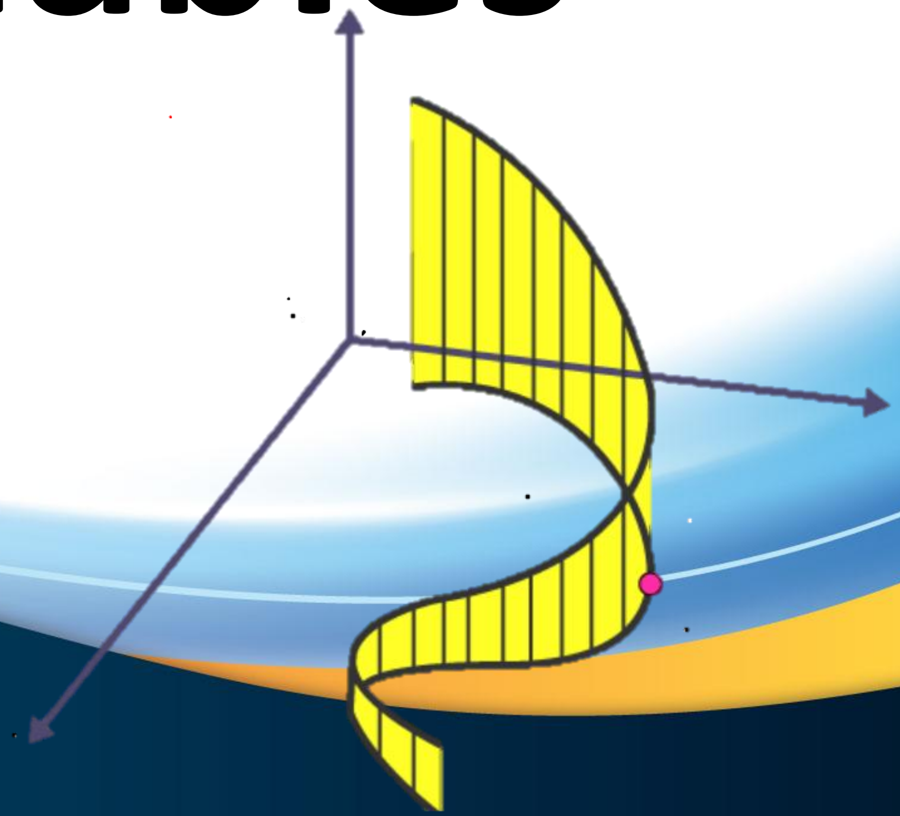
Undergraduate Course in Mathematics

Complex Variables

Topic: Complex Integration



Conducted By

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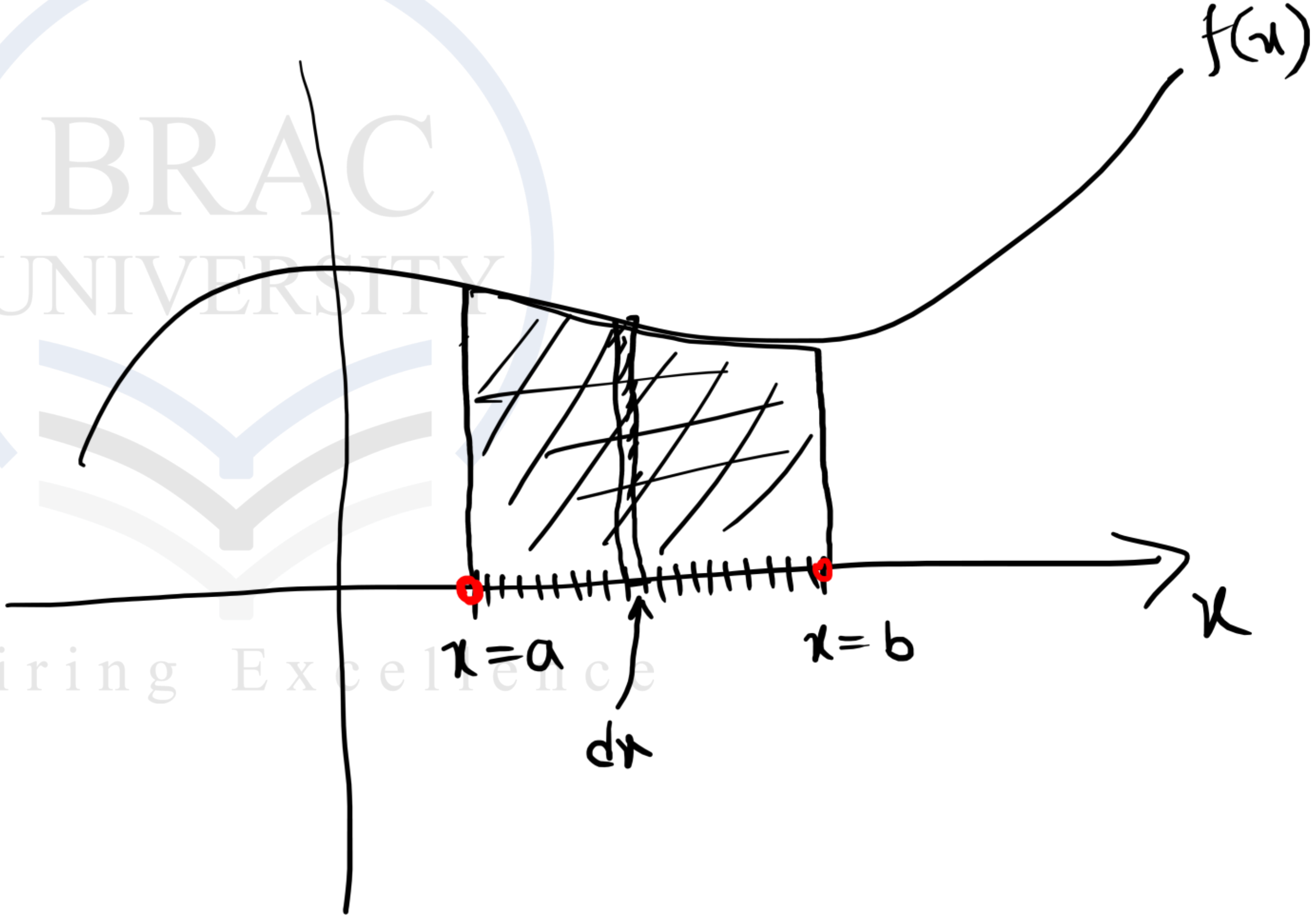
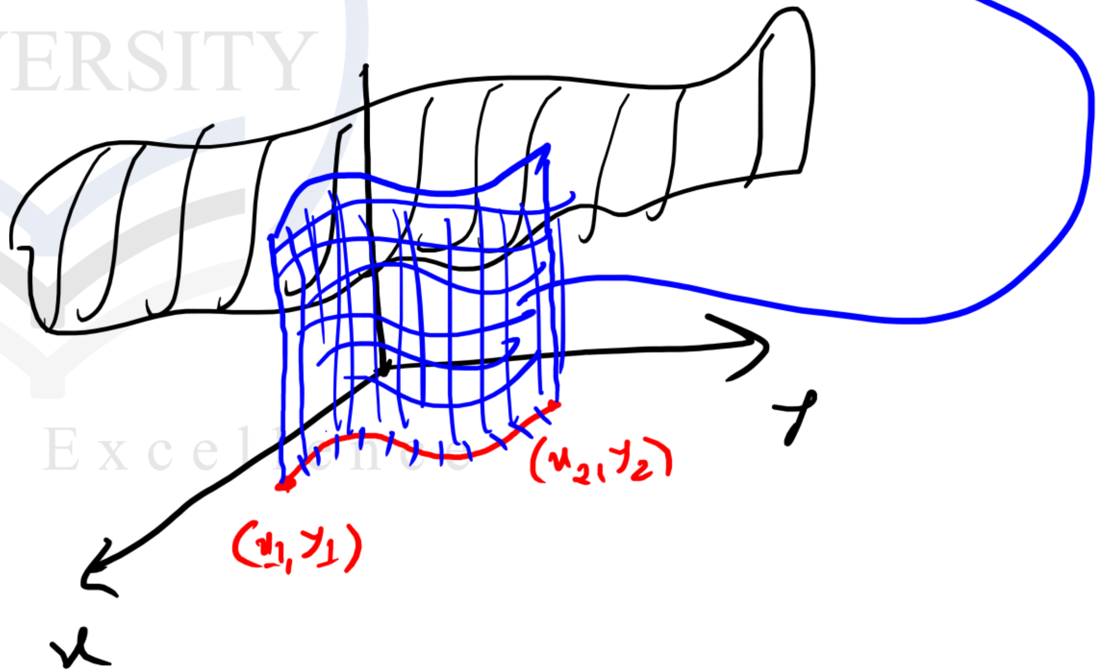
Single Integration on Real Line

