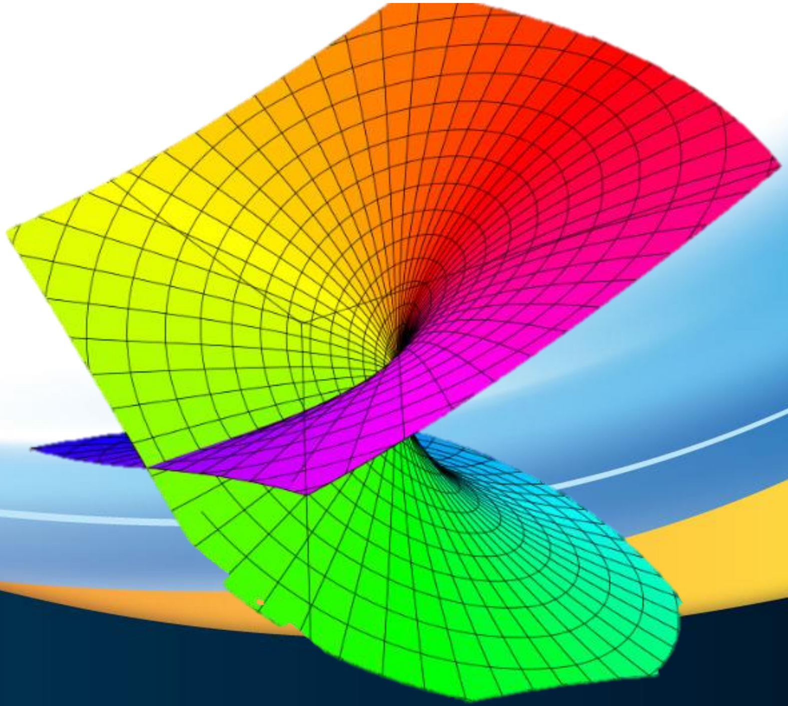
Undergraduate Course in Mathematics

Complex Variables

Topic: Harmonic Functions & Conjugate



Conducted By

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Harmonic Function Definition

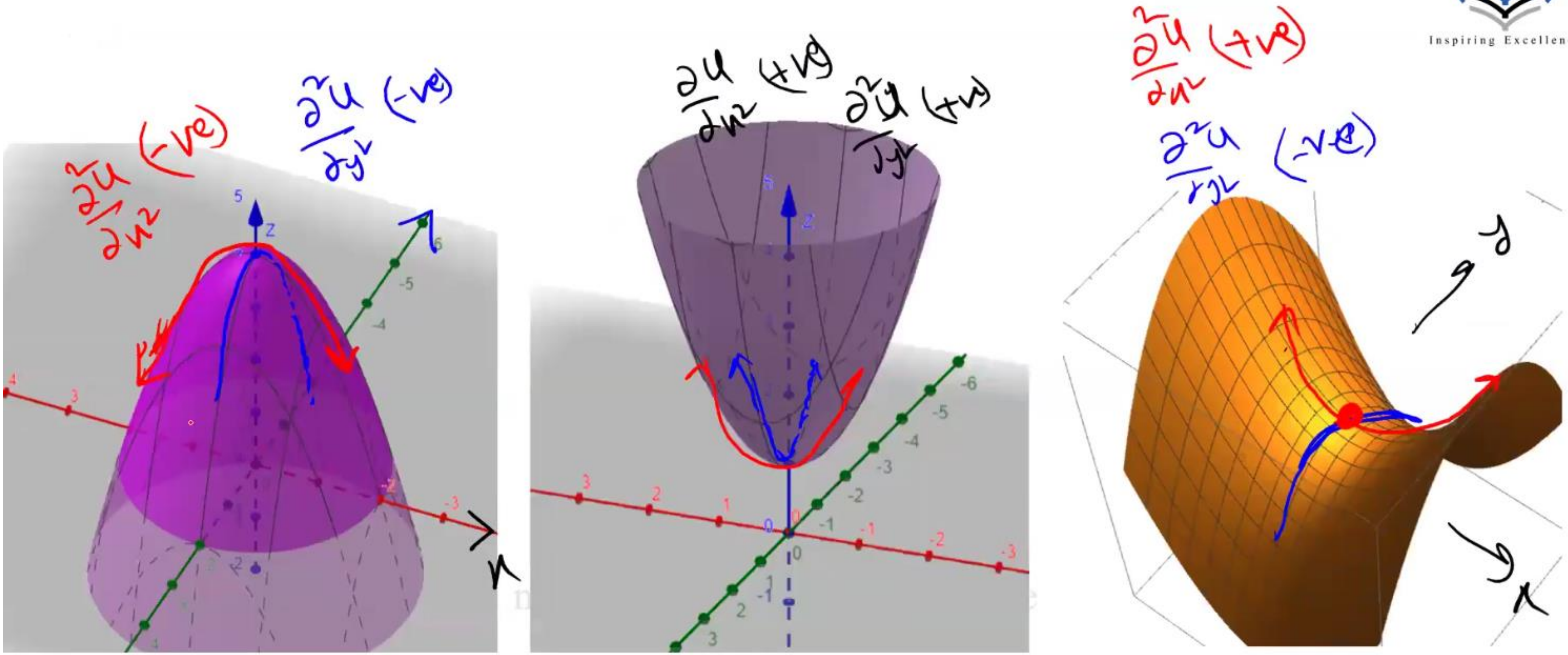
A function $u(x, y)$ is known as harmonic function when it is twice continuously differentiable and also satisfies the Laplace partial differential equation

