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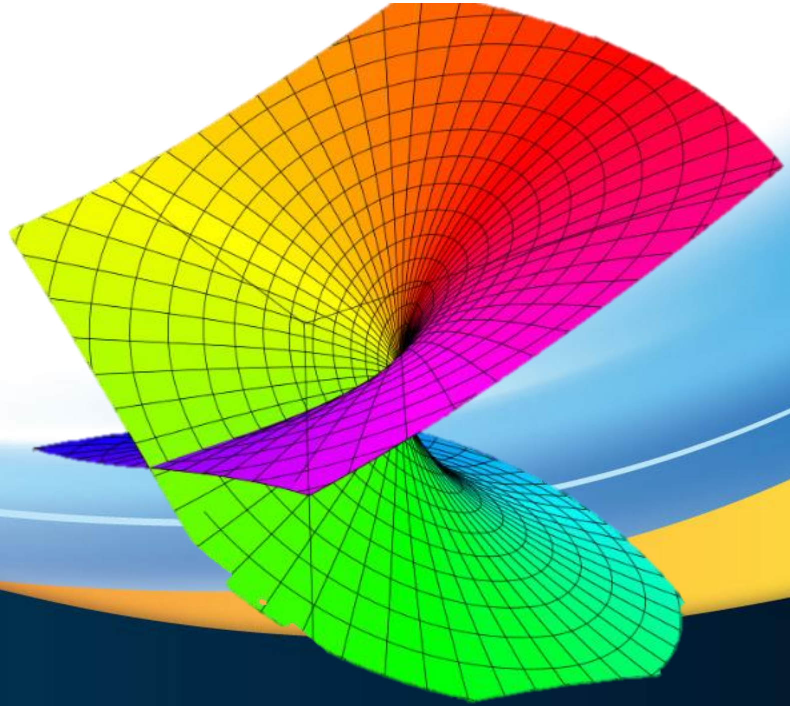
Undergraduate Course in Mathematics

Complex Variables

Topic: Complex Differentiation



Conducted By

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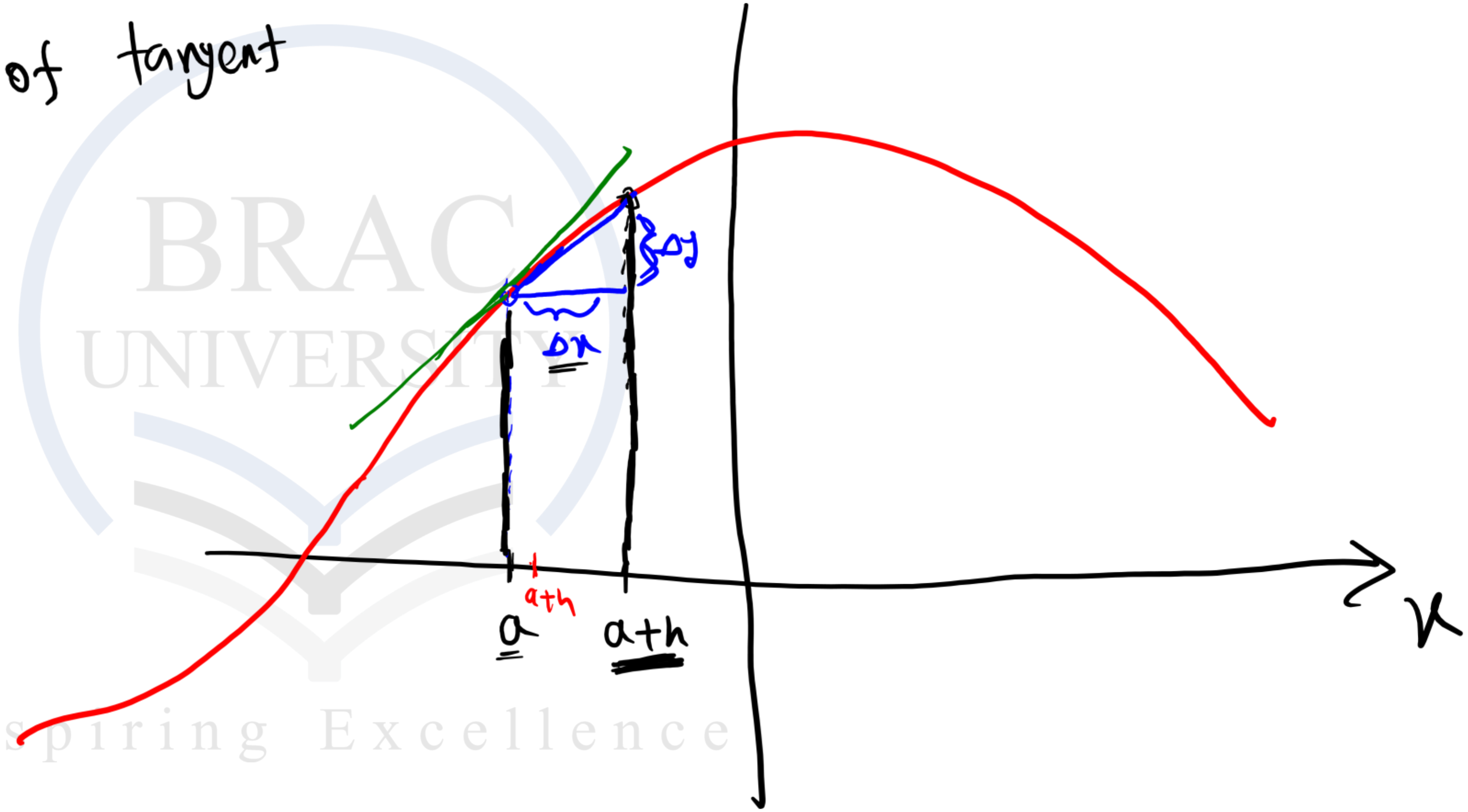
Differentiation in Real Valued Function (Physical Meaning)

