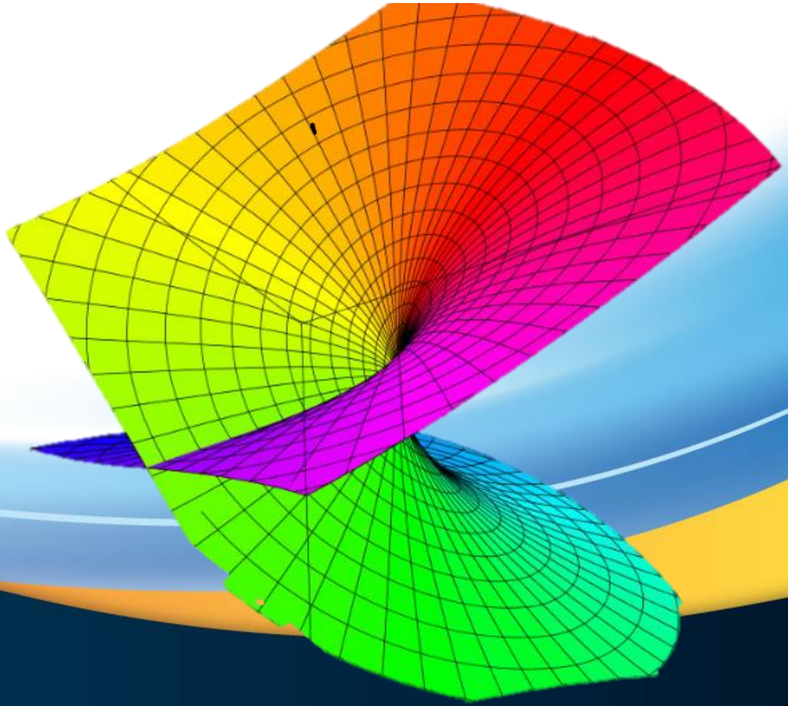
Undergraduate Course in Mathematics

Complex Variables

Topic: Cauchy-Riemann Equations



Conducted By

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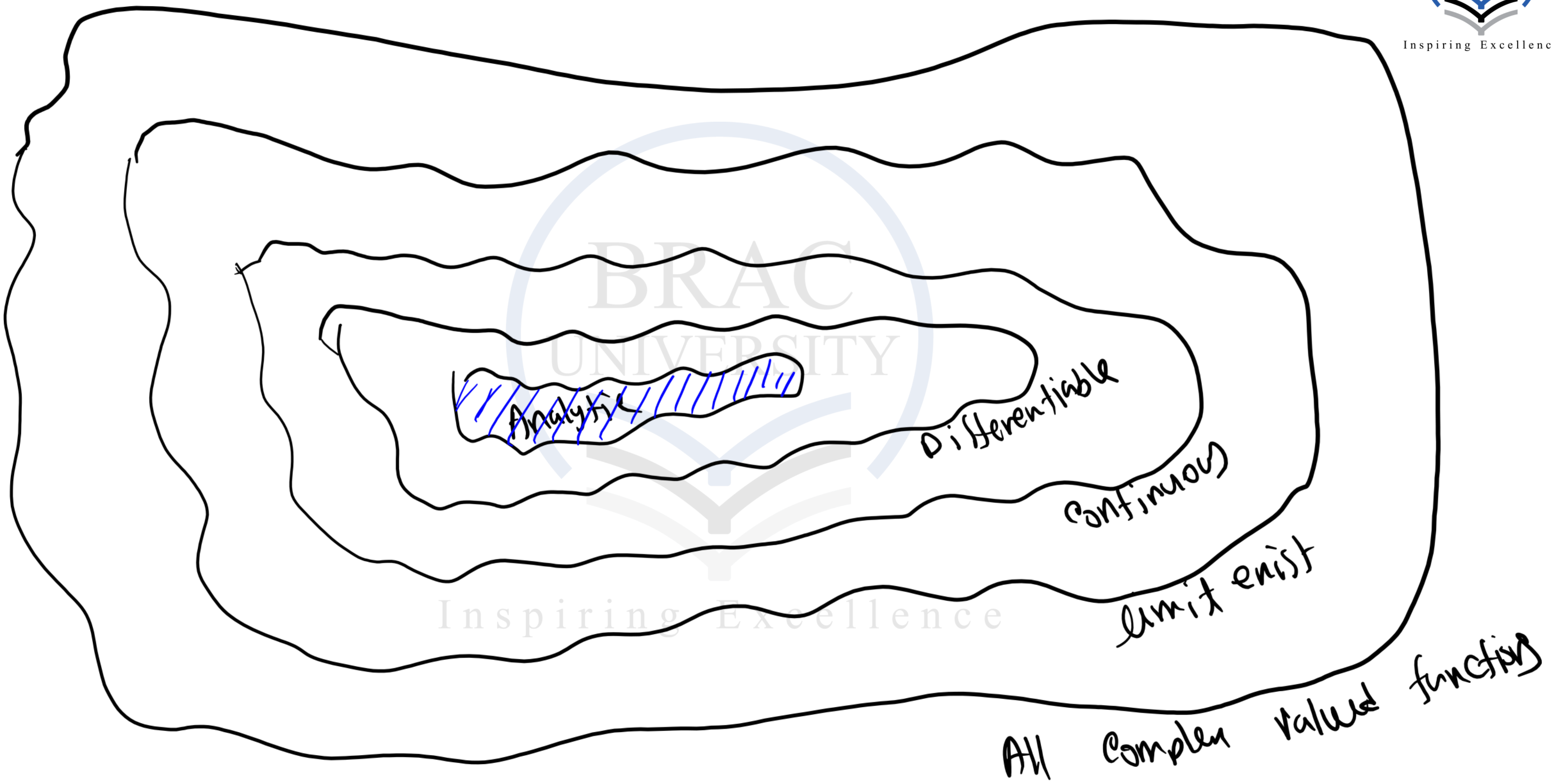
Inspiring Excellence


Analytic
Differentiable
Continuous
limit exist
All complex valued functions

Analytic $\Rightarrow$ Differentiable $\Rightarrow$ continuous $\Rightarrow$ limit exist

