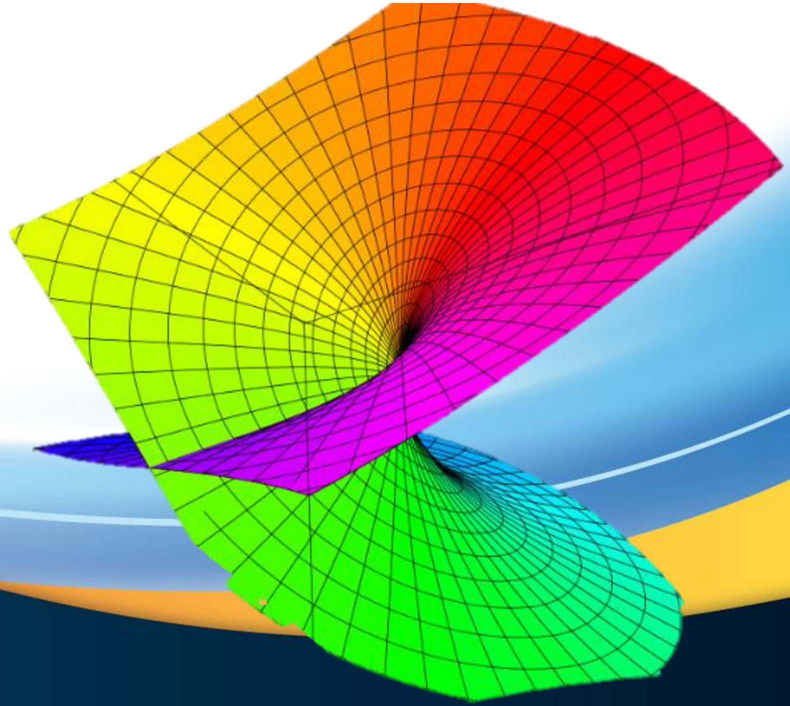
Undergraduate Course in Mathematics

Complex Variables

Topic: Limit and Continuity



Conducted By

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Undefined and Indeterminate Forms

$f(x) = \frac{1}{x}$

$f(0) = \frac{1}{0}$ = big number

$= \infty$

$\frac{1}{0}$ = − big num

$= -\infty$

$f(x) = 2 - x$

$f(3) = ?$

$f(0) =$

$$\frac{1}{0}, \frac{2}{0}, \frac{\text{Something}}{0} = \text{undefined}$$

$$\frac{15}{3} = 5$$

$$\frac{0}{2} = 0 \text{ (defined)}$$

What about $\frac{0}{0}$ = undefined, 0, 1, anything

Indeterminate (অনির্ণেয়)

$\frac{1}{0}, \frac{2}{0}, \frac{3}{0}, \cdots, \frac{10}{0}$, undefined

$\frac{0}{0} \to$ undefined + Indeterminate.

Some other indeterminate forms

$$\frac{0}{0}, \quad \frac{\infty}{\infty}, \quad \infty - \infty, \quad 0 \cdot \infty, \quad 1^{\infty}, \quad 0^{0}$$

