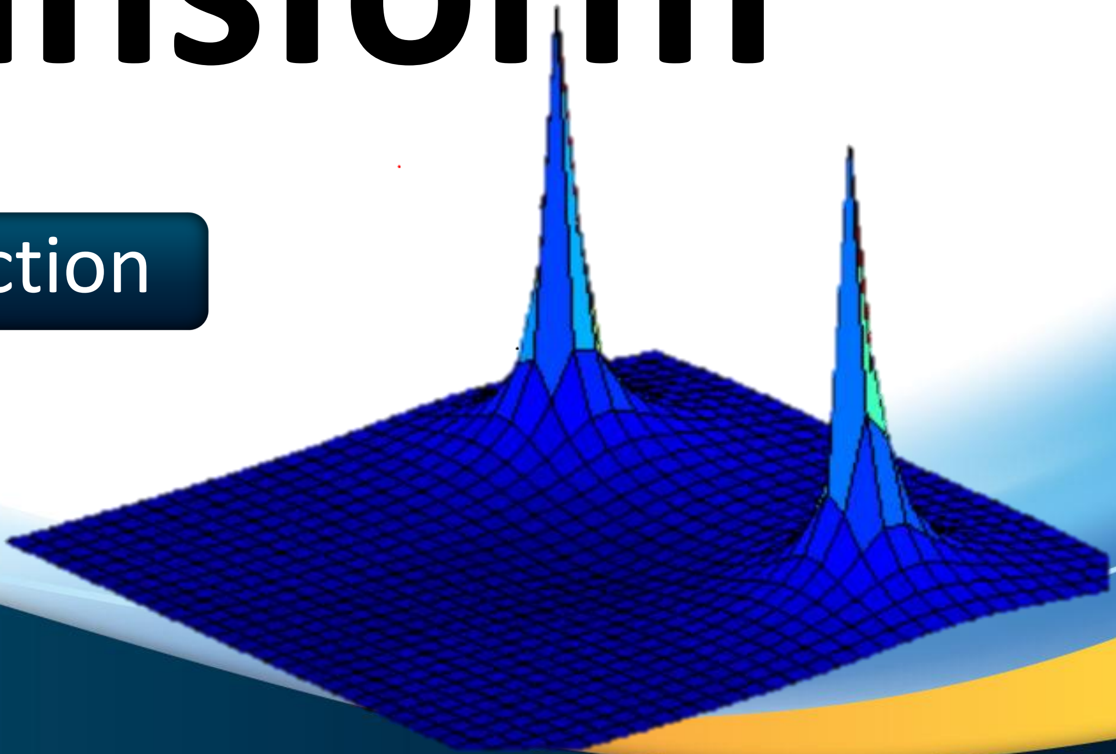
Undergraduate Course in Mathematics

Laplace Transform

Inverse Laplace using Partial Fraction



Conducted By

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Partial Fraction

$$\underbrace{\frac{2}{s-1}+\frac{3}{s-2}}=\frac{2s-4+3s-3}{(s-1)(s-2)}=\frac{5s-7}{\underbrace{s^2-3s+2}}$$

$$\frac{5s-7}{s^2-3s+2}=\frac{?}{s-1}+\frac{?}{s-2}$$

Linear factor without Repeatation

$$\frac{2s}{(s-2)(s-5)} \equiv \frac{A}{s-2} + \frac{B}{s-5}$$

$$\frac{3+2}{(s-2)(s+4)(s-3)} \equiv \frac{A}{s-2} + \frac{B}{s+4} + \frac{C}{s-3}$$

Linear factor with Repeat

$$\frac{6s-3}{(s-2)(s-3)^2} \equiv \frac{A}{s-2} + \frac{B}{s-3} + \frac{C}{(s-3)^2}$$

$$\frac{6s+3}{(s-2)(s-3)^2(s-5)^3} = \frac{A}{s-2} + \frac{B}{s-3} + \frac{C}{(s-3)^2} + \frac{D}{s-5} + \frac{E}{(s-5)^2} + \frac{F}{(s-5)^3}$$

