Lecture 11
Sequences and Summations

Topics:

1. Sequences and their Summation
2. Recurrence Relations

Sequences

Definition: A sequence is a function from a subset of the set of integers (usually either the set {0, 1, 2, …} or the set {1, 2, 3, …}) to a set S. We use the notation an to denote the image of the integer n. We call a_n a term of the sequence.

Example: Consider the sequence $\{a_n\}$, where

$a_n = 1 /n$.

The list of the terms of this sequence, beginning with a_1, namely,

$a_1, a_2, a_3, a_4, \ldots ,$

starts with 1, 1/ 2 , 1 /3 , 1 /4 , . …

Sequences: Geometric Progression

Definition: A geometric progression is a sequence of the form

$a, ar, ar^2, \ldots, ar^n, \ldots$

where the initial term *a* and the common ratio *r* are real numbers.

Remark: A geometric progression is a discrete analogue of the exponential function $f(x) = ar^x$.

Sequences: Geometric Progression

Example: The sequences$\{b_n\}$ with $b_n = (-1)^n$, $\{c_n\}$ with $c_n = 2 \cdot 5^n$, and $\{d_n\}$ with $d_n = 6 \cdot (1/3)^n$ are geometric progressions with initial term and common ratio equal to 1 and −1; 2 and 5; and 6 and 1/3, respectively, if we start at n = 0. The list of terms $b_0, b_1, b_2, b_3, b_4, \ldots$begins with

$1, -1, 1, -1, 1, \ldots;$

the list of terms $c_0, c_1, c_2, c_3, c_4, \ldots$begins with

$2, 10, 50, 250, 1250, \ldots;$

and the list of terms $d_0, d_1, d_2, d_3, d_4, \ldots$begins with

$6, 2, 2/3, 2/9, 2/27, \ldots.$

Sequences: Arithmetic Progression

Definition: An Arithmetic Progression is a sequence of the form
$a, a+d, a+2d, \ldots, a+nd, \ldots$

where the initial term *a* and the common difference *d* are real numbers.

Sequences: Arithmetic Progression

Example: The sequences $\{s_n\}$ with $s_n = -1 + 4n$ and $\{t_n\}$ with $t_n = 7 - 3n$ are both arithmetic progressions with initial terms and common differences equal to −1 and 4, and 7 and −3, respectively, if we start at n = 0. The list of terms $s_0, s_1, s_2, s_3, \ldots$ begins with
$-1, 3, 7, 11, \ldots ,$
and the list of terms $t_0, t_1, t_2, t_3,, \ldots$ begins with
$7, 4, 1, -2, \ldots .$

Sequences of the form $a_1, a_2, \ldots , a_n$ are often used in computer science. These finite sequences are also called *strings*. This string is also denoted by $a_1\, a_2 \ldots a_n$. The length of a string is the number of terms in this string. The empty string, denoted by λ, is the string that has no terms. The *empty string* has length zero. For example, The string abcd is a string of length four.

Recurrence Relations

Definition: A recurrence relation for the sequence $\{a_n\}$ is an equation that expresses an in terms of one or more of the previous terms of the sequence, namely, $a_0, a_1, \ldots, a_{n-1}$, for all integers n with $n \geq n_0$, where n_0 is a nonnegative integer. A sequence is called a solution of a recurrence relation if its terms satisfy the recurrence relation. (A recurrence relation is said to recursively define a sequence.)

Recurrence Relations

Example: Let $\{a_n\}$ be a sequence that satisfies the recurrence relation $a_n = a_{n-1} + 3$ for $n = 1, 2, 3, \ldots$, and suppose that $a_0 = 2$. What are a_1, a_2, and a_3?

Solution: We see from the recurrence relation that $a_1 = a_0 + 3 = 2 + 3 = 5$.

It then follows that $a_2 = 5 + 3 = 8$

and $a_3 = 8 + 3 = 11$.

Recurrence Relations: Fibonacci sequence

Definition: The Fibonacci sequence, f_0, f_1, f_2, … , is defined by the initial conditions $f_0 = 0$, $f_1 = 1$, and the recurrence relation

$$f_n = f_{n-1} + f_{n-2}$$

for $n = 2, 3, 4, \ldots$.

Recurrence Relations: Fibonacci sequence

Example: Find the Fibonacci numbers f_2, f_3, f_4, f_5, and f_6.

<u>Solution:</u> The recurrence relation for the Fibonacci sequence tells us that we find successive terms by adding the previous two terms. Because the initial conditions tell us that $f_0 = 0$ and $f_1 = 1$, using the recurrence relation in the definition we find that,

$f_2 = f_1 + f_0 = 1 + 0 = 1$

$f_3 = f_2 + f_1 = 1 + 1 = 2$

$f_4 = f_3 + f_2 = 2 + 1 = 3$

$f_5 = f_4 + f_3 = 3 + 2 = 5$

$f_6 = f_5 + f_4 = 5 + 3 = 8$

Recurrence Relations: Compound Interest

Example: Suppose that a person deposits \$10,000 in a savings account at a bank yielding 11% per year with interest compounded annually. How much will be in the account after 30 years?