Limit

$$f(x) = \frac{x^2-4}{x-2}$$

$$f(2) = \frac{0}{0} = \text{undefined}$$

$$\lim_{x \to 2} \frac{x^2-4}{x-2} = 4$$

$$\boxed{\frac{z^2-4}{z-2} \neq z+2}$$

$$\frac{z^2-4}{z-2}$$

$$= \frac{(z+2)\cancel{(z-2)}}{\cancel{(z-2)}}$$

$$= z+2$$

$\frac{\cancel{0}}{\cancel{0}} = 1$

undefined at $z=2$

defined at $z=2$

$$f(x) = (1+x)^{\frac{1}{x}}$$

$$f(0) = 1^{\frac{1}{0}}$$

$$= e$$

$$g(x) = (1+2x)^{\frac{1}{x}}$$

$$g(0) = 1^{\frac{1}{0}}$$

$$= e^2$$

$$\lim_{z \to 3}(z^2 - 5)$$

$$= (3)^2 - 5$$

$$= 9 - 5$$

$$= 4$$

$$\lim_{z\to 2}\frac{z^2-4}{z-2}$$

$$= \lim_{z\to 2}\frac{\cancel{(z-2)}(z+2)}{\cancel{(z-2)}}$$

$z \to 2$

z tends to 2

$$= \lim_{z\to 2}(z+2)$$

$$= 2+2$$

$$= 4 \text{ Ans}$$

$$\lim_{x \to 2} \frac{1}{x-2}$$

$$= \cancel{\frac{1}{0}}$$

$=$ limit does not exist.

$$\lim_{z\to 2i} \frac{1-2z}{z^2+4}$$

$$= \frac{1-2\cdot 2i}{(2i)^2+4}$$

$$= \frac{1-2i}{0}$$

= undefined

= limit does not exist.

L'Hospital Rule

$$\lim_{z \to a} \frac{f(z)}{g(z)} \longrightarrow \frac{0}{0} \text{ or } \frac{\infty}{\infty}$$

$$= \lim_{z \to a} \frac{f'(z)}{g'(z)}$$

$$\lim_{x \to 2} \frac{x^3 - 8}{x^2 - 4}$$

$$= \lim_{x \to 2} \frac{3x^2}{2x}$$

$$= \frac{3 \cdot 2^2}{2 \cdot 2} = 3$$

Find $\lim_{z \to e^{\pi i/3}} \left(z - e^{\pi i/3}\right)\left(\frac{z}{z^3+1}\right)$.

$$= \lim_{z \to e^{i\frac{\pi}{3}}} \frac{\left(z - e^{i\frac{\pi}{3}}\right) z}{z^3+1} = \lim_{z \to e^{i\frac{\pi}{3}}} \frac{z^2 - e^{i\frac{\pi}{3}} \cdot z}{z^3+1}$$

$$= \lim_{z \to e^{i\frac{\pi}{3}}} \frac{2z - e^{i\frac{\pi}{3}}}{3z^2}$$

$$= \frac{2 \cdot e^{i\frac{\pi}{3}} - e^{i\frac{\pi}{3}}}{3 \cdot \left(e^{i\frac{\pi}{3}}\right)^2} = \frac{e^{i\frac{\pi}{3}}}{3 \cdot e^{i\frac{2\pi}{3}}} = \frac{1}{3} \cdot e^{-i\frac{\pi}{3}}$$

$$\lim_{z\to 0} \frac{z-\sin z}{z^3}.$$

$$= \lim_{z\to 0} \frac{1-\cos z}{3z^2}$$

$$= \lim_{z\to 0} \frac{0+\sin z}{6z}$$

$$= \lim_{z\to 0} \frac{\cos z}{6}$$

$$= \frac{1}{6}$$

$$\lim_{z\to 0} \frac{\tan z - \sin z}{z^3}$$

$$= \lim_{z\to 0} \frac{\frac{\sin z}{\cos z} - \sin z}{z^3}$$

$$= \lim_{z\to 0} \frac{\sin z - \sin z \cdot \cos z}{z^3 \cdot \cos z}$$

$$= \lim_{z\to 0} \frac{\cos z - \cos z \cdot \cos z - \sin z \cdot (-\sin z)}{3z^2 \cos z + z^3 \cdot (-\sin z)}$$

$$= \lim_{z\to 0} \frac{\cos z - \cos^2 z + \sin^2 z}{3z^2 \cos z - z^3 \sin z}$$

$$= \lim_{z\to 0} \frac{-\sin z - 2\cos z \cdot (-\sin z) + 2 \sin z \cdot \cos z}{6z\cos z + 3z^2 \cdot (-\sin z) - 3z^2 \sin z - z^3 \cdot \cos z}$$