$\nLeftarrow$ $\qquad$ $\nLeftarrow$ $\qquad$ $\nLeftarrow$

Open Set

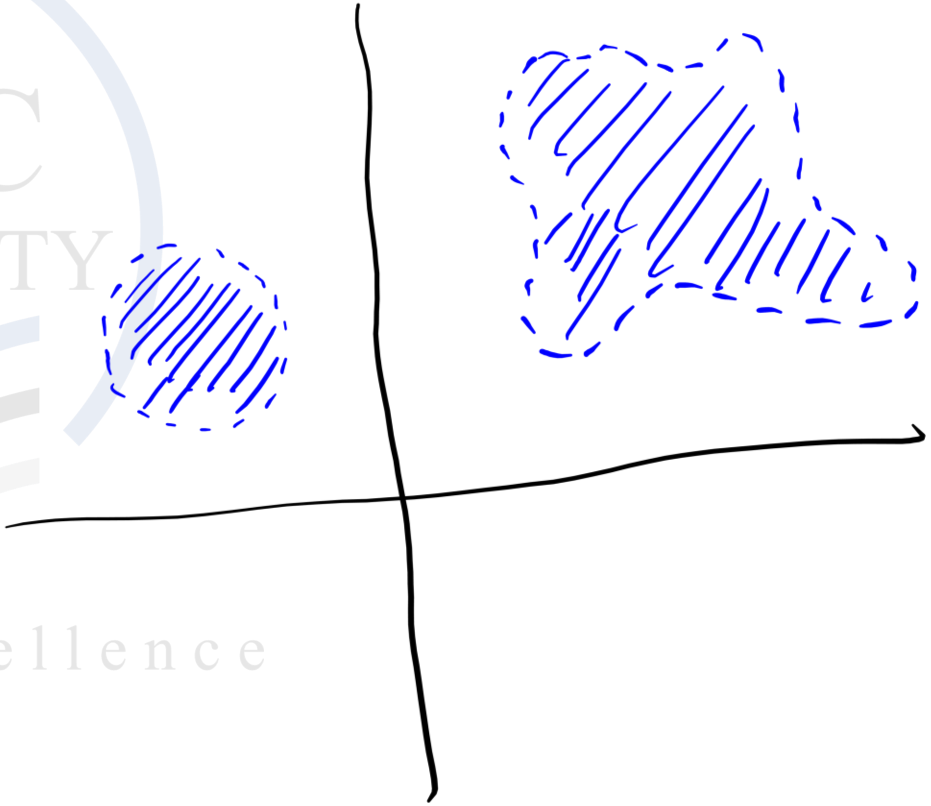
without Boundary

[2, 5]

(2, 5)

open

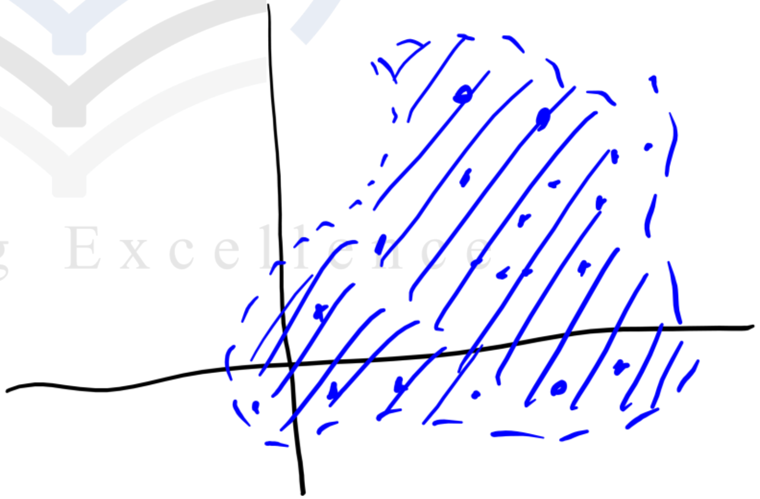
Complex



Analytic / Holomorphic Function Definition

A function $f(z)$ is called Anay on a open set or Region if $f(z)$ is differentiable at all points on R.

