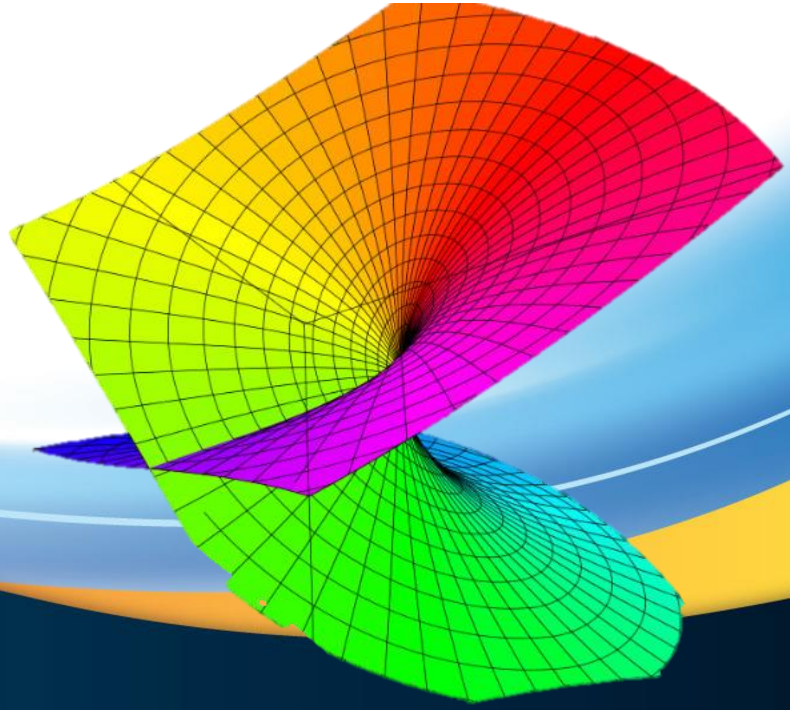
Undergraduate Course in Mathematics

# Complex Variables

## Topic: The Residue Theorem




Conducted By

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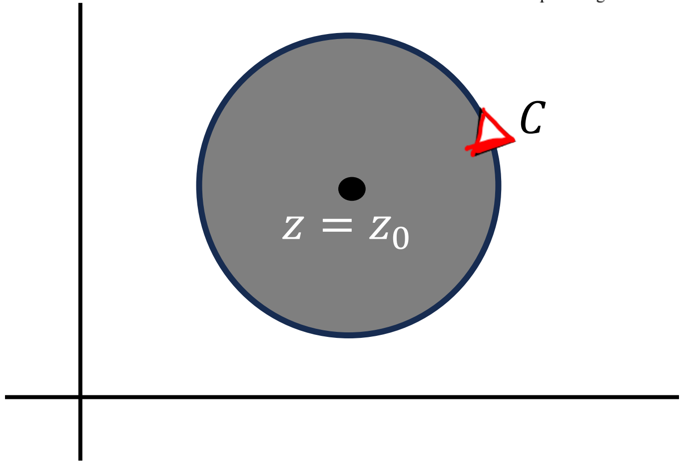
# Residues

Let $f(z)$ be single-valued and analytic inside and on a circle $C$ except at the point $z = z_0$ a chosen as the center of C. Then, $f(z)$ has a Laurent series about $z = z_0$ given by



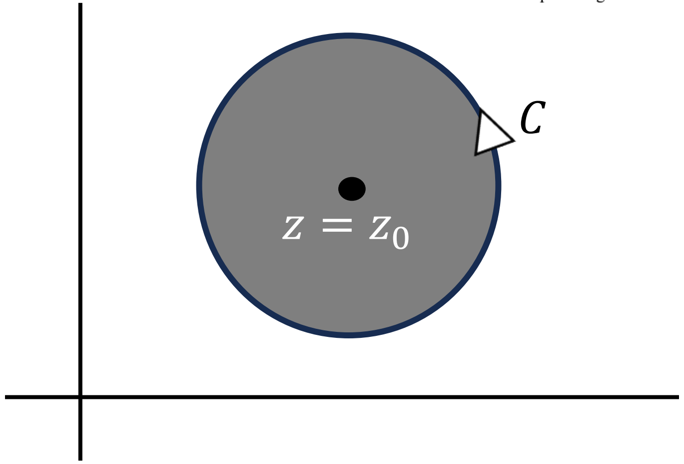

$$f(z) = a_0 + a_1(z - z_0) + a_2(z - z_0)^2 + \cdots + \frac{a_{-1}}{(z - z_0)} + \frac{a_{-2}}{(z - z_0)^2} + \cdots$$

$$where \quad a_n = \frac{1}{2\pi i} \oint_C \frac{f(z)}{(z - z_0)^{n+1}} dz \quad for\ n = 0, \pm 1, \pm 2, \ldots$$

