# MAT120

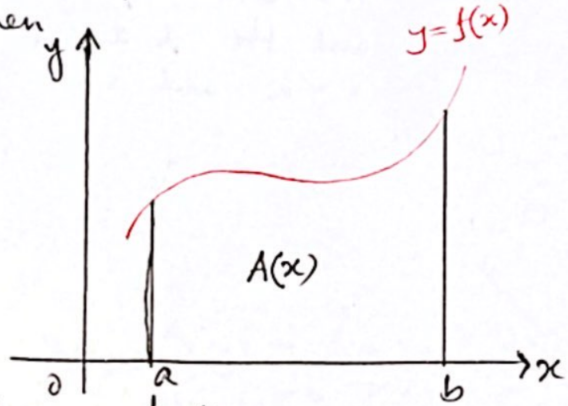
## Integral Calculus and Differential Equations

**Lecture 1:**

### AREA and DERIVATIVES

Isaac Newton and Gottfried Leibniz independently discovered a fundamental relationship between areas and derivatives. They showed that if $f$ is a nonnegative continuous function on the interval $[a,b]$, and if $A(x)$ denotes the area under the graph of $f$ over the interval $[a,x]$ where $x$ is any point in the interval $[a,b]$, then

$$A'(x) = f(x).$$




If we increase $x$ by a small amount $\Delta x$, the change in the area is approximately the height of the function $f(x)$ times the small width $\Delta x$. i.e.,

$$\Delta A \approx f(x) \cdot \Delta x$$

To get the exact rate of change, we take $\Delta x \to 0$,

$$\frac{dA}{dx} = \lim_{\Delta x \to 0} \frac{\Delta A}{\Delta x} = \lim_{\Delta x \to 0} \frac{f(x) \cdot \Delta x}{\Delta x} = f(x)$$

which means,

$$\boxed{A'(x) = f(x)}$$

Here we call $A(x)$ is an antiderivative of $f(x)$. see next page ...

## Antiderivative :

A function $F$ is called an antiderivative of the function $f$ on a given interval $I$ if $F'(x) = f(x) \ \forall x \in I$.

* previous discussion we used
$$A'(x) = f(x)$$
both are same.

so,
$$F'(x) = f(x)$$
$$\frac{dF(x)}{dx} = f(x)$$
$$dF(x) = f(x)\,dx$$
$$\boxed{\cancel{F(x) = \int f(x)\,dx}}$$
$$\int dF(x) = \int f(x)\,dx$$
$$\boxed{\int f(x)\,dx = F(x) + c}$$

$\hookrightarrow$ antiderivative

e.g. $F(x) = \frac{x^3}{3}$ ~~+5~~ is an antiderivative of $f(x) = x^2$

Similarly, $F(x) = \frac{x^3}{3} + 2$ or $\frac{x^3}{3} - 5$ are also antiderivative of $f(x) = x^2$.

which means ~~$\int f(x)\,dx = F(x)+c$~~ $F(x) = \frac{x^3}{3} + c$ is ~~an~~ also an antiderivative of $f(x) = x^2$.

Thus $\int f(x)\,dx = F(x) + c$ , $c$ is called integral constant

or. $\int x^2\,dx = \frac{x^3}{3} + c$ .

$\boxed{\text{F(x)+c is called family of curve !!!}}$

in MATH 1208 ("INTEGRAL Calculus")

QUESTION: "How do you calculate the "AREA" under a CURVE?"

ANSWER: the DEFINITE INTEGRAL

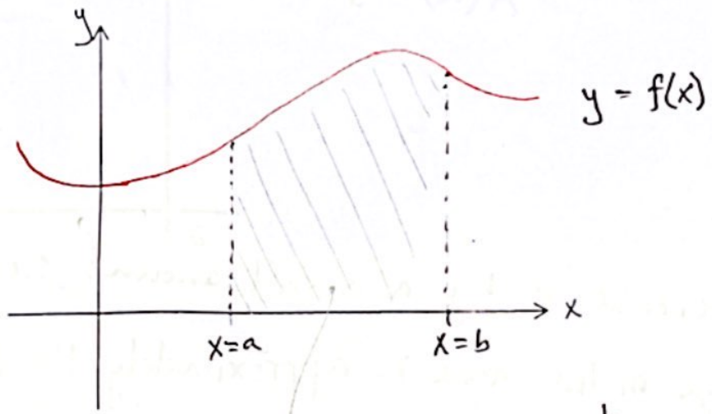
given a curve: $y = f(x)$.