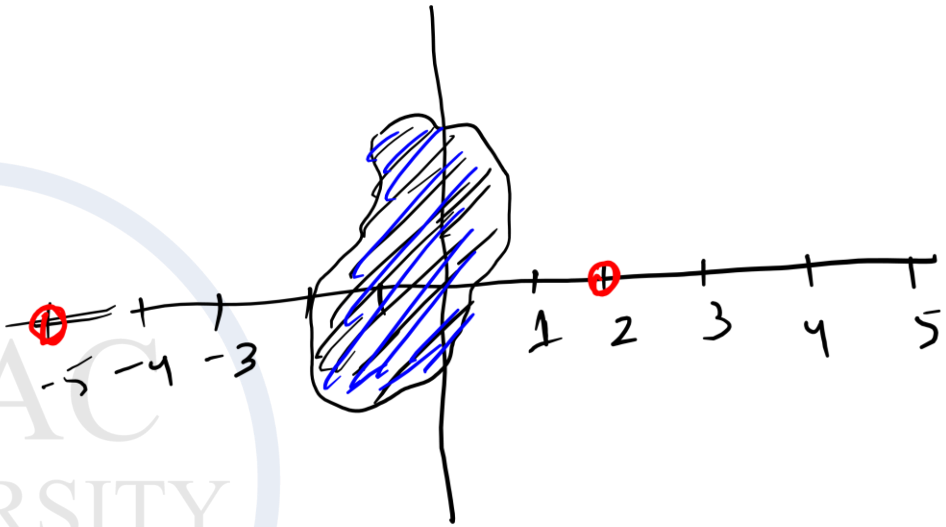
$f(z) = ??$



Examples of some Analytic functions

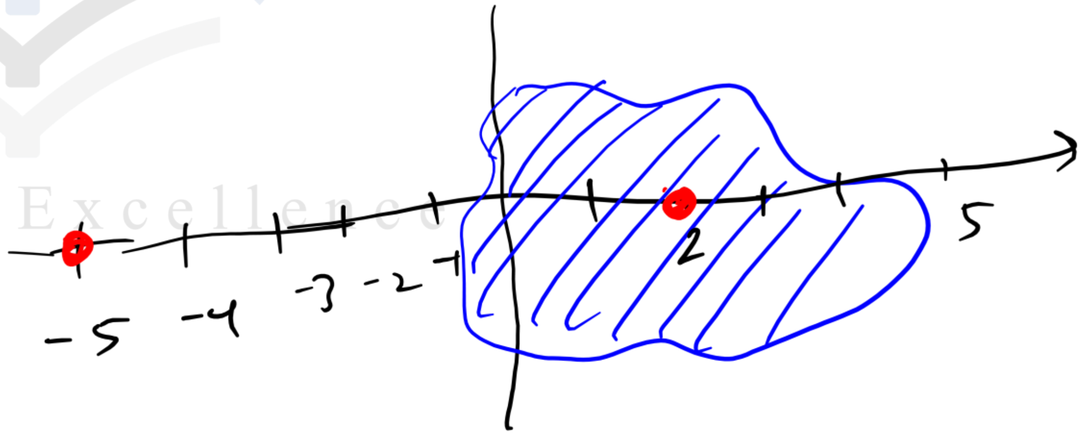
$z = \cancel{2}, \cancel{-5}$

$$f(z) = \frac{z}{(z-2)(z+5)}$$



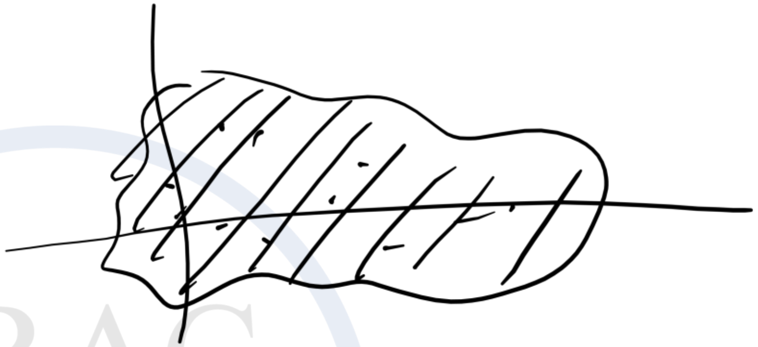
YES Analytic

$$f(z) = \frac{z}{(z-2)(z+5)}$$



Not Analytic

$f(z) = \sin z$

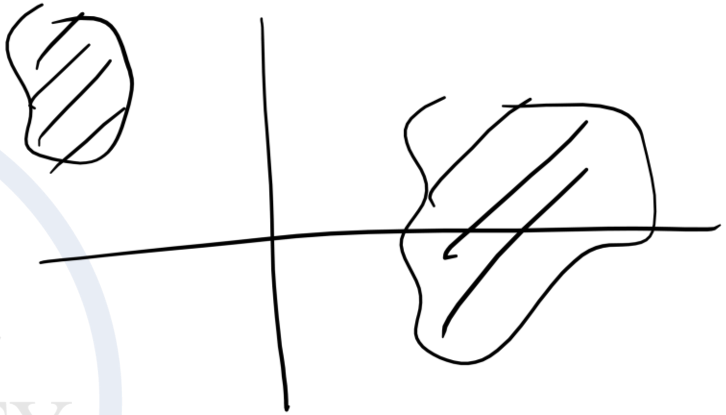


YES

$\tan z$

No

Entire Function



$$f(z) = \sin z$$

$\sin z$, $\cos z$, $\sinh z$, $\cosh z$, e^z, any polynomial

z^2+1, z^3+5z-2

$\cancel{\tan z}$, $\cancel{\frac{1}{z}}$, - -

What is Complex Differentiation?

$$f'(z) = \lim_{\Delta z \to 0} \frac{f(z+\Delta z) - f(z)}{\Delta z}$$

Differentiation of a function $f: \mathbb{R} \to \mathbb{R}$

$f(x) = x^2$

$f'(x) = 2x$

$$\lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$= --$

$= --$

$= 2x$

Differentiation of a function $f: \mathbb{R}^n \to \mathbb{R}$

$$f(x, y, z) = x^3 - 2y^2 + 3z$$

$$f\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \underbrace{x^3 - 2y^2 - 3z}$$

$$f' = [3x^2, -4y, -3]$$

Differentiation of a function $\boldsymbol{f}: \mathbb{R}^n \to \mathbb{R}^m$

$$f\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x^2 - y^2 + z \\ x + y - z^3 \\ z^2 - x^2 \end{pmatrix}$$

$f: R^3 \to R^3$

$$f' = \begin{pmatrix} 2x & -2y & 1 \\ 1 & 1 & -3z^2 \\ -2x & 0 & 2z \end{pmatrix}$$

$$f\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x^2 - y^2 \\ x + y + z^2 \end{pmatrix}$$

$$f\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = \begin{pmatrix} -3 \\ 12 \end{pmatrix}$$

$$f : R^7 \to R^9$$

$$f' = \begin{pmatrix} 2x & -2y & 0 \\ 1 & 1 & 2z \end{pmatrix}$$

$$f' = \begin{pmatrix} \quad \end{pmatrix}_{9\times 7}$$

Differentiation of a function $\boldsymbol{f}: \mathbb{R}^2 \to \mathbb{R}^2$

$$f\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} u \\ v \end{pmatrix}$$

$$f' = \begin{pmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{pmatrix}$$

The function $\mathbf{F}: \mathbb{C} \to \mathbb{C}$ can be represented as $\boldsymbol{f}: \mathbb{R}^2 \to \mathbb{R}^2$

4D

$$\boxed{F(z) = z^2}$$

$$f(x+iy) = (x+iy)^2$$

$$= x^2 + 2xiy + i^2y^2$$

$$= (x^2 - y^2) + i(2xy)$$

4D

$$\boxed{f\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x^2 - y^2 \\ 2xy \end{pmatrix}}$$

$$f(3+4i) = (3+4i)^2 = -7+24i$$

$$f\begin{pmatrix} 3 \\ 4 \end{pmatrix} = \begin{pmatrix} 3^2 - 4^2 \\ 2\cdot 3\cdot 4 \end{pmatrix} = \begin{pmatrix} -7 \\ 24 \end{pmatrix}$$

V.V.g

$$a + ib \Rightarrow \begin{pmatrix} a & -b \\ b & a \end{pmatrix}$$

Matrix Representation of Complex Field

$$3+4i \longrightarrow \begin{pmatrix} 3 & -4 \\ 4 & 3 \end{pmatrix}$$

$$-5-7i \longrightarrow \begin{pmatrix} -5 & 7 \\ -7 & -5 \end{pmatrix}$$

Matrix Representation of Complex Numbers

$$3-5i \longrightarrow \begin{pmatrix} 3 & 5 \\ -5 & 3 \end{pmatrix}$$

$$\begin{pmatrix} 2 & 3 \\ -3 & 2 \end{pmatrix} \longrightarrow 2-3i$$

$$\begin{pmatrix} 4 & -6 \\ 6 & 4 \end{pmatrix} \longrightarrow 4+6i$$

$$\begin{pmatrix} 2 & -7 \\ 3 & 5 \end{pmatrix} \longrightarrow \times$$

Algebra using Matrix

$(2+3i)(3+4i)$

$= 6+8i+9i-12$

$= -6+17i$

$$\begin{pmatrix} 2 & -3 \\ 3 & 2 \end{pmatrix} \cdot \begin{pmatrix} 3 & -4 \\ 4 & 3 \end{pmatrix}$$

$$= \begin{pmatrix} -6 & -17 \\ 17 & -6 \end{pmatrix} \rightarrow -6+17i$$

$$(2+3i)+(3+7i)$$

$$= 5+10i$$

$$\begin{pmatrix} 2 & -3 \\ 3 & 2 \end{pmatrix} + \begin{pmatrix} 3 & -7 \\ 7 & 3 \end{pmatrix}$$

$$= \begin{pmatrix} 5 & -10 \\ 10 & 5 \end{pmatrix}$$

$$\longrightarrow 5+10i$$

Algebra using Matrix $\dfrac{26+7i}{3-4i} = ?$

$= 2+5i$

$$\begin{pmatrix} 26 & -7 \\ 7 & 26 \end{pmatrix} \cdot \begin{pmatrix} 3 & 4 \\ -4 & 3 \end{pmatrix}^{-1}$$

