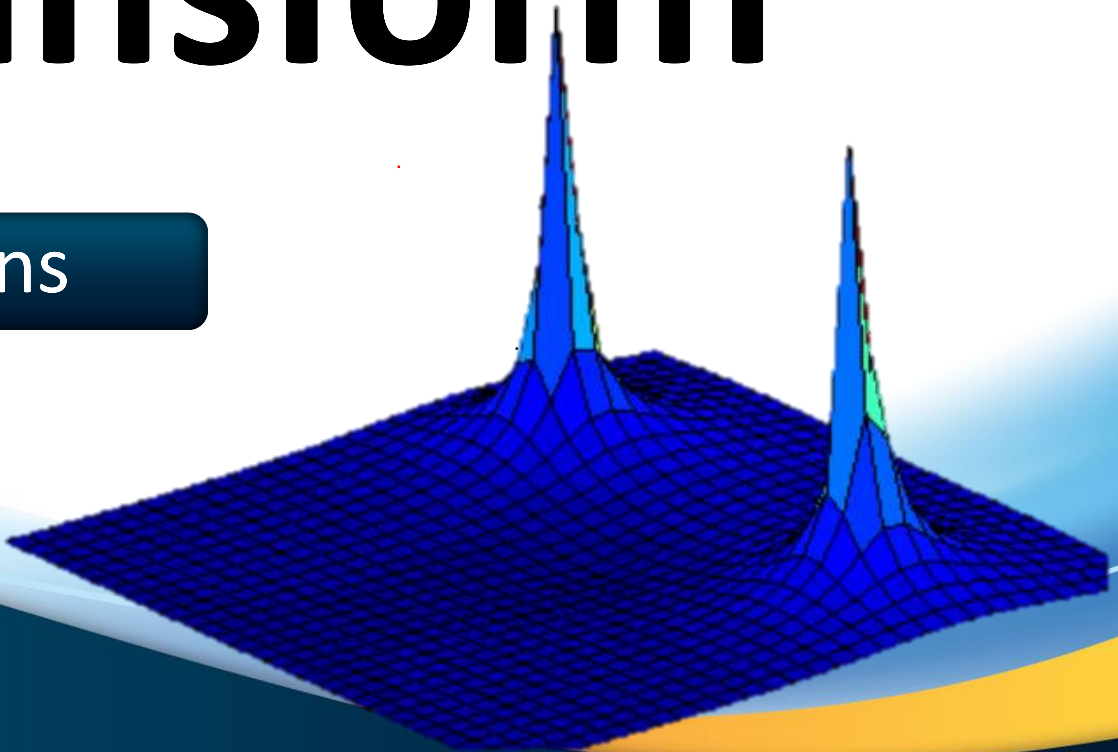
Undergraduate Course in Mathematics

Laplace Transform

Inverse Laplace Transformations



Conducted By

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Inverse Laplace Transform

Let $F(s)$ be the Laplace Transform of some unknown function $f(t)$ defined for $t \geq 0$. Then

$$f(t) = \mathcal{L}^{-1}\{F(s)\} = \frac{1}{2\pi i} \lim_{T\to\infty} \int_{\gamma-iT}^{\gamma+iT} e^{st} F(s)ds$$

where the integration is done along the vertical line $Re(s) = \gamma$ in the complex plane such that γ is greater than the real part of all singularities of $F(s)$ and $F(s)$ is bounded on the line, for example if the contour path is in the region of convergence.

