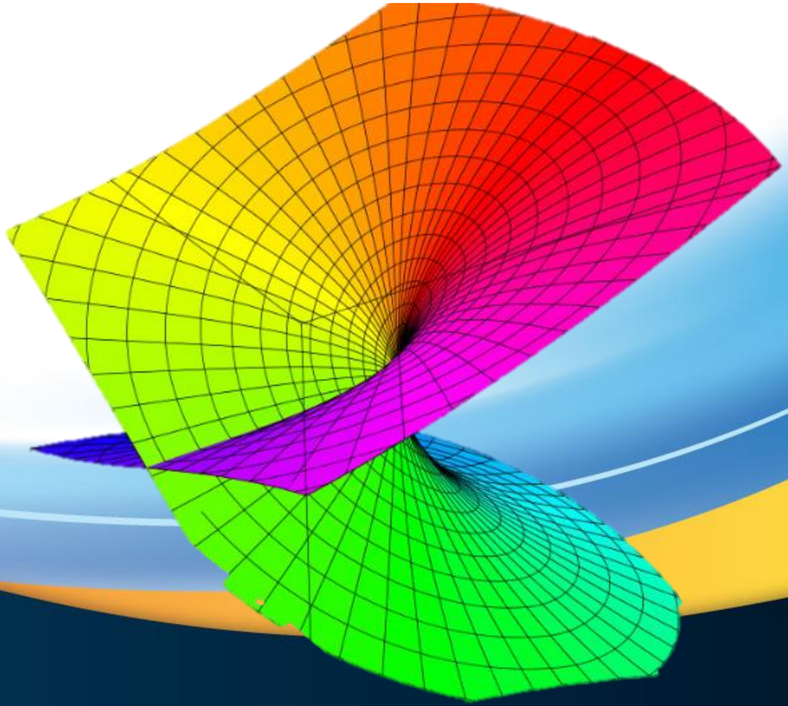
Undergraduate Course in Mathematics

Complex Variables

Topic: Cauchy Integral Formula



Conducted By

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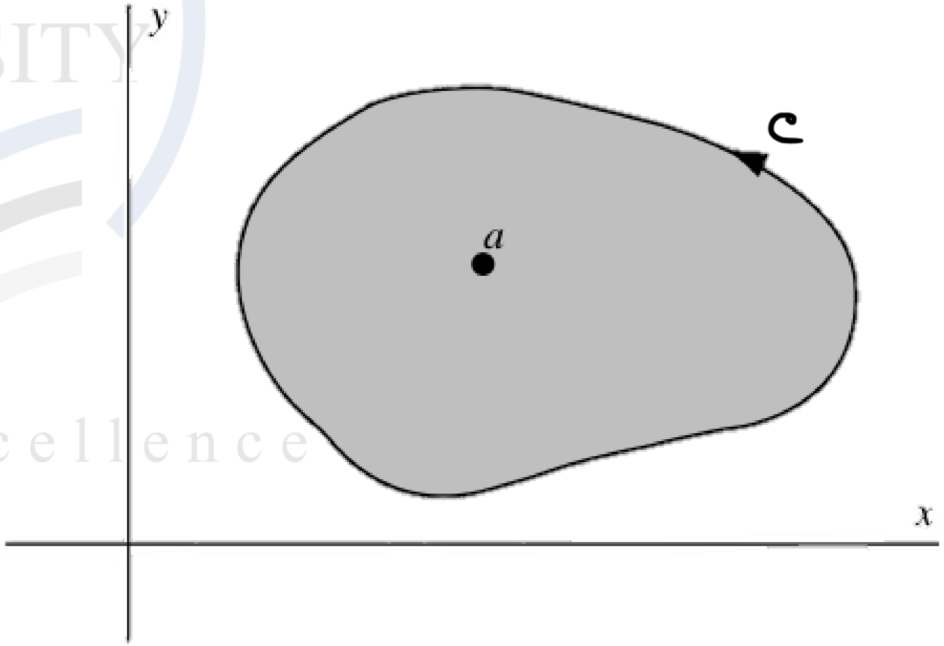
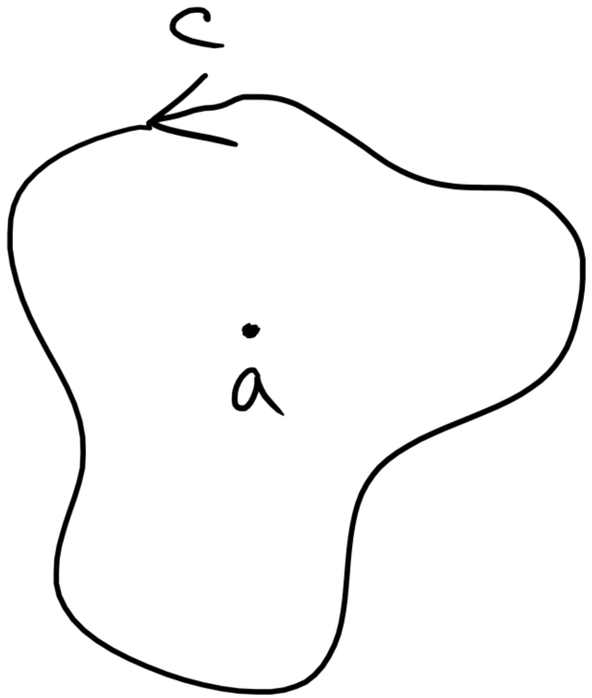
Cauchy Integral Formula

Let $f(z)$ be analytic inside and on a simple closed curve C and let a be any point inside C. Then

$$f(a) = \frac{1}{2\pi i}\oint_C \frac{f(z)}{z-a}\,dz$$



$$\oint_C \frac{f(z)}{(z-a)}\,dz = 2\pi i \cdot f(a)$$



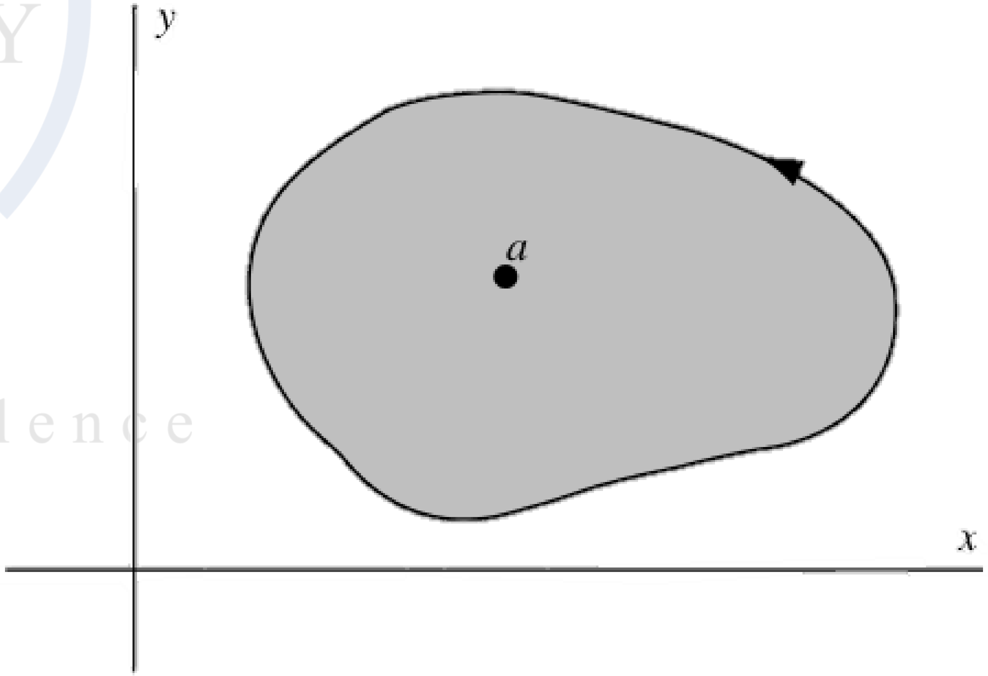
$$\oint \frac{f(z)}{z-a}\,dz = 0$$

Cauchy Integral Formula (General Version)

