5. Neural Networks

GEV6135 Deep Learning for Visual Recognition and Applications

Kibok Lee

Assistant Professor of

Applied Statistics / Statistics and Data Science

Sep 29, 2022

Assignment 3

- Due **Monday 10/10, 11:59pm KST**
- Training linear classifiers (Lec 3) with
  - SVM/Softmax loss (Lec 3)
  - SGD (Lec 4)
- If you feel difficult, consider to take **option 2**.
- Please read the instruction carefully!
  - Do **not write or modify any code outside** of the designated blocks.
  - Do **not add or delete cells** from the notebook.
  - Do **not import** additional libraries.
    - + Do not use torch.nn unless instructed.
  - **Run all cells**, and do **not clear out the outputs**, before submitting.
  - Do **not zip by yourself**, run the provided code.

Assignment 4

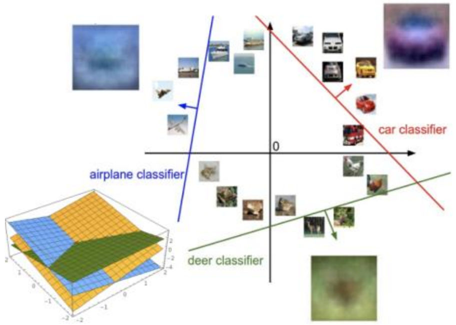
- Will be released around **Tuesday 10/4**
- Expected due **Monday 10/17**
- Training two-layer neural networks (Lec 5) with
  - Softmax loss (Lec 3)
  - SGD (Lec 4)

- A1 grading by this weekend?

Where we are:

1. Use **Linear Models** for image classification problems
2. Use **Loss Functions** to express preferences over different choices of weights
3. Use **Regularization** to prevent overfitting to training data
4. Use **Stochastic Gradient Descent** to minimize our loss functions and train the model

$$s = f(x; W) = Wx$$



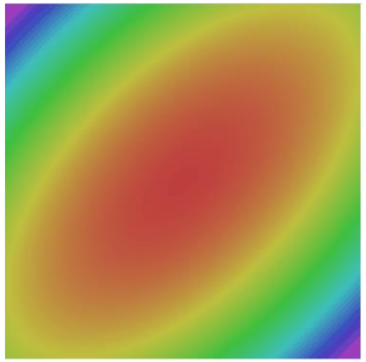
$$L_i = -\log\left(\frac{e^{s_{y_i}}}{\sum_j e^{s_j}}\right)$$ Softmax

SVM

$$L_i = \sum_{j \neq y_i} \max(0, s_j - s_{y_i} + 1)$$

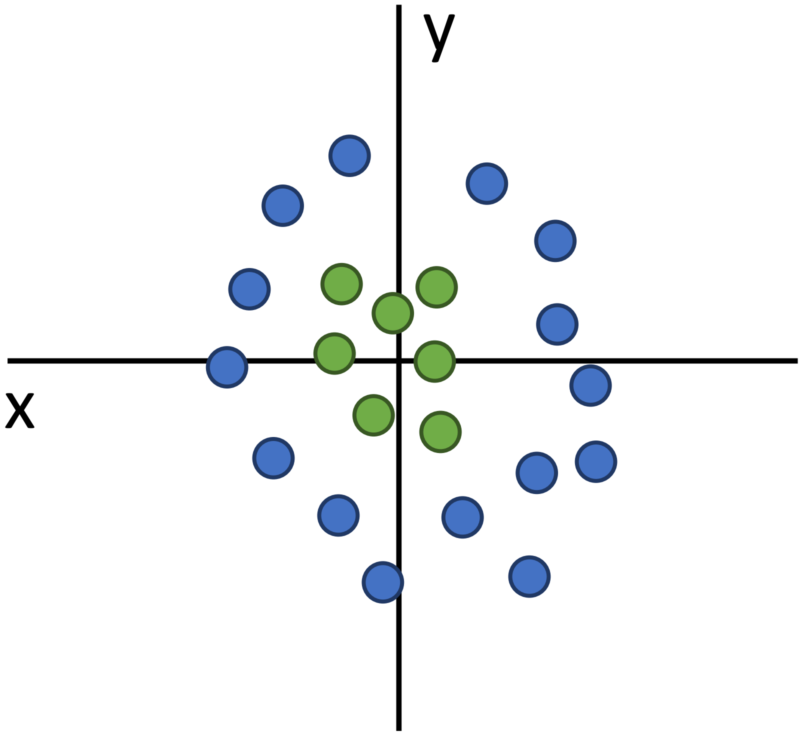
$$L = \frac{1}{N} \sum_{i=1}^{N} L_i + R(W)$$

```
v = 0
for t in range(num_steps):
  dw = compute_gradient(w)
  v = rho * v + dw
  w -= learning_rate * v
```