differentiation = slope of tangent

$$\frac{\Delta y}{\Delta x} = \frac{f(a+h) - f(a)}{h}$$

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$



GeoGebra: https://www.geogebra.org/m/hnbr9yzz

Differentiation in Real Valued Function (Definition)

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

$$f'(a) = \lim_{\Delta x \to 0} \frac{f(a+\Delta x) - f(a)}{\Delta x}$$

Differentiation in Real Valued Function (Demo)

Consider the function $f: \mathbb{R} \to \mathbb{R}$

$$f(x) = x^2$$

Find the derivative at $x = 3$

✓

Consider the function $f: \mathbb{R} \to \mathbb{R}$

$$f(x) = 3 - 2 \cdot |x - 3|$$

Find the derivative at $x = 3$

✗

Consider the function $f: \mathbb{R} \to \mathbb{R}$ such that $f(x) = x^2$

Find the derivative at $x = 3$

$\frac{df}{dx} = 2x \qquad x = 3 \qquad 6$

$$f'(3) = \lim_{\Delta x \to 0} \frac{f(3+\Delta x) - f(3)}{\Delta x}$$

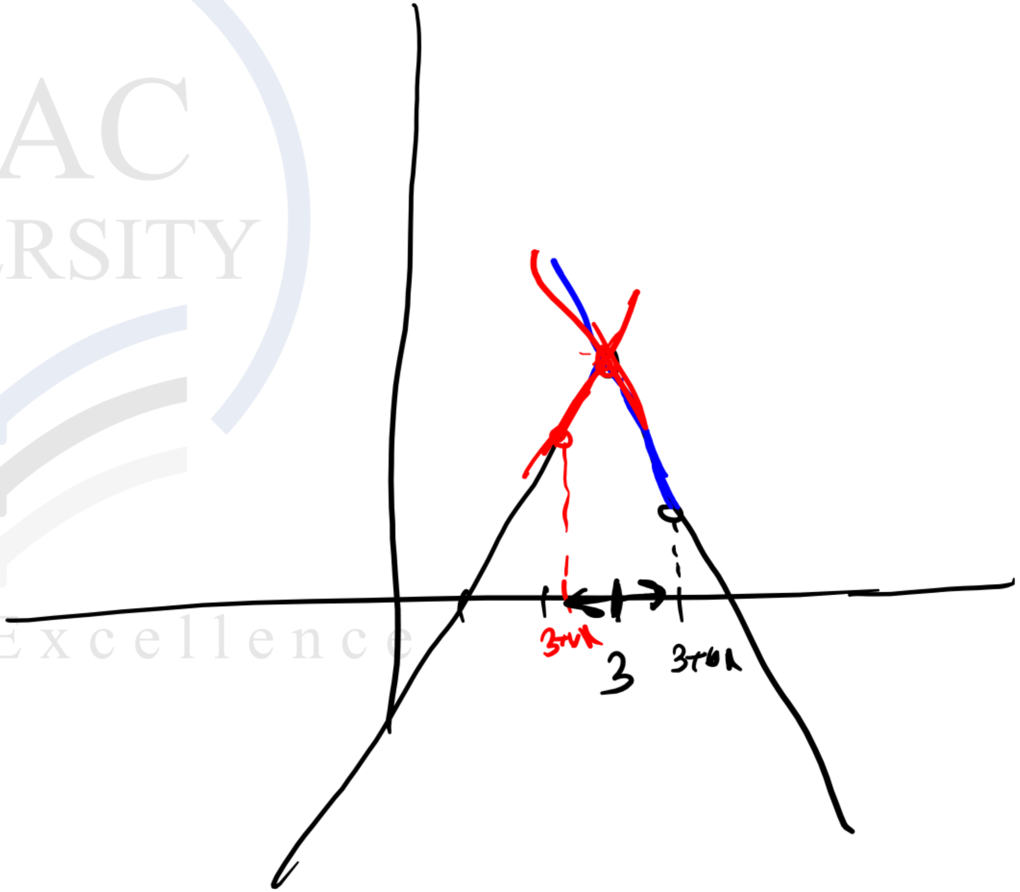
$$f'(3) = 6$$

Consider the function $f: \mathbb{R}$ such that $f(x) = 3 - 2 \cdot |x - 3|$

Find the derivative at $x = 3$

$f'(3) = -2$ // R.H.D

$f'(3) = +2$ L.H.D



Differentiation in Complex Valued Function (Physical Meaning)

2D

2D

Differentiation in Complex Valued Function (Definition)

$$f(z) = ?$$

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(h)}{h}$$

$$\boxed{f'(z) = \lim_{\Delta z \to 0} \frac{f(z+\Delta z) - f(z)}{\Delta z}}$$

$$f'(z_0) = \lim_{\Delta z \to 0} \frac{f(z_0+\Delta z) - f(z_0)}{\Delta z}$$

Differentiation in Complex Valued Function (Demo)

Consider the function $f: \mathbb{C} \to \mathbb{C}$

$$f(z) = z^2 = (x^2 - y^2) + (2xy)i$$

Find the derivative at $x = 3 + 2i$

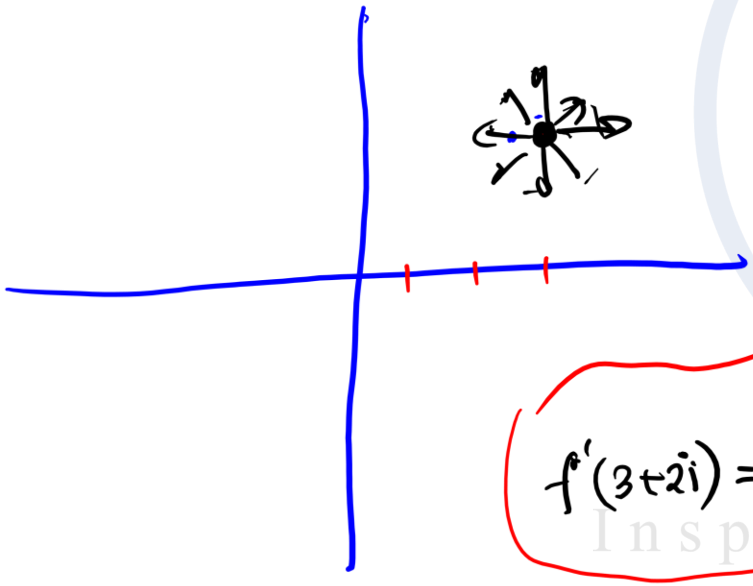
Consider the function $f: \mathbb{C} \to \mathbb{C}$

