Undergraduate Course in Mathematics

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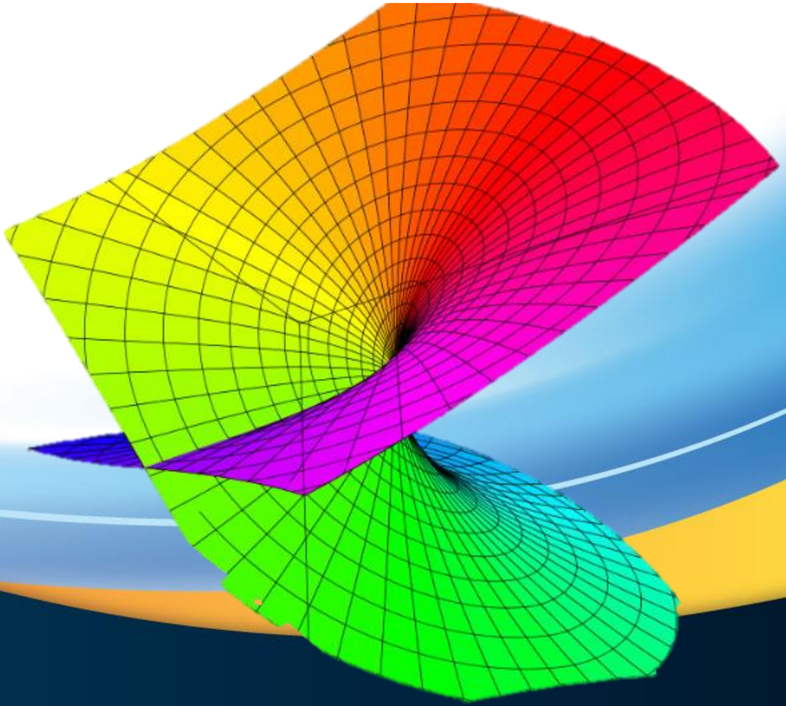
# Complex Variables

## Topic: Cauchy-Goursat Theorem




Conducted By

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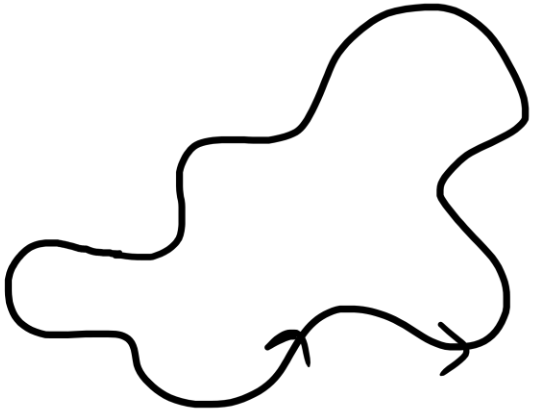
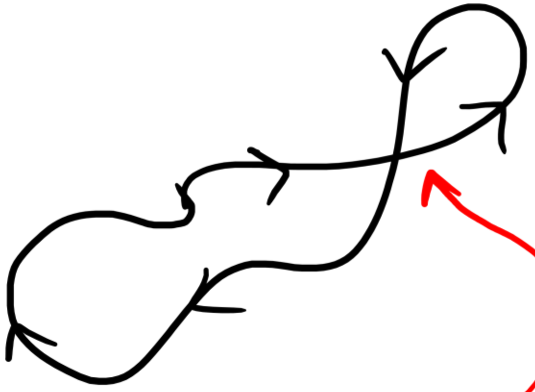
# Simple Closed Contour / Curve

Contour

Closed contour

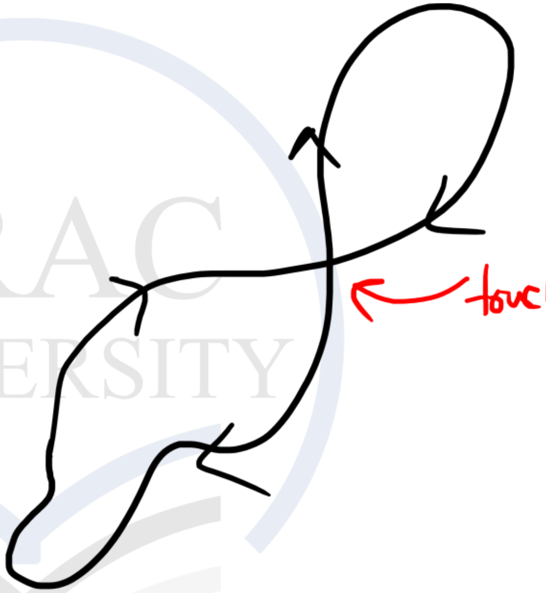


Simple closed contour



not simple

there is a self intersection



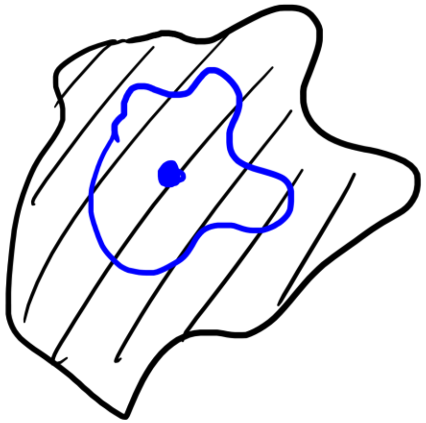
touch

simple closed contour

# Simply and Multiply Connected Regions



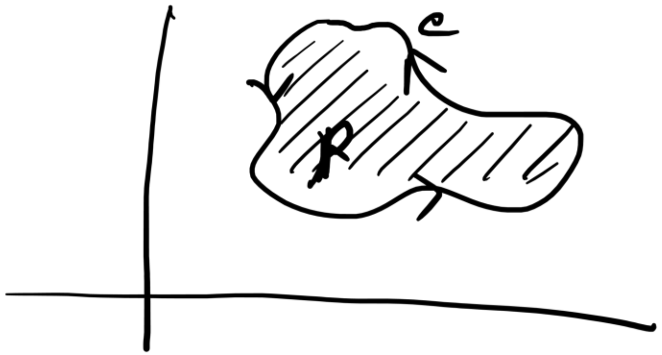
A region $R$ is called simply-connected if any simple closed curve, which lies in $R$, can be shrunk to a point without leaving R. A region $R$, which is not simply-connected, is called multiply connected.