Let $f(z)$ be analytic inside and on a simple closed curve C and let a be any point inside C. Then

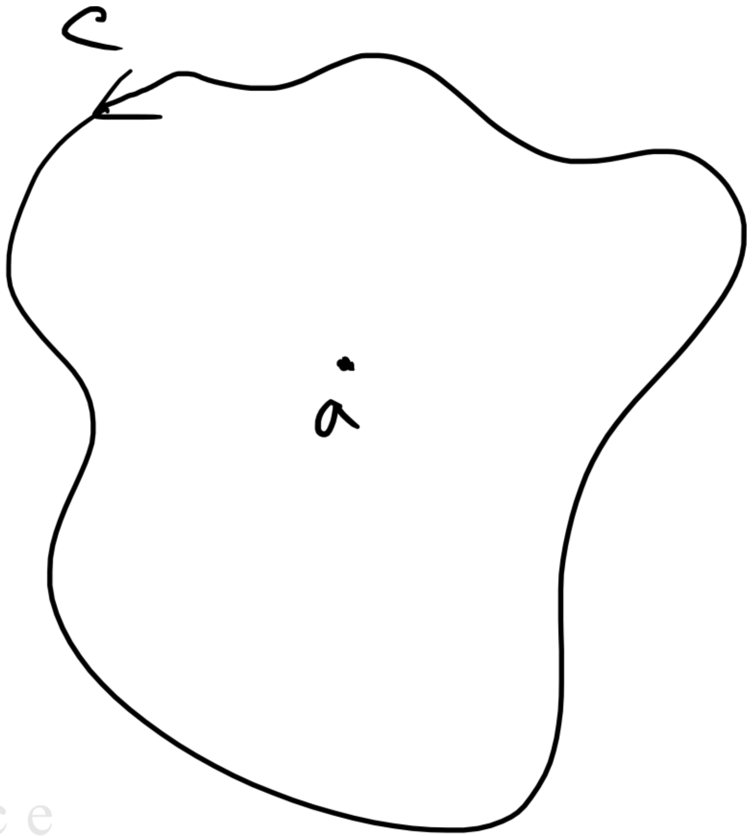
$$f^{n}(a) = \frac{n!}{2\pi i}\oint_{C} \frac{f(z)}{(z-a)^{n+1}}\,dz$$

where $n = 1,2,3,\ldots$

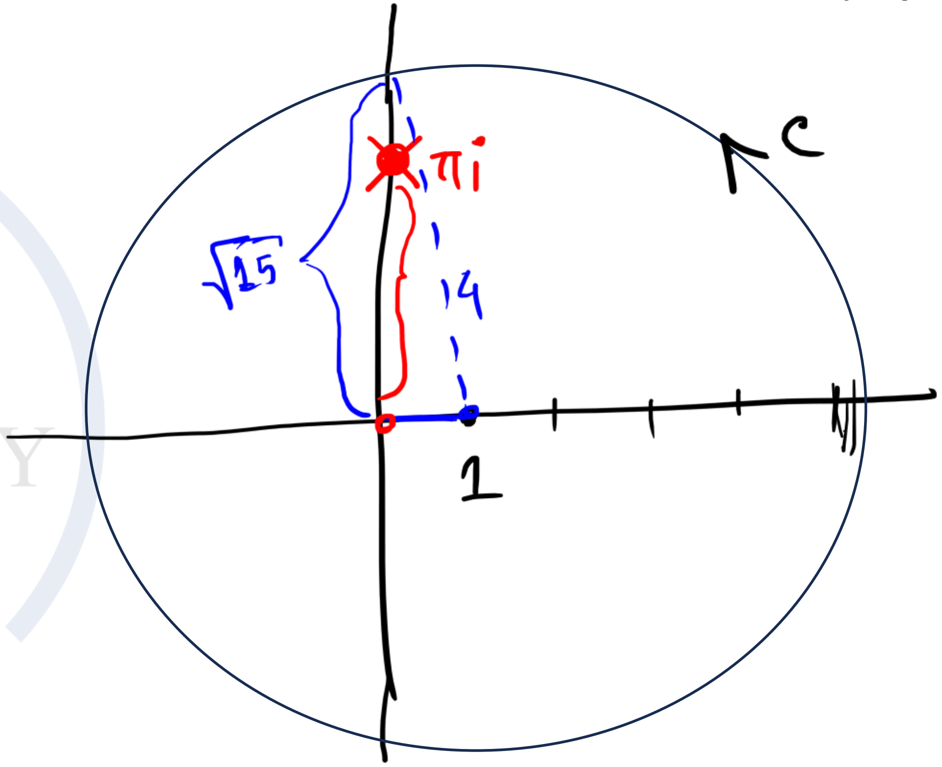


$$\oint_C \frac{f(z)}{(z-a)^{n+1}}\,dz = \frac{2\pi i}{n!} f^{n}(a)$$

$$= \frac{2\pi i}{n!}\left[\frac{d^n}{dz^n}\big(f(z)\big)\right]_{z=a}$$



Evaluate $\oint_C \frac{e^{3z}}{z-\pi i}\,dz$ where C is the circle $|z-1| = 4$.



$f(z) = e^{3z}$ $\quad a = \pi i$

$f(z)$ is analytic inside and on C.

Also $z = a = \pi i$ is inside C

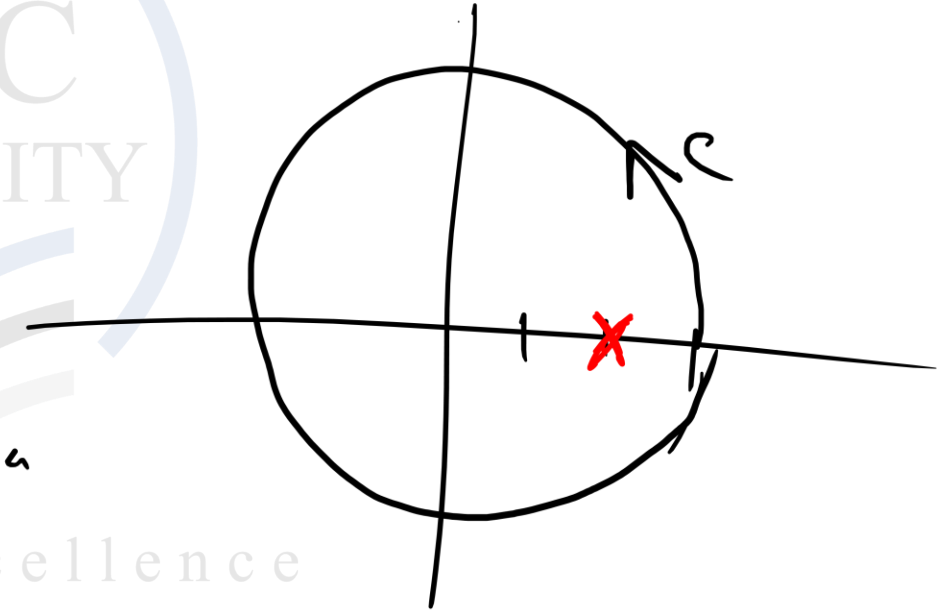
$$\oint_C \frac{f(z)}{z-\pi i}\,dz = 2\pi i \cdot f(\pi i)$$

$$= 2\pi i\, e^{3\pi i} = 2\pi i\left(\cos 3\pi + i \sin 3\pi\right) = -2\pi i \ \checkmark$$