$$=$$

$$= \lim_{z \to 0} \frac{-\sin z + 4 \sin z \cos z}{6z \cos z - 6z^2 \sin z - z^3 \cos z}$$

$$= \lim_{z \to 0} \frac{-\cos z + 4 \cos z \cdot \cos z + 4 \sin z\,(-\sin z)}{6 \cos z + 6z\,(-\sin z) - 12 z \sin z - 6z^2 \cos z - 3z^2 \cos z - z^3\,(-\sin z)}$$

$$= \frac{-1+4}{6}$$

$$= \frac{1}{2}$$

$$\lim_{z\to 0}\left(\frac{\sin z}{z}\right)^{\frac{1}{z^2}}$$

let $W = \lim_{z\to 0}\left(\frac{\sin z}{z}\right)^{\frac{1}{z^2}}$

$\Rightarrow \ln W = \ln\left[\lim_{z\to 0}\left(\frac{\sin z}{z}\right)^{\frac{1}{z^2}}\right]$

$= \lim_{z\to 0}\left[\ln\left(\frac{\sin z}{z}\right)^{\frac{1}{z^2}}\right]$

$= \lim_{z\to 0}\frac{1}{z^2}\cdot\ln\left(\frac{\sin z}{z}\right)$

$= \lim_{z\to 0}\frac{\ln\left(\frac{\sin z}{z}\right)}{z^2}$

$= \lim_{z\to 0}\frac{\ln(\sin z)-\ln(z)}{z^2}$

$$= \lim_{z\to 0} \frac{\frac{1}{\sin z}\cdot \cos z \quad \frac{1}{z}}{2z}$$

$$= \lim_{z\to 0} \frac{z\cdot \cos z - \sin z}{2\cdot z^2\cdot \sin z}$$

$$= \lim_{z\to 0} \frac{1\cdot \cancel{\cos z} + z\cdot(-\sin z)) - \cancel{\cos z}}{4z\sin z + 2z^2\cos z}$$

$$= \lim_{z\to 0} \frac{-z\sin z}{4z\sin z + 2z^2\cos z}$$

$$= \lim_{z\to 0} \frac{-1.\sin z - z.\cos z}{4\sin z + 4z\cos z + 4z\cos z + 2z^2(-\sin z)}$$

$$= \lim_{z\to 0} \frac{-\sin z - z\cos z}{4\sin z + 8z\cos z - 2z^2\sin z}$$

$$= \lim_{z\to 0} \frac{-\cos z - 1\cdot\cos z - z(-\sin z)}{4\cos z + 8\cos z + 8z(-\sin z) - 4z\sin z - 2z^2(\cos z)}$$

$$= \frac{-1-1}{4+8} = \frac{-1}{6}.$$

Now

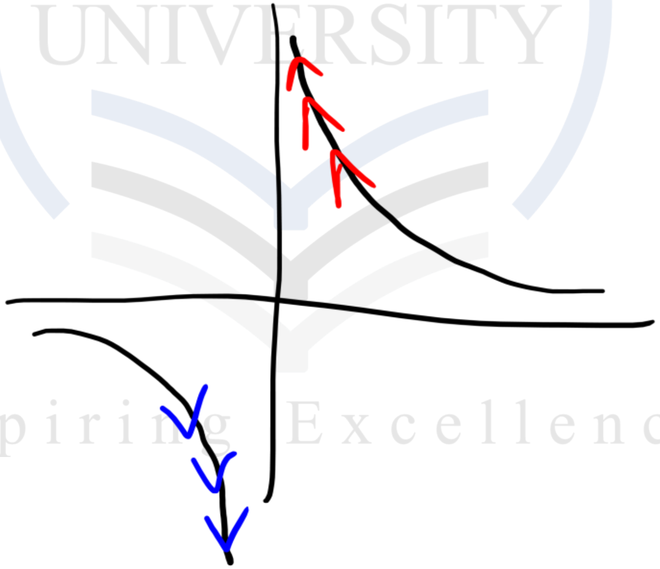
$$\ln w = \frac{-1}{6}$$

$$w = e^{-\frac{1}{6}}$$

$$\therefore \lim_{z \to 0} \left(\frac{\sin z}{z}\right)^{\frac{1}{z^2}} = e^{-\frac{1}{6}}$$