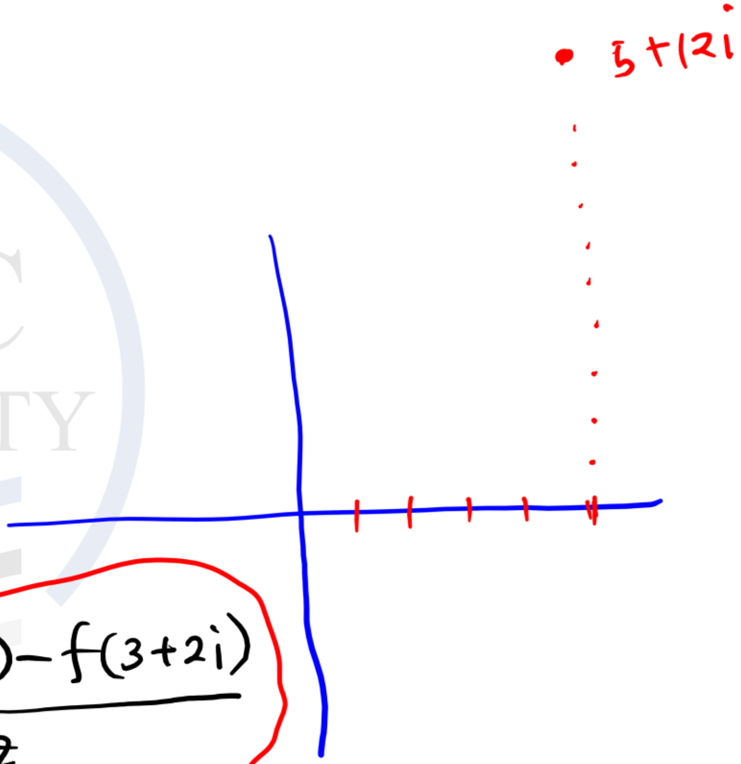
$$f(z) = \bar{\mathbf{z}}^2 = (x^2 - y^2) - (2xy)i$$

Find the derivative at $x = 3 + 2i$

$f(z) = z^2 = (x^2 - y^2) + (2xy)\, i$, Find the derivative at $x = 3 + 2i$

$3^2 - 2^2$





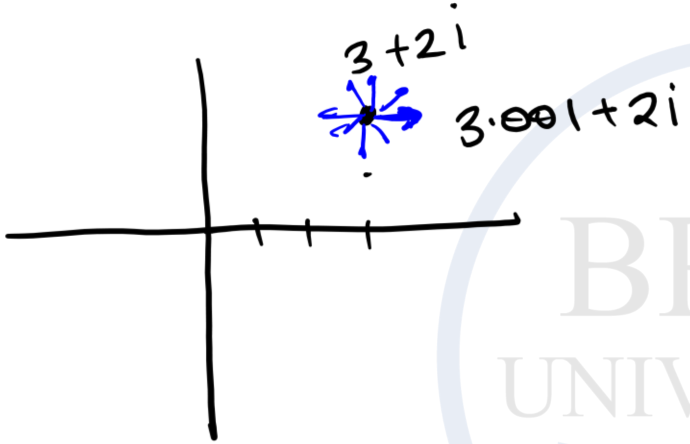
$$f'(3+2i) = \lim_{\Delta z \to 0} \frac{f(3+2i+\Delta z) - f(3+2i)}{\Delta z}$$