Evaluate $\oint_C \frac{z^2 + \cos^2 \pi z}{z - 2}\, dz$ If (a) C: $|z| = 3$ (b) C: $|z| = 1$

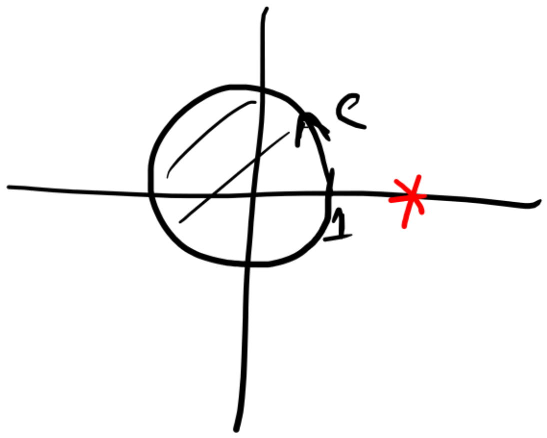
(a) $f(z) = z^2 + \cos^2 \pi z$ $a = 2$

$f(z)$ is analytic inside and on $c: |z| = 3$. Also $z = a = 2$ is inside c. using cauchy integral formula



$$\oint_c \frac{f(z)}{z-2}\, dz = 2\pi i \cdot f(2) = 2\pi i \cdot \left[2^2 + \cos^2 2\pi\right] = 10\pi i \checkmark$$

Evaluate $\oint_C \dfrac{z^2 + \cos^2 \pi z}{z-2}\, dz$ If (a) C: $|z| = 3$ (b) C: $|z| = 1$



$f(z) = \dfrac{z^2 + \cos^2 \pi z}{z-2}$ is analytic inside

and on $c: |z| = 1$

using cauchy-goursat theorem

$$\oint_c \frac{z^2 + \cos^2 \pi z}{z-2}\, dz = 0$$

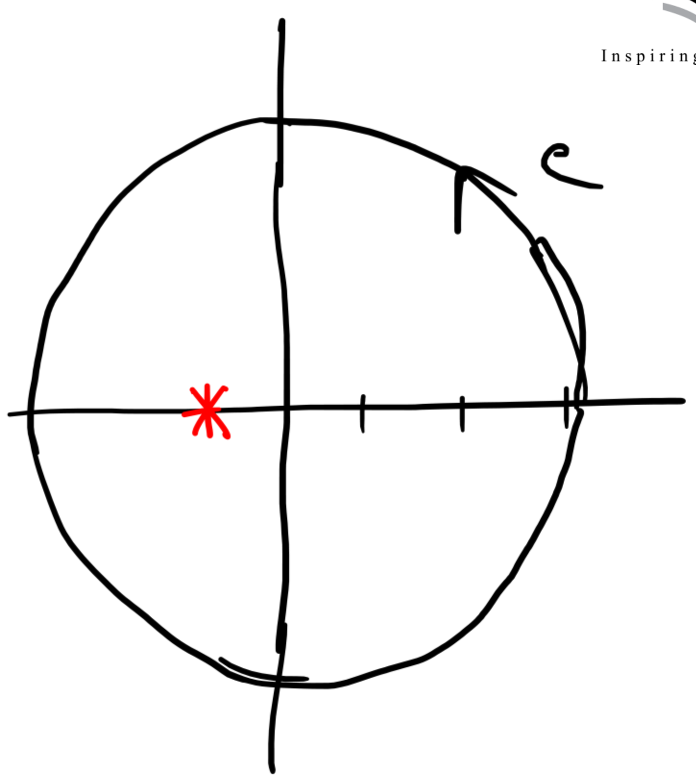
Evaluate $\oint_C \frac{e^{2z}}{(z+1)^4}\,dz$ where C is the circle $|z| = 3$.

$f(z) = e^{2z}$ and $a = -1$

$f(z)$ is analytic inside and on C.

Also $z = a = -1$ is inside C.

$$\oint_C \frac{f(z)}{(z-a)^{n+1}}\,dz = \frac{2\pi i}{n!} f^{n}(a)$$



$$\Rightarrow \oint_C \frac{f(z)}{(z+1)^{3+1}}\,dz = \frac{2\pi i}{3!} f^3(-1)$$

$$= \frac{2\pi i}{6} \cdot 8e^{-2}$$

$$= \frac{8\pi i}{3e^2}$$

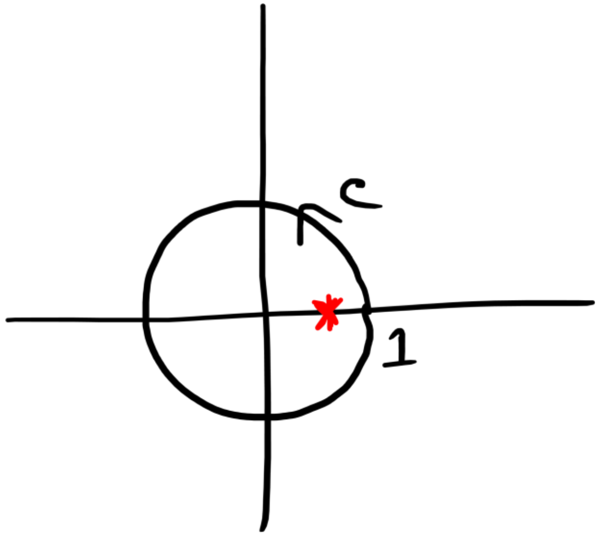
$$f(z) = e^{2z}$$

$$f'(z) = 2e^{2z}$$

$$f'''(z) = 8e^{2z}$$

Evaluate $\oint_C \frac{\sin^6 z}{\left(z-\frac{\pi}{6}\right)^4} dz$ where C is the circle $|z| = 1$

$f(z) = \sin^6 z$ and $a = \frac{\pi}{6}$



$f(z)$ is analytic inside and on c. Also

$z = a = \frac{\pi}{6}$ is inside c.

$$\oint_c \frac{f(z)}{(z-a)^{n+1}} dz = \frac{2\pi i}{n!} f^n(a)$$

$$\oint \frac{\sin^6 z}{\left(z-\frac{\pi}{6}\right)^4} dz = \frac{2\pi i}{3!}\left[\frac{d^3}{dz^3}\left(\sin^6 z\right)\right]_{z=\frac{\pi}{6}}$$

$$= \frac{2\pi i}{6}\left[\frac{d^2}{dz^2}\left(6\sin^5 z \cdot \cos z\right)\right]_{z=\frac{\pi}{6}}$$

$$= \frac{2\pi i}{3}\left[\frac{d}{dz}\left(30\sin^4 z \cdot \cos z \cdot \cos z - 6\sin^5 z \cdot \sin z\right)\right]$$

$$= \frac{2\pi i}{3}\left[\frac{d}{dz}\left(30\sin^4 z \cos^2 z - 6\sin^6 z\right)\right]_{z=\frac{\pi}{6}}$$

$$= \frac{2\pi i}{3}\left[\frac{d}{dz}\left(30\sin^4 z\cos^2 z - 6\sin^6 z\right)\right]_{z=\frac{\pi}{6}}$$

$$= \frac{2\pi i}{3}\left[120\sin^3 z\cos^3 z + 30\sin^4 z\cdot 2\cos z\,(-\sin z) - 36\sin^5 z\cdot\cos z\right]_{z=\frac{\pi}{6}}$$

$$= \frac{2\pi i}{3}\left[120\cdot\left(\frac{1}{2}\right)^3\cdot\left(\frac{\sqrt{3}}{2}\right)^3 - 30\left(\frac{1}{2}\right)^5\left(\frac{\sqrt{3}}{2}\right) - 36\left(\frac{1}{2}\right)^5\frac{\sqrt{3}}{2}\right]$$

