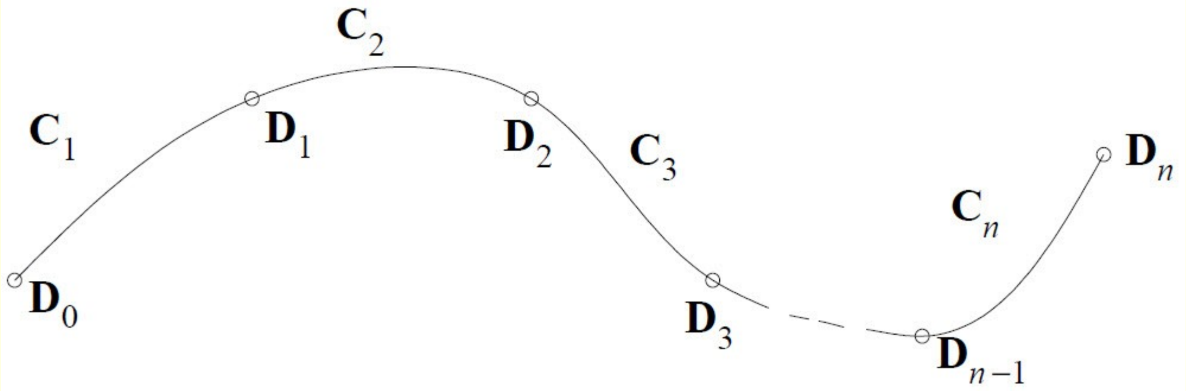
3.1.9 Curve Interpolation using Uniform Cubic B-Spline Curves

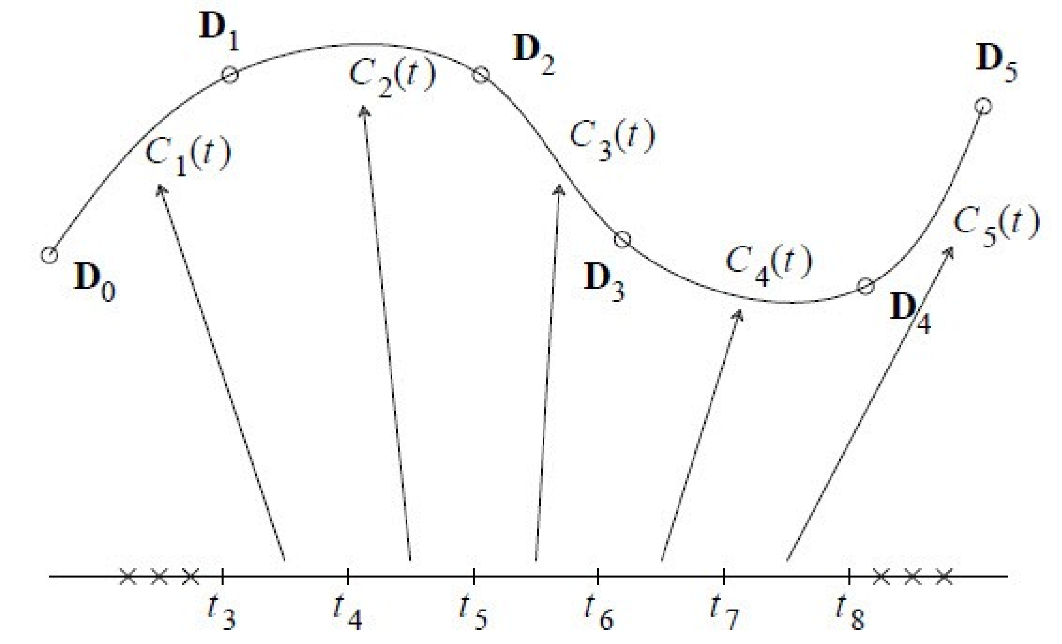
- Give a set of data points $\mathbf{D}_i = (x_i, y_i)$, $i = 0, 1, \cdots, n$, $(n \geq 2)$, how can a cubic B-spline curve that interpolates these points be constructed?



- The cubic B-spline curve has n segments $\mathbf{C}_1(t)$, $\mathbf{C}_2(t)$, ..., $\mathbf{C}_n(t)$ with $\mathbf{D}_{i-1}$ and $\mathbf{D}_i$ being the start and end points of $\mathbf{C}_i(t)$

An analysis of the problem:

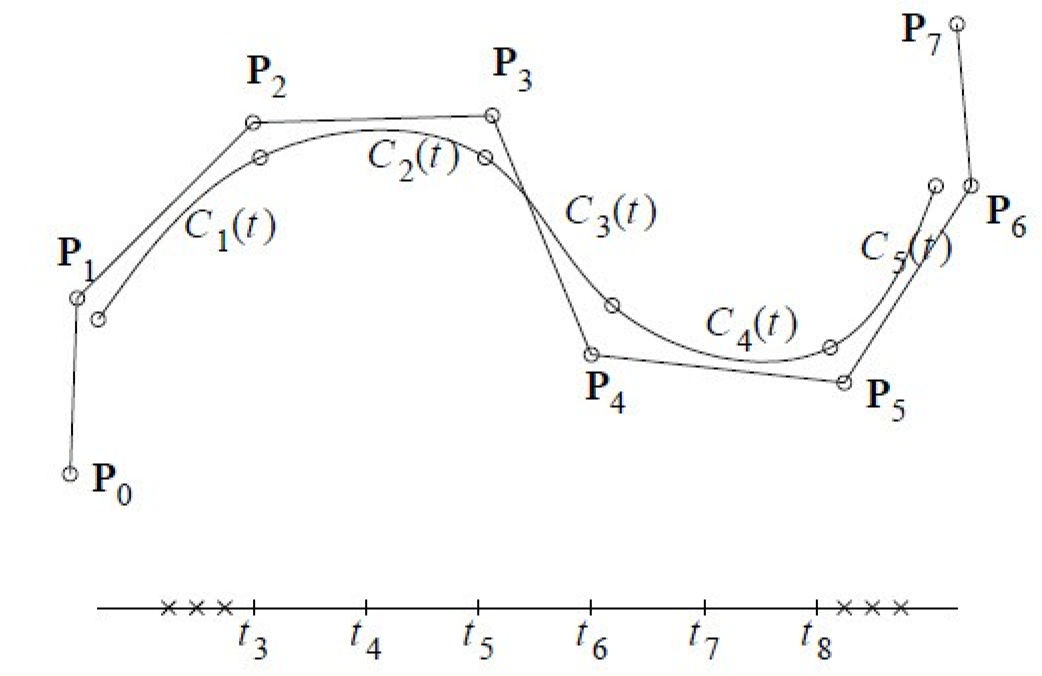
- To get the curve constructed, how many knots are needed? Consider the following case:



So, to interpolate (n +1) data points, one needs (n +7) knots, t_0, $t_1, \ldots, t_{n+6}$, for a uniform cubic B-spline interpolating curve.

3.1.9 Curve Interpolation using Uniform Cubic B-Spline Curves

- To get the curve constructed, how many control points are needed? Consider the following case:



So, to interpolate (n +1) points, one needs (n +3) control points, $P_0, P_1, \ldots, P_{n+2}$, for a uniform cubic B-spline interpolating curve.

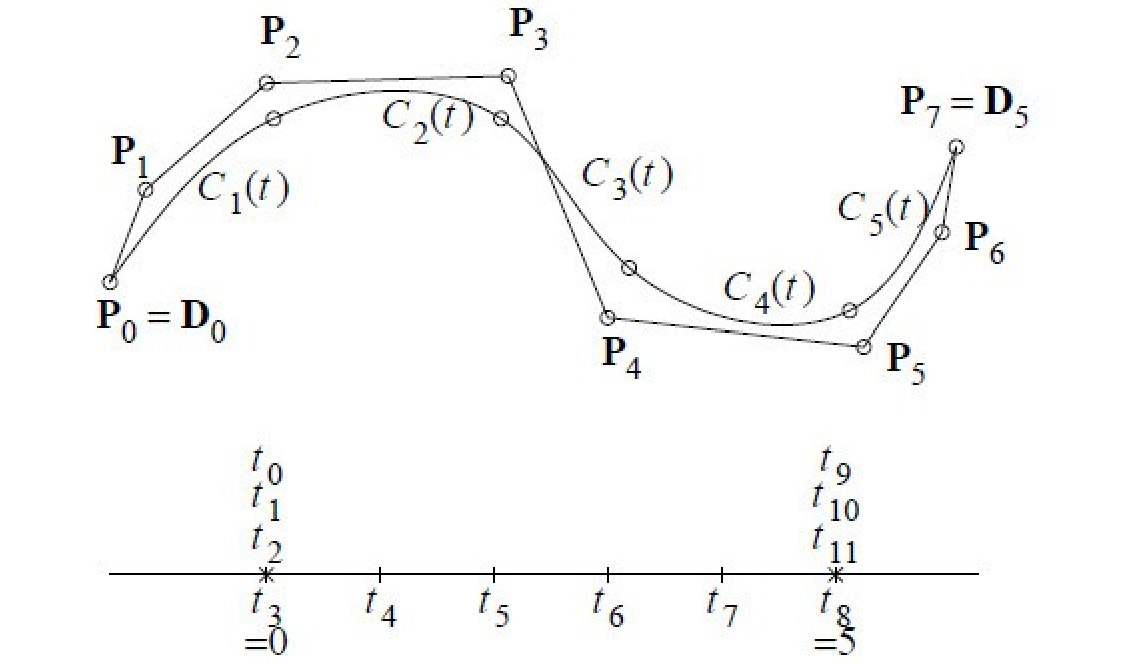
3.1.9 Curve Interpolation using Uniform Cubic B-Spline Curves

- To make things easier, we shall assume that

$$t_i = i - 3, \quad i = 3,4,\ldots,n+3$$

with $t_0 = t_1 = t_2 = t_3$ and $t_{n+3} = t_{n+4} = t_{n+5} = t_{n+6}$.

Consequently, we have $P_0 = D_0$ and $P_{n+2} = D_n$.



Still, we need to find $P_1 = P_2 = \ldots = P_{n+1}$.

How?