Problem: Linear Classifiers aren't that powerful

Geometric Viewpoint

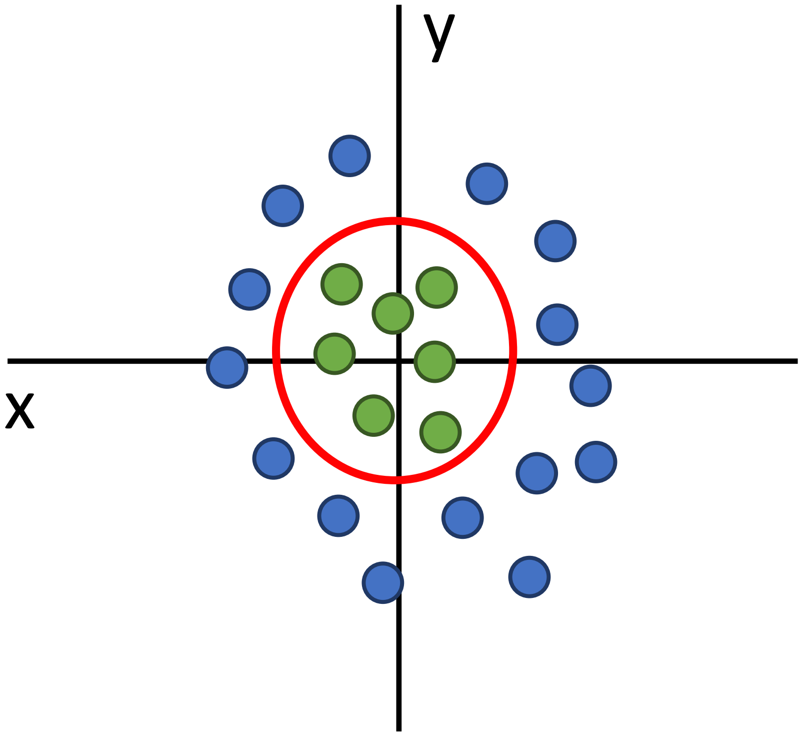


Visual Viewpoint

One template per class:
Can't recognize different
modes of a class

One solution: **Feature Transforms**

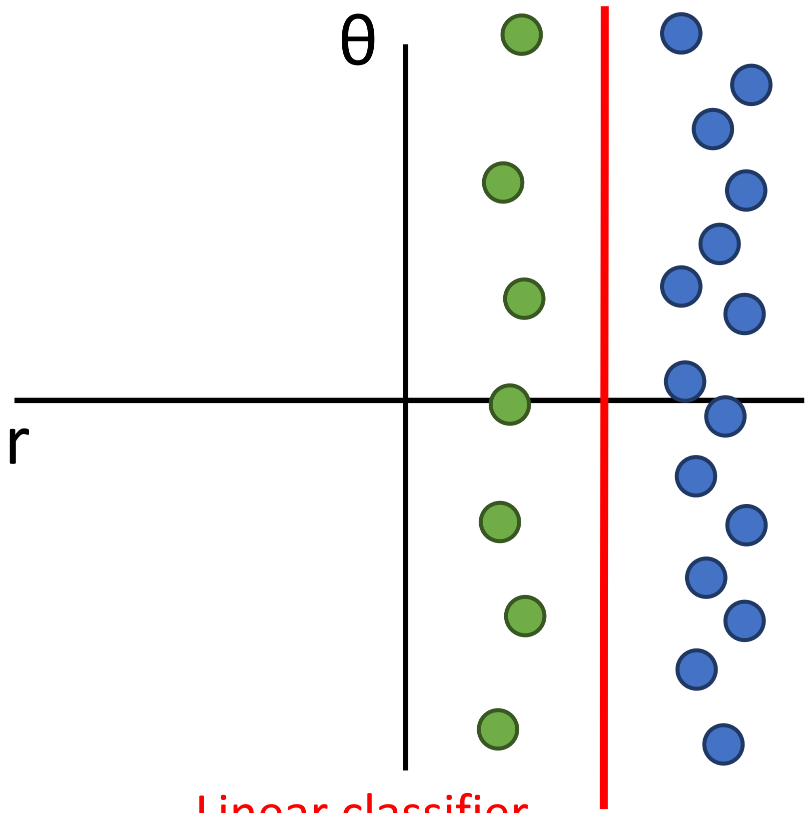
Original space



$$r = (x^2 + y^2)^{1/2}$$
$$\theta = \tan^{-1}(y/x)$$

Feature transform

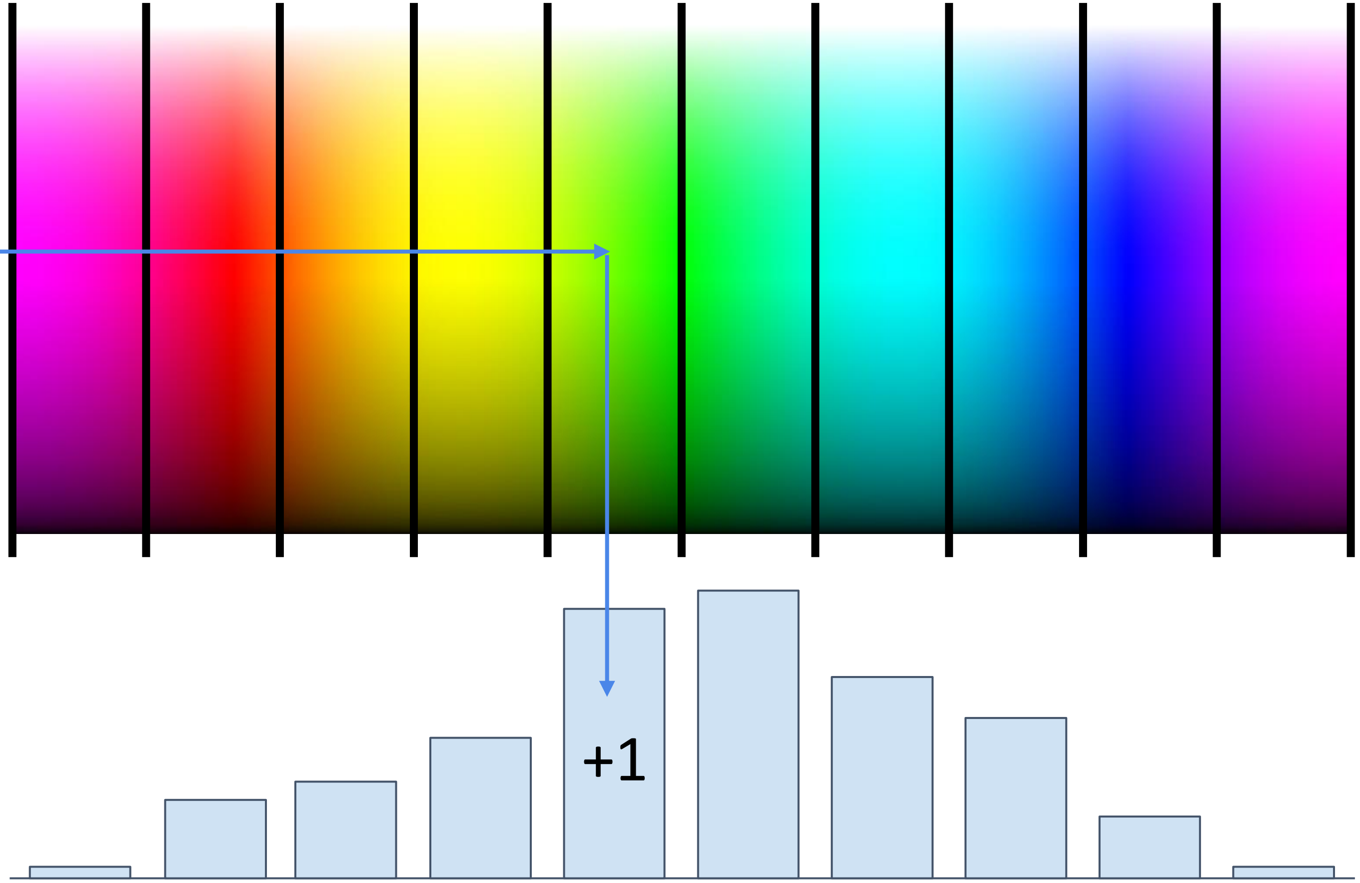
Feature space



Nonlinear classifier
in original space!

Linear classifier
in feature space

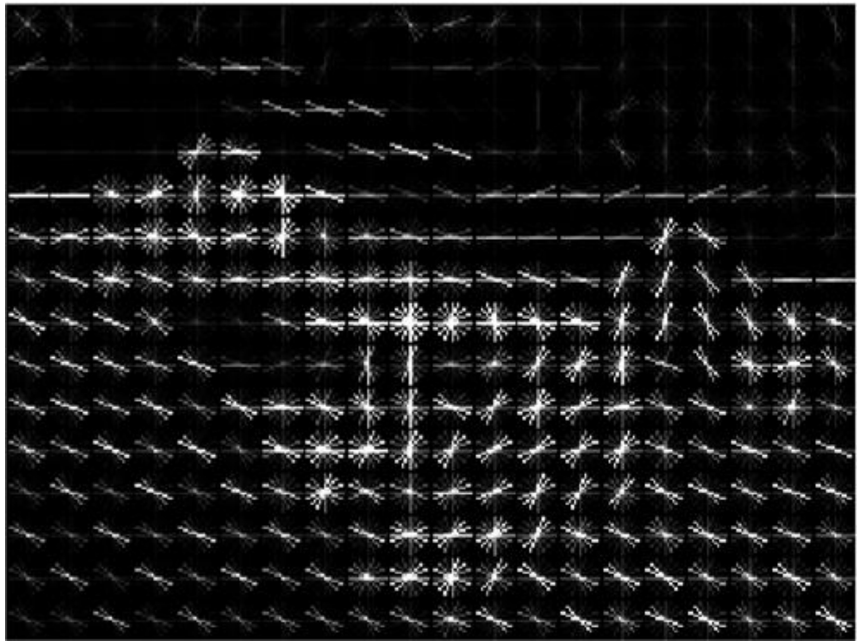
Image Features: Color Histogram



Ignores texture,
spatial positions

Frog image is in the public domain

Image Features: Histogram of Oriented Gradients (HoG)

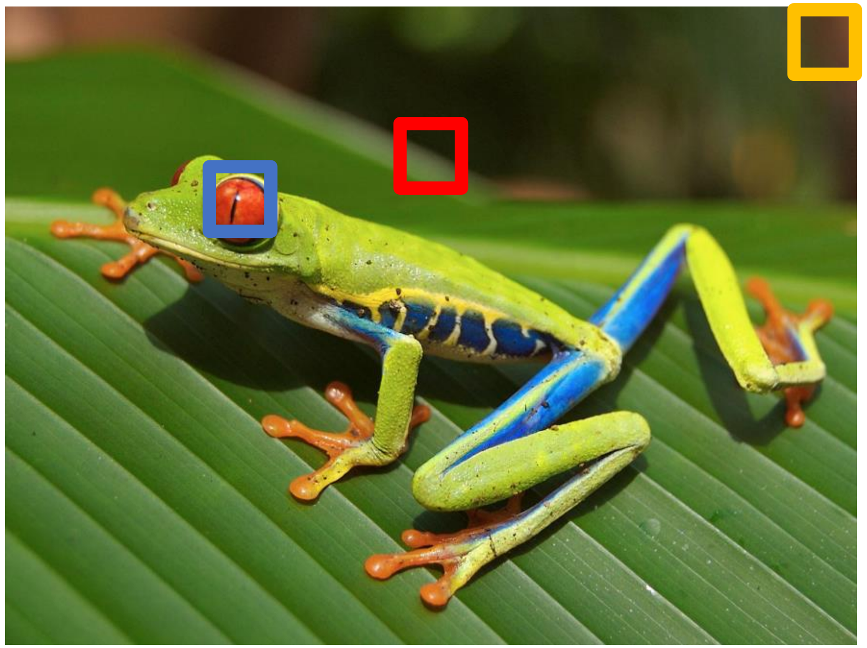


1. Compute edge direction / strength at each pixel
2. Divide image into 8x8 regions
3. Within each region compute a histogram of edge directions weighted by edge strength

Example: 320x240 image gets divided into 40x30 bins; 8 directions per bin; feature vector has 30*40*9 = 10,800 numbers

Lowe, “Object recognition from local scale-invariant features”, ICCV 1999
Dalal and Triggs, "Histograms of oriented gradients for human detection," CVPR 2005

