Lecture 19

Topics:

1. Rules of counting
   a. Sum Rule
   b. Product Rule
   c. Subtraction Rule
2. The Pigeonhole Principle

6.1 The Basics of Counting

We first present two basic counting principles, the product rule and the sum rule. Then we will show how they can be used to solve many different counting problems.

The product rule applies when a procedure is made up of separate tasks.

THE PRODUCT RULE Suppose that a procedure can be broken down into a sequence of two tasks. If there are n_1 ways to do the first task and for each of these ways of doing the first task, there are n_2 ways to do the second task, then there are $n_1 * n_2$ ways to do the procedure.

Product Rule Examples:

EXAMPLE 1: A new company with just two employees, Sanchez and Patel, rents a floor of a building with 12 offices. How many ways are there to assign different offices to these two employees?

Solution: The procedure of assigning offices to these two employees consists of assigning an office to Sanchez, which can be done in 12 ways, then assigning an office to Patel different from the office assigned to Sanchez, which can be done in 11 ways.

By the product rule, there are 12 * 11 = 132 ways to assign offices to these two employees.

EXAMPLE 4: How many different bit strings of length seven are there?

Solution: Each of the seven bits can be chosen in two ways, because each bit is either 0 or 1. Therefore, the product rule shows there are a total of 27 = 128 different bit strings of length seven.

THE SUM RULE:

If a task can be done either in one of n_1 ways or in one of n_2 ways, where none of the set of n_1 ways is the same as any of the set of n_2 ways, then there are $n_1 + n_2$ ways to do the task.

EXAMPLE 13 A student can choose a computer project from one of three lists. The three lists contain 23, 15, and 19 possible projects, respectively. No project is on more than one list. How many possible projects are there to choose from?

Solution: The student can choose a project by selecting a project from the first list, the second list, or the third list. Because no project is on more than one list, by the sum rule there are $23 + 15 + 19 = 57$ ways to choose a project.

More Complex Examples:

EXAMPLE 16 Each user on a computer system has a password, which is six to eight characters long, where each character is an uppercase letter or a digit. Each password must contain at least one digit. How many possible passwords are there?

Solution: Let P be the total number of possible passwords, and let P_6, P_7, and P_8 denote the number of possible passwords of length 6, 7, and 8, respectively.

By the sum rule, $P = P_6 + P_7 + P_8$. We will now find P_6, P_7, and P_8 . Finding P_6 directly is difficult. To find P_6 it is easier to find the number of strings of uppercase letters and digits that are six characters long, including those with no digits, and subtract from this the number of strings with no digits. By the product rule, the number of strings of six characters is 366, and the number of strings with no digits is 266.

P_6 = 366 − 266 = 2,176,782,336 − 308,915,776 = 1,867,866,560.

Similarly, we have

P_7 = 367 − 267 = 78,364,164,096 − 8,031,810,176 = 70,332,353,920

and

P_8 = 368 − 268 = 2,821,109,907,456 − 208,827,064,576

= 2,612,282,842,880.

Consequently,

$P = P_6 + P_7 + P_8$ = 2,684,483,063,360.

