Lecture 8

Gamma and Beta Function

Formula

1. $\Gamma(n)=\int_0^{\infty} e^{-x}x^{n-1}dx, \quad where \quad n>0$, Euler's integral of the second kind.

2. $\beta(m,n)=\int_0^{1} x^{m-1}(1-x)^{n-1}dx, \quad where \quad m>0, n>0.$ Euler's integral of the 1st kind.

3. $\int_0^{\pi/2} \sin^p x\cos^q x dx=\dfrac{\Gamma(\frac{p+1}{2})\,\Gamma(\frac{q+1}{2})}{2\,\Gamma(\frac{p+q+2}{2})}$

4. $\beta(m,n)=\dfrac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}$

5. $\Gamma(n)=(n-1)!$

6. $\Gamma(n+1)=n\Gamma(n)=n!$

7. $\Gamma(\frac{1}{2})=\sqrt{\pi}$

8. $\Gamma(1)=1$

9. $\Gamma(\frac{p}{2})=(\frac{p}{2}-1)(\frac{p}{2}-2)(\frac{p}{2}-3)\ldots\ldots\ldots\ldots\frac{1}{2}.\Gamma(\frac{1}{2})$

10. $\Gamma(n)=(n-1)(n-2)\ldots\ldots\ldots 3.2.1$

11. $\int_0^{\pi/2} 2\sin^{2x-1}(t)\cos^{2y-1}(t)dt=\beta(x,y)$

Example: Evaluate: $\int_0^1 \left(1-\frac{1}{x}\right)^{1/3} dx$

$$= \int_0^1 \left(\frac{x-1}{x}\right)^{1/3} dx$$

$$= \int_0^1 x^{-1/3} (x-1)^{1/3} dx$$

$$= \cancel{\int_0^1 x^{2/3-1/3}}$$

$$= -\int_0^1 x^{2/3-1} (1-x)^{4/3-1} dx$$

$$= -\beta(2/3, 4/3) = -\frac{\Gamma 2/3 \, \Gamma 4/3}{\Gamma 2}$$

Example: $\int_0^1 x^7 (1-x)^3 dx = \int_0^1 x^{8-1} (1-x)^{4-1} dx = \beta(8,4) = \frac{\Gamma 8 \, \Gamma 4}{\Gamma 12}$

Example: $\int_0^\infty x^5 e^{-4x} dx = \int_0^\infty \left(\frac{1}{4}u\right)^5 e^{-u} \frac{1}{4} du$

$4x = u$, $x = \frac{1}{4}u$
$dx = \frac{1}{4} du$

x	0	∞
u	0	∞

$$= \frac{1}{4^5} \int_0^\infty u^5 e^{-u} du$$

$$= \frac{1}{4^5} \int_0^\infty e^{-u} u^{6-1} du$$

$$= \frac{1}{4^5} \Gamma 6$$

Gamma and Beta Function

Example: Evaluate : $\int_0^4 x^{3/2}(4-x)^{5/2}\,dx$

$$= \int_0^4 x^{3/2}\, 4^{5/2}\left(1-\frac{x}{4}\right)^{5/2} dx$$

$$= 4^{5/2}\int_0^1 (4u)^{3/2}(1-u)\,4\,du$$

$$= 4^5\int_0^1 u^{3/2}(1-u)^{5/2}\,du$$

Let $u = \frac{x}{4}$

$du = \frac{1}{4}dx$

x	0	4
u	0	1

$4^{5/2}.4.4^{3/2}$

4^5

Since $\beta(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}\,dx$

So, $\int_0^4 x^{3/2}(4-x)^{5/2}\,dx = 4^5\int_0^1 u^{5/2-1}(1-u)^{7/2-1}\,du$

$$= 4^5\,\beta(5/2, 7/2)$$

$$= 4^5\,\frac{\Gamma 5/2\;\Gamma 7/2}{\Gamma\left(\frac{5}{2}+7/2\right)}$$

$$= 4^5\,\frac{\Gamma 5/2\;\Gamma 7/2}{\Gamma 6}$$

$$\int_0^{\pi/6} \sin^2(6x)\cos^4(3x)\,dx$$

$\sin^2 x = 1 - \cos 2x$

$$\sin^2(6x) = \frac{1-\cos(12x)}{2}$$

~~$\sin^2 2(3x) = \frac{1}{2}(1-\cos 2$~~

$$(\sin 2(3x))^2 = (2\sin 3x\cos 3x)^2$$

$$= 4\sin^2 3x\cos^2 3x$$

$3x = u$

$x = \frac{1}{3}u$

$dx = \frac{1}{3}du$

x	0	$\pi/6$
u	0	$\pi/2$

$$= \int_0^{\pi/6} 4\sin^2 3x\cos^2 3x.\cos^4 3x\,dx$$

$$= \int_0^{\pi/6} 4\sin^2 3x\cos^6 3x\,dx$$

$$= \frac{4}{3}\int_0^{\pi/4} \sin^2 u\cos^6 u\,du$$

$$= \frac{4}{3}\int_0^{\pi/2}$$

$$= \frac{4}{3}\,\frac{\Gamma\left(\frac{2+1}{2}\right)\Gamma\left(\frac{6+1}{2}\right)}{2\Gamma\left(\frac{2+6+2}{2}\right)}$$

$$= \frac{4}{3}\,\frac{\Gamma 3/2\,\Gamma 7/2}{2\Gamma 5}$$

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