# Lecture 14 [Homogeneous Linear Eq with constant coefficient, continue ...]

**Example 2:** Solve $4y''+4y'+17y=0$, $y(0)=-1$, $y'(0)=2$.

**Sol$^n$:**

$$4y''+4y'+17y=0 \quad \text{------ (1)}$$

Let $y=e^{mx}$ be the trial sol$^n$ of eq$^n$ (1)

So the auxiliary eq$^n$

$$4m^2+4m+17=0 \quad , \text{ since } e^{mx}\neq 0$$

$$m=\frac{-4\pm\sqrt{16-4.17.4}}{2.4}$$

$$=\frac{-4\pm\sqrt{16-272}}{2.4}=\frac{-4\pm\sqrt{-256}}{2.4}=\frac{-1\pm\sqrt{16}\,i}{2}=\frac{-1\pm 4i}{2}$$

$$m_1=-\tfrac{1}{2}+2i \ , \ m_2=-\tfrac{1}{2}-2i$$

$$\alpha=-\tfrac{1}{2}, \ \beta=2$$

The sol$^n$:

$$y=e^{-\frac{1}{2}x}\left(A\cos 2x+B\sin 2x\right),$$

For initial condition.

when $x=0$, $y=-1$, so $\quad -1=1.(A\cos 0+B\underbrace{\sin 0}_{0})$

$$A=-1$$

Now
$$y=e^{-\frac{1}{2}x}(-\cos 2x+B\sin 2x)$$

$$y'=-\tfrac{1}{2}e^{-\frac{1}{2}x}(-\cos 2x+B\sin 2x)+e^{-\frac{1}{2}x}(2\sin 2x+2B\cos 2x)$$

when $x=0$, $y'=2$

$$2=-\tfrac{1}{2}.1(-1+B.0)+1.(2.0+2B)=\tfrac{1}{2}+2B$$

$$2B=\tfrac{3}{2} \quad \therefore B=\tfrac{3}{4}$$

The required sol$^n$:

$$y=-e^{-\frac{1}{2}x}\left(-\cos 2x+\tfrac{3}{4}\sin 2x\right)$$

Example : Solve $\frac{d^4y}{dx^4} + 2\frac{d^2y}{dx^2} + y = 0$ ——— (1)

Solⁿ let $y = e^{mx}$ be the trial solⁿ of (1)

so, the auxiliary eqⁿ

$$(m^4 + 2m^2 + 1)e^{mx} = 0$$

$$m^4 + 2m^2 + 1 = 0 \qquad , e^{mx} \neq 0$$

$$(m^2+1)^2 = 0$$

$$m^2 + 1 = 0$$
$$m^2 = -1$$
$$m = \pm\sqrt{-1} = \pm i$$

∴ Here are have four roots

$$m_1 = m_3 = i$$
$$m_2 = m_4 = -i$$

∴ The solⁿ

$$y = c_1 e^{ix} + c_2 e^{-ix} + c_3 x e^{ix} + \cancel{c_4 e^{-i}} c_4 x e^{-ix}$$

## TRY YOURSELF

1. $y'' + 8y' + 16y = 0$

2. $y'' - 10y' + 25y = 0$

3. $y'' + 9y = 0$

4. $y''' - y = 0$

5. $y''' + 3y'' + 3y' + y = 0$

6. $y'' + 16y = 0$ , $y(0) = 2$, $y'(0) = -2$

7. $\frac{d^2y}{d\theta^2} + y = 0$, $y(\pi/3) = 0$, $y'(\pi/3) = 2$

## Undetermined coefficients- superposition approach

Now introducing a nonhomogeneous linear differential eq$^n$ is

$$a_n y^{(n)} + a_{n-1} y^{(n-1)} + \cdots + a_2 y'' + a_1 y' + a_0 y = g(x) \quad \text{——} (1)$$

where $a_i$, $i = 0, 1, 2, \cdots$, $a_n \neq 0$ and $g(x) \neq 0$.

The general sol$^n$ of eq$^n$ (1) is $y = y_c + y_p$

Here $y_c$ represents the complementary function which ~~and $y_p$ represents~~ we already introduced (last lecture) a solution of homogeneous linear eq$^n$ with constant coefficient.

$y_p$ represents a particular solution which actually related to $g(x)$ in eq$^n$ (1).

$y_p$ - Particular solution depends on what types of function of $g(x)$ in eq$^n$ (1).

Example:

Solve $y'' + 4y' - 2y = 2x^2 - 3x + 6$. —— (1)

Sol$^n$: First we work for homogeneous eq$^n$ of (1) i.e.

$$y'' + 4y' - 2y = 0 \quad \text{—— (2)}$$

Let $y = e^{mx}$ be the trial sol$^n$ of (2)

So, auxiliary eq$^n$.

$$m^2 + 4m - 2 = 0, \quad e^{mx} \neq 0$$

$$m = \frac{-4 \pm \sqrt{16+8}}{2} = \frac{-4 \pm \sqrt{24}}{2} = -2 \pm \sqrt{6}$$

~~So, $m_1 = -2 + \sqrt{6}$, $m$~~

$$m_1 = -2 - \sqrt{6}, \quad m_2 = -2 + \sqrt{6}$$

$$\therefore y_c = c_1 e^{m_1 x} + c_2 e^{m_2 x} = c_1 e^{(-2-\sqrt{6})x} + c_2 e^{(-2+\sqrt{6})x}$$

Now for particular sol$^n$.

$$y_p = Ax^2 + Bx + C$$

$$y_p' = 2Ax + B$$

$$y_p'' = 2A$$

From (1)

$$2A + 4(2Ax + B) - 2(Ax^2 + Bx + C) = 2x^2 - 3x + 6$$

$$2A + 4(2Ax+B) ~~~~ 2A + 4B - 2C = 2x^2 - 3x + 6$$

$$-2Ax^2 + (8A - 2B)x + 2A + 4B - 2C = 2x^2 - 3x + 6$$

Equating both sides.