Some Special Limit Problems

$$\lim_{x \to 0} \frac{1}{x} = \text{limit does not exist}$$



$$\lim_{x \to 0} \frac{1}{x^2} = +\infty$$

$$= \text{undefined}$$

Show that the limit, $\lim\limits_{z\to 0}\frac{\bar{z}}{z}$ does not exist

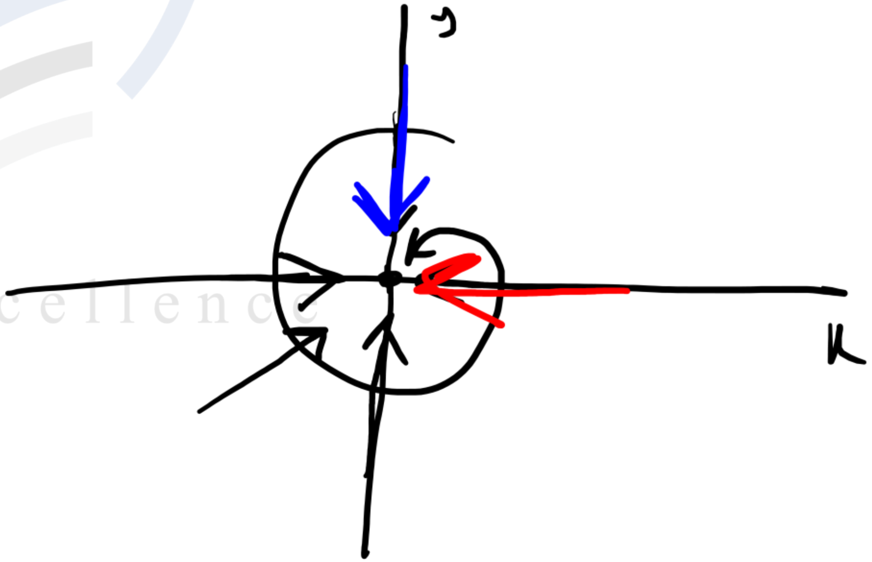
$z = x+iy$

$$= \lim_{\substack{x\to 0\\ y\to 0}} \frac{\overline{x+iy}}{x+iy}$$

$z \to 0$

$x+iy \to 0+i\cdot 0$

$$= \lim_{\substack{x\to 0\\ y\to 0}} \frac{x-iy}{x+iy}$$



In the direction $y=0$

$$\lim_{\substack{x\to 0\\ y\to 0}} \frac{x-iy}{x+iy}$$

$$= \lim_{x\to 0} \left(\frac{x-i\cdot 0}{x+i\cdot 0}\right)$$

$$= \lim_{x\to 0} (1) = 1$$

In the direction $x=0$

$$\lim_{\substack{x\to 0\\ y\to 0}} \frac{x-iy}{x+iy}$$

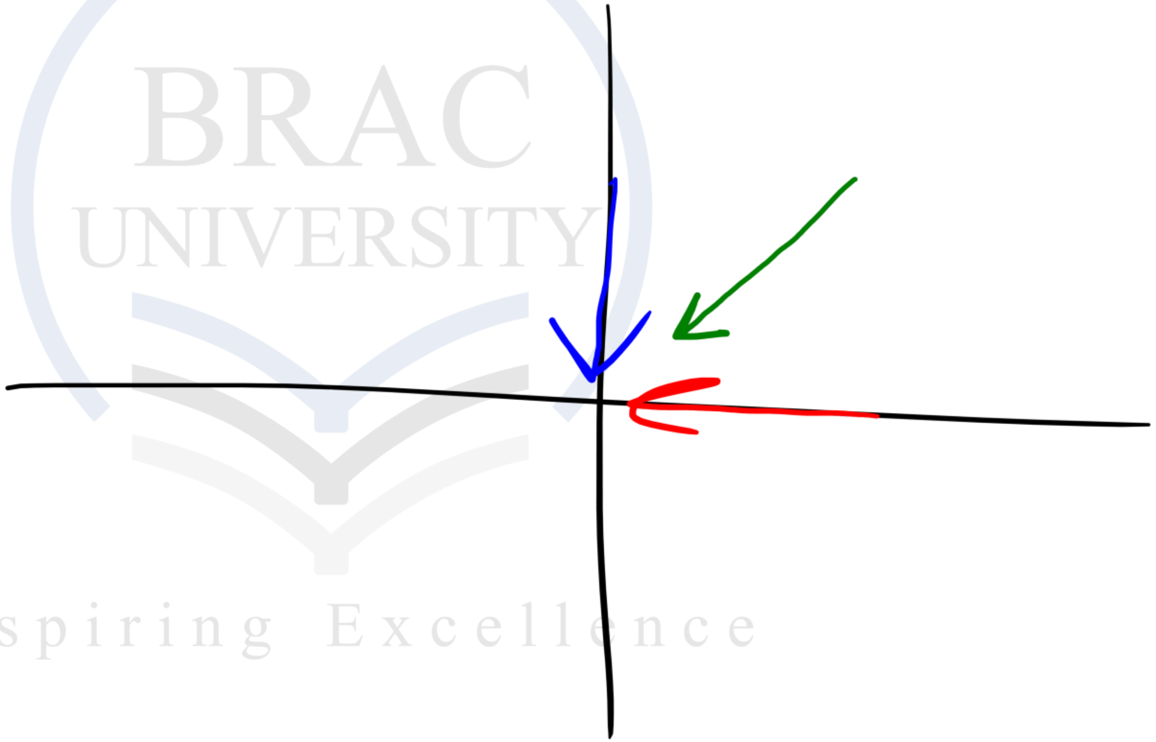
$$= \lim_{y\to 0} \frac{0-iy}{0+iy}$$

$$= \lim_{y\to 0} (-1) = -1$$

$\therefore$ limit does not exist. ✓

Show that the limit, $\lim_{z \to 0} \frac{xy}{x^2+y^2}$ does not exist

