

Lecture 20

Topics:

1. Permutations
2. Combinations

6.3 Permutations

EXAMPLE 1 In how many ways can we select three students from a group of five students to stand in line for a picture? In how many ways can we arrange all five of these students in a line for a picture?

Solution: First, note that the order in which we select the students matters. There are five ways to select the first student to stand at the start of the line. Once this student has been selected, there are four ways to select the second student in the line. After the first and second students have been selected, there are three ways to select the third student in the line. By the product rule, there are $5 * 4 * 3 = 60$ ways to select three students from a group of five students to stand in line for a picture. To arrange all five students in a line for a picture, we select the first student in five ways, the second in four ways, the third in three ways, the fourth in two ways, and the fifth in one way. Consequently, there are $5 * 4 * 3 * 2 * 1 = 120$ ways to arrange all five students in a line for a picture.

Permutations

Example 1 Illustrates how ordered arrangements of distinct objects can be counted. This leads to some terminology. A permutation of a set of distinct objects is an ordered arrangement of these objects. We also are interested in ordered arrangements of some of the elements of a set. An ordered arrangement of r elements of a set is called an r -permutation.

EXAMPLE 2 Let $S = \{1, 2, 3\}$. The ordered arrangement 3, 1, 2 is a permutation of S . The ordered arrangement 3, 2 is a 2-permutation of S .

The number of r -permutations of a set with n elements is denoted by $P(n, r)$. We can find $P(n, r)$ using the product rule.

Theorem 1:

If n is a positive integer and r is an integer with $1 \leq r \leq n$, then there are $P(n, r) = n(n - 1)(n - 2) \cdots (n - r + 1)$ r -permutations of a set with n distinct elements.

Proof: We will use the product rule to prove that this formula is correct. The first element of the permutation can be chosen in n ways because there are n elements in the set. There are $n - 1$ ways to choose the second element of the permutation, because there are $n - 1$ elements left in the set after using the element picked for the first position. Similarly, there are $n - 2$ ways to choose the third element, and so on, until there are exactly $n - (r - 1) = n - r + 1$ ways to choose the r th element. Consequently, by the product rule, there are $n(n - 1)(n - 2) \cdots (n - r + 1)$ r -permutations of the set.

COROLLARY 1: If n and r are integers with $0 \leq r \leq n$, then $P(n, r) = n! / (n - r)!$.

Proof: When n and r are integers with $1 \leq r \leq n$, by Theorem 1 we have

$$P(n, r) = n(n - 1)(n - 2) \cdots (n - r + 1) = n! / (n - r)!$$

EXAMPLE 4 How many ways are there to select a first-prize winner, a second-prize winner, and a third-prize winner from 100 different people who have entered a contest?

Solution: Because it matters which person wins which prize, the number of ways to pick the three prize winners is the number of ordered selections of three elements from a set of 100 elements, that is, the number of 3-permutations of a set of 100 elements. Consequently, the answer is

$$P(100, 3) = 100 * 99 * 98 = 970200.$$

EXAMPLE 5 Suppose that there are eight runners in a race. The winner receives a gold medal, the second place finisher receives a silver medal, and the third-place finisher receives a bronze medal. How many different ways are there to award these medals, if all possible outcomes of the race can occur and there are no ties?

Solution: The number of different ways to award the medals is the number of 3-permutations of a set with eight elements. Hence, there are $P(8, 3) = 8 * 7 * 6 = 336$ possible ways to award the medals.

EXAMPLE 7 How many permutations of the letters ABCDEFGH contain the string ABC ?

Solution: Because the letters ABC must occur as a block, we can find the answer by finding the number of permutations of six objects, namely, the block ABC and the individual letters D, E, F, G, and H. Because these six objects can occur in any order, there are $6! = 720$ permutations of the letters ABCDEFGH in which ABC occurs as a block.

6.3.3 Combinations

We now turn our attention to counting unordered selections of objects. We begin by solving a question posed in the introduction to this section of the chapter. **EXAMPLE 8** How many different committees of three students can be formed from a group of four students?