Quadratic factor

$$\frac{6s+2}{(s-2)(s^2+4)} = \frac{A}{s-2} + \frac{Bs+C}{s^2+4}$$

$$\frac{- - -}{(s-2)(s+3)(s^2+4)(s^2+9)} = \frac{A}{s-2} + \frac{B}{s+3} + \frac{Cs+D}{s^2+4} + \frac{Es+F}{s^2+9}$$

$$\frac{5}{s-2}+\frac{3}{s^2+4}=\frac{5s^2+20+3s-6}{(s-2)(s^2+4)}=\frac{5s^2+3s+14}{(s-2)(s^2+14)}$$

$$\frac{5}{s-2}+\frac{3s}{s^2+4}=\frac{5s^2+20+3s^2-6s}{(s-2)(s^2+4)}=\frac{8s^2-6s+20}{(s-2)(s^2+4)}$$

$$\frac{5}{s-2}+\frac{3s+2}{s^2+4}=\frac{5s^2+20+3s^2-6s+2s-4}{(s-2)(s^2+4)}=\frac{8s^2-4s+16}{(s-2)(s^2+4)}$$

Quadratic with Repeat (General

$$\frac{---}{(s-2)(s-3)^3(s^2+4)(s^2+9)^2} \equiv \frac{A}{s-2} + \frac{B}{s-3} + \frac{C}{(s-3)^2} + \frac{D}{(s-3)^3}$$

$$+ \frac{Es+F}{s^2+4} + \frac{Gs+H}{(s^2+9)} + \frac{Is+J}{(s^2+9)^2}$$

$$\frac{5s-3}{(s-2)^2} = \frac{5(s-2)+10-3}{(s-2)^2} = \frac{5(s-2)}{(s-2)^2} + \frac{7}{(s-2)^2}$$

$$= \frac{5}{s-2} + \frac{7}{(s-2)^2}$$

$=$ Vs $\equiv$

$\underbrace{2x+1=5}_{x=2}$

$\underbrace{x^2-5x+6=0}_{x=2,3}$

$\underbrace{x^2+2x \equiv (x+1)^2-1}$

$x=2$

$x=2,3,\ldots$

any number.

$\underbrace{(a+1)^2 \equiv a^2+2a+1}$

Decompose into partial fractions and then find the Inverse Laplace Transform of

$$\frac{1}{s(s+1)}$$

$$\frac{1}{s(s+1)} \equiv \frac{A}{s} + \frac{B}{s+1}$$

$$\Rightarrow 1 \equiv A(s+1) + B(s)$$

$s=0$

$$1 = A(0+1) + B \cdot 0$$

$$\Rightarrow A = 1$$

$s=-1$

$$1 = A(0) + B(-1)$$

$$\Rightarrow B = -1$$

$$\frac{1}{s(s+1)} = \frac{1}{s} + \frac{-1}{s+1}$$

$$\mathcal{L}^{-1}\left\{\frac{1}{s(s+1)}\right\} = \mathcal{L}^{-1}\left\{\frac{1}{s}\right\} - \mathcal{L}^{-1}\left\{\frac{1}{s+1}\right\}$$

$$= 1 - e^{-t}$$

Decompose into partial fractions and then find the Inverse Laplace Transform of

Ans

$$\frac{1}{s(s+1)}e^{-s}$$

$$\frac{1}{s(s+1)} \equiv \frac{A}{s} + \frac{B}{s+1}$$

$$\Rightarrow 1 \equiv A(s+1) + B(s)$$

$s = 0$

$1 = A(0+1) + B \cdot 0$

$\Rightarrow A = 1$

$s = -1$

$1 = A(0) + B(-1)$

$\Rightarrow B = -1$

$$f(t) = \mathcal{L}^{-1}\left\{\frac{1}{s(s+1)}\right\}$$

$$= \mathcal{L}^{-1}\left\{\frac{1}{s}\right\} - \mathcal{L}^{-1}\left\{\frac{1}{s+1}\right\}$$

$$= 1 - e^{-t}.$$

$$\mathcal{L}^{-1}\left\{\frac{1}{s(s+1)}e^{-s}\right\}$$

$$= f(t-1)\cdot u(t-1)$$

$$= \left[1 - e^{-(t-1)}\right]\cdot u(t-1)$$

$$= \left(1 - e^{-t+1}\right)\cdot u(t-1)$$

Decompose into partial fractions and then find the Inverse Laplace Transform of

$$\frac{2s+3}{(s+1)(2s^3+3s^2-3s-2)}$$

$s = -2, 1, -\frac{1}{2}$

$(s+2)$ $(s-1)$ $(2s+1)$

$$= \frac{2s+3}{(s+1)(s+2)(s-1)(2s+1)}$$

$$\frac{2s+3}{(s+1)(s+2)(s-1)(2s+1)} \equiv \frac{A}{s+1} + \frac{B}{s+2} + \frac{C}{s-1} + \frac{D}{2s+1}$$

$$\Rightarrow 2s+3 \equiv A(s+2)(s-1)(2s+1) + B(s+1)(s-1)(2s+1) + C(s+1)(s+2)(2s+1) + D(s+1)(s+2)(s-1)$$