$$\int \cos x \, dx = \sin x \qquad \frac{d}{dx}(\sin x) = \cos x$$

$$\int (2x)\, dx = x^2 + (\text{Constant})$$

Inverse Laplace Transforms of some Algebraic functions

- $\mathcal{L}\{1\} = \frac{1}{s}$
- $\mathcal{L}^{-1}\left\{\frac{1}{s}\right\} = 1$
- $\mathcal{L}\{t\} = \frac{1}{s^2}$
- $\mathcal{L}^{-1}\left\{\frac{1}{s^2}\right\} = t$
- $\mathcal{L}\{t^n\} = \frac{n!}{s^{n+1}}$, n is non-negative integer
- $\mathcal{L}^{-1}\left\{\frac{n!}{s^{n+1}}\right\} = t^n$
- $\mathcal{L}\{t^n\} = \frac{\Gamma(n+1)}{s^{n+1}}$
- $\mathcal{L}^{-1}\left\{\frac{\Gamma(n+1)}{s^{n+1}}\right\} = t^n$
- $\mathcal{L}\{e^{at}\} = \frac{1}{s-a}$
- $\mathcal{L}^{-1}\left\{\frac{1}{s-a}\right\} = e^{at}$

Transforms of some Trigonometric and Hyperbolic functions

- $\mathcal{L}\{\sin at\} = \frac{a}{s^2+a^2}$
- $\mathcal{L}\{\cos at\} = \frac{s}{s^2+a^2}$
- $\mathcal{L}\{\sinh at\} = \frac{a}{s^2-a^2}$
- $\mathcal{L}\{\cosh at\} = \frac{s}{s^2-a^2}$

- $\mathcal{L}^{-1}\left\{\frac{a}{s^2+a^2}\right\} = \sin at$
- $\mathcal{L}^{-1}\left\{\frac{s}{s^2+a^2}\right\} = \cos at$
- $\mathcal{L}^{-1}\left\{\frac{a}{s^2-a^2}\right\} = \sinh at$
- $\mathcal{L}^{-1}\left\{\frac{s}{s^2-a^2}\right\} = \cosh at$

Linearity of Inverse Laplace Transformation

- $\mathcal{L}^{-1}\{ F(s) \pm G(s) \} = \mathcal{L}^{-1}\{ F(s)\} \pm \mathcal{L}^{-1}\{ G(s)\}$

- $\mathcal{L}^{-1}\{k \cdot F(s)\} = k \cdot \mathcal{L}^{-1}\{F(s)\}$

Evaluate

$$\mathcal{L}^{-1}\left\{\frac{1}{s^4}\right\}$$

$$= \mathcal{L}^{-1}\left\{\frac{1}{s^{3+1}}\right\}$$

$$= \frac{1}{3!}\,\mathcal{L}^{-1}\left\{\frac{3!}{s^{3+1}}\right\}$$

$$= \frac{1}{6}\,t^3$$

Evaluate

$$\mathcal{L}^{-1}\left\{\frac{12}{4-3s}\right\}$$

$$= 12 \cdot \mathcal{L}^{-1}\left\{\frac{1}{4-3s}\right\}$$

$$= -12\,\mathcal{L}^{-1}\left\{\frac{1}{3s-4}\right\}$$

$$= \frac{-12}{3}\,\mathcal{L}^{-1}\left\{\frac{1}{s-\frac{4}{3}}\right\}$$

$$= \frac{-12}{3} \cdot e^{\frac{4}{3}t}.$$

$$= -4e^{\frac{4}{3}t}.$$

Evaluate

$$\mathcal{L}^{-1}\left\{\frac{5}{s^2+9}\right\}$$

$$= 5 \cdot \mathcal{L}^{-1}\left\{\frac{1}{s^2+9}\right\}$$

$$= \frac{5}{3}\,\mathcal{L}^{-1}\left\{\frac{3}{s^2+3^2}\right\}$$

$$= \frac{5}{3}\sin 3t$$

Evaluate

$$\mathcal{L}^{-1}\left\{\frac{23s-15}{s^2+5}\right\}$$

$$= \mathcal{L}^{-1}\left\{\frac{23s}{s^2+5} - \frac{15}{s^2+5}\right\}$$

$$= \mathcal{L}^{-1}\left\{\frac{23s}{s^2+5}\right\} - \mathcal{L}^{-1}\left\{\frac{15}{s^2+5}\right\}$$

