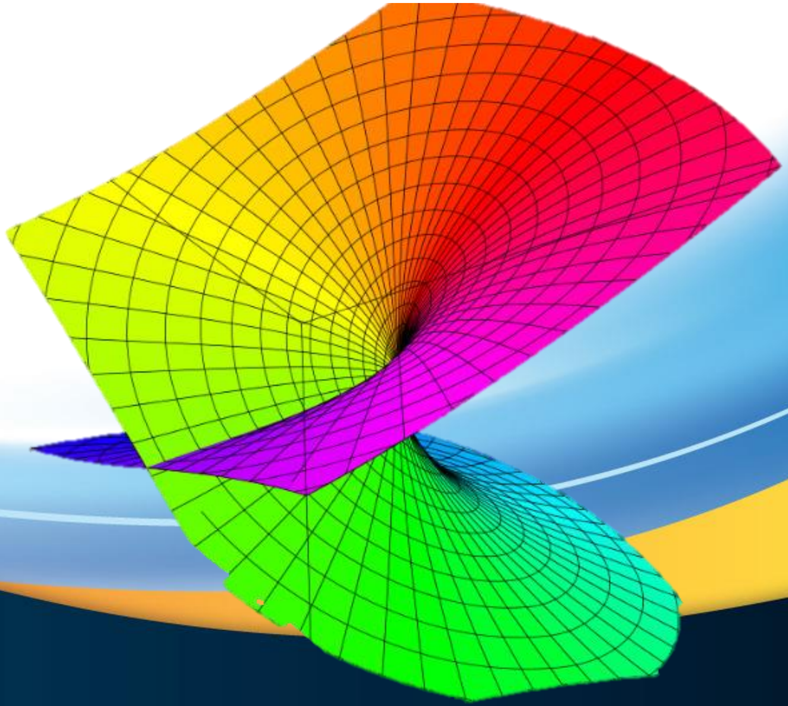
Undergraduate Course in Mathematics

Complex Variables

Topic: Complex Numbers



Conducted By

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The Real Number System

0, 1, 2, 3, - ——

-1, -5, -6, ——

2.5, 6.5, 1.39, ——

1.3333 - - -

$\sqrt{6}$, $\sqrt{2}$

Real numbers

Why Complex Numbers?

$\sqrt{4} = 2$

$\sqrt{9} = 3$

$2^2 = 4$

$3^2 = 9$

$\sqrt{-9} = ?$ Not a Real number

$x \cdot x = -9$

$3 \cdot 3 = 9$

$(-3)(-3) = 9$

Solving Quadratic Equations

Solve the equation $\boxed{x^2 - 6x + 10 = 0}$ → No solution in Real

$ax^2+bx+c=0$

$$x = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$$

$$x = \frac{6 \pm \sqrt{6^2 - 4\cdot 1\cdot 10}}{2}$$

$$= \frac{6 \pm \sqrt{-4}}{2}$$

× not possible

$=$

$$x^2 - 6x + 10 = 0$$

$$y = x^2 - 6x + 10$$

$y = 0$

x axis

solution x axis cut করে কোথাও ?

$$ax^3+bx^2+cx+d=0$$

$$x=$$

Solving Cubic Equations

If $x^3 = 3px + 2q$ then,

$$x = \sqrt[3]{q + \sqrt{q^2 - p^3}} + \sqrt[3]{q - \sqrt{q^2 - p^3}}$$

$p = 5$

$q = 2$

$x = 4$

Solve the equation $x^3 = 15x + 4$

$$x = \sqrt[3]{2 + \sqrt{4 - 125}} + \sqrt[3]{2 - \sqrt{4 - 125}}$$

Girolamo Cardano (1501-1576)

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$$x = \sqrt[3]{2+\sqrt{-121}} + \sqrt[3]{2-\sqrt{-121}}$$

$$= \sqrt[3]{2+11\sqrt{-1}} + \sqrt[3]{2-11\sqrt{-1}}$$

$$= \sqrt[3]{8+12\sqrt{-1}-6-\sqrt{-1}} + \sqrt[3]{8-12\sqrt{-1}-6+\sqrt{-1}}$$

$$= \sqrt[3]{2^3+3\cdot 2^2\sqrt{-1}+3\cdot 2\cdot(\sqrt{-1})^2+(\sqrt{-1})^3} + \sqrt[3]{2^3-3\cdot 2^2\sqrt{-1}+3\cdot 2\cdot(\sqrt{-1})^2-(\sqrt{-1})^3}$$

$$\sqrt{-121}$$

$$= \sqrt{121\times(-1)}$$

$$= \sqrt{121}\cdot\sqrt{-1}$$

$$= 11\sqrt{-1}$$

$$= \sqrt[3]{\left(2+\sqrt{-1}\right)^3} + \sqrt[3]{\left(2-\sqrt{-1}\right)^3}$$