$$\nabla^2 u = 0 \quad \textbf{or} \quad \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

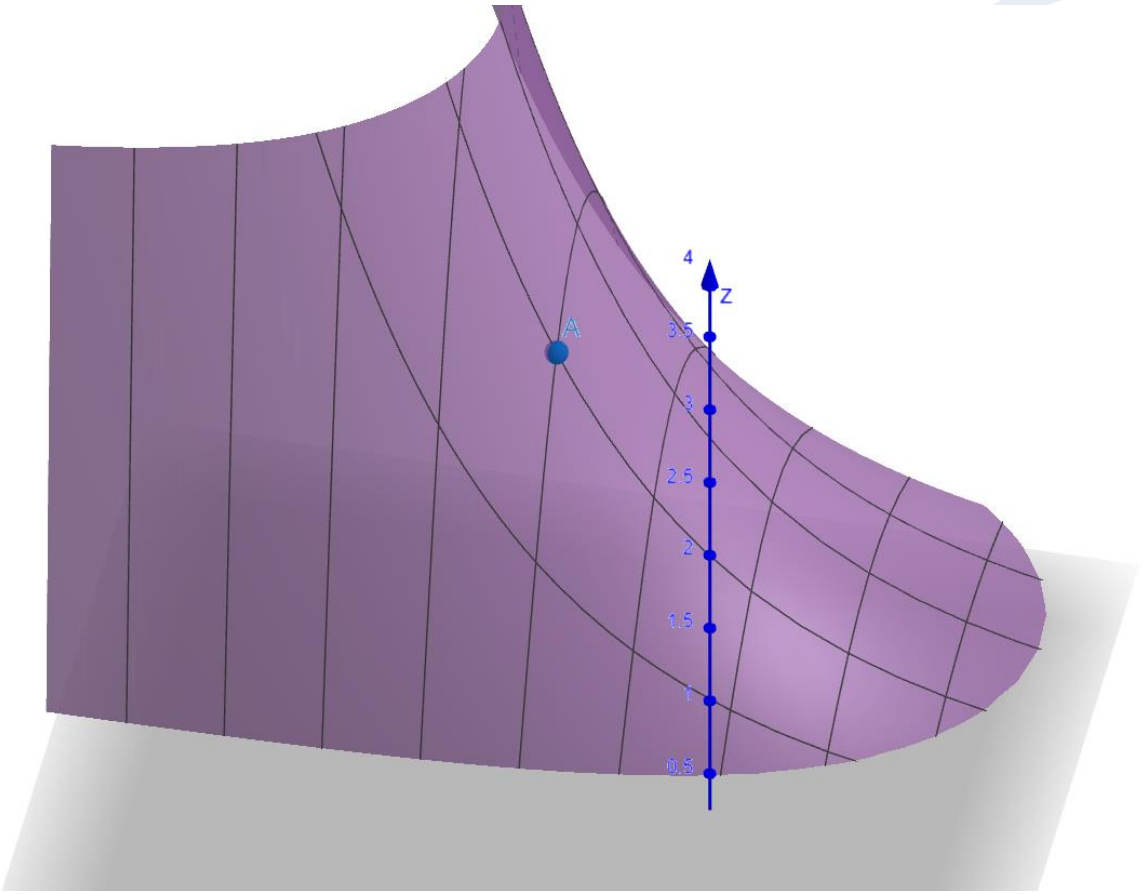
Physical Significance of Harmonic Function