$$= 23 \cdot \mathcal{L}^{-1}\left\{\frac{s}{s^2+5}\right\} - 15\,\mathcal{L}^{-1}\left\{\frac{1}{s^2+5}\right\}$$

$$= 23 \cdot \mathcal{L}^{-1}\left\{\frac{s}{s^2+(\sqrt{5})^2}\right\} - \frac{15}{\sqrt{5}}\,\mathcal{L}^{-1}\left\{\frac{\sqrt{5}}{s^2+(\sqrt{5})^2}\right\}$$

$$= 23 \cdot \cos(\sqrt{5}\,t) - \frac{15}{\sqrt{5}} \cdot \sin(\sqrt{5}\,t)$$

Ans.

Evaluate

$$\mathcal{L}^{-1}\left\{\frac{2s-5}{s^2-9}\right\}$$

$$= \mathcal{L}^{-1}\left\{\frac{2s}{s^2-9}\right\} - \mathcal{L}^{-1}\left\{\frac{5}{s^2-9}\right\}$$

$$= 2\cdot \mathcal{L}^{-1}\left\{\frac{s}{s^2-3^2}\right\} - \frac{5}{3}\,\mathcal{L}^{-1}\left\{\frac{3}{s^2-3^2}\right\}$$

$$= 2\cdot \cosh(3t) - \frac{5}{3}\cdot \sinh(3t)$$

First Translation Theorem

If $\mathcal{L}\{\, f(t)\} = F(s)$ and a is any real number, then

$$\mathcal{L}\{e^{at}\, f(t)\} = \mathcal{L}\{f(t)\}\Big|_{s\to s-a} = F(s-a)$$

If $\mathcal{L}^{-1}\{\, F(s)\} = f(t)$ and a is any real number, then

$$\mathcal{L}^{-1}\{\, F(s)\} = e^{at} \cdot \mathcal{L}^{-1}\{\, F(s+a)\}$$

Find the Inverse Laplace Transform of

$$\mathcal{L}^{-1}\left\{\frac{6}{(s-2)^6}\right\}$$

$$= e^{2t} \cdot \mathcal{L}^{-1}\left\{\frac{6}{(s+2-2)^6}\right\}$$

$$= e^{2t} \cdot \mathcal{L}^{-1}\left\{\frac{6}{s^6}\right\}$$

$$= e^{2t} \cdot 6 \cdot \mathcal{L}^{-1}\left\{\frac{1}{s^6}\right\}$$

$$= e^{2t} \cdot \frac{6}{5!} \mathcal{L}^{-1}\left\{\frac{5!}{s^{5+1}}\right\}$$

$$= e^{2t} \cdot \frac{6}{120} \cdot t^5$$

$$= \frac{1}{20} t^5 \cdot e^{2t} \cdot$$

Find the Inverse Laplace Transform of

$$\mathcal{L}^{-1}\left\{\frac{5}{(s-3)^2+15}\right\}$$

$$= e^{3t}\,\mathcal{L}^{-1}\left\{\frac{5}{(s+3-3)^2+15}\right\}$$

$$= e^{3t}\cdot\mathcal{L}^{-1}\left\{\frac{5}{s^2+15}\right\}$$

$$= e^{3t}\cdot\frac{5}{\sqrt{15}}\cdot\mathcal{L}^{-1}\left\{\frac{\sqrt{15}}{s^2+(\sqrt{15})^2}\right\}$$

$$= \frac{5}{\sqrt{15}}e^{3t}\cdot\sin(\sqrt{15}\,t)$$

Ans.