3.1.9 Curve Interpolation using Uniform Cubic B-Spline Curves

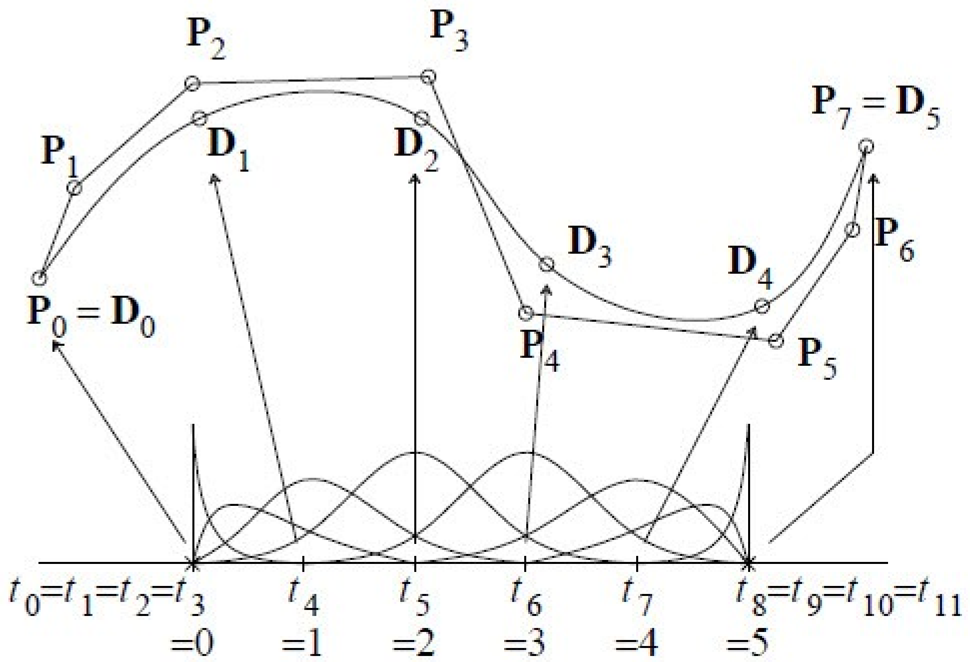
The interpolating curve to be constructed must be of the following form:

$$C(t) = \sum_{i=0}^{n+2} N_{i,3}(t)\,\mathbf{P}_i, \quad t \in [0, n]$$

and satisfies the following conditions:

$$C(i) = \sum_{i=0}^{n+2} N_{i,3}(i)\,\mathbf{P}_i = \mathbf{D}_i,$$

$$i = 0, 1, \ldots, n \qquad (*)$$



3.1.9 Curve Interpolation using Uniform Cubic B-Spline Curves

Note that at each knot there are at most 3 cubic B-spline basis functions which are non-zero.
Therefore, equations in (*) are of the following form:

$$N_{i,3}(i)\mathbf{P}_i + N_{i+1,3}(i)\mathbf{P}_{i+1} + N_{i+2,3}(i)\mathbf{P}_{i+2} = \mathbf{D}_i, \quad i=0,1,\ldots,n$$

or

$$\mathbf{P}_0 = \mathbf{D}_0$$

$$\# \quad \frac{1}{4}\mathbf{P}_1 + \frac{7}{12}\mathbf{P}_2 + \frac{1}{6}\mathbf{P}_3 = \mathbf{D}_1$$

$$\# \quad \frac{1}{6}\mathbf{P}_2 + \frac{2}{3}\mathbf{P}_3 + \frac{1}{6}\mathbf{P}_4 = \mathbf{D}_2$$

$$\# \quad \ldots$$

$$\# \quad \frac{1}{6}\mathbf{P}_{n-2} + \frac{2}{3}\mathbf{P}_{n-1} + \frac{1}{6}\mathbf{P}_n = \mathbf{D}_{n-2}$$

$$\# \quad \frac{1}{6}\mathbf{P}_{n-1} + \frac{7}{12}\mathbf{P}_n + \frac{1}{4}\mathbf{P}_{n+1} = \mathbf{D}_{n-1}$$

$$\mathbf{P}_{n+2} = \mathbf{D}_n$$

3.1.9 Curve Interpolation using Uniform Cubic B-Spline Curves

So, actually, only $P_1, P_2, \ldots, P_{n+1}$ are unknown. By ignoring the 1st and the last equations, we have a system of n -1 equations (those marked with "#") in n +1 unknowns. We need two extra conditions to get this system solved.

One option is to set the second derivative of the curve at the start and end points to zero:

$$C''(0) = N_{0,3}''(0)\mathbf{P}_0 + N_{1,3}''(0)\mathbf{P}_1 + N_{2,3}''(0)\mathbf{P}_2 = 0$$

$$C''(n) = N_{n,3}''(n)\mathbf{P}_n + N_{n+1,3}''(n)\mathbf{P}_n + N_{n+2,3}''(n)\mathbf{P}_{n+2} = 0$$

3.1.9 Curve Interpolation using Uniform Cubic B-Spline Curves

or

$$6\mathbf{P}_0 - 9\mathbf{P}_1 + 3\mathbf{P}_2 = 0$$

$$3\mathbf{P}_n - 9\mathbf{P}_{n+1} + 6\mathbf{P}_{n+2} = 0$$

Note that $\mathbf{P}_0$ and $\mathbf{P}_{n+2}$ are known to us ($\mathbf{P}_0 = \mathbf{D}_0$ and $\mathbf{P}_{n+2} = \mathbf{D}_n$). Hence, the above equations can be written as:

$$3\mathbf{P}_1 - \mathbf{P}_2 = 2\mathbf{D}_0$$

$$-\mathbf{P}_n + 3\mathbf{P}_{n+1} = 2\mathbf{D}_n$$

3.1.9 Curve Interpolation using Uniform Cubic B-Spline Curves

By combining these two equations with the equations on page 6 marked with #, we have a system of n +1 equations in n +1 unknowns:

$$3\mathbf{P}_1 - \mathbf{P}_2 = 2\mathbf{D}_0$$

$$\frac{1}{4}\mathbf{P}_1 + \frac{7}{12}\mathbf{P}_2 + \frac{1}{6}\mathbf{P}_3 = \mathbf{D}_1$$

$$\frac{1}{6}\mathbf{P}_2 + \frac{2}{3}\mathbf{P}_3 + \frac{1}{6}\mathbf{P}_4 = \mathbf{D}_2$$

$$\ldots$$

$$\frac{1}{6}\mathbf{P}_{n-2} + \frac{2}{3}\mathbf{P}_{n-1} + \frac{1}{6}\mathbf{P}_n = \mathbf{D}_{n-2}$$

$$\frac{1}{6}\mathbf{P}_{n-1} + \frac{7}{12}\mathbf{P}_n + \frac{1}{4}\mathbf{P}_{n+1} = \mathbf{D}_{n-1}$$

$$-\mathbf{P}_n + 3\mathbf{P}_{n+1} = 2\mathbf{D}_n$$

3.1.9 Curve Interpolation using Uniform Cubic B-Spline Curves

The matrix form of this system is:

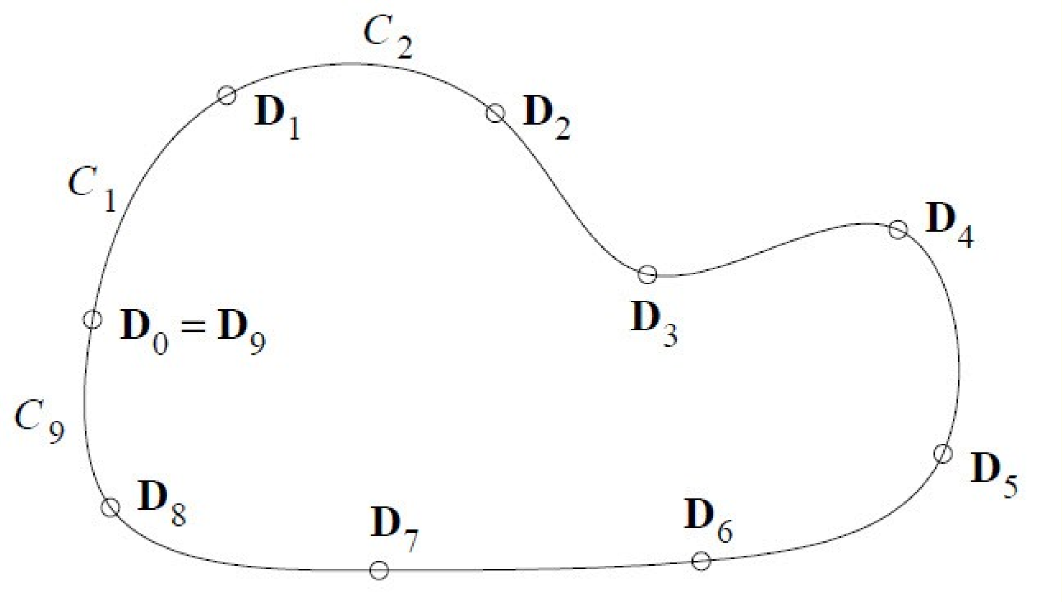
$$
\begin{bmatrix}
3 & -1 & & & & & \\
1/4 & 7/12 & 1/6 & & & & \\
 & 1/6 & 2/3 & 1/6 & & & \\
 & & & \cdot & & & \\
 & & & \cdot & & & \\
 & & & 1/6 & 2/3 & 1/6 & \\
 & & & & 1/6 & 7/12 & 1/4 \\
 & & & & & -1 & 3
\end{bmatrix}
\begin{bmatrix}
\mathbf{P}_1 \\ \mathbf{P}_2 \\ \mathbf{P}_3 \\ \cdot \\ \cdot \\ \mathbf{P}_{n-1} \\ \mathbf{P}_n \\ \mathbf{P}_{n+1}
\end{bmatrix}
=
\begin{bmatrix}
2\mathbf{D}_0 \\ \mathbf{D}_1 \\ \mathbf{D}_2 \\ \cdot \\ \cdot \\ \mathbf{D}_{n-2} \\ \mathbf{D}_{n-1} \\ 2\mathbf{D}_n
\end{bmatrix}
$$

can be solved using Gaussian elimination without pivoting. Most of the curve drawing programs in commercial packages (such as xfig, …) are implemented using this approach.

Question:

- What should we do to generate smooth, closed interpolating curves?

