Undergraduate Course in Mathematics

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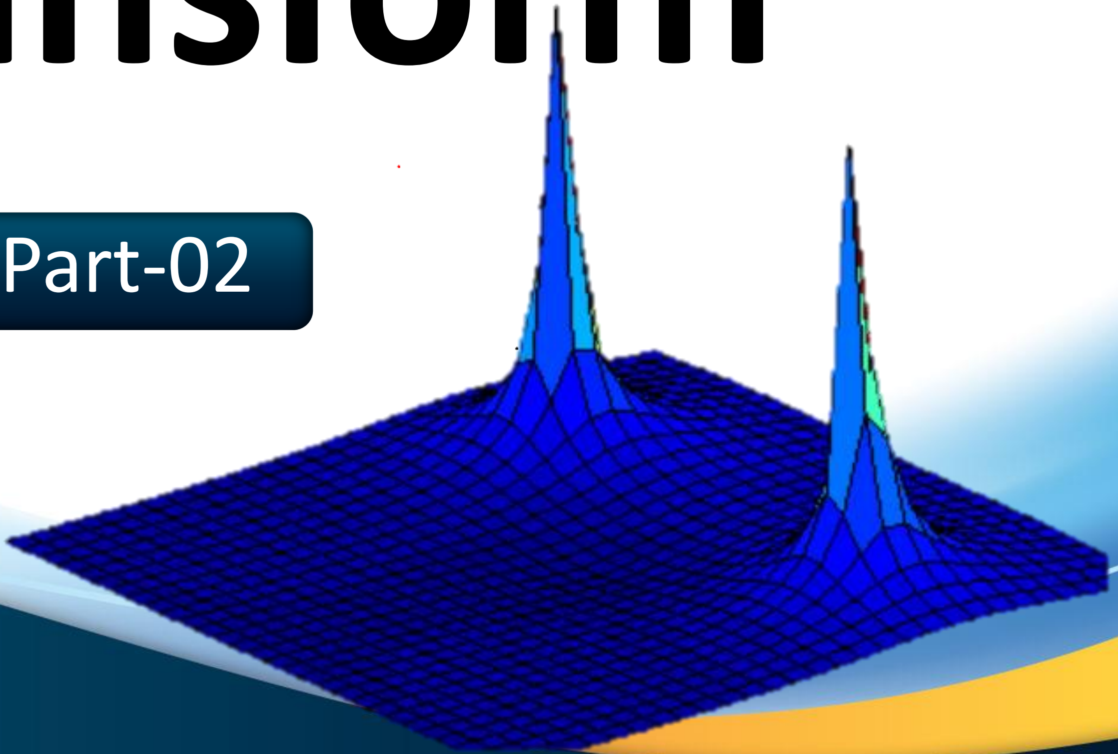
Laplace Transform

Solving Differential Equations | Part-02



Conducted By

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Laplace Transform of Derivatives

If $\mathcal{L}\{y(t)\} = Y(s)$ then

- $\mathcal{L}\{y'(t)\} = s \cdot Y(s) - y(0)$
- $\mathcal{L}\{y''(t)\} = s^2 \cdot Y(s) - s \cdot y(0) - y'(0)$
- $\mathcal{L}\{y'''(t)\} = s^3 \cdot Y(s) - s^2 \cdot y(0) - s \cdot y'(0) - y''(0)$

➢ $\mathcal{L}\{y^n(t)\} = s^n \cdot Y(s) - s^{n-1} \cdot y(0) - s^{n-2} \cdot y'(0) - s^{n-3} y''(0) - \cdots - y^{n-1}(0)$

