Lecture 10

Function

Topics

- Invertibility and Inverse Functions
- Function Compositions

Inverse Functions and Compositions of Functions

Let ***f*** be a one-to-one correspondence from the set ***A*** to the set ***B***. The ***inverse function of f*** is the function that assigns to an element ***b*** belonging to ***B*** to the unique element ***a*** in ***A*** such that ***f(a) = b***. The inverse function of ***f*** is denoted by f^{-1}. Hence, $f^{-1}(b) = a$ when $f(a) = b$.

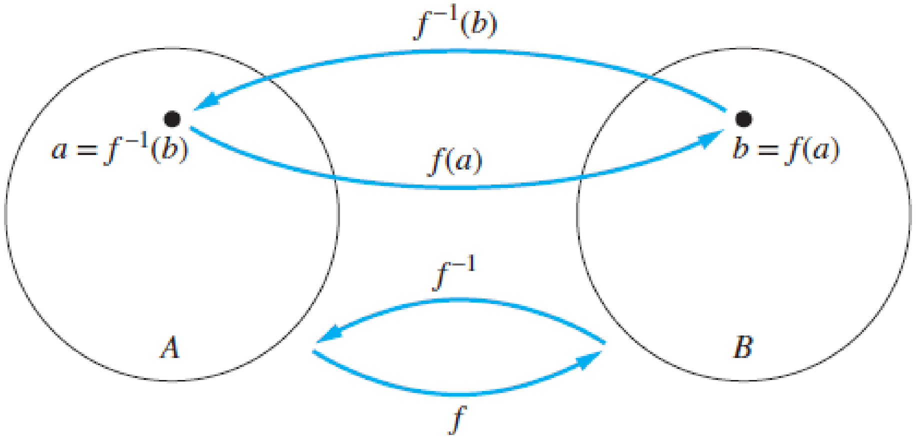

Fig : The function f^{-1} is the inverse function of f

Inverse Functions and Compositions of Functions

A **one-to-one correspondence** is called **invertible** because we can **define** an inverse of this function. A function is **not invertible** if it is **not a one-to-one correspondence**, because the inverse of such a function **does not exist**.

EXAMPLE: Let $\boldsymbol{f}$ be the function from $\boldsymbol{\{a, b, c\}}$ to $\boldsymbol{\{1, 2, 3\}}$ such that $\boldsymbol{f(a) = 2}$, $\boldsymbol{f(b) = 3}$, and $\boldsymbol{f(c) = 1}$. Is $\boldsymbol{f}$ invertible, and if it is, what is its inverse?

SOLUTION: The function f is invertible because it is a one-to-one correspondence. The inverse function f^{-1} reverses the correspondence given by f, so $f^{-1}(1) = c$, $f^{-1}(2) = a$, and $f^{-1}(3) = b$.

Examples

EXAMPLE: Let ***f*** be the function from ***R*** to ***R*** with $\boldsymbol{f(x) = x^2}$. Is f invertible?

SOLUTION: Because $\boldsymbol{f(-2) = f(2) = 4}$, ***f*** is not one-to-one. If an inverse function were defined, it would have to assign ***two*** elements to ***4***. Hence, ***f*** is not invertible. (Note we can also show that ***f*** is not invertible because it is not onto.)

Compositions of Functions

Let ***g*** be a function from the set ***A*** to the set ***B*** and let ***f*** be a function from the set ***B*** to the set ***C***. The composition of the functions ***f*** and g, denoted for all ***a*** $\in$ ***A*** by ***f∘g***, is the function from ***A*** to ***C*** defined by ***(f∘g)(a) = f(g(a))***