$s = -2$

$-1 = B(-1)(-3)(-3)$

$B = \frac{1}{9}.$

$s = 1$

$5 = C \cdot 2 \cdot 3 \cdot 3$

$C = \frac{5}{18}.$

$s = -1$

$1 = A(1)(-2)(-1)$

$A = \frac{1}{2}.$

$s = -\frac{1}{2}$

$2 = D \cdot \left(\frac{1}{2}\right) \cdot \frac{3}{2} \cdot \left(-\frac{3}{2}\right)$

$D = \frac{-16}{9}.$

$$\mathcal{L}^{-1}\left\{\frac{2s+3}{(s+1)(s+2)(s-1)(2s+1)}\right\}$$

$$= \mathcal{L}^{-1}\left\{\frac{\frac{1}{9}}{s+1} + \frac{\frac{5}{18}}{s+2} + \frac{\frac{1}{2}}{s-1} + \frac{\frac{-16}{9}}{2s+1}\right\}$$

$$= \frac{1}{9}\underline{\mathcal{L}^{-1}\left\{\frac{1}{s+1}\right\}} + \frac{5}{18}\cdot\mathcal{L}^{-1}\left\{\frac{1}{s+2}\right\} + \frac{1}{2}\mathcal{L}^{-1}\left\{\frac{1}{s-1}\right\} - \frac{16}{9\cdot 2}\mathcal{L}^{-1}\left\{\frac{1}{s+\frac{1}{2}}\right\}$$

$$= \frac{1}{9}\cdot e^{-t} + \frac{5}{18}\cdot e^{-2t} + \frac{1}{2}\cdot e^{t} - \frac{8}{9}\cdot e^{-\frac{1}{2}t}.$$

Decompose into partial fractions and then find the Inverse Laplace Transform of

$$\frac{4s^2 - 5s}{(s+1)(s-2)^2}$$

$$\frac{4s^2-5s}{(s+1)(s-2)^2} = \frac{A}{s+1} + \frac{B}{s-2} + \frac{C}{(s-2)^2}$$

$$\Rightarrow 4s^2-5s \equiv A(s-2)^2 + B(s+1)(s-2) + C(s+1)$$

$$\Rightarrow 4s^2-5s \equiv As^2 - 4As + 4A + Bs^2 - Bs - 2B + Cs + C$$

$\underline{s=-1}$

$$4+5 = A(-3)^2 + 0 + 0$$

$$\Rightarrow A = 1 \checkmark$$

$\underline{s=2}$

$$6 = 0 + 0 + C \cdot 3 .$$

$$C = 2 \checkmark$$

equating the coefficient of (s^2) on both sides.

$$4 = A + B$$

$$\Rightarrow 4 = 1 + B$$

$$\Rightarrow B = 3 .$$

$$\frac{4s^2 - 5s}{(s+1)(s-2)^2} = \frac{1}{s+1} + \frac{3}{s-2} + \frac{2}{(s-2)^2}$$

$$\mathcal{L}^{-1}\left\{\frac{4s^2-5s}{(s+1)(s-2)^2}\right\}$$

$$= \mathcal{L}^{-1}\left\{\frac{1}{s+1}\right\} + \mathcal{L}^{-1}\left\{\frac{3}{s-2}\right\} + \mathcal{L}^{-1}\left\{\frac{2}{(s-2)^2}\right\}$$

$$= e^{-t} + 3 \cdot e^{2t} + e^{2t} \cdot \mathcal{L}^{-1}\left\{\frac{2}{s^2}\right\}$$