# Cauchy-Goursat Theorem

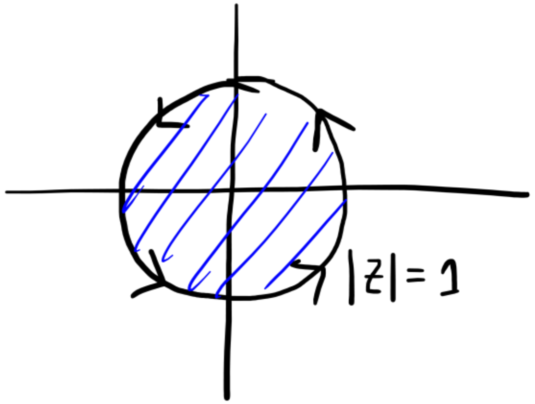
Let $f(z)$ be analytic in a region $R$ and on its boundary $C$. Then

$$\oint_C f(z)dz = 0$$



This fundamental theorem, often called Cauchy's integral theorem or simply Cauchy's theorem, is valid for both simply- and multiply-connected regions. It was first proved by use of Green's theorem with the added restriction that $f'(z)$ be continuous in R. However, Goursat gave a proof which removed this restriction. For this reason, the theorem is sometimes called the Cauchy–Goursat theorem when one desires to emphasize the removal of this restriction.

Evaluate $\oint_C \left(5z^4 - z^3 + 2\right) dz$ around the circle $|z| = 1$.




$f(z) = 5z^4 - z^3 + 2$

There is no singularity inside and on the boundary of $|z| = 1$.

$\Rightarrow f(z)$ is analytic inside and on the boundary of $c: |z| = 1$

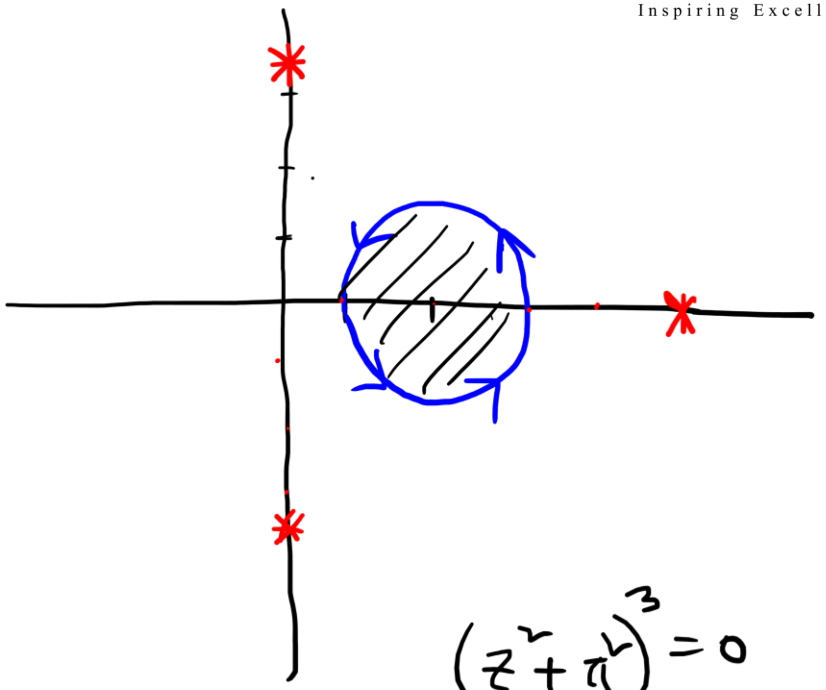
$\Rightarrow \oint_c f(z)\,dz = 0$ (by using cauchy-goursat thm)

$\Rightarrow \oint_c \left(5z^4 - z^3 + 2\right) dz = 0$.

Evaluate

$$\oint_C \frac{e^{3z}\cos(z)}{(z^2+\pi^2)^3(z-5)}\,dz$$

Where $C$ is the circle $|Z-2|=1$



$$f(z)=\frac{e^{3z}\cos(z)}{(z^2+\pi^2)^3\,(z-5)}$$ has singularity at

$$z = 5,\ i\pi,\ -i\pi$$

$(z^2+\pi^2)^3=0$

$z^2+\pi^2=0$

$\Rightarrow z=\pm i\pi$

$\Rightarrow$ $f(z)$ has no singularity inside and on the boundary of $C: |z-2|=1$.

$\Rightarrow f(z)$ is analytic inside and on the boundary of $C$.

$$\oint_C f(z)\,dz = 0$$

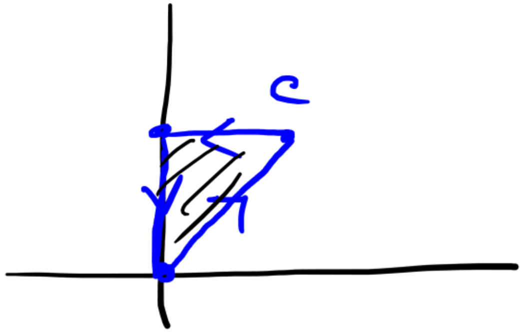
$$\Rightarrow \oint \frac{e^{3z}\cos(z)}{(z^2+\pi^2)^3(z-5)}\,dz = 0 \quad \checkmark$$

Verify the Cauchy-Goursat theorem for

$$\oint_C z^2 dz$$

where $C$ is the boundary of the triangle with vertices $(0,0)$, $(1,1)$ and $(0,1)$.