$$= 2+\sqrt{-1} + 2-\sqrt{-1}$$

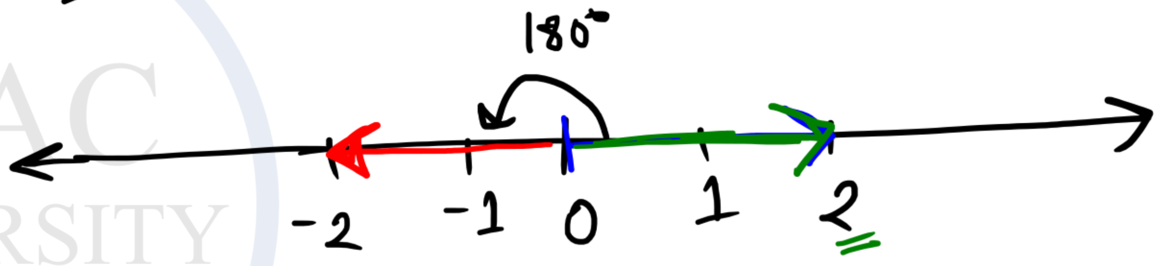
$$= \underline{\underline{4}}$$

What is 'i' (form different perspectives)

def^n

$$i = \sqrt{-1}$$

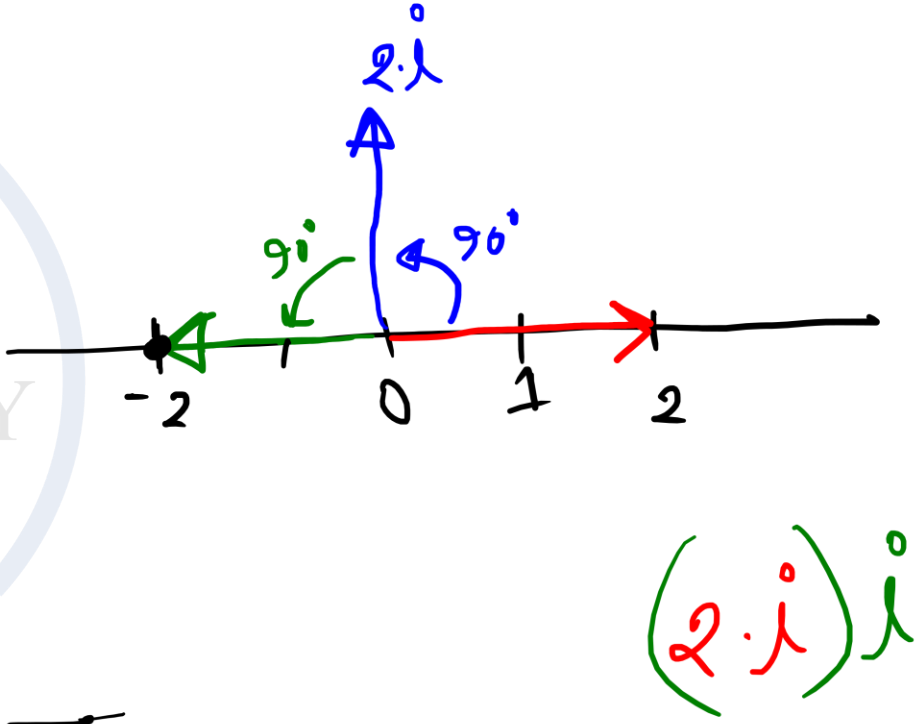
Rotation



$$-(-2) = 2$$

90° Rotation

assumption : i



$\therefore\ 2\cdot i\cdot i = -2$

$\Rightarrow\ i^2 = -1 \Rightarrow i = \sqrt{-1}$

Complex Numbers Notation

$z = a + bi$ wher $a, b \in \mathbb{R}$

$a \longrightarrow$ Real Part denoted by $Re\{z\}$

$b \longrightarrow$ Imaginary Part denoted by $Im\{z\}$

Is 5 a complex number or Real number?

$$5 = 5 + 0 \cdot i$$

Real

Complex

$2+3i$

Real Part $= 2$

Im Part $= 3$ Not $3i$

$2-3i$

Real $= 2$

Im $= -3$

Find real numbers x and y such that $3x + 2iy - ix + 5y = 7 + 5i$.

$$\Rightarrow (3x+5y) + (2y-x)i = 7 + 5i$$

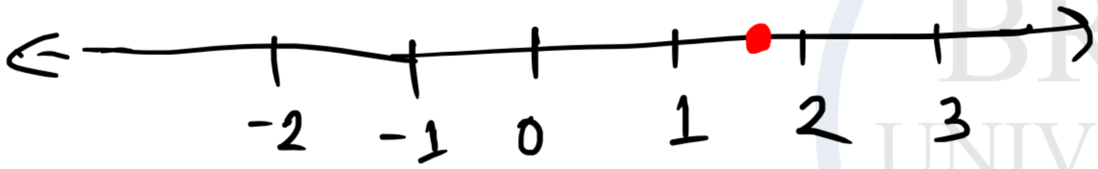
$$\therefore \left.\begin{aligned} 3x+5y &= 7 \\ 2y-x &= 5 \end{aligned}\right\}$$