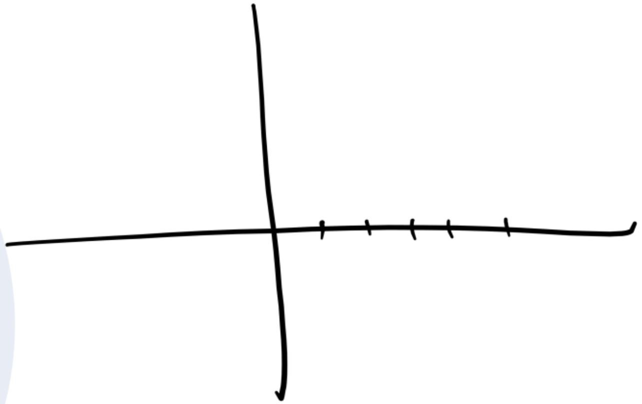
$= 6+4i$ w

z plane (input)

f(z) plane (output)

$f(z) = \bar{\mathbf{z}}^2 = (x^2 - y^2) - (2xy)\,i$, Find the derivative at $x = 3 + 2i$





$f'(3+2i) = \text{undefined}$

$f' = -6+4i$

$f' = -4-6i$

$f' = 6-4i$

$6-4i$

$-6+4i$

$5-12i$

$$f'(z_0) = \lim_{\Delta z \to 0} \frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z}$$

Using Definition show that $f(z) = \frac{2z-3i}{3z-2i}$ is differentiable at $z = -i$.

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$$f'(z_0) = \lim_{\Delta z \to 0} \frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z}$$

$$\therefore f'(-i) = \lim_{\Delta z \to 0} \frac{f(-i + \Delta z) - f(-i)}{\Delta z}$$

$$= \lim_{\Delta z \to 0} \frac{\frac{2(-i+\Delta z) - 3i}{3\cdot(-i+\Delta z) - 2i} - \frac{2(-i) - 3i}{3(-i) - 2i}}{\Delta z}$$

$$= \lim_{\Delta z \to 0} \frac{\dfrac{2\Delta z - 5i}{3\Delta z - 5i} - 1}{\Delta z}$$

$$= \lim_{\Delta z \to 0} \frac{\dfrac{2\Delta z - 5i - 3\Delta z + 5i}{3\Delta z - 5i}}{\Delta z}$$

$$= \lim_{\Delta z \to 0} \frac{-\Delta z}{\Delta z \cdot (3\Delta z - 5i)} = \lim_{\Delta z \to 0} \frac{-1}{3\Delta z - 5i} = \frac{-1}{-5i} = \frac{1}{5i}$$

Using Definition find the derivative of $f(z) = z^2$ at $z = z_0$.

or

Using Definition show that $f(z) = z^2$ is differentiable at all points.

$$f'(z_0) = \lim_{\Delta z \to 0} \frac{f(z_0+\Delta z) - f(z_0)}{\Delta z}$$

$$= \lim_{\Delta z \to 0} \frac{(z_0+\Delta z)^2 - z_0^2}{\Delta z}$$

$$= \lim_{\Delta z \to 0} \frac{\cancel{z_0^2} + 2\cdot z_0 \cdot \Delta z + \Delta z^2 - \cancel{z_0^2}}{\Delta z}$$

$$f(z) = z^2$$

$$f(z_0+\Delta z)$$

$$= (z_0+\Delta z)^2$$

$$= \lim_{\Delta z \to 0} \frac{2z_0 \cdot \Delta z + \Delta z^2}{\Delta z}$$

$$= \lim_{\Delta z \to 0} (2z_0 + \Delta z)$$

$$= 2z_0 + 0$$

$$= 2z_0 .$$

$\therefore f'(z_0) = 2z_0 .$

for any $z = z_0$, we have a derivative.

$\Rightarrow f(z) = z^2$ is differentiable at all points.

Using Definition show that $f(z) = \bar{z}$ is not differentiable at $z = 0$.