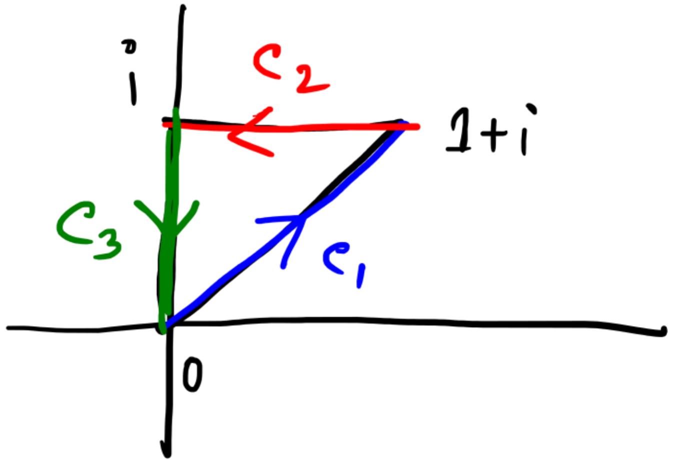
1st Part: $f(z) = z^2$ is analytic inside and on the boundary of the triangular contour $c$.




$$\oint_c f(z)\,dz = 0 \quad \checkmark$$

## 2nd Part :

$$\oint_C z^2 dz = \int_{C_1} z^2 dz + \int_{C_2} z^2 dz + \int_{C_3} z^2 dz$$




for $C_1$ :

$$z(t) = 0 + \{(1+i) - 0\} t = t + it = (t) + i(t)$$

$x = t$

$y = t$

$$dz = (1+i)dt$$

$t \to$ end $= 1$

$t \to$ start $= 0$

$$\int_{C_1} z^2\,dz = \int_0^1 (t+it)^2(1+i)\,dt = (1+i)\int_0^1 (t^2+2it^2-t^2)\,dt$$

$$= (1+i)\int_0^1 2it^2\,dt = (1+i)2i\left[\frac{t^3}{3}\right]_0^1 = (1+i)2i\frac{1}{3}$$

$$= \frac{-2+2i}{3} \quad \checkmark$$

<u>for $C_2$</u>: $z(t) = (1+i) + \{i - (1+i)\}t = 1+i-t = (1-t) + (1)i$

$dz = -dt$

$x = 1-t$

$y = 1$

$$\int_{C_2} z^2 dz = \int_0^1 [(1-t)+i]^2 (-dt)$$

$$= \int_0^1 [\cancel{1} + t^2 + \cancel{i^2} - 2t - 2it + 2i](-dt)$$

$$= \int_0^1 (t^2 - 2t - 2it + 2i)\,dt$$

$$= \int_0^1 (-t^2 + 2t + 2it - 2i)\,dt$$

$$= \left[ -\frac{t^3}{3} + t^2 + it^2 - 2it \right]_0^1$$

$$= -\frac{1}{3} + 1 + i - 2i$$

$$= \frac{2}{3} - i$$

for $c_3$:  $z(t) = i + (0-i)t = i - it = 0 + (1-t)i$

$dz = -i\,dt$

$x = 0$

$y = 1-t$

$$\int_{c_3} z^2 dz = \int_0^1 (1-t)^2 i^2 (-i\,dt)$$

$$= i\int_0^1 (1-2t+t^2)\,dt$$

$$= i\left[t - t^2 + \frac{t^3}{3}\right]_0^1$$

$$= i\left(1-1+\frac{1}{3}\right)$$

$$= \frac{i}{3}$$

$$\oint_C z^2\,dz = \int_{C_1} z^2\,dz + \int_{C_2} z^2\,dt + \int_{C_3} z^2\,dt = \frac{-2+2i}{3} + \frac{2}{3} - i + \frac{i}{3}$$

$$= 0$$

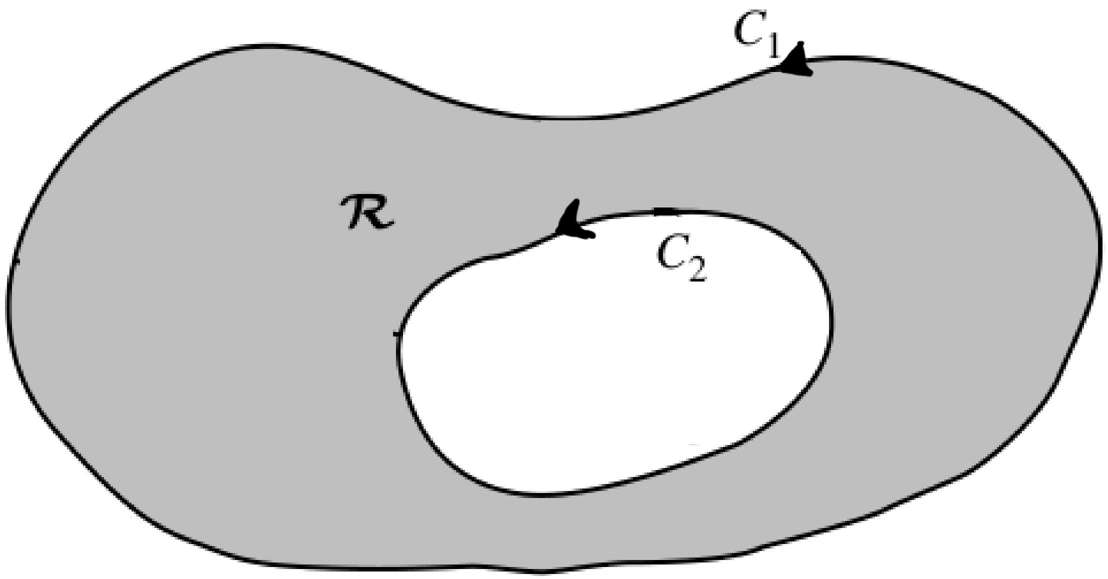
$\therefore$ Cauchy-Goursat Thm is verified

# Consequences of Cauchy's Theorem

Let $f(z)$ be analytic in a region $R$ bounded by two simple closed curves $C_1$ and $C_2$ and also on $C_1$ and $C_2$. Then

$$\oint_{C_1} f(z)dz = \oint_{C_2} f(z)dz$$

where $C_1$ and $C_2$ are both traversed in the positive sense.

