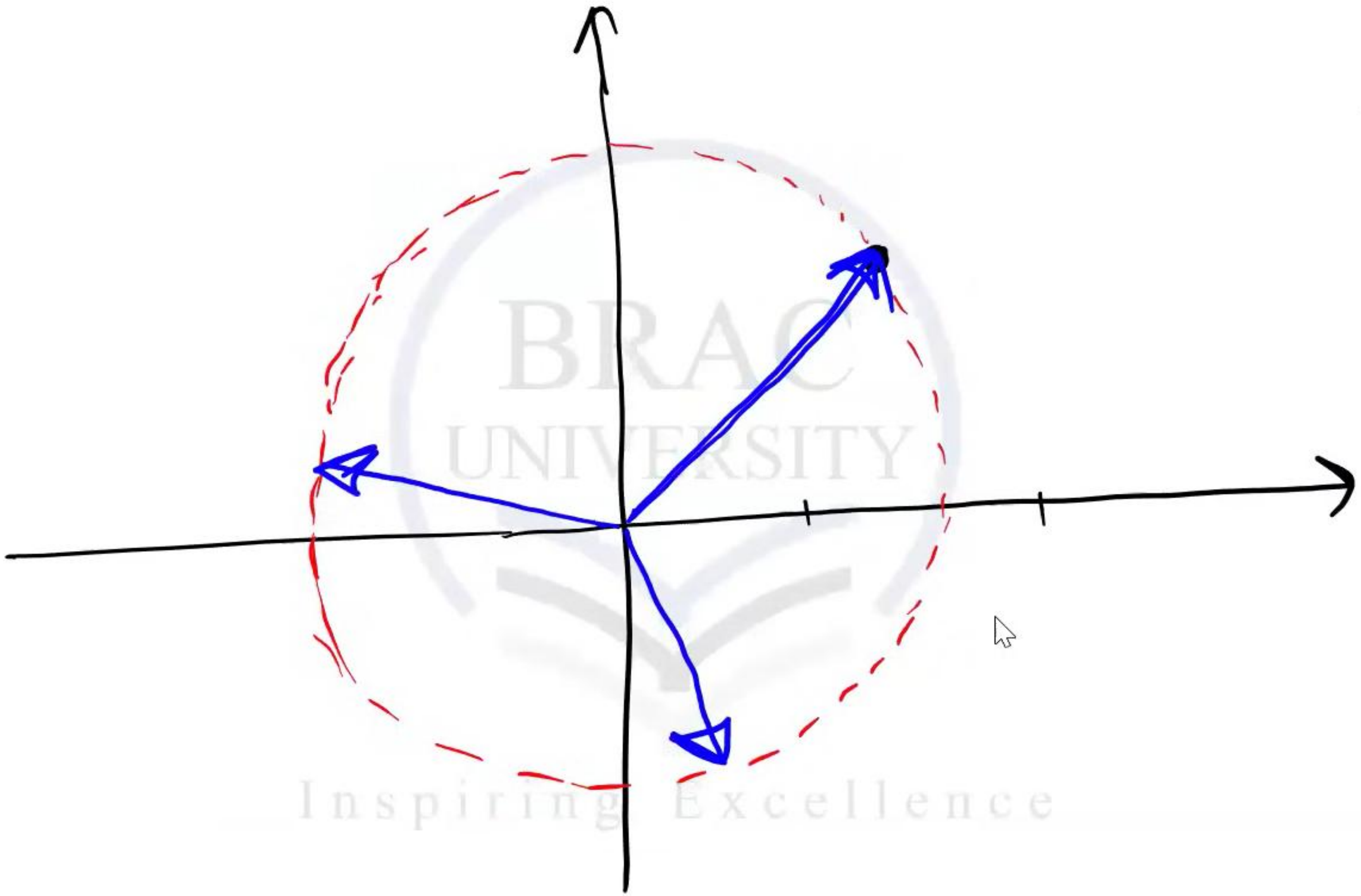
$$= 2^{\frac{1}{6}}\sqrt{2} + i\,2^{\frac{1}{6}}\cdot\sqrt{2}$$

$\underline{n=1}$

$$(-1+i)^{\frac{1}{3}} = 2^{\frac{1}{6}}\left(\cos\frac{11\pi}{12} + i\sin\frac{11\pi}{12}\right)$$

$\underline{\underline{n=2}}$

$$(-1+i)^{\frac{1}{3}} = 2^{\frac{1}{6}}\left(\cos\frac{19\pi}{12} + i\sin\frac{19\pi}{12}\right)$$



$$\frac{360°}{3} = 120°$$

Find all the cube roots of unity

$$1 = 1\cdot\left(\cos(2n\pi) + i\sin(2n\pi)\right)$$

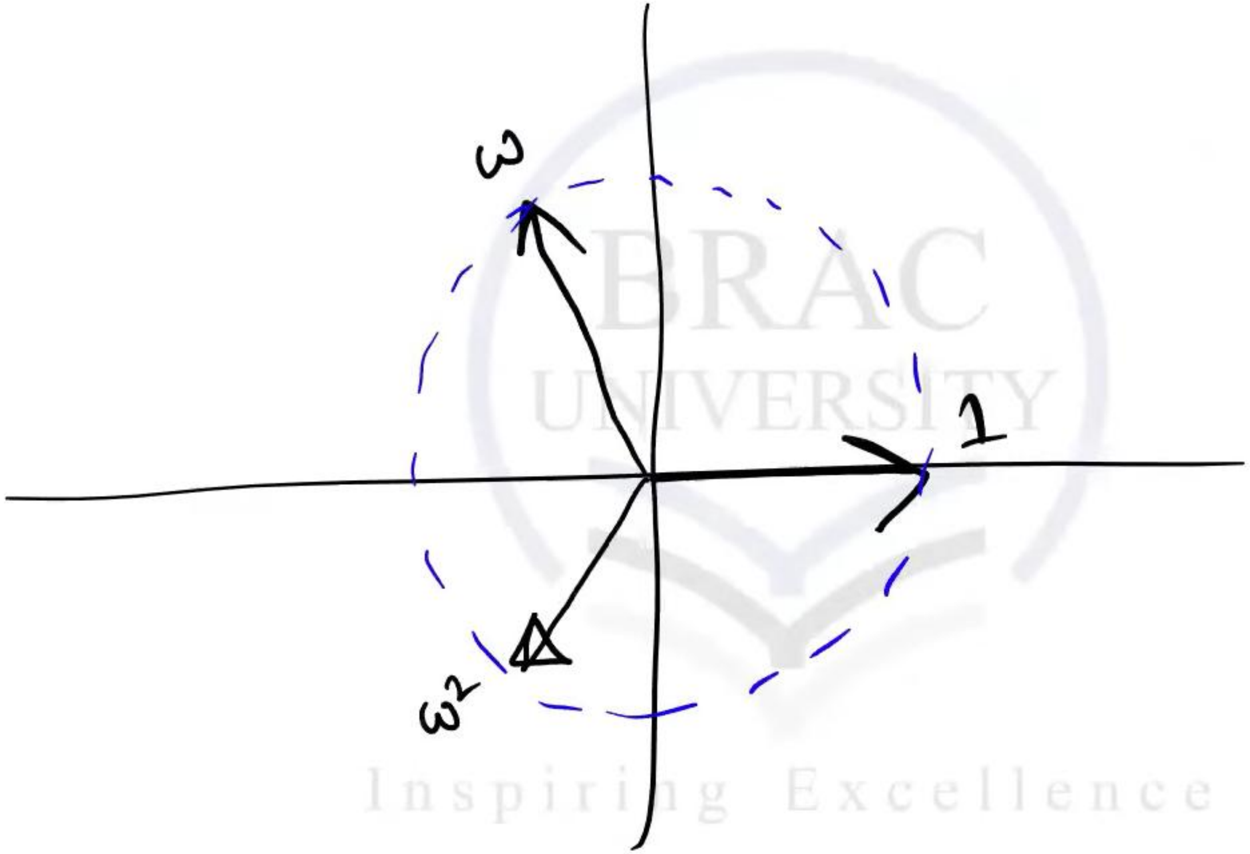
$$(1)^{\frac{1}{3}} = \cos\left(\frac{2n\pi}{3}\right) + i\sin\left(\frac{2n\pi}{3}\right)$$