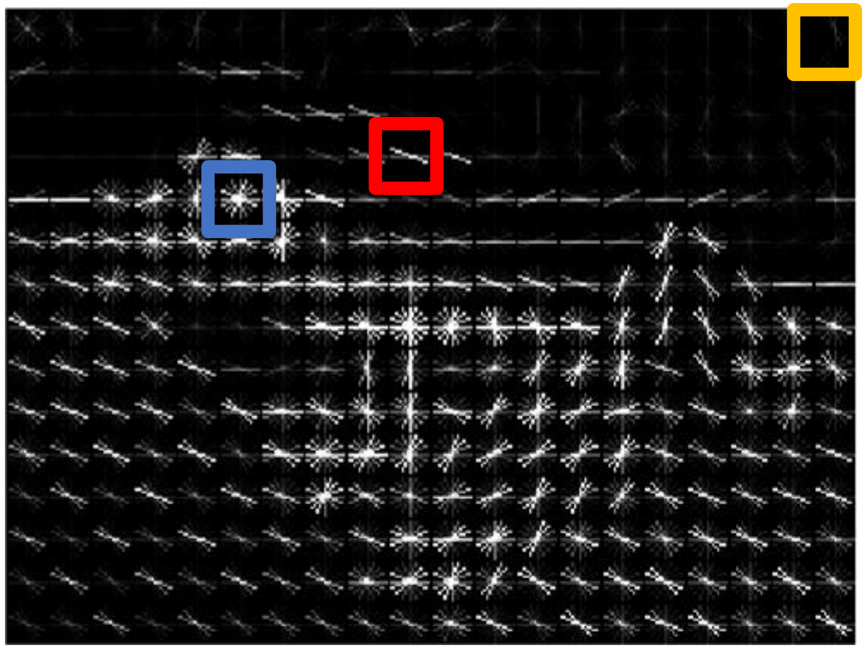
Image Features: Histogram of Oriented Gradients (HoG)



Weak edges

Strong diagonal edges

Edges in all directions



1. Compute edge direction / strength at each pixel
2. Divide image into 8x8 regions
3. Within each region compute a histogram of edge directions weighted by edge strength

Captures texture and position, robust to small image changes

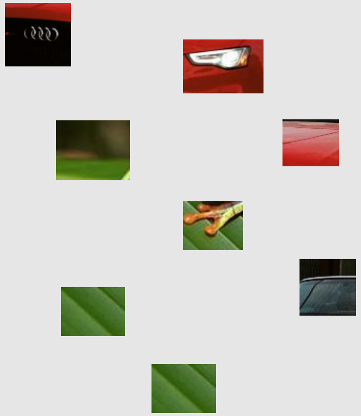
Example: 320x240 image gets divided into 40x30 bins; 8 directions per bin; feature vector has 30*40*9 = 10,800 numbers

Lowe, "Object recognition from local scale-invariant features", ICCV 1999
Dalal and Triggs, "Histograms of oriented gradients for human detection," CVPR 2005

Image Features: Bag of Words (Data-Driven!)

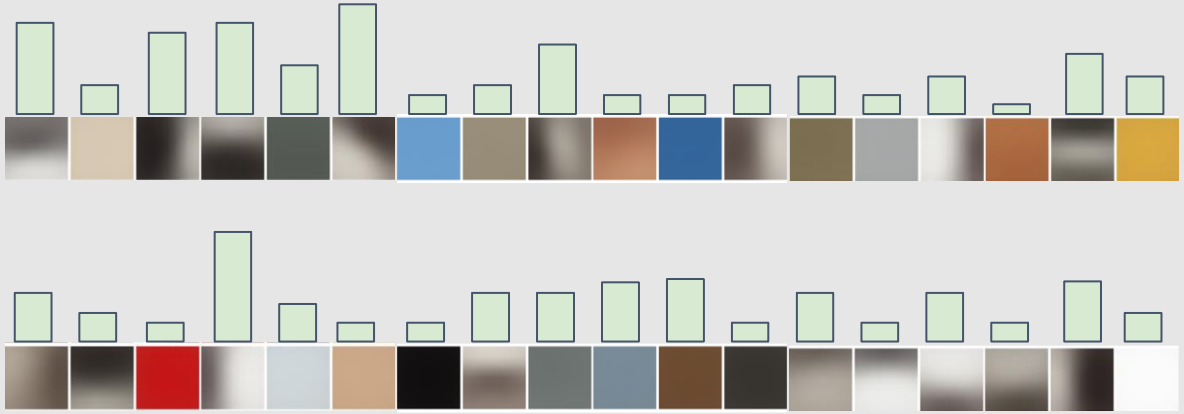
Step 1: Build codebook

Extract random patches



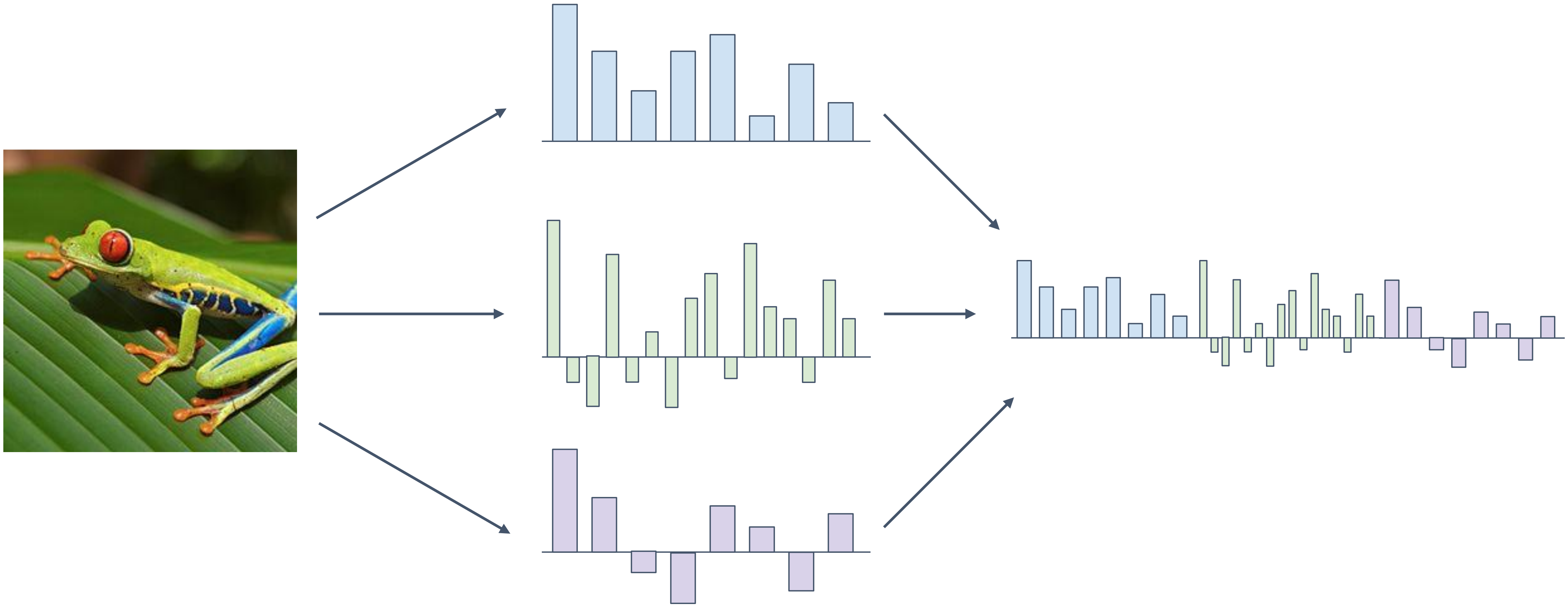
Cluster patches to form "codebook" of "visual words"

Step 2: Encode images



Fei-Fei and Perona, "A bayesian hierarchical model for learning natural scene categories", CVPR 2005

Image Features



Example: Winner of 2011 ImageNet challenge

Low-level feature extraction ≈ 10k patches per image

- SIFT: 128-dim
- color: 96-dim

$\Big\}$ reduced to 64-dim with PCA

FV extraction and compression:

- N=1,024 Gaussians, R=4 regions ⇨ 520K dim x 2
- compression: G=8, b=1 bit per dimension