$(x, y) = ?$

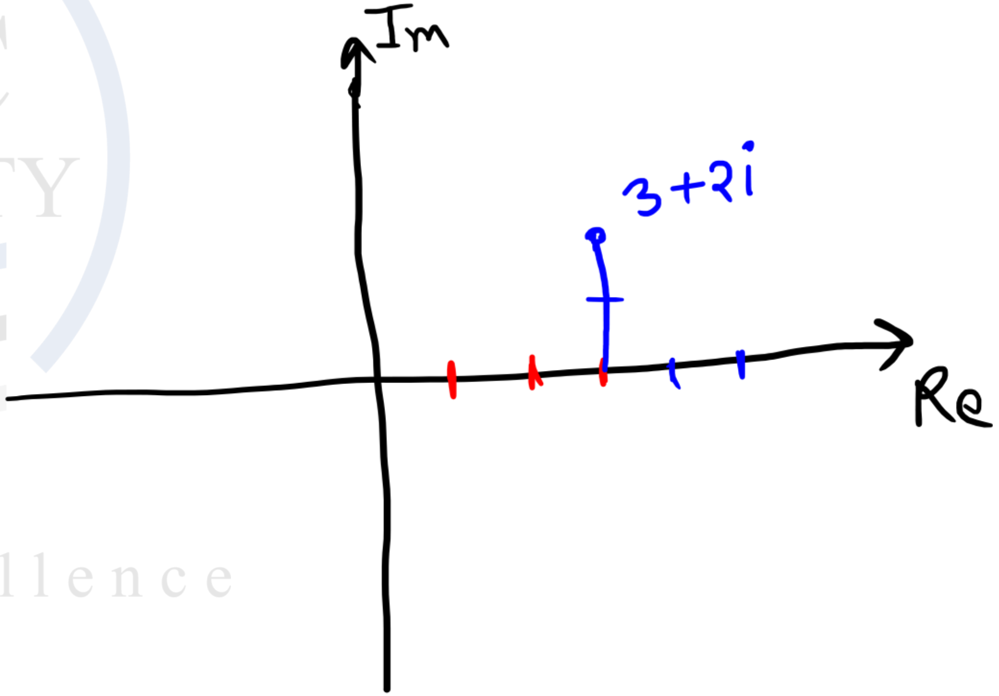
Hw

Graphical Representation of Complex Numbers

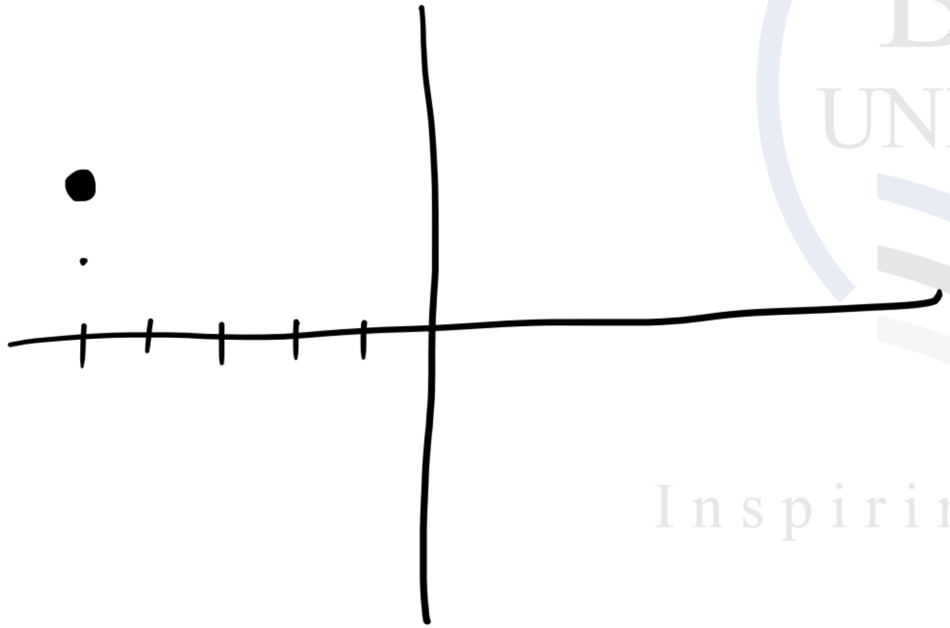
$x = 1.67$

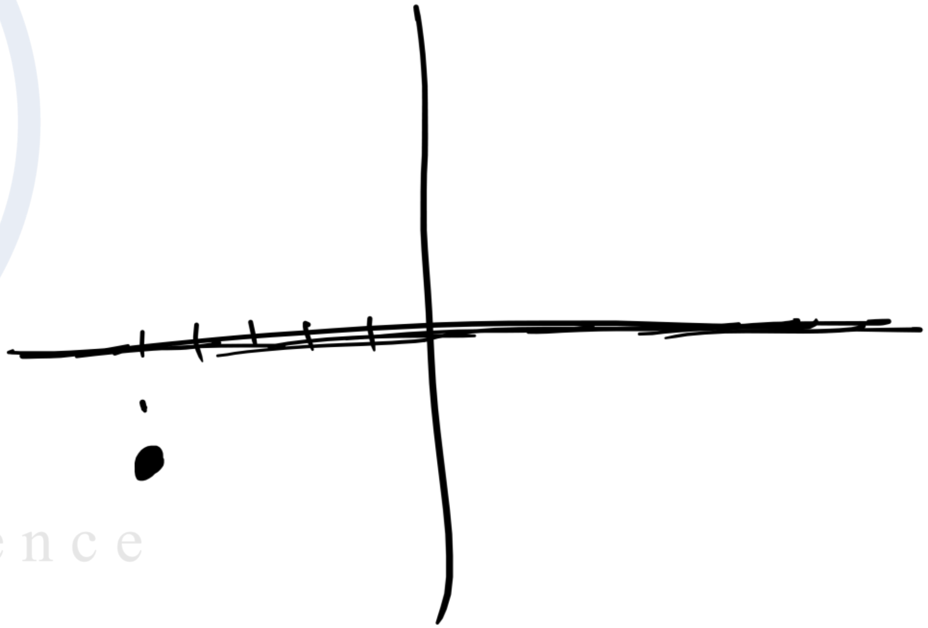


$z = \underline{\underline{3}} + \underline{2}i$



$-5+2i$ $\qquad\qquad\qquad\qquad$ $-5-2i$





Operations between Complex Numbers

$$(2+3i)+(3-5i) = 5-2i$$

$$i^2 = -1$$

$$(2+3i)\cdot(3-5i) = 6-10i+9i-15i^2$$

$$= 21 - i$$

Complex Conjugate

$$z = a + ib$$

$$\bar{z} = a - ib$$

$$z = 2 - 3i$$