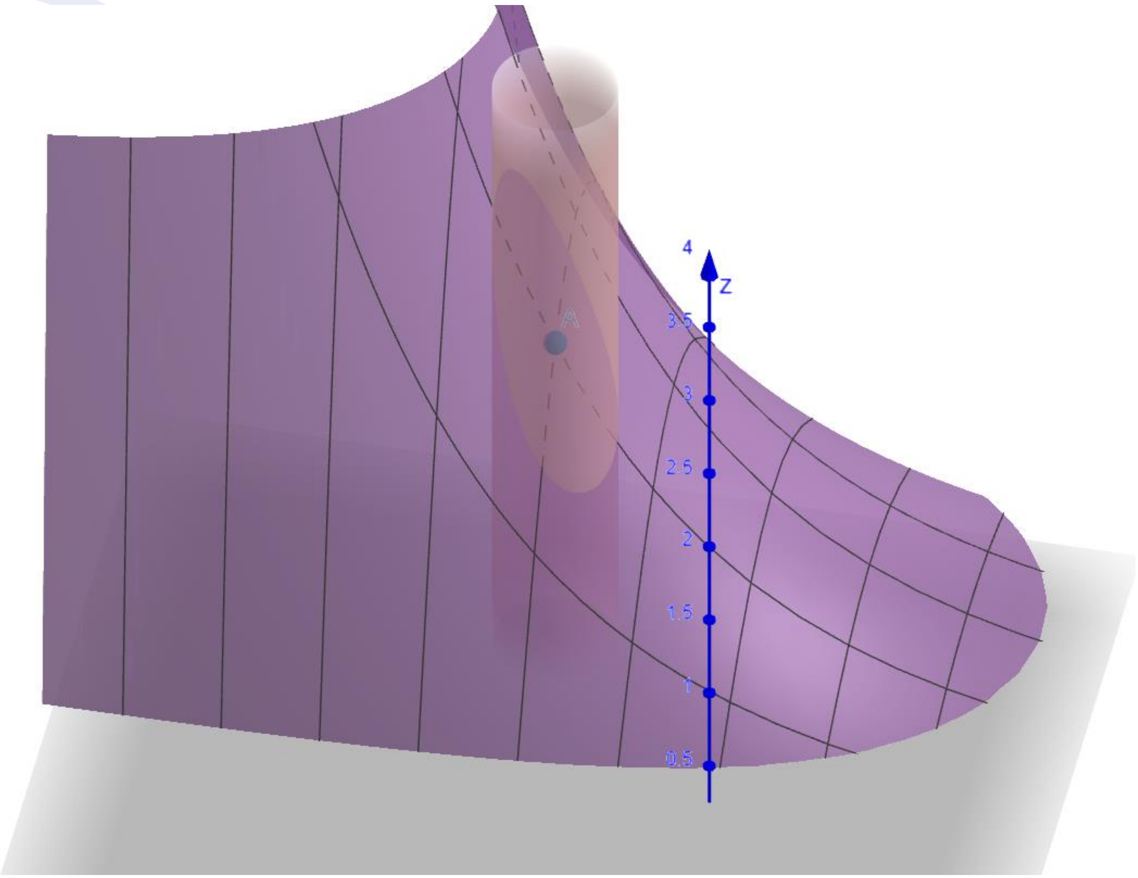
$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

There is no relative Max / Min or Every point is Saddle



At each point, function value is the average of surroundings





Show that $u(x, y) = 3x^2y + 2x^2 - y^3 - 2y^2$ is harmonic

$$\frac{\partial u}{\partial x} = 6xy + 4x - 0 - 0$$

$$\frac{\partial^2 u}{\partial x^2} = 6y + 4$$

$$\frac{\partial u}{\partial y} = 3x^2 + 0 - 3y^2 - 4y$$

$$\frac{\partial^2 u}{\partial y^2} = 0 - 6y - 4$$

$$\therefore \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 6y + 4 - 6y - 4 = 0$$

$\therefore$ u is harmonic. ✓

Show that $\boldsymbol{u}(\boldsymbol{x}, \boldsymbol{y}) = \tan^{-1}\left(\frac{y}{x}\right)$ is harmonic

$$\frac{\partial u}{\partial x} = \frac{1}{1+\left(\frac{y}{x}\right)^2} \cdot \frac{\partial}{\partial x}\left(\frac{y}{x}\right) = \frac{1}{1+\frac{y^2}{x^2}} \cdot \frac{x \cdot \frac{\partial}{\partial x}(y) - y \cdot \frac{\partial}{\partial x}(x)}{x^2}$$

$$= \frac{x^2}{x^2+y^2} \cdot \frac{x \cdot 0 - y}{x^2} = \frac{-y}{x^2+y^2}$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{(x^2+y^2) \cdot \frac{\partial}{\partial x}(-y) - (-y)\frac{\partial}{\partial x}(x^2+y^2)}{(x^2+y^2)^2} = \frac{2xy}{(x^2+y^2)^2}$$

