Lecture 9

Function

Topics

- Function Definition
- Domain and Range
- One-to-one and Onto Functions
- One-to-one Correspondence
- Cardinality of Sets
- Countable and Uncountable Sets

Function Definition

Let ***A*** and ***B*** be nonempty sets. A function ***f*** from ***A*** to ***B*** is an assignment of exactly one element of ***B*** to each element of ***A***. We write ***f(a) = b*** if ***b*** is the unique element of ***B*** assigned by the function ***f*** to the element ***a*** of ***A***. If ***f*** is a function from ***A*** to ***B***, we write ***f : A*** → ***B***.

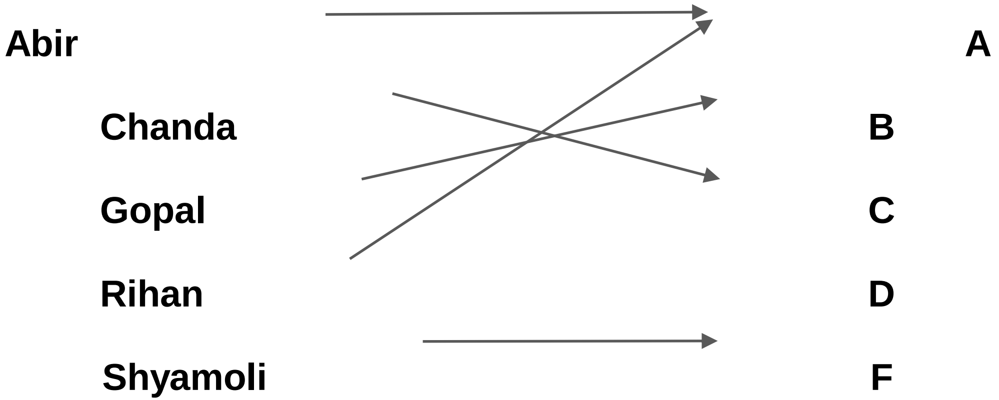


Fig : Assignment of grades in a discrete mathematics class

Domain and Range

If ***f*** is a function from ***A*** to ***B***, we say that ***A*** is the ***domain*** of ***f*** and ***B*** is the ***codomain*** of ***f***. If ***f(a) = b***, we say that ***b*** is the ***image*** of ***a*** and ***a*** is a ***preimage*** of ***b***. The ***range***, or ***image***, of ***f*** is the ***set of all images*** of elements of ***A***. Also, if ***f*** is a function from ***A*** to ***B***, we say that ***f maps A to B***.

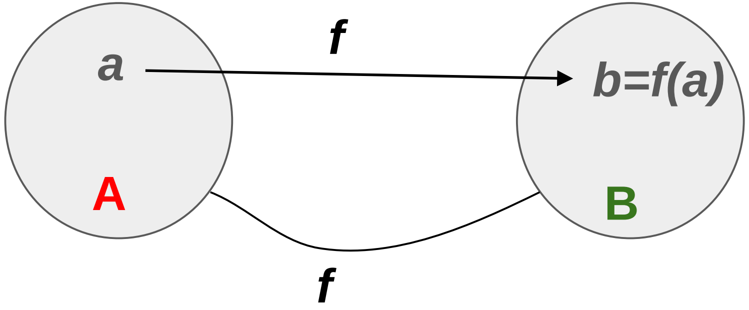


Fig : The function *f* maps *A* to *B*

Examples

EXAMPLE: Let **R** be the relation with ordered pairs (Abdul, 22), (Bidyut, 24), (Chanda, 21), (Dayem, 22), (Emon, 24), and (Fahim, 22). Here each pair consists of a graduate student and this student's age. Specify a function determined by this relation.

SOLUTION: If ***f*** is a function specified by **R**, then ***f(***Abdul***)*** = 22, ***f(***Bidyut***)*** = 24 and so on. For domain we take the set **{Abdul, Bidyut, Chanda, Dayem, Emon, Fahim}**. We also need to specify the codomain, which needs to contain all possible ages of the students. Because it is highly likely that all students are less than 100 years old, we can take the **set of positive integers less than 100** as the codomain.

