Lecture 12

Initial-Value Problem

Example: Solve $\frac{dy}{dx} = \frac{xy^2 - \cos x \sin x}{y(1-x^2)}$, $y(0) = 2$.

Sol^n: We rewrite the DEs.

$$(\cos x \sin x - xy^2)\,dx + y(1-x^2)\,dy = 0$$

Now for exact form,

$$\frac{\partial M}{\partial y} = -2xy \qquad \frac{\partial N}{\partial x} = -2xy$$

Thus, $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, which is exact.

Now, $\frac{\partial f}{\partial x} = M(x,y) = \cos x \sin x - xy^2$, $\frac{\partial f}{\partial y} = N(x,y) = y(1-x^2)$

$$\frac{\partial f}{\partial y} = y(1-x^2)$$

$$f = \int y(1-x^2)\,dy + g(x) \quad [\text{Integrating w.r. to } y]$$

$$f(x,y) = \frac{y^2}{2}(1-x^2) + g(x) \quad \text{———} \; (*)$$

$$\frac{\partial f}{\partial x} = \frac{y^2}{2}\cdot(-2x) + g'(x) = \cos x \sin x - xy^2$$

$$\Rightarrow -xy^2 + g'(x) = \cos x \sin x - xy^2$$

Comparing both sides. $\quad g'(x) = \cos x \sin x$

$$g(x) = -\frac{1}{2}\cos^2 x$$

Therefore, Eq^n $(*)$ $\Rightarrow$

$$\frac{y^2}{2}(1-x^2) - \frac{1}{2}\cos^2 x = c_1$$

$$y^2(1-x^2) - \cos^2 x = c$$

when $x=0$, $y=2$ $\quad 2^2(1-0) - 1 = c \Rightarrow c = 3$

The required sol^n is

$$y^2(1-x^2) - \cos^2 x = 3$$

Extra Problem:

1. Solve : $(4x^3y^3 - 2xy)\,dx + (3x^4y^2 - x^2)\,dy = 0$

2. Solve : $(3e^{3x}y - 2x)\,dx + e^{3x}\,dy = 0$

~~3. Solve : 2~~

3. Show that the following eqn. is exact and solve it.

$$2x(ye^{x^2} - 1)\,dx + e^{x^2}\,dy = 0$$

4. Determine whether the given differential eqn. is exact. If it is exact, solve it.

(a) $(2xy^2 - 3)\,dx + (2x^2y + 4)\,dy = 0$

(b) $(x^3 + y^3)\,dx + 3xy^2\,dy = 0$

5. Solve the ~~th~~ given initial-value problem

(a) $(e^x + y)\,dx + (2 + x + ye^y)\,dy = 0$, $y(0) = 1$

(b) $(4y + 2t - 5)\,dt + (6y + 4t - 1)\,dy = 0$, $y(-1) = 2$

Non exact DE

If the DE. $M(x,y)dx + N(x,y)dy = 0$ is not exact form i.e.

$$\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$$

Then we want to introduce an integrating factor for the differential equation $M(x,y)dx + N(x,y)dy = 0$. ——— (1)

$\Rightarrow$ If $\frac{M_y - N_x}{N}$ is a function of x alone, then an integrating factor,

$$I.F = e^{\int \frac{M_y - N_x}{N} dx}$$

$\Rightarrow$ If $\frac{N_x - M_y}{M}$ is a function of y alone, then an integrating factor,

$$I.F. = e^{\int \frac{N_x - M_y}{M} dy}$$

Multiplying I.F in eqn (1), then it will be an exact.

After proofing exact form, similar way to solve the Differential equation.

Example: solve $xy\,dx + (2x^2+3y^2-20)\,dy = 0$. ——— (1)

Solⁿ Here $M(x,y) = xy$, $N(x,y) = 2x^2+3y^2-20$

$M_y = x$ $\qquad$ $N_x = 4x$

So, $M_y \neq N_x$, Not an exact

Now, $\dfrac{M_y - N_x}{N} = \dfrac{x-4x}{2x^2+3y^2-20} = \dfrac{-3x}{2x^2+3y^2-20}$ Not x or y alone.

again, $\dfrac{N_x - M_y}{M} = \dfrac{4x-x}{xy} = \dfrac{3x}{xy} = \dfrac{3}{y}$, only y depends.

So, Integrating factor, I.F. $= e^{\int \frac{3}{y}dy} = e^{3\ln y} = e^{\ln y^3} = y^3$.

Now multiplying y^3 in eqⁿ (1) both sides,

$$xy^4\,dx + (2x^2y^3+3y^5-20y^3)\,dy = 0$$

Say, $M = xy^4 = \dfrac{\partial f}{\partial x}$, $N = 2x^2y^3+3y^5-20y^3 = \dfrac{\partial f}{\partial y}$

$M_y = 4xy^3$ $\qquad$ $N_x = 4xy^3$

$$\frac{\partial f}{\partial x} = M(x,y) = xy^4$$

$$f(x,y) = \int xy^4\,dx + g(y) = \frac{x^2}{2}y^4 + g(y) \quad \text{——— (2)}$$

$$\frac{\partial f}{\partial y} = \frac{x^2}{2}4y^3 + g'(y) = 2x^2y^3 + g'(y)$$

Since $\dfrac{\partial f}{\partial y} = N(x,y)$

So, $2x^2y^3 + g'(y) = 2x^2y^3+3y^5-20y^3$

$\therefore g'(y) = 3y^5 - 20y^3$ ——— (3)

Integrating ③ w.r.to y then

$$g(y) = \cancel{3}\frac{y^6}{\cancel{6}_2} - \overset{5}{\cancel{20}}\frac{y^4}{\cancel{4}} = \frac{1}{2}y^6 - 5y^4$$

Eqn. ② becomes,

$$f(x,y) = \frac{1}{2}x^2y^4 + \frac{1}{2}y^6 - 5y^4$$

Therefore, the soln of the above differential eqn is

$$\frac{1}{2}x^2y^4 + \frac{1}{2}y^6 - 5y^4 = c$$

<u>Do yourself</u>: Solve the given differential equation.

1. $(2y^2+3x)dx + 2xy\,dy = 0$
2. $6xy\,dx + (4y+9x^2)dy = 0$
3. $(10-6y+e^{-3x})\,dx - 2dy = 0$
4. $(x^2+y^2-5)\,dx = (y+xy)dy$.