$$f'(0) = \lim_{\Delta z \to 0} \frac{f(0+\Delta z) - f(0)}{\Delta z}$$

$$= \lim_{\Delta z \to 0} \frac{f(\Delta z) - \bar{0}}{\Delta z}$$

$$= \lim_{\Delta z \to 0} \frac{\overline{\Delta z}}{\Delta z}$$

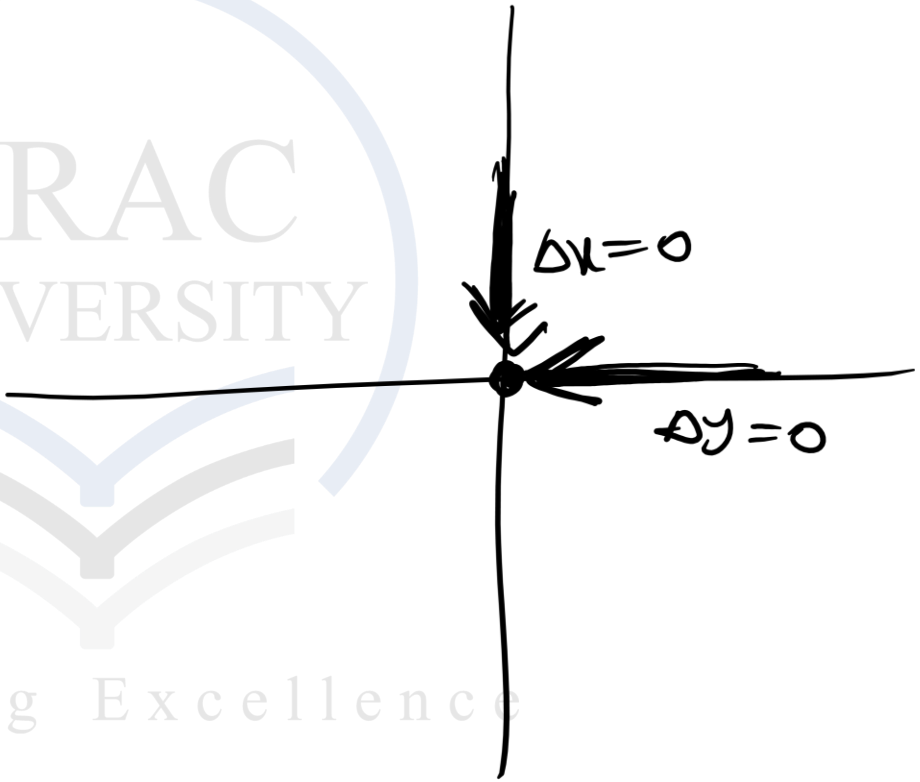
let $\Delta z = \Delta x + i\Delta y$

$\overline{\Delta z} = \Delta x - i\Delta y$

$$\left\{ \begin{array}{l} \Delta z \to 0 \\ \Delta x \to 0 \\ \Delta y \to 0 \end{array} \right.$$

$$= \lim_{\Delta z \to 0} \frac{\overline{\Delta z}}{\Delta z}$$

$$= \lim_{\substack{\Delta x \to 0 \\ \Delta y \to 0}} \frac{\Delta x - i\Delta y}{\Delta x + i\Delta y}$$



In direction $\Delta x = 0$

$$f'(0) = \lim_{\substack{\Delta x \to 0 \\ \Delta y \to 0}} \frac{0 - i\Delta y}{0 + i\Delta y}$$

$$= \lim_{\substack{\Delta x \to 0 \\ \Delta y \to 0}} (-1)$$

$$= -1$$

In direction $\Delta y = 0$

$$f'(0) = \lim_{\substack{\Delta x \to 0 \\ \Delta y \to 0}} \frac{\Delta x - i \cdot 0}{\Delta x + i \cdot 0}$$