$n = 0, 1, 2$

$\underline{\underline{n=0}}$ $\quad (1)^{\frac{1}{3}} = \cos(0) + i\sin(0) = 1$

$\underline{\underline{n=1}}$ $\quad (1)^{\frac{1}{3}} = \cos\left(\frac{2\pi}{3}\right) + i\sin\left(\frac{2\pi}{3}\right) = -\frac{1}{2} + \frac{\sqrt{3}}{2}i = \omega$

$\underline{\underline{n=2}}$ $\quad (1)^{\frac{1}{3}} = \cos\left(\frac{4\pi}{3}\right) + i\sin\left(\frac{4\pi}{3}\right) = -\frac{1}{2} - \frac{\sqrt{3}}{2}i = \omega^2$



$$1+\omega+\omega^2=0$$

Find all the values of z for which $z^5 = 32\sqrt{-1}$, and locate them in the complex plane.

$$z^5 = 32i$$

$$\Rightarrow z = (32i)^{\frac{1}{5}}$$

$$= \left[2^5 \cdot \left(\cos\left(\frac{\pi}{2} + 2n\pi\right) + i \sin\left(\frac{\pi}{2} + 2n\pi\right) \right) \right]^{\frac{1}{5}}$$

$$= 2 \cdot \left(\cos \frac{\pi + 4n\pi}{10} + i \sin \frac{\pi + 4n\pi}{10} \right)$$

$$n = 0, 1, 2, 3, 4$$

$\underline{n=0}$

$$z = 2\cos\left(\frac{\pi}{5}\right) + 2i\sin\frac{\pi}{5}$$

$\underline{n=1}$

$$z = 2\cos\left(\frac{5\pi}{10}\right) + i\,2\sin\left(\frac{5\pi}{10}\right) = 2i$$

$\underline{n=2}$

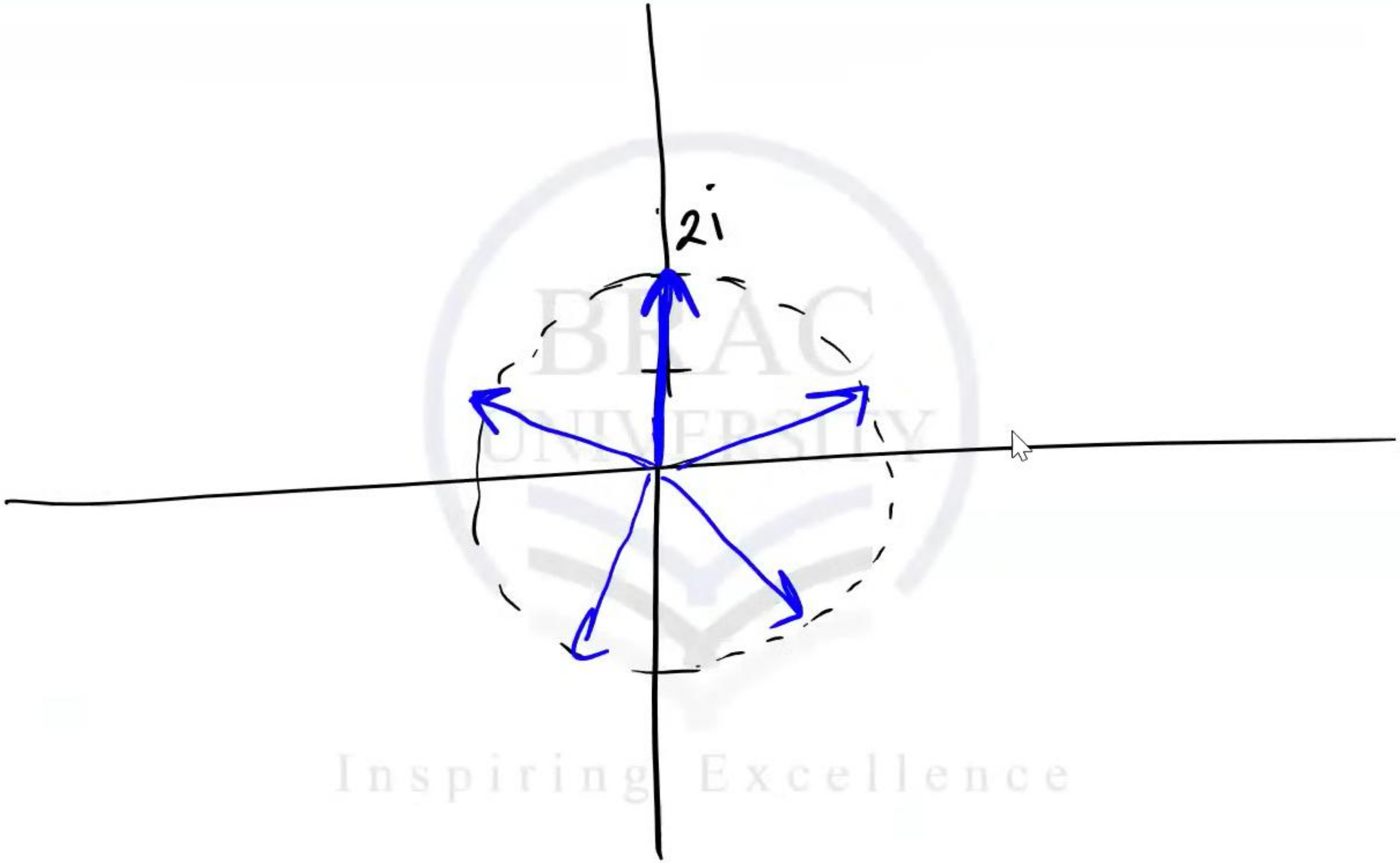
$$z = 2\cos\left(\frac{9\pi}{10}\right) + i\,2\sin\left(\frac{9\pi}{10}\right)$$

$\underline{n=3}$

$$z = 2\cos\left(\frac{13\pi}{10}\right) + i\,2\sin\left(\frac{13\pi}{10}\right)$$

$\underline{n=4}$

$$z = 2\cos\left(\frac{17\pi}{10}\right) + i\,2\sin\left(\frac{17\pi}{10}\right)$$


2i

Graphing

$|z| \rightarrow$ modulus of z.

$\text{Arg}(z) \rightarrow$ Argument of z

$(x-a)^2 + (y-b)^2 = r^2$.

centre (a, b)