$$\bar{z} = 2 + 3i$$

$$z = -2 + 3i$$

$$\bar{z} = -2 - 3i$$

Properties of Conjugate

$\rightarrow \; z + \bar{z} = \text{Real}$

$z \cdot \bar{z} = \text{Real}$

$(2+3i) + (2-3i) = 4 \quad (\text{Real})$

$(2+3i) \cdot (2-3i) = 13 \quad (\text{Real})$

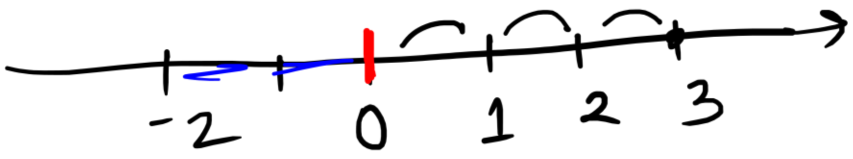
$$\frac{2+3i}{3+4i} = \frac{(2+3i)(3-4i)}{(3+4i)(3-4i)} = \frac{18+i}{25}$$

$$= \frac{18}{25} + \frac{1}{25}i$$

Modulus

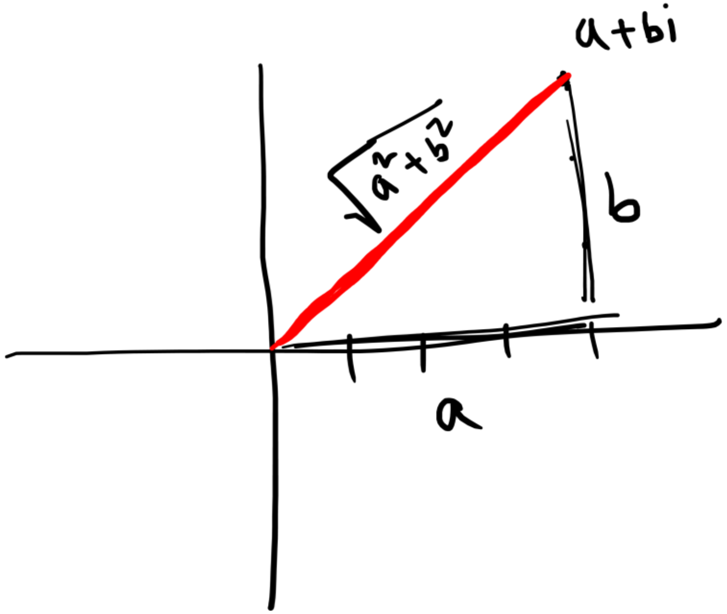
$|3| = 3$

$|-2| = 2$



Modulus: Distance from origin.