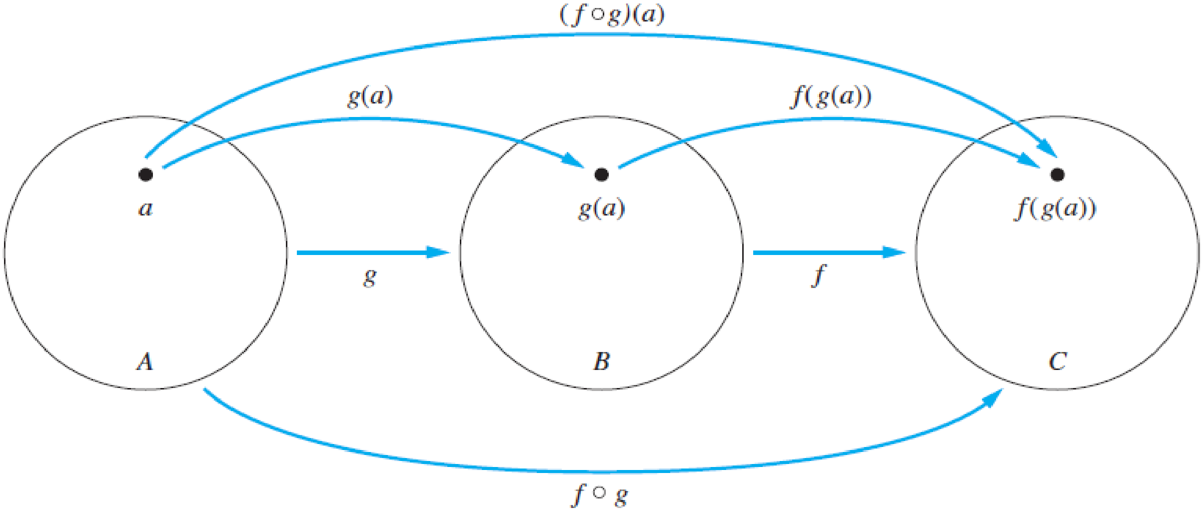

Fig : The composition of the functions *f* and *g*

Examples

EXAMPLE: Let ***g*** be the function from the set ***{a, b, c}*** to ***itself*** such that ***g(a) = b, g(b) = c, and g(c) = a.*** Let ***f*** be the function from the set ***{a, b, c}*** to the set ***{1, 2, 3}*** such that ***f(a) = 3, f(b) = 2, and f(c) = 1***. What is the composition of ***f*** and ***g***, and what is the composition of ***g*** and ***f*** ?

SOLUTION: The composition $f \circ g$ is defined by $(f \circ g)(a) = f(g(a)) = f(b) = 2$, $(f \circ g)(b) = f(g(b)) = f(c) = 1$, and $(f \circ g)(c) = f(g(c)) = f(a) = 3$.

Note that $g \circ f$ is not defined, because the range of f is not a subset of the domain of g.

Examples

EXAMPLE: Let f and g be the functions defined by $f : R \rightarrow R^{+} \cup \{0\}$ with $f(x) = x^2$ and $g : R^{+} \cup \{0\} \rightarrow R$ with $g(x) = \sqrt{x}$ (where $\sqrt{x}$ is the nonnegative square root of x). What is the function $(f \circ g)(x)$?

SOLUTION: The domain of $(f \circ g)(x) = f(g(x))$ is the domain of g, which is $R^{+} \cup \{0\}$, the set of nonnegative real numbers. If x is a ***nonnegative real number***, we have $(f \circ g)(x) = f(g(x)) = f(\sqrt{x}) = (\sqrt{x})^2 = x$. The range of $f \circ g$ is the image of the range of g with respect to the function f. This is the set $R^{+} \cup \{0\}$, the set of nonnegative real numbers. Summarizing, $f : R^{+} \cup \{0\} \rightarrow R^{+} \cup \{0\}$ and $f(g(x)) = x$ for all x.