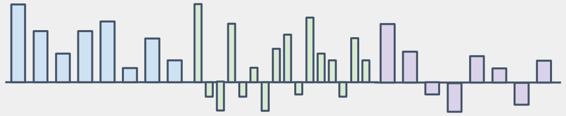
One-vs-all SVM learning with SGD

Late fusion of SIFT and color systems

F. Perronnin, J. Sánchez, “Compressed Fisher vectors for LSVRC”, PASCAL VOC / ImageNet workshop, ICCV, 2011.

Image Features

Feature Extraction



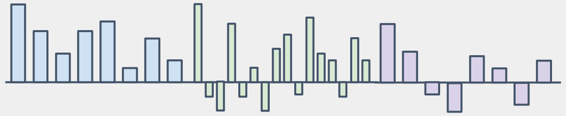
f

10 numbers giving scores for classes

training

Image Features vs Neural Networks

Feature Extraction



f

10 numbers giving scores for classes

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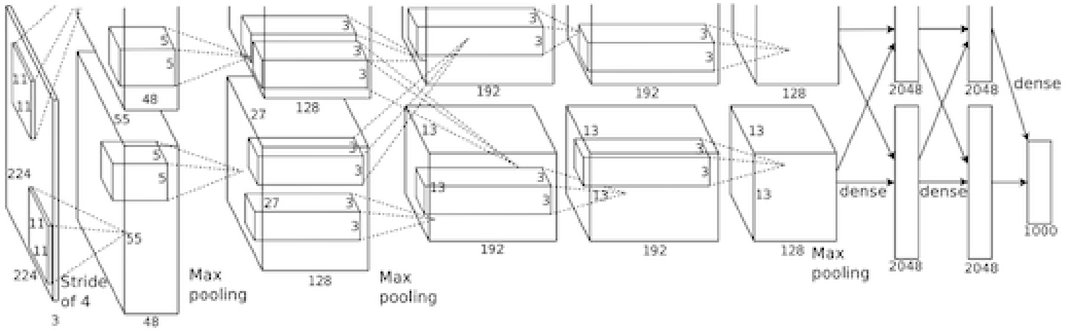

Krizhevsky, Sutskever, and Hinton, "Imagenet classification with deep convolutional neural networks", NIPS 2012.
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10 numbers giving scores for classes

training

Neural Networks

Linear classifiers

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Neural Networks

Input: $x \in \mathbb{R}^D$ **Output**: $s(x) \in \mathbb{R}^C$ **Activation function**: f

Before: Linear Classifier: $s(x) = Wx + b$
Learnable parameters: $W \in \mathbb{R}^{C \times D}, b \in \mathbb{R}^C$

Now: Two-Layer Neural Network: $s(x) = W_2 f(W_1 x + b_1) + b_2$
Learnable parameters: $W_1 \in \mathbb{R}^{H \times D}, b_1 \in \mathbb{R}^H, W_2 \in \mathbb{R}^{C \times H}, b_2 \in \mathbb{R}^C$

Neural Networks

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Feature Extraction
Linear Classifier

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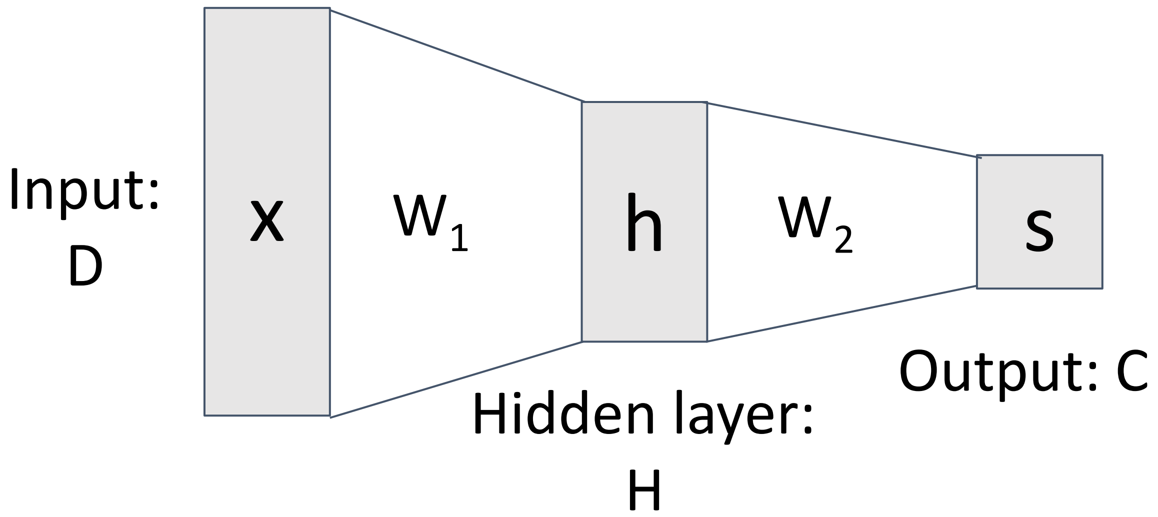
Learnable parameters: $W_1 \in \mathbb{R}^{H \times D}, b_1 \in \mathbb{R}^H, W_2 \in \mathbb{R}^{C \times H}, b_2 \in \mathbb{R}^C$

Or Three-Layer Neural Network:
$s(x) = W_3\, f(\, W_2\, f(W_1 x + b_1) + b_2) + b_3$

Neural Networks

Before: Linear classifier $s(x) = Wx + b$

Now: 2-layer Neural Network $s(x) = W_2 f(W_1 x + b_1) + b_2$



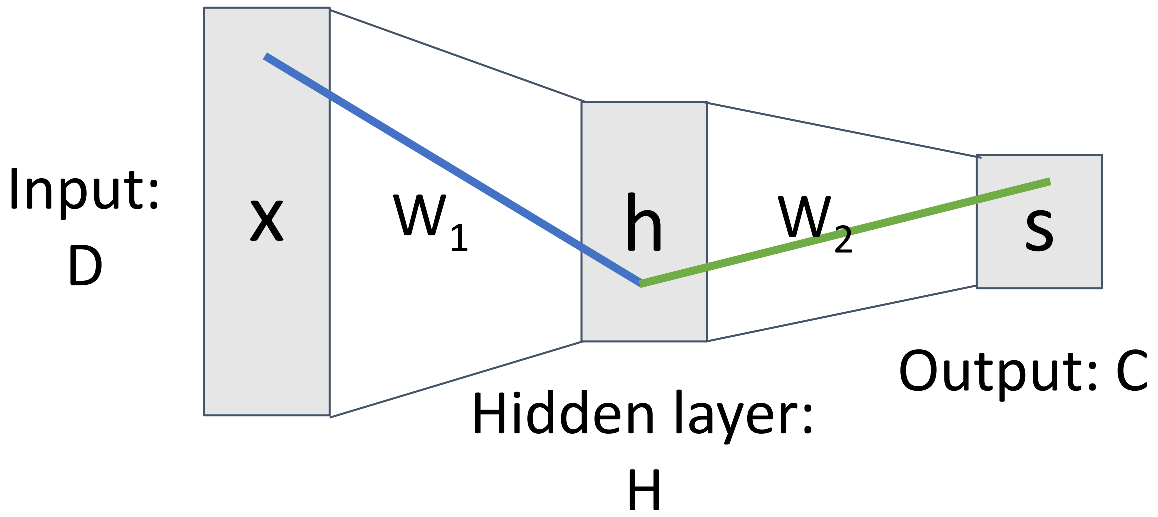
$$x \in \mathbb{R}^D, W_1 \in \mathbb{R}^{H \times D}, W_2 \in \mathbb{R}^{C \times H}$$

Neural Networks

Before: Linear classifier $s(x) = Wx + b$

Now: 2-layer Neural Network $s(x) = W_2 f(W_1 x + b_1) + b_2$

Element (i, j) of W_1 gives the effect on h_i from x_j



Element (i, j) of W_2 gives the effect on s_i from h_j

$$x \in \mathbb{R}^D, W_1 \in \mathbb{R}^{H \times D}, W_2 \in \mathbb{R}^{C \times H}$$