$|4+3i| = 5$



$z = a+bi$

$$|z| = \sqrt{a^2+b^2}$$

Properties of Modulus

$$|z_1 \cdot z_2| = |z_1| \cdot |z_2|$$

$$\left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|}$$

$$|z^m| = |z|^m$$

$$z = \frac{(2+i)(3+4i)}{3+2i}$$

$$|z| = \left| \frac{(2+i)\,(3+4i)}{(3+2i)} \right| = \frac{|(2+i)(3+4i)|}{|3+2i|}$$

$$= \frac{|2+i| \cdot |3+4i|}{|3+2i|} = \frac{\sqrt{2^2+1^2} \cdot \sqrt{3^2+4^2}}{\sqrt{3^2+2^2}}$$

$$= \frac{\sqrt{3} \cdot 5}{13}$$

Examples of Modulus

$$z = \frac{3}{2} + \frac{\sqrt{3}}{2} i$$

$$|z| = \sqrt{\left(\frac{3}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2} = \sqrt{\frac{9}{4} + \frac{3}{4}} = \sqrt{\frac{12}{4}}$$

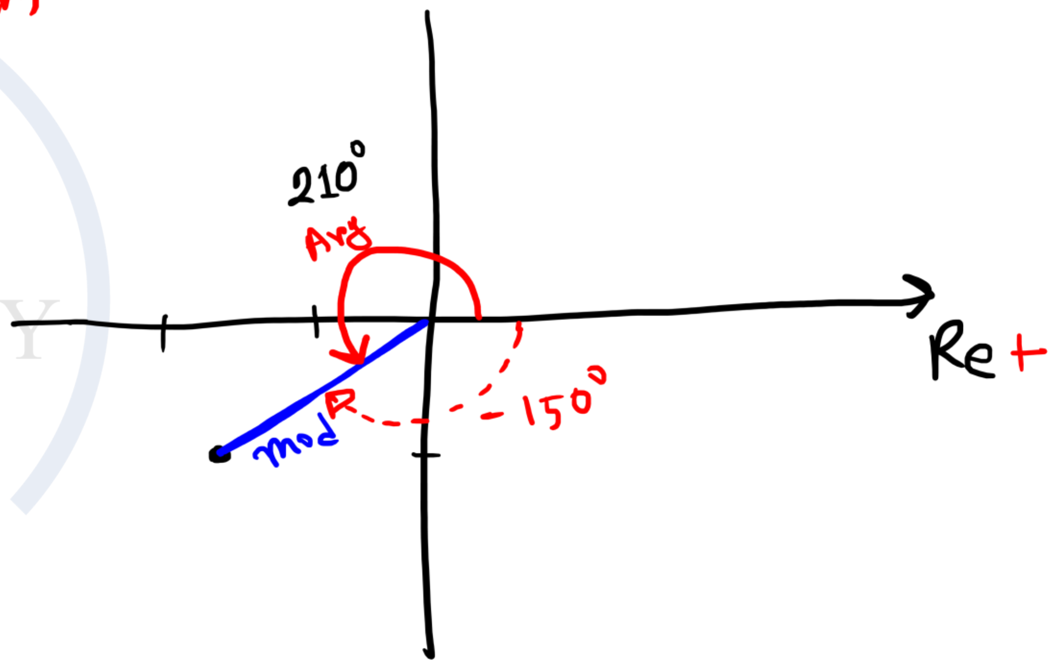
$$= \sqrt{3}.$$

Arguments and Principal Argument

$$z = -\sqrt{3} - i$$

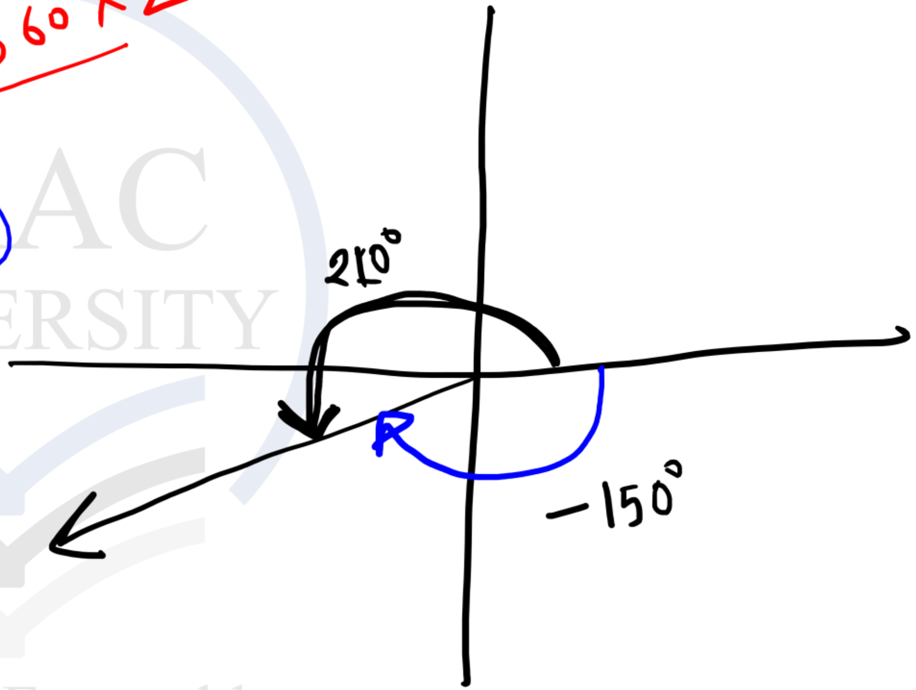
Principal Argument

$$Arg(z) = 210^\circ, -150^\circ, 570^\circ, \ldots$$



$$210^\circ = -150^\circ$$

$$-150^\circ + 360^\circ \times 2 = 570^\circ$$

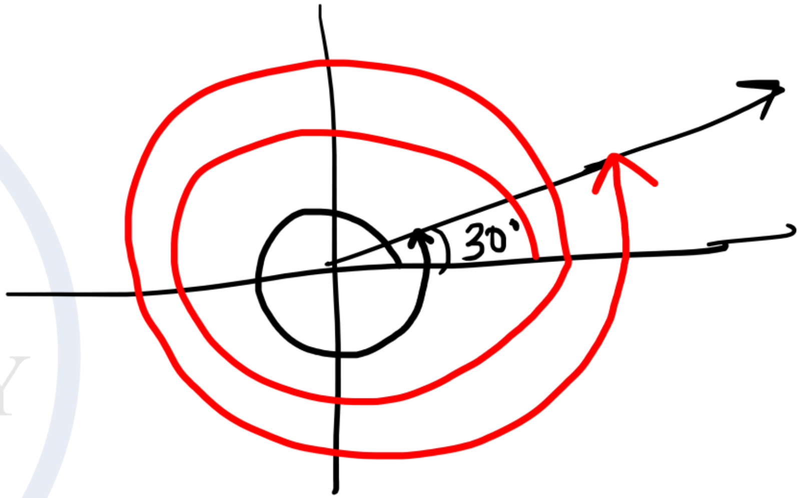


$$-150^\circ + 360^\circ \times 1 = 210^\circ$$

$$30^\circ \equiv 390^\circ \equiv 750^\circ$$

$$30^\circ \pm 360^\circ \cdot n$$

integer



$$2\pi \cdot n$$

Principal Arguments Formulae

$z = a + bi$

$$\theta = \pi - \tan^{-1}\left|\frac{b}{a}\right| \qquad \theta = \tan^{-1}\left|\frac{b}{a}\right|$$

$$\theta = -\pi + \tan^{-1}\left|\frac{b}{a}\right| \qquad \theta = -\tan^{-1}\left|\frac{b}{a}\right|$$

Arguments Formulae

$z = a + bi$, $n \to$ integer