Solve the given differential equation:

let $\mathcal{L}\{y(t)\} = Y(s)$

$$\frac{dy}{dt} + 3y = 13\, sin\, 2t, \quad y(0) = 6$$

$$\Rightarrow y' + 3y = 13 \sin 2t$$

$$\mathcal{L}\{y' + 3y\} = \mathcal{L}\{13 \sin 2t\}$$

$$\Rightarrow s \cdot Y(s) - y(0) + 3Y(s) = 13 \cdot \frac{2}{s^2+2^2}$$

$$\Rightarrow sY(s) - 6 + 3Y(s) = \frac{26}{s^2+4}$$

$$(s+3)\,Y(s) = \frac{26}{s^2+4} + 6 = \frac{6s^2+50}{s^2+4}$$

$$\Rightarrow Y(s) = \frac{6s^2+50}{(s+3)(s^2+4)}$$

$$\therefore y(t) = \mathcal{L}^{-1}\left\{ \frac{6s^2+50}{(s+3)(s^2+4)} \right\}$$

Now,

$$\frac{6s^2+50}{(s+3)(s^2+4)} = \frac{A}{s+3} + \frac{Bs+C}{s^2+4}$$

$$\Rightarrow 6s^2+50 = A(s^2+4) + (Bs+C)(s+3)$$

$$\Rightarrow 6s^2+50 = As^2+4A+Bs^2+3Bs+Cs+3C$$

$s=-3$

$6\cdot(-3)^2+50 = A(9+4)$

$\Rightarrow 13A = 104 \Rightarrow A=8.$

$[s^2]$: $6 = A+B \Rightarrow B=-2$

$[s]$: $0 = 3B+C$

$\Rightarrow C = -3B = -3\cdot(-2)$

$= 6.$

$$y(t) = \mathcal{L}^{-1}\left\{ \frac{8}{s+3} + \frac{-2s+6}{s^2+4} \right\}$$

$$= 8 \cdot \mathcal{L}^{-1}\left\{ \frac{1}{s+3} \right\} - 2 \cdot \mathcal{L}^{-1}\left\{ \frac{s}{s^2+2^2} \right\} + \frac{6}{2} \cdot \mathcal{L}^{-1}\left\{ \frac{2}{s^2+2^2} \right\}$$

$$= 8e^{-3t} - 2 \cdot \cos 2t + 3 \sin 2t$$

Solve the given differential equation:

$$y'' + 2y' + 5y = e^{-t}\sin(t), \quad y(0) = 0, \quad y'(0) = 1$$

$$\mathcal{L}\{y'' + 2y' + 5y\} = \mathcal{L}\{e^{-t}\sin t\}$$

$$\Rightarrow s^2Y(s) - \cancel{sy(0)} - y'(0) + 2[sY(s) - \cancel{y(0)}] + 5Y(s) = \frac{1}{(s+1)^2+1}$$

$$\Rightarrow s^2Y(s) - 1 + 2sY(s) + 5Y(s) = \frac{1}{s^2+2s+2}$$

$$\Rightarrow (s^2+2s+5)Y(s) = \frac{1}{s^2+2s+2} + 1 = \frac{s^2+2s+3}{s^2+2s+2}$$

$$Y(s) = \frac{s^2+2s+3}{(s^2+2s+2)(s^2+2s+5)}$$

$$y(t) = \mathcal{L}^{-1}\left\{ \frac{s^2+2s+3}{(s^2+2s+2)(s^2+2s+5)} \right\}$$

Let $s^2+2s=u$

$$\therefore \frac{u+3}{(u+2)(u+5)} = \frac{A}{u+2} + \frac{B}{u+5}$$

$$\Rightarrow u+3 = A(u+5) + B(u+2)$$

$$\therefore \frac{s^2+2s+3}{(s^2+2s+2)(s^2+2s+5)} = \frac{\frac{1}{3}}{s^2+2s+2} + \frac{\frac{2}{3}}{s^2+2s+5}$$

$u=-2$

$1 = 3A \Rightarrow A = \frac{1}{3}$

$u=-5$

$-2 = A(0) + B(-3)$

$B = \frac{2}{3}$.