Find the Inverse Laplace Transform of

$$\mathcal{L}^{-1}\left\{\frac{4s}{(s-3)^2+25}\right\}$$

$$= e^{3t}\cdot \mathcal{L}^{-1}\left\{\frac{4(s+3)}{(s+3-3)^2+25}\right\}$$

$$= e^{3t}\cdot \mathcal{L}^{-1}\left\{\frac{4s+12}{s^2+25}\right\}$$

$$= e^{3t}\cdot \mathcal{L}^{-1}\left\{\frac{4s}{s^2+5^2}\right\} + e^{3t}\cdot \mathcal{L}^{-1}\left\{\frac{12}{s^2+5^2}\right\}$$

$$= e^{3t} \cdot 4\, \mathcal{L}^{-1}\left\{\frac{s}{s^2+5^2}\right\} + e^{3t} \cdot \frac{12}{3} \cdot \mathcal{L}^{-1}\left\{\frac{3}{s^2+5^2}\right\}$$

$$= e^{3t} \cdot 4 \cdot \cos(5t) + e^{3t} \cdot 4 \cdot \sin(5t)$$

$$= 4e^{3t}\left(\cos 5t + \sin 5t\right)$$

Find the Inverse Laplace Transform of

$$\mathcal{L}^{-1}\left\{\frac{5s}{s^2+2s+5}\right\}$$

$s \longrightarrow s-1$

$$= \mathcal{L}^{-1}\left\{\frac{5s}{(s^2+2s+1)+4}\right\}$$

$$= \mathcal{L}^{-1}\left\{\frac{5s}{(s+1)^2+2^2}\right\}$$

$$= e^{-1\cdot t}\,\mathcal{L}^{-1}\left\{\frac{5(s-1)}{(s-1+1)^2+2^2}\right\}$$

$$= e^{-t}\cdot\mathcal{L}^{-1}\left\{\frac{5s-5}{s^2+2^2}\right\}$$

$$= e^{-t} \cdot \mathcal{L}^{-1}\left\{\frac{5s}{s^2+2^2}\right\} - e^{-t} \cdot \mathcal{L}^{-1}\left\{\frac{5}{s^2+2^2}\right\}$$

$$= 5e^{-t} \cdot \mathcal{L}^{-1}\left\{\frac{s}{s^2+2^2}\right\} - \frac{5}{2}e^{-t} \cdot \mathcal{L}^{-1}\left\{\frac{5}{s^2+2^2}\right\}$$

$$= 5e^{-t} \cdot \cos(2t) - \frac{5}{2}e^{-t} \cdot \sin(2t)$$

Find the Inverse Laplace Transform of

$$\mathcal{L}^{-1}\left\{\frac{6s-4}{s^2-8s-9}\right\}$$

$$= \mathcal{L}^{-1}\left\{\frac{6s-4}{s^2-2\cdot s\cdot 4\cdot+16-25}\right\}$$

$$= \mathcal{L}^{-1}\left\{\frac{6s-4}{(s-4)^2-5^2}\right\}$$

$$= e^{4t}\cdot\mathcal{L}^{-1}\left\{\frac{6(s+4)-4}{s^2-5^2}\right\}$$

$$= e^{4t}\cdot\mathcal{L}^{-1}\left\{\frac{6s+20}{s^2-5^2}\right\}$$

$$= 6\cdot e^{4t}\cdot\mathcal{L}^{-1}\left\{\frac{s}{s^2-5^2}\right\}+\frac{20}{5}e^{4t}\cdot\mathcal{L}^{-1}\left\{\frac{5}{s^2-5^2}\right\}$$

$$= 6e^{4t}\cdot\cosh(5t)+4\cdot e^{4t}\cdot\sinh(5t)$$

Second Translation Theorem

If $\mathcal{L}\{ f(t)\} = F(s)$ and a is any real number, then

$$\mathcal{L}\{ f(t) \cdot u(t-a)\} = \mathcal{L}\{f(t+a)\} \cdot e^{-as}$$

If$\mathcal{L}^{-1}\{ F(s)\} = f(t)$ and a is any real number, then

$$\mathcal{L}^{-1}\{ F(s)e^{-as}\} = f(t-a) \cdot u(t-a)$$

$$
\begin{tikzcd}
\boxed{f(t)}_{\text{original}} \arrow[rr, leftrightarrow] \arrow[d, leftrightarrow] & & F(s) \arrow[d, leftrightarrow] \\
\underline{f(t-a)} \arrow[rr, leftrightarrow, "\text{No}", "\text{connetin}"'] & & \underline{F(s-a)}
\end{tikzcd}
$$

