Lecture 7

Topics:

Set Terminologies

a. Set and Set Notations
b. Empty Set, Subsets, Power sets, Cartesian Products
c. Infinite Sets
d. Venn Diagrams

CHAPTER 2 : SETS

Definition: A set is an unordered collection of distinct objects, called elements or members of the set. A set is said to contain its elements. We write a $\in$ A to denote that a is an element of the set A. The notation a $\notin$ A denotes that a is not an element of the set A.

There are several ways to describe a set. One way is to list all the members of a set, when this is possible. We use a notation where all members of the set are listed between braces. For example, the notation {a, b, c, d} represents the set with the four elements a, b, c, and d. This way of describing a set is known as the roster method.

Examples:

EXAMPLE 1: The set V of all vowels in the English alphabet can be written as **V = {a, e, i, o, u}**.

EXAMPLE 2: The set O of odd positive integers less than 10 can be expressed by **O = {1, 3, 5, 7, 9}**.

EXAMPLE 3: The set L of Nobel Laureates in chemistry in Bangladesh, **L = { }**, known as empty set often denoted by ϕ.

Elements of a set may even be inconceivably unrelated like {a, 2, Sabuj, Rajshahi}

Set Builder Notation:

Another way to describe a set is to use set builder notation. We characterize all those Extra elements in the set by stating the property or properties they must have to be members. General form of this notation is ***{ x | x has property P}*** and is read “the set of all x such that x has property P.” For instance, the set O of all even positive integers less than 10 can be written as ***O = { x | x is an even positive integer less than 10 }***, that can also be enumerated as ***{0, 2, 4, 6, 8}***. However, if it were 100 instead of 10, it would have been difficult to do so.

Set Builder Notation:

Some of the accepted notations are as follows:

- **N** = {1, 2, 3, …}, the set of all **Natural numbers**
- **Z** = {… ,−2,−1, 0, 1, 2, …}, the set of all **Integers**
- $\mathbf{Z}^+$ = {1, 2, 3, …}, the set of all **Positive Integers**
- **Q** = {p/q | p ∈ **Z,** q ∈ **Z,** and q ≠ 0}, the set of all **Rational Numbers**
- **R**, the set of all **Real Numbers**
- $\mathbf{R}^+$, the set of all **Positive Real Numbers**
- **C**, the set of all **Complex Numbers.**

If a and b are real numbers with $a \leq b$, we denote these intervals by:

- $[a, b] = \{ x \mid a \leq x \leq b \}$
- $[a, b) = \{ x \mid a \leq x < b \}$
- $(a, b] = \{ x \mid a < x \leq b \}$
- $(a, b) = \{ x \mid a < x < b \}$

Note that $[a, b]$ is called the closed interval from a to b and (a, b) is called the open interval from a to b.

Set Equality:

Definition : Two sets are equal if and only if they have the same elements. Therefore, if A and B are sets, then A and B are equal if and only if

$\forall x\ (x \in A \leftrightarrow x \in B)$. We write A = B, if A and B are equal sets.

EXAMPLE: The sets {1, 3, 7} and {3, 7, 1} are equal, because they have the same elements. Note that the order in which the elements of a set are listed does not matter. Note also that it does not matter if an element of a set is listed more than once, so {2, 3, 3, 3, 5, 5, 5} is the same as the set {2, 3, 5} because they have the same elements.

Empty Set:

There is a special set that has no elements. This set is called the empty set, or null set, and is denoted by $\boldsymbol{\phi}$ or **{ }**.

A set with one element is called a **singleton set.**

A common error is to confuse the empty $\{\boldsymbol{\phi}\}$ has one more element than $\boldsymbol{\phi}$. Set $\boldsymbol{\phi}$ with the set $\{\boldsymbol{\phi}\}$, which is a **singleton set**.

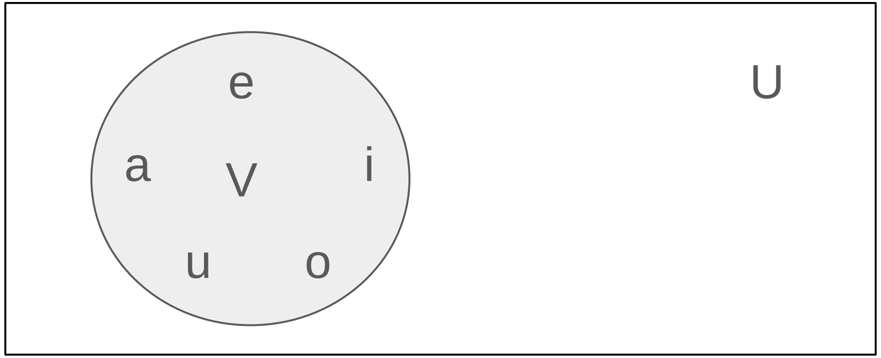
The single element of the set $\{\boldsymbol{\phi}\}$ is the **empty set** itself!