radius $= r$

Graph the Lines

i) $|Z| = 2$

ii) $|z - 2i| = 3$

iii) $\left|\frac{z-3}{z+3}\right| = 2$

iv) $Im\{z^2\} = 4$

v) $Re\{z^2\} = 4$

vi) $Re\left\{\frac{1}{z}\right\} = 1$

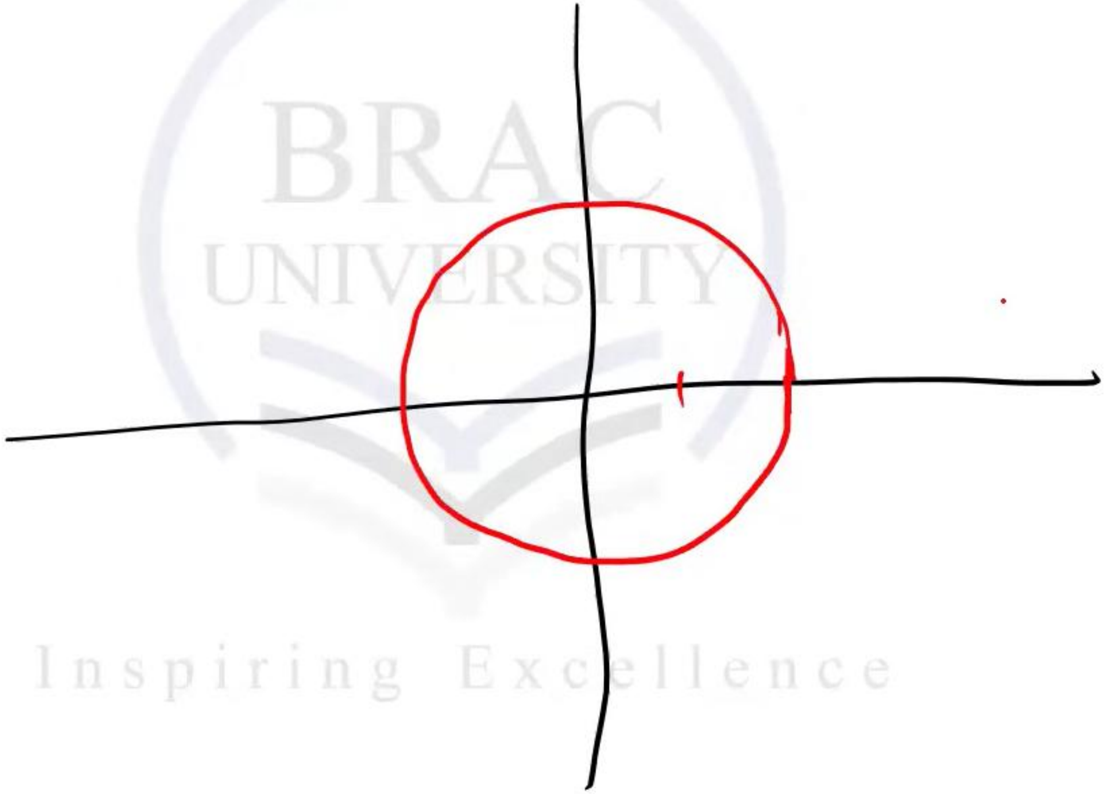
vii) $Arg\{Z\} = \frac{\pi}{3}$

$z = x + iy$

i) $|Z| = 2$

$\Rightarrow \sqrt{x^2+y^2} = 2$

$\Rightarrow x^2+y^2 = 2^2$



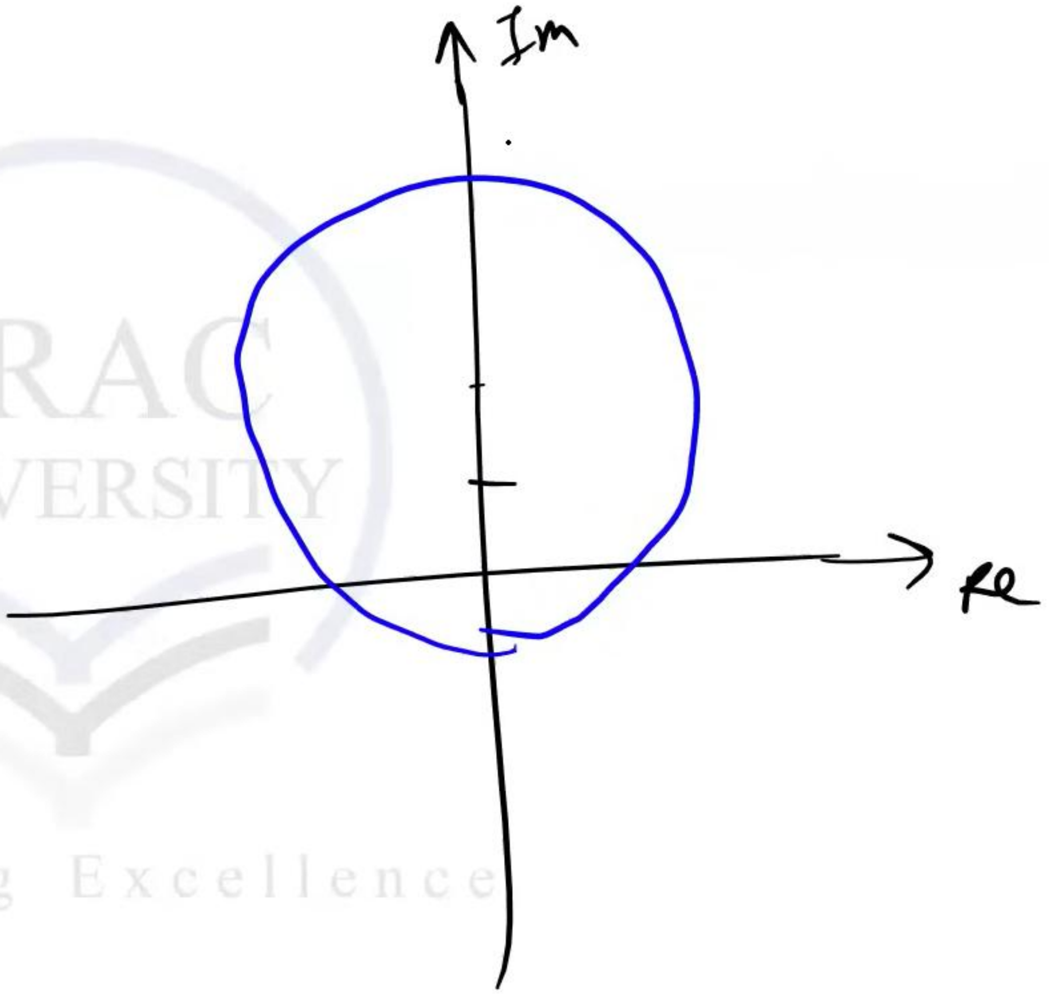
ii) $|z - 2i| = 3$

$\Rightarrow |x + iy - 2i| = 3$

$\Rightarrow |x + (y-2)i| = 3$

$\Rightarrow \sqrt{x^2 + (y-2)^2} = 3$

$\Rightarrow (x-0)^2 + (y-2)^2 = 3^2$



$$iii)\ \left|\frac{z-3}{z+3}\right| = 2$$

$$\Rightarrow \frac{|z-3|}{|z+3|} = 2$$