$$\int_a^b f(x)\,dx = \text{Area}$$

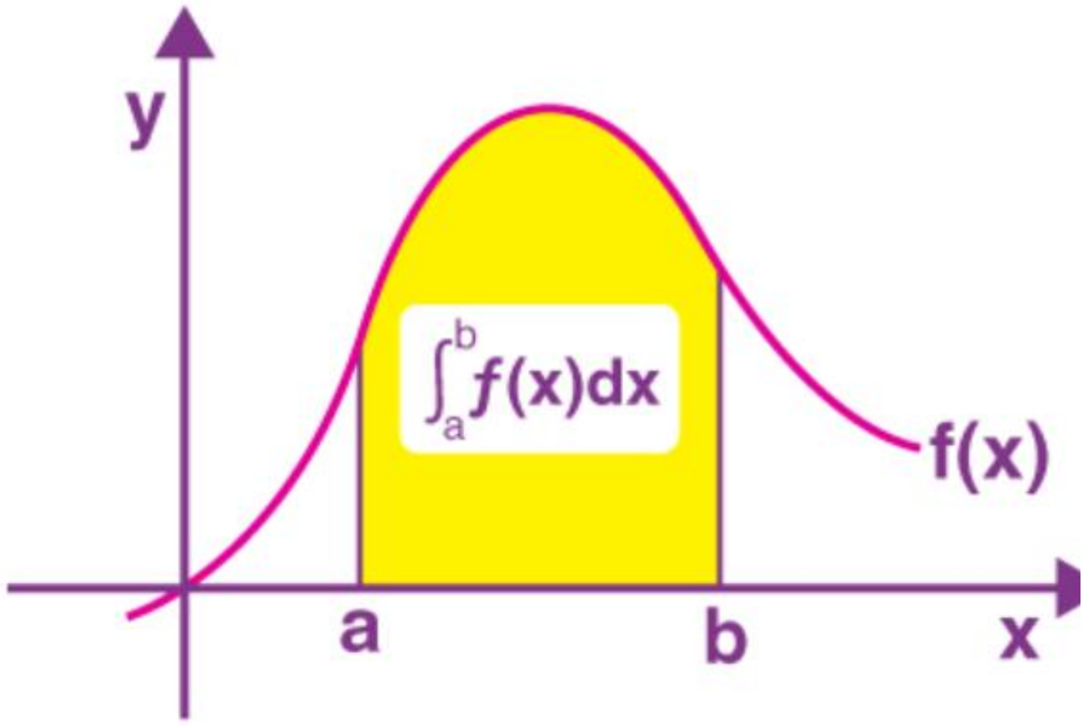


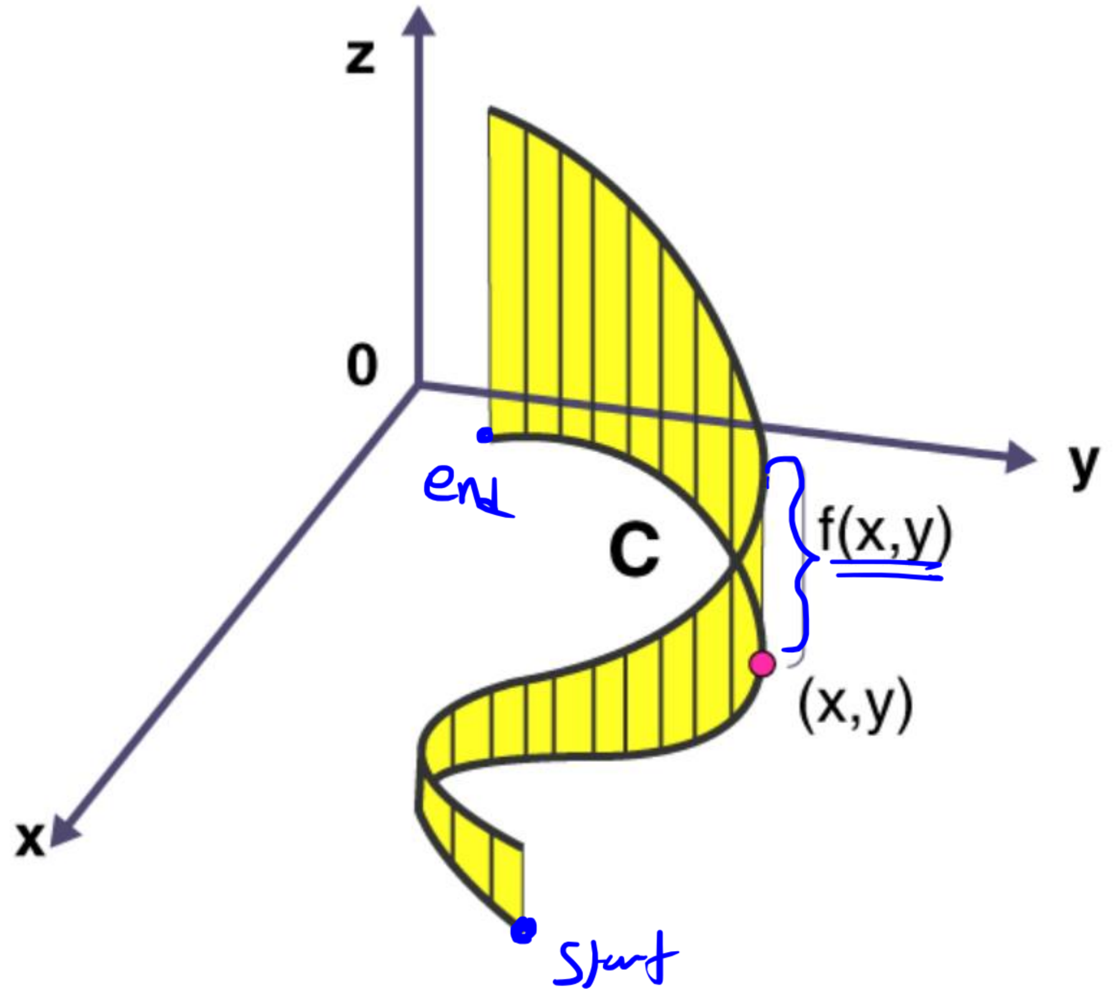
Single / Line / Path Integration on $\mathbb{R}^2$