$s \to s-1$

Now, $$y(t) = \mathcal{L}^{-1}\left\{\frac{\frac{1}{3}}{s^2+2s+2} + \frac{2/3}{s^2+2s+5}\right\}$$

$$= \frac{1}{3}\cdot \mathcal{L}^{-1}\left\{\frac{1}{(s+1)^2+1^2}\right\} + \frac{2}{3}\mathcal{L}^{-1}\left\{\frac{1}{(s+1)^2+2^2}\right\}$$

$$= \frac{1}{3}\cdot e^{-t}\mathcal{L}^{-1}\left\{\frac{1}{s^2+1^2}\right\} + \frac{2}{2\cdot 3}e^{-t}\mathcal{L}^{-1}\left\{\frac{2}{s^2+2^2}\right\}$$

$$= \frac{1}{3}e^{-t}\cdot \sin t + \frac{1}{3}e^{-t}\sin 2t \quad \text{tt}$$

Solve the given differential equation:

$$y' + y = f(t), \quad y(0) = 5$$

$$f(t) = \begin{cases} 0, & 0 \le t < \pi \\ 2\cos(t), & t \ge \pi \end{cases}$$

$$f(t) = \cancel{0 \cdot [u(t-0) - u(t-\pi)]} + 2\cos t\,[u(t-\pi) - \cancel{u(t-\infty)}]$$

$$= 2\cos t \cdot u(t-\pi)$$

Now, $y' + y = 2\cos t \cdot u(t-\pi)$ $\qquad y(0) = 5$ —— (1)

$$\mathcal{L}\{y' + y\} = \mathcal{L}\{2\cos t \cdot u(t-\pi)\}$$

$$\Rightarrow sY(s) - y(0) + Y(s) = \mathcal{L}\{2 \cdot \cos(t+\pi)\} e^{-\pi s}$$

$$\Rightarrow (s+1)Y(s) - 5 = \mathcal{L}\{-2\cos t\} e^{-\pi s}$$

$$\Rightarrow (s+1)Y(s) = -2 \cdot \frac{s}{s^2+1} e^{-\pi s} + 5$$

$$Y(s) = \frac{-2s}{(s+1)(s^2+1)} e^{-\pi s} + \frac{5}{s+1}$$

$$y(t) = \mathcal{L}^{-1}\left\{ \frac{-2s}{(s+1)(s^2+1)} e^{-\pi s} \right\} + \mathcal{L}^{-1}\left\{ \frac{5}{s+1} \right\}$$

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$$g(t) = \mathcal{L}^{-1}\left\{ \frac{-2s}{(s+1)(s^2+1)} \right\}$$

$$\frac{-2s}{(s+1)(s^2+1)} = \frac{A}{s+1} + \frac{Bs+C}{s^2+1}$$

$$\Rightarrow -2s = A(s^2+1) + (Bs+C)(s+1)$$

$$\Rightarrow -2s = As^2 + A + Bs^2 + Bs + Cs + C$$

$\underline{s=-1}$

$$2 = 2A \Rightarrow A = 1$$

$[s^2]$: $0 = A + B$

$$\Rightarrow B = -1$$

$[s]$: $-2 = B + C$

$$\Rightarrow C = -1.$$

$$y(t) = \mathcal{L}^{-1}\left\{\frac{1}{s+1} + \frac{-1s-1}{s^2+1}\right\}$$

$$= \mathcal{L}^{-1}\left\{\frac{1}{s+1}\right\} - \mathcal{L}^{-1}\left\{\frac{s}{s^2+1}\right\} - \mathcal{L}^{-1}\left\{\frac{1}{s^2+1}\right\}$$

$$= e^{-t} - \cos t - \sin t$$

$$y(t) = \mathcal{L}^{-1}\left\{\frac{-2s}{(s+1)(s^2+1)}e^{-\pi s}\right\} + \mathcal{L}^{-1}\left\{\frac{5}{s+1}\right\}$$