HW

$$= \begin{pmatrix} 26 & -7 \\ 7 & 26 \end{pmatrix} \cdot \begin{pmatrix} \frac{3}{25} & \frac{-4}{25} \\ \frac{4}{25} & \frac{3}{25} \end{pmatrix}$$

$$= \begin{pmatrix} 2 & 5 \\ 5 & 2 \end{pmatrix} \longrightarrow 2+5i$$

Derivative of $\mathbf{F}: \mathbb{C} \to \mathbb{C}$ using $\boldsymbol{f}: \mathbb{R}^2 \to \mathbb{R}^2$

$$f(z) = z^2$$

$$f(x+iy) = (x+iy)^2$$

$$= (x^2-y^2) + i(2xy)$$

$$f\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \underline{x^2-y^2} \\ \underline{2xy} \end{pmatrix}$$

$$f' = \begin{pmatrix} 2x & -2y \\ 2y & 2x \end{pmatrix}$$

$$= 2x + i\,2y$$

$$f(z) = z^2$$

$$f'(z) = 2z$$

$$= 2(x+iy)$$

$$= 2x + i\,2y$$

Can all derivative of $f: \mathbb{R}^2 \to \mathbb{R}^2$ be represented as derivative of $\mathbf{F}: \mathbb{C} \to \mathbb{C}$

$$f\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x^2 - y^3 + 2x \\ x - y^2 \end{pmatrix}$$

$$\begin{pmatrix} a & -b \\ b & a \end{pmatrix}$$

$$f' = \begin{pmatrix} 2x+2 & -3y^2 \\ 1 & -2y \end{pmatrix} \to \text{not possible}$$

$$f\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} u \\ v \end{pmatrix}$$

$$f' = \begin{pmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{pmatrix}$$

$$\begin{cases} ① \quad \dfrac{\partial u}{\partial x} = \dfrac{\partial v}{\partial y} \\ ② \quad \dfrac{\partial u}{\partial y} = -\dfrac{\partial v}{\partial x} \end{cases}$$

Cauchy-Riemann Equations

A necessary condition that $w = \boxed{f(z) = u(x, y) + iv(x, y)}$ be analytic in a region $\mathcal{R}$ is that, in $\mathcal{R}$, u and v satisfy the *Cauchy–Riemann equations*

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \qquad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \tag{3.2}$$

If the partial derivatives in (3.2) are continuous in $\mathcal{R}$, then the Cauchy–Riemann equations are sufficient conditions that $f(z)$ be analytic in $\mathcal{R}$.

Necessary Condition

A necessary condition that $w = f(z) = u(x, y) + iv(x, y)$ be analytic in a region $\mathcal{R}$ is that, in $\mathcal{R}$, u and v satisfy the *Cauchy–Riemann equations*

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \qquad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \tag{3.2}$$

$f(z) = u + iv$ analytic $\Longrightarrow$ $\boxed{\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \text{ and } \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}}$

Sufficient Condition

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \qquad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \tag{3.2}$$

If the partial derivatives in (3.2) are continuous in $\mathcal{R}$, then the Cauchy–Riemann equations are sufficient conditions that $f(z)$ be analytic in

$f(z) = u + iv$

$$\boxed{\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \text{ and } \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}} \Longrightarrow f(z) \text{ is } \underline{\text{Analytic}}$$

$+$

$$\boxed{\frac{\partial u}{\partial x}, \frac{\partial v}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial v}{\partial y} \text{ are cont.}}$$

p is prime $\Longrightarrow$ $p > 1$

$p > 1$ $\Longrightarrow$ p is prime

$+$

only 2 divisors

Analytic $\Rightarrow$ C-R

Proof: Given $f(z) = u(x,y) + i\,v(x,y)$ is analytic

$\Rightarrow f'(z)$ exists.

$$f'(z) = \lim_{\Delta z \to 0} \frac{\overbrace{f(z+\Delta z)} - \overbrace{f(z)}}{\Delta z}$$

$$= \lim_{\substack{\Delta x \to 0 \\ \Delta y \to 0}} \frac{\overbrace{u(x+\Delta x, y+\Delta y) + i\,v(x+\Delta x, y+\Delta y)} - \overbrace{u(x,y) - i\,v(x,y)}}{\Delta x + i\,\Delta y}$$

In $\Delta y = 0$ direction

$$f'(z) = \text{limit} = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$$

In $\Delta x = 0$ direction

$$f'(z) = \text{limit} = \frac{\partial v}{\partial y} - i\frac{\partial u}{\partial y}$$

but $f'(z)$ exists

$\Rightarrow$ limit also exists

$$\therefore \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = \frac{\partial v}{\partial y} - i\frac{\partial u}{\partial y}$$

$$\therefore \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$$

$$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

How to prove a functions is Analytic

$$f(z) = u + iv$$

$$\left.\begin{array}{l} ① \quad \dfrac{\partial u}{\partial x} = \dfrac{\partial v}{\partial y}, \quad \dfrac{\partial u}{\partial y} = -\dfrac{\partial v}{\partial x} \\[2em] ② \quad \underline{\dfrac{\partial u}{\partial x}, \dfrac{\partial u}{\partial y}, \dfrac{\partial v}{\partial x}, \dfrac{\partial v}{\partial y}} \ \text{cont} \end{array}\right\} \Longrightarrow f(z) \text{ is Analytic}$$

Show that $f(z) = z^2$ is analytic

$$f(x+iy) = (x+iy)^2$$

$$= x^2 + 2xiy + i^2y^2$$

$$= (x^2 - y^2) + i(2xy)$$

$$u = x^2 - y^2 \qquad v = 2xy$$

$u = x^2 - y^2 \qquad v = 2xy$

$$\frac{\partial u}{\partial x} = 2x$$

$$\frac{\partial u}{\partial y} = -2y$$

$$\frac{\partial v}{\partial x} = 2y$$

$$\frac{\partial v}{\partial y} = 2x$$

Now. $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ ✓ $\qquad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$

$\therefore$ C-R equations are satisfied ✓

$\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial v}{\partial x}, \frac{\partial v}{\partial y}$ all are cont.