Explanation

When the composition of a function and its inverse is formed, in either order, an identity function is obtained. To see this, suppose that f is a ***one-to-one correspondence*** from the set A to the set B. Then the inverse function f^{-1} exists and is a ***one-to-one correspondenc***e from B to A. The inverse function reverses the correspondence of the original function, so $f^{-1}(b) = a$ when $f(a) = b$, and $f(a) = b$ when $f^{-1}(b) = a$. Hence, $(f^{-1} \circ f)(a) = f^{-1}(f(a)) = f^{-1}(b) = a$, and $(f \circ f^{-1})(b) = f(f^{-1}(b)) = f(a) = b$. Consequently $f^{-1} \circ f = I_A$ and $f \circ f^{-1} = I_B$, where I_A and I_B are the identity functions on the sets A and B, respectively. That is, $(f^{-1})^{-1} = f$.

Some Important Functions

The ***floor function*** assigns to the real number ***x*** the ***largest*** integer that is ***less than or equal to x***. The value of the floor function at ***x*** is denoted by $\lfloor x \rfloor$.

The ***ceiling function*** assigns to the real number ***x*** the ***smallest*** integer that is ***greater than or equal to x***. The value of the ceiling function at ***x*** is denoted by $\lceil x \rceil$.

TABLE 1 Useful Properties of the Floor and Ceiling Functions.
(n is an integer, x is a real number)

(1a)	$\lfloor x \rfloor = n$ if and only if $n \leq x < n + 1$
(1b)	$\lceil x \rceil = n$ if and only if $n - 1 < x \leq n$
(1c)	$\lfloor x \rfloor = n$ if and only if $x - 1 < n \leq x$
(1d)	$\lceil x \rceil = n$ if and only if $x \leq n < x + 1$
(2)	$x - 1 < \lfloor x \rfloor \leq x \leq \lceil x \rceil < x + 1$
(3a)	$\lfloor -x \rfloor = -\lceil x \rceil$
(3b)	$\lceil -x \rceil = -\lfloor x \rfloor$
(4a)	$\lfloor x + n \rfloor = \lfloor x \rfloor + n$
(4b)	$\lceil x + n \rceil = \lceil x \rceil + n$

Examples

EXAMPLE: Prove that if x is a ***real number***, then $\lfloor 2x \rfloor = \lfloor x \rfloor + \lfloor x + 1/2 \rfloor$.

SOLUTION: To prove this statement we let $x = n + \epsilon$, where n is an integer and $0 \leq \epsilon < 1$. There Examples are two cases to consider, depending on whether ϵ is l***ess than, or greater than or equal to 1/2*** . (The reason we choose these two cases will be made clear in the proof.)

We first consider the case when $0 \leq \epsilon < 1/2$. In this case, $2x = 2n + 2\epsilon$ and $\lfloor 2x \rfloor = 2n$ because $0 \leq 2\epsilon < 1$. Similarly, $x + 1/2 = n + (1/2 + \epsilon)$, so $\lfloor x + 1/2 \rfloor = n$, because $0 < 1/2 + \epsilon < 1$. Consequently, $\lfloor 2x \rfloor = 2n$ and $\lfloor x \rfloor + \lfloor x + 1/2 \rfloor = n + n = 2n$.

Examples

Next, we consider the case when $1/2 \leq \epsilon < 1$. In this case, $2x = 2n + 2\epsilon = (2n + 1) + (2\epsilon - 1)$. Because $0 \leq 2\epsilon - 1 < 1$, it follows that $\lfloor 2x \rfloor = 2n + 1$. Because $\lfloor x + 1/2 \rfloor = \lfloor n + (1/2 + \epsilon) \rfloor = \lfloor n + 1 + (\epsilon - 1/2) \rfloor$ and $0 \leq \epsilon - 1/2 < 1$, it follows that $\lfloor x + 1/2 \rfloor = n + 1$. Consequently, $\lfloor 2x \rfloor = 2n + 1$ ***and*** $\lfloor x \rfloor + \lfloor x + 1/2 \rfloor = n + (n + 1) = 2n + 1$. This concludes the proof.

Thank You