Solution: To answer this question, we need only find the number of subsets with three elements from the set containing the four students. We see that there are four such subsets, one for each of the four students, because choosing three students is the same as choosing one of the four students to leave out of the group. This means that there are four ways to choose the three students for the committee, where the order in which these students are chosen does not matter.

An r -combination of elements of a set is an unordered selection of r elements from the set. Thus, an r -combination is simply a subset of the set with r elements.

EXAMPLE 9 Let S be the set $\{1, 2, 3, 4\}$. Then $\{1, 3, 4\}$ is a 3-combination from S . (Note that $\{4, 1, 3\}$ is the same 3-combination as $\{1, 3, 4\}$, because the order in which the elements of a set are listed does not matter.)

The number of r -combinations of a set with n distinct elements is denoted by $C(n, r)$. Note that $C(n, r)$ is also denoted by $\binom{n}{r}$ and is called a binomial coefficient. We will learn where this terminology comes from in Section 6.4.

EXAMPLE 10 We see that $C(4, 2) = 6$, because the 2-combinations of $\{a, b, c, d\}$ are the six subsets $\{a, b\}$, $\{a, c\}$, $\{a, d\}$, $\{b, c\}$, $\{b, d\}$, and $\{c, d\}$.

COROLLARY 2

Let n and r be nonnegative integers with $r \leq n$. Then $C(n, r) = C(n, n - r)$.

Proof:

From Theorem 2 it follows that,

$$C(n, r) = \frac{n!}{r! (n - r)!} \text{ And}$$

$$C(n, n - r) = \frac{n!}{(n - r)! [n - (n - r)]!} = \frac{n!}{(n - r)! r!} .$$

Hence, $C(n, r) = C(n, n - r)$.

Definition 1 A combinatorial proof of an identity is a proof that uses counting arguments to prove that both sides of the identity count the same objects but in different ways or a proof that is based on showing that there is a bijection between the sets of objects counted by the two sides of the identity. These two types of proofs are called **double counting proofs** and **bijection proofs**, respectively.

Proof: We will use a bijective proof to show that $C(n, r) = C(n, n - r)$ for all integers n and r with $0 \leq r \leq n$. Suppose that S is a set with n elements. The function that maps a subset A of S to A^c is a bijection between subsets of S with r elements and subsets with $n - r$ elements (as the reader should verify).

The identity $C(n, r) = C(n, n - r)$ follows because when there is a bijection between two finite sets, the two sets must have the same number of elements.

Alternatively, we can reformulate this argument as a double counting proof.

By definition, the number of subsets of S with r elements equals $C(n, r)$. But each subset A of S is also determined by specifying which elements are not in A , and so are in A . Because the complement of a subset of S with r elements has $n - r$ elements, there are also $C(n, n - r)$ subsets of S with r elements.

It follows that $C(n, r) = C(n, n - r)$.

EXAMPLE 13 A group of 30 people have been trained as astronauts to go on the first mission to Mars. How many ways are there to select a crew of six people to go on this mission (assuming that all crew members have the same job)?

Solution: The number of ways to select a crew of six from the pool of 30 people is the number of 6-combinations of a set with 30 elements, because the order in which these people are chosen does not matter. By Theorem 2, the number of such combinations is $C(30, 6) = 30! / (6! * 24!)$

$$= (30 * 29 * 28 * 27 * 26 * 25) / (6 * 5 * 4 * 3 * 2 * 1) = 593,775.$$

EXAMPLE 15: Suppose that there are 9 faculty members in the mathematics department and 11 in the computer science department. How many ways are there to select a committee to develop a discrete mathematics course at a school if the committee is to consist of three faculty members from the mathematics department and four from the computer science department?

Solution: By the product rule, the answer is the product of the number of 3-combinations of a set with nine elements and the number of 4-combinations of a set with 11 elements. By Theorem 2, the number of ways to select the committee is

$$C(9, 3) * C(11, 4) = (9! / 3!6!) * (11! / 4!7!) = 84 \cdot 330 = 27,720.$$

End of the Final Syllabus