6.1.4 The Subtraction Rule (Inclusion-Exclusion for Two Sets)

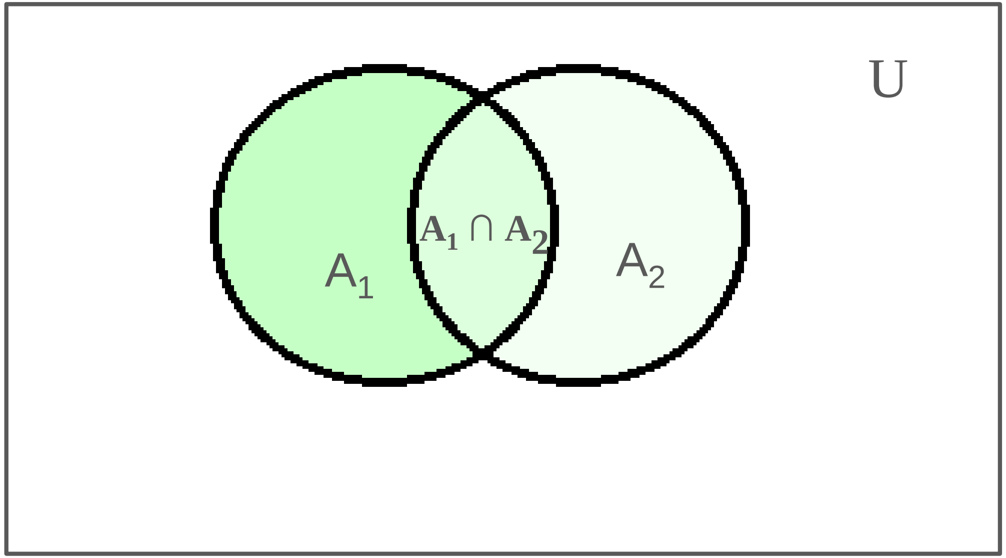
If a task can be done in either n1 ways or n2 ways, then the number of ways to do the task is $n_1 + n_2$ minus the number of ways to do the task that are common to the two different ways.

The subtraction rule is also known as the principle of **inclusion–exclusion,** especially when it is used to count the number of elements in the union of two sets.

Subtraction Rules:

Suppose that A_1 and A are sets. Then, there are $|A_1|$ ways to select an element from A_1 and $|A_2|$ ways to select an element from A_2. The number of ways to select an element from A_1 or from A_2, that is, the number of ways to select an element from their union, is the sum of the number of ways to select an element from A_1 and the number of ways to select an element from A_2, minus the number of ways to select an element that is in both A_1 and A_2. Because there are $|A_1 \cup A_2|$ ways to select an element in either A_1 or in A_2, and $|A_1 \cap A_2|$ ways to select an element common to both setsm we have $\mathbf{|A_1 \cup A_2| = |A_1| + |A_2| - |A_1 \cap A_2|}$

EXAMPLE 19 A computer company receives 350 applications from college graduates for a job planning a line of new web servers. Suppose that 220 of these applicants majored in computer science, 147 majored in business, and 51 majored both in computer science and in business. How many of these applicants majored neither in computer science nor in business?



$|A_1| = 220$

$|A_2| = 147$

$|A_1 \cap A_2| = 51$

Solution:

To find the number of these applicants who majored neither in computer science nor in business, we can subtract the number of students who majored either in computer science or in business (or both) from the total number of applicants. Let A_1 be the set of students who majored in computer science and A_2 the set of students who majored in business. Then $A_1 \cup A_2$ is the set of students who majored in computer science or business (or both), and $A_1 \cap A_2$ is the set of students who majored both in computer science and in business. By the subtraction rule

the number of students who majored either in computer science or in business (or both) equals

$|A_1 \cup A_2| = |A_1| + |A_2| - |A_1 \cap A_2| = 220 + 147 - 51 = 316.$

We conclude that $350 - 316 = 34$ of the applicants majored neither in computer science nor in business.

6.2 The Pigeonhole Principle

THEOREM 1 : If k is a positive integer and $k + 1$ or more objects are placed into k boxes, then there is at least one box containing two or more of the objects.

Proof: We prove the pigeonhole principle using a proof by contraposition. Suppose that none of the k boxes contains more than one object. Then the total number of objects would be at most k. This is a contradiction, because there are at least $k + 1$ objects.

COROLLARY 1 A function f from a set with k + 1 or more elements to a set with k elements is not one-to-one.

Proof: Suppose that for each element y in the codomain of f we have a box that contains all elements x of the domain of f such that **f(x) = y.** Because the domain contains **k + 1** or more elements and the codomain contains only k elements, the pigeonhole principle tells us that one of these boxes contains two or more elements x of the domain. This means that f cannot be one-to-one.

EXAMPLE 2 In any group of 27 English words, there must be at least two that begin with the same letter, because there are 26 letters in the English alphabet.

EXAMPLE 4 Show that for every integer n there is a multiple of n that has only 0s and 1s in its decimal expansion.