$$\int f(x,y)\,dx \qquad \int f(x,y)\,dy \qquad \int f(x,y)\,ds$$



Geometric Meaning





Line Integal by Partho Sutra Dhor

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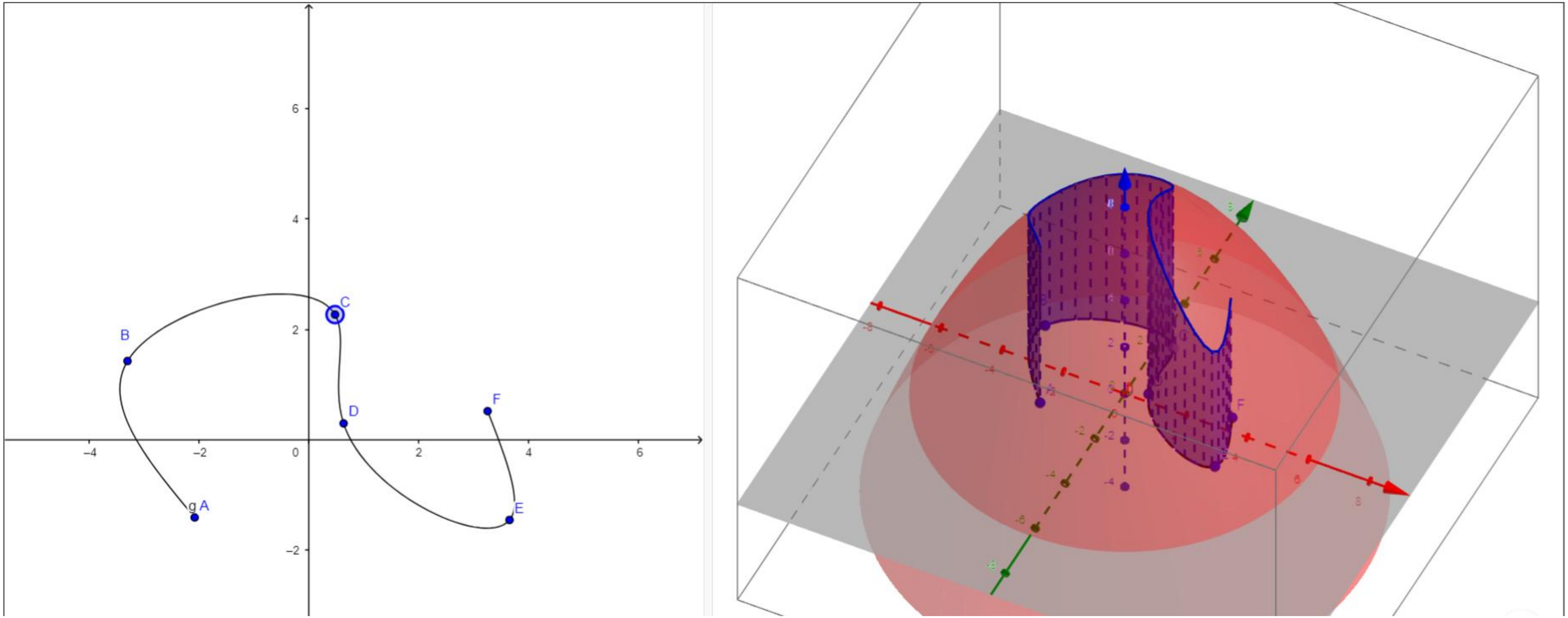
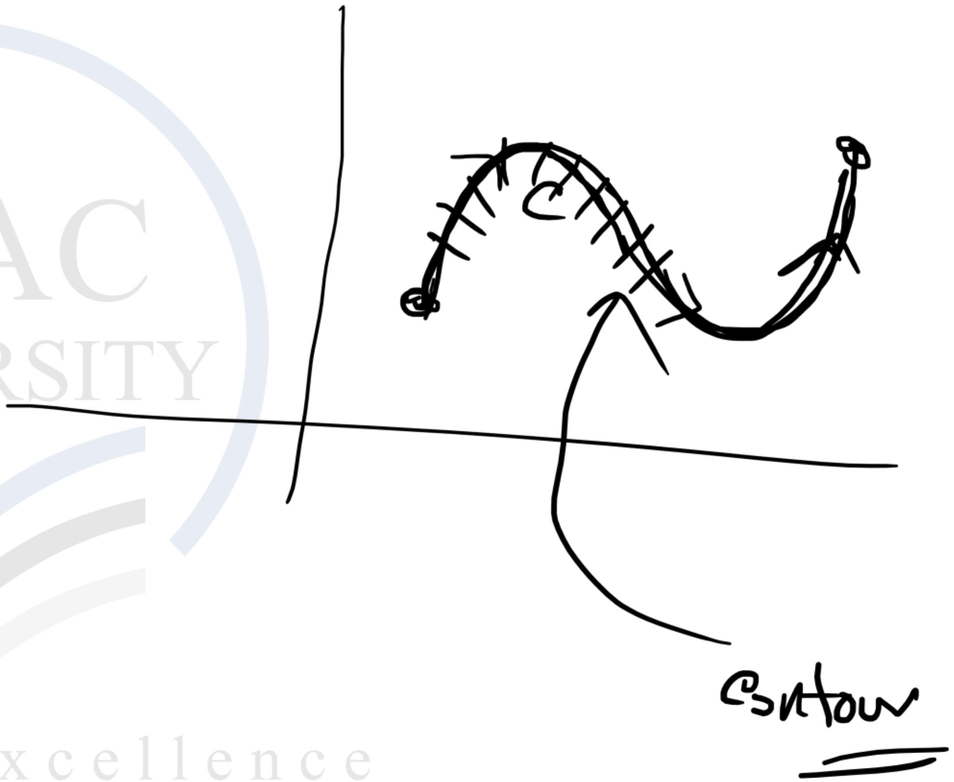