$$= g(t-\pi)\cdot u(t-\pi) + 5\cdot e^{-t}$$

$$= \left[e^{-(t-\pi)} - \cos(t-\pi) - \sin(t-\pi)\right]u(t-\pi) + 5e^{-t}$$

$$= \left[e^{-t+\pi} + \cos t + \sin t\right]u(t-\pi) + 5e^{-t}$$

Solve the given differential equation:

$$y'' + 4y = \sin t \cdot u(t-2\pi), \quad y(0) = 1, \; y'(0) = 0$$

$$\mathcal{L}\{y'' + 4y\} = \mathcal{L}\{\sin t \cdot u(t-2\pi)\}$$

$$\Rightarrow s^2Y(s) - s\cdot y(0) - \cancel{y'(0)} + 4Y(s) = \mathcal{L}\{\sin(t+2\pi)\}e^{-2\pi s}$$

$$\Rightarrow (s^2+4)Y(s) - s = \mathcal{L}\{\sin t\}e^{-2\pi s}$$

$$\Rightarrow (s^2+4)Y(s) = \frac{1}{s^2+1}e^{-2\pi s} + s.$$

$$Y(s) = \frac{1}{(s^2+1)(s^2+4)} e^{-2\pi s} + \frac{s}{s^2+4}$$

$$y(t) = \mathcal{L}^{-1}\left\{ \frac{1}{(s^2+1)\,(s^2+4)} e^{-2\pi s} \right\} + \mathcal{L}^{-1}\left\{ \frac{s}{s^2+4} \right\}$$

(1)

$$g(t) = \mathcal{L}^{-1}\left\{ \frac{1}{(s^2+1)(s^2+4)} \right\}$$

let $s^2 = u$

$$\frac{1}{(u+1)(u+4)} = \frac{A}{u+1} + \frac{B}{u+4}$$

$$\Rightarrow 1 = A(u+4) + B(u+1)$$

$u = -1$

$$1 = 3A \Rightarrow A = \frac{1}{3}$$

$u = -4$

$$1 = B(-3)$$

$$B = \frac{-1}{3}$$

$$g(t) = \mathcal{L}^{-1}\left\{\frac{\frac{1}{3}}{s^2+1} + \frac{-\frac{1}{3}}{s^2+4}\right\}$$

$$= \frac{1}{3}\mathcal{L}^{-1}\left\{\frac{1}{s^2+1}\right\} \frac{-1}{2\cdot 3}\mathcal{L}^{-1}\left\{\frac{2}{s^2+2^2}\right\}$$

$$= \frac{1}{3}\sin t - \frac{1}{6}\sin 2t$$ ✓

$$y(t) = \mathcal{L}^{-1}\left\{ \frac{1}{(s^2+1)(s^2+4)} e^{-2\pi s} \right\} + \mathcal{L}^{-1}\left\{ \frac{s}{s^2+4} \right\}$$

$$= g(t-2\pi)\cdot u(t-2\pi) + \cos 2t$$

$$= \left[\frac{1}{3}\sin(t-2\pi) - \frac{1}{6}\sin\left(2(t-2\pi)\right) \right] u(t-2\pi) + \cos 2t$$

$$= \left[\frac{1}{3}\sin t - \frac{1}{6}\sin 2t \right] u(t-2\pi) + \cos t$$

Solve the given differential equation:

$$y'' + 9y = \cos 2t, \quad y(0) = 1, y\left(\frac{\pi}{2}\right) = -1$$

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Let $y'(0) = c$

$$\mathcal{L}\{y'' + 9y\} = \mathcal{L}\{\cos 2t\}$$

$$\Rightarrow s^2 Y(s) - s\,y(0) - y'(0) + 9Y(s) = \frac{s}{s^2+4}$$

$$\Rightarrow s^2 Y(s) - s - c + 9Y(s) = \frac{s}{s^2+4}$$

$$(s^2+9)Y(s) = \frac{s}{s^2+4} + s + c$$

$$\Rightarrow Y(s) = \frac{s}{(s^2+4)(s^2+9)} + \frac{s}{s^2+9} + \frac{c}{s^2+9}$$