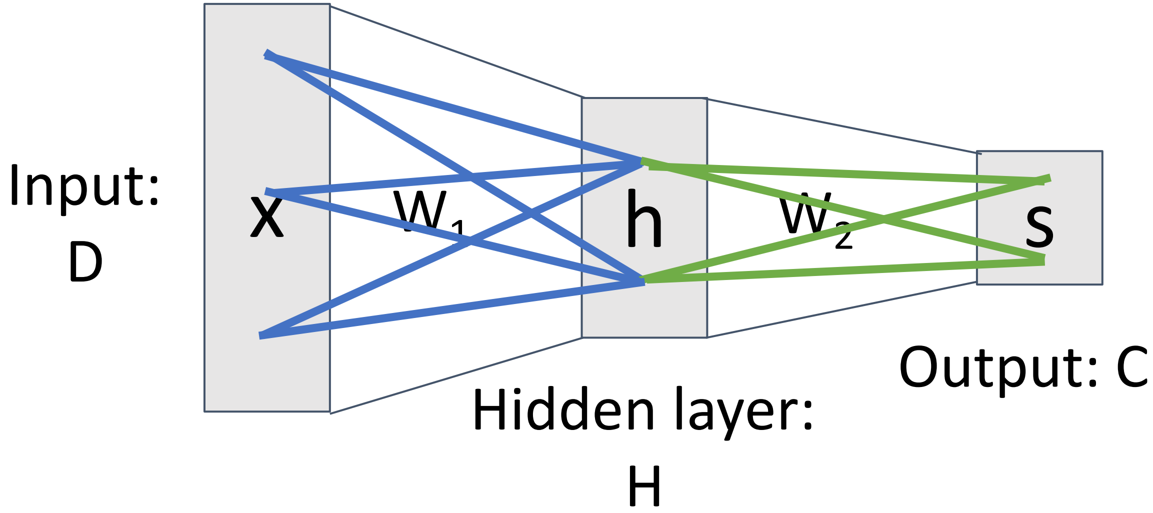
Neural Networks

Before: Linear classifier $s(x) = Wx + b$

Now: 2-layer Neural Network $s(x) = W_2 f(W_1 x + b_1) + b_2$

Element (i, j) of W_1 gives the effect on h_i from x_j

All elements of x affect all elements of h



Element (i, j) of W_2 gives the effect on s_i from h_j

All elements of h affect all elements of s

Fully-connected neural network

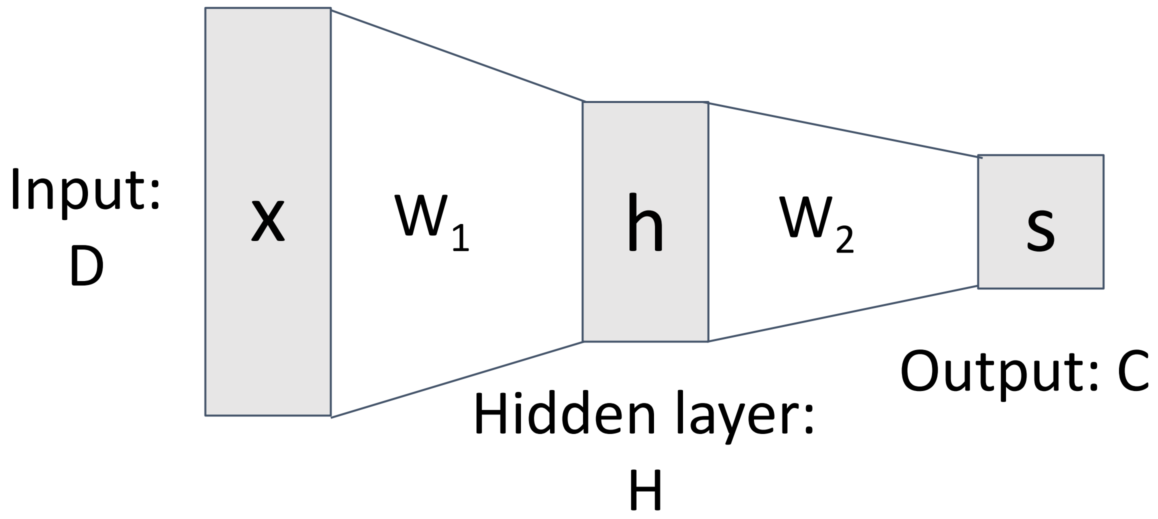
Also “Multi-Layer Perceptron” (MLP)

Neural Networks

(**Before**) Linear score function:

(**Now**) 2-layer Neural Network

Linear classifier: One template per class



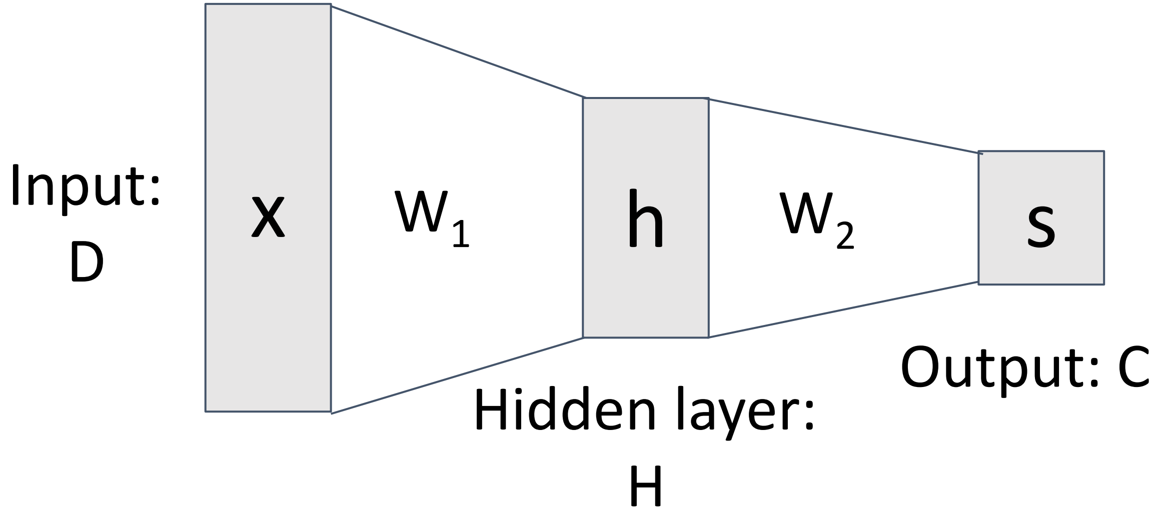
$$x \in \mathbb{R}^D, W_1 \in \mathbb{R}^{H \times D}, W_2 \in \mathbb{R}^{C \times H}$$

Neural Networks

Neural net: first layer is bank of templates;
Second layer recombines templates

(**Before**) Linear score function:

(**Now**) 2-layer Neural Network



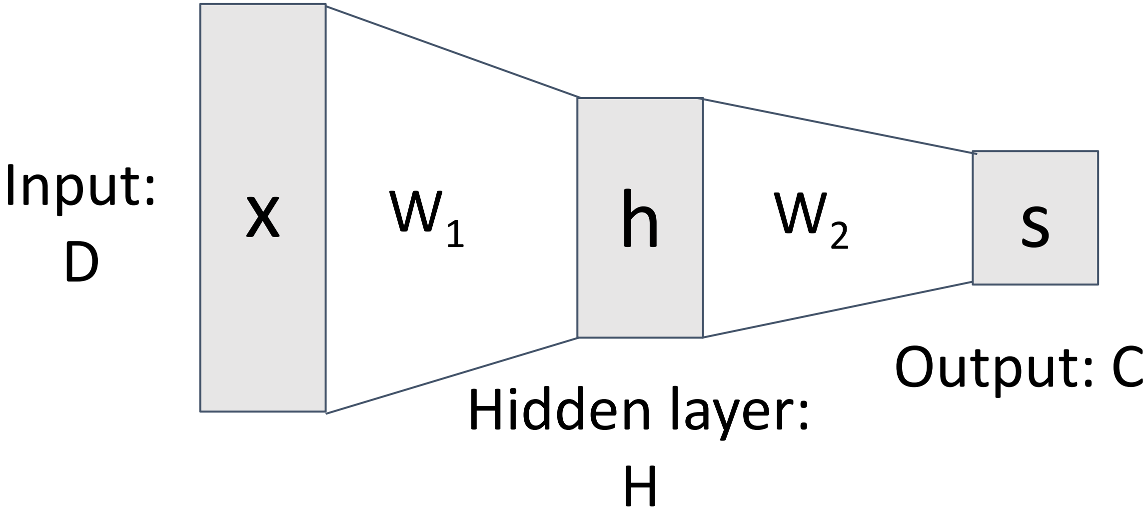
$$x \in \mathbb{R}^D, W_1 \in \mathbb{R}^{H \times D}, W_2 \in \mathbb{R}^{C \times H}$$

Neural Networks

Can use different templates to cover multiple modes of a class!