**we call $a_{-1}$ the residue of $f(z)$ at $z = z_0$, denoted by $Res(f, z_0)$**

# Why $a_{-1}$ is so special?

Let $f(z)$ be single-valued and analytic inside and on a circle $C$ except at the point $z = z_0$ a chosen as the center of C. Then, $f(z)$ has a Laurent series about $z = z_0$ given by




$$f(z) = \underbrace{a_0 + a_1(z - z_0) + a_2(z - z_0)^2 + \cdots}_{} + \frac{a_{-1}}{(z - z_0)} + \frac{a_{-2}}{(z - z_0)^2} + \cdots$$

Now the interesting fact is $\oint_C \boldsymbol{f(z)dz} = \boxed{\boldsymbol{2\pi i \cdot a_{-1}}}$

*Proof:*

$$\oint f(z) = \underbrace{\oint_C \left(a_0 + a_1(z-z_0) + a_2(z-z_0)^2 + \cdots\right) dz}_{=0}$$

$$+ \oint_C \frac{a_{-1}}{z-z_0} + \oint_C \frac{a_{-2}}{(z-z_0)^2} + \oint_C \frac{a_{-3}}{(z-z_0)^3} + \cdots$$

$$= \boxed{\oint_C \frac{a_{-1}}{z-z_0} dz} + \underbrace{\oint \frac{a_{-2}}{(z-z_0)^2} dz + \oint \frac{a_{-3}}{(z-z_0)^3} dz + \cdots}_{=0}$$

$$= a_{-1} \cdot \oint_C \frac{1}{z-z_0} dz \quad = 2\pi i \cdot a_{-1}.$$

$$\oint_c \frac{1}{(z-z_0)^n} = \begin{cases} \underline{\underline{2\pi i}} & n=1 \\ 0 & n\neq 1 \end{cases}$$

(chap-4 part-2)

# Calculation of Residue of a Pole



The residue of a function $f(z)$ at $z = z_0$, is the constant $\boldsymbol{a_{-1}}$.

However, in the case where $\boldsymbol{z = z_0}$ **is a pole of order n**, there is a simple formula for $a_{-1}$ given by

$$a_{-1} = \frac{1}{(n-1)!} \lim_{z \to z_0} \left\{ \frac{d^{n-1}}{dz^{n-1}} \left( (z - z_0)^n f(z) \right) \right\}$$

If $n = 1$ (simple pole), then the result is especially simple and is given by

$$a_{-1} = \lim_{z \to z_0} \{(z - z_0) f(z)\}$$

*Proof:*

pole $\underline{\text{Order} = 4}$

$$f(z) = \frac{a_{-4}}{(z-z_0)^4} + \frac{a_{-3}}{(z-z_0)^3} + \frac{a_{-2}}{(z-z_0)^2} + \frac{a_{-1}}{(z-z_0)} + a_0 + a_1(z-z_0) + a_2(z-z_0)^2 + \cdots$$

$$\Rightarrow (z-z_0)^4 f(z) = a_{-4} + a_{-3}(z-z_0) + a_{-2}(z-z_0)^2 + \boxed{a_{-1}(z-z_0)^3} + a_0(z-z_0)^4 + a_1(z-z_0)^5 + \cdots$$

$$\Rightarrow \frac{d^3}{dz^3}\left( \quad \right) = 0 + 0 + 0 + 3!\, a_{-1} + 4\cdot3\cdot2\cdot a_0(z-z_0) + 5\cdot4\cdot3\, a_1(z-z_0)^2 + \cdots$$

$$\lim_{z \to z_0} \frac{d^3}{dz^3}\left[(z-z_0)^4 f(z)\right] = 3!\, a_{-1}$$

$$\Rightarrow a_{-1} = \frac{1}{3!} \lim_{z \to z_0} \frac{d^3}{dz^3}\left((z-z_0)^4 f(z)\right)$$

$$a_{-1} = \frac{1}{(n-1)!} \cdot \lim_{z \to z_0} \cdot \frac{d^{n-1}}{dz^{n-1}}\left((z-z_0)^n \cdot f(z)\right)$$