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$$\frac{\partial u}{\partial y} = \frac{\partial}{\partial y}\left(\tan^{-1}\frac{y}{x}\right) = \frac{1}{1+\left(\frac{y}{x}\right)^2}\cdot\frac{\partial}{\partial y}\left(\frac{y}{x}\right) = \frac{\frac{1}{x}}{1+\frac{y^2}{x^2}} = \frac{x}{x^2+y^2}$$

$$\frac{\partial^2 u}{\partial y^2} = \frac{(x^2+y^2)\frac{\partial}{\partial y}(x) - x\cdot\frac{\partial}{\partial y}(x^2+y^2)}{(x^2+y^2)^2} = \frac{-2xy}{(x^2+y^2)^2}$$

$\therefore \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \implies u$ is harmonic &

Connection between Analytic Function and Harmonic Function

$f(z) = u + iv$ analytic

$+$

second derivative condition

$\Longrightarrow$

u harmonic

v harmonic

Given $f(z) = u + iv$ is analytic in a region R. Prove that u and v are harmonic if they have continuous second partial derivatives in R.

given $f(z) = u + iv$ analytic

$\Rightarrow$ C-R equations will be satisfied

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \text{---} (1) \qquad \text{and} \qquad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \quad \text{---} (2)$$

(1)

$$\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right) = \frac{\partial}{\partial x}\cdot\frac{\partial v}{\partial y}$$

$$\Rightarrow \frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 v}{\partial x \partial y}$$

$$\frac{\partial}{\partial y}\left(\frac{\partial u}{\partial y}\right) = -\frac{\partial}{\partial y}\left(\frac{\partial v}{\partial x}\right)$$

$$\Rightarrow \frac{\partial^2 u}{\partial y^2} = -\frac{\partial^2 v}{\partial x \partial y}$$

$$\Rightarrow \frac{\partial^2 u}{\partial y^2} = -\frac{\partial^2 v}{\partial x \partial y}$$

$$\therefore \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

$\Rightarrow$ u is harmonic ✓

②

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$$

$$\Rightarrow \frac{\partial v}{\partial y} = \frac{\partial u}{\partial x}$$

$$\Rightarrow \frac{\partial^2 v}{\partial y^2} = \frac{\partial^2 u}{\partial y \partial x}$$

$$= \frac{\partial^2 u}{\partial x \partial y}$$

$$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

$$\Rightarrow \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}$$

$$\Rightarrow \frac{\partial^2 v}{\partial x^2} = -\frac{\partial^2 u}{\partial x \partial y}$$