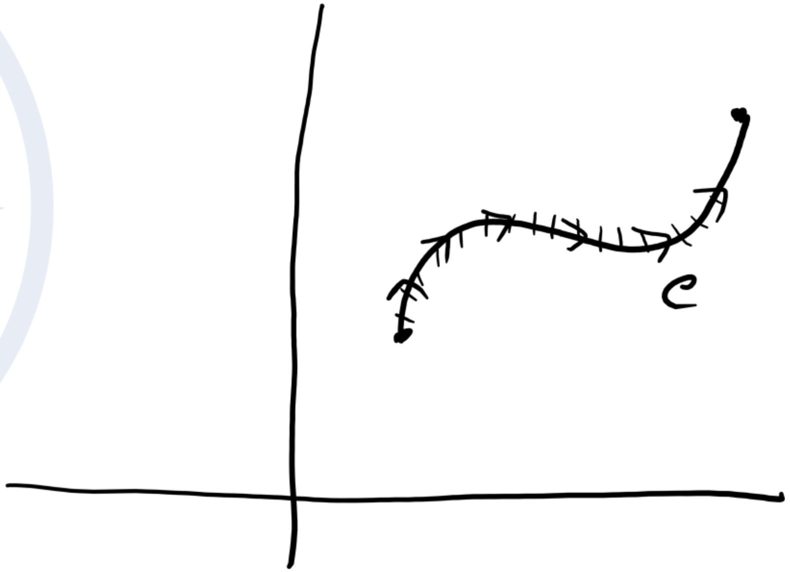
Contour

$$\int_C f(z)\,dz$$

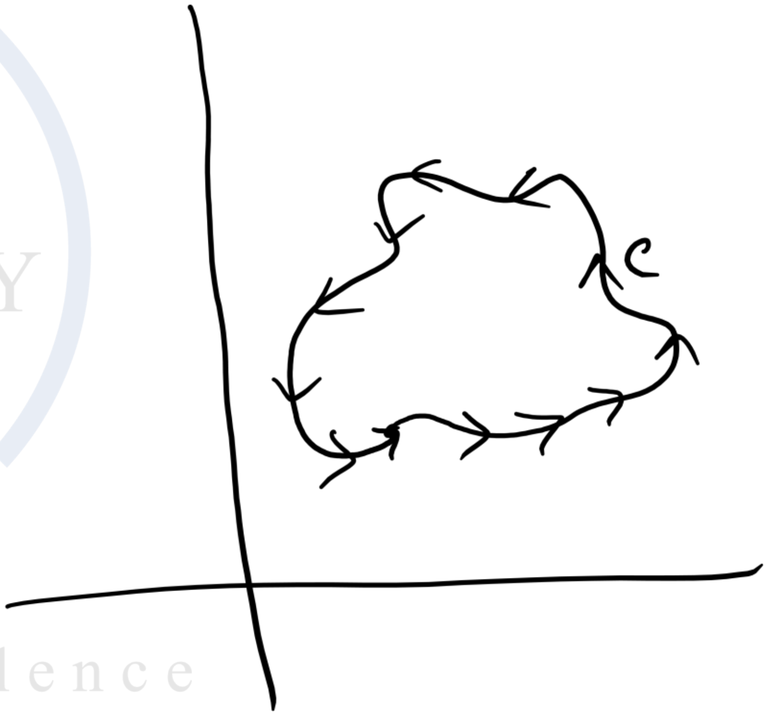


Complex Line/Contour Integral

$$\int_C f(z)\,dz$$



$$\oint_C f(z)\, dz$$



Parametrization of Strait Line

find the Parametric equation of St. line Joining

from $(1,2)$ to $(2,8)$

Slope $m = \frac{8-2}{2-1} = 3$

$$y - y_1 = m(x - x_1)$$

$$\Rightarrow y - 2 = 3(x-1)$$

$$\Rightarrow y = 3x - 3 + 2$$

$$\Rightarrow \boxed{y = 3x - 1}$$

$$\begin{cases} x = t \\ y = 3t - 1 \end{cases}$$

2nd method

from $(1,2)$ to $(2,8)$

$$1+2i \quad z_1 \qquad 2+8i \quad z_2$$

$$\boxed{z = z_1 + (z_2 - z_1)t}$$

$$= 1+2i + \left((2+8i) - (1+2i)\right)t$$

$$= 1+2i + (1+6i)t$$

$$= 1+2i+t+6ti$$

$$z = (1+t) + i(2+6t)$$

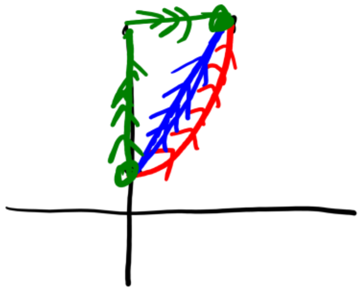
$$\begin{cases} x = 1+t \\ y = 2+6t \end{cases}$$

Evaluate $\int_{(0,1)}^{(2,5)} (3x+y)dx + (2y-x)dy$

a) along the parabola $y = x^2 + 1$

b) along the straight line from (0,1) to (2,5)

c) along the straight lines from (0,1) to (0,5) and then from (0,5) to (2,5)



$x = t \rightarrow dx = dt$

$y = t^2 + 1 \rightarrow dy = 2t\,dt$

limit

$x=t$	$y=t^2+1$
$2=t$	$5=t^2+1$
$\Rightarrow t=2$	$\Rightarrow t=\pm 2$

(x,y)	t
$(2,5)$	$t=2$
$(0,1)$	$t=0$

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$$\int_{(0,1)}^{(2,5)} (3x+y)\,dx + (2y-x)\,dy$$

$$= \int_0^2 (3t+t^2+1)\,dt + (2t^2+2-t)\,2t\,dt$$

$$= \int_0^2 (3t+t^2+1+4t^3+4t-2t^2)\,dt$$

$$= \int_0^2 (4t^3-t^2+7t+1)\,dt$$

$$= \left[\frac{4t^4}{4} - \frac{t^3}{3} + 7\frac{t^2}{2} + t\right]_0^2$$

$$= \left(16 - \frac{8}{3} + 14 + 2\right) - (0)$$

$$= \frac{88}{3}.$$

Evaluate $\int_{(0,1)}^{(2,5)} (3x+y)dx + (2y-x)dy$

a) along the parabola $y = x^2 + 1$

b) along the straight line from $(0,1)$ to $(2,5)$

$0 + i = i$

$2 + 5i$

c) along the straight lines from $(0,1)$ to $(0,5)$ and then from $(0,5)$ to $(2,5)$

$$z = z_1 + (z_2 - z_1)t$$
$$= i + (2+5i - i)t$$
$$= i + 2t + 4it$$
$$= (2t) + (1+4t)i$$