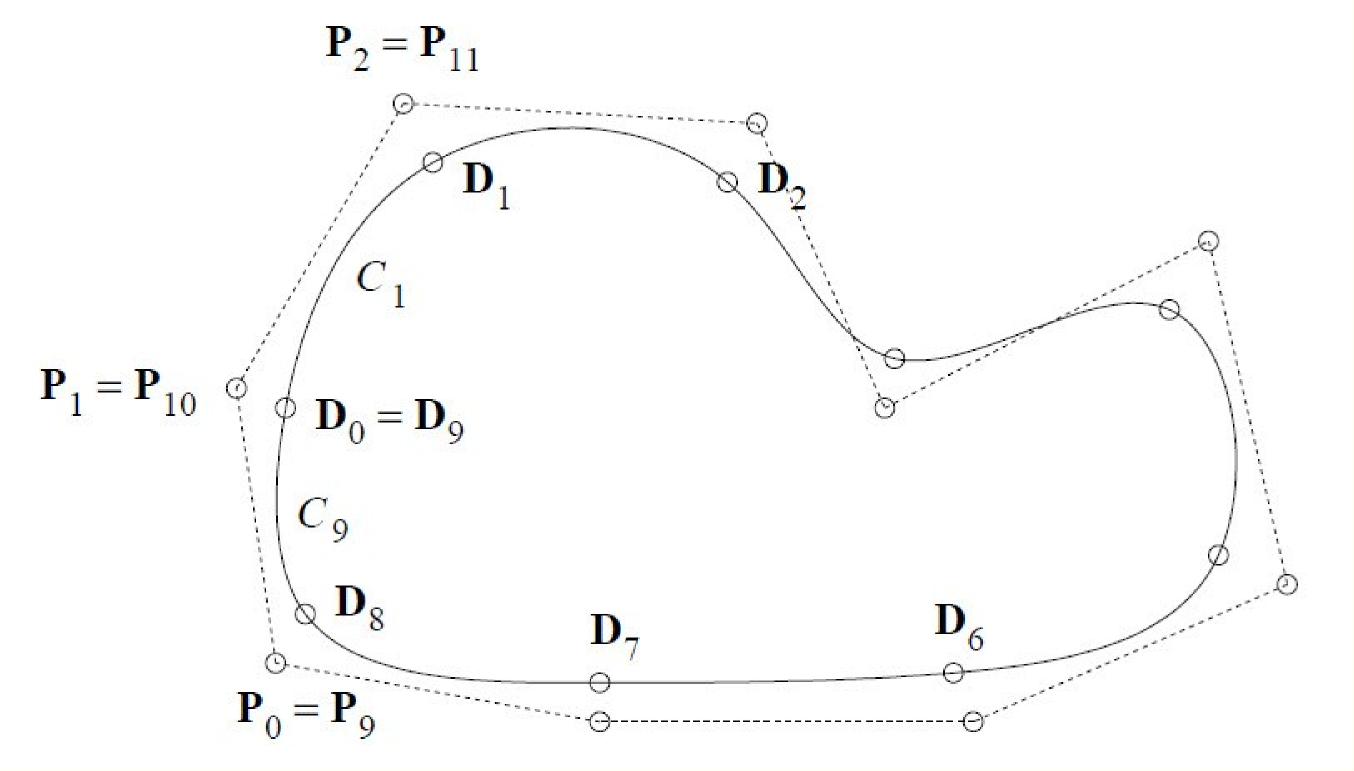
For instance, give a set of data points $D_i = (x_i, y_i),\ i = 0,1,\ldots,n$, with $D_0 = D_n$, how can a closed, smooth (C^2-continuous) cubic B-spline curve that interpolates these points be constructed?



The closed cubic B-spline curve has n segments $C_1(t), C_2(t), \ldots, C_n(t)$, with D_{i-1} and D_i being the start and end points of $C_i(t)$

3.1.9 Curve Interpolation using Uniform Cubic B-Spline Curves

- From previous discussion, we know such a curve must have (n +3) control points: $P_0, P_1, \ldots, P_{n+2}$



To guarantee C2 continuity at $C_0 = C_n$, control points must satisfy the following conditions:

$$P_n = P_0,$$
$$P_{n+1} = P_1,$$
$$P_{n+2} = P_2$$

3.1.9 Curve Interpolation using Uniform Cubic B-Spline Curves

- Such a curve needs (n +7) knots: $t_0, t_1, \ldots, t_{n+6}$
- To make things easier, we shall assume that

  $t_i = i - 3$, i = 0,1,...,n +6
- Such a (cyclic) curve can be defined as follows

$$C(t) = \sum_{i=0}^{n+2} N_{i,3}(t)\,\mathbf{P}_{(i \bmod n)}, \quad t \in [t_3, t_{n+3}] = [0, n]$$

such that

$$C(t_{i+3}) = C(i) = \sum_{i=0}^{n+2} N_{i,3}(i)\,\mathbf{P}_{(i \bmod n)} = \mathbf{D}_i, \quad \text{(C1)}$$

$$i = 0, 1, \ldots, n$$

3.1.9 Curve Interpolation using Uniform Cubic B-Spline Curves

There are n unknows and n equations in the above system (C1):

$$\frac{1}{6}\mathbf{P}_0 + \frac{2}{3}\mathbf{P}_1 + \frac{1}{6}\mathbf{P}_2 = \mathbf{D}_0$$

$$\frac{1}{6}\mathbf{P}_1 + \frac{2}{3}\mathbf{P}_2 + \frac{1}{6}\mathbf{P}_3 = \mathbf{D}_1$$

$$\cdots$$

$$\frac{1}{6}\mathbf{P}_{n-2} + \frac{2}{3}\mathbf{P}_{n-1} + \frac{1}{6}\mathbf{P}_{n} = \mathbf{D}_{n-2}$$

$$\frac{1}{6}\mathbf{P}_{n-1} + \frac{2}{3}\mathbf{P}_{n} + \frac{1}{6}\mathbf{P}_{n+1} = \mathbf{D}_{n-1}$$

$$\frac{1}{6}\mathbf{P}_{n} + \frac{2}{3}\mathbf{P}_{n+1} + \frac{1}{6}\mathbf{P}_{n+2} = \mathbf{D}_{n}$$

(The last equation is the same as the first equation and, hence, can be ignored.)