<u>Notation</u>:

$$\int_a^b f(x)\, dx$$

this is: "the definite integral, from $x=a$ to $x=b$, of $f(x)$"

this calculates the "AREA" between the curve $y = f(x)$ and the x-axis, between the values of $x = a$ and $x = b$:




$$\text{AREA} = \int_a^b f(x) \cdot dx$$

okay, so how do we <u>actually</u> calculate this?

we'll start with two approaches:

(1) using GEOMETRY

(2) APPROXIMATING / ESTIMATING the area using a finite number of RECTANGLES

## 5.1 - Areas | Estimating Areas with Finite Sums

ex: (1) Calculate the area between the curve
$y = \sqrt{49 - x^2} + 1$ and the x-axis,
between $x = -7$ and $x = 7$

i.e. Calculate the value of : $\int_{-7}^{7} \left(\sqrt{49-x^2} + 1\right)\, dx$

we'll do this by DRAWING the curve:

$$y = \sqrt{49 - x^2} + 1$$

$$y - 1 = \sqrt{49 - x^2}$$

$$(y-1)^2 = 49 - x^2$$

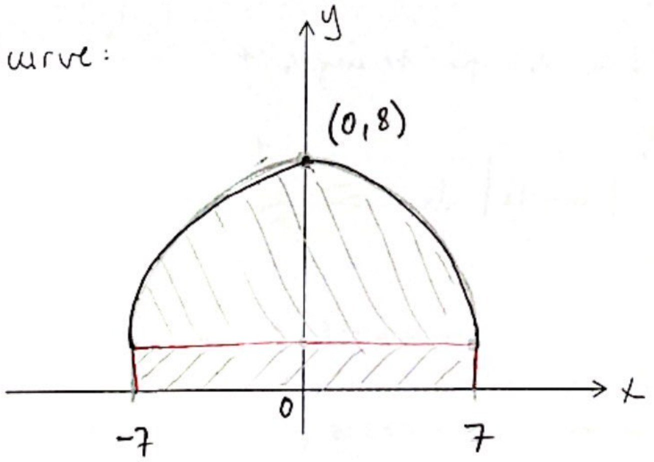
$$x^2 + (y-1)^2 = 49$$

$$(x-0)^2 + (y-1)^2 = (7)^2$$

this is a CIRCLE CENTERED at $(x,y) = (0,1)$
with RADIUS $r = 7$

Since the curve has $y = \oplus\sqrt{\cdots}$
we draw the TOP HALF of the circle.

the curve:




the area is a semicircle of radius 7, + a rectangle:

$$\frac{1}{2}\pi(7)^2 + 14(1) = \frac{49\pi}{2} + 14$$

(2) Repeat the same exercise for: $\int_0^7 |16-4x|\, dx$

i.e. Calculate the <u>AREA</u> between the curve

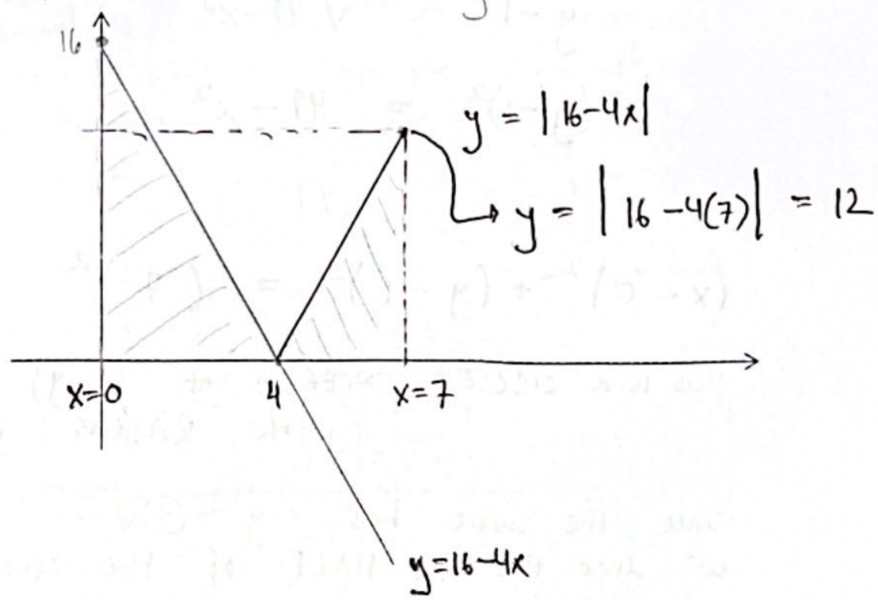
$y =$ <u>$|16-4x|$</u> and the <u>x-axis</u>