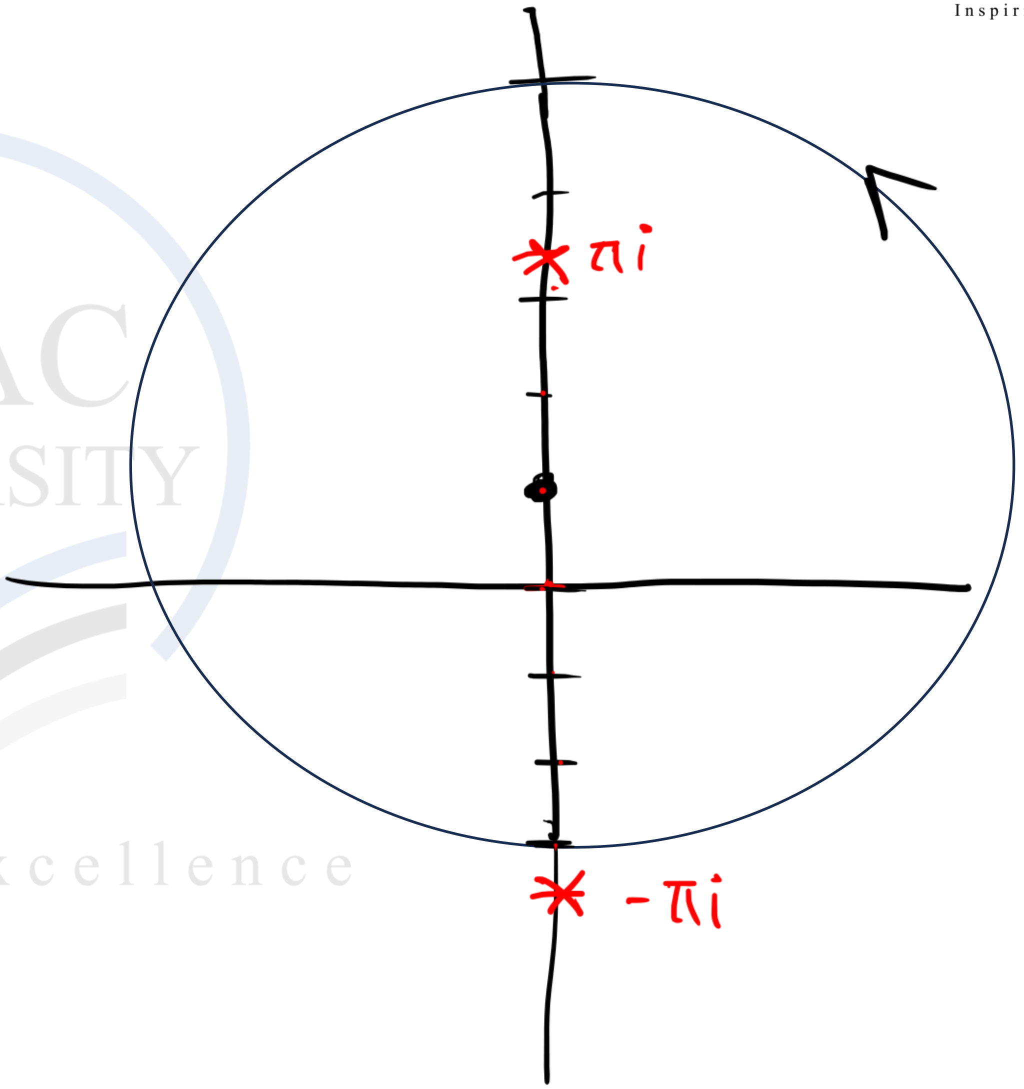
Evaluate $\oint_C \frac{e^z}{(z^2+\pi^2)^2}\,dz$ where C is the circle $|z-i|=4$

$$(z^2+\pi^2)^2=0$$

$$\Rightarrow z^2+\pi^2=0$$

$$\Rightarrow z^2=-\pi^2$$

$$\Rightarrow z=\pm\pi i$$



$$\oint_C \frac{e^z}{(z^2+\pi^2)^2}\,dz$$

$$= \oint \frac{e^z}{\{(z+\pi i)(z-\pi i)\}^2}\,dz$$

$$= \oint_c \frac{e^z}{(z+\pi i)^2\underbrace{(z-\pi i)^2}_{\times}}\,dz$$

$$= \oint \frac{\dfrac{e^z}{(z+\pi i)^2}}{(z-\pi i)^2}\,dz$$

$f(z) = \dfrac{e^z}{(z+\pi i)^2}$ and $a = \pi i$

$f(z)$ is analytic inside and on $C: |z-i| = 4$. Also $z = a = \pi i$ is in C.

$$\oint \frac{\frac{e^z}{(z+\pi i)^2}}{(z-\pi i)^2}\,dz = \frac{2\pi i}{1!}\left[\frac{d'}{dz'}\left(\frac{e^z}{(z+\pi i)^2}\right)\right]_{z=\pi i}$$

$$= 2\pi i\left[\frac{(z+\pi i)^2\cdot e^z - e^z\cdot 2(z+\pi i)}{(z+\pi i)^4}\right]_{z=\pi i}$$

$$= 2\pi i\,\frac{(2\pi i)^2 e^{\pi i} - 2\cdot e^{\pi i}(2\pi i)}{(2\pi i)^4}$$

The cases when we have more than a single singularity

Partial fraction

Residu Thm

(chapter-7)

Partial Fraction

$$\frac{1}{(x-1)(x-3)^2(x-5)^3} = \frac{A}{x-1} + \frac{B}{x-3} + \frac{C}{(x-3)^2} + \frac{D}{(x-5)} + \frac{E}{(x-5)^2} + \frac{F}{(x-5)^3}$$

$$\frac{1}{(x-1)(x-3)} = \frac{A}{x-1} + \frac{B}{x-3}$$

Type-I (no repetition of roots)

$$\frac{1}{(z-1)(z-2)} \equiv \frac{A}{z-1} + \frac{B}{z-2} \quad \text{——} \ (1)$$

$$\Rightarrow 1 \equiv A(z-2) + B(z-1) \quad \text{——} \ (2)$$

$\underline{\underline{z=2}}$

$1 = B$

$\Rightarrow B = 1$

$\underline{\underline{z=1}}$

$1 = A(-1)$

$\Rightarrow A = -1$

$$\therefore \frac{1}{(z-1)(z-2)} \equiv \frac{-1}{z-1} + \frac{1}{z-2}$$

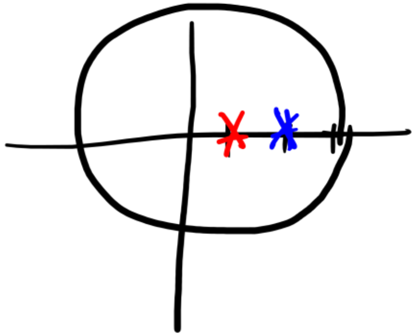
$x^2 - 1 = 3$

$\Rightarrow x = 2, -2$

$(x+1)^2 - 1 \equiv x^2 + 2x$

$\Rightarrow$

Evaluate $\oint_C \dfrac{\sin \pi z^2 + \cos \pi z^2}{(z-1)(z-2)}\,dz$ where C is the circle $|z| = 3$.