3.1.9 Curve Interpolation using Uniform Cubic B-Spline Curves

The matrix form of this system is:

$$
\begin{bmatrix}
2/3 & 1/6 & & & & & 1/6 \\
1/6 & 2/3 & 1/6 & & & & \\
 & 1/6 & 2/3 & 1/6 & & & \\
 & & & \cdot & & & \\
 & & & \cdot & & & \\
 & & & 1/6 & 2/3 & 1/6 & \\
 & & & & 1/6 & 2/3 & 1/6 \\
1/6 & & & & & 1/6 & 2/3
\end{bmatrix}
\begin{bmatrix}
\mathbf{P}_1 \\ \mathbf{P}_2 \\ \mathbf{P}_3 \\ \cdot \\ \cdot \\ \mathbf{P}_{n-1} \\ \mathbf{P}_n
\end{bmatrix}
=
\begin{bmatrix}
\mathbf{D}_0 \\ \mathbf{D}_1 \\ \mathbf{D}_2 \\ \cdot \\ \cdot \\ \mathbf{D}_{n-2} \\ \mathbf{D}_{n-1}
\end{bmatrix}
$$

This system of equaitons can be solved using Gaussian elimination without pivoting as well.

3.1.9 Curve Interpolation using Uniform Cubic B-Spline Curves

Curve Design Procedure:

1. Specify a set of points $\mathbf{D}_0, \mathbf{D}_1, \ldots, \mathbf{D}_n$ which lie approximately on the desired curve

2. Generate a cubic B-spline or composite Bezier curve that interpolates these points

3. Adjust control points of the interpolating curve to "sculpt" it into a more satisfactory shape.

3.1.10 Non-Uniform B-Spline Surfaces

Definition: A B-spline surface of degree p in u direction and degree q in v direction isdefined as follows

$$S(u,v) = \sum_{i=0}^{m} \sum_{j=0}^{n} N_{i,p}(u) N_{j,q}(v) \mathbf{P}_{i,j}$$

where $P_{i,j}$ are 3D control points, $N_{i,p}(u)$ and $N_{j,q}(v)$ are B-spline basis functions of degree p and q , respectively, defined with respect to the knot vectors

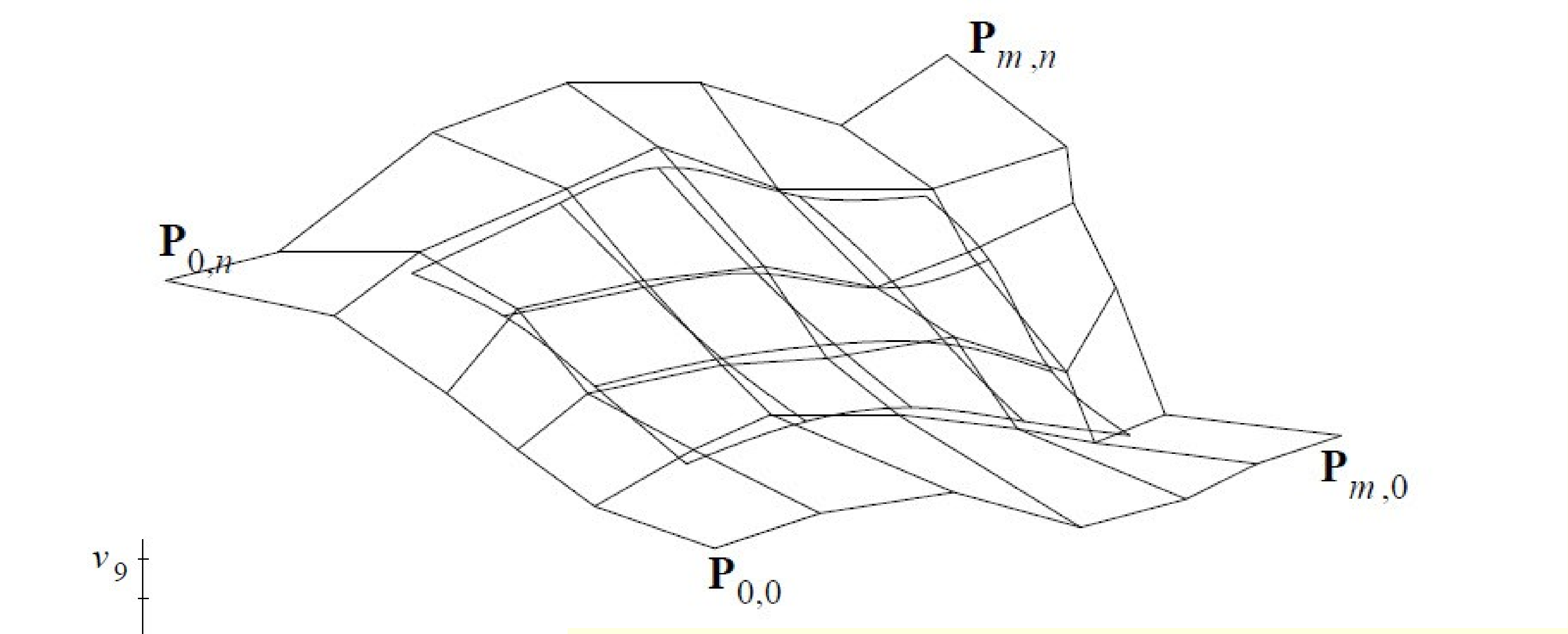
$U = \{u_0, u_1, \cdots, u_{m+p+1}\},$ and $V = \{v_0, v_2, \cdots, v_{n+q+1}\},$