between $x =$ <u>0</u> and $x =$ <u>7</u>

we'll do this by drawing the graph of:

$$y = |16-4x|$$

start with the LINE: $y = 16-4x$




the area is two triangles:

$$\int_0^7 |16-4x| \cdot dx = \frac{1}{2}(4)(16) + \frac{1}{2}(3)(12)$$

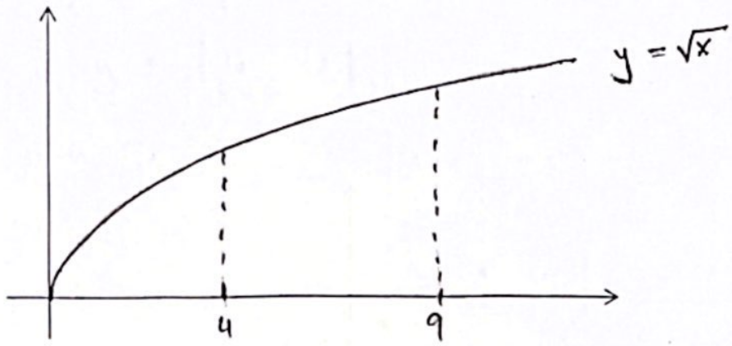
$$= 32 + 18$$

$$= \underline{\underline{50}}$$

Another approach: APPROXIMATE the area using a finite number of RECTANGLES

ex: (1) Approximate the value of $\int_4^9 \sqrt{x}\, dx$ using rectangles

the graph of $y = \sqrt{x}$ is:




Suppose we use "n" rectangles

then each rectangle has a WIDTH we'll call:

$$\Delta x = \frac{b-a}{n} = \frac{9-4}{n} = \frac{5}{n}$$

$x_n = b$, $x_n - a = n\Delta x$, $\boxed{x_n = a + n\Delta x}$

we now use $\Delta x$ to find all values of $x$ between $a$ & $b$ ($x=4$ & $x=9$) at which we'll measure the HEIGHT of each rectangle

we define: $X_0 = a = 4$

$$X_1 = a + \Delta X = 4 + \frac{5}{n}$$

$$X_2 = a + 2\cdot\Delta X = 4 + \frac{10}{n}$$

$$X_3 = a + 3\cdot\Delta X = 4 + \frac{15}{n}$$

$\vdots$

the general pattern is: $X_i = a + i\cdot\Delta X = 4 + \frac{5i}{n}$

$\vdots$

$$X_n = a + n\cdot\Delta X = 4 + 5 = 9 = b$$

we now have $n$ "sub-intervals" between $x=a$ & $x=b$

$[x_0, x_1]$

$[x_1, x_2]$

$[x_2, x_3]$

$\vdots$

$[x_{n-1}, x_n]$

we have to DECIDE: which value of $x$ do we use to measure the HEIGHT of each rectangle?

Do we use the LEFT ENDPOINTS?

or do we use the RIGHT ENDPOINTS?

i will use the RIGHT ENDPOINTS so the height of the rectangles will be:

$$f(x_i) \quad \text{with} \quad i = 1, 2, 3, 4, \ldots, n$$

so

$$f(x_1) = \sqrt{x_1} = \sqrt{4 + \frac{5}{n}}$$

$$f(x_2) = \sqrt{x_2} = \sqrt{4 + \frac{10}{n}}$$

$$f(x_3) = \sqrt{x_3} = \sqrt{4 + \frac{15}{n}}$$

$$\vdots$$

$$f(x_i) = \sqrt{x_i} = \sqrt{4 + \frac{5i}{n}}$$

Left endpoint

$$x_i = a + (i-1)\Delta x$$

Right endpoint

$$x_i = a + i\Delta x$$

Midpoint

$$x_i = a + \left(i - \frac{1}{2}\right)\Delta x$$

Here I used to use right endpont rest of the examples.