$\Rightarrow f(z) = z^2$ is analytic

Show that $f(z) = |z|^2 - \bar{z}$ is not analytic

$$f(z) = |x+iy|^2 - \overline{(x+iy)}$$

$$= x^2+y^2 - x - iy$$

$$= (x^2+y^2-x) + i(y)$$

$$u = x^2+y^2-x \qquad v = y$$

$$\frac{\partial u}{\partial x} = 2x-1$$

$$\frac{\partial u}{\partial y} = 2y$$

$$\frac{\partial v}{\partial x} = 0$$

$$\frac{\partial v}{\partial y} = 1$$

$$\frac{\partial u}{\partial x} \neq \frac{\partial v}{\partial y}$$

$$\frac{\partial u}{\partial y} \neq -\frac{\partial v}{\partial x}$$

$\therefore f(z)$ is not analytic

Show that $f(z) = \frac{1}{z-3}$ is not analytic except at $z = 3$

$$f(z) = \frac{1}{(x+iy)-3}$$

$$= \frac{1}{(x-3)+iy}$$

$$= \frac{1\cdot[(x-3)-iy]}{[(x-3)+iy]\cdot[(x-3)-iy]}$$

$$= \frac{x-3-iy}{(x-3)^2+y^2}$$

$$= \frac{x-3}{(x-3)^2+y^2} + i\,\frac{-y}{(x-3)^2+y^2}$$

$$u = \frac{x-3}{(x-3)^2+y^2} \qquad v = \frac{-y}{(x-3)^2+y^2}$$

$$\frac{\partial u}{\partial x} = \frac{[(x-3)^2+y^2]\cdot 1 - (x-3)\cdot 2(x-3)}{[(x-3)^2+y^2]^2} = \frac{-(x-3)^2+y^2}{[(x-3)^2+y^2]^2}$$

$$\frac{\partial u}{\partial y} = \frac{[(x-3)^2+y^2)\cdot 0 - (x-3)\cdot 2y}{[(x-3)^2+y^2]^2} = \frac{-2(x-3)y}{[(x-3)^2+y^2]^2}$$

$$\frac{\partial v}{\partial x} = \frac{[(x-3)^2+y^2]\cdot 0 - (-y)\cdot 2\cdot(x-3)}{[(x-3)^2+y^2]^2} = \frac{2(x-3)y}{[(x-3)^2+y^2)}$$

$$\frac{\partial v}{\partial y} = \frac{[(x-3)^2+y^2]\cdot(-1) - (-y)2y}{[(x-3)^2+y^2]^2} = \frac{-(x-3)^2+y^2}{[(x-3)^2+y^2]^2}$$

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \&$$

$$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

$\therefore$ CR equation are satisfied &

$\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial v}{\partial x}, \frac{\partial v}{\partial y}$ all are continuous except $(x,y) = (3,0)$

" " " " $z = 3 + 0i$

$\Rightarrow z = 3.$

$\Rightarrow f(z) = \frac{1}{z-3}$ is analytic except $z = 3.$

Show that $f(z) = \sinh(2z)$ is analytic

$$f(z) = \frac{e^{2z} - e^{-2z}}{2}$$

$$f(x+iy) = \frac{e^{2x+i2y} - e^{-2x-i2y}}{2}$$

$$= \frac{e^{2x} \cdot e^{i2y} - e^{-2x} \cdot e^{i(-2y)}}{2}$$

$$= \frac{e^{2x}}{2}(\cos 2y + i \sin 2y) - \frac{e^{-2x}}{2}(\cos 2y - i \sin 2y)$$

$$= \frac{1}{2}(e^{2x} - e^{-2x}) \cos 2y + i \frac{1}{2}(e^{2x} + e^{-2x}) \sin 2y$$

$$u = \frac{1}{2}(e^{2x} - e^{-2x}) \cos 2y$$

$$v = \frac{1}{2}(e^{2x} + e^{-2x}) \sin 2y$$

$$u = \frac{1}{2}\left(e^{2x} - e^{-2x}\right)\cos 2y \qquad v = \frac{1}{2}\left(e^{2x} + e^{-2x}\right)\sin 2y$$

$$\frac{\partial u}{\partial x} = \left(e^{2x} + e^{-2x}\right)\cos 2y$$

$$\frac{\partial u}{\partial y} = -\left(e^{2x} - e^{-2x}\right)\sin 2y$$

$$\frac{\partial v}{\partial x} = \left(e^{2x} - e^{-2x}\right)\sin 2y$$

$$\frac{\partial v}{\partial y} = \left(e^{2x} + e^{-2x}\right)\cos 2y$$

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \checkmark \qquad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \checkmark$$

All the partial derivatives are continuous.