Definition

Let $\boldsymbol{f_1}$ and $\boldsymbol{f_2}$ be functions from $\boldsymbol{A}$ to $\boldsymbol{R}$. Then $\boldsymbol{f_1+f_2}$ and $\boldsymbol{f_1 f_2}$ are also functions from $\boldsymbol{A}$ to $\boldsymbol{R}$ defined for all $x \in A$ by

- $(\boldsymbol{f_1+f_2})(x) = \boldsymbol{f_1}(x) + \boldsymbol{f_2}(x)$
- $(\boldsymbol{f_1 f_2})(x) = \boldsymbol{f_1}(x)\boldsymbol{f_2}(x)$

EXAMPLE: Let $\boldsymbol{f_1}$ and $\boldsymbol{f_2}$ be functions from $\boldsymbol{R}$ to $\boldsymbol{R}$ such that $\boldsymbol{f_1(x) = x^2}$ and $\boldsymbol{f_2(x) = x - x^2}$**.** What are the functions $\boldsymbol{f_1 + f_2}$ and $\boldsymbol{f_1 f_2}$?

SOLUTION: From the definition of the sum and product of functions, it follows that

- ➔ $(f_1+ f_2)(x) = f_1(x) + f_2(x) = x^2 + (x - x^2) = x$
- ➔ $(f_1f_2)(x) = x^2 (x - x^2) = x^3 - x^4$.

Definition

Let ***f*** be a function from ***A*** to ***B*** and let ***S*** be a subset of ***A***. The ***image of S*** under the function ***f*** is the ***subset of B*** that consists of the images of the elements of ***S***. We denote the image of ***S*** by ***f(S)***, so

$$f(S) = \{t \mid \exists s \in S\ (t = f(s))\}.$$

We also use the shorthand $\{f(s) \mid s \in S\}$ to denote this set.

EXAMPLE: Let **A** = {a, b, c, d, e} and **B** = {1, 2, 3, 4} with *f(a)* = 2, *f(b)* = 1, *f(c)* = 4, *f(d)* = 1, and *f(e)* = 1. The image of the subset **S** = {b, c, d} is the set *f(S*) = {1, 4}.

One-to-one Functions

A function ***f*** is said to be ***one-to-one***, or an ***injection***, if and only if ***f(a) = f(b)*** implies that ***a = b*** for all ***a*** and ***b*** in the domain of ***f***.
A function is said to be injective if it is one-to-one.

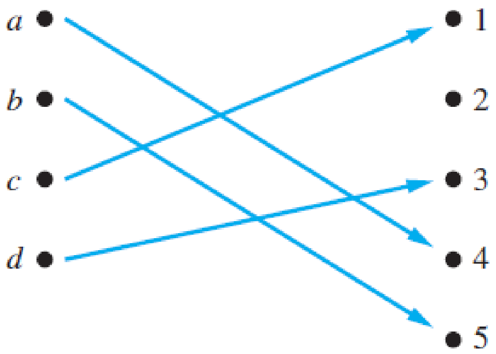
Contrapositive of the previous implication: A function ***f*** is ***one-to-one*** if and only if ***f(a) ≠ f(b)*** whenever ***a ≠ b***.

Expressing the previous implication using quantifiers: ***∀a∀b(f(a) = f(b) → a = b)*** or equivalently ***∀a∀b(a ≠ b → f(a) ≠ f(b))***, where the universe of discourse is the domain of the function.

Examples

EXAMPLE: Determine whether the function ***f*** from **{a, b, c, d}** to **{1, 2, 3, 4, 5}** with ***f(a) = 4, f(b) = 5, f(c) = 1,*** and ***f(d) = 3*** is one-to-one.

SOLUTION: The function ***f*** is one-to-one because ***f*** takes on different values at the four elements of its domain.



Examples

EXAMPLE: Determine whether the function $\boldsymbol{f(x) = x^2}$ from the set of integers to the set of integers is one-to-one.

SOLUTION: The function $f(x) = x^2$ is not one-to-one because, for instance, $f(1) = f(-1) = 1$, but $1 \neq -1$.