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## 5.3 the DEFINITE INTEGRAL

Def'n. let $f(x)$ be a function that is defined on a finite interval $[a, b]$

let $\Delta X = \frac{b-a}{n}$

and let $x_i = a + i \cdot \Delta X$

then the RIEMANN SUM is:

$$R_n = \sum_{i=1}^{n} f(x_i)\,\Delta X$$

Riemann proved that the Riemann Sum Approximation becomes EXACT as the value of $n$ gets LARGE

we do this by letting $n \longrightarrow \infty$ using: LIMITS

and we define:

$$\lim_{n\to\infty}\left(\sum_{i=1}^{n} f(x_i)\,\Delta X\right) = \int_a^b f(x)\,dx$$

ie. the DEFINITE INTEGRAL is defined as

the LIMIT of a RIEMANN SUM

notice that as $n \to \infty$ $\quad \Sigma \longrightarrow \int$

$\Delta X \longrightarrow dx$

ex: Express the following definite integrals as the LIMIT of a RIEMANN Sum

(1) $\int_2^8 \frac{1}{\sqrt{x}}\, dx$ $\quad a = 2 \quad b = 8 \quad f(x) = \frac{1}{\sqrt{x}}$

$$\Delta x = \frac{b-a}{n} = \frac{8-2}{n} = \frac{6}{n}$$

($\Delta x$ IS IN TERMS OF $n$)

$$x_i = a + i \cdot \Delta x = 2 + i \cdot \frac{6}{n} = 2 + \frac{6i}{n}$$

($x_i$ IS IN TERMS OF $n$, AND $i$)

the Riemann sum is:

$$R_n = \sum_{i=1}^{n} f(x_i) \cdot \Delta x$$

$$= \sum_{i=1}^{n} \frac{1}{\sqrt{x_i}} \cdot \Delta x$$

$$= \sum_{i=1}^{n} \frac{1}{\sqrt{2 + \frac{6i}{n}}} \cdot \frac{6}{n}$$

So: $\int_2^8 \frac{1}{\sqrt{x}} \cdot dx = \lim_{n \to \infty} \left( \sum_{i=1}^{n} \frac{1}{\sqrt{2 + \frac{6i}{n}}} \cdot \frac{6}{n} \right)$

(a) $\int_0^2 (x^2 - 3x + 2) \cdot dx$

~~this was supposed to be~~ ~~~~ ~~~~

$$a = 0 \qquad b = 2 \qquad f(x) = x^2 - 3x + 2$$

$$\Delta x = \frac{b-a}{n} = \frac{2-0}{n} = \frac{2}{n}$$

$$x_i = a + i \cdot \Delta x = 0 + i \cdot \frac{2}{n} = \frac{2i}{n}$$

$$f(x_i) = (x_i)^2 - 3(x_i) + 2$$

$$= \left(\frac{2i}{n}\right)^2 - 3\left(\frac{2i}{n}\right) + 2$$

$$= \frac{4 \cdot i^2}{n^2} - \frac{6 \cdot i}{n} + 2$$

the Riemann sum is:

$$R_n = \sum_{i=1}^{n} f(x_i) \cdot \Delta x$$

WARNING! $\Delta x$ has no "$i$" in it!

$$= \sum_{i=1}^{n} \left( \frac{4i^2}{n^2} - \frac{6i}{n} + 2 \right) \cdot \frac{2}{n}$$

so:

$$\int_0^2 (x^2 - 3x + 2) \cdot dx = \lim_{n \to \infty} \left( \sum_{i=1}^{n} \left( \frac{4i^2}{n^2} - \frac{6i}{n} + 2 \right) \cdot \frac{2}{n} \right)$$

Follow-up : EVALUATE THAT LIMIT!