$$\therefore \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0$$

$\Rightarrow$ v is harmonic

The Converse may not be true

$\underline{u}, \underline{v}$ harmonic $\Longrightarrow$ $u+iv$ analytic

$+$

C-R

Harmonic Conjugate

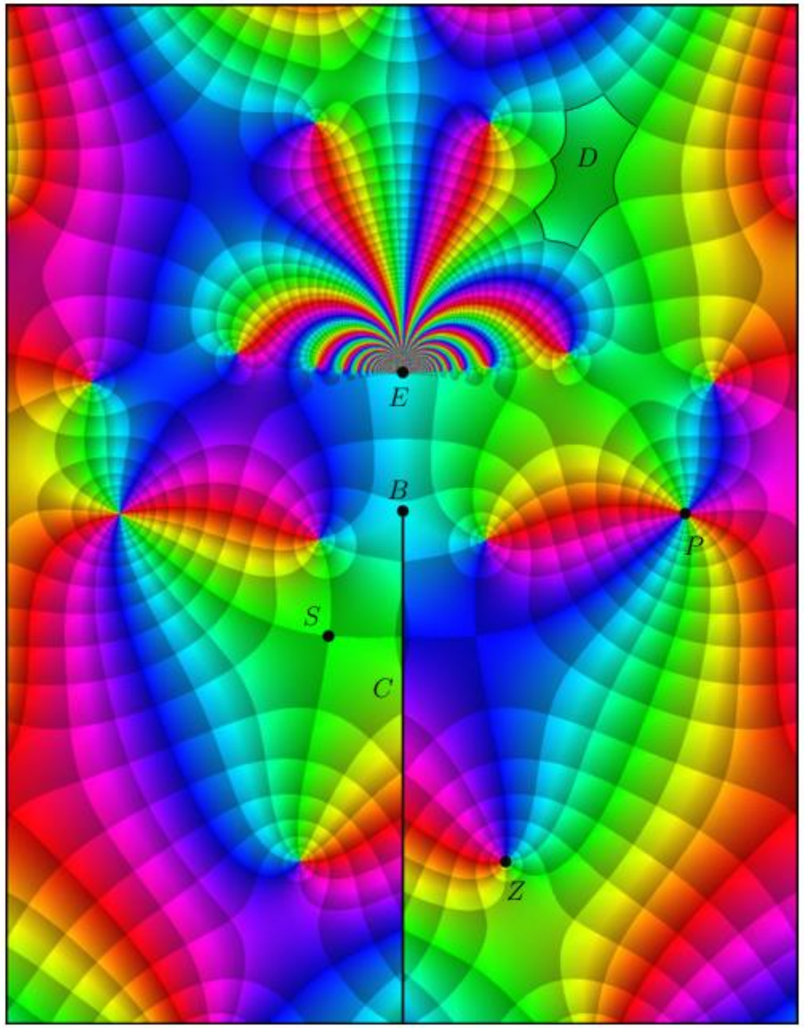
$u \longrightarrow$ harmonic

$u + iv$

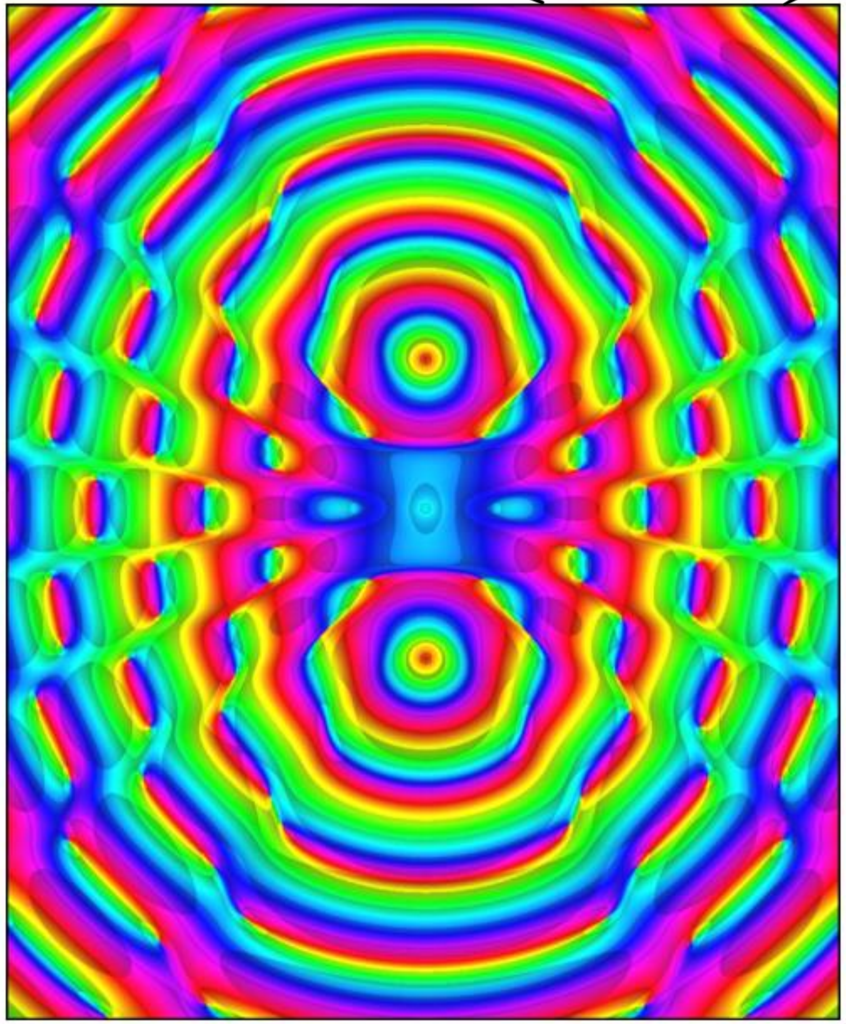
harmonic conjugate of u.

Physical Significance of Harmonic Conjugate

u + iv analytic



u + iv (not analytic) ✗



Show that $u(x,y) = 3x^2y + 2x^2 - y^3 - 2y^2$ is harmonic. Also find the harmonic conjugate $v(x,y)$ such that $u + iv$ is analytic.