$$-2A = 2 \qquad 8A - 2B = -3 \qquad 2A + 4B - 2C = 6$$

$$A = -1 \qquad -2B = 5 \qquad -2 - 10 - 2C = 6$$

$$B = -\frac{5}{2} \qquad -2C = 18$$

$$C = -9$$

$$\therefore y_p = -x^2 - \frac{5}{2}x - 9$$

General sol$^n$:

$$y = c_1 e^{(-2-\sqrt{6})x} + c_2 e^{(-2+\sqrt{6})x} - x^2 - \frac{5}{2}x - 9$$

**Example:** Solve $y'' - y' + y = 2\sin 3x$ ——— (1)

**Solⁿ:** General solⁿ:

$$y = y_c + y_p$$

For $y_c$: $y'' - y' + y = 0$

$$(m^2 - m + 1) = 0 \quad , \quad e^{mx} \neq 0$$

$$m = \frac{1 \pm \sqrt{1-4}}{2} = \frac{1 \pm \sqrt{3}\, i}{2} = \frac{1}{2} \pm \frac{\sqrt{3}}{2} i$$

So, $m_1 = \frac{1}{2} - \frac{\sqrt{3}}{2} i \quad , m_2 = \frac{1}{2} + \frac{\sqrt{3}}{2} i$

Therefore, $y_c = e^{\frac{1}{2}x}\left(A\cos\frac{\sqrt{3}}{2}x + B\sin\frac{\sqrt{3}}{2}x\right)$

For $y_p$: let $y_p = E\cos 3x + F\sin 3x$

$$y_p' = -3E\sin 3x + 3F\cos 3x$$

$$y_p'' = -9E\cos 3x - 9F\sin 3x$$

From (1)

$$-9E\cos 3x - 9F\sin 3x + 3E\sin 3x - 3F\cos 3x + E\cos 3x + F\sin 3x = 2\sin 3x$$

$$(-8E - 3F)\cos 3x + (3E - 8F)\sin 3x = 2\sin 3x$$

Equating both sides

$$-8E - 3F = 0$$
$$8E = -3F$$
$$E = -\frac{3}{8}F$$

$$3E - 8F = 2$$
$$-\frac{9}{8}F - 8F = 2$$

$$-\frac{73}{8}F = 2$$
$$F = -\frac{16}{73}$$
$$E = -\frac{3}{8}\cdot\left(-\frac{16}{73}\right) = \frac{6}{73}$$

General solⁿ:

$$y = e^{x/2}\left(A\cos\frac{\sqrt{3}}{2}x + B\sin\frac{\sqrt{3}}{2}x\right) + \frac{6}{73}\cos 3x - \frac{16}{73}\sin 3x$$

Example: Solve $y'' - 2y' - 3y = 4x - 5 + 6xe^{2x}$ —— (1)

Sol$^n$: General sol$^n$ of eq$^n$ (1)

$$y = y_c + y_p$$

For $y_c$: We know how to solve $y_c$ (see previous example or lecture)

$$y_c = c_1 e^{-x} + c_2 e^{3x}$$

For $y_p$: let $y_p = Ax + B + Exe^{2x} + Fe^{2x}$

$$y_p' = A + Ee^{2x} + 2Exe^{2x} + 2Fe^{2x}$$

$$y_p'' = \underline{2Ee^{2x} + 2Ee^{2x}} + 4Exe^{2x} + 4Fe^{2x}$$

From (1)

$$4Ee^{2x} + \cancel{4Exe^{2x}} + \cancel{4Fe^{2x}} - 2A - 2Ee^{2x} - \cancel{4Exe^{2x}} - \cancel{4Fe^{2x}} - 3Ax - 3B - 3Exe^{2x} - 3Fe^{2x} = 4x - 5 + 6xe^{2x}$$

$$(2E - 3F)e^{2x} - 3Exe^{2x} - 3Ax - (2A + 3B) = 4x - 5 + 6xe^{2x}$$

Equating both sides.

$$-3E = 6 \qquad E = -2$$

$$2E - 3F = 0 \qquad 3F = 2E \qquad F = -\frac{4}{3}$$

$$-3A = 4 \qquad A = -\frac{4}{3}$$

$$2A + 3B = 5 \qquad 3B = 5 + \frac{8}{3} \qquad B = \frac{23}{9}$$

General sol$^n$ of eq$^n$ (1)

$$y = c_1 e^{-x} + c_2 e^{3x} - \frac{4}{3}x + \frac{23}{9} - 2xe^{2x} - \frac{4}{3}e^{2x}$$

~~Nonhomogeneous DEs~~

## For Particular Solutions

| $g(x)$ | $y_p$ |
|---|---|
| 1. Any constant | $A$ |
| 2. $5x+7$ | $Ax+B$ |
| 3. $3x^2-2$ | $Ax^2+Bx+C$ |
| 4. $x^3-x+1$ | $Ax^3+Bx^2+Cx+D$ |
| 5. $\sin ax$ | $A\cos ax+B\sin ax$ |
| 6. $\cos bx$ | $A\cos bx+B\sin bx$ |
| 7. $e^{ax}$ | $Ae^{ax}$ |
| 8. $(9x+2)e^{bx}$ | $(Ax+B)e^{bx}$ |
| 9. $x^2e^{bx}$ | $(Ax^2+Bx+C)e^{bx}$ |
| 10. $e^{bx}\sin ax$ | $Ae^{bx}\cos ax+Be^{bx}\sin ax$ |