Solution: To solve this problem, let P_n denote the amount in the account after *n* years. Because the amount in the account after *n* years equals the amount in the account after *n−1* years plus interest for the *n*th year, we see that the sequence $\{P_n\}$ satisfies the recurrence relation

$$P_n = P_{n-1} + 0.11P_{n-1} = (1.11)P_{n-1}.$$

The initial condition is $P_0 = 10{,}000$. We can use an iterative approach to find a formula for P_n . Note that

$$P_1 = (1.11)P_0$$

$$P_2 = (1.11)P_1 = (1.11)^2P_0$$

$$P_3 = (1.11)P_2 = (1.11)^3P_0$$

$$\vdots$$

$$P_n = (1.11)P_{n-1} = (1.11)^nP_0.$$

Recurrence Relations: Compound Interest

When we insert the initial condition $P_0 = 10{,}000$, the formula $P_n = (1.11)^n 10{,}000$ is obtained.

Inserting $n = 30$ into the formula $P_n = (1.11)^n 10{,}000$ shows that after 30 years the account contains

$P_{30} = (1.11)^{30} 10{,}000 = \$228{,}922.97$.

Special Integer Sequences

Example 1: Find formulae for the sequences with the following first five terms: (a) 1, 1/2, 1/4, 1/8, 1/16 (b) 1, 3, 5, 7, 9 (c) 1, −1, 1, −1, 1.

Solution: (a) We recognize that the denominators are powers of 2. The sequence with $a_n = 1/2^n$, n = 0, 1, 2, … is a possible match. This proposed sequence is a geometric progression with a = 1 and r = 1/2.

(b) We note that each term is obtained by adding 2 to the previous term. The sequence with $a_n = 2n + 1$, n = 0, 1, 2, … is a possible match. This proposed sequence is an arithmetic progression with a = 1 and d = 2.

(c) The terms alternate between 1 and −1. The sequence with $a_n = (-1)^n$, n = 0, 1, 2… is a possible match. This proposed sequence is a geometric progression with a = 1 and r = −1.

Special Integer Sequences

Example 2: How can we produce the terms of a sequence if the first 10 terms are 1, 2, 2, 3, 3, 3, 4, 4, 4, 4?

Solution: In this sequence, the integer 1 appears once, the integer 2 appears twice, the integer 3 appears three times, and the integer 4 appears four times. A reasonable rule for generating this sequence is that the integer n appears exactly n times, so the next five terms of the sequence would all be 5, the following six terms would all be 6, and so on. The sequence generated this way is a possible match.

Summations

Next, we consider the addition of the terms of a sequence. For this we introduce summation notation. We begin by describing the notation used to express the sum of the terms

$a_m, a_{m+1}, \dots , a_n$

from the sequence $\{a_n\}$. We use the notation

$$\sum_{j=m}^{n} a_j \,, \qquad \textstyle\sum_{j=m}^{n} a_j \,, \qquad \text{or} \qquad \textstyle\sum_{m \le j \le n} a_j$$

(read as the sum from $j = m$ to $j = n$ of a_j) to represent

$a_m + a_{m+1} + \dots + a_n$

Summations

Here, the variable *j* is called the index of summation, and the choice of the letter *j* as the variable is arbitrary; that is, we could have used any other letter, such as *i* or *k*. Or, in notation,

$$\sum_{j=m}^{n} a_j = \sum_{i=m}^{n} a_i = \sum_{k=m}^{n} a_k.$$

Sum of terms of a geometric progression

Theorem:

If a and r are real numbers and $r \neq 0$, then

$$\sum_{j=0}^{n} ar^j = \begin{cases} \dfrac{ar^{n+1} - a}{r - 1} & \text{if } r \neq 1 \\ (n+1)a & \text{if } r = 1. \end{cases}$$

Sum of terms of a geometric progression

Proof: Let $S_n = \sum_{j=0}^{n} ar^j.$

To compute S, first multiply both sides of the equality by r and then manipulate the resulting sum as follows:

$$
\begin{aligned}
rS_n &= r\sum_{j=0}^{n} ar^j && \text{substituting summation formula for } S \\
&= \sum_{j=0}^{n} ar^{j+1} && \text{by the distributive property} \\
&= \sum_{k=1}^{n+1} ar^k && \text{shifting the index of summation, with } k = j+1 \\
&= \left(\sum_{k=0}^{n} ar^k\right) + (ar^{n+1} - a) && \text{removing } k = n+1 \text{ term and adding } k = 0 \text{ term} \\
&= S_n + (ar^{n+1} - a) && \text{substituting } S \text{ for summation formula}
\end{aligned}
$$

Sum of terms of a geometric progression

From these equalities, we see that

$$rS_n = S_n + (ar^{n+1} - a).$$

Solving for S_n shows that if $r \neq 1$, then

$$S_n = \frac{ar^{n+1} - a}{r - 1}.$$

If $r = 1$, then the $S_n = \sum_{j=0}^{n} ar^j = \sum_{j=0}^{n} a = (n + 1)a.$