Lecture 3

Integration by Parts

Our primary goal is how to integrate product of two functions.

For example,

$$\int f(x)\, g(x)\, dx$$

Here we use a formula to integrate product of two functions which we call it integration by parts.

Say, $u = f(x)$ $\quad du = f'(x)\,dx$

$g(x)\,dx = dv$

$$\boxed{\int u\,dv = uv - \int v\,du}$$

Example: Evaluate: $\int x \cos x\, dx$

Say, $u = x$ and $dv = \cos x\, dx$

$du = dx$ $\qquad v = \sin x$

$$\int x\cos x\, dx = x\sin x - \int \sin x\, dx$$

$$= x\sin x - (-\cos x) + C$$

$$= x\sin x + \cos x + C$$

Example: Evaluate $\int e^x \cos x \, dx$

$\left[\begin{array}{l} L - \text{Logarithmic} \\ I - \text{Inverse trigonometric} \\ A - \text{Algebric} \\ T - \text{Trigonometric} \\ E - \text{Exponential} \end{array}\right.$

Sol

Say, $u = \cos x \quad \Big| \quad dv = e^x dx$

$du = -\sin x \, dx \quad \Big| \quad v = e^x$

Thus,

$$\int e^x \cos x \, dx = e^x \cos x - \int e^x(-\sin x) dx$$

$$\therefore I = e^x \cos x + \int e^x \sin x \, dx \quad \text{——— (1)}$$

Say, $I = \int e^x \cos x dx$

Now let $\sin x = u \qquad dv = e^x dx$

$\cos x dx = du \qquad v = e^x$

$$\int e^x \sin x \, dx = e^x \sin x - \int e^x \cos x \, dx = e^x \sin x - I$$

From (1)

$$I = e^x \cos x + e^x \sin x - I$$

$$2I = e^x \cos x + e^x \sin x$$

$$I = \frac{e^x}{2}(\cos x + \sin x)$$

Therefore $\int e^x \cos x \, dx = \frac{1}{2} e^x (\sin x + \cos x) + C$

We can also use short cut techniques for ~~pro~~ integration by parts.

Example: Evaluate: $\int x^2 \cdot e^x \, dx$

Solⁿ: $\int x^2 e^x dx = x^2 e^x - 2x e^x + 2e^x + C$

$\times$.

D		I
x^2	$+$	e^x
$2x$	$-$	e^x
2	$+$	e^x
0		e^x

Example: Evaluate: $\int (x^2 - x) \cos x \, dx$

Solⁿ:

$\int (x^2 - x) \cos x \, dx$

$= (x^2 - x) \sin x + (2x-1)(-)(-\cos x)$

$\quad + 2(+) \cdot (-\sin x)$

$= (x^2 - x) \sin x + (2x-1) \cos x - 2\sin x + C$

$\times$.

D		I
$x^2 - x$	$+$	$\cos x$
$2x - 1$	$-$	$\sin x$
2	$+$	$-\cos x$
0		$-\sin x$

Multiply ~~under~~ inside redcolor's function with sign and use '+' inbetween two ~~red~~ bounded red color's mark.

TRY YOURSELF

(4) Evaluate:

1. $\int x \ln x \, dx$

2. $\int e^{3x} \cos 2x \, dx$

3. $\int x \tan^2 x \, dx$

4. $\int (x + x\cos x) \, dx$

5. $\int x^3 e^{x^2} \, dx$

Fundamental Theorem of Calculus

<u>Theorem</u>: If f is continuous on $[a,b]$ and F is any antiderivative of f on $[a,b]$, then

$$\int_a^b f(x)\,dx = F(b) - F(a).$$

<u>Example</u>: Evaluate : $\int_1^2 x\,dx$

<u>Sol^n</u>: The function $F(x) = \frac{x^2}{2}$ is an antiderivative of $f(x) = x$ on $[1,2]$, thus,

$$\int_1^2 x\,dx = \frac{x^2}{2}\Big|_1^2 = \frac{1}{2}2^2 - \frac{1}{2}1^2 = 2 - \frac{1}{2} = \frac{3}{2}$$

Theorem:

If f is continuous on an interval, then f has an antiderivative on that interval. In particular, if a is any point in the interval, then the function F defined by

$$F(x) = \int_a^x f(t)\,dt$$

is an antiderivative of f; that is, $F'(x) = f(x)$ for each x in the interval, we can write alternative notation

$$\boxed{\frac{d}{dx}\left[\int_a^x f(t)\,dt\right] = f(x).}$$

Example: Find $\frac{d}{dx}\left[\int_1^x t^3\,dt\right]$, since t^3 is continuous on $[1, x]$.

$$\int_1^x t^3\,dt = \frac{t^4}{4}\Big|_1^x$$

$$= \frac{x^4}{4} - \frac{1}{4}$$

$$\frac{d}{dx}\left[\frac{x^4}{4} - \frac{1}{4}\right] = \frac{1}{4}\cdot 4x^3 - 0$$

$$= x^3.$$

Using theorem,

$$\frac{d}{dx}\left[\int_1^x t^3\,dt\right] = x^3.$$

Extra Problem

Evaluate:

① $\int_{-1}^{2} 4x(1-x^2)\,dx$

② $\int_{0}^{\pi/3} (2x - \sec x \tan x)\,dx$

③ $\int_{1}^{4} \left(\frac{1}{\sqrt{t}} - 3\sqrt{t}\right) dt$

④ $\int_{\pi/6}^{\pi/2} \left(x + \frac{2}{\sin^2 x}\right) dx$

⑤ Define $F(x)$ by $F(x) = \int_{1}^{x} (3t^2 - 3)\,dt$.

Find $F'(x)$.

⑥ Define $F(x)$ by $F(x) = \int_{\pi/4}^{x} \cos 2t\,dt$.

Find $F'(x)$.