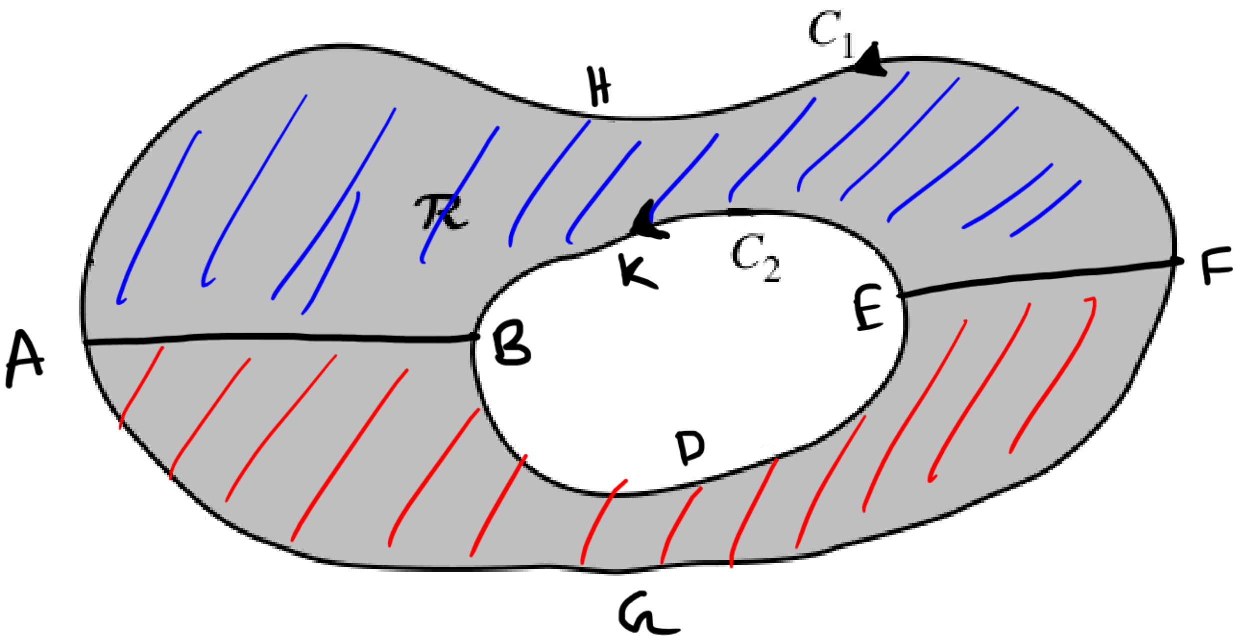


***Proof:***

$$\oint_{AGFEDBA} f(z)\,dz = 0 \qquad \oint_{ABKEFHA} f(z)\,dz = 0$$

$$\int_{AGF} + \int_{FE} + \int_{EDB} + \int_{BA} = 0$$

$$\int_{AB} + \int_{BKE} + \int_{EF} + \int_{FHA} = 0$$



$$\int_{AGF} + \cancel{\int_{FE}} + \int_{EDB} + \cancel{\int_{BA}} + \cancel{\int_{AB}} + \int_{BKE} + \cancel{\int_{EF}} + \int_{FHA} = 0$$

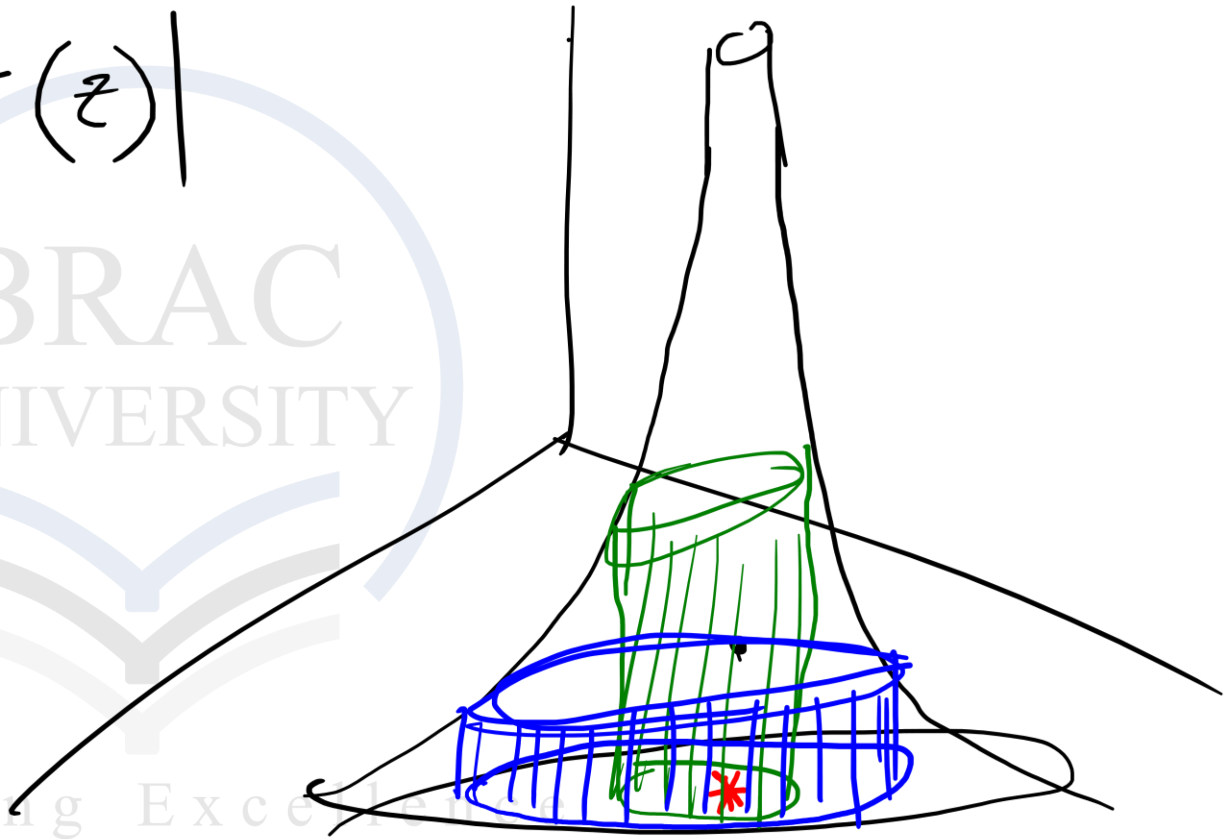
$$\Rightarrow \int_{AGF} + \int_{FHA} = -\int_{EDB} - \int_{BKE}$$