Calculate the residue of $f(z)$ at $z = 0$

$$f(z) = e^{-\frac{1}{z}}$$

Residu $(z-z_0)^{-1} \to$ coefficient

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

$$e^{-\frac{1}{z}} = 1 - \frac{1}{z} + \frac{1}{2!z^2} - \frac{1}{3!z^3} + \cdots$$

$$= 1 + (-1)z^{-1} + \frac{1}{2!}z^{-2} + \left(-\frac{1}{3!}\right)z^{-3} + \cdots$$

$z^{-1}$ ??

Residue $= -1$

Calculate the residue of $f(z)$ at $z = 1$

$$f(z) = \frac{z}{(z-1)(z+1)^2}$$

$z=1$ is a pole of order 1 or simple pole.

$$\therefore \operatorname{Res}(f, z=1) = \lim_{z \to 1} (z-1) \cdot f(z) = \lim_{z \to 1} \frac{z}{(z+1)^2}$$

$$= \frac{1}{4} \quad \text{Ans}$$

Calculate the residue of $f(z)$ at $z = -1$

$$f(z) = \frac{z}{(z-1)(z+1)^2}$$

$z = -1$ is a pole of order $= 2$.

$$\text{Res}(f, z=-1) = \frac{1}{(2-1)!} \lim_{z \to -1} \frac{d'}{dz'}\left((z+1)^2 f(z)\right)$$

$$= \lim_{z \to -1} \frac{d}{dz}\left(\frac{z}{z-1}\right)$$

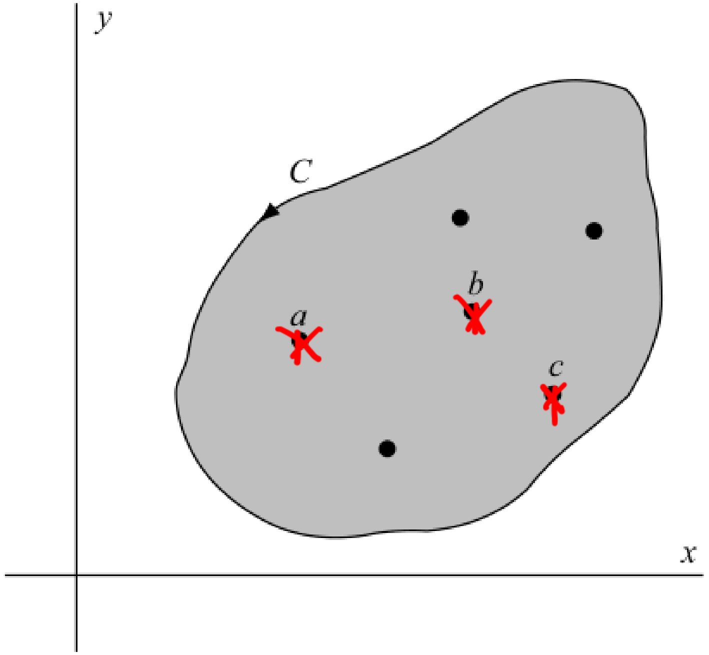
$$= \lim_{z \to -1} \frac{(z-1)\cdot 1 - z\cdot 1}{(z-1)^2}$$

$$= \lim_{z \to -1} \frac{-1}{(z-1)^2}$$

$$= \frac{-1}{4}$$

# The Residue Theorem

Let $f(z)$ be single-valued and analytic inside and on a simple closed curve $C$ except at the singularities $a, b, c, \ldots$ inside C, which have residues given by $a_{-1}$, $b_{-1}$, $c_{-1}$, … . Then, the residue theorem states that




$$\oint_C f(z)dz = \underline{2\pi i} \cdot (\underbrace{a_{-1} + b_{-1} + c_{-1}}_{\text{Sum of Residues.}} + \cdots)$$

# The Cauchy-Goursat Theorem and the Cauchy Integral Formula are special cases of the Residue Theorem

✓✓ $\boxed{\text{Residue Thm}}$ → multiple singularity

✓✓ Cauchy Integral formula → only one singularity / Pole at a time

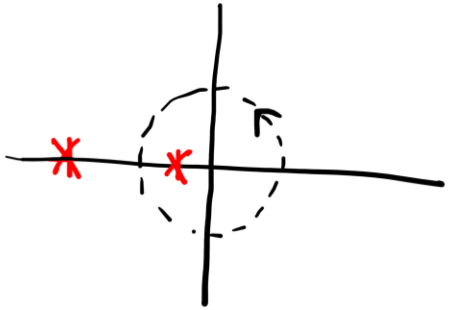
✓ Cauchy-Goursat Thm → no singularity.