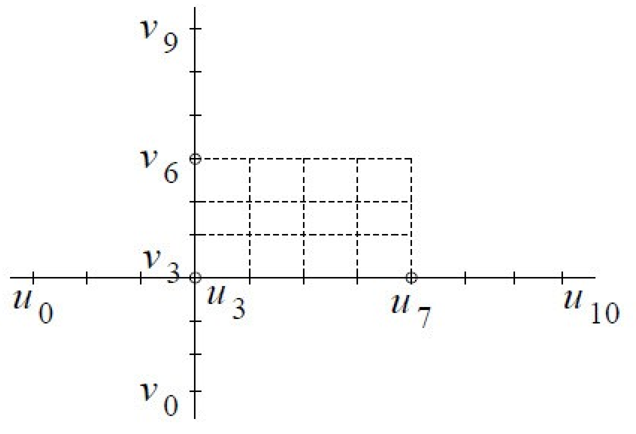
respectively. The parameter space of the surface is

$$[u_p, u_{m+1}] \times [v_q, q_{n+1}].$$

3.1.10 Non-Uniform B-Spline Surfaces

An example of a bicubic B-spline surface:





3.1.10 Non-Uniform B-Spline Surfaces

- The grid defined by the control points of a B-spline surface is called a control net or a control polyhedron.
- The image of each $[u_i, u_{i+1}] \times [v_j, v_{j+1}]$ in the domain of a B-spline surface is called a Bspline patch.
- The B-spline surface defined on page 17 is $C^{(p-1)}$-continuous in u direction and $C^{(q-1)}$-continuous in v direction.
- A B-spline surface usually does not interpolate its control points. However, if the knots satisfy the following condition

$$u_0 = u_1 = \cdots = u_p; \qquad v_0 = v_1 = \cdots = v_q$$

and

$$u_{m+1} = \cdots = u_{m+p+1}; \qquad v_{n+1} = \cdots = v_{n+q+1}$$

then the surface interpolates the corner points of the control net.

3.1.10 Non-Uniform B-Spline Surfaces

- Each patch of the the B-spline surface defined on page 17 is contained in the convex hull of $(p+1)\times(q+1)$ control points.

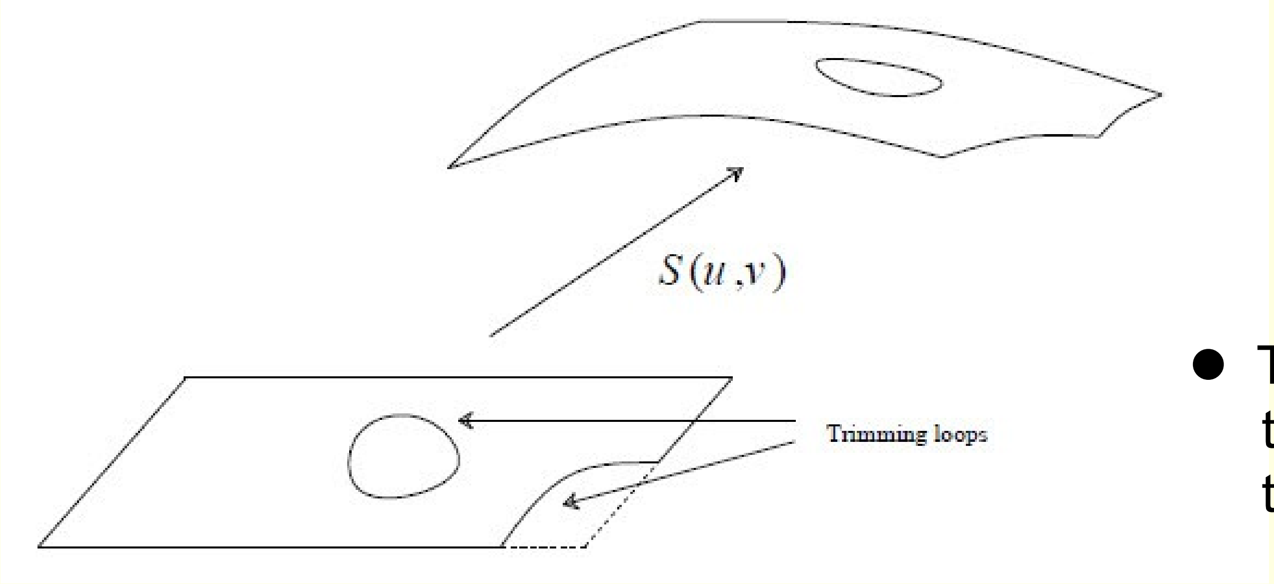
- A B-spline surface can be converted into a composite Bezier surface using a technique similar to the curve case.

- Surface Design, in some cases, follows the same procedure as the curve design process. More complicated shape design techniques will be introduced in the next chapter.

- Sometime it is necessary to remove certain portions of a surface to get a special shape. The resulting surface is called a *trimmed surface*.