Part-1:

$$\frac{\partial u}{\partial x} = 6xy + 4x \qquad \frac{\partial u}{\partial y} = 3x^2 - 3y^2 - 4y$$

$$\frac{\partial^2 u}{\partial x^2} = 6y + 4 \qquad \frac{\partial^2 u}{\partial y^2} = -6y - 4$$

$$\therefore \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

$\Rightarrow u$ is harmonic

Part-02

since $u+iv$ is analytic.

$$\Rightarrow \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \text{—— (1)} \qquad \text{and} \qquad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \quad \text{—— (2)}$$

from (1), $\dfrac{\partial u}{\partial x} = \dfrac{\partial v}{\partial y}$

$$\Rightarrow \frac{\partial v}{\partial y} = \frac{\partial u}{\partial x}$$

$$\frac{\partial v}{\partial y} = \frac{\partial u}{\partial x}$$

$$\Rightarrow \frac{\partial v}{\partial y} = 6xy + 4x$$

integrating with respect to y,

$$\Rightarrow v(x,y) = \int (6xy + 4x)\,dy = 6x\frac{y^2}{2} + 4xy + g(x)$$

$$\therefore v(x,y) = 3xy^2 + 4xy + g(x) \quad \text{--- (3)}$$

from (2) $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$

$$\Rightarrow 3x^2 - 3y^2 - 4y = -\frac{\partial}{\partial x}\left(3xy^2 + 4xy + g(x)\right)$$

$$\Rightarrow 3x^2 - \cancel{3y^2} \cancel{-4y} = \cancel{-3y^2} \cancel{-4y} - g'(x)$$

$$\Rightarrow g'(x) = -3x^2$$

$$\Rightarrow g(x) = \int (-3x^2)\,dx = -3\cdot\frac{x^3}{3} + c = -x^3 + c.$$

$$\therefore \quad v(x,y) = 3xy^2 + 4xy + g(x)$$

$$= 3xy^2 + 4xy - x^3 + c.$$

$$\int (2x)\,dx = \underbrace{x^2 + c}$$

$$\int (2xy)\,dy = \underbrace{2 \cdot x \frac{y^2}{2} + g(x)}$$

$$\frac{\partial}{\partial y}$$

$$\int (2xy)\,dx = \underbrace{2 \cdot \frac{x^2}{2} \cdot y + g(y)}$$

$$\frac{\partial}{\partial x}$$

$$\int (2xyz)\,dy = 2\cdot x\frac{y^2}{2}\cdot z + g(x,z)$$

$$\frac{\partial}{\partial y}$$

Show that $\mathbf{v}(x, y) = e^{-2x} \, sin(2y)$ is harmonic. Also find the harmonic conjugate $\mathrm{u}(x, y)$ such that $u + iv$ is analytic.

$$\frac{\partial v}{\partial x} = -2e^{-2x} \sin(2y)$$

$$\frac{\partial v}{\partial y} = 2e^{-2x} \cos(2y)$$

$$\frac{\partial v}{\partial x} = 4e^{-2x} \sin(2y)$$

$$\frac{\partial^2 v}{\partial y} = -4e^{-2x} \sin(2y)$$

$$\therefore \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0$$

$\Rightarrow$ v is harmonic

since $u+iv$ is analytic

$$\Rightarrow \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \text{and} \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

(1) (2)

from ①,

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$$

$$\Rightarrow \frac{\partial u}{\partial x} = 2e^{-2x}\cos(2y)$$

$$\Rightarrow u(x,y) = \int 2e^{-2x}\cos(2y)\,dx$$

$$\therefore u(x,y) = -e^{-2x}\cos(2y) + g(y) \longrightarrow ③$$

Again $\dfrac{\partial u}{\partial y} = -\dfrac{\partial v}{\partial x}$

$$\Rightarrow \frac{\partial}{\partial y}\left(-e^{-2x}\cos(2y) + g(y)\right) = -\left(-2e^{-2x}\sin(2y)\right)$$

$$\Rightarrow \cancel{2e^{-2x}}\sin(2y) + g'(y) = 2\cancel{e^{-2x}\sin(2y)}$$

$$\Rightarrow g'(y) = 0$$

$$\Rightarrow g(y) = c$$

$$\therefore \quad u(x,y) = -e^{-2x}\cos(2y) + g(y)$$

$$= -e^{-2x}\cos(2y) + c$$

Show that $\boldsymbol{u(x, y) = ln(x^2 + y^2)}$ is harmonic. Also find the harmonic conjugate $\boldsymbol{v(x, y)}$ such that $\boldsymbol{u + iv}$ is analytic.