$$= e^{-t} \cdot 3 \cdot e^{2t} + e^{2t} \cdot 2 \cdot t$$

Ans

Decompose into partial fractions and then find the Inverse Laplace Transform of

$$\frac{-s}{(s^2+1)(s+1)}$$

$$\frac{-s}{(s^2+1)(s+1)} \equiv \frac{As+B}{s^2+1} + \frac{C}{s+1}$$

$$\Rightarrow -s \equiv (As+B)(s+1) + C\cdot(s^2+1)$$

$$\Rightarrow -s = As^2 + As + Bs + B + Cs^2 + C.$$

$s = -1$

$1 = 0 + C\cdot(2)$

$\Rightarrow C = \frac{1}{2}.$

$[s^2]: \quad 0 = A + C \implies 0 = A + \frac{1}{2} \implies A = -\frac{1}{2}$ ✓

$[s]: \quad -1 = A + B \implies B = -1 - A = -1 - \left(-\frac{1}{2}\right) = -\frac{1}{2}$ ✓

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$$\mathcal{L}^{-1}\left\{\frac{-s}{(s^2+1)(s+1)}\right\}$$

$$= \mathcal{L}^{-1}\left\{\frac{-\frac{1}{2}s-\frac{1}{2}}{s^2+1}+\frac{\frac{1}{2}}{s+1}\right\}$$

$$= -\frac{1}{2}\mathcal{L}^{-1}\left\{\frac{s}{s^2+1}\right\}-\frac{1}{2}\mathcal{L}^{-1}\left\{\frac{1}{s^2+1}\right\}+\frac{1}{2}\mathcal{L}^{-1}\left\{\frac{1}{s+1}\right\}$$

$$= -\frac{1}{2}\cos t-\frac{1}{2}\sin t+\frac{1}{2}\cdot e^{-t}$$

Decompose into partial fractions and then find the Inverse Laplace Transform of

$$\frac{-s}{(s^2+1)(s+1)}e^{-\pi s}$$

$$\frac{-s}{(s^2+1)(s+1)} \equiv \frac{As+B}{s^2+1} + \frac{c}{s+1}$$

$$\Rightarrow \quad -s \equiv (As+B)(s+1) + c\cdot(s^2+1)$$

$$\Rightarrow \quad -s = As^2 + As + Bs + B + cs^2 + c.$$

$s=-1$

$1 = 0 + c\cdot(2)$

$\Rightarrow c = \frac{1}{2}.$

$[s^2]: \quad 0 = A + C \implies 0 = A + \frac{1}{2} \implies A = -\frac{1}{2}$ ✓

$[s]: \quad -1 = A + B \implies B = -1 - A = -1 - \left(-\frac{1}{2}\right) = -\frac{1}{2}$ ✓

$$f(t) = \mathcal{L}^{-1}\left\{\frac{-s}{(s^2+1)(s+1)}\right\}$$

$$= \mathcal{L}^{-1}\left\{\frac{-\frac{1}{2}s-\frac{1}{2}}{s^2+1} + \frac{\frac{1}{2}}{s+1}\right\}$$

$$= -\frac{1}{2}\mathcal{L}^{-1}\left\{\frac{s}{s^2+1}\right\} - \frac{1}{2}\mathcal{L}^{-1}\left\{\frac{1}{s^2+1}\right\} + \frac{1}{2}\mathcal{L}^{-1}\left\{\frac{1}{s+1}\right\}$$

$$= -\frac{1}{2}\cos t - \frac{1}{2}\sin t + \frac{1}{2}\cdot e^{-t}$$

$$\mathcal{L}^{-1}\left\{\frac{-s}{(s^2+1)(s+1)}e^{-\pi s}\right\}$$

$$= f(t-\pi)\cdot u(t-\pi)$$

$$= \left[-\frac{1}{2}\cos(t-\pi) - \frac{1}{2}\sin(t-\pi) + \frac{1}{2}e^{-(t-\pi)}\right]\cdot u(t-\pi)$$

$$= \left[\frac{1}{2}\cos t + \frac{1}{2}\sin t + \frac{1}{2}e^{-t+\pi}\right]\cdot u(t-\pi).$$