Trigonometric Identity

$$\sin(\boldsymbol{\theta} \pm \boldsymbol{odd} \cdot \boldsymbol{\pi}) = -\boldsymbol{sin}(\boldsymbol{\theta})$$
$$\sin(\boldsymbol{\theta} \pm \boldsymbol{even} \cdot \boldsymbol{\pi}) = \boldsymbol{sin}(\boldsymbol{\theta})$$

$$\cos(\boldsymbol{\theta} \pm \boldsymbol{odd} \cdot \boldsymbol{\pi}) = -\boldsymbol{cos}(\boldsymbol{\theta})$$
$$\cos(\boldsymbol{\theta} \pm \boldsymbol{even} \cdot \boldsymbol{\pi}) = \boldsymbol{cos}(\boldsymbol{\theta})$$

মুখস্থবিদ্যা প্রতিভাকে ধ্বংস করে কিন্তু সফলতাকে ত্বরান্বিত করে।

Evaluate

$$\mathcal{L}^{-1}\left\{\frac{1}{s^4}e^{-2s}\right\}$$

Now. $F(s) = \dfrac{1}{s^4}$

$$f(t) = \mathcal{L}^{-1}\left\{\frac{1}{s^4}\right\}$$

$$= \frac{1}{3!}\mathcal{L}^{-1}\left\{\frac{3!}{s^{3+1}}\right\}$$

$\therefore f(t) = \frac{1}{6}t^3.$

$a = 2.$

Now, $\mathcal{L}^{-1}\left\{\frac{1}{s^4}e^{-2s}\right\} = f(t-2)\cdot u(t-2)$

$$= \left[\frac{1}{6}(t-2)^3\right]\cdot u(t-2)$$

Evaluate

$$\mathcal{L}^{-1}\left\{\frac{1}{s-3}e^{-2s}\right\}$$

$F(s) = \frac{1}{s-3}$.