$$\frac{\partial u}{\partial x} = \frac{1}{x^2+y^2} \cdot 2x = \frac{2x}{x^2+y^2}$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{(x^2+y^2)\cdot 2 - 2x \cdot 2x}{(x^2+y^2)^2} = \frac{2(y^2-x^2)}{(x^2+y^2)^2}$$

$$\frac{\partial u}{\partial y} = \frac{2y}{(x^2+y^2)}$$

$$\frac{\partial^2 u}{\partial y^2} = \frac{(x^2+y^2)\cdot 2 - 2y \cdot 2y}{(x^2+y^2)^2} = \frac{2(x^2-y^2)}{(x^2+y^2)}$$

$$\therefore \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

$\Rightarrow$ u is harmonic

since $u+iv$ is analytic $\Longrightarrow \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ ——— (1) and $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$ ——— (2)

from (1),

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$$

$$\Rightarrow \frac{\partial v}{\partial y} = \frac{\partial u}{\partial x} = \frac{2x}{x^2+y^2}$$

$$\therefore v(x,y) = \int \frac{2x}{x^2+y^2}\,dy$$

$$v(x,y) = 2x \cdot \int \frac{1}{y^2+x^2}\,dy$$

$$= 2x \cdot \frac{1}{x} \tan^{-1}\left(\frac{y}{x}\right) + g(x)$$

$$v(x,y) = 2\tan^{-1}\left(\frac{y}{x}\right) + g(x) \quad \text{——— (3)}$$

Again

$$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

$$\Rightarrow \frac{2y}{(x^2+y^2)} = -\frac{\partial}{\partial x}\left(2 \cdot \tan^{-1}\frac{y}{x} + g(x)\right)$$

$$\Rightarrow \frac{2y}{\cancel{x^2+y^2}} = -2 \cdot \frac{\frac{-y}{x^2}}{1+\left(\frac{y}{x}\right)^2} + g'(x)$$

$$= -2 \cdot \frac{\cancel{-y}}{x^2+y^2} + g'(x) \quad \Rightarrow g'(x) = 0$$

$$\Rightarrow g(x) = c$$

$\therefore \quad v = 2\tan^{-1}\left(\frac{y}{x}\right) + c$

$$\frac{d}{dx}(\sinh x) = \cosh x \qquad \int \sinh x \, dx = \cosh x$$

$$\frac{d}{dx}(\cosh x) = \sinh x \qquad \int \cosh x \, dx = \sinh x$$

Show that $\mathbf{v}(x, y) = \boldsymbol{cos}(\boldsymbol{x})\ \boldsymbol{sinh}(\boldsymbol{y})$ is harmonic. Also find the harmonic conjugate $\mathrm{u}(\boldsymbol{x}, \boldsymbol{y})$ such that $\boldsymbol{u} + \boldsymbol{iv}$ is analytic.

$$v(x,y) = \cos x \cdot \left(\frac{e^{y} - e^{-y}}{2} \right) = \frac{1}{2} e^{y} \cos x - \frac{1}{2} e^{-y} \cos x$$

$$\frac{\partial v}{\partial x} = -\frac{1}{2} e^{y} \sin x + \frac{1}{2} e^{-y} \sin x$$

$$\frac{\partial^2 v}{\partial x^2} = -\frac{1}{2} e^{y} \cos x + \frac{1}{2} e^{-y} \cos x$$

$$\frac{\partial v}{\partial y} = \frac{1}{2} e^{y} \cos x + \frac{1}{2} e^{-y} \cos x$$

$$\frac{\partial^2 v}{\partial y^2} = \frac{1}{2} e^{y} \cos x - \frac{1}{2} e^{-y} \cos x$$

$$\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0$$

$\therefore$ v is harmonic

since $u+iv$ is analytic $\Rightarrow \dfrac{\partial u}{\partial x}=\dfrac{\partial v}{\partial y}$ —— (1) and $\dfrac{\partial u}{\partial y}=-\dfrac{\partial v}{\partial x}$ —— (2)

from (1)

$$\frac{\partial u}{\partial x}=\frac{\partial v}{\partial y}$$

$$\Rightarrow \frac{\partial u}{\partial x}=\frac{1}{2}e^{y}\cos x+\frac{1}{2}e^{-y}\cos x$$

$$\Rightarrow u(x,y)=\int\left(\frac{1}{2}e^{y}\cos x+\frac{1}{2}e^{-y}\cos x\right)dx$$

$$=\frac{1}{2}e^{y}\sin x+\frac{1}{2}e^{-y}\sin x+g(y)$$

Again $\quad \dfrac{\partial u}{\partial y} = -\dfrac{\partial v}{\partial x}$

$$\Rightarrow \frac{\partial}{\partial y}\left(\frac{1}{2}e^{y}\sin x + \frac{1}{2}e^{-y}\sin x + g(y)\right) = -\left(-\frac{1}{2}e^{y}\sin x + \frac{1}{2}e^{-y}\sin x\right)$$