3.1.10 Non-Uniform B-Spline Surfaces

Definition: A trimmed B-spline surface is a B-spline surface whose actual extent is specified by a set of closed loops defined in the parameter space of the surface.

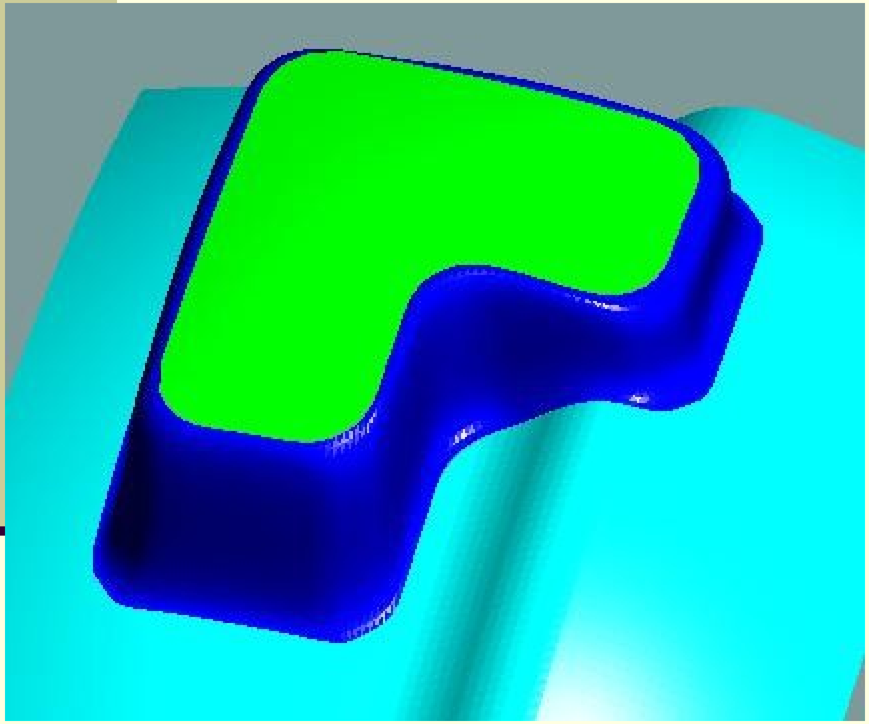


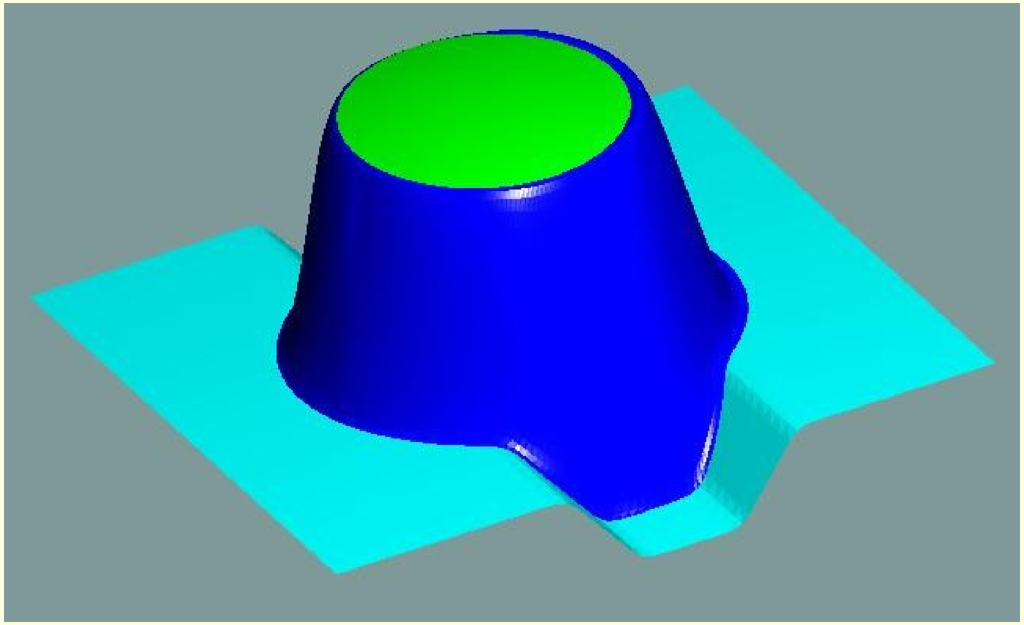
- The loops are called trimming loops or trimming curves.

- An ordinary B-spline surface can be considered as a special case of a trimmed Bspline surface by viewing the boundary of the parameter space as a trimming loop.

3.1.10 Non-Uniform B-Spline Surfaces

Examples of trimmed NURBS surfaces:





3.1.10 Non-Uniform B-Spline Surfaces

- The trimming loops are typically produced by the intersections between two or more untrimmed surfaces.
- Each of the trimming loops may be defined by one or more components. Each component is defined by a B-spline curve.
- The interior (trimming area) of a trimmed surface is usually defined by the *odd winding rule* or *curve handedness rule*.
- *Curve handedness rule*: the increasing parameter value of a trimming curve must corresponding to counterclock motion around the enclosed (trimming) region.
- The trimming curves are not permitted to intersect each other, nor is a trimming curve permitted to intersect itself.

End of Section 3.1.10