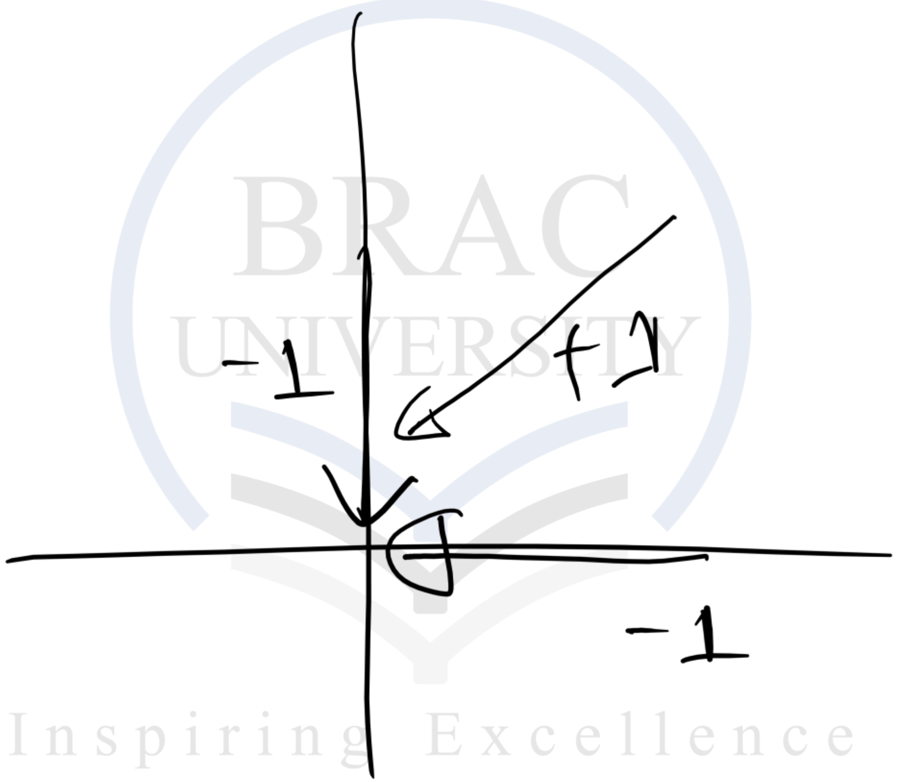
$$= \lim_{\substack{\Delta x \to 0 \\ \Delta y \to 0}} (1)$$

$$= 1$$

Since $-1 \neq 1$.

$\therefore f'(0)$ does not exist.

$f(z) = \bar{z}$ is not differentiable at $z = 0$.


-1
+1
-1

Using Definition show that the function

$$f(z) = z \cdot \bar{z} \;\; or \;\; \boxed{f(z) = |z|^2}$$

is not differentiable other than $z = 0$.

$|z|^2 = z \cdot \bar{z}$

let $\quad z_0 \neq 0$

$$f'(z_0) = \lim_{\Delta z \to 0} \frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z}$$

$$= \lim_{\Delta z \to 0} \frac{|z_0 + \Delta z|^2 - |z_0|^2}{\Delta z} = \lim_{\Delta z \to 0} \frac{(z_0 + \Delta z) \cdot \overline{(z_0 + \Delta z)} - z_0 \cdot \overline{z_0}}{\Delta z}$$

$$= \lim_{\Delta z \to 0} \frac{(z_0 + \Delta z)(\bar{z}_0 + \overline{\Delta z}) - z_0 \cdot \bar{z}_0}{\Delta z}$$

$$= \lim_{\Delta z \to 0} \frac{\cancel{z_0 \cdot \bar{z}_0} + z_0 \cdot \overline{\Delta z} + \Delta z \cdot \bar{z}_0 + \Delta z \cdot \overline{\Delta z} - \cancel{z_0 \cdot \bar{z}_0}}{\Delta z}$$

$$= \lim_{\Delta z \to 0} \left(\frac{z_0 \cdot \overline{\Delta z}}{\Delta z} + \frac{\cancel{\Delta z} \cdot \bar{z}_0}{\cancel{\Delta z}} + \frac{\cancel{\Delta z} \cdot \overline{\Delta z}}{\cancel{\Delta z}} \right)$$

$$= \lim_{\Delta z \to 0} \left(z_0 \frac{\overline{\Delta z}}{\Delta z} + \bar{z}_0 + \overline{\Delta z} \right)$$

$$= \lim_{\Delta z \to 0} \left(z_0 \cdot \frac{\overline{\Delta z}}{\Delta z} + \overline{z_0} + \overline{\Delta z} \right)$$

Let

$$\Delta z = \Delta x + i\Delta y$$

$$\overline{\Delta z} = \Delta x - i\Delta y$$

$$\Delta z \to 0$$

$$\Delta x \to 0$$

$$\Delta y \to 0$$

$$= \lim_{\substack{\Delta x \to 0 \\ \Delta y \to 0}} \left(z_0 \cdot \frac{\Delta x - i\Delta y}{\Delta x + i\Delta y} + \overline{z_0} + \Delta x - i\Delta y \right)$$