$$\Rightarrow \cancel{\frac{1}{2}e^{y}\sin x} - \cancel{\frac{1}{2}e^{-y}\sin x} + g'(y) = \cancel{\frac{1}{2}e^{y}\sin x} - \cancel{\frac{1}{2}e^{-y}\sin x}$$

$$\Rightarrow g'(y) = 0 \quad \Rightarrow g(y) = c.$$

$$u(x,y) = \frac{1}{2}e^{y}\sin x + \frac{1}{2}e^{-y}\sin x + c$$

$$= \frac{(e^{y} + e^{-y})}{2}\sin x + c$$

$$= \cosh y \sin x + c$$

$$= \sin x \cdot \cosh y + c$$

Show that $u(x,y) = xe^{-x}\sin(y) - e^{-x}y\cos(y)$ is harmonic. Also find the harmonic conjugate $v(x,y)$ such that $u + iv$ is analytic.

1st Part: we have to show $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$.

$$\frac{\partial u}{\partial x} = e^{-x}\sin y - xe^{-x}\sin y + e^{-x}y\cos y$$

$$\frac{\partial^2 u}{\partial x^2} = -e^{-x}\sin y - e^{-x}\sin y + xe^{-x}\sin y - e^{-x}y\cos y$$

$$= xe^{-x}\sin y - 2e^{-x}\sin y - e^{-x}y\cos y$$

$$u = xe^{-x}\sin y - e^{-x}y\cos y$$

$$\frac{\partial u}{\partial y} = xe^{-x}\cos y - e^{-x}\left(y(-\sin y) + 1.\cos y\right)$$

$$= xe^{-x}\cos y + e^{-x}y\sin y - e^{-x}\cos y$$

$$\frac{\partial^2 u}{\partial y^2} = -xe^{-x}\sin y + e^{-x}y\cos y + e^{-x}\sin y + e^{-x}\sin y$$

$$= -xe^{-x}\sin y + 2e^{-x}\sin y + e^{-x}y\cos y.$$

$$\therefore \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

$\Rightarrow u$ is harmonic.

2nd Part: Given $u+iv$ is analytic

$\Rightarrow$ C-R equations are satisfied.

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \text{--- (1)}$$

$$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \quad \text{--- (2)}$$

from 1,

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$$

$$\Rightarrow \frac{\partial v}{\partial y} = \frac{\partial u}{\partial x}$$

$$\Rightarrow \frac{\partial v}{\partial y} = e^{-x}\sin y - xe^{-x}\sin y + e^{-x}y\cos y$$

$$\Rightarrow v(x,y) = \int \left(e^{-x}\sin y - xe^{-x}\sin y + e^{-x}y\cos y\right) dy$$

$$= e^{-x}\int \sin y\,dy - x e^{-x}\int \sin y\,dy + e^{-x}\int y\cos y\,dy$$

$$= -e^{-x}\cos y + x e^{-x}\cos y + e^{-x}\left[y\sin y + \cos y\right] + g(x)$$

$$= \cancel{-e^{-x}\cos y} + x e^{-x}\cos y + e^{-x}y\sin y + \cancel{e^{-x}\cos y} + g(x)$$

$$V(x,y) = x e^{-x}\cos y + e^{-x}y\sin y + g(x).$$

from ②,

$$\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}$$

$$\Rightarrow \frac{\partial}{\partial x}\left(x e^{-x} \cos y + e^{-x} y \sin y + g(x)\right) = -\left(x e^{-x}\cos y + e^{-x} y \sin y - e^{-x}\cos y\right)$$

$$\Rightarrow \cancel{e^{-x}\cos y} - \cancel{x e^{-x}\cos y} - \cancel{e^{-x} y \sin y} + g'(x) = -\cancel{x e^{-x}\cos y} - \cancel{e^{-x} y\sin y} + \cancel{e^{-x}\cos y}$$

$$g'(x) = 0$$

$$\Rightarrow g(x) = \int 0\,dx$$

$$= c$$

$\therefore v(x,y) = xe^{-x}\cos y + e^{-x}y\sin y + c$, where c is a constant.

Integration UV formula.

$$\int x^3 \sin(2x)\,dx$$

$$= \left(\frac{-x^3}{2}\cos 2x\right) - \left(\frac{-3x^2}{4}\sin 2x\right) + \left(\frac{6x}{8}\cos 2x\right)$$

$$- \left(\frac{6}{16}\sin 2x\right) + C$$

x^3	$\sin 2x$
$3x^2$	$\frac{-1}{2}\cos 2x$
$6x$	$\frac{-1}{4}\sin 2x$
6	$\frac{1}{8}\cos 2x$
0	$\frac{1}{16}\sin 2x$


Thank you!