$$\Rightarrow \frac{|x+iy-3|}{|x+iy+3|} = 2$$

$$\Rightarrow \frac{|(x-3)+yi|}{|(x+3)+yi|} = 2$$

$$\Rightarrow \frac{\sqrt{(x-3)^2+y^2}}{\sqrt{(x+3)^2+y^2}} = 2$$

$$\Rightarrow \frac{(x-3)^2+y^2}{(x+3)^2+y^2} = 4$$

$$4x^2+24x+36+4y^2 = x^2-6x+9+y^2$$

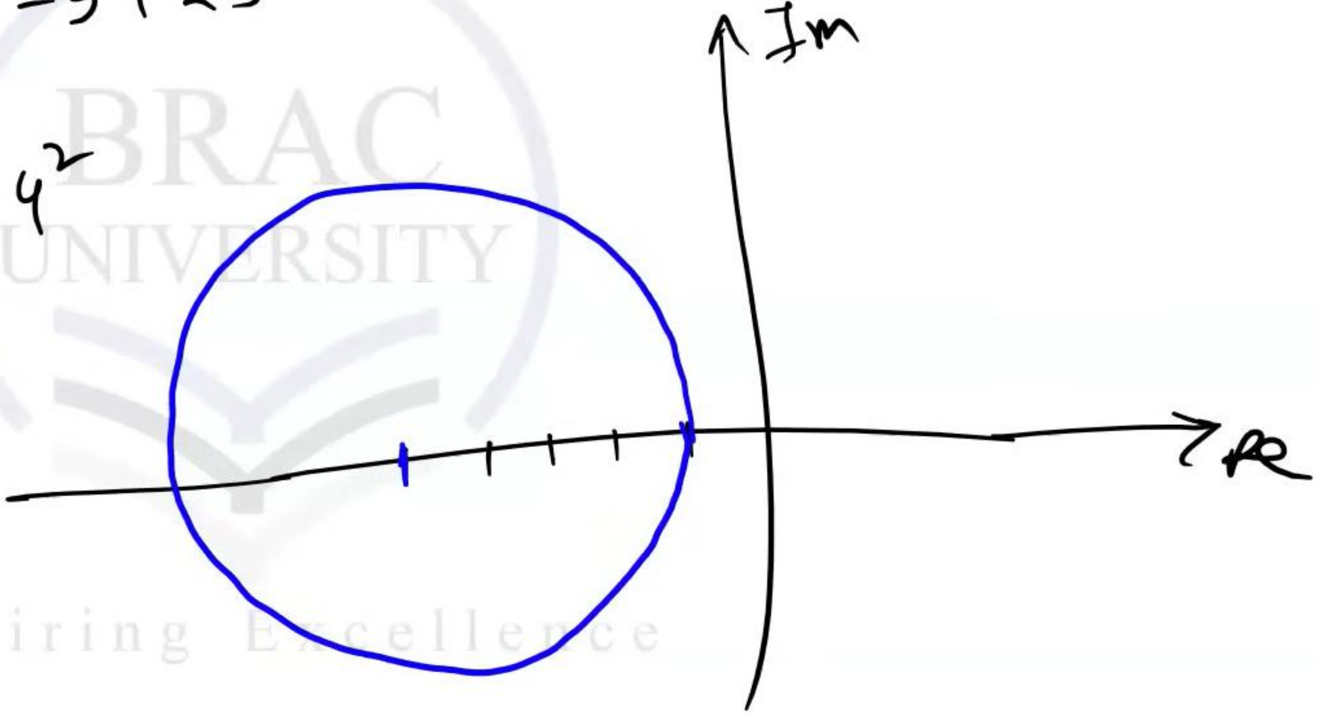
$$\Rightarrow 4x^2+24x+36+4y^2$$

$$\Rightarrow 3x^2+30x+3y^2 = -27$$

$$\Rightarrow \quad x^2 + 10x + y^2 = -9$$

$$\Rightarrow \left(x^2 + 10x + 25\right) + y^2 = -9 + 25$$

$$\Rightarrow \quad (x+5)^2 + (y-0)^2 = 4^2$$



iv) $\operatorname{Im}\{z^2\} = 4$

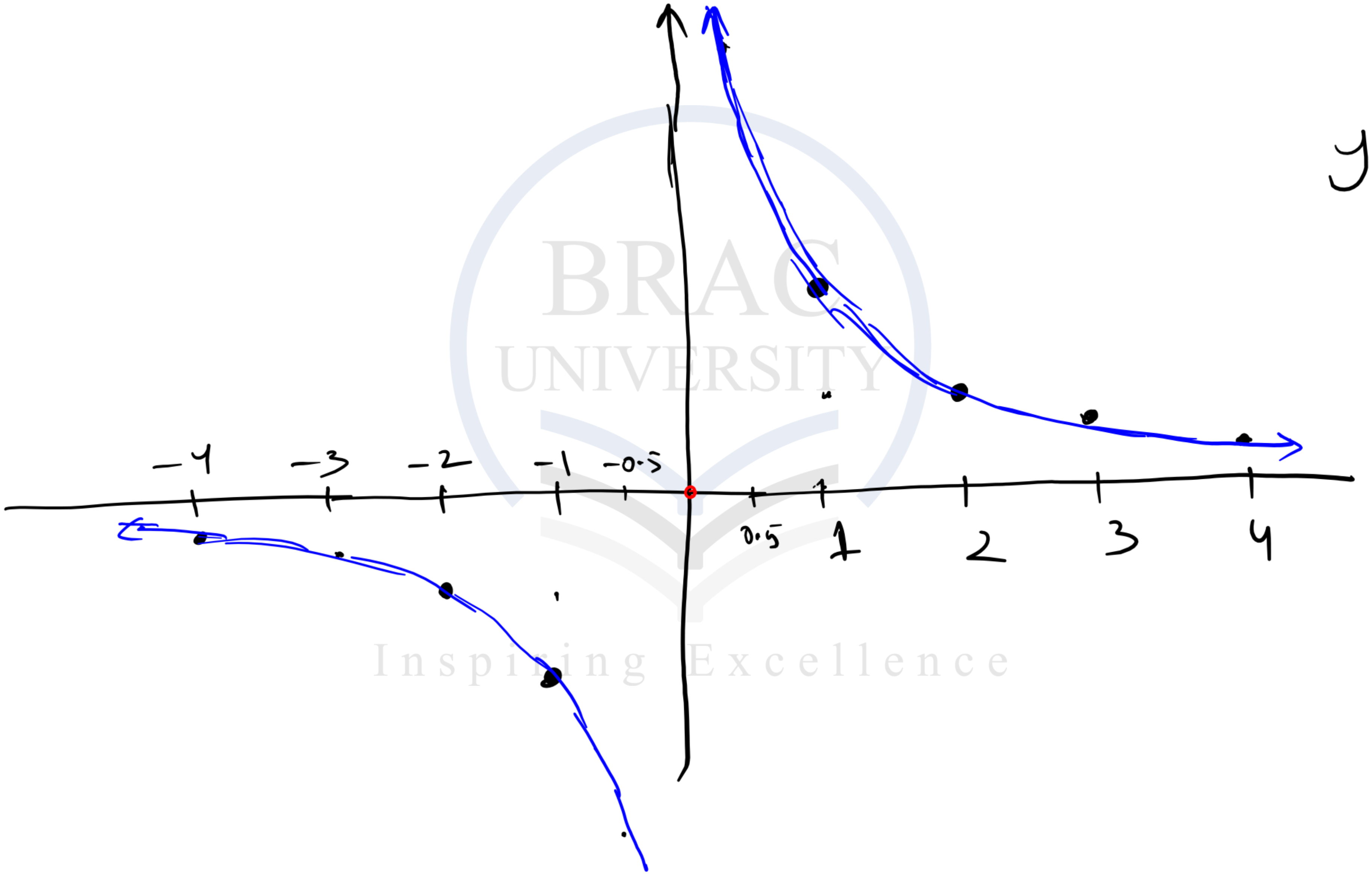
$\Rightarrow \operatorname{Im}\{(x+iy)^2\} = 4$