(Start by simplifying the Riemann Sum using the properties of Sigma notation to turn every "$i$" into an "$n$")

the Riemann is:

$$R_n = \sum_{i=1}^{n}\left(\frac{4i^2}{n^2} - \frac{6i}{n} + 2\right)\cdot\frac{2}{n}$$

$$= \sum_{i=1}^{n}\left(\frac{8i^2}{n^3} - \frac{12i}{n^2} + \frac{4}{n}\right)$$

$$= \sum_{i=1}^{n}\frac{8i^2}{n^3} - \sum_{i=1}^{n}\frac{12i}{n^2} + \sum_{i=1}^{n}\frac{4}{n}$$

$$= \frac{8}{n^3}\cdot\sum_{i=1}^{n} i^2 - \frac{12}{n^2}\cdot\sum_{i=1}^{n} i + \frac{4}{n} * n$$

$$= \frac{\overset{4}{\cancel{8}}}{\cancel{n^3}\,{}_2}\;\frac{n(n+1)(2n+1)}{{}^3\cancel{6}} - \frac{{}^6\cancel{12}}{n^2}\;\frac{n(n+1)}{\cancel{2}} + 4$$

$$= \frac{4\cdot(n+1)\cdot(2n+1)}{3n^2} - \frac{6\cdot(n+1)}{n} + 4$$

FINALLY!

$$\int_0^2 (x^2-3x+2)\cdot dx = \lim_{n\to\infty}\left(\frac{4(n+1)(2n+1)}{3n^2} - \frac{6(n+1)}{n} + 4\right)$$

$$= \lim_{n\to\infty}\left(\frac{8\cancel{n^2}}{3\cancel{n^2}} - \frac{6\cancel{n}}{\cancel{n}} + 4\right)$$

$$= \frac{8}{3} - 6 + 4$$

$$= \frac{2}{3}$$

ex: Express $\int_0^4 (5-x^3)\,dx$ * as the limit of a Riemann sum, then use the properties of Sigma notation to simplify the Riemann sum, and evaluate the limit.

Soln: $a = 0 \qquad b = 4 \qquad f(x) = 5 - x^3$

$$\Delta x = \frac{b-a}{n} = \frac{4-0}{n} = \frac{4}{n}$$

$$x_i = a + i \cdot \Delta x = 0 + i \cdot \frac{4}{n} = \frac{4i}{n}$$

$$f(x_i) = 5 - (x_i)^3$$

$$= 5 - \left(\frac{4i}{n}\right)^3$$

$$= 5 - \frac{64 i^3}{n^3}$$

the Riemann sum is:

$$R_n = \sum_{i=1}^{n} f(x_i) \cdot \Delta x$$

$$= \sum_{i=1}^{n} \left(5 - \frac{64 i^3}{n^3}\right) \cdot \frac{4}{n}$$

$$= \sum_{i=1}^{n} \left(\frac{20}{n} - \frac{256 i^3}{n^4}\right)$$

$$= \sum_{i=1}^{n} \frac{20}{n} - \sum_{i=1}^{n} \frac{256 i^3}{n^4}$$

$$= \frac{20}{n} \sum_{i=1}^{n} 1 - \frac{256}{n^4} \sum_{i=1}^{n} i^3$$

$$= 20 - \frac{256}{n^4} \cdot \left(\frac{n(n+1)}{2}\right)^2$$

$$= 20 - \frac{\overset{64}{\cancel{256}}}{\cancel{n^4}\,2} \cdot \frac{\cancel{n^2}(n+1)^2}{\cancel{4}}$$

$$= 20 - \frac{64(n+1)^2}{n^2}$$

So: $$\int_0^4 (5-x^3)\cdot dx = \lim_{n\to\infty}\left(20 - \frac{64(n+1)^2}{n^2}\right)$$