EXAMPLE: Determine whether the function $\boldsymbol{f(x) = x + 1}$ from the set of real numbers to itself is one-to-one.
SOLUTION: Suppose that $\boldsymbol{x}$ and $\boldsymbol{y}$ are real numbers with $\boldsymbol{f(x) = f(y)}$, so,

$$f(x) = f(y)$$
$$\rightarrow x + 1 = y + 1$$
$$\rightarrow x = y.$$

Hence, $\boldsymbol{f(x) = x + 1}$ is a one-to-one function from R to R.

Definition

A function ***f*** whose domain and codomain are subsets of the set of real numbers is called ***increasing*** if ***f(x) ≤ f(y)***, and ***strictly increasing*** if ***f(x) < f(y)***, whenever ***x < y*** and ***x*** and ***y*** are in the domain of ***f***. Similarly, ***f*** is called ***decreasing*** if ***f(x) ≥ f(y)***, and ***strictly decreasing*** if ***f(x) > f(y)***, whenever ***x < y*** and ***x*** and ***y*** are in the domain of ***f***. (The word strictly in this definition indicates a strict inequality.)

Examples

EXAMPLE: The function $f(x) = x^2$ from R^+ to R^+ is strictly increasing. To see this, suppose that x and y are positive real numbers with $x < y$. Multiplying both sides of this inequality by x gives $x^2 < xy$. Similarly, multiplying both sides by y gives $xy < y^2$. Hence, $f(x) = x^2 < xy < y^2 = f(y)$. However, the function $f(x) = x^2$ from R to the set of ***nonnegative real numbers*** is not strictly increasing because $-1 < 0$, but $f(-1) = (-1)^2 = 1$ is not less than $f(0) = 0^2 = 0$.

Onto Functions

A function ***f*** from ***A*** to ***B*** is called ***onto***, or a ***surjection***, if and only if for every element ***b*** $\in$ ***B*** there is an element ***a*** $\in$ ***A*** with ***f(a) = b***. A function ***f*** is called surjective if it is onto.

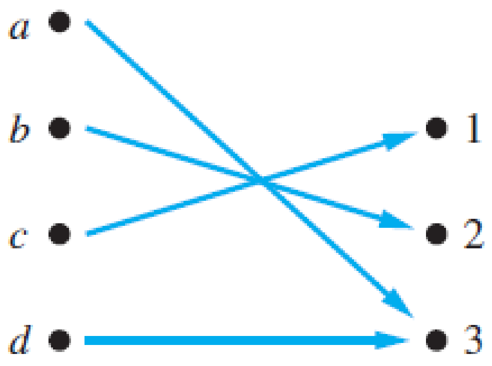


Fig: An onto Function

Examples

EXAMPLE: Is the function ***f(x) = x+1*** from the set of integers to the set of integers onto?

SOLUTION: This function is onto, because for every integer y there is an integer x such that $f(x) = y$. To see this, note that $f(x) = y$ if and only if $x+1 = y$, which holds if and only if $x = y-1$. (Note that $y-1$ is also an integer, and so, is in the domain of f)

EXAMPLE: Is the function $\boldsymbol{f(x) = x^2}$ from the set of integers to the set of integers onto?
SOLUTION: The function f is not onto because there is no integer x with $x^2 = -1$, for instance.

One-to-one Correspondence

The function ***f*** is a ***one-to-one correspondence***, or a ***bijection***, if it is both ***one-to-one*** and ***onto***. We also say that such a function is bijective.

EXAMPLE: Let ***f*** be the function from **{a, b, c, d}** to **{1, 2, 3, 4}** with ***f(a) = 4, f(b) = 2, f(c) = 1,*** and ***f(d) = 3***. Is ***f*** a bijection?

SOLUTION: The function ***f*** is one-to-one and onto. It is one-to-one because no two values in the domain are assigned the same function value. It is onto because all four elements of the codomain are images of elements in the domain. Hence, ***f*** is a bijection.