$$\frac{1}{(z-1)(z-2)} = \frac{-1}{z-1} + \frac{1}{z-2}$$

$$\Rightarrow \oint_C \frac{\sin \pi z^2 + \cos \pi z^2}{(z-1)(z-2)} = -\oint_C \frac{\sin \pi z^2 + \cos \pi z^2}{z-1}\,dz + \oint_C \frac{\sin \pi z^2 + \cos \pi z^2}{z-2}\,dz$$

$$= -2\pi i \cdot \left[\sin \pi 1^2 + \cos \pi 1^2\right] + 2\pi i \cdot \left[\sin \pi 2^2 + \cos \pi 2^2\right]$$

$$= -2\pi i\left[-1\right] + 2\pi i\left[1\right]$$

$$= 4\pi i$$

Type-II (with repetition of roots)

$$\frac{1}{(z-1)(z-2)^2} \equiv \frac{A}{z-1} + \frac{B}{z-2} + \frac{C}{(z-2)^2}$$

$$\Rightarrow \quad 1 \equiv A(z-2)^2 + B(z-1)(z-2) + C(z-1)$$

$$\Rightarrow \quad \underbrace{1} \equiv Az^2 - 4Az + 4A + Bz^2 - 3Bz + 2B + Cz - C$$

$\underline{z=1}$

$1 = A$

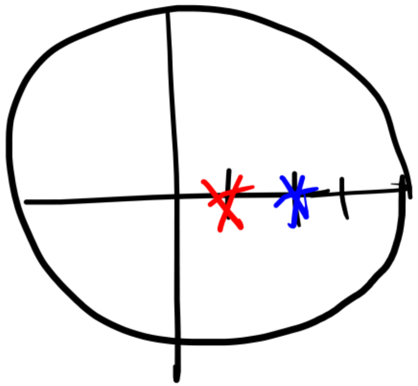
$\underline{\underline{z=2}}$

$1 = C$

equating the coefficients of z^2

$0 = A + B \Rightarrow B = -1$

Evaluate $\oint_C \frac{e^{2z}}{(z-1)(z-2)^2}\,dz$ where C is the circle $|z| = 4$.



$$\oint \frac{e^{2z}}{(z-1)(z-2)^2}\,dz = \oint \frac{e^{2z}}{z-1}\,dz - \oint \frac{e^{2z}}{z-2}\,dz + \oint \frac{e^{2z}}{(z-2)^{1+1}}\,dz$$

$$= 2\pi i \cdot \left[e^{2z}\right]_{z=1} - 2\pi i \cdot \left[e^{2z}\right]_{z=2} + \frac{2\pi i}{1!} \cdot \left[\frac{d}{dz}\left(e^{2z}\right)\right]_{z=2}$$

$$= 2\pi i e^{2} - 2\pi i e^{4} + 2\pi i\, 2 \cdot e^{4}$$

Evaluate $\oint_C \frac{e^{2z}}{(z^2+\pi^2)^2}\,dz$ where C is the circle $|z| = 4$.

$$\frac{1}{(z^2+\pi^2)^2} = \frac{1}{(z+i\pi)^2 (z-i\pi)^2} = \text{—} + \text{—} + \text{—} + \text{—}$$

$$\frac{1}{(z^2+\pi^2)^2} = \frac{1}{(z+\pi i)^2(z-\pi i)^2}$$

$$\frac{1}{(z+\pi i)^2(z-\pi i)^2} = \frac{A}{(z+\pi i)} + \frac{B}{(z+\pi i)^2} + \frac{B}{(z-\pi i)} + \frac{D}{(z-\pi i)^2}$$

$$\frac{1e^{2t}}{(z+\pi i)^2(z-\pi i)^2} = \int\frac{\frac{i}{4\pi^3}e^{2t}}{(z+\pi i)} + \int\frac{\frac{-1}{4\pi^2}e^{2t}}{(z+\pi i)^2} + \int\frac{\frac{-i}{4\pi^3}e^{2t}}{(z-\pi i)} + \int\frac{\frac{-1}{4\pi^2}e^{2t}}{(z-\pi i)^2}$$

Hw

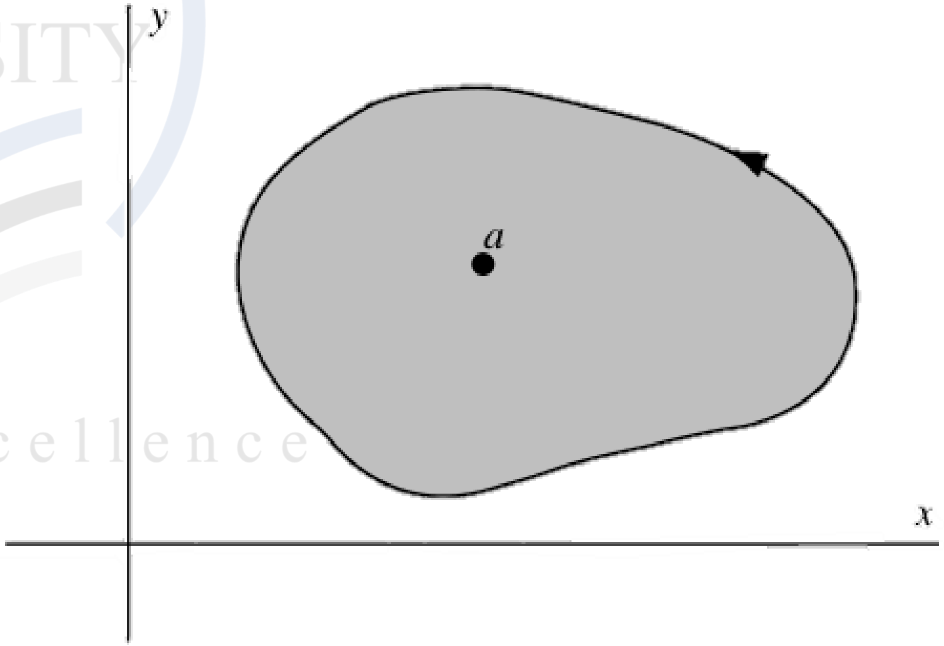
Evaluate $\frac{1}{2\pi i}\oint_C \frac{e^{zt}}{(z^2+1)^2}dz$ if $t>0$ and C is the circle $|z|=3$.