Evaluate $\oint_C \frac{z^2}{2z^2+5z+2}\,dz$ using the residue at the poles, where $C$ is the unit circle $|z|=1$.

$$f(z) = \frac{z^2}{2z^2+5z+2}$$

singularity at $2z^2+5z+2=0$

$$\Rightarrow z = -\frac{1}{2}, -2$$



$$= \frac{z^2}{2\left(z+\frac{1}{2}\right)^1 (z+2)^1}$$

$$\operatorname{Res}\left(f, z=-\frac{1}{2}\right) = \lim_{z\to -\frac{1}{2}} \left(z-\left(-\frac{1}{2}\right)\right)\cdot f(z) = \lim_{z\to -\frac{1}{2}} \frac{z^2}{2(z+2)}$$

$$= \frac{\frac{1}{4}}{2\cdot\frac{3}{2}} = \frac{1}{12}$$

$$\oint_C \frac{z^2}{2z^2+5z+2}\,dz = 2\pi i \cdot \left(\text{sum of residues}\right)$$

$$= 2\pi i \left(\frac{1}{12}\right) = \frac{\pi i}{6}$$

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Evaluate $\oint_C \frac{z^2+4}{z^3+2z^2+2z}\,dz$ using the residue at the poles, around the circle $|z|=3$

$$f(z) = \frac{z^2+4}{z^3+2z^2+2z}$$ has singularities at $z^3+2z^2+2z=0$

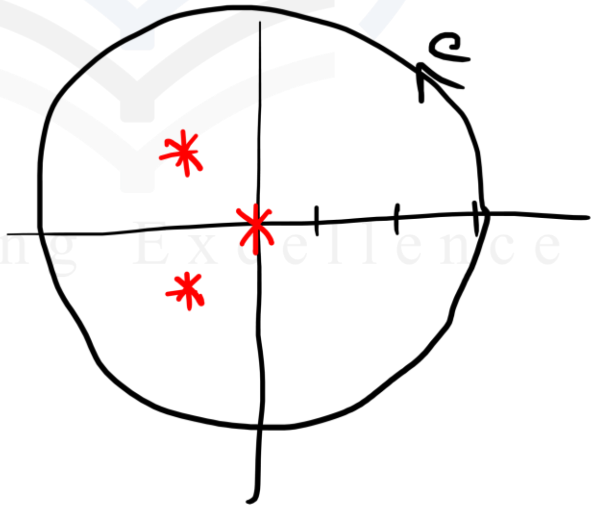
$$z(z^2+2z+2)=0$$

$$= \frac{z^2+4}{z(z+1-i)(z+1+i)}$$

$z=0$ | $z^2+2z+2=0$

$\Rightarrow z=-1+i,\ -1-i$

$z-(-1+i)$



for $\operatorname{Res}(f, z=0) = \lim_{z\to 0} \left( (z-0)\, f(z) \right)$

$$= \lim_{z\to 0} \frac{z^2+4}{(z+1-i)(z+1+i)}$$

$$= \frac{4}{(1-i)(1+i)}$$

$$= 2.$$

$$\operatorname{Res}(f, z=-1+i) = \lim_{z\to -1+i} \left( \left(z-(-1+i)\right) \cdot f(z) \right)$$

$$= \lim_{z\to -1+i} \frac{z^2+4}{z(z+1+i)}$$

$$= \frac{(-1+i)^2+4}{(-1+i)(2i)}$$

$$= \frac{-1}{2} + \frac{3i}{2}.$$

$$\text{Res}(f, z=-1-i) = \lim_{z \to -1-i} \left( \left(z-(-1-i)\right) \cdot f(z) \right)$$

$$= \lim_{z \to -1-i} \frac{z^2+4}{z(z+1-i)}$$

$$= \frac{(-1-i)^2+4}{(-1-i)(-2i)}$$

$$= \frac{-1}{2} + \frac{-3i}{2}.$$

$$\oint_C \frac{z^2+4}{z^3+2z^2+z}\,dz = 2\pi i\,(\text{sum of Residue inside } C)$$