$f(t) = \mathcal{L}^{-1}\left\{\frac{1}{s-3}\right\}$

$= e^{3t}$

$\mathcal{L}^{-1}\left\{\frac{1}{s-3}e^{-2s}\right\}$

$= f(t-2)\cdot u(t-2)$

$= e^{3(t-2)}\cdot u(t-2)$

Evaluate

$$\mathcal{L}^{-1}\left\{\frac{1}{s^2+4}e^{-\pi s}\right\}$$

$$F(s) = \frac{1}{s^2+4}$$

$$f(t) = \mathcal{L}^{-1}\left\{\frac{1}{s^2+4}\right\}$$

$$= \frac{1}{2}\mathcal{L}^{-1}\left\{\frac{2}{s^2+2^2}\right\}$$

$$= \frac{1}{2}\sin(2t)$$

$$\therefore \mathcal{L}^{-1}\left\{\frac{1}{s^2+4}e^{-\pi s}\right\}$$

$$= f(t-\pi)\,u(t-\pi)$$

$$= \frac{1}{2}\sin\left(2(t-\pi)\right)\cdot u(t-\pi)$$

$$= \frac{1}{2}\sin(2t-2\pi)\cdot u(t-\pi) = \frac{1}{2}\sin(2t)\cdot u(t-\pi)$$

Evaluate

$$\mathcal{L}^{-1}\left\{\frac{s}{s^2+9}e^{-\pi s}\right\}$$

$$F(s) = \frac{s}{s^2+9}$$

$$f(t) = \mathcal{L}^{-1}\left\{\frac{s}{s^2+9}\right\}$$

$$= \cos(3t)$$

$$\mathcal{L}^{-1}\left\{\frac{s}{s^2+9}e^{-\pi s}\right\}$$

$$= f(t-\pi)\cdot u(t-\pi)$$

$$= \cos(3(t-\pi))\, u(t-\pi)$$

$$= \cos(3t-3\pi)\cdot u(t-\pi) = -\cos(3t)\cdot u(t-\pi)$$

Evaluate

$$\mathcal{L}^{-1}\left\{\frac{6s-3}{s^2+4}e^{-\pi s}\right\}$$

$$F(s) = \frac{6s-3}{s^2+4}$$

$$f(t) = \mathcal{L}^{-1}\left\{\frac{6s-3}{s^2+4}\right\}$$

$$= 6\,\mathcal{L}^{-1}\left\{\frac{s}{s^2+4}\right\} - \frac{3}{2}\cdot\mathcal{L}^{-1}\left\{\frac{2}{s^2+4}\right\}$$

$$= 6\cos(2t) - \frac{3}{2}\sin(2t).$$

$$\mathcal{L}^{-1}\left\{\frac{6s-3}{s^2+4}e^{-\pi s}\right\}$$

$$= f(t-\pi)\cdot u(t-\pi)$$

$$= 6\cos(2(t-\pi)) - \frac{3}{2}\sin(2(t-\pi))$$

$$= 6\cos(2t-2\pi) - \frac{3}{2}\sin(2t-2\pi)$$

$$= 6\cos(2t) - \frac{3}{2}\sin(2t)$$

Evaluate

$$\mathcal{L}^{-1}\left\{\frac{6s-4}{s^2-8s+25}e^{-\pi s}\right\}$$

$s \to s+4$

$$f(t) = \mathcal{L}^{-1}\left\{\frac{6s-4}{s^2-8s+25}\right\}$$

$$= \mathcal{L}^{-1}\left\{\frac{6s-4}{s^2-8s+16+9}\right\}$$

$$= \mathcal{L}^{-1}\left\{\frac{6s-4}{(s-4)^2+3^2}\right\}$$

$$= e^{4t} \cdot \mathcal{L}^{-1}\left\{\frac{6(s+4)-4}{(s+4-4)^2+3^2}\right\}$$

$$= e^{4t} \cdot \mathcal{L}^{-1}\left\{\frac{6s+20}{s^2+3^2}\right\}$$

$$= e^{4t} \cdot 6 \cdot \mathcal{L}^{-1}\left\{\frac{s}{s^2+3^2}\right\} + \frac{20}{3} e^{4t} \cdot \mathcal{L}^{-1}\left\{\frac{3}{s^2+3^2}\right\}$$

$$= \underbrace{6\, e^{4\cancel{t}} \cdot \cos(3\cancel{t}) + \frac{20}{3} \cdot e^{4\cancel{t}} \cdot \sin(3\cancel{t})}$$

Now

$$\mathcal{L}^{-1}\left\{ \frac{6s-4}{s^2-8s+25} e^{-\pi s} \right\}$$

$$= f(t-\pi) \cdot u(t-\pi)$$

$$= \left[6 e^{4(t-\pi)} \cos(3(t-\pi)) + \frac{20}{3} \cdot e^{4(t-\pi)} \sin(3(t-\pi)) \right] \cdot u(t-\pi) .$$

$$= \left[6 e^{4t-4\pi} \cdot \cos(3t-3\pi) + \frac{20}{3} e^{4t-4\pi} \sin(3t-3\pi) \right] u(t-\pi)$$

$$= \left[-6 e^{4t-4\pi} \cos 3t - \frac{20}{3} \cdot e^{4t-4\pi} \sin 3t \right] \cdot u(t-\pi) \quad \text{↵}$$


Thank you!