Hw

Proof of Cauchy Integral Formula

Let $f(z)$ be analytic inside and on a simple closed curve C and let a be any point inside C. Then

$$f(a) = \frac{1}{2\pi i}\oint_C \frac{f(z)}{z-a}\,dz$$



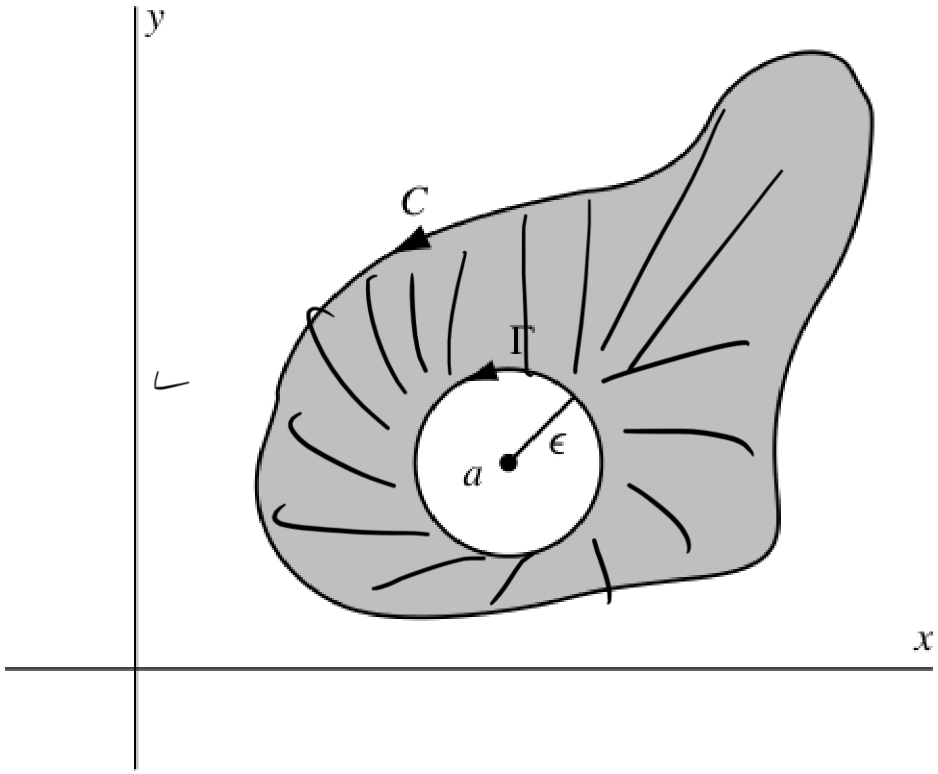
Proof:

The function $\frac{\boldsymbol{f(z)}}{\boldsymbol{z-a}}$ is analytic inside and on C except at the point $\boldsymbol{z=a}$.

Then by a consequence of Cauchy-Goursat theorem, we have

$$\oint_C \frac{f(z)}{z-a}dz = \oint_\Gamma \frac{f(z)}{z-a}dz$$

where we can choose Γ as a circle of radius ϵ with center at a.



Then an equation for Γ is $|z-a| = \epsilon$

$\Rightarrow z - a = \epsilon e^{i\theta}$ where $0 \leq \theta \leq 2\pi$.

$$\oint_{\Gamma} \frac{f(z)}{z-a}\,dz = \int_0^{2\pi} \frac{f(a+\epsilon e^{i\theta})\,i\epsilon e^{i\theta}}{\epsilon e^{i\theta}}\,d\theta = i\int_0^{2\pi} f(a+\epsilon e^{i\theta})\,d\theta$$

Thus $$\oint_{C} \frac{f(z)}{z-a}\,dz = i\int_0^{2\pi} f(a+\epsilon e^{i\theta})\,d\theta$$

Taking the limit of both sides of and making use of the continuity of $f(z)$

$$\lim_{\epsilon \to 0} \oint_C \frac{f(z)}{z-a} dz = \lim_{\epsilon \to 0} i \int_0^{2\pi} f(a + \epsilon e^{i\theta})\, d\theta$$

$$\oint_C \frac{f(z)}{z-a} dz = \lim_{\epsilon \to 0} i \int_0^{2\pi} f(a + \epsilon e^{i\theta})\, d\theta$$

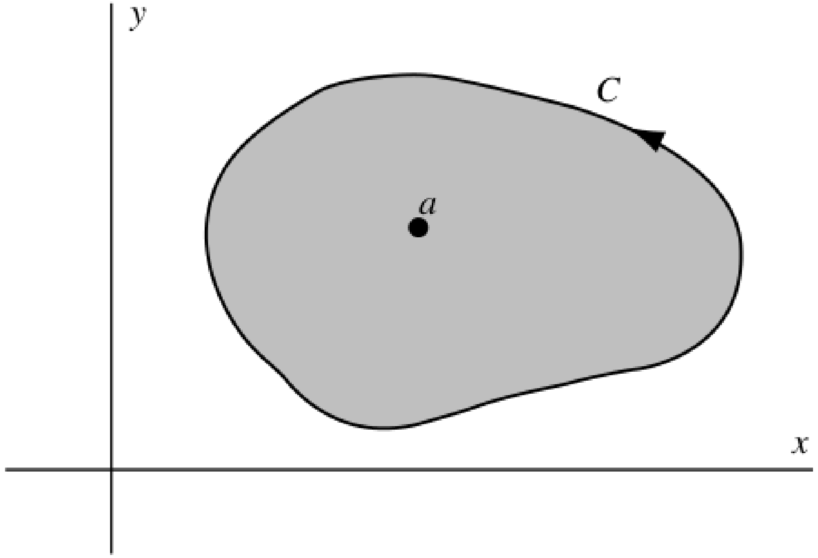
$$= i \int_0^{2\pi} \lim_{\epsilon \to 0} f(a + \cancel{\epsilon e}^{i\theta})\, d\theta = i \int_0^{2\pi} \underline{f(a)}\, d\theta = 2\pi i\, f(a)$$

So, $$f(a) = \frac{1}{2\pi i} \oint_C \frac{f(z)}{z-a} dz$$

Proof of Cauchy Integral Formula (General Version)

Let $f(z)$ be analytic inside and on a simple closed curve C and let a be any point inside C. Then

$$f^{(n)}(a) = \frac{n!}{2\pi i}\oint_C \frac{f(z)}{(z-a)^{n+1}}\,dz \quad n = 1.\ 2.\ 3.\ldots$$



Proof:

$$\frac{d}{da}\left(\frac{1}{z-a}\right) = \frac{d}{da}\left((z-a)^{-1}\right) = -1(z-a)^{-2}(-1)$$

$$= \frac{1!}{(z-a)^2}$$

$$\frac{d^2}{da^2}\left(\frac{1}{z-a}\right) = \frac{2!}{(z-a)^3}$$

$$\frac{d^3}{da^3}\left(\frac{1}{z-a}\right) = \frac{3!}{(z-a)^4}$$

$$\frac{d^n}{da^n}\left(\frac{1}{z-a}\right) = \frac{n!}{(z-a)^{n+1}}$$

$$f(a) = \frac{1}{2\pi i} \oint_c \frac{f(z)}{z-a}\, dz$$

$$\frac{d^n}{da^n} f(a) = \frac{1}{2\pi i} \oint_c \frac{\partial^n}{\partial a^n} \frac{f(z)}{(z-a)}\, dz$$

$$f^n(a) = \frac{1}{2\pi i} \oint \frac{f(z) \cdot n!}{(z-a)^{n+1}}\, dz$$

$$\Rightarrow f^n(a) = \frac{n!}{2\pi i} \oint_c \frac{f(z)}{(z-a)^{n+1}}\, dz$$

Liouville's theorem

Suppose that for all z in the entire complex plane,

(i) $f(z)$ is analytic and

(ii) $f(z)$ is bounded, i.e. $|f(z)| \leq M$ for some constant M.

$\Rightarrow$ Then $f(z)$ must be a constant.

Fundamental Theorem of Algebra

Every polynomial equation

$$P(z) = a_0 + a_1 z + a_2 z^2 + \cdots + a_n z^n = 0$$

with degree $n \geq 1$ and $a_n \neq 0$ has at least one root over $\mathbb{C}$.

From this it follows that $P(z) = 0$ has exactly n roots, due attention being paid to multiplicities of roots.