Decompose into partial fractions and then find the Inverse Laplace Transform of

$$\frac{1}{(s^2+1)(s^2+4)}$$

$$\frac{1}{(s^2+1)(s^2+4)} \equiv \frac{As+B}{s^2+1} + \frac{Cs+D}{s^2+4}$$

$$1 \equiv (As+B)(s^2+4) + (Cs+D)(s^2+1)$$

Hw

Decompose into partial fractions and then find the Inverse Laplace Transform of

$$\frac{1}{(s^2+1)(s^2+4)}$$

$$= \frac{1}{(u+1)(u+4)} \quad [\text{when } u = s^2].$$

$$\therefore \frac{1}{(u+1)(u+4)} \equiv \frac{A}{u+1} + \frac{B}{u+4}$$

$$\Rightarrow 1 \equiv A(u+4) + B(u+1)$$

$u=-1$: $1 = 3A \Rightarrow A = \frac{1}{3}$

$u=-4$: $1 = B(-3) \Rightarrow B = \frac{-1}{3}$.

$$\frac{1}{(u+1)(u+4)} = \frac{\frac{1}{3}}{u+1} + \frac{-\frac{1}{3}}{u+4}$$

$$\Rightarrow \frac{1}{(s^2+1)(s^2+4)} = \frac{\frac{1}{3}}{s^2+1} + \frac{-\frac{1}{3}}{s^2+4}.$$

$$\therefore \mathcal{L}^{-1}\left\{\frac{1}{(s^2+1)(s^2+4)}\right\} = \frac{1}{3}\mathcal{L}^{-1}\left\{\frac{1}{s^2+1^2}\right\} - \frac{1}{6}\mathcal{L}^{-1}\left\{\frac{2}{s^2+2^2}\right\}$$

$$= \frac{1}{3}\sin(t) - \frac{1}{6}\sin(2t)$$

Decompose into partial fractions and then find the Inverse Laplace Transform of

$$\frac{1}{(s^2+1)(s^2+4)}e^{-2\pi s}$$

$$= \frac{1}{(u+1)(u+4)} \quad [\text{when } u = s^2].$$

$$\therefore \frac{1}{(u+1)(u+4)} \equiv \frac{A}{u+1} + \frac{B}{u+4}$$

$$\Rightarrow 1 \equiv A(u+4) + B(u+1)$$

$u = -1$: $1 = 3A \Rightarrow A = \frac{1}{3}$

$u = -4$: $1 = B(-3) \Rightarrow B = \frac{-1}{3}$.

$$\frac{1}{(u+1)(u+4)} = \frac{\frac{1}{3}}{u+1} + \frac{-\frac{1}{3}}{u+4}$$

$$\Rightarrow \frac{1}{(s^2+1)(s^2+4)} = \frac{\frac{1}{3}}{s^2+1} + \frac{-\frac{1}{3}}{s^2+4}.$$

$$\therefore f(t) = \mathcal{L}^{-1}\left\{\frac{1}{(s^2+1)(s^2+4)}\right\} = \frac{1}{3}\mathcal{L}^{-1}\left\{\frac{1}{s^2+1^2}\right\} - \frac{1}{6}\mathcal{L}^{-1}\left\{\frac{2}{s^2+2^2}\right\}$$

$$= \frac{1}{3}\sin(t) - \frac{1}{6}\sin(2t)$$

$$\mathcal{L}^{-1}\left\{\frac{1}{(s^2+1)(s^2+4)}e^{-\pi s}\right\}$$

$$= f(t-\pi)\cdot u(t-\pi)$$

$$= ??$$

HW

Decompose into partial fractions and then find the Inverse Laplace Transform of

$$\frac{s^2+2s+3}{(s^2+2s+2)(s^2+2s+5)}$$

$$= \frac{u+3}{(u+2)(u+5)} \quad [\text{when } s^2+2s=u].$$

$$\frac{u+3}{(u+2)(u+5)} = \frac{A}{u+2} + \frac{B}{u+5}$$

$$\Rightarrow u+3 = A(u+5) + B(u+2)$$

$\underline{\underline{u=-2}}$ $\quad 1 = 3A \Rightarrow A = \frac{1}{3}$

$\underline{\underline{u=-5}}$ $\quad -2 = -3B \Rightarrow B = \frac{2}{3}.$

$$\frac{u+3}{(u+2)(u+5)} = \frac{\frac{1}{3}}{u+2} + \frac{\frac{2}{3}}{u+5}$$

$$\mathcal{L}^{-1}\left\{\frac{s^2+2s+3}{(s^2+2s+2)(s^2+2s+5)}\right\} = \frac{1}{3}\cdot\mathcal{L}^{-1}\left\{\frac{1}{s^2+2s+2}\right\} + \frac{2}{3}\cdot\mathcal{L}^{-1}\left\{\frac{1}{s^2+2s+5}\right\}$$