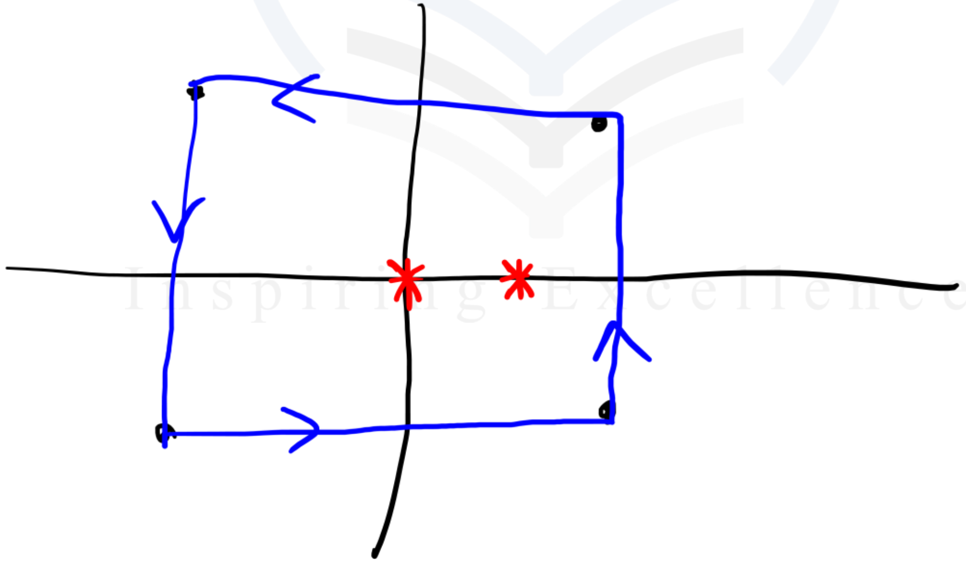
$$= 2\pi i \cdot \left(2 + \frac{-1}{2} + \cancel{\frac{3i}{2}} + \frac{-1}{2} + \cancel{\frac{-3i}{2}}\right)$$

$$= 2\pi i \quad \checkmark$$

Evaluate $\oint_C \frac{2+3\sin \pi z}{z(z-1)^2} dz$ where $C$ is a square having vertices at $3+3i, 3-3i, -3+3i, -3-3i$.

$$f(z) = \frac{2+3\cdot \sin \pi z}{z(z-1)^2}$$ has singularities at $z(z-1)^2 = 0$

$\Rightarrow z = 0,\ z = 1.$



$$\operatorname{Res}(f, z=0) = \lim_{z \to 0} (z-0) \cdot f(z)$$

$$= \lim_{z \to 0} \frac{2+3\sin \pi z}{(z-1)^2}$$

$$= \frac{2}{1}$$

$$= 2$$

$$\operatorname{Res}(f, z=1) = \frac{1}{(2-1)!} \cdot \lim_{z\to 1} \cdot \frac{d}{dz}\left((z-1)^2 \cdot f(z)\right)$$

$$= \lim_{z\to 1} \cdot \frac{d}{dz}\left(\frac{2+3\sin \pi z}{z}\right)$$

$$= \lim_{z\to 1} \frac{z \cdot (3\pi \cos \pi z) - (2+3\sin \pi z) \cdot 1}{z^2}$$

$$= \frac{1(-3\pi) - (2)}{1} = -2-3\pi \checkmark$$

$$\oint \frac{2+3\sin\pi z}{z(z-1)^2}\,dz$$

$$= 2\pi i\left(\cancel{2} + \left(-\cancel{2} - 3\pi\right)\right)$$

$$= -6\pi^2 i \checkmark$$

Evaluate $\dfrac{1}{2\pi i}\oint_C \dfrac{z^2-z+2}{z^4+10z^2+9}\,dz$ using the residue at the poles, around the circle $C$ with the equation $|z|=4$.