$$\boxed{\begin{aligned} x &= 2t \\ y &= 4t+1 \end{aligned}}$$

$dx = 2\,dt$

$dy = 4\,dt$

(x,y)	t
$(2,5)$	$t=1$
$(0,1)$	$t=0$

$$\int_{(0,1)}^{(2,5)} (3x+y)\,dx + (2y-x)\,dy$$

$$= \int_0^1 (6t+4t+1)\cdot 2\,dt + (8t+2-2t)\cdot 4\,dt$$

$$= \int_0^1 (12t + \cancel{8t} + 2 + 32t + 8 - \cancel{8t})\,dt$$

$$= \int_0^1 (44t+10)\,dt$$

$$= \left[44\frac{t^2}{2} + 10t\right]_0^1$$

$$= \left(\frac{44}{2}+10\right) - (0)$$

$$= 32.$$

Ans

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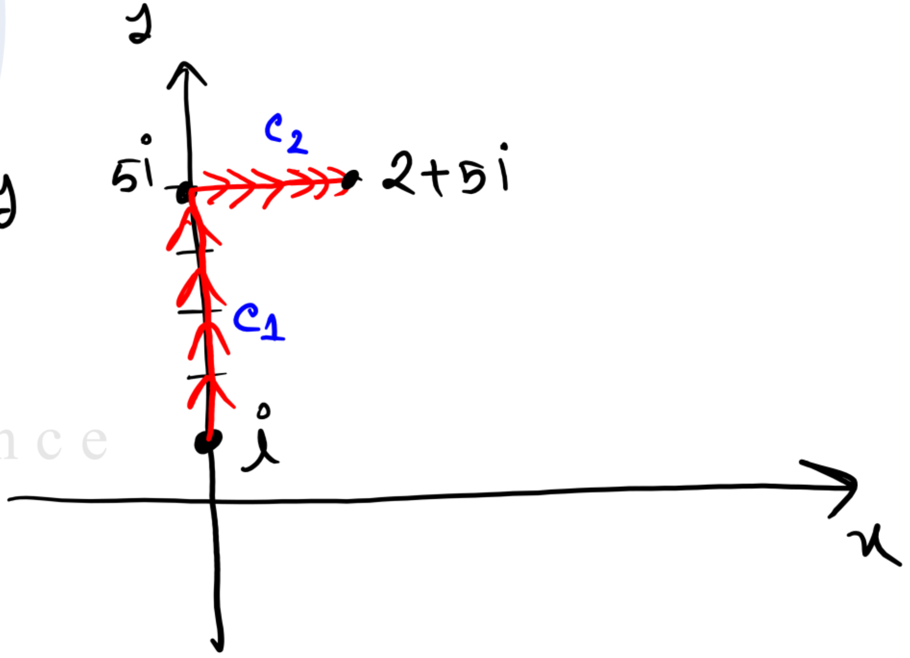
Evaluate $\int_{(0,1)}^{(2,5)} (3x + y)dx + (2y - x)dy$

a) along the parabola $y = x^2 + 1$

b) along the straight line from $(0,1)$ to $(2,5)$

c) along the straight lines from $(0,1)$ to $(0,5)$ and then from $(0,5)$ to $(2,5)$

$$\text{Integral} = \int_{(0,1)}^{(0,5)} (3x+y)dx + (2y-x)dy + \int_{(0,5)}^{(2,5)} (3x+y)dx + (2y-x)dy$$



Now,

C_1: straight line from i to $5i$

$$z = z_1 + (z_2 - z_1)t$$

$$= i + (5i - i)t$$

$$= i + 4it$$

$$= (0) + (1+4t)i$$

$x = 0$ X

$y = 1+4t$

$dx = 0$

$dy = 4dt$

limit

(x,y)	t
$(0,5)$	$t=1$
$(0,1)$	$t=0$

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$$\int_{(0,1)}^{(0,5)} (3x+y)\,dx + (2y-x)\,dy$$

$$= \int_0^1 (3\cdot 0 + 4t+1)\cdot 0 + (2+8t-0)\cdot 4\,dt$$

$$= \int_0^1 (2+8t)\,dt$$

$$= \left[2t + 8\frac{t^2}{2}\right]_0^1$$

$$= (2+4) - (0)$$

$$= 6.$$

Now,

C_2: $5i$ to $2+5i$

$$z = z_1 + (z_2 - z_1)t$$

$$= 5i + (2+5i - 5i)t$$

$$= 5i + 2t$$

$$= (2t) + (5)i$$

$x = 2t$

$y = 5$ ✗

$dx = 2\,dt$

$dy = 0$

limit

(x,y)	t
$(2,5)$	$t=1$
$(0,5)$	$t=0$

$z = 2t$

$$\int_{(0,5)}^{(2,5)} (3x+y)\,dx + (2y-x)\,dy$$

$$= \int_0^1 (6t+5)\cdot 2\,dt + (10-2t)\cdot 0$$

$$= \int_0^1 (12t+10)\,dt$$

$$= \left[6t^2 + 10t\right]_0^1$$

$$= (6+10) - (0)$$

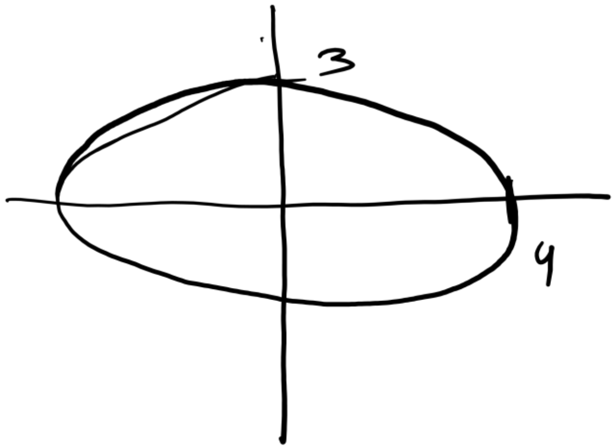
$$= 16.$$

$$\int_{(0,1)}^{(2,5)} - - - -$$

$$= \int_{(0,1)}^{(0,5)} - - - + \int_{(0,5)}^{(2,5)}$$

$$= 6 + 16 = 22$$

Evaluate $\oint_C (x+2y)\,dx + (y-2x)dy$ around the ellipse C defined by $x = 4\cos\theta$, $y = 3\sin\theta$, $0 \le \theta \le 2\pi$ if C is described in a counterclockwise direction.



$dx = -4\sin\theta$

$dy = 3\cos\theta\, d\theta$

limit = from 0 to 2π.

$$\oint_C (x+2y)\,dx + (y-2x)\,dy$$

$$= \int_0^{2\pi} \left(4\cos\theta + 6\sin\theta\right)(-4\sin\theta)\,d\theta + \left(3\sin\theta - 8\cos\theta\right)\cdot 3\cos\theta\,d\theta$$

$$= \int_0^{2\pi} \left(-16\sin\theta\cos\theta - 24\sin^2\theta + 9\sin\theta\cos\theta - 24\cos^2\theta\right)d\theta$$

$$= \int_0^{2\pi} \left(-7\sin\theta\cos\theta - 24\right)d\theta$$

$$= \int_0^{2\pi} \left(-\frac{7}{2} \cdot 2\sin\theta\cos\theta - 24 \right) d\theta$$

$$= \int_0^{2\pi} \left(-\frac{7}{2} \sin 2\theta - 24 \right) d\theta$$

$$= \left[+\frac{7}{2} \cdot \frac{+\cos 2\theta}{2} - 24\theta \right]_0^{2\pi}$$

$$= \left(\frac{7}{4} \cos 4\pi - 24 \cdot 2\pi \right) - \left(\frac{7}{2} \cos 0 - 0 \right)$$

$$= -48\pi$$

Evaluate $\int_C (x^2 - iy^2)dz$

a) along the parabola $y = 2x^2$ from $(1,2)$ to $(2,8)$

b) along the straight line from $(1,2)$ to $(2,8)$

c) along the straight lines from $(1,2)$ to $(1,8)$ and then from $(1,8)$ to $(2,8)$

$x = t$

$y = 2t^2$

$z = x + iy$

$= t + i\,2t^2$

$x = t \;\Rightarrow\; 1 = t \;\Rightarrow\; t = 1$

$y = 2t^2 \;\Rightarrow\; 2 = 2t^2 \;\Rightarrow\; t^2 = 1 \;\Rightarrow\; t = \pm 1$

limit

(x,y)	t
$(2,8)$	$t = 2$
$(1,2)$	$t = 1$

$dz = dt + i\,4t\,dt = (1 + 4it)\,dt$

$$\int_C (x^2 - iy^2)\,dz$$

$$= \int_1^2 \left[t^2 - i(2t^2)^2\right]\cdot(1+4it)\,dt$$

$$= \int_1^2 \left(t^2 + 4it^3 - 4it^4 + 16t^5\right)dt$$

$$= \left[\frac{t^3}{3} + 4i\frac{t^4}{4} - 4i\frac{t^5}{5} + 16\cdot\frac{t^6}{6}\right]_1^2$$

$$= \left(\frac{8}{3} + 16i - \frac{128i}{5} + \frac{256}{3}\right) - \left(\frac{1}{3} + i - \frac{i}{5} + \frac{4}{3}\right)$$

$$= \frac{511}{3} - \frac{49}{5}i$$

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Evaluate $\int_C (x^2 - iy^2)dz$

a) along the parabola $y = 2x^2$ from (1,2) to (2,8)

b) along the straight line from (1,2) to (2,8)

c) along the straight lines from (1,2) to (1,8) and then from (1,8) to (2,8)

$$z = z_1 + (z_2 - z_1)t$$
$$= (1+2i) + ((2+8i) - (1+2i))t$$
$$= 1+2i + (1+6i)t$$
$$= (1+t) + (2+6t)i$$

$$dz = dt + 6i\,dt$$
$$= (1+6i)dt$$

limit from 0 to 1.