$$= \lim_{\substack{x \to 0 \\ y \to 0}} \frac{xy}{x^2+y^2}$$



In $y=0$ direction

$$\lim_{\substack{x\to 0\\ y\to 0}} \frac{xy}{x^2+y^2}$$

$$= \lim_{x\to 0} \frac{x\cdot 0}{x^2+0^2}$$

$$= \lim_{x\to 0} (0)$$

$$= 0$$

In $x=0$ direction

$$\lim_{\substack{x\to 0\\ y\to 0}} \frac{xy}{x^2+y^2}$$

$$= \lim_{y\to 0} \frac{0\cdot y}{0^2+y^2}$$

$$= \lim_{y\to 0} (0)$$

$$= 0$$

In $y=x$ direct.

$$\lim_{\substack{x\to 0\\ y\to 0}} \frac{xy}{x^2+y^2}$$

$$= \lim_{x\to 0} \frac{x\cdot x}{x^2+x^2}$$

$$= \lim_{x\to 0} \frac{1}{2}$$

$$= \frac{1}{2}$$

Continuity

Smoothness of function

function Value = limit

Let $f(z) = \dfrac{z^2+4}{z-2i}$ if $z \neq 2i$, while $f(2i) = 3+4i$. (a) Prove that $\lim_{z\to i} f(z)$ exists and determine its value. (b) Is $f(z)$ continuous at $z = 2i$? Explain. (c) Is $f(z)$ continuous at points $z \neq 2i$? Explain. (d) redefine the function to make it continuous.

