

## Lecture 3

### Integration by Parts

Our primary goal is how to integrate product of two functions.

For example,

$$\int f(x) g(x) dx$$

Here we use a formula to integrate product of two functions which we call it integration by parts.

Say,  $u = f(x) \quad du = f'(x) dx$

$$g(x) dx = dv$$

$$\boxed{\int u dv = uv - \int v du}$$

Example: Evaluate:  $\int x \cos x dx$

Say,  $u = x$  and  $dv = \cos x dx$

$$du = dx$$

$$v = \sin x$$

$$\int x \cos x dx = x \sin x - \int \sin x dx$$

$$= x \sin x - (-\cos x) + C$$

$$= x \sin x + \cos x + C$$

✱

Example: Evaluate  $\int e^x \cos x \, dx$

Sol<sup>n</sup>:

$$\text{Say, } u = \cos x \quad \left| \begin{array}{l} dv = e^x dx \\ u = e^x \end{array} \right.$$
$$du = -\sin x \, dx$$

L - Logarithmic  
I - Inverse trigonometric  
A - Algebraic  
T - Trigonometric  
E - Exponential

Thus,

$$\int e^x \cos x \, dx = e^x \cos x - \int e^x (-\sin x) \, dx$$

$$\therefore I = e^x \cos x + \int e^x \sin x \, dx \quad \text{--- ①}$$

∅ Say,  $I = \int e^x \cos x \, dx$

Now let  $\sin x = u$   $dv = e^x \, dx$   
 $\cos x \, dx = du$   $dv = e^x$

$$\int e^x \sin x \, dx = e^x \sin x - \int e^x \cos x \, dx = e^x \sin x - I$$

From ①

$$I = e^x \cos x + e^x \sin x - I$$

$$2I = e^x \cos x + e^x \sin x$$

$$I = \frac{e^x}{2} (\cos x + \sin x)$$

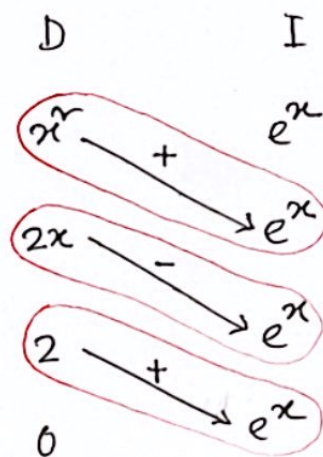
Therefore  $\int e^x \cos x \, dx = \frac{1}{2} e^x (\sin x + \cos x) + C$

✱

We can also use shortcut techniques for ~~pro~~ integration by parts.

Example: Evaluate:  $\int x^2 \cdot e^x dx$

Sol<sup>n</sup>:  $\int x^2 e^x dx = x^2 e^x - 2x e^x + 2e^x + C$  X.



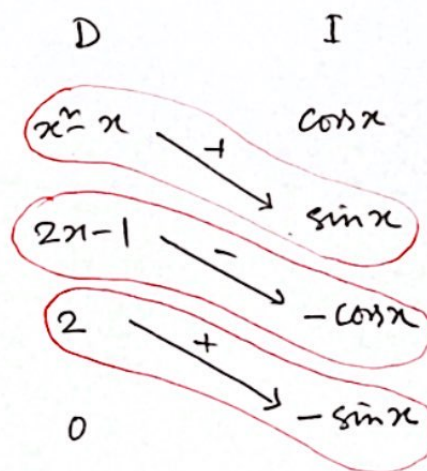
Example: Evaluate:  $\int (x^2 - x) \cos x dx$

Sol<sup>n</sup>:  $\int (x^2 - x) \cos x dx$

$$= (x^2 - x) \sin x + (2x - 1) (-) (-\cos x) + 2 (+) (-\sin x)$$

$$= (x^2 - x) \sin x + (2x - 1) \cos x - 2 \sin x + C$$

X.



Multiply ~~under~~ inside red color's function with sign and use '+' inbetween two ~~red~~ bounded red color's mark.



TRY YOURSELF (7) Evaluate:

1.  $\int x \ln x dx$

2.  $\int e^{3x} \cos 2x dx$

3.  $\int x \tan^{-1} x dx$

4.  $\int (x + x \cos x) dx$

5.  $\int x^3 e^{x^2} dx$

## Fundamental Theorem of Calculus

Theorem: If  $f$  is continuous on  $[a, b]$  and  $F$  is any antiderivative of  $f$  on  $[a, b]$ , then

$$\int_a^b f(x) dx = F(b) - F(a).$$

Example: Evaluate:  $\int_1^2 x dx$

Sol<sup>n</sup>: The function  $F(x) = \frac{x^2}{2}$  is an antiderivative of  $f(x) = x$  on  $[1, 2]$ , thus,

$$\int_1^2 x dx = \left. \frac{x^2}{2} \right|_1^2 = \frac{1}{2} 2^2 - \frac{1}{2} 1^2 = 2 - \frac{1}{2} = \frac{3}{2} \quad \times$$

Theorem:

If  $f$  is continuous on an interval, then  $f$  has an antiderivative on that interval. In particular, if  $a$  is any point in the interval, then the function  $F$  defined by

$$F(x) = \int_a^x f(t) dt$$

is an antiderivative of  $f$ ; that is,  $F'(x) = f(x)$  for each  $x$  in the interval, we can write alternative notation

$$\boxed{\frac{d}{dx} \left[ \int_a^x f(t) dt \right] = f(x).}$$

Example: Find  $\frac{d}{dx} \left[ \int_1^x t^3 dt \right]$ , since  $t^3$  is continuous on  $[1, x]$ .

$$\begin{aligned} \int_1^x t^3 dt &= \left. \frac{t^4}{4} \right|_1^x \\ &= \frac{x^4}{4} - \frac{1}{4} \end{aligned}$$

$$\begin{aligned} \frac{d}{dx} \left[ \frac{x^4}{4} - \frac{1}{4} \right] &= \frac{1}{4} \cdot 4x^3 - 0 \\ &= x^3. \end{aligned} \quad *$$

Using theorem,

$$\frac{d}{dx} \left[ \int_1^x t^3 dt \right] = x^3. \quad *$$

### Extra Problem

Evaluate:

$$\textcircled{1} \int_{-1}^2 4x(1-x^2) dx$$

$$\textcircled{2} \int_0^{\pi/3} (2x - \sec x \tan x) dx$$

$$\textcircled{3} \int_1^4 \left( \frac{1}{\sqrt{t}} - 3\sqrt{t} \right) dt$$

$$\textcircled{4} \int_{\pi/6}^{\pi/2} \left( x + \frac{2}{\sin x} \right) dx$$

$$\textcircled{5} \text{ Define } F(x) \text{ by } F(x) = \int_1^x (3t^2 - 3) dt.$$

Find  $F'(x)$ .

$$\textcircled{6} \text{ Define } F(x) \text{ by } F(x) = \int_{\pi/4}^x \cos 2t dt.$$

Find  $F'(x)$ .