$\boldsymbol{\phi}$, $\{\boldsymbol{\phi}\}$ and $\{\{\boldsymbol{\phi}\}\}$ are all different sets.

Venn Diagram:

Sets can be represented graphically using Venn diagrams, named after the English mathematician John Venn, who introduced their use in 1881. In Venn diagrams the universal set U, which contains all the objects under consideration, is represented by a rectangle.

EXAMPLE: Draw a Venn diagram that represents V, the set of vowels in the English alphabet.



Subsets:

Definition : The set A is a subset of B, and B is a superset of A, if and only if every element of A is also an element of B. We use the notation $\mathbf{A \subseteq B}$ to indicate that A is a subset of the set B. If, instead, we want to stress that B is a superset of A, we use the equivalent notation $\mathbf{B \supseteq A}$**.** (So, $\mathbf{A \subseteq B}$ and $\mathbf{B \supseteq A}$ are equivalent statements.)

We see that $\mathbf{A \subseteq B}$ if and only if the quantification $\mathbf{\forall x(x \in A \rightarrow x \in B)}$

Showing that A is a Subset of B To show that $\mathbf{A \subseteq B}$, show that if x belongs to A then x also belongs to B.

Showing that A is Not a Subset of B To show that $\mathbf{A \nsubseteq B}$, find a single $\mathbf{x \in A}$ such that $\mathbf{x \notin B}$

Theorems:

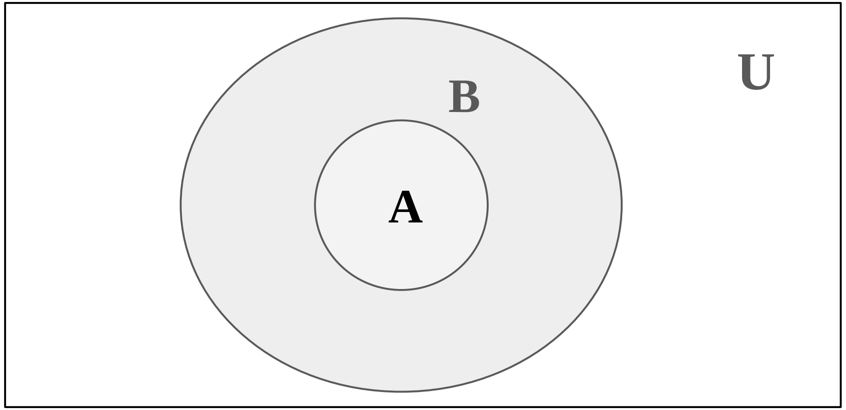
THEOREM 1: For every set S, (i) $\emptyset \subseteq S$ and (ii) $S \subseteq S$.

Proof: We will prove **(i)** and leave the proof of **(ii)** as an exercise.
Let **S** be a set. To show that $\emptyset \subseteq S$, we must show that $\forall x(x \in \emptyset \rightarrow x \in S)$ is true. Because the empty set contains no elements, it follows that $x \in \emptyset$ is always false. It follows that the conditional statement $x \in \emptyset \rightarrow x \in S$ is always true, because its hypothesis is always false and a conditional statement with a false hypothesis is true.
Therefore, $\forall x(x \in \emptyset \rightarrow x \in S)$ is true. This completes the proof of **(i)**.
[Note that this is an example of a vacuous proof.]

Proper Subset:

When we wish to emphasize that a set A is a subset of a set B but that **$A \neq B$**, we write **$A \subset B$** and say that A is a proper subset of B. For **$A \subset B$** to be true, it must be the case that **$A \subseteq B$** and there must exist an element x of B that is not an element of A. That is, A is a proper subset of B if and only if

$$\forall X(x \in A \rightarrow x \in B) \wedge \exists x(x \in B \wedge x \notin A)$$



The Size of a Set:

Sets may have other sets as members. For instance, we have the sets **A = {∅, {a}, {b}, {a, b}}** and **B = {x | x is a subset of the set {a, b}}.**
Note that these two sets are equal, that is, **A = B.** Also note that **{a} ∈ A**, but **a ∉ A.**
Definition : Let S be a set. If there are exactly n distinct elements in S where n is a nonnegative integer, we say that S is a finite set and that n is the cardinality of S. The cardinality of S is denoted by **|S|.**

EXAMPLE 10: Let A be the set of odd positive integers less than 10. Then **|A| = 5.**

EXAMPLE 11: Let S be the set of letters in the English alphabet. Then **|S| = 26.**

EXAMPLE 12: Because the null set has no elements, it follows that **|∅| = 0.**

Infinite sets and power sets:

Definition : A set is said to be infinite if it is not finite.
EXAMPLE : The set of positive integers is infinite.

Definition : Given a set S, the power set of S is the set of all subsets of the set S. The power set of S is denoted by (S).