$$\Rightarrow \oint_{C_1} = \int_{BDE} + \int_{EKB} = \oint_{C_2}$$

$$\therefore \oint_{C_1} f(z)\,dz = \oint_{C_2} f(z)\,dz$$

$$\Gamma(n) = (n-1)!$$

$$|f(z)|$$



$$\Gamma(n) = \int_0^\infty x^{n-1} e^{-x}\, dx$$

Evaluate

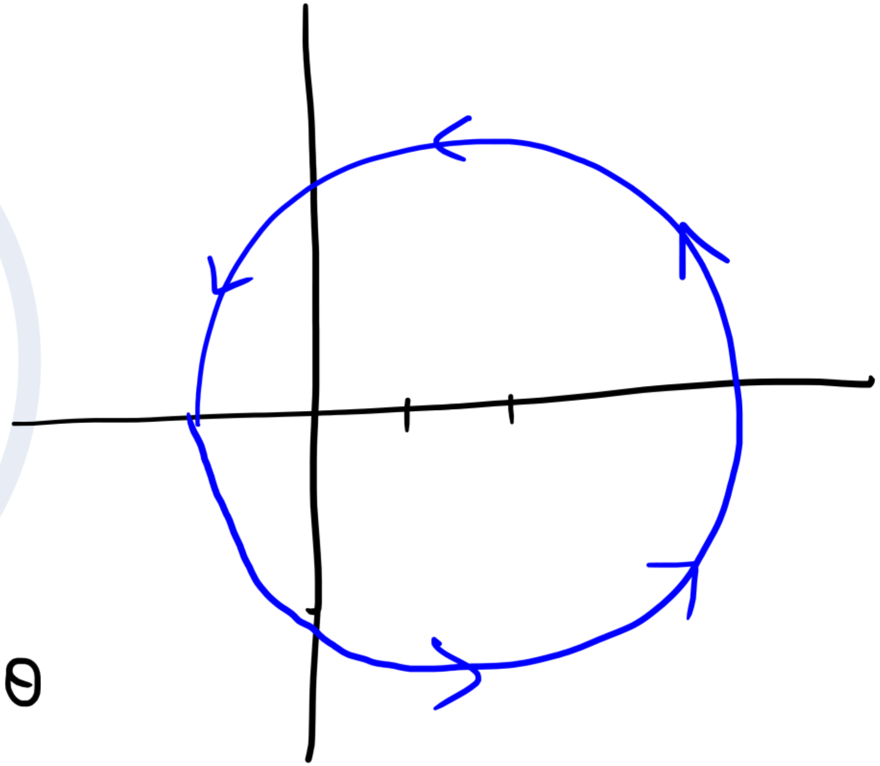
$$\oint_C \frac{z^2}{z-2}\,dz$$

Where $C$ is the circle $|Z-2|=3$

$$\Rightarrow z-2 = 3e^{i\theta}$$

$$\Rightarrow dz = i\,3e^{i\theta}\,d\theta$$

$$\oint \frac{z^2}{z-2}dz = \int_0^{2\pi} \frac{(2+3e^{i\theta})^2}{\cancel{3e^{i\theta}}} \cdot i\,\cancel{3e^{i\theta}}\,d\theta$$



$$= i\int_0^{2\pi}\left[4+12e^{i\theta}+9e^{2i\theta}\right)d\theta$$

$e^{2\pi i} = \cos 2\pi + i\sin 2\pi$

$= 1 + i0 = 1$

$$= i\left[4\theta+\frac{12}{i}e^{i\theta}+\frac{9}{2i}e^{2i\theta}\right)_0^{2\pi}$$

$$= i\left[8\pi+\cancel{\frac{12}{i}\underbrace{e^{2\pi i}}_{1}}+\cancel{\frac{9}{2i}\underbrace{e^{4\pi i}}_{1}}\right) - i\left[0+\cancel{\frac{12}{i}(1)}+\cancel{\frac{9}{2i}(1)}\right]$$