$$\theta = \pi - \tan^{-1}\left|\frac{b}{a}\right| + 2\pi n$$

$$\theta = \tan^{-1}\left|\frac{b}{a}\right| + 2\pi n$$

$$\theta = -\pi + \tan^{-1}\left|\frac{b}{a}\right| + 2\pi n$$

$$\theta = -\tan^{-1}\left|\frac{b}{a}\right| + 2\pi n$$

Examples of finding Arguments

$$z = 1 + \sqrt{3}\, i$$

$$\operatorname{Arg}(z) = \tan^{-1}\left|\frac{\sqrt{3}}{1}\right| + 2n\pi$$

$$= \frac{\pi}{3} + 2n\pi$$

$$z = -\frac{3}{2} + \frac{\sqrt{3}}{2} i$$

$$Arg(z) = \pi - \tan^{-1}\left|\frac{\frac{\sqrt{3}}{\cancel{2}}}{-\frac{3}{\cancel{2}}}\right| + 2n\pi$$

$$= \pi - \tan^{-1}\left|\frac{\sqrt{3}}{3}\right| + 2n\pi$$

$$= \pi - \tan^{-1}\left|\frac{1}{\sqrt{3}}\right| + 2n\pi$$

$$= \pi - \frac{\pi}{6} + 2n\pi$$

$$= \frac{5\pi}{6} + 2n\pi$$

$$\boxed{\theta = \frac{5\pi}{6} + 2n\pi}$$

$$P.A = \frac{5\pi}{6}$$

$$\text{Arg} = \frac{5\pi}{6}, \quad \frac{5\pi}{6} + 2\pi, \quad \frac{5\pi}{6} + 4\pi, \quad \frac{5\pi}{6} + 6\pi, \; - - -$$

$$\frac{5\pi}{6} - 2\pi, \quad \frac{5\pi}{6} - 4\pi, \; - - -$$

$-2\sqrt{3}-2i$

$$\text{Arg} = -\pi + \tan^{-1}\left|\frac{-2}{-2\sqrt{3}}\right| + 2n\pi$$

$$= -\pi + \tan^{-1}\left|\frac{1}{\sqrt{3}}\right| + 2n\pi$$

$$= -\pi + \frac{\pi}{6} + 2n\pi \quad = \frac{-5\pi}{6} + 2n\pi$$

Properties of Arguments

$$\text{Arg}(z_1 \cdot z_2) = \text{Arg}(z_1) + \text{Arg}(z_2)$$

$$\text{Arg}\left(\frac{z_1}{z_2}\right) = \text{Arg}(z_1) - \text{Arg}(z_2)$$

$$\text{Arg}(z^m) = m \cdot \text{Arg}(z)$$

$$\operatorname{Arg}\left((1+i)\left(\frac{3}{2}+\frac{\sqrt{3}}{2}i\right)\right)$$