$$= \lim_{n\to\infty}\left(20 - \frac{64n^2}{n^2}\right)$$

$$= 20 - 64$$

$$= -44$$

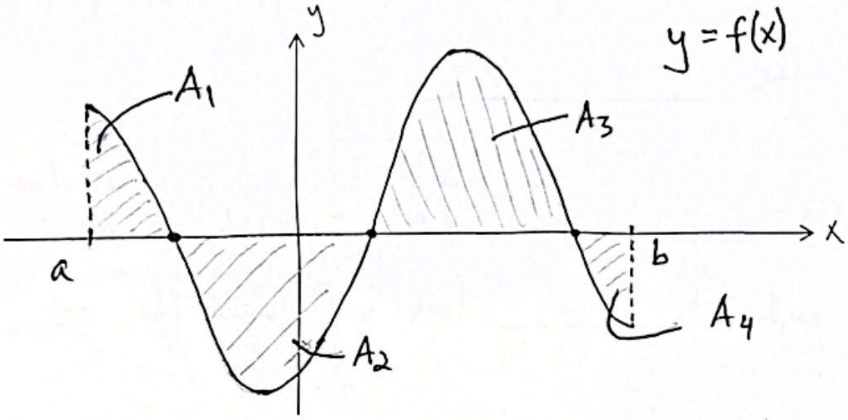
NEGATIVE 44 ?

hmm ...

Hmm ... why is it NEGATIVE ?

A definite integral measures the total SIGNED AREAS between $y = f(x)$ and $y = 0$

Areas BELOW the x-axis are SUBTRACTED FROM the areas ABOVE the x-axis




$$\int_a^b f(x)\,dx = +A_1 - A_2 + A_3 - A_4$$

(where each area is $> 0$)

this number could be: $> 0$

$< 0$

OR $= 0$

$\equiv \ldots \quad \int_0^2 (x^3 - 3x + 2)\, dx$

$a = 0 \qquad b = 2 \qquad f(x) = x^3 - 3x + 2$

- $\Delta x = \dfrac{b-a}{n} = \dfrac{2-0}{n} = \dfrac{2}{n}$

- $x_i = a + i \cdot \Delta x = 0 + i \cdot \dfrac{2}{n} = \dfrac{2i}{n}$

- $f(x_i) = (x_i)^3 - 3(x_i) + 2$

$$= \left(\frac{2i}{n}\right)^3 - 3\left(\frac{2i}{n}\right) + 2$$

$$= \frac{8i^3}{n^3} - \frac{6i}{n} + 2$$

the Riemann Sum is:

$$R_n = \sum_{i=1}^{n} f(x_i)\, \Delta x$$

$$= \sum_{i=1}^{n} \left(\frac{8i^3}{n^3} - \frac{6i}{n} + 2\right) \cdot \frac{2}{n}$$

$$= \sum_{i=1}^{n} \left(\frac{16i^3}{n^4} - \frac{12i}{n^2} + \frac{4}{n}\right)$$

$$= \frac{16}{n^4} \cdot \sum_{i=1}^{n} i^3 - \frac{12}{n^2} \cdot \sum_{i=1}^{n} i + \sum_{i=1}^{n} \frac{4}{n}$$

$$= \frac{16}{n^4} \cdot \frac{n^2(n+1)^2}{4} - \frac{12}{n^2} \cdot \frac{n(n+1)}{2} + \frac{4}{n} \cdot n$$

$$= \frac{4(n+1)^2}{n^2} - \frac{6(n+1)}{n} + 4$$

this is still just $R_n$

now...

$$\int_0^2 (x^3 - 3x + 2)\, dx = \lim_{n \to \infty} \left( \frac{4(n+1)^2}{n^2} - \frac{6(n+1)}{n} + 4 \right)$$

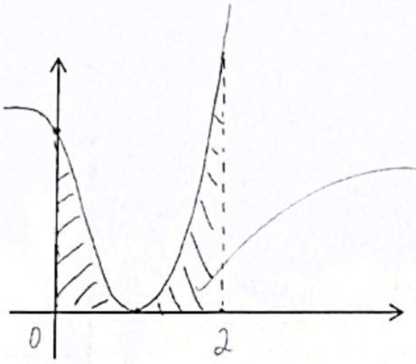
$$= \lim_{n \to \infty} \left( \frac{4n^2}{n^2} - \frac{6n}{n} + 4 \right)$$

$$= 4 - 6 + 4$$

$3x^2 - 3$

$$= \underline{\underline{2}}$$

and what does this mean?




this area is <u>EXACTLY</u> $\underline{\underline{2}}$