$$\int_c (x^2 - iy^2)\,dz$$

$$= \int_0^1 \left((1+t)^2 - i(2+6t)^2 \right)(1+6i)\,dt$$

$$= \int_0^1 \left(1+2t+t^2-4i-24i-36it^2 \right)(1+6i)\,dt$$

$$= \int_0^1 \left((217-30i)t^2 + (146-12i)t + (25-2i) \right) dt$$

$$= \left[(217-30i)\frac{t^3}{3} + (146-12i)\frac{t^2}{2} + (25-2i)t \right]_0^1$$

$$= \frac{(217-30i)}{3} + \frac{(146-12i)}{2} + (25-2i)$$

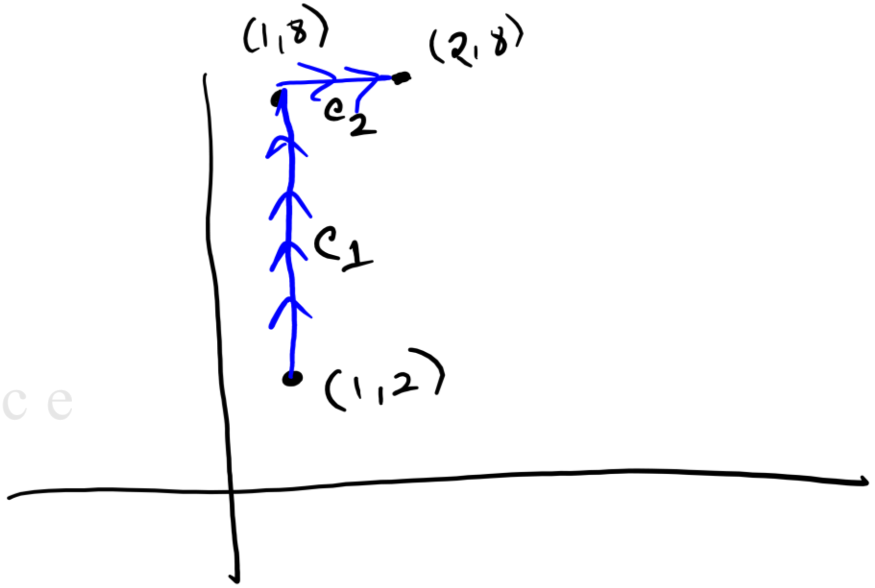
Evaluate $\int_C (x^2 - iy^2)dz$

a) along the parabola $y = 2x^2$ from (1,2) to (2,8)

b) along the straight line from (1,2) to (2,8)

c) along the straight lines from (1,2) to (1,8) and then from (1,8) to (2,8)

$$\int_C = \int_{(1,2)}^{(1,8)} + \int_{(1,8)}^{(2,8)}$$



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Now,

C_1: from $(1,2)$ to $(1,8)$

$$z = z_1 + (z_2 - z_1)t$$

$$= (1+2i) + \big(1+8i - (1+2i)\big)t$$

$$= (1+2i) + (6i)t$$

$$= (1) + (2+6t)i$$

$x = 1$

$y = 2 + 6t$

$dz = 6i\,dt$

limit t → End $= 1$, Start $= 0$

$$\int_{(1,2)}^{(1,8)} (x^2 - iy^2)\,dz$$

$$= \int_0^1 \left[1^2 - i(2+6t)^2\right] \cdot 6i\,dt$$

$$= \int_0^1 (216t^2 + 144t + 24 + 6i)\,dt$$

$$= \left[216\frac{t^3}{3} + 144\frac{t^2}{2} + 24t + 6it\right]_0^1$$

$$= \left(\frac{216}{3} + \frac{144}{2} + 24 + 6i\right) - (0)$$

$$= 72 + 72 + 24 + 6i$$

$$= 168 + 6i$$

Now,

C_2: from $(1,8)$ to $(2,8)$

$$z = z_1 + (z_2 - z_1)t$$

$$= (1+8i) + (2+8i - (1+8i))t$$

$$= (1+8i) + t$$

$$= (1+t) + (8)i$$

$x = 1+t$

$y = 8$

$dz = dt$

limit t → End $= 1$, Start $= 0$

$$\int_{(1,8)}^{(2,8)} (x^2 - iy^2)\,dz$$

$$= \int_0^1 \left[(1+t)^2 - i\,8^2\right] \cdot dt$$

$$= \int_0^1 \left(1+2t+t^2-64i\right) dt$$

$$= \left[t + 2\frac{t^2}{2} + \frac{t^3}{3} - 64it\right]_0^1$$

$$= \left(1 + 1 + \frac{1}{3} - 64i\right) - (0)$$

$$= \frac{7}{3} - 64i$$

find Arg $= (168 + 6i) + \left(\frac{7}{3} - 64i\right)$

$= ??$

Evaluate $\int_{i}^{2-i}(3xy + iy^2)dz$

a) along the straight line joining from $z = i$ to $z = 2 - i$

b) along the parabola from $x = 2t - 2$ and $y = 1 + t - t^2$

$z = z_1 + (z_2 - z_1)t$

$= i + (2 - i - i)t$

$\rightarrow = i + 2t - 2it$

$= (2t) + (1 - 2t)i$

$x = 2t$

$y = 1 - 2t$

$dz = 2dt - 2i\,dt$

$= (2 - 2i)dt$

limit $t \to 1$, $t \to 0$

$$\int_{1}^{2-i} \left(3xy + iy^2\right) dz$$

$$= \int_0^1 \left(3\cdot(2t)(1-2t) + i(1-2t)^2\right)(2-2i)\, dt$$

$$= \int_0^1 \left(-16t^2 + 4t + 2 + 32it^2 - 20it + 2i\right) dt$$

$$= \left[-\frac{16}{3}t^3 + 2t^2 + 2t + \frac{32}{3}i\,t^3 - 10it^2 + 2it\right]_0^1$$

$$= \left(-\frac{16}{3} + 2 + 2 + \frac{32}{3}i - 10i + 2i\right)$$

Evaluate $\int_{i}^{2-i}(3xy + iy^2)dz$

a) ~~along the straight line joining from $z = i$ to $z = 2 - i$~~

b) along the parabola from $x = 2t - 2$ and $y = 1 + t - t^2$

$z = x + iy$

$$\boxed{z = (2t-2) + i(1+t-t^2)}$$

$dz = (2)dt + i(1-2t)dt$

$$\boxed{dz = (2+i-2it)\,dt}$$

$0 = 2t-2$

$\Rightarrow 2t = 2$

$\Rightarrow t = 1$

$1 = 1+t-t^2$

$t^2 - t = 0$

$t(t-1) = 0$

$t = 0, 1$

limit

	(x,y)	t
$2-i \to$	$(2,-1)$	$t=2$
$i \to$	$(0,1)$	$t=1$

$$\int_{i}^{2-i} \left(3xy + iy^2\right) dz$$

$$= \int_{1}^{2} \left[3\cdot(2t-2)\cdot\left(1+t-t^2\right) + i\left(1+t-t^2\right)^2 \right] \cdot \left(2+i-2it\right) dt$$

$$= \int_{1}^{2} \left[2t^5 - (5-14i)t^4 - (12+34i)t^3 + (29+10i)t^2 + 16it - (13+4i) \right] dt$$

$$= -\frac{1}{3} + \frac{79\,i}{30}$$

Evaluate $\oint_C |z|^2 \, dz$ around the square with vertices at $(0,0), (1,0), (1,1), (0,1)$.

