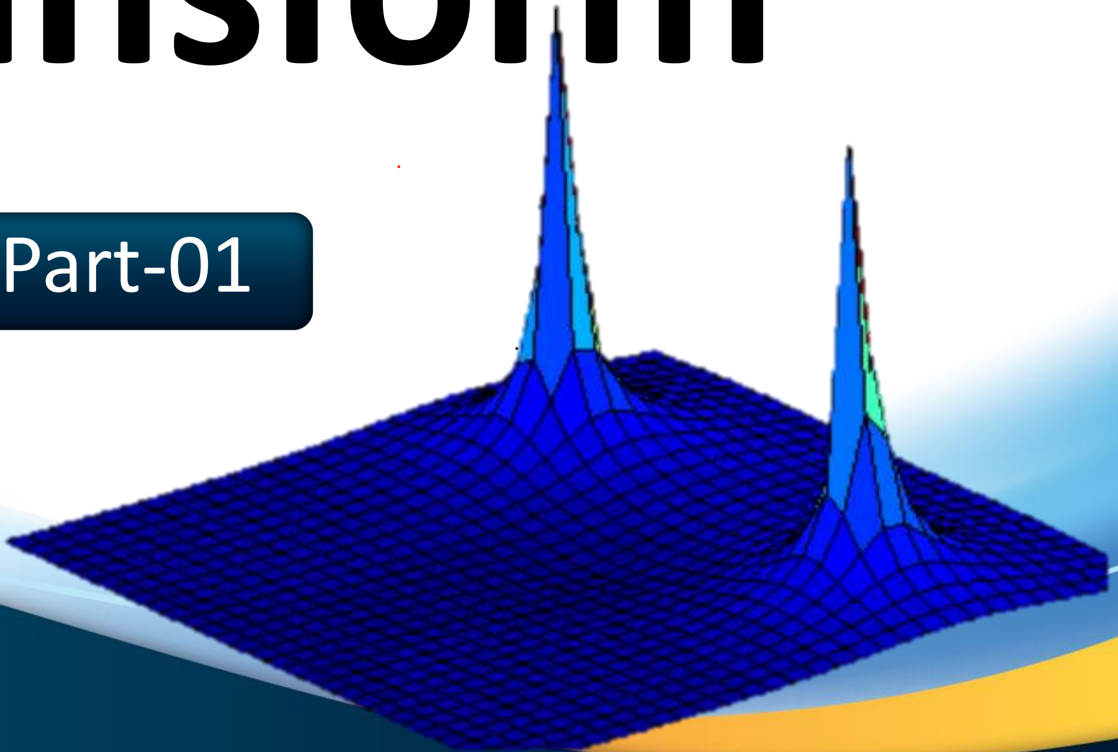
Undergraduate Course in Mathematics

Laplace Transform

Solving Differential Equations | Part-01



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Solving Differential Equations

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$*$ $\frac{dy}{dx} = x^2$ or $y' = x^2$

$\Rightarrow y = \frac{x^3}{3} + c$

$\boxed{y = \frac{x^3}{3} + c.}$

$y' + 2xy = e^x$

Initial value problem of ODE

second ord

$$5y'' + 3y' + 6y = e^t \sin 2t$$

Laplace Transform of Derivatives

If $\mathcal{L}\{y(t)\} = Y(s)$ then

- $\mathcal{L}\{y'(t)\} = s \cdot Y(s) - y(0)$
- $\mathcal{L}\{y''(t)\} = s^2 \cdot Y(s) - s \cdot y(0) - y'(0)$
- $\mathcal{L}\{y'''(t)\} = s^3 \cdot Y(s) - s^2 \cdot y(0) - s \cdot y'(0) - y''(0)$

➢ $\mathcal{L}\{y^n(t)\} = s^n \cdot Y(s) - s^{n-1} \cdot y(0) - s^{n-2} \cdot y'(0) - s^{n-3} y''(0) - \cdots - y^{n-1}(0)$