Solution: Let n be a positive integer. Consider the n + 1 integers 1, 11, 111, … , 11 … 1 (where the last integer in this list is the integer with n + 1 1s in its decimal expansion). Note that there are n possible remainders when an integer is divided by n. Because there are n + 1 integers in this list, by the pigeonhole principle there must be two with the same remainder when divided by n. The larger of these integers less the smaller one is a multiple of n, which has a decimal expansion consisting entirely of 0s and 1s.

6.2.2 The Generalized Pigeonhole Principle

- If N objects are placed into k boxes, then there is at least one box containing $\lceil N/k \rceil$ objects
- **Proof by contradiction**:
- Suppose that none of the boxes contains more than $\lceil N/k \rceil$-1 objects. Then, the total number of objects is at most:

$$k\left(\left\lceil \frac{N}{k} \right\rceil - 1\right) < k\left(\left(\frac{N}{k} + 1\right) - 1\right) = N,$$

  - * $\lceil N/k \rceil < (N/k) + 1$

EXAMPLE 6 What is the minimum number of students required in a discrete mathematics class to be sure that at least six will receive the same grade, if there are five possible grades, A, B, C, D, and F?

Solution: The minimum number of students needed to ensure that at least six students receive Examples the same grade is the smallest integer N such that

$\lceil N/5 \rceil = 6$. The smallest such integer is $N = 5 \cdot 5 + 1 = 26$.

If you have only 25 students, it is possible for there to be five who have received each grade so that no six students have received the same grade. Thus, 26 is the minimum number of students needed to ensure that at least six students will receive the same grade.

6.2.3 Some Elegant Applications of the Pigeonhole Principle

EXAMPLE 10 During a month with 30 days, a baseball team plays at least one game a day, but no more than 45 games. Show that there must be a period of some number of consecutive days during which the team must play exactly 14 games.

Solution: Let aj be the number of games played on or before the jth day of the month. Then a_1, a_2, … , a_{30} is an increasing sequence of distinct positive integers, with $1 \leq aj \leq 45$. Moreover, $a_1 + 14$, $a_2 + 14$, … , $a_{30} + 14$ is also an increasing sequence of distinct positive integers, with $15 \leq a_j + 14 \leq 59$. The 60 positive integers a_1, a_2, … , a_{30}, $a_1 + 14$, $a_2 + 14$, … , $a_{30} + 14$ are all less than or equal to 59. Hence, by the pigeonhole principle two of these integers are equal. Because the integers a_j, $j = 1, 2, \ldots, 30$ are all distinct and the integers $a_j + 14$, $j = 1, 2, \ldots, 30$ are all distinct, there must be indices i and j with $a_i = a_j + 14$. This means that exactly 14 games were played from day $j + 1$ to day i.

EXAMPLE 11 Show that among any $n + 1$ positive integers not exceeding $2n$ there must be an integer that divides one of the other integers.

Solution: Write each of the $n + 1$ integers $a_1, a_2, \ldots, a_{n+1}$ as a power of 2 times an odd integer. In other words, let $\mathbf{a_j = 2a^{k}{}_{i}\, q_j}$ for $j = 1, 2, \ldots, n + 1$, where q_j is a nonnegative integer and q_j is odd. The integers $a_1, a_2, \ldots, a_{n+1}$ are all odd positive integers less than $2n$. Because there are only n odd positive integers less than $2n$, it follows from the pigeonhole principle that two of the integers $a_1, a_2, \ldots, a_{n+1}$ must be equal. Therefore, there are distinct integers i and j such that $q_i = q_j$. Let q be the common value of a_i and a_j. Then, $\mathbf{a_i = 2^{k}{}_{i}q}$ and $\mathbf{a_j = 2^{k}{}_{j}q}$**.**

It follows that if $k_i < k_j$, then a_i divides a_j ; while if $k_i > k_j$, then a_j divides a_i .

Self Study

THEOREM 3: Every sequence of $n^2 + 1$ distinct real numbers contains a subsequence of length $n + 1$ that is either strictly increasing or strictly decreasing. We give an example before presenting the proof of Theorem 3.