$$= \operatorname{Arg}\left(\frac{3}{2}+\frac{\sqrt{3}}{2}i+\frac{3}{2}i-\frac{\sqrt{3}}{2}\right)$$

$$= \operatorname{Arg}\left(\left(\frac{3}{2}-\frac{\sqrt{3}}{2}\right)+i\left(\frac{\sqrt{3}}{2}+\frac{3}{2}\right)\right)$$

$$= \tan^{-1}\left(\frac{\frac{\sqrt{3}}{2}+\frac{3}{2}}{\frac{3}{2}-\frac{\sqrt{3}}{2}}\right)+2n\pi$$

$$= \frac{5\pi}{12} + 2n\pi$$

$$\operatorname{Arg}\left((1+i)\left(\frac{3}{2}+\frac{\sqrt{3}}{2}i\right)\right)$$

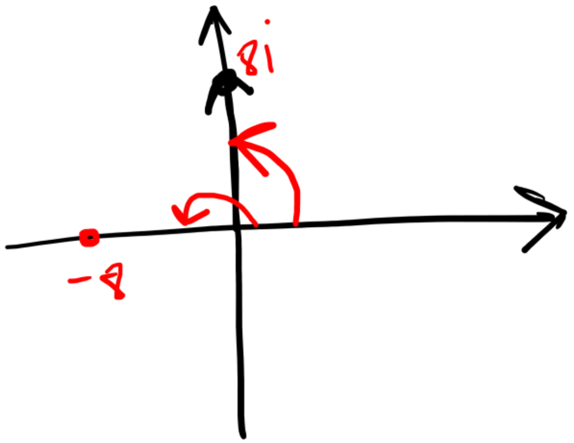
$$= \operatorname{Arg}(1+i) + \operatorname{Arg}\left(\frac{3}{2}+\frac{\sqrt{3}}{2}i\right)$$

$$= \frac{\pi}{4} + \frac{\pi}{6} + 2n\pi = \frac{5\pi}{12} + 2n\pi$$

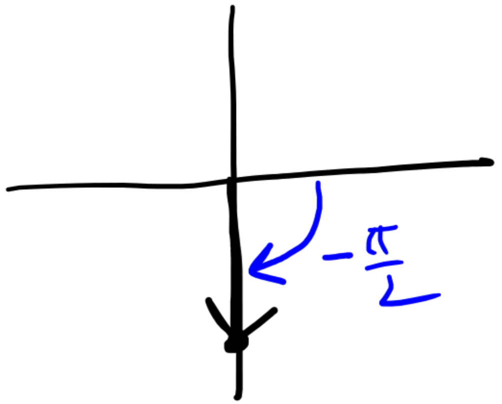
Arguments when on axis

~~x+iy~~

$z_1 = 8i \qquad \operatorname{Arg}(z_1) = \frac{\pi}{2} + 2n\pi$



$z_2 = -8i \qquad \operatorname{Arg}(z_2) = -\frac{\pi}{2} + 2n\pi$



$z_3 = 8 \qquad \operatorname{Arg}(z_3) = 0 + 2n\pi$

$z_4 = -8 \qquad \operatorname{Arg}(z_4) = \pi + 2n\pi$

Finding Square Roots

Find the square root of $-15 - 8i$

$$\sqrt{-15-8i} = x + iy$$

$$\Rightarrow -15 - 8i = (x+iy)^2 = x^2 + 2xiy + i^2y^2$$

$$= x^2 - y^2 + (2xy)i$$

$$x^2 - y^2 = -15 \quad \text{---} (1)$$

$$2xy = -8 \quad \text{---} (2)$$

$$\Rightarrow y = \frac{-4}{x}$$

$$(x^2+y^2)^2 = (x^2-y^2)^2 + 4\cdot x^2\cdot y^2$$

$$= (-15)^2 + (2xy)^2$$

$$= 225 + (-8)^2$$

$$= 225 + 64 = 289$$

$$\therefore \quad x^2+y^2 = 17 \quad \text{———} \quad (3)$$

$$x^2+y^2=17$$
$$x^2-y^2=-15$$
$$2x^2=2$$
$$\Rightarrow x^2=1$$
$$\Rightarrow x=\pm 1$$

$x=1$

$$y=\frac{-4}{1}=-4$$

Sol^n-1

$$x+iy=1-4i$$

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$$x = -1$$

$$y = \frac{-4}{-1} = 4$$

Sol^n-2 $x + iy = -1 + 4i$

$$\sqrt{-15-8i} = 1 - 4i, \; -1 + 4i$$

Solve the equation $z^2 + (2i - 3)z + 5 - i = 0$.

$$z = \frac{-(2i-3) \pm \sqrt{(2i-3)^2 - 4 \cdot 1 \cdot (5-i)}}{2(1)}$$

$$= \frac{-2i+3 \pm \sqrt{-4-12i+9-20+4i}}{2}$$

$$= \frac{-2i+3 \pm \sqrt{-15-8i}}{2}$$

$$= \frac{-2i+3 \pm (1-4i)}{2}$$

$$z_1 = \frac{-2i+3+1-4i}{2}$$

$$z_2 = \frac{-2i+3-1+4i}{2}$$

Euler's Formula

$$\boxed{e^{i\theta} = \cos\theta + i\sin\theta}$$

Proof by calculus

Taylor series.