$\Rightarrow \operatorname{Im}\{x^2+2ixy+i^2y^2\} = 4$

$\Rightarrow \operatorname{Im}\{(x^2-y^2)+i(2xy)\} = 4$

$\Rightarrow 2xy = 4$

$\Rightarrow y = \frac{2}{x}$



$$y = \frac{2}{x}$$

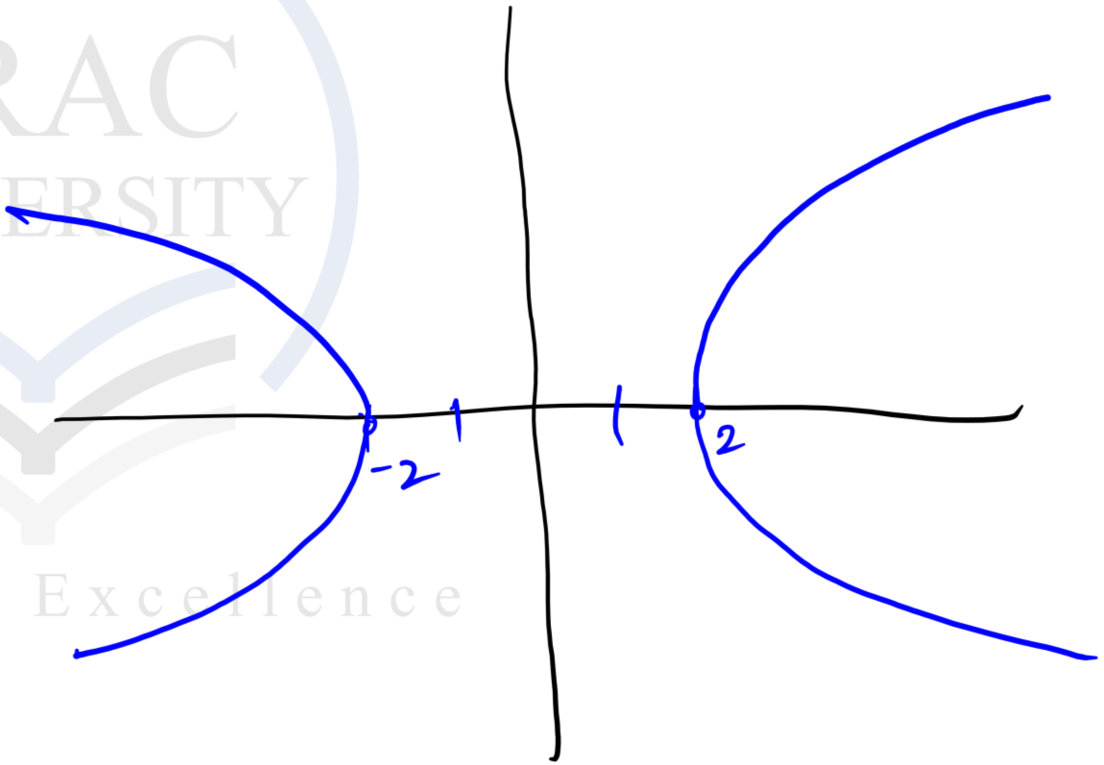
$$\frac{2}{1} = 2$$

v) $Re\{z^2\} = 4$

$\Rightarrow Re\{(x^2-y^2)+i(2xy)\} = 4$

$\Rightarrow x^2-y^2=4$

$\Rightarrow x^2-y^2=2^2$



vi) $\mathrm{Re}\left\{\frac{1}{z}\right\}=1$

$$\Rightarrow \mathrm{Re}\left\{\frac{1}{x+iy}\right\}=1$$

$$\Rightarrow \mathrm{Re}\left\{\frac{1.(x-iy)}{(x+iy)(x-iy)}\right\}=1$$

$$\Rightarrow \mathrm{Re}\left\{\frac{x-iy}{x^2+y^2}\right\}=1$$

$$\Rightarrow \mathrm{Re}\left\{\frac{x}{x^2+y^2}+i\frac{-y}{x^2+y^2}\right\}=1$$

$$\Rightarrow \frac{x}{x^2+y^2}=1$$

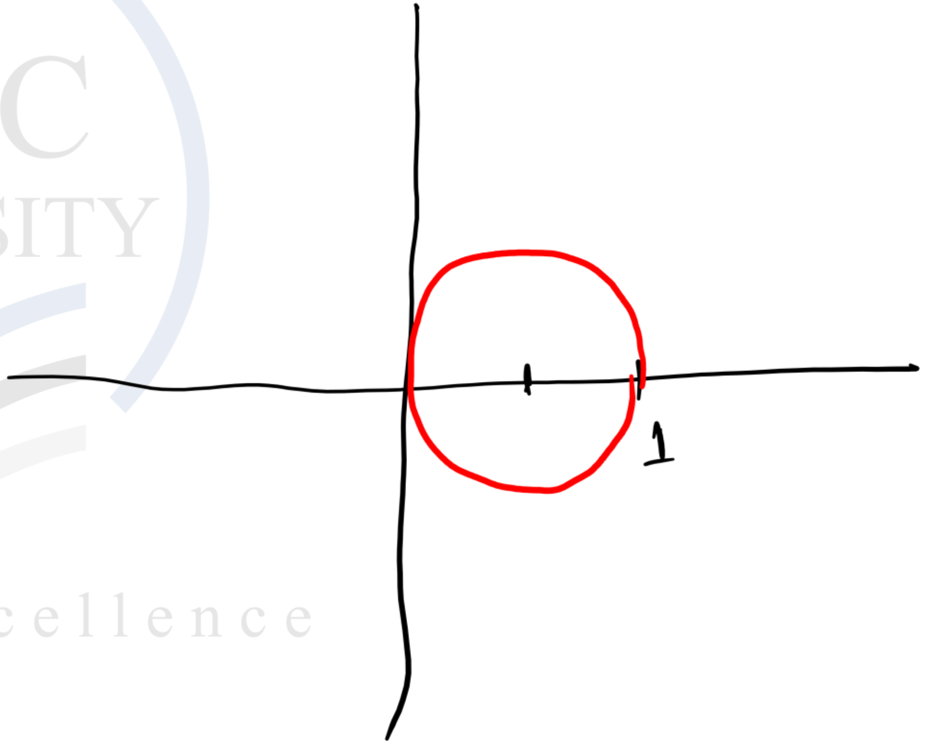
$$\Rightarrow x^2+y^2=x$$

$$\Rightarrow x^2 + y^2 = x$$

$$\Rightarrow x^2 - x + y^2 = 0$$

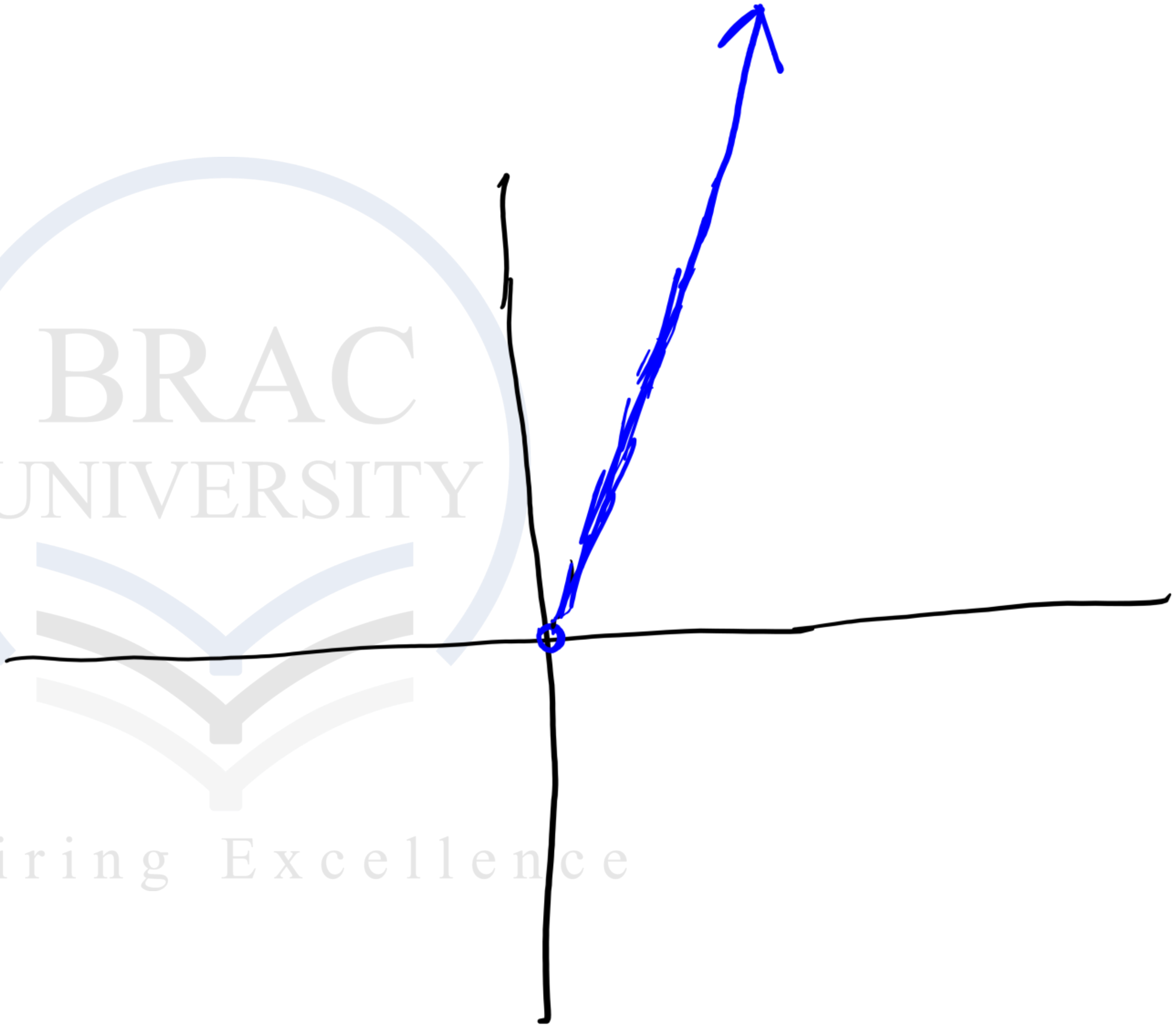
$$\Rightarrow x^2 - 2 \cdot x \cdot \frac{1}{2} + \left(\frac{1}{2}\right)^2 + y^2 = \left(\frac{1}{2}\right)^2$$