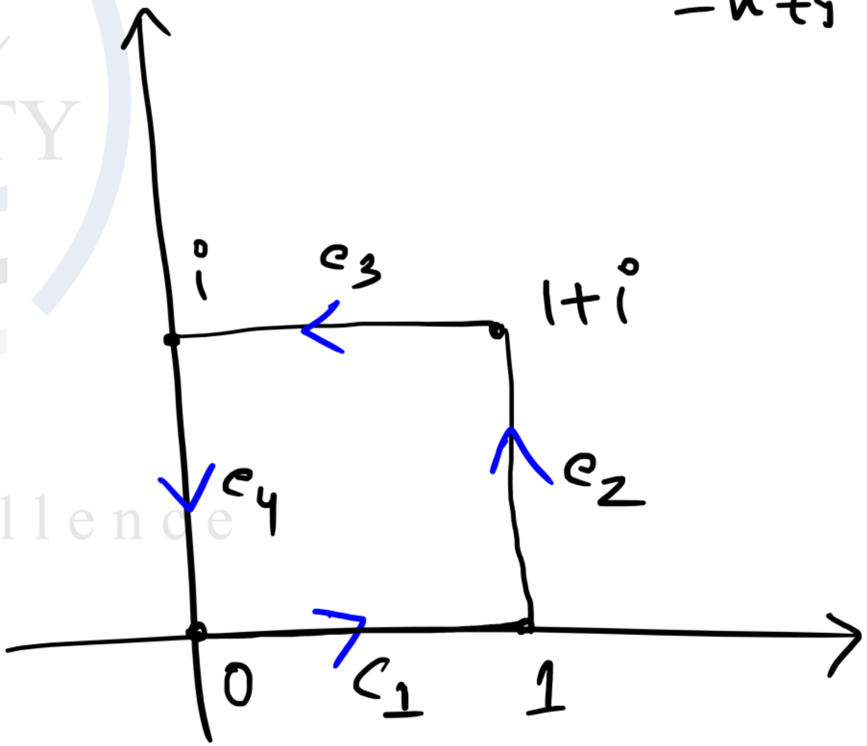
V.V.g

$$|z|^2 = \left(\sqrt{x^2+y^2}\right)^2$$

$$= x^2+y^2$$

$$= \oint_C (x^2+y^2)\, dz$$

$$= \int_{C_1} + \int_{C_2} + \int_{C_3} + \int_{C_4}$$



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for c_1 : from 0 to 1 along st. line.

$$z(t) = 0 + (1-0)t = t = t + (0)i$$

$$dz = dt$$

$x = t$

$y = 0$

$$\int_{c_1} (x^2+y^2)\,dz$$

$$= \int_0^1 (t^2+0^2)\,dt = \frac{1}{3} \checkmark$$

$t \rightarrow 0$

$t \rightarrow 1$

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For C_2: from 1 to $1+i$

$$z(t) = 1 + (1+i-1)t$$

$$= 1 + it$$

$$= 1 + (t)i$$

$x = 1$

$y = t$

$dz = i\,dt$

$t \to 0, 1$

$$\int_{C_2} (x^2+y^2)\,dz$$

$$= \int_0^1 (1+t^2)\, i\,dt = i\left[t + \frac{t^3}{3}\right]_0^1$$

$$= \frac{4i}{3}$$ ✓

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<u>for c_3</u>: from $1+i$ to i

$$z(t) = 1+i+\{i-(1+i)\}t$$

$$= 1+i-t = (1-t)+(1)i$$

$$x = 1-t$$

$$y = 1$$

$$dz = -dt$$

$$t = 0,1$$

$$\int_{c_3} (x^2+y^2)\,dz$$

$$= \int_0^1 \{(1-t)^2 + 1^2\}(-dt)$$

$$= -\int_0^1 (1-2t+t^2+1)\,dt = -\frac{4}{3} \checkmark$$

for C_4: from i to 0.

$$z = i + (0-i)t = 0 + (1-t)i$$

$$x = 0$$

$$y = 1-t$$

$$dz = -i\,dt$$

$$\int_{C_4} (x^2+y^2)\,dz$$

$$= \int_0^1 \left(0 + (1-t)^2\right)(-i\,dt)$$

$$= \frac{-i}{3}.$$

$$\oint_{C} (x^2+y^2)\,dz$$

$$= \int_{C_1} (x^2+y^2)\,dz + \int_{C_2} (x^2+y^2)\,dz + \int_{C_3} (x^2+y^2)\,dz + \int_{C_4} (x^2+y^2)\,dz$$

