Lecture-16

LINEAR MODELS

Growth and Decay [Initial-value problem]

$$\frac{dx}{dt} = Kx \quad , \quad x(t_0) = x_0$$

where K is a constant of proportionality.

The solution of the above differential eqn. is

$$x(t) = Ce^{Kt}$$

Case I: $K > 0$ i.e. K is a growth constant.

Case II: $K < 0$ i.e. K is a decay constant.

Example: A culture initially has P_0 number of bacteria. At $t = 1\,h$ the number of bacteria is measured to be $\frac{3}{2}P_0$. If the rate of growth is proportional to the number of bacteria $P(t)$ present at time t, determine the time necessary for the number of bateria to triple.

Soln. According to the question, we have

$$\frac{dP}{dt} = KP$$ where P is number of bacteria present time at t.

$$\frac{dP}{dt} - KP = 0 \quad \text{[linear form]}$$

$$\text{I.F.} = e^{-\int K dt} = e^{-Kt}$$

$$\frac{d}{dt}\left(e^{-Kt}P(t)\right) = 0$$

$$e^{-Kt}P(t) = C$$

$$P(t) = Ce^{Kt} \qquad \text{——— (1)}$$

Now for initial condition, $t=0$, $P(0)=P_0$

From ① $P_0 = P(0) = Ce^0 = C$

Eqn. ① becomes

$$P = P_0 e^{Kt} \qquad \text{———} ②$$

at $t=1$, $P(1) = \frac{3}{2}P_0$

Eqn. ②

$$\frac{3}{2}P_0 = P_0 e^K$$

$$e^K = \frac{3}{2} \Rightarrow K = \ln\frac{3}{2} = 0.4055$$

Now, required time $t = t_r$, $P(t_r) = 3P_0$

Eqn ② becomes,

$$3P_0 = P_0 e^{0.4055t}$$

$$0.4055t = \ln 3$$

$$t = \frac{\ln 3}{0.4055} \approx 2.71 \text{ h.}$$

X

Half life: The half-life is a measure of the stability of a radioactive subtance that the time it takes for one-half of the atoms in an initial amount A_0 to disintegrate, or transmute, into the atoms of another elements.
e.g. the half-life of radium, Ra-226, is about 1700 years.

Problem: The half-lif of a radioactive substance is simply the time it takes for one half of the atoms in an initial amount X_0 to disintegrate into the atoms of another element. A breeder reactor converts relatively stable U-238 into the isotope Pu-239. $\boxed{\text{After 15 years}}$ it is determined that 0.043% of the initial amount of Pu-239 has disintegrated. Find the half-life of this isotope if the rate of disintegration is proportional to the amount remaining.

Solution: Let $X(t)$ denote the amount of Pu-239 remaining at time t.

According to the question (red line):

$$\frac{dX}{dt} = KX \quad , \quad X(0) = X_0$$

Sol^n. $\quad X(t) = X_0 e^{Kt}$

If 0.043% of the atoms ~~di~~ of X_0 have disintegrated, then $(100-0.043)\% = 99.957\%$ of the substance remains.

So, $X(15) = 0.99957 X_0$

$$\Rightarrow 0.99957 \cancel{X_0} = \cancel{X_0} e^{K \cdot 15} \Rightarrow K = \frac{1}{15} \ln(0.99957)$$
$$= -0.00002867$$

Therefore,
$$X(t) = X_0 e^{-0.00002867t}$$

Now for half-life, $X(t) = \frac{1}{2}X_0$

$$\therefore \quad \frac{1}{2}X_0 = X_0 e^{-0.00002867t}$$

$$\Rightarrow -0.00002867t = -\ln 2$$

$$\Rightarrow \quad t = \frac{\ln 2}{0.00002867} \approx 24{,}180 \text{ yrs.}$$

Problem: A fossilized bone is found to contain one-thousandth of the C-14 level found in living matter. Estimate the age of the fossil. Since, the half-life of radioactive C-14 is approximately 5600 years.

Solⁿ TRY YOURSELF:

$A(t) = A_0 e^{kt}$, $A(t)$ – ~~starting~~ Present

K – decay constant

C-14: half-life $t=5600$

Newton's Law of Cooling/Warming

$$\frac{dT}{dt} = K(T - T_m)$$

where $T(t)$ - the temperature of the object for $t > 0$

T_m - the ambient temperature (surrounding temperature)

K - a constant of proportionality

Problem: A thermometer is taken from an inside room to the outside, where the air temperature is 5°F. After 1 min, the thermometer reads 55°F, and after 5 min, it reads 30°F. What is the initial temperature of the inside room?

Sol^n:

$$\frac{dT}{dt} = K(T-5) \quad , \quad T(1) = 55°F$$
$$T(5) = 30°F$$

$$\frac{dT}{T-5} = K dt$$

$$\ln|T-5| = Kt + C$$

$$T - 5 = e^{Kt+C} = C_1 e^{Kt}$$

$$\therefore T(t) = 5 + C_1 e^{Kt} \quad \text{——— (1)}$$

Since, $T(1) = 55$

From (1)

$$55 = 5 + C_1 \cancel{e^{Kt}} e^{K \cdot 1} = 5 + C_1 e^{K}$$

$$C_1 e^{K} = 50$$

$$C_1 = 50 e^{-K}$$

again after 5 min,

$$T(5) = 30$$

$$30 = 5 + C_1 e^{5K}$$

$$30 = 5 + 50 e^{-K} e^{5K} \qquad \left[C_1 = 50 e^{-K} \right]$$

$$50 e^{4K} = 25$$

$$e^{4K} = \frac{1}{2}$$

$$4K = \ln \frac{1}{2}$$

$$K = \frac{\ln 1/2}{4} = -0.1732$$

Now,

~~T(0)=~~

Therefore $C_1 = 50 e^{+0.1732} = 59.461$

Now, for initial temperature, $t=0$

$$T(0) = 5 + 59.461\, e^{(-0.1732)\cdot 0} = 5 + 59.461$$

$$= 64.461^\circ F$$

$\divideontimes$

DO Yourself

Extra problems:

Books: Differential Equations (Seventh edition)
— Dennis G. Zill

page: 89

Exercises: 3.1

Problems: 1, 3, 5, 13, 15, 21, ~~20~~ 31.

Examples: Example: 5 [page: 86]

Example: 6 [page: 88]