$\Rightarrow f(z)$ is analytic everywhere

Show that $f(z) = ze^{-z}$ is analytic

$$f(x+iy) = (x+iy)\, e^{-x-iy}$$

$$= (x+iy)\, e^{-x} \cdot e^{-iy}$$

$$= (x+iy)\, e^{-x} (\cos y - i \sin y)$$

$$= e^{-x} (x \cos y - i x \sin y + i y \cos y + y \sin y)$$

$$= e^{-x} (x \cos y + y \sin y) + i\, e^{-x} (-x \sin y + y \cos y)$$

$$u = e^{-x} (x \cos y + y \sin y)$$

$$v = e^{-x} (-x \sin y + y \cos y)$$

Hw

Graph of Complex valued Functions

$$f(z) = z^2$$

$$z \downarrow \quad z^2 \downarrow$$

$x+iy \qquad u+iv$

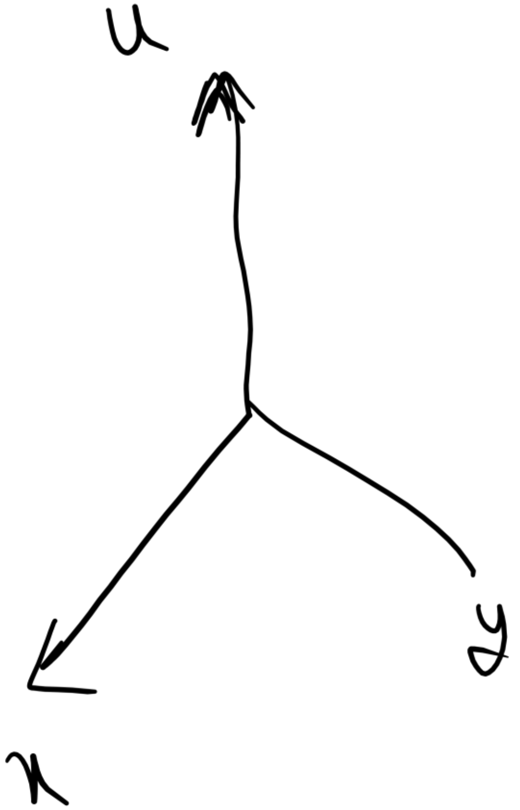
$$\underline{\underline{4D}}$$

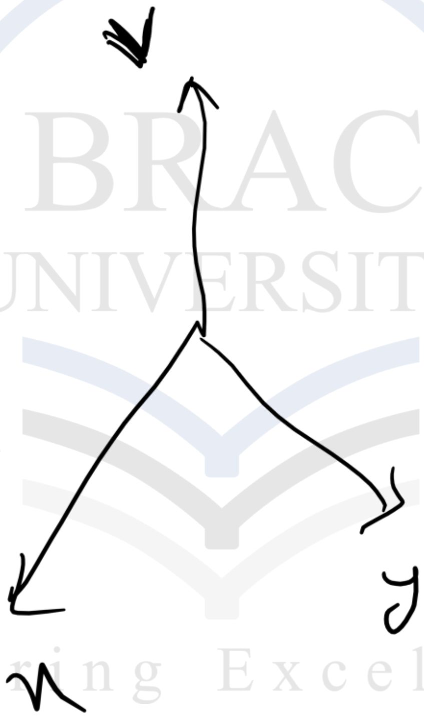
$$f\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x^2-y^2 \\ 2xy \end{pmatrix}$$