(**Before**) Linear score function:

(**Now**) 2-layer Neural Network



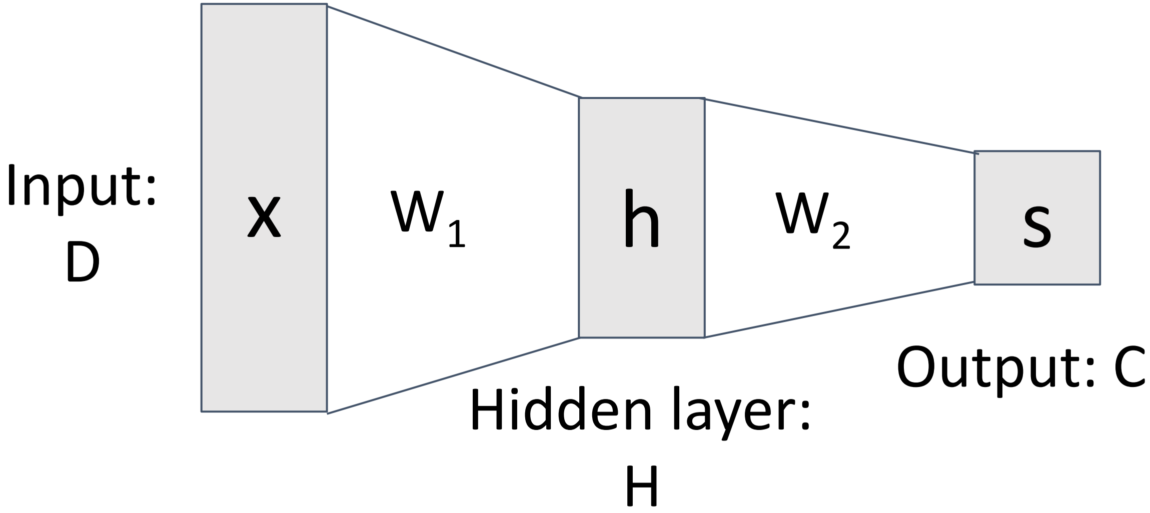
$$x \in \mathbb{R}^D, W_1 \in \mathbb{R}^{H \times D}, W_2 \in \mathbb{R}^{C \times H}$$

Neural Networks

"Distributed representation":
Most templates not interpretable!

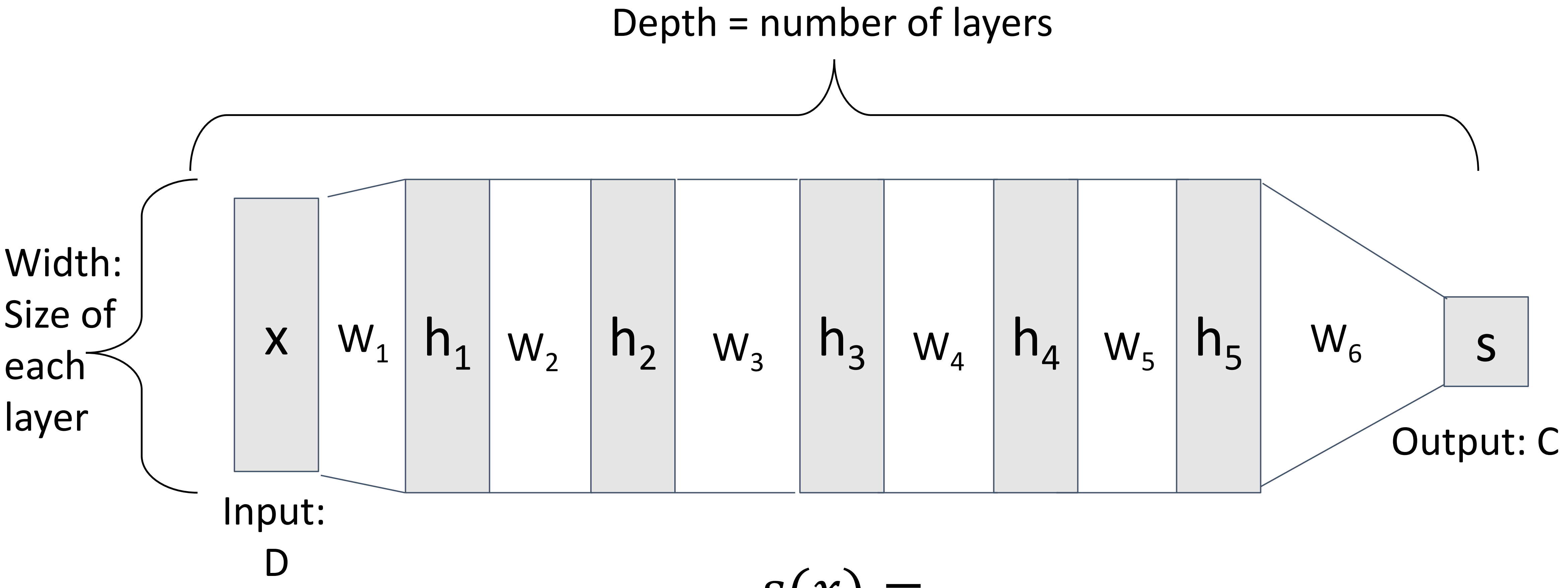
(**Before**) Linear score function:

(**Now**) 2-layer Neural Network



$$x \in \mathbb{R}^D, W_1 \in \mathbb{R}^{H \times D}, W_2 \in \mathbb{R}^{C \times H}$$

Deep Neural Networks



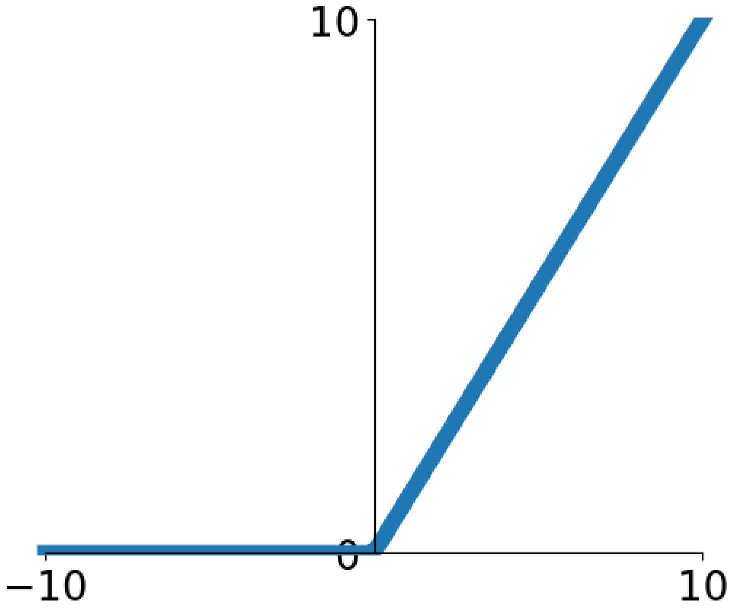
$$s(x) =$$
$$W_6\, f(W_5\, f(W_4\, f(W_3\, f(W_2\, f(W_1 x + b_1) + b_2) + b_3) + b_4) + b_5) + b_6$$

Activation Functions

2-layer Neural Network

$$s(x) = W_2 \boxed{f}(W_1 x + b_1) + b_2$$

The function $ReLU(z) = \max(0, z)$ is called "Rectified Linear Unit"



This is called the **activation function** of the neural network

Q: What happens if we build a neural network with no activation function?

$$s(x) = W_2(W_1 x + b_1) + b_2$$

Activation Functions

2-layer Neural Network

$$s(x) = W_2 \boxed{f}(W_1 x + b_1) + b_2$$

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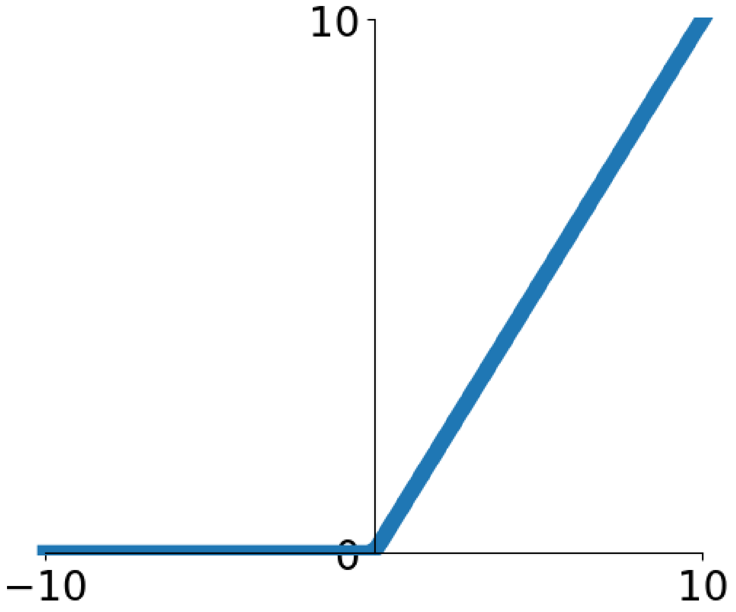


This is called the **activation function** of the neural network

Q: What happens if we build a neural network with no activation function?

$$\begin{aligned} s(x) &= W_2(W_1 x + b_1) + b_2 \\ &= (W_1 W_2)x + (W_2 b_1 + b_2) \end{aligned}$$

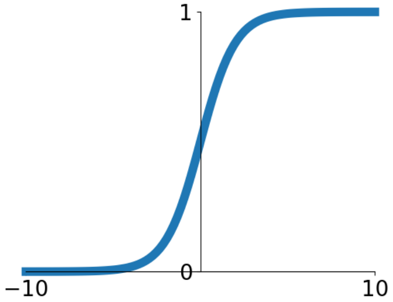
A: We end up with a linear classifier!