$$f(z)=\frac{z^2-z+2}{z^4+10z^2+9}$$ has singularities at $z^4+10z^2+9=0$

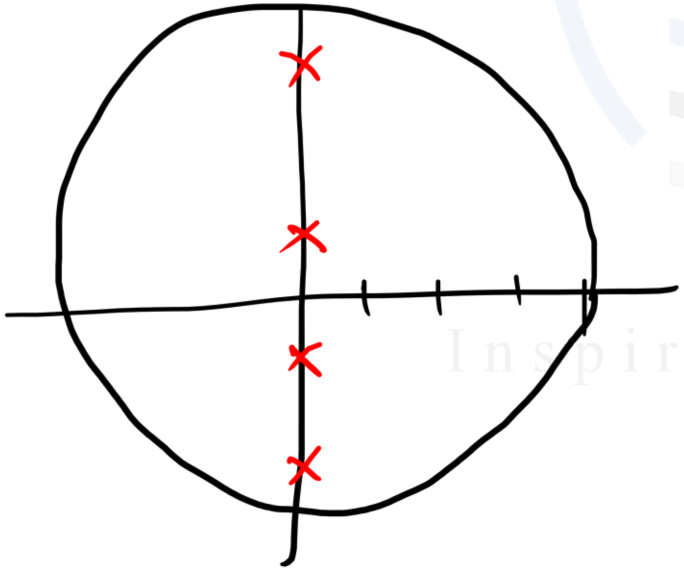
$$=\frac{z^2-z+2}{(z-i)(z+i)(z-3i)(z+3i)}$$

$\Rightarrow z^4+9z^2+z^2+9=0$

$\Rightarrow z^2(z^2+9)+1(z^2+9)=0$

$\Rightarrow (z^2+9)(z^2+9)$

$\Rightarrow z=\pm 3i, \pm i$



$$\operatorname{Res}(f, z=i) = \lim_{z \to i} (z-i) f(z)$$

$$= \lim_{z \to i} \frac{z^2 - z + 2}{(z+i)(z-3i)(z+3i)}$$

$$= \frac{-1-i+2}{2i(-2i)(4i)}$$

$$= \frac{-1}{16} + \frac{-i}{16}$$

$$\operatorname{Res}(f, z=-i) = \lim_{z\to -i} \left(z-(-i)\right) f(z)$$

$$= \lim_{z\to -i} (z+i) \frac{z^2-z+2}{z^4+10z^2+9}$$

$$= \frac{-1}{16} + \frac{i}{16} \checkmark$$

$$\operatorname{Res}(f, z=3i) = \lim_{z \to 3i} (z-3i) f(z)$$

$$= \frac{1}{16} + \frac{-7i}{48}$$

$$\operatorname{Res}(f, z=-3i) = \lim_{z \to -3i} (z+3i) f(z)$$

$$= \frac{1}{16} + \frac{7i}{48}.$$

$$\oint_C \frac{z^2-z+2}{z^4+10z^2+9}\,dz = 2\pi i\left(\cancel{\frac{-1}{16}}+\cancel{\frac{-i}{16}}+\cancel{\frac{-1}{16}}+\cancel{\frac{i}{16}}+\cancel{\frac{1}{16}}+\cancel{\frac{-7i}{48}}+\cancel{\frac{1}{16}}+\cancel{\frac{7i}{48}}\right)$$

$$=0$$

$$\Rightarrow \frac{1}{2\pi i}\oint_C \frac{z^2-z+2}{z^4+10z^2+9}\,dz = 0 \quad \checkmark$$

Evaluate $\frac{1}{2\pi i}\oint_C \frac{e^{zt}}{(z^2+1)^2}dz$ if $t>0$ and $C$ is the circle $|z|=3$.

$$f(z)=\frac{e^{zt}}{(z^2+1)^2}$$

$$=\frac{e^{zt}}{(z-i)^2(z+i)^2}$$

$$(z^2+1)^2=0$$

$$\Rightarrow \left((z+i)(z-i)\right)^2=0$$

$$\Rightarrow (z-i)^2(z+i)^2=0$$

$$\operatorname{Res}(f, z=i) = \frac{1}{(2-1)!} \lim_{z \to i} \frac{d}{dz}\left((z-i)^2 \cdot f(z)\right)$$

$$= \lim_{z \to i} \frac{d}{dz}\left(\frac{e^{zt}}{(z+i)^2}\right)$$

$$= \lim_{z \to i} \frac{(z+i)^2 \cdot t \cdot e^{zt} - e^{zt} \cdot 2(z+i)}{(z+i)^4}$$