$$= \frac{1}{3}\mathcal{L}^{-1}\left\{\frac{1}{(s+1)^2+1^2}\right\} + \frac{2}{3}\mathcal{L}^{-1}\left\{\frac{1}{(s+1)^2+2^2}\right\}$$

$$= \frac{1}{3}e^{-t}\mathcal{L}^{-1}\left\{\frac{1}{(s-1+1)^2+1^2}\right\} + \frac{2}{3}\cdot e^{-t}\cdot\mathcal{L}^{-1}\left\{\frac{1}{(s-1+1)^2+2^2}\right\}$$

$$= \frac{1}{3} e^{-t} \cdot \mathcal{L}^{-1}\left\{ \frac{1}{s^2+1^2} \right\} + \frac{1}{3} e^{-t} \cdot \mathcal{L}^{-1}\left\{ \frac{2}{s^2+2^2} \right\}$$

$$= \frac{1}{3} e^{-t} \cdot \sin t + \frac{1}{3} e^{-t} \cdot \sin(2t)$$

Decompose into partial fractions and then find the Inverse Laplace Transform of

$$\frac{s^2 + 2s + 3}{(s^2 + 2s + 2)(s^2 + 2s + 5)} e^{-3\pi s}$$

$$= \frac{u+3}{(u+2)(u+5)} \quad \left[\text{when } s^2 + 2s = u\right].$$

$$\frac{u+3}{(u+2)(u+5)} = \frac{A}{u+2} + \frac{B}{u+5}$$

$$\Rightarrow u+3 = A(u+5) + B(u+2)$$

$\underline{\underline{u=-2}}$ $\quad 1 = 3A \Rightarrow A = \frac{1}{3}$

$\underline{\underline{u=-5}}$ $\quad -2 = -3B \Rightarrow B = \frac{2}{3}.$

$$\frac{u+3}{(u+2)(u+5)} = \frac{\frac{1}{3}}{u+2} + \frac{\frac{2}{3}}{u+5}$$

$$\mathcal{L}^{-1}\left\{\frac{s^2+2s+3}{(s^2+2s+2)(s^2+2s+5)}\right\} = \frac{1}{3}\cdot\mathcal{L}^{-1}\left\{\frac{1}{s^2+2s+2}\right\} + \frac{2}{3}\cdot\mathcal{L}^{-1}\left\{\frac{1}{s^2+2s+5}\right\}$$

$$= \frac{1}{3}\mathcal{L}^{-1}\left\{\frac{1}{(s+1)^2+1^2}\right\} + \frac{2}{3}\mathcal{L}^{-1}\left\{\frac{1}{(s+1)^2+2^2}\right\}$$

$$= \frac{1}{3}e^{-t}\mathcal{L}^{-1}\left\{\frac{1}{(s-1+1)^2+1^2}\right\} + \frac{2}{3}\cdot e^{-t}\cdot\mathcal{L}^{-1}\left\{\frac{1}{(s-1+1)^2+2^2}\right\}$$

$$= \frac{1}{3} e^{-t} \cdot \mathcal{L}^{-1}\left\{ \frac{1}{s^2+1^2} \right\} + \frac{1}{3} e^{-t} \cdot \mathcal{L}^{-1}\left\{ \frac{2}{s^2+2^2} \right\}$$

$$\therefore f(t) = \frac{1}{3} e^{-t} \cdot \sin t + \frac{1}{3} e^{-t} \cdot \sin(2t)$$

$$\mathcal{L}^{-1}\left\{ - - - \, e^{-3\pi s} \right\}$$

$$= f(t-3\pi)\cdot u(t-3\pi) \,.$$

$$= \left[\frac{1}{3} e^{-(t-3\pi)} \cdot \sin(t-3\pi) + \frac{2}{3} e^{-(t-3\pi)} \cdot \sin\left(2(t-3\pi)\right) \cdot \right] u(t-3\pi)$$

$$= \left[-\frac{1}{3} \cdot e^{-t+3\pi} \cdot \sin t + \frac{2}{3} e^{-t+3\pi} \sin(2t) \right] u(t-3\pi)$$


Thank you!