EXAMPLE : What is the power set of the set {0, 1, 2}?

Solution: The power set ({0, 1, 2}) is the set of all subsets of {0, 1, 2}. Hence, P ({0, 1, 2}) = {∅, {0}, {1}, {2}, {0, 1}, {0, 2}, {1, 2}, {0, 1, 2}}.

Note that the empty set and the set itself are members of this set of subsets.

Power set of empty set:

EXAMPLE : What is the power set of the empty set? What is the power set of the set **{ Ø }**?

Solution: The empty set has exactly one subset, namely, itself. Consequently,

P(Ø) = {Ø}

The set **{Ø}** has exactly two subsets, namely, **Ø** and the set **{Ø}** itself. Therefore,

P({Ø}) = {Ø, {Ø} }

If a set has n elements, then its power set has 2^n elements.

Cartesian Products:

Definition: The ordered n-tuple $(\mathbf{a_1, a_2, \ldots, a_n})$ is the ordered collection that has $\mathbf{a_1}$ as its first element, $\mathbf{a_2}$ as its second element, … , and $\mathbf{a_n}$ as its nth element.

We say that two ordered n-tuples are equal if and only if each corresponding pair of their elements is equal. In other words, $(\mathbf{a_1, a_2, \ldots, a_n}) = (\mathbf{b_1, b_2, \ldots, b_n})$ if and only if $\mathbf{a_i = b_i}$ **, for i = 1, 2, … , n.**

Definition: Let A and B be sets. The Cartesian product of A and B, denoted by A × B, is the set of all ordered pairs (a, b), where a ∈ A and b ∈ B.

Hence, $A \times B = \{(a, b) \mid a \in A \land b \in B\}$.

Examples:

EXAMPLE: What is the Cartesian product of $A = \{1, 2\}$ and $B = \{a, b, c\}$?

Solution:

The Cartesian product $A \times B$ is: $A \times B = \{(1, a), (1, b), (1, c), (2, a), (2, b), (2, c)\}$.

Note that the Cartesian products $A \times B$ and $B \times A$ are not equal unless $A = \emptyset$ or $B = \emptyset$ (so that $A \times B = \emptyset$) or $A = B$

Definition: The Cartesian product of the sets $A_1, A_2, \cdots, A_n$, denoted by $A_1 \times A_2 \times \cdots \times A_n$, is the set of ordered n-tuples $(a_1, a_2, \ldots, a_n)$, where a_i belongs to A_i for $i = 1, 2, \ldots, n$. In other words, $A_1 \times A_2 \times \cdots \times A_n = \{(a_1, a_2, \ldots, a_n) \mid a_i \in A_i \text{ for } i = 1, 2, \ldots, n\}$.

What is the Cartesian product **A × B × C**, where **A = {0, 1}, B = {1, 2}, and C = {0, 1, 2}?**

Solution: The Cartesian product **A × B × C** consists of all ordered triples (a, b, c), where **a ∈ A, b ∈ B**, and **c ∈ C.**

Hence, **A × B × C = {(0, 1, 0), (0, 1, 1), (0, 1, 2), (0, 2, 0), (0, 2, 1), (0, 2, 2), (1, 1, 0), (1, 1, 1), (1, 1, 2), (1, 2, 0), (1, 2, 1), (1, 2, 2)}.**

We use the notation A^2 to denote A × A, the Cartesian product of the set A with itself. Similarly, $A^3 = A \times A \times A$, $A^4 = A \times A \times A \times A$, and so on. More generally, $A^n = \{(a_1, a_2, \ldots, a_n) \mid a_i \in A \text{ for } i = 1, 2, \ldots, n\}$.

EXAMPLE: Suppose that A = {1, 2}. It follows that:

A^2 = {(1, 1), (1, 2), (2, 1), (2, 2)} and

A^3 = {(1, 1, 1), (1, 1, 2), (1, 2, 1), (1, 2, 2), (2, 1, 1), (2, 1, 2), (2, 2, 1), (2, 2, 2)}.

EXAMPLE: What are the ordered pairs in the less than or equal to relation, which contains **(a, b) if a $\leq$ b, on the set {0, 1, 2, 3}?**

Solution: The ordered pair (a, b) belongs to R if and only if both a and b belong to **{0, 1, 2, 3} and a $\leq$ b.**
Consequently, **R = {(0, 0), (0, 1), (0, 2), (0, 3), (1, 1), (1, 2), (1, 3), (2, 2), (2, 3), (3, 3)}.**

EXAMPLE: What do the statement **$\forall x \in R\ (x^2 \geq 0)$** mean?

Solution: The statement **$\forall x \in R\ (x^2 \geq 0)$** states that for every real number x, **$x^2 \geq 0$.** This statement can be expressed as "The square of every real number is nonnegative."

This is a true statement.

The End