$$\int uv\,dt = u\int v\,dt - \int\left(u'\int v\,dt\right)dt.$$

Prove that

$$\rightarrow \quad \mathcal{L}\{y'(t)\} = s \cdot Y(s) - y(0)$$

given $\mathcal{L}\{y(t)\} = Y(s)$

$$\mathcal{L}\{y'(t)\} = \int_0^\infty y'(t)\, e^{-st}\, dt$$

$$= \int_0^\infty e^{-st} \cdot y'(t)\, dt$$

$$= \left[e^{-st} y(t) \right]_0^\infty - \int_0^\infty (-s) e^{-st} y(t)\, dt$$

$$= 0 - e^{-0} y(0) + s \cdot \int_0^\infty e^{-st} y(t)\, dt$$

$$= -y(0) + s \int_0^\infty y(t)\, e^{-st}\, dt$$

$$= -y(0) + s \cdot \mathcal{L}\{y(t)\}$$

$$= -y(0) + s \cdot Y(s)$$

$$= s \cdot Y(s) - y(0)$$

Prove that

$$\mathcal{L}\{f''(t)\} = s^2 \cdot F(s) - s \cdot f(0) - f'(0)$$

given $\mathcal{L}\{f(t)\} = F(s)$

$$\therefore \mathcal{L}\{f''(t)\} = \int_0^\infty f''(t)\, e^{-st}\, dt$$

$$= \int_0^\infty e^{-st} f''(t)\, dt$$

$$= \left[e^{-st} \cdot f'(t) \right]_0^\infty - \int_0^\infty (-s)\, e^{-st} \cdot f'(t)\, dt$$

$$= 0 - e^{-0} \cdot f'(0) + s \cdot \int_0^\infty e^{-st} \cdot f'(t)\, dt$$

$$= -f'(0) + s \cdot \left\{ \left[e^{-st} \cdot f(t) \right]_0^\infty - \int_0^\infty (-s)\, e^{-st} \cdot f(t)\, dt \right\}$$

$$= -f'(0) + s \cdot \left\{ 0 - e^{-0} f(0) + s \cdot \int_0^\infty e^{-st} f(t) \right\}$$

$$= -f'(0) - s \cdot f(0) + s^2 \cdot \int_0^\infty f(t)\, e^{-st}\, dt$$

$$= -f'(0) - s\, f(0) + s^2 \cdot \mathcal{L}\{f(t)\}$$

$$= s^2 \cdot F(s) - s \cdot f(0) - f'(0)$$

Prove that

$$\mathcal{L}\{y'''(t)\} = s^3 \cdot Y(s) - s^2 \cdot y(0) - s \cdot y'(0) - y''(0)$$

$$\mathcal{L}\{y'''(t)\} = \int_0^\infty y'''(t)\, e^{-st}\, dt$$

$$= \int_0^\infty e^{-st}\, y'''(t)\, dt$$

$$= \left[e^{-st}\, y''(t) \right]_0^\infty - \int_0^\infty (-s)\, e^{-st} \cdot y''(t)\, dt$$

$$= 0 - y''(0) + s \cdot \int_0^\infty e^{-st} y''(t)\, dt$$

$$= -y''(0) + s \cdot \left\{ \left[e^{-st} \cdot y'(t) \right]_0^\infty - \int_0^\infty (-s) e^{-st} \cdot y'(t)\, dt \right\}$$

$$= -y''(0) + s \cdot \left\{ 0 - y'(0) + s \cdot \int_0^\infty e^{-st} y'(t)\, dt \right\}$$

$$= -y''(0) - s \cdot y'(0) + s^2 \cdot \int_0^\infty e^{-st} y'(t)\, dt$$

$$= -y''(0) - s\cdot y'(0) + s^2\cdot\left\{ \left[e^{-st}\cdot y(t) \right]_0^\infty - \int_0^\infty (-s) e^{-st}\cdot y(t)\, dt \right\}$$

$$= -y''(0) - s\cdot y'(0) + s^2 \left\{ 0 - y(0) + s\cdot \int_0^\infty y(t) e^{-st} dt \right\}$$

$$= -y''(0) - s\cdot y'(0) - s^2\, y(0) + s^3\cdot Y(s)$$

$$= s^3\, Y(s) - s^2 y(0) - s\cdot y'(0) - y''(0)$$

Solve the given differential equation:

$$y'' - 4y' + 4y = t^3 e^{2t}, \quad y(0) = 0, \quad y'(0) = 0 \qquad \text{—(1)}$$

let $\mathcal{L}\{y(t)\} = Y(s)$

Applying Laplace transform on both sides of the given DE.

$\mathcal{L}\{t^3\} = \frac{3!}{s^4}$

$\mathcal{L}\{t^3 e^{2t}\} = \frac{3!}{(s-2)^4}$

$$\mathcal{L}\{y'' - 4y' + 4y\} = \mathcal{L}\{t^3 \cdot e^{2t}\}$$

$$\Rightarrow s^2 Y(s) - s\cancel{y(0)} - \cancel{y'(0)} - 4\cdot\left[s\cdot Y(s) - \cancel{y(0)}\right] + 4\cdot Y(s) = \frac{3!}{(s-2)^4}$$

$$\Rightarrow \quad s^2 Y(s) - 4s \cdot Y(s) + 4Y(s) = \frac{6}{(s-2)^4}$$

$$\Rightarrow \quad \underline{(s^2 - 4s + 4)} \, Y(s) = \frac{6}{(s-2)^4}$$

$$\Rightarrow \quad (s-2)^2 \, Y(s) = \frac{6}{(s-2)^4}$$

$$\Rightarrow \quad \underline{\underline{Y(s)}} = \frac{6}{(s-2)^6}$$

$$\mathcal{L}^{-1}\{Y(s)\} = \mathcal{L}^{-1}\left\{\frac{6}{(s-2)^6}\right\}$$

$$\Rightarrow y(t) = e^{2t} \cdot \mathcal{L}^{-1}\left\{\frac{6}{(s+2-2)^6}\right\}$$