In $\Delta x = 0$ direction

$$f'(z_0) = \lim_{\substack{\Delta x \to 0 \\ \Delta y \to 0}} \left(z_0 \frac{-i\Delta y}{i\Delta y} + \bar{z}_0 + 0 - i\Delta y \right)$$

$$= \lim_{\substack{\Delta x \to 0 \\ \Delta y \to 0}} \left(-z_0 + \bar{z}_0 - i\Delta y \right)$$

$$= -z_0 + \bar{z}_0$$

In direction $\Delta y = 0$

$$f'(z_0) = \lim_{\substack{\Delta x \to 0 \\ \Delta y \to 0}} \left(z_0 \frac{\Delta x}{\Delta x} + \overline{z_0} + \Delta x \right)$$

$$= \lim_{\substack{\Delta x \to 0 \\ \Delta y \to 0}} \left(z_0 + \overline{z_0} + \Delta x \right)$$

$$= z_0 + \overline{z_0}$$

Since

$$\boxed{-z_0 + \overline{z_0} \neq z_0 + \overline{z_0}}$$

for all $z_0 \neq 0$

$\Rightarrow f(z) = z \cdot \bar{z}$ or $|z|^2$

is not differentiable

other than 0.

Differentiation of a function of the form $f(z) = u + iv$

$$f'(z_0) = \lim_{\Delta z \to 0} \frac{f(z+\Delta z) - f(z)}{\Delta z}$$

$$f(z) = u + iV$$

$$f(x+iy) = u(x,y) + i\,v(x,y)$$

$$= \lim_{\substack{\Delta x \to 0 \\ \Delta y \to 0}} \frac{u(x+\Delta x, y+\Delta y) + i\,v(x+\Delta x, y+\Delta y) - u(x,y) - i\,v(x,y)}{\Delta x + i\Delta y}$$

$\Delta y = 0$

$$f'(z_0) = \lim_{\substack{\Delta x \to 0 \\ \Delta y \to 0}} \frac{u(x+\Delta x, y) + i\,v(x+\Delta x, y) - u(x,y) - i\,v(x,y)}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{u(x+\Delta x, y) - u(x,y)}{\Delta x} + i \lim_{\Delta x \to 0} \frac{v(x+\Delta x, y) - v(x,y)}{\Delta x}$$

$$= \frac{\partial u}{\partial x} + i \frac{\partial v}{\partial x}$$

$$\boxed{f'(z) = \frac{\partial u}{\partial x} + i \frac{\partial v}{\partial x}}$$

If $\boldsymbol{f(z) = u(x, y) + i\, v(x, y)}$ is differentiable, then find $\boldsymbol{f'(z)}$.

$$\boxed{f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}}$$

$$f'(z) = \frac{\partial v}{\partial y} - i\frac{\partial u}{\partial y}$$

Consider the function

$$f(z) = (x^2 - y^2 + x) + (2xy + y)i$$

If $f(z)$ is differentiable, then find $f'(z)$.

$$u = x^2 - y^2 + x$$

$$v = 2xy + y$$

$$f'(z) = \frac{\partial u}{\partial x} + i\,\frac{\partial v}{\partial x}$$

$$= (2x+1) + i\,(2y+0)$$

$$= 2x + i2y + 1 + i = 2(x+iy) + (1+0)$$

$$= \boxed{2z+1}$$

Consider the function

$$f(z) = (x^2 - y^2 + x) + (2xy + y)i$$

If $f(z)$ is differentiable, then find $f'(z)$.

$$f(z) = \underbrace{(x^2-y^2)+i(2xy)}_{} + (x+iy)$$

$$= (x+iy)^2 + (x+iy)$$

$$= z^2 + z$$

$$f'(z) = 2z+1$$

Consider the function

$$f(z) = (x^2 - y^2 - x) + (-2xy + y)i$$

find $f'(z)$.

$f'(z) = (2x-1) + i(-2y)$

not possible

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