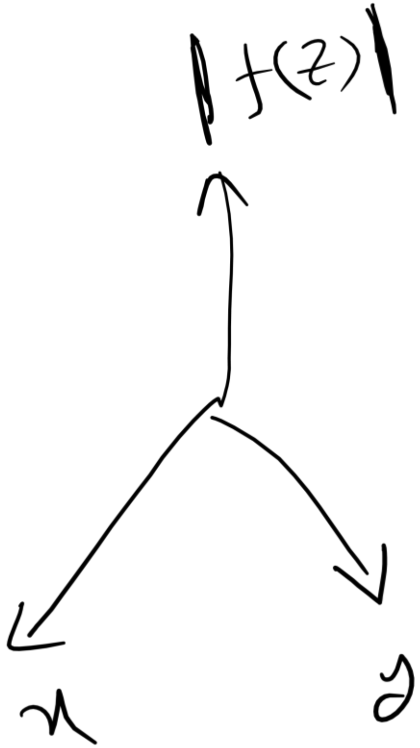
$$\underline{\underline{4D}}$$

$$f(x+iy) = u+iv$$

$$f(z) = z^2 = (x^2-y^2) + i(2xy)$$



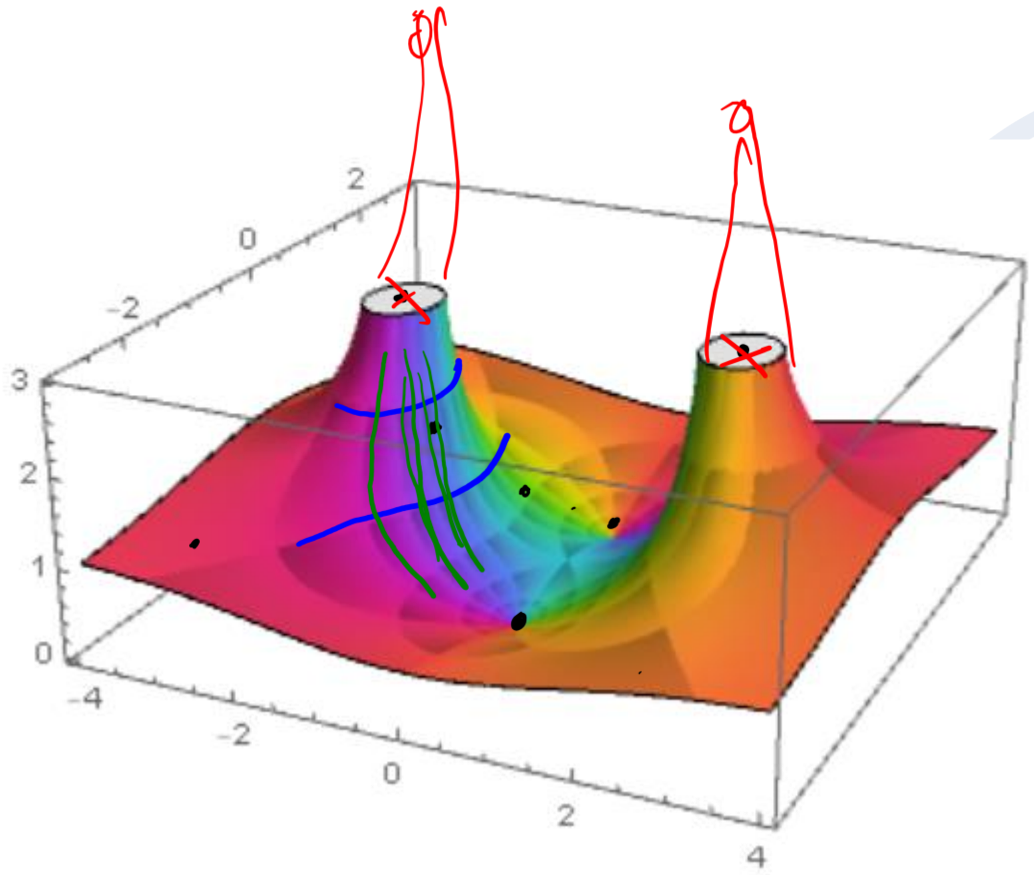


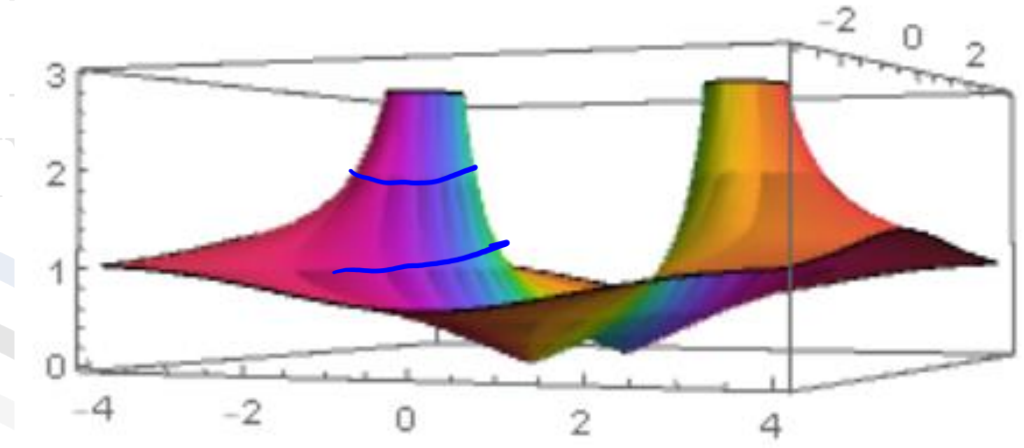


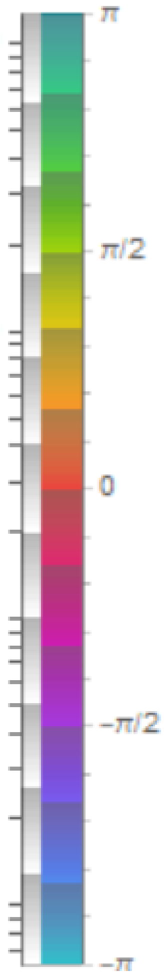
3D Mod-Arg approach

$$f(z) = \frac{z^2+1}{z^2-4}$$

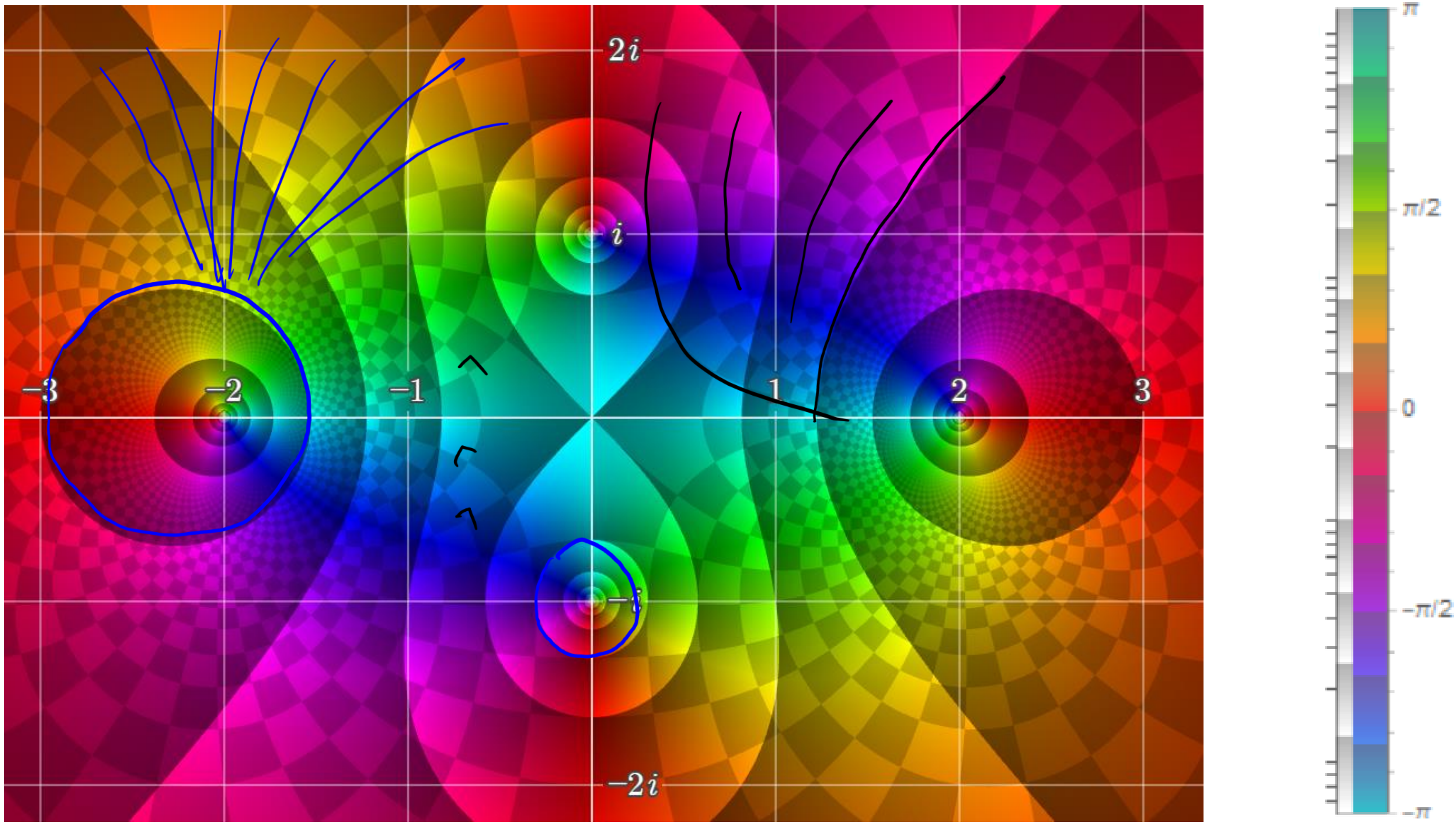
$z = i, -i$





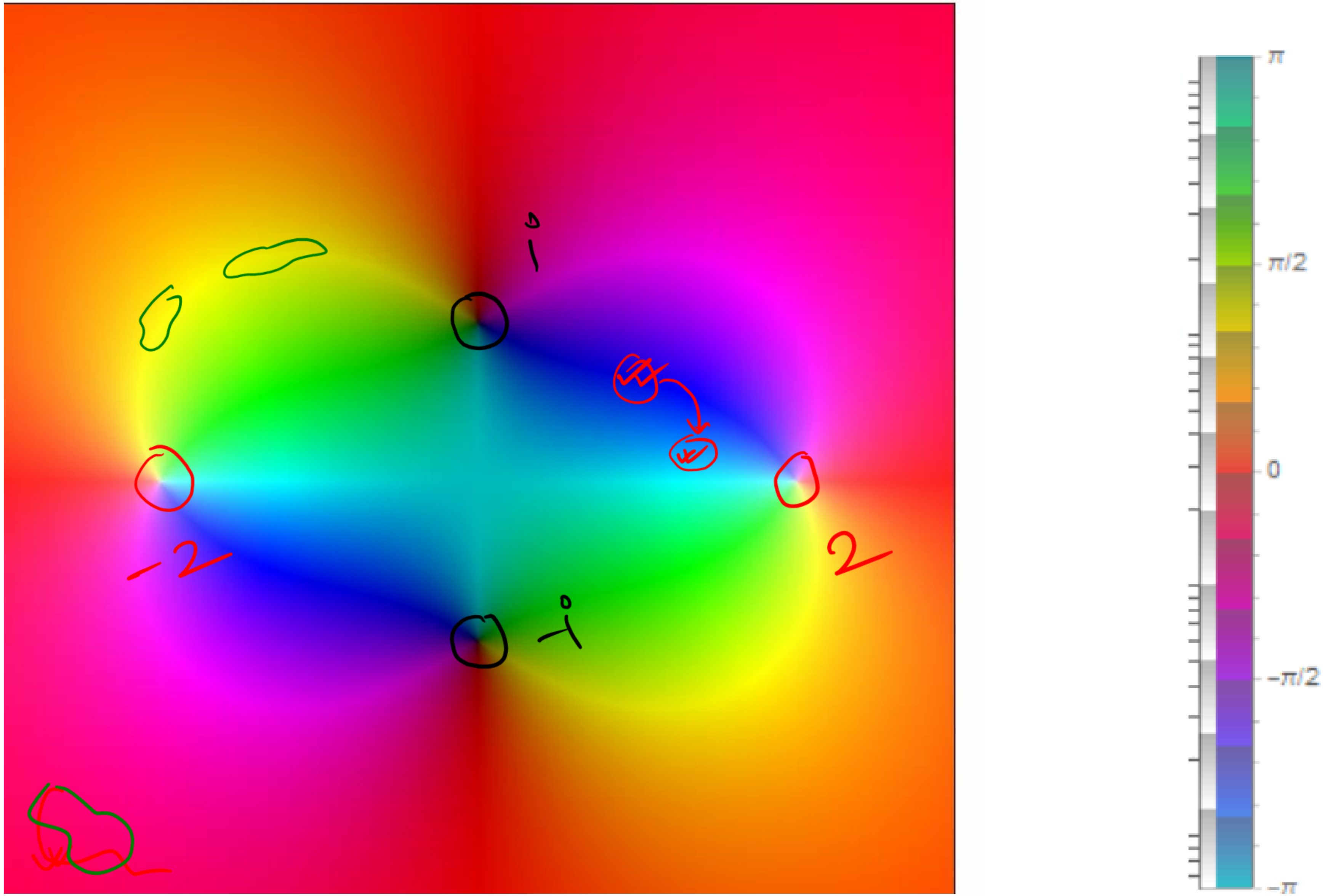


2D Color-Contour approach $f(z) = \dfrac{z^2+1}{z^2-4}$



2D Color-Contour approach $f(z) = \dfrac{z^2+1}{z^2-4}$

Black = 0
White = ∞



Zeros of a Function

$$f(z) = \frac{(z-1)^2 (z-3)(z-5)^3}{(z+2)(z-2)^4 (z-4)^2}$$