$\mathbb{C}$

$\underline{\underline{\mathbb{R}}}$

$p(z) = 3z$ ✓

$p(z) = 5$ ✓ $= 5 \cdot z^0$

$p(z) = 0$ ✓ $= 0 \cdot z^2$

$p(z) = 3 + z^{-2}$ ✗

$p(z) = 3 + \sqrt{z}$ ✗

$\boxed{a \cdot z^n}$

nonnegative integer.

complex

$$\underbrace{1 + z + z^2 + z^3 + \cdots}\;=\;\underbrace{\frac{1}{1-z}}$$

$z = 1$ ✓

Abel–Ruffini Theorem (from Galois Theory)

There is no general formula in radicals (addition, subtraction, multiplication, division, power, n^{th} root etc.) of the equation

$$P(z) = a_0 + a_1 z + a_2 z^2 + \cdots + a_n z^n = 0$$

with degree $n \geq 5$ and $a_n \neq 0$.

$ax^3 + bx^2 + cx + d = 0$

$$x_1 = -\frac{b}{3a} - \frac{1}{3a}\sqrt[3]{\frac{1}{2}\left[2b^3 - 9abc + 27a^2d + \sqrt{\left(2b^3 - 9abc + 27a^2d\right)^2 - 4\left(b^2 - 3ac\right)^3}\right]} - \frac{1}{3a}\sqrt[3]{\frac{1}{2}\left[2b^3 - 9abc + 27a^2d - \sqrt{\left(2b^3 - 9abc + 27a^2d\right)^2 - 4\left(b^2 - 3ac\right)^3}\right]}$$

$$x_2 = -\frac{b}{3a} + \frac{1 + i\sqrt{3}}{6a}\sqrt[3]{\frac{1}{2}\left[2b^3 - 9abc + 27a^2d + \sqrt{\left(2b^3 - 9abc + 27a^2d\right)^2 - 4\left(b^2 - 3ac\right)^3}\right]} + \frac{1 - i\sqrt{3}}{6a}\sqrt[3]{\frac{1}{2}\left[2b^3 - 9abc + 27a^2d - \sqrt{\left(2b^3 - 9abc + 27a^2d\right)^2 - 4\left(b^2 - 3ac\right)^3}\right]}$$

$$x_3 = -\frac{b}{3a} + \frac{1 - i\sqrt{3}}{6a}\sqrt[3]{\frac{1}{2}\left[2b^3 - 9abc + 27a^2d + \sqrt{\left(2b^3 - 9abc + 27a^2d\right)^2 - 4\left(b^2 - 3ac\right)^3}\right]} + \frac{1 + i\sqrt{3}}{6a}\sqrt[3]{\frac{1}{2}\left[2b^3 - 9abc + 27a^2d - \sqrt{\left(2b^3 - 9abc + 27a^2d\right)^2 - 4\left(b^2 - 3ac\right)^3}\right]}$$

e

$ax^2 + bx + c = 0$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$ax^4+bx^3+cx^2+dx+e=0\ (a\neq 0)$$

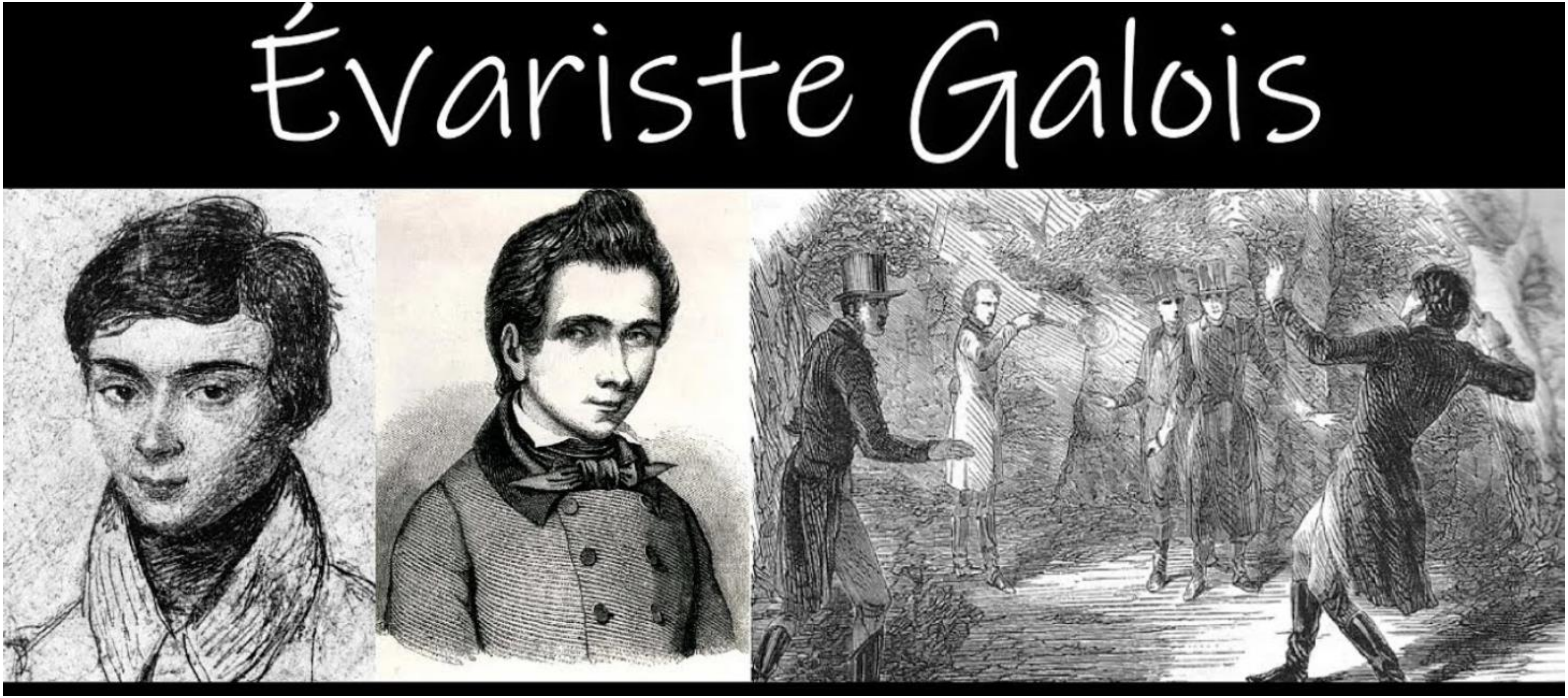
$$x_1=-\frac{b}{4a}+\frac{1}{2}\sqrt{\frac{b^2}{4a^2}-\frac{2c}{3a}+\frac{\sqrt[3]{2}\,(c^2-3bd+12ae)}{3a\sqrt[3]{2c^3-9bcd+27ad^2+27b^2e-72ace+\sqrt{-4(c^2-3bd+12ae)^3+(2c^3-9bcd+27ad^2+27b^2e-72ace)^2}}}+\frac{\sqrt[3]{2c^3-9bcd+27ad^2+27b^2e-72ace+\sqrt{-4(c^2-3bd+12ae)^3+(2c^3-9bcd+27ad^2+27b^2e-72ace)^2}}}{3\sqrt[3]{2}a}}-\frac{1}{2}\sqrt{\frac{b^2}{2a^2}-\frac{4c}{3a}-\frac{\sqrt[3]{2}\,(c^2-3bd+12ae)}{3a\sqrt[3]{2c^3-9bcd+27ad^2+27b^2e-72ace+\sqrt{-4(c^2-3bd+12ae)^3+(2c^3-9bcd+27ad^2+27b^2e-72ace)^2}}}-\frac{\sqrt[3]{2c^3-9bcd+27ad^2+27b^2e-72ace+\sqrt{-4(c^2-3bd+12ae)^3+(2c^3-9bcd+27ad^2+27b^2e-72ace)^2}}}{3\sqrt[3]{2}a}+\frac{-b^3+4abc-8a^2d}{4a^3\sqrt{\frac{b^2}{4a^2}-\frac{2c}{3a}+\frac{\sqrt[3]{2}\,(c^2-3bd+12ae)}{3a\sqrt[3]{2c^3-9bcd+27ad^2+27b^2e-72ace+\sqrt{-4(c^2-3bd+12ae)^3+(2c^3-9bcd+27ad^2+27b^2e-72ace)^2}}}+\frac{\sqrt[3]{2c^3-9bcd+27ad^2+27b^2e-72ace+\sqrt{-4(c^2-3bd+12ae)^3+(2c^3-9bcd+27ad^2+27b^2e-72ace)^2}}}{3\sqrt[3]{2}a}}}}$$