Activation Functions

ReLU is a good default choice for most problems

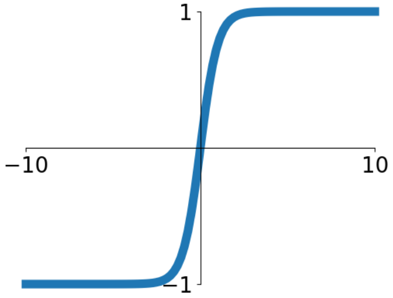
Sigmoid

$$\sigma(x) = \frac{1}{1 + e^{-x}}$$



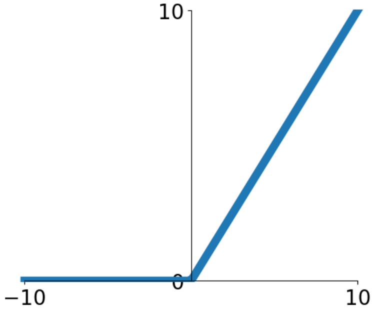
tanh

$$\tanh(x) = \frac{e^{2x} - 1}{e^{2x} + 1}$$



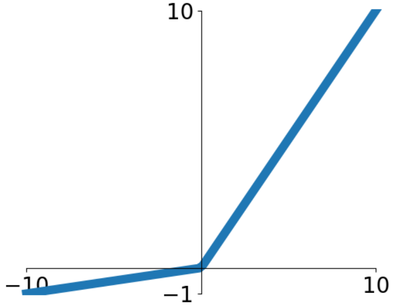
ReLU

$$\max(0, x)$$



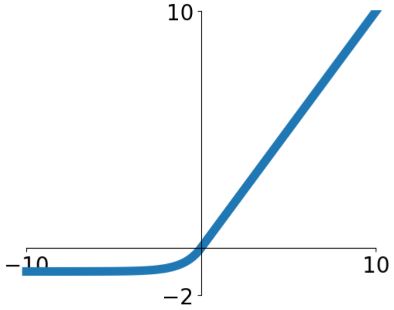
Leaky ReLU

$$\max(0.1x, x)$$



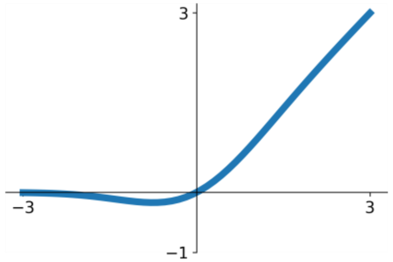
ELU

$$\begin{cases} x & x > 0 \\ \alpha(e^{x} - 1) & x \le 0 \end{cases}$$

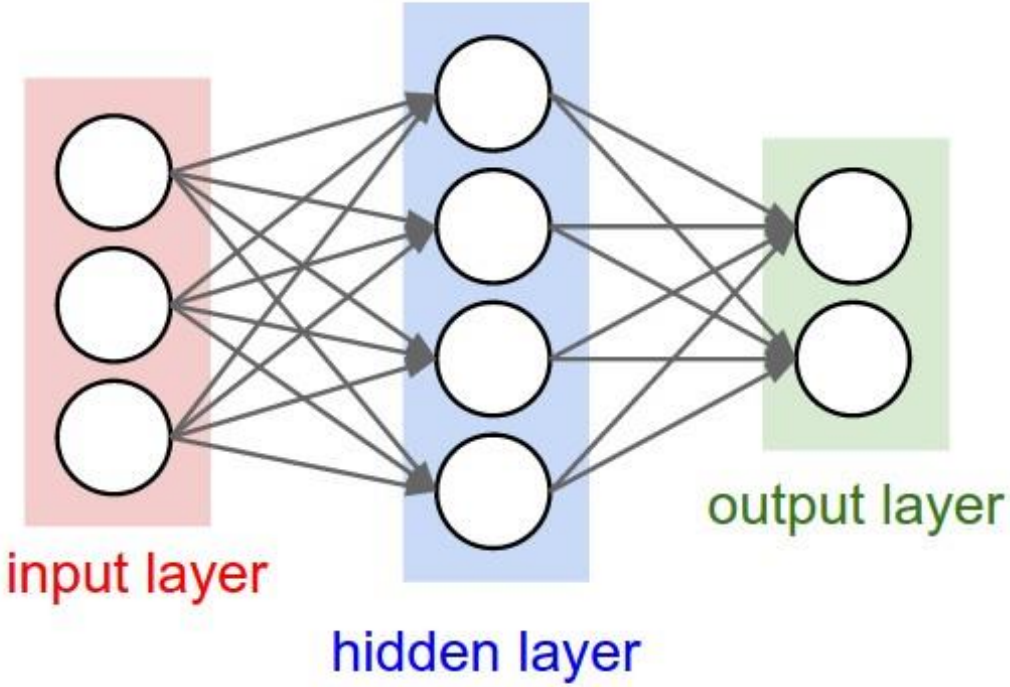


GELU

$$= 0.5x\left[1 + \mathrm{erf}(x/\sqrt{2})\right]$$
$$\approx x\sigma(1.702x)$$



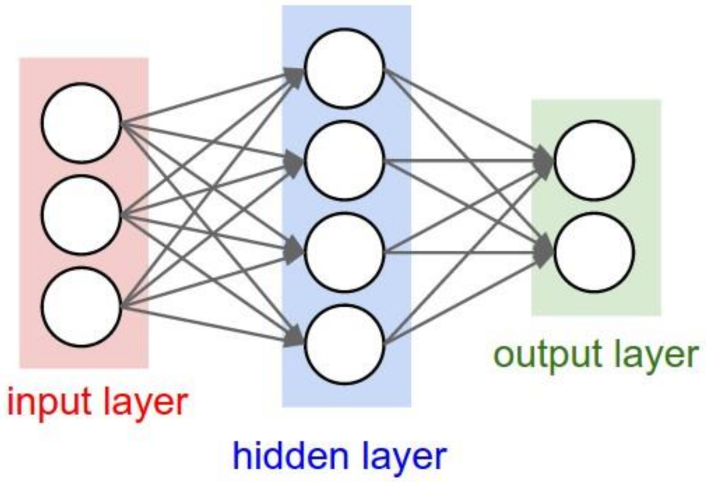
Neural Net in <20 lines!




```python
import numpy as np
from numpy.random import randn

N, Din, H, Dout = 64, 1000, 100, 10
x, y = randn(N, Din), randn(N, Dout)
w1, w2 = randn(Din, H), randn(H, Dout)
for t in range(10000):
  h = 1.0 / (1.0 + np.exp(-x.dot(w1)))
  y_pred = h.dot(w2)
  loss = np.square(y_pred - y).sum()
  dy_pred = 2.0 * (y_pred - y)
  dw2 = h.T.dot(dy_pred)
  dh = dy_pred.dot(w2.T)
  dw1 = x.T.dot(dh * h * (1 - h))
  w1 -= 1e-4 * dw1
  w2 -= 1e-4 * dw2
```

Neural Net in <20 lines!