$$= \frac{1}{3} + \frac{4i}{3} + \frac{-4}{3} + \frac{-i}{3} = -1 + i$$

Parametrization of Circle

Cente (3,4) radius 5

$$\boxed{(x-3)^2+(y-4)^2=5^2}$$

Cartesian

Cente 3+4i radius 5

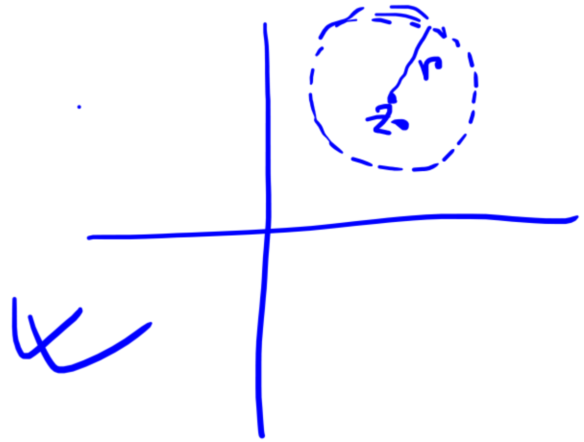
$$\boxed{|z-(3+4i)|=5}$$

$$\boxed{z-(3+4i)=5e^{i\theta}}$$

A circle centred at z_0 with radius r can be expressed as

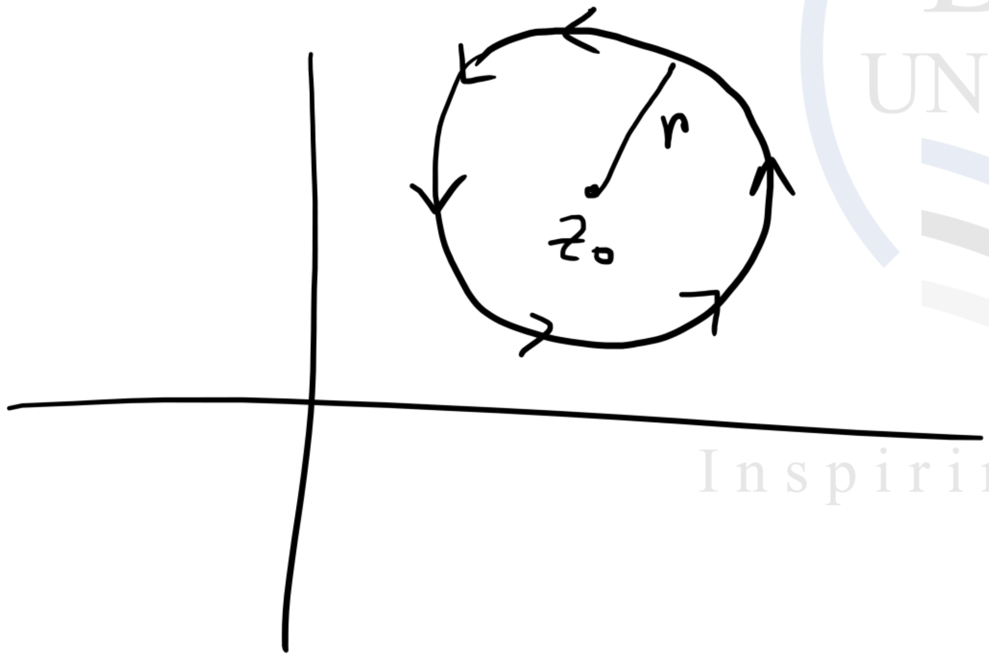
$$|z - z_0| = r$$

$$\Rightarrow \quad z - z_0 = re^{i\theta}$$



$$|z - z_0| = r \implies z - z_0 = re^{i\theta}$$

$\theta \to$ Stat $= 0$ end $= 2\pi$



for anti-clockwise

Counter-clockwise

Positive orientation.

Evaluate $\oint_C (\bar{z})^2 dz$ around the circles (a) $|z| = 1$ and (b) $|z-1| = 1$.

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(a) $|z| = 1$

$\Rightarrow z = 1 \cdot e^{i\theta}$

$\Rightarrow z = e^{i\theta}$

$\bar{z} = e^{-i\theta}$

$dz = i e^{i\theta} d\theta$

$\theta \longrightarrow 0 \text{ to } 2\pi$

$$\oint_C (\bar{z})^2 dz$$

$$= \int_0^{2\pi} (e^{-i\theta})^2 \, i e^{i\theta} \, d\theta$$

$$= \int_0^{2\pi} i e^{-2i\theta} e^{i\theta} \, d\theta$$

$$= i\int_0^{2\pi} e^{-i\theta}\, d\theta$$

$$= i\left[\frac{e^{-i\theta}}{-i}\right]_0^{2\pi}$$

$$= -\left[e^{-i\theta}\right)_0^{2\pi}$$

$$= -\left[e^{i(-2\pi)} - e^{0}\right]$$

$$= -\left[\cos(-2\pi) + i\sin(-2\pi) - 1\right)$$

$$= -\left[1 + i\,0 - 1\right) = 0$$

Evaluate $\oint_C (\bar{z})^2 dz$ around the circles (a) $|z| = 1$ and (b) $|z-1| = 1$.

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(b) $|z-1| = 1$

$\Rightarrow z - 1 = 1e^{i\theta}$

$\Rightarrow z = 1 + e^{i\theta}$

$\Rightarrow \bar{z} = 1 + e^{-i\theta}$

$dz = ie^{i\theta} d\theta$

$\theta \rightarrow 0, 2\pi$

$\oint_c (\bar{z})^2 dz$

$= \int_0^{2\pi} (1 + e^{-i\theta})^2 \, i e^{i\theta} d\theta$

$$= \int_0^{2\pi} \left[1 + 2e^{-i\theta} + e^{-2i\theta}\right] i e^{i\theta}\, d\theta$$

$$= i \int_0^{2\pi} \left(e^{i\theta} + 2 + e^{-i\theta}\right) d\theta$$

$$= i \left[\frac{e^{i\theta}}{i} + 2\theta + \frac{e^{-i\theta}}{-i}\right]_0^{2\pi}$$

$$= i \left[\frac{\overset{=1}{e^{i2\pi}}}{i} + 2\cdot 2\pi + \frac{\overset{=1}{e^{i(-2\pi)}}}{-i}\right]$$

$$- i \left[\frac{e^{0}}{i} + 2\cdot(0) + \frac{e^{0}}{-i}\right]$$

$$= 4\pi i \quad \checkmark$$

Evaluate $\int_C (z^2 + 3z)\,dz$

along the circle $|z| = 2$ from $(2,0)$ to $(0,2)$ in a counter clockwise direction.

$|z| = 2$

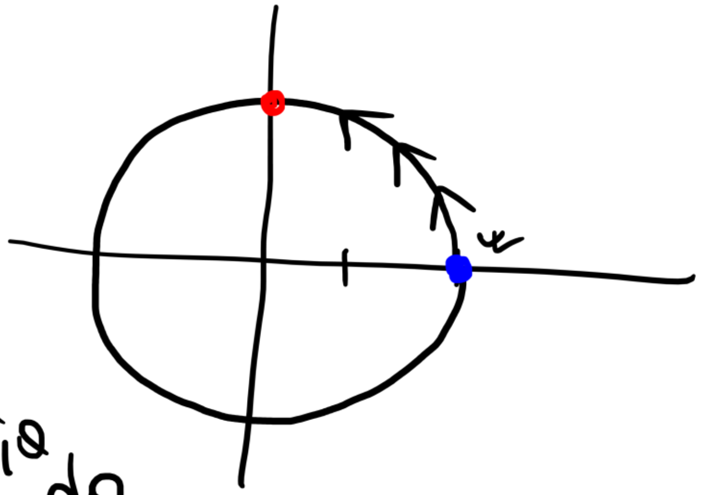
$\Rightarrow \boxed{z = 2e^{i\theta}}$

$\Rightarrow dz = 2ie^{i\theta}$

$\theta = 0$ to $\frac{\pi}{2}$.

$$\oint_C (z^2 + 3z)\,dz$$

$$= \int_0^{\frac{\pi}{2}} \left(4e^{2i\theta} + 6e^{i\theta}\right) 2ie^{i\theta}\,d\theta$$



$$= i\int_0^{\frac{\pi}{2}} \left(8e^{3i\theta} + 12e^{2i\theta}\right) d\theta$$

$$= i\left[8\cdot\frac{e^{3i\theta}}{3i} + 12\cdot\frac{e^{2i\theta}}{2i}\right]_0^{\frac{\pi}{2}}$$

$$= \left[\frac{8}{3}e^{3i\theta} + 6e^{2i\theta}\right]_0^{\frac{\pi}{2}}$$

$$= \left(\frac{8}{3}e^{i\frac{3\pi}{2}} + 6e^{i\pi}\right) - \left(\frac{8}{3}+6\right)$$

$$= \frac{8}{3}(0-i) + 6(-1) - \frac{8}{3} - 6$$

= u !! u

Evaluate $\int_C \left(z^2+3z\right)dz$

along the circle $|z| = 2$ ~~from (2,0) to (0,2)~~ in a ~~counter~~ clockwise direction.

bw

$$\int_{2\pi}^{0}$$

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