$$\Rightarrow y(t) = \mathcal{L}^{-1}\left\{\frac{s}{(s^2+4)(s^2+9)}\right\} + \mathcal{L}^{-1}\left\{\frac{s}{s^2+9}\right\} + \mathcal{L}^{-1}\left\{\frac{c}{s^2+9}\right\}$$

$$\frac{s}{(s^2+4)(s^2+9)} = \frac{As+B}{s^2+4} + \frac{Cs+D}{s^2+9}$$

$$\Rightarrow s = (As+B)(s^2+9) + (Cs+D)(s^2+4)$$

$$\Rightarrow s = As^3 + 9As + Bs^2 + 9B$$

$$+ Cs^3 + 4Cs + Ds^2 + 4D$$

$[s^3]: \quad 0 = A + C \Rightarrow C = -A$

$[s^2]: \quad 0 = B + D$

$[s]: \quad 1 = 9A + 4C$

$1 = 9A - 4A$

$[\text{Const}]: \quad 0 = 9B + 4D$

$A = \frac{1}{5} \qquad C = \frac{-1}{5}$

$B = 0 \qquad D = 0$

$$y(t) = \mathcal{L}^{-1}\left\{ \frac{\frac{1}{5}s + 0}{s^2+4} + \frac{-\frac{1}{5}s + 0}{s^2+9} \right\} + \mathcal{L}^{-1}\left\{ \frac{s}{s^2+9} \right\} + \mathcal{L}^{-1}\left\{ \frac{c}{s^2+9} \right\}$$

$$= \frac{1}{5}\mathcal{L}^{-1}\left\{ \frac{s}{s^2+2^2} \right\} - \frac{1}{5}\mathcal{L}^{-1}\left\{ \frac{s}{s^2+3^2} \right\} + \mathcal{L}^{-1}\left\{ \frac{s}{s^2+3^2} \right\} + \frac{c}{3}\mathcal{L}^{-1}\left\{ \frac{3}{s^2+3^2} \right\}$$

$$y(t) = \frac{1}{5}\cos 2t - \frac{1}{5}\cos 3t + \cos 3t + \frac{c}{3}\sin 3t \, .$$

Now, given $y\left(\frac{\pi}{2}\right) = -1$

$$\Rightarrow \frac{1}{5}\cos\left(2\frac{\pi}{2}\right) - \frac{1}{5}\cos\left(3\frac{\pi}{2}\right) + \cos\left(3\frac{\pi}{2}\right) + \frac{c}{3}\sin\left(3\cdot\frac{\pi}{2}\right) = -1$$

$$\Rightarrow \frac{1}{5}(-1) - 0 + 0 + \frac{c}{3}\cdot(-1) = -1$$

$$\Rightarrow \frac{1}{5} + \frac{c}{3} = 1 \Rightarrow \frac{c}{3} = \frac{4}{5} \Rightarrow c = \frac{12}{5} \checkmark$$

$$y(t) = \frac{1}{5}\cos 2t - \frac{1}{5}\cos 3t + \cos 3t + \frac{\frac{12}{5}}{3}\sin 3t$$