$$\Rightarrow \left(x - \frac{1}{2}\right)^2 + y^2 = \left(\frac{1}{2}\right)^2$$



vii)

$$\mathrm{Arg}(z) = \frac{\pi}{3}$$



Graph the Regions

i) $|z| > 2$

ii) $|z - 2i| \leq 3$

iii) $\left|\frac{z-3}{z+3}\right| < 2$

iv) $Im\{z^2\} > 4$

v) $Re\{z^2\} < 4$

vi) $\frac{\pi}{4} < Arg\{Z\} \leq \frac{2\pi}{3}$

i) $|z| > 2$

$1+i$

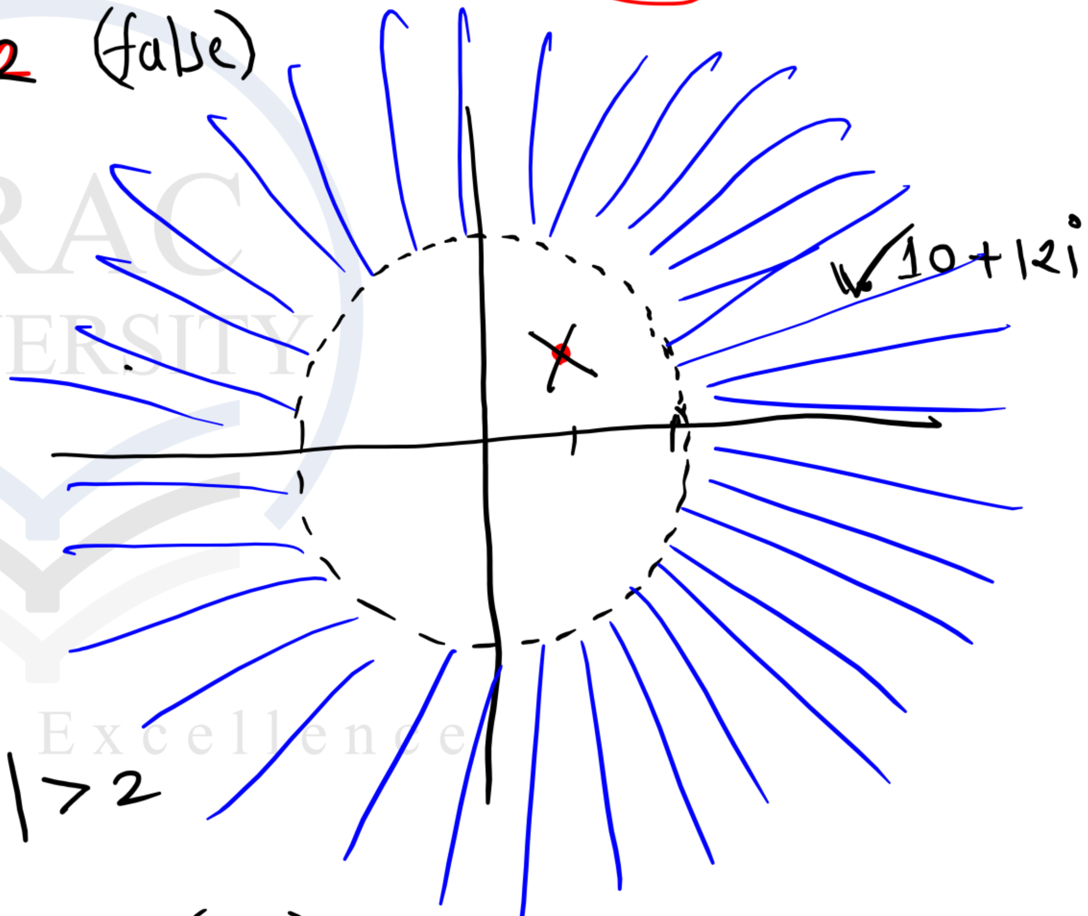
$|1+i| > 2$

$\Rightarrow \sqrt{2} > 2$ (false)

Consider $|z| = 2$

$\Rightarrow \sqrt{x^2+y^2} = 2$

$\Rightarrow x^2+y^2 = 2^2$



$|10+12i| > 2$

$\Rightarrow \sqrt{244} > 2$ (True)

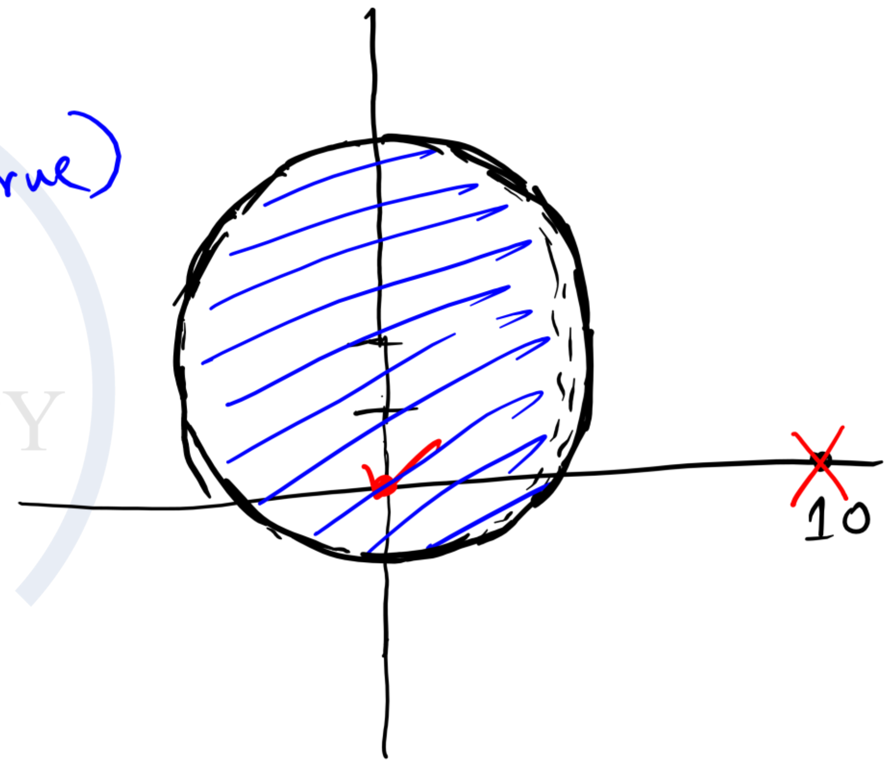
ii) $|z-2i| \le 3$

Consider $|z-2i| = 3$

$$\Rightarrow (x-0)^2+(y-2)^2=3^2$$

$|0-2i| \le 3$

$\Rightarrow 2 \le 3$ (True)



