Lecture 4

Integrating Powers of Sine and Cosine

1. $$\int \sin^n x\,dx = -\frac{1}{n}\sin^{n-1}x\cos x + \frac{n-1}{n}\int \sin^{n-2}x\,dx$$

2. $$\int \cos^n x\,dx = \frac{1}{n}\cos^{n-1}x\sin x + \frac{n-1}{n}\int \cos^{n-2}x\,dx$$

We can prove the above integration by reduction formulas.

$\boxed{\text{For } 2}$

$$\int \cos^n x\,dx = \int \cos^{n-1}x\cos x\,dx$$

Say, $u = \cos^{n-1}x$ $\qquad dv = \cos x\,dx$

$du = (n-1)\cos^{n-2}x\,(-\sin x)\,dx \qquad v = \sin x$

So that

$$\int \cos^n x\,dx = \cos^{n-1}x\sin x + (n-1)\int \sin^2 x\cos^{n-2}x\,dx$$

$$= \cos^{n-1}x\sin x + (n-1)\int (1-\cos^2 x)\cos^{n-2}x\,dx$$

$$= \cos^{n-1}x\sin x + (n-1)\int \cos^{n-2}x\,dx - (n-1)\int \cos^n x\,dx$$

$$(n\cancel{-1}\cancel{+1})\int \cos^n x\,dx = \cos^{n-1}x\sin x + (n-1)\int \cos^{n-2}x\,dx$$

$$\int \cos^n x\,dx = \frac{1}{n}\cos^{n-1}x\sin x + \frac{n-1}{n}\int \cos^{n-2}x\,dx$$

Example: $\int \sin^3 x\,dx \qquad , n = 3$

$$= -\frac{1}{3}\sin^2 x\cos x + \frac{3-1}{3}\int \sin x\,dx$$

$$= -\frac{1}{3}(1-\cos^2 x)\cos x + \frac{2}{3}(-\cos x) + C$$

$$= -\frac{1}{3}\cos x + \frac{1}{3}\cos^3 x - \frac{2}{3}\cos x + C$$

$$= \frac{1}{3}\cos^3 x - \cos x + C$$

Integrating Products of Sines and Cosines

$$\boxed{\int \sin^m x \cos^n x\,dx}$$

If n odd then $\cos^2 x = 1 - \sin^2 x$; say, $u = \sin x$

If m odd then $\sin^2 x = 1 - \cos^2 x$; say, $u = \cos x$

If m even, n even then $\sin^2 x = \frac{1}{2}(1-\cos 2x)$, $\cos^2 x = \frac{1}{2}(1+\cos 2x)$

Example : Evaluate $\int \sin^4 x \cos^5 x \, dx$

Solⁿ:

$$\int \sin^4 x \cos^5 x \, dx = \int \sin^4 x (1-\sin^2 x)^2 \underline{\cos x \, dx}$$

let $\cancel{u = \cos x}$, $\cancel{du = \sin}$

let $u = \sin x$, $du = \cos x \, dx$

$$= \int u^4 (1-u^2)^2 \, du$$

$$= \int u^4 (1 - 2u^2 + u^8) \, du$$

$$= \int (u^4 - 2u^6 + u^8) \, du$$

$$= \frac{u^5}{5} - 2\frac{u^7}{7} + \frac{u^9}{9} + C$$

$$= \frac{1}{5}\sin^5 x - \frac{2}{7}\sin^7 x + \frac{1}{9}\sin^9 x + C \qquad \square$$

Extra Problem:

1. $\int \sin^2 t \cos^3 t \, dt$

2. $\int \sin^3 x \cos^3 x \, dx$

3. $\int \sin^4 3x \cos^3 3x \, dx$

4.

Trigonometric Identities

$$2 \sin\alpha \cos\beta = \sin(\alpha-\beta) + \sin(\alpha+\beta)$$

$$2 \sin\alpha \sin\beta = \cos(\alpha-\beta) - \cos(\alpha+\beta)$$

$$2 \cos\alpha \cos\beta = \cos(\alpha-\beta) + \cos(\alpha+\beta)$$

Example: Evaluate $\int \sin 7x \cos 3x \, dx$

$$= \frac{1}{2} \int \left[\sin(7x-3x) + \sin(7x+3x)\right] dx$$

$$= \frac{1}{2} \int \left[\sin 4x + \sin 10x\right] dx$$

$$= \cancel{\frac{1}{2}\Big[-\cos 4}$$

$$= \frac{1}{2}\left[-\frac{\cos 4x}{4} - \frac{\cos 10x}{10}\right] + C$$

$$= -\frac{1}{8}\cos 4x - \frac{1}{20}\cos 10x + C$$

Trigonometric Substitution

		Restriction
if $\sqrt{a^2-x^2}$	say, $x = a\sin\theta$	$-\pi/2 \le \theta \le \pi/2$
if $\sqrt{a^2+x^2}$	say, $x = a\tan\theta$	$-\pi/2 < \theta < \pi/2$
if $\sqrt{x^2-a^2}$	say, $x = a\sec\theta$	$\begin{cases} \boxed{0 \le \theta < \pi/2 \text{ if } x \ge a} \\ \boxed{\pi/2 < \theta \le \pi \text{ if } x \le -a} \end{cases}$

Example: Evaluate $\int \frac{dx}{x^2\sqrt{4-x^2}}$

say, $x = 2\sin\theta$

$dx = 2\cos\theta\, d\theta$

$4-4\sin^2\theta$

$4(1-\sin^2\theta)$

$4\cos^2\theta$

Thus, $\int \frac{dx}{x^2\sqrt{4-x^2}} = \int \frac{2\cos\theta\, d\theta}{4\sin^2\theta\sqrt{4-4\sin^2\theta}}$

$= \int \frac{2\cos\theta\, d\theta}{4\sin^2\theta\sqrt{4\cos^2\theta}}$

$= \int \frac{2\cos\theta\, d\theta}{4\sin^2\theta\; 2\cos\theta}$

$= \frac{1}{4}\int \csc^2\theta\, d\theta = -\frac{1}{4}\cot\theta + C$