Non exact DE

If the DE. $M(x,y)dx + N(x,y)dy = 0$ is not exact form i.e.

$$\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$$

Then we want to introduce an integrating factor for the differential equation $M(x,y)dx + N(x,y)dy = 0$. ——— (1)

⇒ If $\frac{M_y - N_x}{N}$ is a function of x alone, then an integrating factor,

$$I.F = e^{\int \frac{M_y - N_x}{N} dx}$$

⇒ If $\frac{N_x - M_y}{M}$ is a function of y alone, then an integrating factor,

$$I.F. = e^{\int \frac{N_x - M_y}{M} dy}$$

Multiplying I.F in eqn (1), then it will be an exact. After ~~the~~ proofing exact form, similar way to solve the Differential equation.

Example: solve $xy\,dx + (2x^2+3y^2-20)\,dy = 0$. ——— (1)

Solⁿ: Here $M(x,y) = xy$, $N(x,y) = 2x^2+3y^2-20$

$M_y = x$ $\qquad$ $N_x = 4x$

So, $M_y \neq N_x$, Not an exact

Now, $\dfrac{M_y - N_x}{N} = \dfrac{x-4x}{2x^2+3y^2-20} = \dfrac{-3x}{2x^2+3y^2-20}$ Not x or y alone.

again, $\dfrac{N_x - M_y}{M} = \dfrac{4x-x}{xy} = \dfrac{3x}{xy} = \dfrac{3}{y}$, only y depends.

So, Integrating factor, $I.F. = e^{\int \frac{3}{y} dy} = e^{3\ln y} = e^{\ln y^3} = y^3$.

Now multiplying y^3 in eqⁿ (1) both sides,

$$xy^4\,dx + (2x^2y^3+3y^5-20y^3)\,dy = 0$$

Say, $M = xy^4 = \dfrac{\partial f}{\partial x}$, $N = 2x^2y^3+3y^5-20y^3 = \dfrac{\partial f}{\partial y}$

$M_y = 4xy^3$ $\qquad$ $N_x = 4xy^3$

$$\frac{\partial f}{\partial x} = M(x,y) = xy^4$$

$$f(x,y) = \int xy^4\,dx + g(y) = \frac{x^2}{2}y^4 + g(y) \quad \text{——— (2)}$$

$$\frac{\partial f}{\partial y} = \frac{x^2}{2}4y^3 + g'(y) = 2x^2y^3 + g'(y)$$

Since $\dfrac{\partial f}{\partial y} = N(x,y)$

So $2x^2y^3 + g'(y) = 2x^2y^3+3y^5-20y^3$

$\therefore g'(y) = 3y^5 - 20y^3$ ——— (3)

Integrating (3) w.r.to y then

$$g(y) = \cancel{3}\frac{y^6}{\cancel{6}_2} - \cancel{20}^{5}\frac{y^4}{\cancel{4}} = \frac{1}{2}y^6 - 5y^4$$

Eq^n: (2) becomes,

$$f(x,y) = \frac{1}{2}x^2y^4 + \frac{1}{2}y^6 - 5y^4$$

Therefore, the sol^n of the above differential eq^n is

$$\frac{1}{2}x^2y^4 + \frac{1}{2}y^6 - 5y^4 = c$$

Do yourself: Solve the given differential equation.

1. $(2y^2+3x)dx + 2xy\,dy = 0$

2. $6xy\,dx + (4y+9x^2)dy = 0$

3. $(10-6y+e^{-3x})\,dx - 2dy = 0$

4. $(x^2+y^2-5)\,dx = (y+xy)dy$.

Lecture:

Homogeneous Linear Equations with Constant Coefficients

In this Lecture, we will discuss ~~about~~ the homogeneous linear higher-order differential equations and to ~~looks~~ produce a process for finding its solution. The ~~homo~~ general equation of homogeneous linear higher-order DEs is

$$a_n y^{(n)} + a_{n-1} y^{(n-1)} + \cdots + a_2 y'' + a_1 y' + a_0 y = 0,$$

where the coefficient a_i, $i = 0, 1, 2, \ldots, n$ are real constant and $a_n \neq 0$.

At the begining, we consider special case of second-order equation

$$ay'' + by' + cy = 0, \quad \text{—(1)} \quad a, b, c \text{ are constants.}$$

Here we say $y = e^{mx}$ be a trial solution of eq^n (1). Since we already know that in linear differential equation there was a solution which is related to exponential function [Integrating factor, I.f. $= e^{\int P dx}$]

Now, $y = e^{mx}$

$y' = me^{mx}$

$y'' = m^2 e^{mx}$

Eq^n (1) becomes, $e^{mx}(am^2 + bm + c) = 0$ since $e^{mx} \neq 0$ —(11)

$\therefore am^2 + bm + c = 0$

$$m = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Eq^n (11) is called auxillary equations.