$$f(z) = 0 \implies z = 1, 3, 5 \longrightarrow \underline{\underline{\text{Dark}}}$$

$z = 1$: order $= 2$

$z = 3$: order $= 1$

$z = 5$: order $= 3$

Singularity / Singular Point

when function is not analytic

$$f(z) = \frac{(z-1)^2 (z-3)(z-5)^3}{(z+2)(z-2)^4 (z-4)^2}$$

undefined

$z = -2, 2, 4$ → Bright

order $=1$

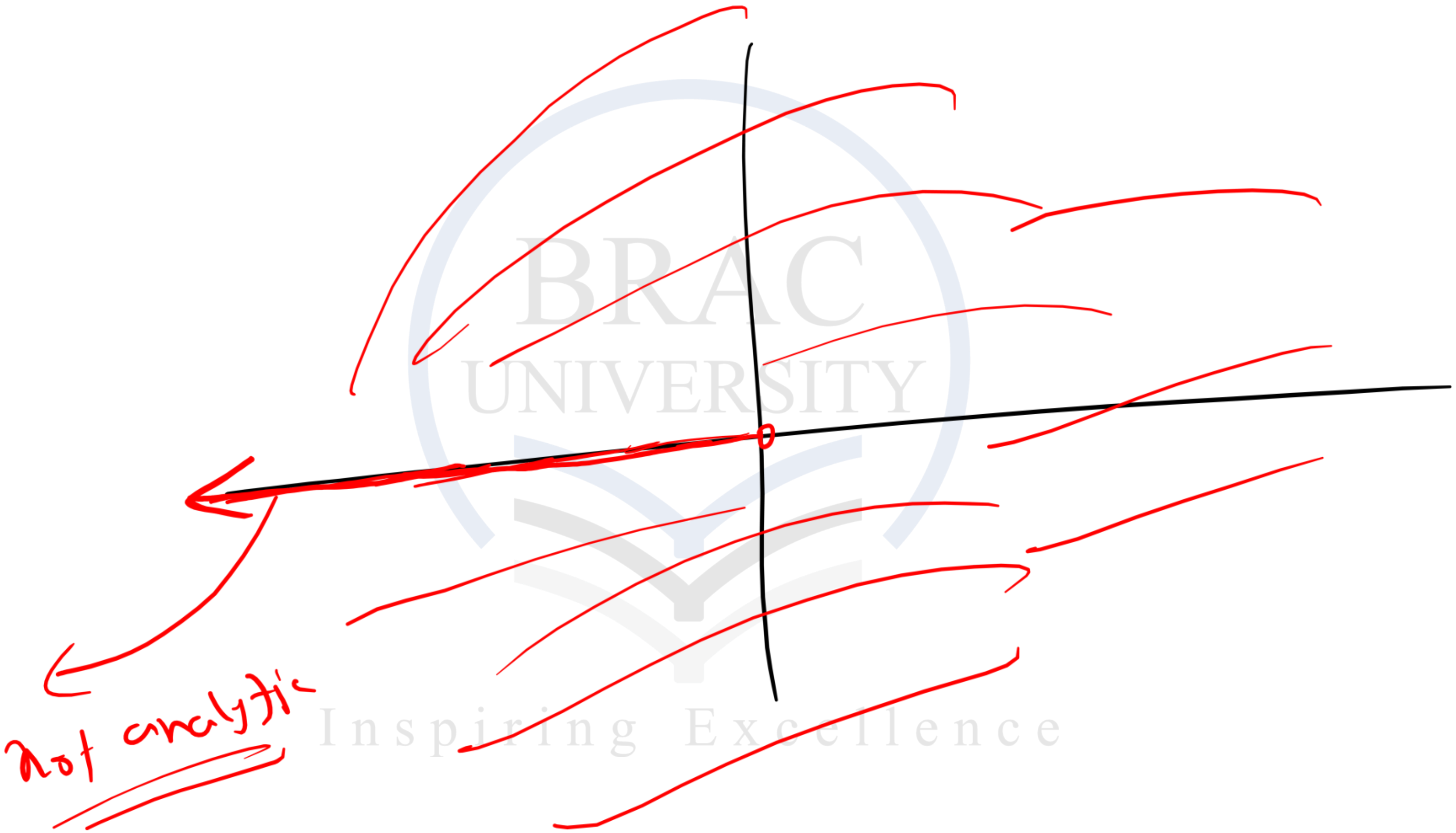
ord $= 4$

ord $= 2$

Classifications of Singularity

- Pole → order $= 1, 2, 3, 4, \ldots$
- Essential → order $= \infty$
- Removable → order $= 0$
- Branch $z^{\frac{1}{2}}$, $\ln z$, $z^{\frac{1}{3}}$

$z^{\frac{1}{2}}$



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