The Most Beautiful Equation of Mathematics

$$e^{i\pi} = \cos\pi + i\sin\pi$$

$$\Rightarrow e^{i\pi} = -1$$

$$\Rightarrow \boxed{e^{i\pi} + 1 = 0}$$

$$\boxed{\begin{array}{c} e \\ \pi \\ i \\ 1 \\ 0 \end{array}}$$

Show that (a) $\cos\theta = \dfrac{e^{i\theta}+e^{-i\theta}}{2}$, (b) $\sin\theta = \dfrac{e^{i\theta}-e^{-i\theta}}{2i}$.

$$e^{i\theta} = \cos\theta + i\sin\theta$$

$$e^{-i\theta} = \cos\theta - i\sin\theta$$

$$e^{i\theta} - e^{-i\theta} = 2i\sin\theta$$

$$\sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i}$$

Polar Form of Complex Numbers (Euler & CiS form)

$$\boxed{z = \underbrace{a+bi}_{\text{Cartesian}} = \underbrace{re^{i\theta}}_{\text{Polar}} = \underbrace{r(\cos\theta + i\sin\theta)}_{\text{Polar}}} = \underbrace{r.\,\text{cis}(\theta)}_{\text{Polar}}$$

Express each of the following complex numbers in polar form.

(a) $2+2\sqrt{3}i$, (b) $-5+5i$, (c) $-\sqrt{6}-\sqrt{2}i$, (d) $-3i$

(a) $z = 2+2\sqrt{3}i$

modulus $r = 4$

Argument $\theta = \frac{\pi}{3} + 2n\pi$

$2+2\sqrt{3}i = 4 \cdot e^{i\left(\frac{\pi}{3}+2n\pi\right)} = 4\left(\cos\left(\frac{\pi}{3}+2n\pi\right) + i\sin\left(\frac{\pi}{3}+2n\pi\right)\right)$

Solve the equation $e^{4z} = i$

$$e^{4z} = 0 + 1i$$

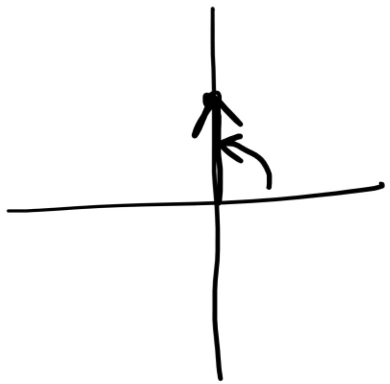
$$\Rightarrow e^{4z} = 1 \cdot e^{i\left(\frac{\pi}{2} + 2n\pi\right)}$$

$$\Rightarrow 4z = i\left(\frac{\pi}{2} + 2n\pi\right)$$

$$\Rightarrow z = i\left(\frac{\pi}{8} + \frac{n\pi}{2}\right)$$

$\text{mod} = 1$

$\text{Arg} = \frac{\pi}{2} + 2n\pi$



Solve for x and y

$$\left(\frac{3}{2}+\frac{\sqrt{3}}{2}i\right)^{2024} = 3^{1012}(x+iy)$$

$$\Rightarrow \left[\sqrt{3}\cdot e^{i\left(\frac{\pi}{6}+2n\pi\right)}\right]^{2024} = 3^{1012}\cdot(x+iy)$$

$$\text{mod} = \sqrt{\left(\frac{3}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2} = \sqrt{\frac{9}{4}+\frac{3}{4}} = \sqrt{3}$$

$$\Rightarrow \cancel{3^{\frac{1}{2}\times 2024}}\cdot e^{i\left(\frac{\pi}{6}+2n\pi\right)\times 2024} = \cancel{3^{1012}}\cdot(x+iy)$$

$$\text{Arg} = \frac{\pi}{6}+2n\pi$$

$$x + iy = e^{i\left(\frac{1012\pi}{3} + 2048n\pi\right)}$$

$$= \cos\left(\frac{1012\pi}{3} + 2048n\pi\right) + i\sin\left(\frac{1012\pi}{3} + 2048n\pi\right)$$

$$= -\frac{1}{2} - \frac{\sqrt{3}}{2}i$$

$$\therefore x = -\frac{1}{2}$$

$$\therefore y = -\frac{\sqrt{3}}{2}$$


Thank you!