Case for m:

1. m_1 and m_2 real and distinct if $b^2-4ac>0$

2. m_1 and m_2 real and equal if $b^2-4ac=0$

3. m_1 and m_2 conjugate (complex number) if $b^2-4ac<0$.

Process:

Step 1: Say $y=e^{mx}$ be the trial sol^n of given DEs.

Step 2: Make an auxiliary eq^n and find the roots (m) of auxiliary eq^n.

Step 3: (a) If roots are real and distinct then write the sol^n of DEs.

$$y = c_1 e^{m_1 x} + c_2 e^{m_2 x}$$

(b) If roots are real and equal then

$$y = c_1 e^{m_1 x} + c_2 x e^{m_1 x}$$

(c) If roots are complex conjugate, then

$$y = c_1 e^{(\alpha+i\beta)x} + c_2 e^{(\alpha-i\beta)x}$$

$$= e^{\alpha x}\left(c_1 e^{i\beta x} + c_2 e^{-i\beta x}\right)$$

$$= e^{\alpha x}\left(c_1 \cos\beta x + c_2 \sin\beta x\right)$$

Example 1: Solve the following DEs.

ⓐ $2y'' - 5y' - 3y = 0$

ⓑ $y'' + 4y' + 7y = 0$

Solⁿ:

ⓐ $2y'' - 5y' - 3y = 0$ ——— ①

Let $y = e^{mx}$ be the trial solⁿ of eqⁿ ①

So, $y' = me^{mx}$; $y'' = m^2 e^{mx}$

Eqⁿ ① becomes

$$(2m^2 - 5m - 3)e^{mx} = 0$$

$$2m^2 - 5m - 3 = 0 \quad \text{since } e^{mx} \neq 0$$

$$2m^2 - 6m + m - 3 = 0$$

$$2m(m-3) + 1(m-3) = 0$$

$$(m-3)(2m+1)$$

$$m_1 = 3,\ m_2 = -\tfrac{1}{2}$$

Hence the solⁿ:

$$y = c_1 e^{3x} + C_2 e^{-\frac{1}{2}x}$$

ⓑ $y'' + 4y' + 7y = 0$ ——— ①

Let $y = e^{mx}$ be the trial solⁿ of ①

$y' = me^{mx}$ $y'' = m^2 e^{mx}$

① becomes, $(m^2 + 4m + 7)e^{mx} = 0$

$$m^2 + 4m + 7 = 0 \quad , e^{mx} \neq 0$$

$$\therefore \; m = \frac{-4 \pm \sqrt{16 - 4.1.7}}{2.1} = \frac{-4 \pm \sqrt{16-28}}{2} = \frac{-4 \pm \sqrt{4.3(-1)}}{2}$$

$\therefore \; m_1 = -2 + \sqrt{3}i \;, \; m_2 = -2 - \sqrt{3}i$ $\qquad i = \sqrt{-1}$

Hence the solⁿ is,

$$y = c_1 e^{(-2+\sqrt{3}i)x} + c_2 e^{(-2-\sqrt{3}i)x} = e^{-2x}\left(c_1 e^{\sqrt{3}ix} + c_2 e^{-\sqrt{3}ix}\right)$$

Since, $e^{i\theta} = \cos\theta + i\sin\theta$

$$\therefore \; e^{i\sqrt{3}x} = \cos\sqrt{3}x + i\sin\sqrt{3}x \;;\; e^{-\sqrt{3}ix} = \cos\sqrt{3}x - i\sin\sqrt{3}x$$

$$c_1 e^{i\sqrt{3}x} = c_1\cos\sqrt{3}x + ic_1\sin\sqrt{3}x \;;\; c_2 e^{-i\sqrt{3}x} = c_2\cos\sqrt{3}x - ic_2\sin\sqrt{3}x$$

$$c_1 e^{i\sqrt{3}x} + c_2 e^{-i\sqrt{3}x} = (c_1+c_2)\cos\sqrt{3}x + i(c_1-c_2)\sin\sqrt{3}x$$
$$= A\cos\sqrt{3}x + B\sin\sqrt{3}x$$

Therefore, the general solⁿ of eqⁿ 1 is

$$y = e^{-2x}\left(A\cos\sqrt{3}x + B\sin\sqrt{3}x\right)$$