$$= \left(\frac{-t}{4} + \frac{-i}{4}\right) e^{ti}$$

$$\operatorname{Res}(f, z=-i) = \frac{1}{(2-1)!} \lim_{z \to -i} \frac{d}{dz}\left((z+i)^2 f(z)\right)$$

$$= \lim_{z \to -i} \frac{d}{dz}\left(\frac{e^{zt}}{(z-i)^2}\right)$$

$$= \lim_{z \to -i} \frac{(z-i)^2 \cdot t e^{zt} - e^{zt} \cdot 2(z-i)}{(z-i)^4}$$

$$= \left(\frac{-t}{4} + \frac{i}{4}\right) e^{-ti}$$

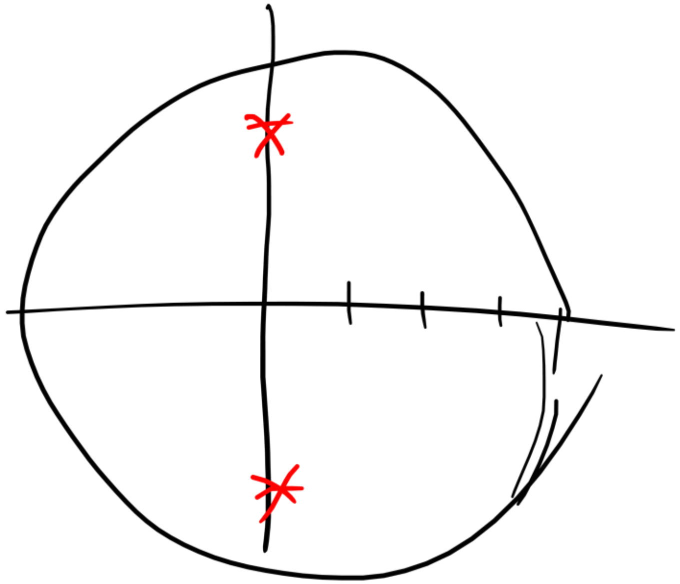
$$\oint_C \frac{e^{zt}}{(z^2+1)^2}\,dz = 2\pi i\left(\left(\frac{-t}{4}+\frac{-i}{4}\right)e^{ti} + \left(\frac{-t}{4}+\frac{i}{4}\right)e^{-ti}\right)$$

$$\therefore \frac{1}{2\pi i}\oint \frac{e^{zt}}{(z^2+1)^2}\,dz = \left(\frac{-t}{4}+\frac{-i}{4}\right)e^{ti} + \left(\frac{-t}{4}+\frac{i}{4}\right)e^{-ti}$$

Evaluate $\oint_C \frac{e^{2z}}{(z^2+\pi^2)^2}\,dz$ where $C$ is the circle $|z| = 4$.

$$f(z) = \frac{e^{2z}}{(z^2+\pi^2)^2}$$

$$= \frac{e^{2z}}{(z-i\pi)^2(z+i\pi)^2}$$



$$\operatorname{Res}(f, z = i\pi) = \frac{1}{(2-1)!} \lim_{z \to i\pi} \frac{d}{dz}\left((z-i\pi)^2 f(z)\right)$$

$$= \lim_{z \to i\pi} \frac{d}{dz}\left(\frac{e^{2z}}{(z+i\pi)^2}\right)$$

$$= \frac{-2\pi - i}{4\pi^3}$$

$$\operatorname{Res}(f, z=-i\pi) = \frac{1}{(2-1)!}\lim_{z\to -i\pi} \frac{d}{dz}\left((z+i\pi)^2 f(z)\right)$$

$$= \lim_{z\to -i\pi} \frac{d}{dz}\left(\frac{e^{2z}}{(z-i\pi)^2}\right)$$

$$= \frac{-2\pi+i}{4\pi^3}.$$

$$\oint_C \frac{e^{2t}}{(z^2+\pi^2)^2}\,dz = 2\pi i \left( \frac{-2\pi - i}{4\pi^3} + \frac{-2\pi + i}{4\pi^3} \right)$$

$$= 2\pi i \, \frac{-4\pi}{4\pi^3}$$

$$= \frac{-2i}{\pi} \checkmark$$


Thank you!