$= 8\pi i$ ✓

Evaluate

$$\oint_C \frac{z^2}{z-1}\,dz$$

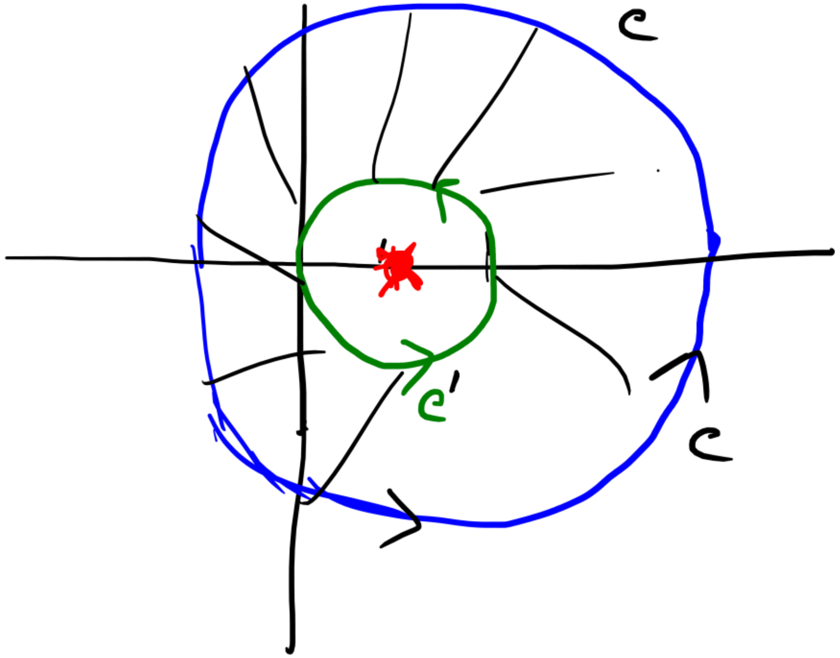
Where $C$ is the circle $|Z-2| = 3$

$f(z) = \dfrac{z^2}{z-1}$, has a singularity at $z=1$

$c: |z-2| = 3$ $\quad$ $c': |z-1| = 1$

$f(z)$ is analytic in between $c$ and $c'$

$\therefore \oint_c f(z)\,dz = \oint_{c'} f(z)\,dz$



$$c' : |z-1| = 1 \Rightarrow z - 1 = e^{i\theta}$$

$$dz = ie^{i\theta}$$

$$\oint_{c'} f(z)\, dz = \oint_{c'} \frac{z^2}{z-1} dz = \int_0^{2\pi} \frac{(1+e^{i\theta})^2}{\cancel{e^{i\theta}}} i \cancel{e^{i\theta}} d\theta$$

$$= i \int_0^{2\pi} \left[ 1 + 2e^{i\theta} + e^{2i\theta} \right) d\theta$$

$$= i\left[\theta + \frac{2e^{i\theta}}{i} + \frac{e^{2i\theta}}{2i}\right]_0^{2\pi}$$

$$= i\left[2\pi + \cancel{\frac{2e^{2\pi i}}{i}} + \cancel{\frac{e^{4\pi i}}{2i}}\right] - i\left[0 + \cancel{\frac{2e^0}{i}} + \cancel{\frac{e^0}{2i}}\right]$$

$$= 2\pi i \quad \checkmark$$

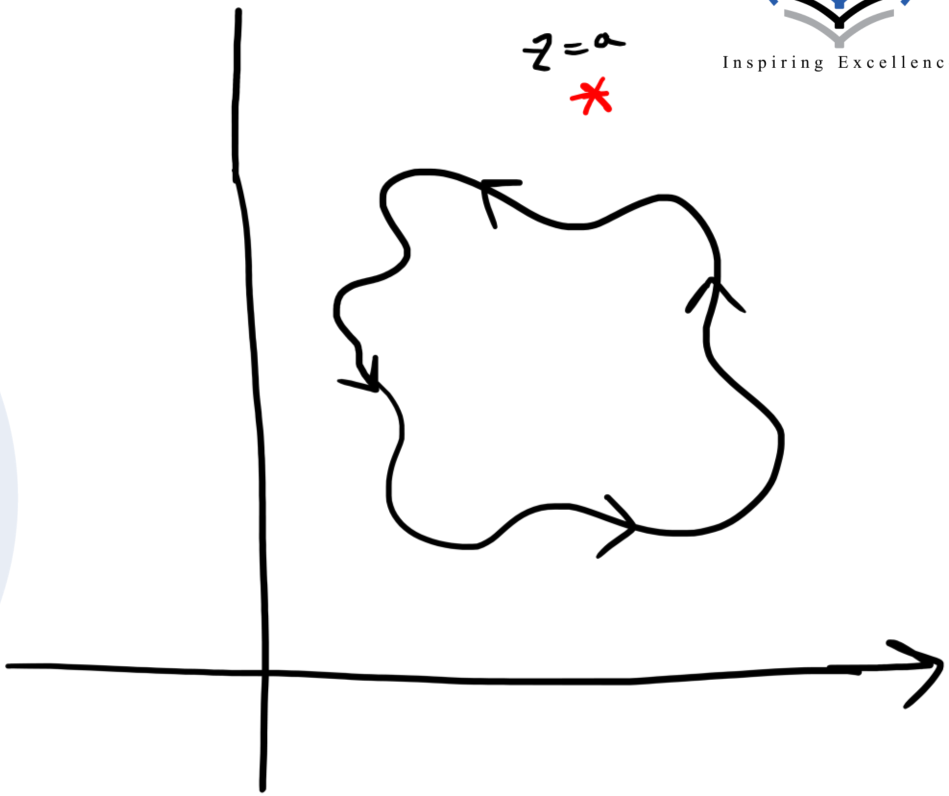
Evaluate

$$\oint_C \frac{1}{z-a}\,dz$$

Where $C$ is any simple closed curve and

*i)* $z = a$ is outside $C$

*ii)* $z = a$ is inside $C$




i) $f(z) = \dfrac{1}{z-a}$.

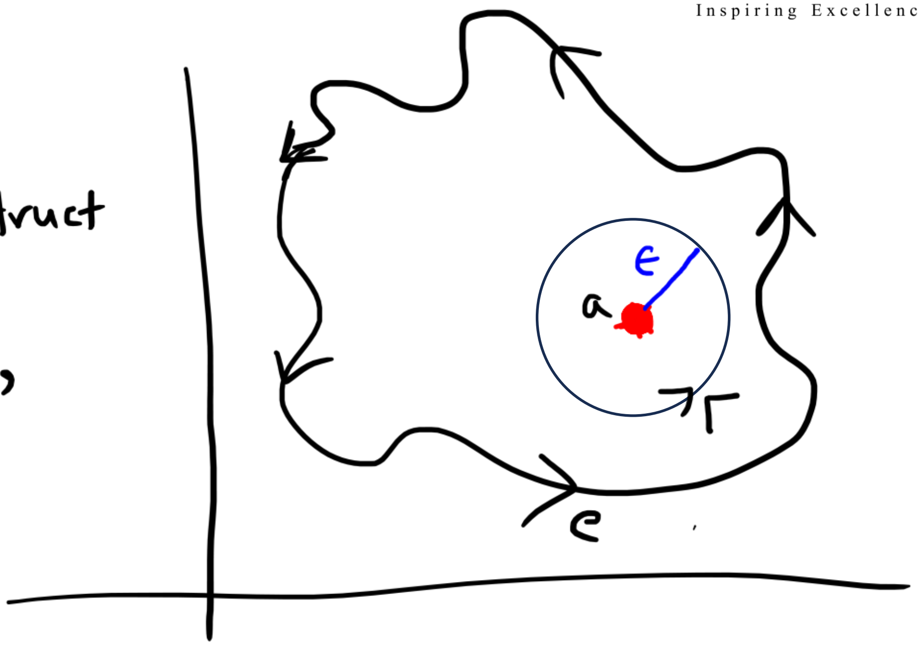
$f(z)$ has the only singularity at $z=a$ which is outside of $C$.