$$x_2=-\frac{b}{4a}+\frac{1}{2}\sqrt{\frac{b^2}{4a^2}-\frac{2c}{3a}+\frac{\sqrt[3]{2}\,(c^2-3bd+12ae)}{3a\sqrt[3]{2c^3-9bcd+27ad^2+27b^2e-72ace+\sqrt{-4(c^2-3bd+12ae)^3+(2c^3-9bcd+27ad^2+27b^2e-72ace)^2}}}+\frac{\sqrt[3]{2c^3-9bcd+27ad^2+27b^2e-72ace+\sqrt{-4(c^2-3bd+12ae)^3+(2c^3-9bcd+27ad^2+27b^2e-72ace)^2}}}{3\sqrt[3]{2}a}}+\frac{1}{2}\sqrt{\frac{b^2}{2a^2}-\frac{4c}{3a}-\frac{\sqrt[3]{2}\,(c^2-3bd+12ae)}{3a\sqrt[3]{2c^3-9bcd+27ad^2+27b^2e-72ace+\sqrt{-4(c^2-3bd+12ae)^3+(2c^3-9bcd+27ad^2+27b^2e-72ace)^2}}}-\frac{\sqrt[3]{2c^3-9bcd+27ad^2+27b^2e-72ace+\sqrt{-4(c^2-3bd+12ae)^3+(2c^3-9bcd+27ad^2+27b^2e-72ace)^2}}}{3\sqrt[3]{2}a}+\frac{-b^3+4abc-8a^2d}{4a^3\sqrt{\frac{b^2}{4a^2}-\frac{2c}{3a}+\frac{\sqrt[3]{2}\,(c^2-3bd+12ae)}{3a\sqrt[3]{2c^3-9bcd+27ad^2+27b^2e-72ace+\sqrt{-4(c^2-3bd+12ae)^3+(2c^3-9bcd+27ad^2+27b^2e-72ace)^2}}}+\frac{\sqrt[3]{2c^3-9bcd+27ad^2+27b^2e-72ace+\sqrt{-4(c^2-3bd+12ae)^3+(2c^3-9bcd+27ad^2+27b^2e-72ace)^2}}}{3\sqrt[3]{2}a}}}}$$

$$x_3=-\frac{b}{4a}-\frac{1}{2}\sqrt{\frac{b^2}{4a^2}-\frac{2c}{3a}+\frac{\sqrt[3]{2}\,(c^2-3bd+12ae)}{3a\sqrt[3]{2c^3-9bcd+27ad^2+27b^2e-72ace+\sqrt{-4(c^2-3bd+12ae)^3+(2c^3-9bcd+27ad^2+27b^2e-72ace)^2}}}+\frac{\sqrt[3]{2c^3-9bcd+27ad^2+27b^2e-72ace+\sqrt{-4(c^2-3bd+12ae)^3+(2c^3-9bcd+27ad^2+27b^2e-72ace)^2}}}{3\sqrt[3]{2}a}}-\frac{1}{2}\sqrt{\frac{b^2}{2a^2}-\frac{4c}{3a}-\frac{\sqrt[3]{2}\,(c^2-3bd+12ae)}{3a\sqrt[3]{2c^3-9bcd+27ad^2+27b^2e-72ace+\sqrt{-4(c^2-3bd+12ae)^3+(2c^3-9bcd+27ad^2+27b^2e-72ace)^2}}}-\frac{\sqrt[3]{2c^3-9bcd+27ad^2+27b^2e-72ace+\sqrt{-4(c^2-3bd+12ae)^3+(2c^3-9bcd+27ad^2+27b^2e-72ace)^2}}}{3\sqrt[3]{2}a}+\frac{-b^3+4abc-8a^2d}{4a^3\sqrt{\frac{b^2}{4a^2}-\frac{2c}{3a}+\frac{\sqrt[3]{2}\,(c^2-3bd+12ae)}{3a\sqrt[3]{2c^3-9bcd+27ad^2+27b^2e-72ace+\sqrt{-4(c^2-3bd+12ae)^3+(2c^3-9bcd+27ad^2+27b^2e-72ace)^2}}}+\frac{\sqrt[3]{2c^3-9bcd+27ad^2+27b^2e-72ace+\sqrt{-4(c^2-3bd+12ae)^3+(2c^3-9bcd+27ad^2+27b^2e-72ace)^2}}}{3\sqrt[3]{2}a}}}}$$

$$x_4=-\frac{b}{4a}-\frac{1}{2}\sqrt{\frac{b^2}{4a^2}-\frac{2c}{3a}+\frac{\sqrt[3]{2}\,(c^2-3bd+12ae)}{3a\sqrt[3]{2c^3-9bcd+27ad^2+27b^2e-72ace+\sqrt{-4(c^2-3bd+12ae)^3+(2c^3-9bcd+27ad^2+27b^2e-72ace)^2}}}+\frac{\sqrt[3]{2c^3-9bcd+27ad^2+27b^2e-72ace+\sqrt{-4(c^2-3bd+12ae)^3+(2c^3-9bcd+27ad^2+27b^2e-72ace)^2}}}{3\sqrt[3]{2}a}}+\frac{1}{2}\sqrt{\frac{b^2}{2a^2}-\frac{4c}{3a}-\frac{\sqrt[3]{2}\,(c^2-3bd+12ae)}{3a\sqrt[3]{2c^3-9bcd+27ad^2+27b^2e-72ace+\sqrt{-4(c^2-3bd+12ae)^3+(2c^3-9bcd+27ad^2+27b^2e-72ace)^2}}}-\frac{\sqrt[3]{2c^3-9bcd+27ad^2+27b^2e-72ace+\sqrt{-4(c^2-3bd+12ae)^3+(2c^3-9bcd+27ad^2+27b^2e-72ace)^2}}}{3\sqrt[3]{2}a}+\frac{-b^3+4abc-8a^2d}{4a^3\sqrt{\frac{b^2}{4a^2}-\frac{2c}{3a}+\frac{\sqrt[3]{2}\,(c^2-3bd+12ae)}{3a\sqrt[3]{2c^3-9bcd+27ad^2+27b^2e-72ace+\sqrt{-4(c^2-3bd+12ae)^3+(2c^3-9bcd+27ad^2+27b^2e-72ace)^2}}}+\frac{\sqrt[3]{2c^3-9bcd+27ad^2+27b^2e-72ace+\sqrt{-4(c^2-3bd+12ae)^3+(2c^3-9bcd+27ad^2+27b^2e-72ace)^2}}}{3\sqrt[3]{2}a}}}}$$

The Epic Duel



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