$|10-2i| \le 3$

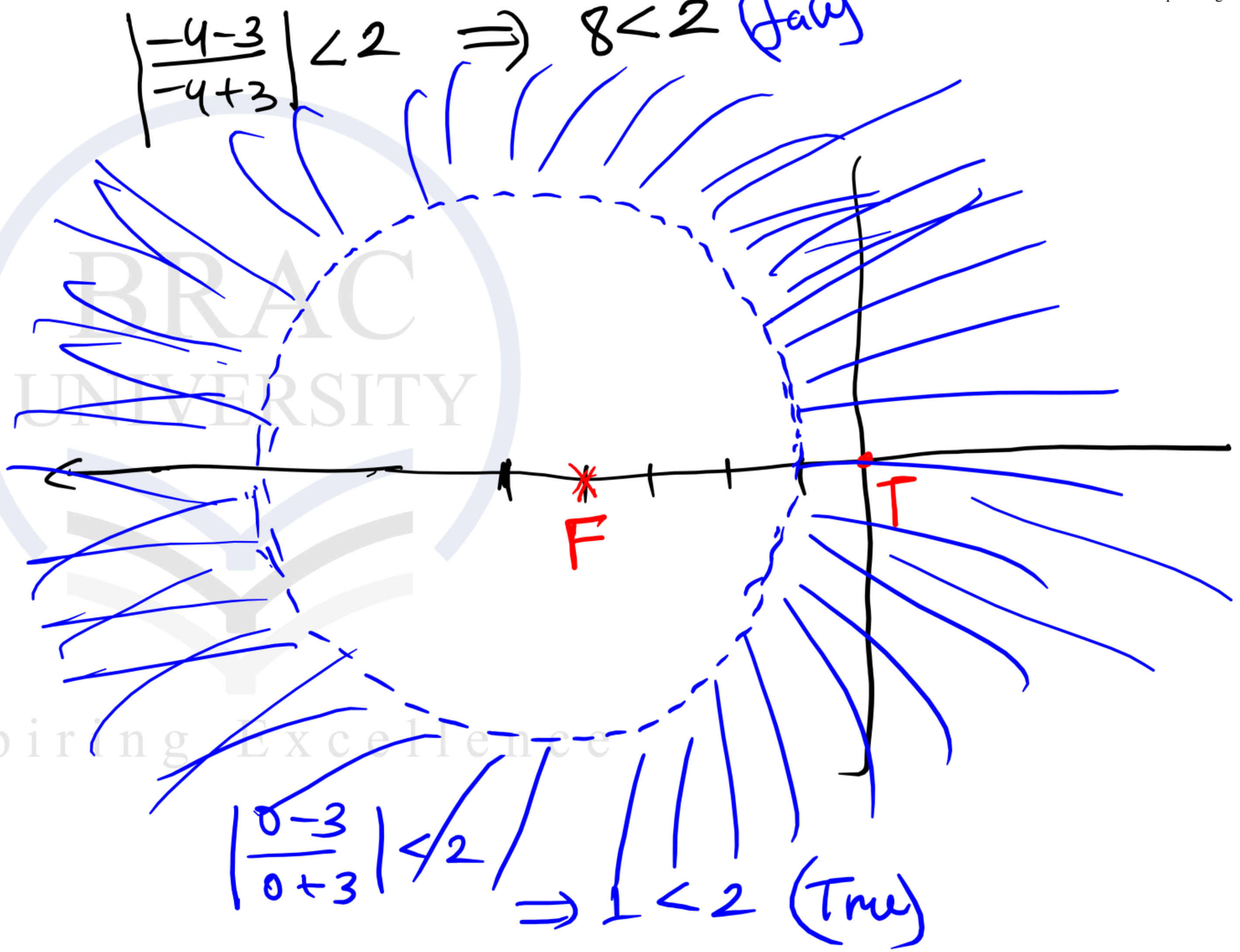
$\Rightarrow \sqrt{104} \le 3$ false

iii) $\left|\frac{z-3}{z+3}\right| < 2$

Consider $\left|\frac{z-3}{z+3}\right| = 2$

⋮

$\Rightarrow (x+5)^2 + y^2 = 4^2$

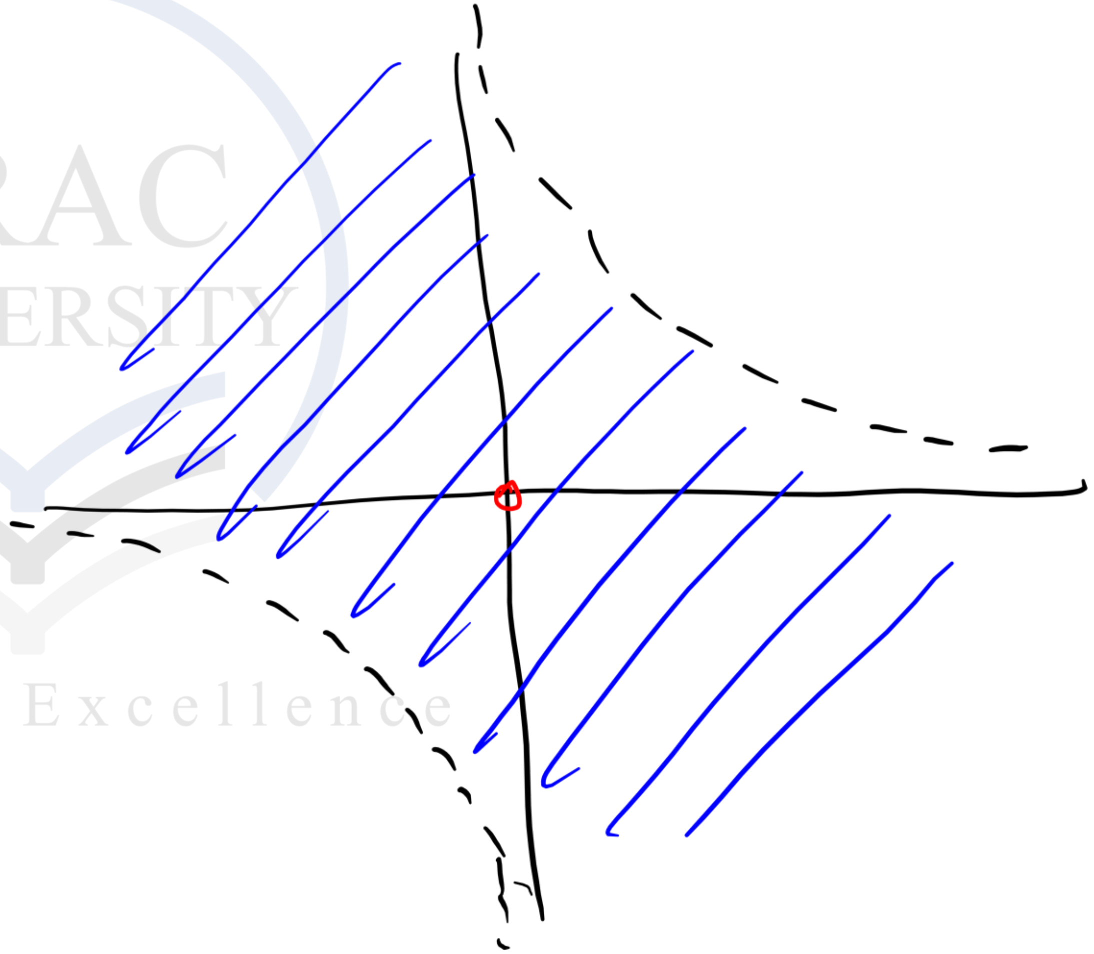


iv) $\operatorname{Im}\{z^2\} < 4$

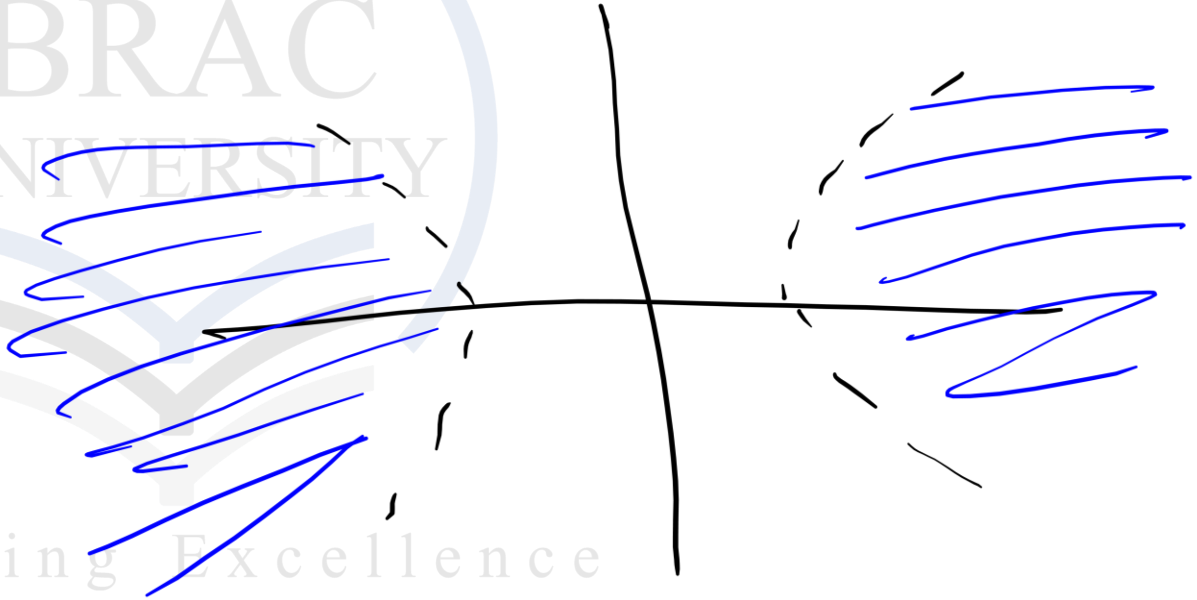
Consider $\operatorname{Im}\{z^2\} = 4$

$2xy = 4$

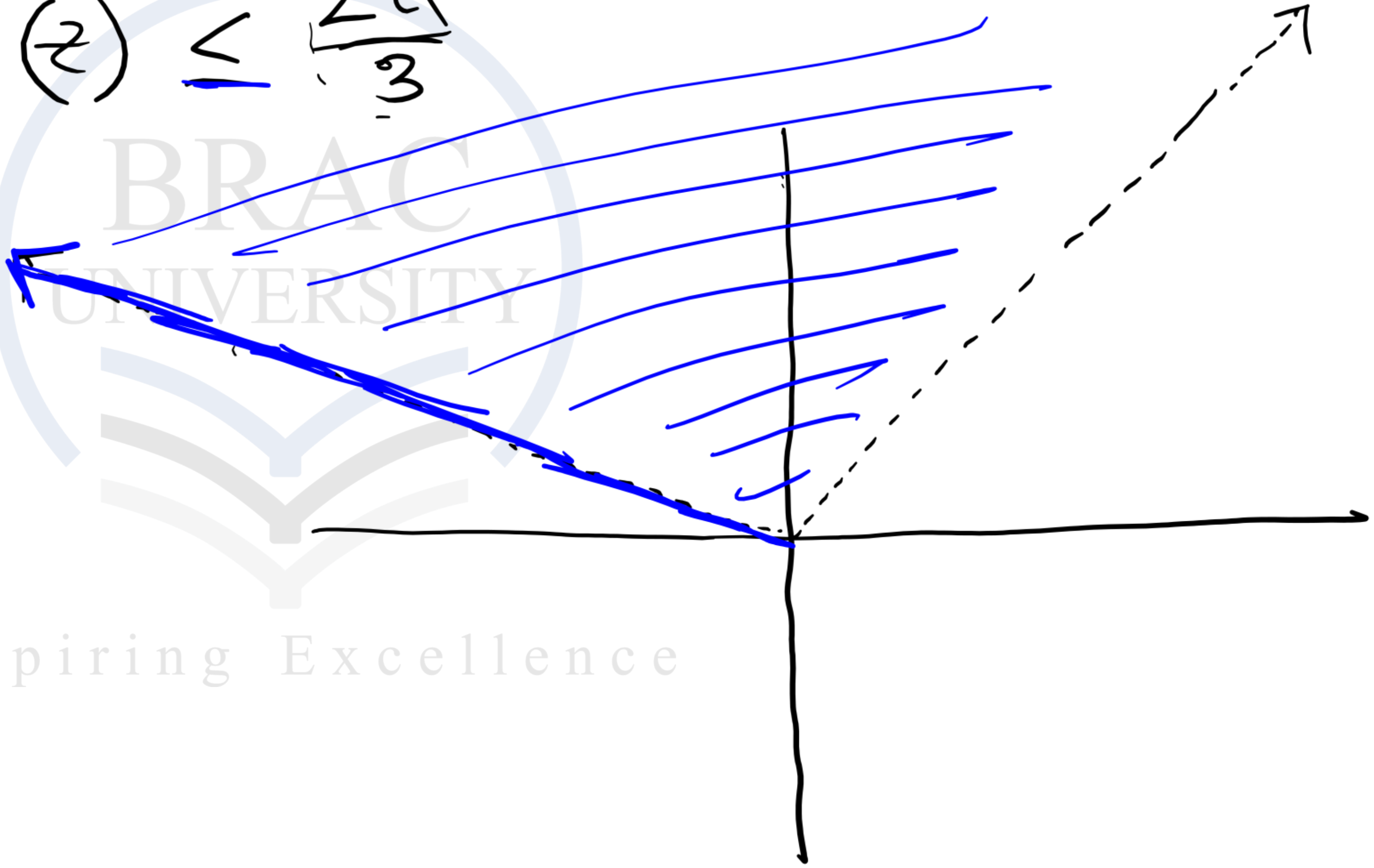
$\Rightarrow y = \frac{2}{x}$



v) $Re\{z^2\} > 4$



vii) $\frac{\pi}{4} < Arg(z) \leq \frac{2\pi}{3}$



Some misconceptions

Modulus

$$|z_1 \cdot z_2| = |z_1| \cdot |z_2|$$

$$\left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|}$$

but $\qquad |z_1 + z_2| \leq |z_1| + |z_2|$

Proof

Square Root may behave like

$$\sqrt{ab} \neq \sqrt{a} \cdot \sqrt{b}$$

$$\sqrt{(-16)(-4)}$$

$$= \sqrt{64}$$

$$= 8$$

$$\sqrt{-16}\ \sqrt{-4}$$

$$= 4i \cdot 2i$$

$$= 8i^2$$

$$= -8$$

Power may behave like

$$(a^b)^c \neq a^{bc}$$

The correct Power Rule

$$\boxed{Z^{w} = e^{w \cdot \underline{\ln(z)}}}$$