EXAMPLE 12 The sequence 8, 11, 9, 1, 4, 6, 12, 10, 5, 7 contains 10 terms. Note that $10 = 3^2 + 1$. There are four strictly increasing subsequences of length four, namely, 1, 4, 6, 12; 1, 4, 6, 7; 1, 4, 6, 10; and 1, 4, 5, 7. There is also a strictly decreasing subsequence of length four, namely, 11, 9, 6, 5.

Self Study

Proof: Let $\mathbf{a_1, a_2, \ldots , a_{n2+1}}$ be a sequence of $\mathbf{n^2 + 1}$ distinct real numbers. Associate an ordered pair with each term of the sequence, namely, associate $\mathbf{(i_k, d_k)}$ to the term a_k, where i_k is the length of the longest increasing subsequence starting at a_k, and d_k is the length of the longest decreasing subsequence starting at a_k. Suppose that there are no increasing or decreasing subsequences of length $\mathbf{n + 1}$. Then i_k and d_k are both positive integers less than or equal to n, for $\mathbf{k = 1, 2, \ldots , n^2 + 1}$. Hence, by the product rule there are n^2 possible ordered pairs for $\mathbf{(i_k, d_k)}$. By the pigeonhole principle, two of these $\mathbf{n^2 + 1}$ ordered pairs are equal. In other words, there exist terms as and at, with $\mathbf{s < t}$ such that $\mathbf{i_s = i_t}$ and $\mathbf{d_s = d_t}$. We will show that this is impossible. Because the terms of the sequence are distinct, either $\mathbf{a_s < a_t}$ **or** $\mathbf{a_s > a_t}$. If $\mathbf{a_s < a_t}$, then, because $\mathbf{i_s = i_t}$, an increasing subsequence of length $\mathbf{i_t + 1}$ can be built starting at a_s , by taking as followed by an increasing subsequence of length it beginning at a_t. This is a contradiction. Similarly, if $\mathbf{a_s > a_t}$, the same reasoning shows that d_s must be greater than d_t, which is a contradiction.

Self Study

The final example shows how the generalized pigeonhole principle can be applied to an important part of combinatorics called **Ramsey theory,** after the English mathematician F. P. Ramsey. In general, Ramsey theory deals with the distribution of subsets of elements of sets.

EXAMPLE 13 Assume that in a group of six people, each pair of individuals consists of two friends or two enemies. Show that there are either three mutual friends or three mutual enemies in the group.

Solution: Let A be one of the six people. Of the five other people in the group, there are either three or more who are friends of A, or three or more who are enemies of A. This follows from:

Self Study

The generalized pigeonhole principle, because when five objects are divided into two sets, one of the sets has at least **$\lceil 5/2 \rceil = 3$** elements. In the former case, suppose that B, C, and D are friends of A. If any two of these three individuals are friends, then these two and A form a group of three mutual friends. Otherwise, B, C, and D form a set of three mutual enemies. The proof in the latter case, when there are three or more enemies of A, proceeds in a similar manner.

The Ramsey number R(m, n), where m and n are positive integers greater than or equal to 2, denotes the minimum number of people at a party such that there are either m mutual friends or n mutual enemies, assuming that every pair of people at the party are friends or enemies. **Example 13** shows that **$R(3, 3) \leq 6$.** We conclude that **$R(3, 3) = 6$** because in a group of five people where every two people are friends or enemies, there may not be three mutual friends or three mutual enemies (see Exercise 28 from the book).

Self Study

It is possible to prove some useful properties about Ramsey numbers, but for the most part it is difficult to find their exact values. Note that by symmetry it can be shown that $R(m, n) = R(n, m)$ (**see Exercise 32**). We also have $R(2, n) = n$ for every positive integer $n \geq 2$ (**see Exercise 31)**. The exact values of only nine Ramsey numbers $R(m, n)$ with $3 \leq m \leq n$ are known, including $R(4, 4) = 18$.

Only bounds are known for many other Ramsey numbers, including $R(5, 5)$, which is known to satisfy **$43 \leq R(5, 5) \leq 49$**. The reader interested in learning more about Ramsey numbers should consult **[MiRo91] or [GrRoSp90]**.

The end