$$f(z) = \begin{cases} \dfrac{z^2+4}{z-2i} & z \neq 2i \\ 3+4i & z = 2i \end{cases}$$

$$\lim_{z\to 2i} f(z)$$

$$= \lim_{z\to 2i} \frac{z^2+4}{z-2i}$$

$$= \lim_{z\to 2i} \frac{2z}{1}$$

$$= 2\cdot 2i$$

$$= 4i$$

(b)

$$\lim_{z \to 2i} f(z) = 4i$$

but function value $f(2i) = 3+4i$

limit $\neq$ function

$\Rightarrow$ Not conts at $z = 2i$

ⓒ $f(z) = \begin{cases} \dfrac{z^2+4}{z-2i} & z \neq 2i \\ 3+4i & z = 2i \end{cases}$

conts at all points other that $2i$,

$$g(z) = \begin{cases} \dfrac{z^2+4}{z-2i} & \text{wh } z \neq 2i \\[2ex] 4i & z = 2i \end{cases}$$

Conts ✓

If $f(z) = \begin{cases} \dfrac{z^2-4}{z^2-3z+2}, & z \neq 2 \\ kz^2 + 6, & z = 2 \end{cases}$, find **k** such that $f(z)$ becomes continuous at $z = 2$.

$f(2) = k \cdot 2^2 + 6 = 4k+6$

$\lim_{z \to 2} f(z)$

$= \lim_{z \to 2} \dfrac{z^2-4}{z^2-3z+2}$

$= \lim_{z \to 2} \dfrac{2z}{2z-3}$

$= \dfrac{4}{4-3}$

$= 4$

for continity

function value = limit

$\Rightarrow 4k+6 = 4$

$\Rightarrow k = \dfrac{-1}{2}$

Finding all the Discontinuities

$$\frac{\text{Something}}{0}, \quad \ln(0)$$

Find all points of discontinuity for the following functions.

(a) $f(z) = \dfrac{2z-3}{z^2+2z+2}$, (b) $f(z) = \dfrac{3z^2+4}{z^4-16}$, (c) $f(z) = \cot z$, (d) $f(z) = \dfrac{1}{z} - \sec z$, (e) $f(z) = \dfrac{\tanh z}{z^2+1}$.

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$z = 0,$

(b) $f(z)$ is discont at $z^4 - 16 = 0$

$\Rightarrow (z^2)^2 - 4^2 = 0$

$\Rightarrow (z^2-4)(z^2+4) = 0$

$\Rightarrow z^2 - 4 = 0$ or $z^2 + 4 = 0$

$\Rightarrow z = \pm 2,$ $z = \pm 2i$

discontinuity at $z = 2, -2, 2i, -2i$

(e) $f(z) = \cot z$

$$= \frac{\cos z}{\sin z}$$

discontinuous at $\sin z = 0$

$$\Rightarrow \frac{e^{iz} - e^{-iz}}{2i} = 0$$

$$\Rightarrow e^{iz} = e^{-iz}$$

$$\Rightarrow \frac{e^{iz}}{e^{-iz}} = 1$$

$$\Rightarrow e^{2iz} = 1$$

$$\Rightarrow e^{2iz} = e^{i(0+2n\pi)}$$

$\therefore 2z = 2n\pi$

$\therefore \boxed{z = n\pi}$

$z = 0, \pi, 2\pi, 3\pi, -\pi, -4\pi, -2\pi, \ldots$

(e) $f(z) = \frac{\tanh z}{z^2+1}$

$= \frac{\sinh z}{\cosh z \cdot (z^2+1)}$

dis at $\cosh z = 0$ or $z^2+1=0$

$\Downarrow$ $\Rightarrow z = \pm i$

H/W


Thank you!