$\Rightarrow f(z)$ is analytic inside and on the boundary of $C$.

$\oint_C f(z)\,dz = 0$ [using Cauchy-Goursat Thm]

ii) $f(z) = \dfrac{1}{z-a}$

Since $z=a$ is an interior point, we can construct a circle centered at $z=a$ with radius $\epsilon$, sufficiently small, such that the circle lies entirely inside $C$.




$\Gamma : |z-a| = \epsilon$

$\Rightarrow z-a = \epsilon e^{i\theta}$

$dz = i\epsilon e^{i\theta}$

Now $f(z)$ is analytic in between $C$ and $\Gamma$.

$$\Rightarrow \oint_C f(z)\,dz = \oint_\Gamma f(z)\,dz$$

$$\oint_{\Gamma} \frac{1}{z-a}\,dz$$

$$= \int_0^{2\pi} \frac{1}{\cancel{\epsilon e^{i\theta}}} \cdot i\cancel{\epsilon e^{i\theta}}\,d\theta$$

$$= i\int_0^{2\pi} d\theta$$

$$= i\left[\theta\right]_0^{2\pi}$$

$$= 2\pi i$$

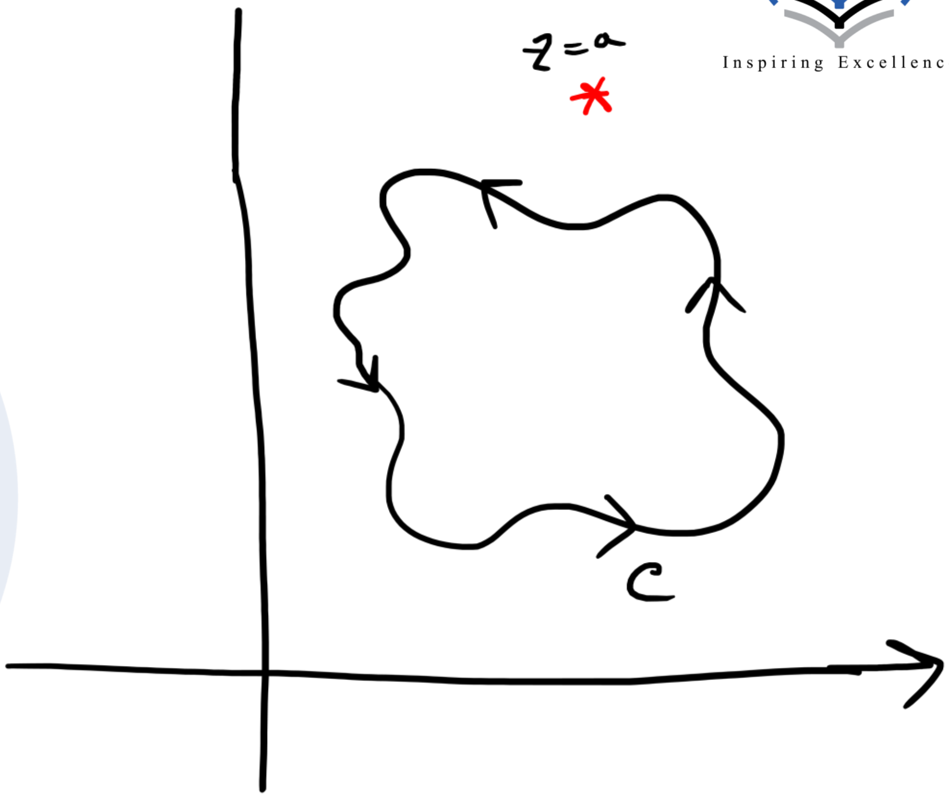
Evaluate

$$\oint_C \frac{1}{(z-a)^n}\, dz$$

Where $n \in \mathbb{N}$, $C$ is any simple closed curve and

*i)* $z = a$ is outside $C$

*ii)* $z = a$ is inside $C$




i) $f(z) = \frac{1}{(z-a)^n}$

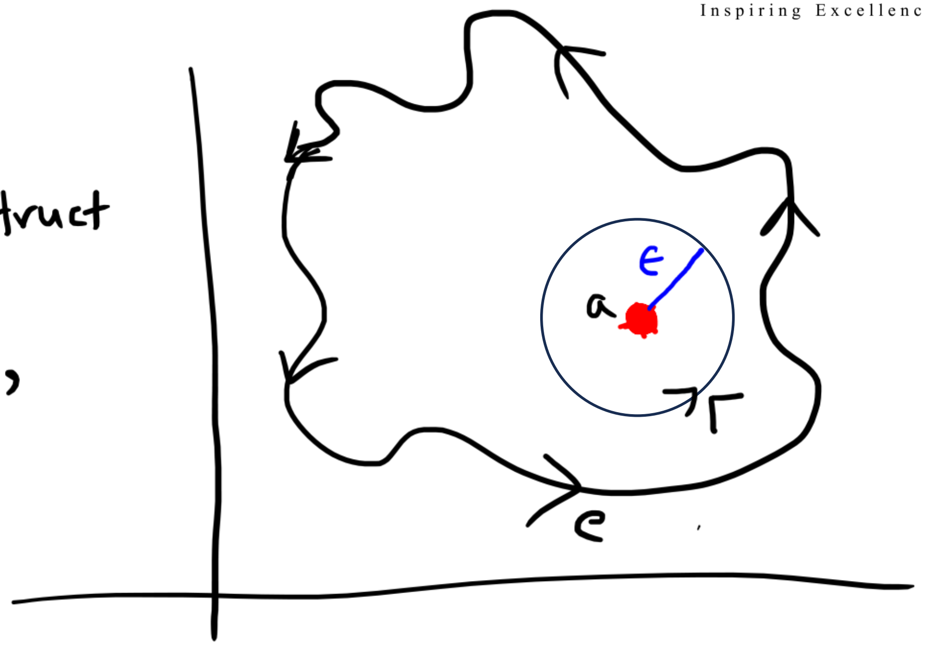
$f(z)$ has the only singularity at $z = a$ which is outside of $C$.