Log may behave like

$$\log_{a} M^{r} \neq r \cdot \log_{a} M$$

The correct Log Rule

$$\boxed{\ln(z) = \ln|r| + i\theta}$$

The i^{th} roots of i or $\sqrt[i]{i}$ in wrong way

$$i = e^{i\left(\frac{\pi}{2}+2n\pi\right)}$$

mod = 1

Arg = $\frac{\pi}{2}+2n\pi$

$$(i)^{\frac{1}{i}} = \left[e^{\cancel{i}\left(\frac{\pi}{2}+2n\pi\right)}\right]^{\frac{1}{\cancel{i}}}$$

$$= e^{\frac{\pi}{2}+2n\pi}$$

The i^{th} roots of i or $\sqrt[i]{i}$ in the correct way

$$z^{\omega} = e^{\omega \ln z}$$

$$\ln z = \ln|r| + i\theta$$

$$(i)^{\frac{1}{i}} = e^{\frac{1}{i}\ln(i)}$$

$$= e^{\frac{1}{i}\left(\ln 1 + i\left(\frac{\pi}{2} + 2n\pi\right)\right)}$$

$$= e^{\frac{1}{i} i \left(\frac{\pi}{2} + 2n\pi\right)} \qquad = e^{\frac{\pi}{2} + 2n\pi}$$

$$\ln(i)$$

$$= \ln(1) + i\left(\frac{\pi}{2} + 2n\pi\right)$$

Multi valued function

$f(z) = z^2$

$f(z) = z^{\frac{1}{3}}$

$f(8) = 8^{\frac{1}{3}} = 2, ?, ?$

$f(8) = 8^{\frac{1}{3}} = 2$

$f(x) = x^2$

$f(x) = x^{\frac{1}{2}}$

$f(x) = x^{\frac{1}{3}}$

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