$$= e^{2t}\, \mathcal{L}^{-1}\left\{\frac{6}{s^6}\right\}$$

$$= e^{2t} \cdot \frac{6}{5!}\, \mathcal{L}^{-1}\left\{\frac{5!}{s^{5+1}}\right\}$$

$$y(t) = e^{2t} \cdot \frac{6}{120} t^5$$

$$= \frac{1}{20} t^5 \cdot e^{2t}$$

Solve the given differential equation:

$$Y'' - 3Y' + 2Y = 4, \quad Y(0) = 1, \;\; Y'(0) = -1$$

Let $\mathcal{L}\{Y(t)\} = y(s)$

Applying Laplace transform both sides,

$$\mathcal{L}\{Y'' - 3Y' + 2Y\} = \mathcal{L}\{4\}$$

$$\Rightarrow s^2 y(s) - s \cdot Y(0) - Y'(0) - 3 \cdot \left[s \cdot y(s) - Y(0)\right] + 2 \cdot y(s) = \frac{4}{s}.$$

$$s^2 y(s) - s + 1 - 3s\,y(s) + 3 + 2\,y(s) = \frac{4}{s}.$$

$$\Rightarrow (s^2 - 3s + 2)\,y(s) = \frac{4}{s} + s - 4$$

$$\Rightarrow (s^2 - 3s + 2)\,y(s) = \frac{4 + s^2 - 4s}{s}$$

$$\Rightarrow \quad y(s) = \frac{s^2 - 4s + 4}{s(s^2 - 3s + 2)} = \frac{(s-2)^2}{s(s-2)(s-1)} = \frac{s-2}{s(s-1)}$$

$$\mathcal{L}^{-1}\{y(s)\} = \mathcal{L}^{-1}\left\{\frac{s-2}{s(s-1)}\right\}$$

$$\Rightarrow Y(t) = \mathcal{L}^{-1}\left\{\frac{s-2}{s(s-1)}\right\}.$$

Now, $$\frac{s-2}{s(s-1)} \equiv \frac{A}{s} + \frac{B}{s-1}$$

$$\Rightarrow s-2 \equiv A(s-1) + B(s)$$

$s=0$: $\quad -2 = -A \Rightarrow A = 2$

$s=1$: $\quad -1 = B(1) \Rightarrow B = -1.$

$$Y(t) = \mathcal{L}^{-1}\left\{ \frac{2}{s} + \frac{-1}{s-1} \right\}$$

$$= 2 \cdot \mathcal{L}^{-1}\left\{ \frac{1}{s} \right\} - \mathcal{L}^{-1}\left\{ \frac{1}{s-1} \right\}$$

$$= 2 - e^{t}$$

Solve the given differential equation:

$$y''' - 3y'' + 3y' - y = e^t t^2, \quad y(0) = 0,\ y'(0) = 1,\ y''(0) = -2$$

Let $\mathcal{L}\{y(t)\} = Y(s)$

$\mathcal{L}\{t^2\} = \frac{2!}{s^3}$

$\mathcal{L}\{e^t \cdot t^2\} = \frac{2!}{(s-1)^3}$.

Applying Laplace transform both side

$$\mathcal{L}\{y''' - 3y'' + 3y' - y\} = \mathcal{L}\{e^t t^2\}$$

$$\Rightarrow s^3 Y(s) - s^2 \cancel{y}(0) - s\cdot y'(0) - y''(0) - 3\cdot\left[s^2 Y(s) - s\cdot \cancel{y}(0) - y'(0)\right] + 3\cdot\left[sY(s) - \cancel{y}(0)\right]$$

$$- Y(s) = \frac{2!}{(s-1)^3}$$

$$\Rightarrow s^3 Y(s) - s + 2 - 3s^2 Y(s) + 3 + 3s\,Y(s) - Y(s) = \frac{2}{(s-1)^3}.$$

$$\Rightarrow \underline{\left(s^3 - 3s^2 + 3s - 1\right)} Y(s) = \frac{2}{(s-1)^3} + s - 5$$

$$\Rightarrow (s-1)^3\, Y(s) = \frac{2}{(s-1)^3} + s - 5$$

$$\Rightarrow Y(s) = \frac{2}{(s-1)^6} + \frac{s-5}{(s-1)^3}.$$

$s \to s+1$

$$y(t) = \mathcal{L}^{-1}\left\{ \frac{2}{(s-1)^6} + \frac{s-5}{(s-1)^3} \right\}$$