$\Rightarrow f(z)$ is analytic inside and on the boundary of $C$.

$\oint_C f(z)\,dz = 0$ [using Cauchy-Goursat Thm]

ii) $f(z) = \dfrac{1}{(z-a)^n}$

Since $z=a$ is an interior point, we can construct a circle centered at $z=a$ with radius $\epsilon$, sufficiently small, such that the circle lies entirely inside $C$.




$\Gamma : |z-a| = \epsilon$

$\Rightarrow z-a = \epsilon e^{i\theta}$

$dz = i\epsilon e^{i\theta}$

Now $f(z)$ is analytic in between $C$ and $\Gamma$.

$$\Rightarrow \oint_C f(z)\,dz = \oint_\Gamma f(z)\,dz$$

$$\oint_\Gamma \frac{1}{(z-a)^n}\,dz$$

$$= \int_0^{2\pi} \frac{1}{(\epsilon e^{i\theta})^n}\, i\epsilon e^{i\theta}\,d\theta$$

$$= i \int_0^{2\pi} \epsilon^{1-n} \cdot e^{i\theta(1-n)}\,d\theta$$

for $n=1$

$$\oint_\Gamma f(z)\,dz = i\int_0^{2\pi} \epsilon^0 e^0\,d\theta = 2\pi i$$

for $n>1$:

$$\oint_\Gamma f(z)\,dz = i\epsilon^{1-n} \cdot \left[\frac{e^{i\theta(1-n)}}{i(1-n)}\right]_0^{2\pi}$$

$$= i\epsilon^{1-n} \cdot \left[ \frac{e^{\overbrace{i(1-n)\cdot 2\pi}^{1}}}{i(1-n)} - \frac{e^{0}}{i(1-n)} \right]$$

$$= i\epsilon^{1-n} \cdot 0$$

$$= 0$$

$$\oint \frac{1}{(z-a)^n}\,dz = \begin{cases} 2\pi i, & n = 1 \\ 0, & n > 1 \end{cases}$$

inside

Evaluate

$$\oint_C \frac{1}{(z-a)^{n+1}}\,dz$$

Where $n = 0, 1, 2, 3 \ldots$, $C$ is any simple closed curve and

*i)* $z = a$ is outside $C$ $= 0$

*ii)* $z = a$ is inside $C$

$$\oint \frac{1}{(z-a)^{n+1}}\,dz = \begin{cases} 2\pi i, & n = 0 \\ 0, & n > 1 \end{cases}$$

$$\oint_c \frac{1}{(z-a)^{-6}}\,dz = \oint_c \underbrace{(z-a)^6}\,dz = 0$$

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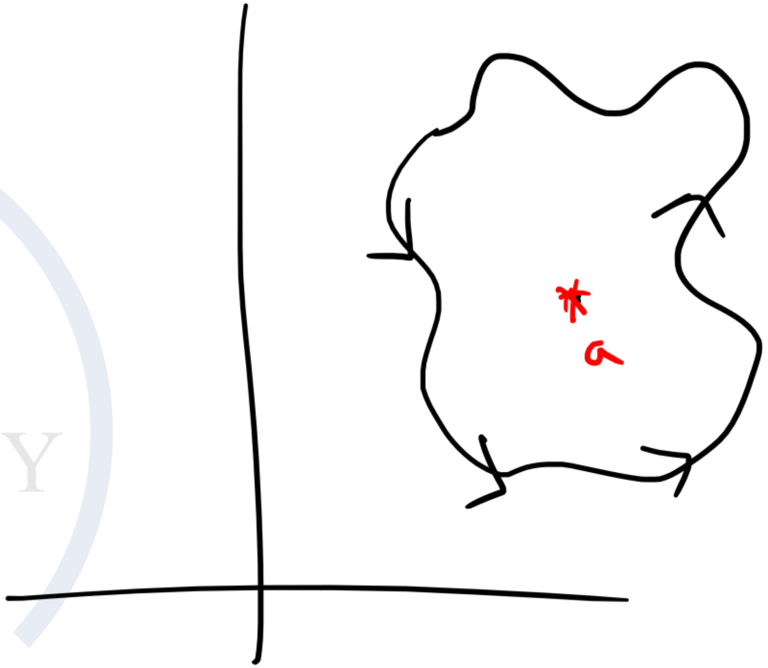
v.v.g

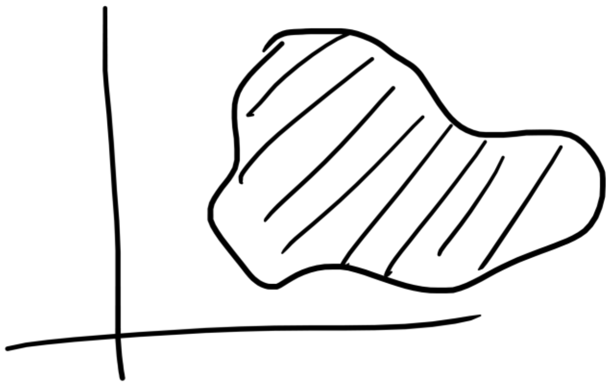
$$\oint \frac{1}{(z-a)^{n+1}}\,dz = \begin{cases} 2\pi i, & n=0 \\ 0 & \text{any other integer.} \end{cases}$$



# Morera's Theorem

Let $f(z)$ be continuous in a simply-connected region $R$ and suppose that

$$\oint_C f(z)dz = 0$$



around every simple closed curve $C$ in $R$. **Then $f(z)$ is analytic in $R$**. This theorem, due to Morera, is often called the converse of Cauchy's theorem. It can be extended to multiply-connected regions.

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