Examples

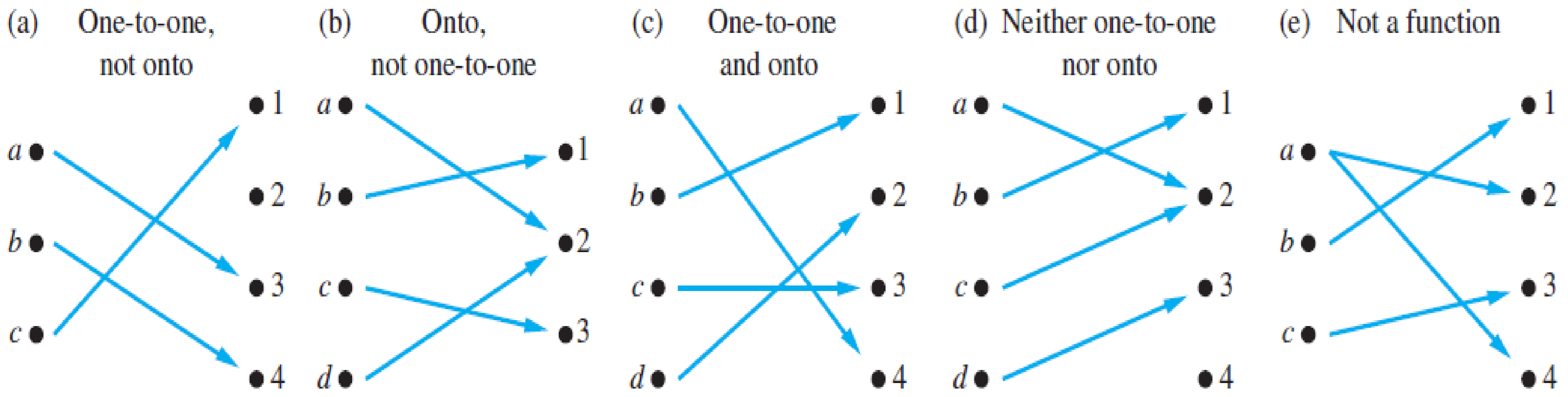


Fig: Examples of different types of correspondences.

Techniques

Suppose that $\boldsymbol{f : A \rightarrow B}$.

- ***To show that f is injective:*** Show that if $f(x) = f(y)$ for arbitrary $x, y \in A$, then $x = y$.
- ***To show that f is not injective:*** Find particular elements $x, y \in A$ such that $x \neq y$ and $f(x) = f(y)$.
- ***To show that f is surjective:*** Consider an arbitrary element $y \in B$ and find an element $x \in A$ such that $f(x) = y$.
- ***To show that f is not surjective:*** Find a particular $y \in B$ such that $f(x) \neq y$ for all $x \in A$.

Cardinality of Sets

The sets A and B have the ***same cardinality*** if and only if there is a ***one-to-one correspondence*** from A to B. When A and B have the same cardinality, we write $|A| = |B|$.

In another words, if there is a one-to-one function from A to B, the cardinality of A is ***less than or the same*** as the cardinality of B and we write $|A| \leq |B|$. Moreover, when $|A| \leq |B|$ and A and B have ***different*** cardinality, we say that the cardinality of A is ***less than*** the cardinality of B and we write $|A| < |B|$.

Countable and Uncountable Sets

A set that is either ***finite*** or has the ***same cardinality*** as ***the set of positive integers*** is called ***countable***. A set that is ***not countable*** is called ***uncountable***.

When an ***infinite*** set ***S*** is ***countable***, we denote the cardinality of ***S*** by $\aleph_0$ (where $\aleph$ is aleph, the first letter of the Hebrew alphabet). We write $|S| = \aleph_0$ and say that ***S*** has cardinality ***"aleph null."***

Examples

EXAMPLE: Show that the set of odd positive integers is a countable set.

SOLUTION: To show that the set of odd positive integers is countable, we will exhibit a one-to-one correspondence between this set and the set of positive integers. Consider the function $f(n) = 2n-1$, from Z^+ to the set of odd positive integers.

We show that ***f*** is a ***one-to-one correspondence*** by showing that it is ***both one-to-one and onto***. To see that it is one-to-one, suppose that $f(n) = f(m)$. Then $2n-1 = 2m-1$, so $n = m$. To see that it is onto, suppose that t is an odd positive integer. Then t is 1 less than an even integer $2k$, where k is a natural number. Hence $t = 2k-1 = f(k)$.

Examples

Please try to understand the examples 4 and 5 from book.

Thank You