$$= e^t \cdot \mathcal{L}^{-1}\left\{ \frac{2}{s^6} + \frac{(s+1)-5}{s^3} \right\}$$

$$= e^t \cdot \mathcal{L}^{-1}\left\{ \frac{2}{s^6} + \frac{s-4}{s^3} \right\}$$

$$= e^t \mathcal{L}^{-1}\left\{ \frac{2}{s^6} + \frac{1}{s^2} - \frac{4}{s^3} \right\}$$

$$= 2e^{t}\mathcal{L}^{-1}\left\{\frac{1}{s^6}\right\} + e^{t}\mathcal{L}^{-1}\left\{\frac{1}{s^2}\right\} - 4e^{t}\mathcal{L}^{-1}\left\{\frac{1}{s^3}\right\}$$

$$= \frac{2}{5!}e^{t}\mathcal{L}^{-1}\left\{\frac{5!}{s^6}\right\} + e^{t}\cdot\mathcal{L}^{-1}\left\{\frac{1}{s^2}\right\} - \frac{4}{2!}e^{t}\mathcal{L}^{-1}\left\{\frac{2!}{s^3}\right\}$$

$$= \frac{2}{120}e^{t}t^{5} + e^{t}\cdot t - 2\cdot e^{t}t^{2}$$

Ans.

Solve the given differential equation:

$$2y''' + 3y'' - 3y' - 2y = e^{-t}, \quad y(0) = 0, y'(0) = 0, y''(0) = 1$$

Let $\mathcal{L}\{y(t)\} = Y(s)$

$$\mathcal{L}\{2y''' + 3y'' - 3y' - 2y\} = \mathcal{L}\{e^{-t}\}$$

$$\Rightarrow 2\left[s^3Y(s) - \cancel{s^2y(0)} - \cancel{sy'(0)} - y''(0)\right] + 3\cdot\left[s^2Y(s) - \cancel{sy(0)} - \cancel{y'(0)}\right]$$

$$-3\cdot\left[sY(s) - \cancel{y(0)}\right] - 2Y(s) = \frac{1}{s+1}$$

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$$\Rightarrow 2s^3 Y(s) - 2 + 3s^2 Y(s) - 3s Y(s) - 2Y(s) = \frac{1}{s+1}$$

$$\Rightarrow (2s^3 + 3s^2 - 3s - 2) Y(s) = \frac{1}{s+1} + 2 = \frac{2s+3}{s+1}.$$

$$\Rightarrow Y(s) = \frac{2s+3}{(s+1)(2s^3+3s^2-3s-2)}.$$

$$= \frac{2s+3}{(s+1)(s-1)(s+2)(2s+1)}$$

$2s+1$

$s+\frac{1}{2}$

$s = -\frac{1}{2}$

$s = 1$

$s = -2$

Now. $$\frac{2s+3}{(s+1)(s-1)(s+2)(2s+1)} = \frac{A}{s+1} + \frac{B}{s-1} + \frac{C}{s+2} + \frac{D}{2s+1}$$

$$2s+3 = A(s-1)(s+2)(2s+1) + B(s+1)(s+2)(2s+1)$$
$$+ C(s+1)(s-1)(2s+1) + D(s+1)(s-1)(s+2)$$

$s=-1$

$1 = A\cdot(-2)(1)(-1)$

$A = \frac{1}{2}.$

$s=1$

$5 = B\cdot 2\cdot 3\cdot 3$

$B = \frac{5}{18}.$

$s=-2$

$-1 = C\cdot(-1)(-3)(-3)$

$C = \frac{1}{9}$

$s=-\frac{1}{2}$

$2 = D\cdot\left(\frac{1}{2}\right)\left(-\frac{3}{2}\right)\left(\frac{3}{2}\right)$

$D = \frac{-16}{9}.$

$$y(t) = \mathcal{L}^{-1}\left\{ \frac{\frac{1}{2}}{s+1} + \frac{\frac{5}{18}}{s-1} + \frac{\frac{1}{9}}{s+2} + \frac{\frac{-16}{9}}{2s+1} \right\}$$

$$= \frac{1}{2}\mathcal{L}^{-1}\left\{\frac{1}{s+1}\right\} + \frac{5}{18}\mathcal{L}^{-1}\left\{\frac{1}{s-1}\right\} + \frac{1}{9}\mathcal{L}^{-1}\left\{\frac{1}{s+2}\right\} - \frac{16}{9\cdot 2}\mathcal{L}^{-1}\left\{\frac{1}{s+\frac{1}{2}}\right\}$$

$$= \frac{1}{2}e^{-t} + \frac{5}{18}e^{t} + \frac{1}{9}e^{-2t} - \frac{8}{9}e^{-\frac{1}{2}t}.$$


Thank you!