$$= \frac{1}{5}\cos 2t + \frac{4}{5}\cos 3t + \frac{4}{5}\sin 3t$$

Solve the system

$$\frac{dx}{dt} = -x + y \quad , \qquad \Rightarrow x' = -x + y$$

$$\frac{dy}{dt} = 2x \quad , \qquad \Rightarrow y' = 2x$$

$$x(0) = 0 \quad , \quad y(0) = 1 \quad .$$

Let $\mathcal{L}\{x(t)\} = X(s)$ $\qquad \mathcal{L}\{y(t)\} = Y(s)$

from ①,

$$x' = -x + y$$

$$\Rightarrow \mathcal{L}\{x'\} = \mathcal{L}\{-x+y\}$$

$$\Rightarrow s \cdot X - x(0) = -X + Y$$

$$\Rightarrow s \cdot X - 0 = -X + Y$$

$$\Rightarrow (s+1)X - Y = 0$$

$$\Rightarrow Y = (s+1)X.$$

from ②,

$$y' = 2x$$

$$\Rightarrow \mathcal{L}\{y'\} = \mathcal{L}\{2x\}$$

$$\Rightarrow sY - y(0) = 2 \cdot X$$

$$\Rightarrow sY - 2X = 1$$

$$\Rightarrow 2X - sY = -1$$

$$\Rightarrow 2X - s\cdot(s+1)X = -1$$

$$\Rightarrow 2X - (s^2+s)X = -1$$

$$\Rightarrow (2-s^2-s)X = -1$$

$$\Rightarrow X = \frac{1}{s^2+s-2}$$

$$X = \frac{1}{(s+2)(s-1)}$$

$$x(t) = \mathcal{L}^{-1}\left\{ \frac{-\frac{1}{3}}{s+2} + \frac{\frac{1}{3}}{s-1} \right\}$$

$$= \frac{-1}{3} \cdot e^{-2t} + \frac{1}{3} e^{t} \quad //$$

Now, $Y = (s+1)X$

$$= \frac{s+1}{(s+2)(s-1)}$$

$$y(t) = \mathcal{L}^{-1}\left\{ \frac{\frac{1}{3}}{s+2} + \frac{\frac{2}{3}}{s-1} \right\}$$

$$= \frac{1}{3}e^{-2t} + \frac{2}{3}e^{t} \quad \text{u}$$

Evaluation of some Improper Integrals

Using Laplace Transform we can easily evaluate this type of integral

$$\int_0^\infty \underline{f(t)}e^{-kt}dt$$

Evaluate

$$\int_0^\infty \underbrace{\sin 3t}\, e^{-2t}\,dt$$

$$\mathcal{L}\{\sin 3t\} = \frac{3}{s^2+9}$$

$$\Rightarrow \int_0^\infty \sin 3t \cdot e^{-st}\,dt = \frac{3}{s^2+9}$$

$$\therefore \int_0^\infty \sin 3t \cdot e^{-2t}\,dt = \frac{3}{2^2+9}$$

$$= \frac{3}{13}$$

Evaluate

$$\int_0^\infty \underbrace{t \sin 2t}\, e^{-t} dt$$

$$\mathcal{L}\{\sin 2t\} = \frac{2}{s^2+4}$$

$$\mathcal{L}\{t \cdot \sin 2t\} = -\frac{d}{ds}\left(\frac{2}{s^2+4}\right)$$

$$\int_0^\infty t \cdot \sin 2t \cdot e^{-st}\, dt = \frac{4s}{(s^2+4)^2}$$

$$\therefore \int_0^\infty t \cdot \sin 2t\, e^{-t}\, dt = \frac{4}{25}$$

Evaluate

$$\int_0^\infty \frac{\sin t}{t}\,dt$$

$$\mathcal{L}\{\sin t\} = \frac{1}{s^2+1}$$

$$\mathcal{L}\left\{\frac{\sin t}{t}\right\} = \int_s^\infty \frac{1}{u^2+1}\,du$$

$$= \left[\tan^{-1}(u)\right]_s^\infty$$

$$= \tan^{-1}(\infty) - \tan^{-1}(s) = \frac{\pi}{2} - \tan^{-1}(s)$$

$$\therefore \mathcal{L}\left\{\frac{\sin t}{t}\right\} = \frac{\pi}{2} - \tan^{-1}(s)$$

$$\Rightarrow \int_0^\infty \frac{\sin t}{t}\cdot e^{-st}\,dt = \frac{\pi}{2} - \tan^{-1}(s)$$

$$\Rightarrow \int_0^\infty \frac{\sin t}{t}\,dt = \frac{\pi}{2} - \tan^{-1}(0) = \frac{\pi}{2}$$

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