Initialize weights and data

Compute loss (sigmoid activation, L2 loss)

Compute gradients

SGD step

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  w1 -= 1e-4 * dw1
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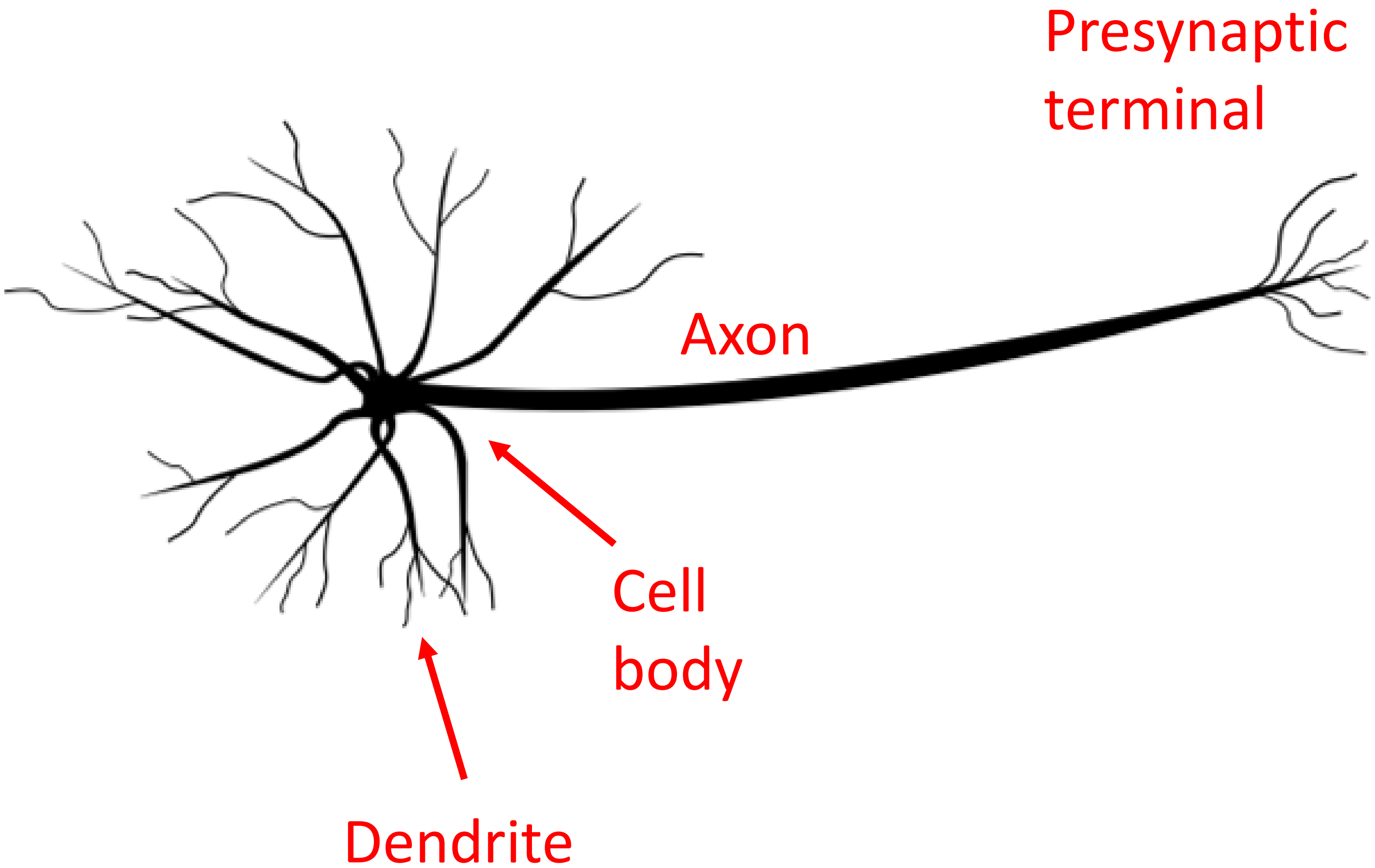
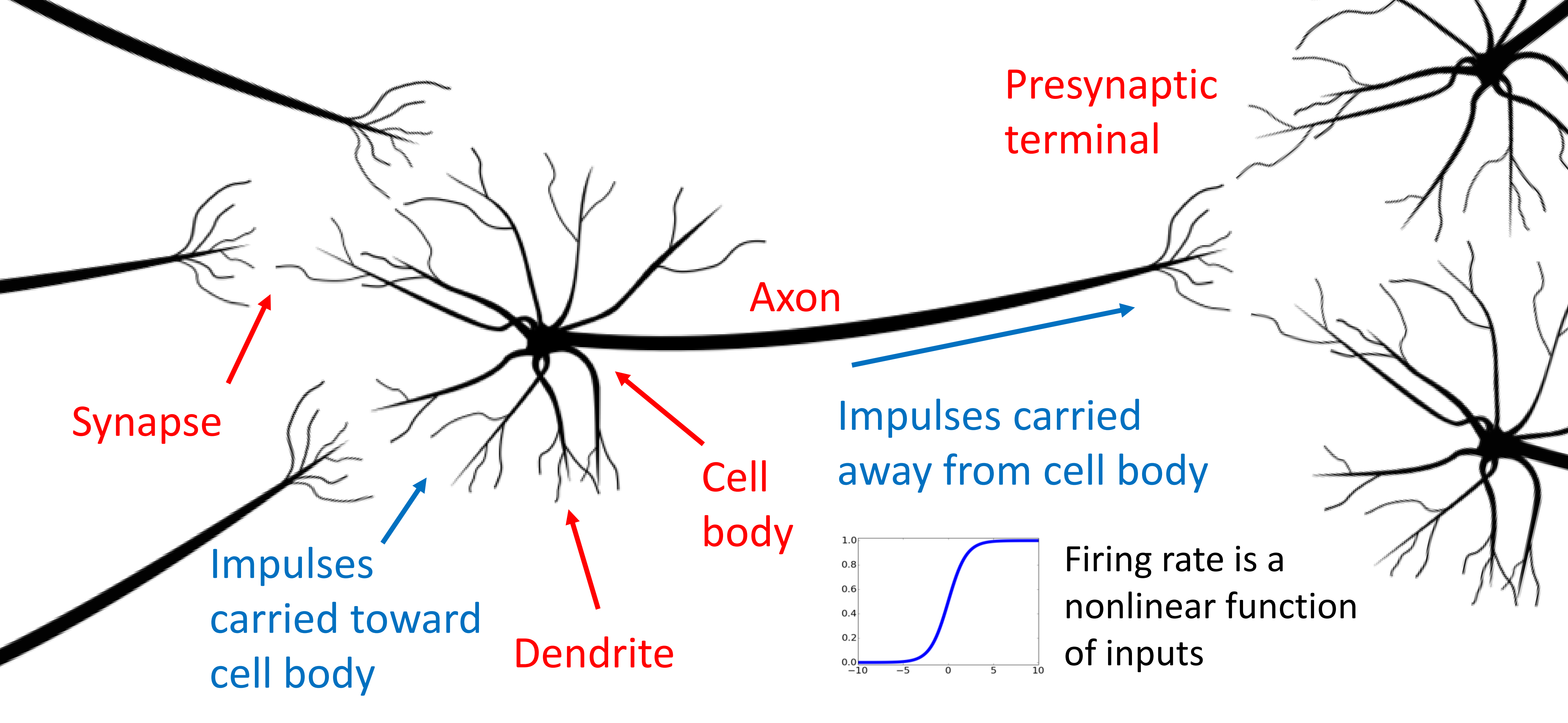
```

Attendance Check

Our brains are made of Neurons

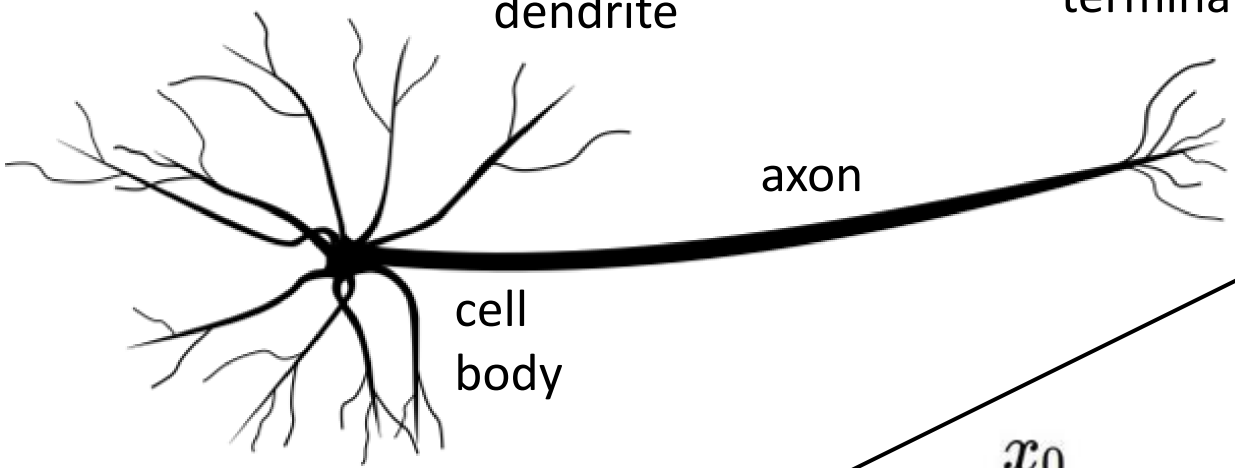


Our brains are made of Neurons

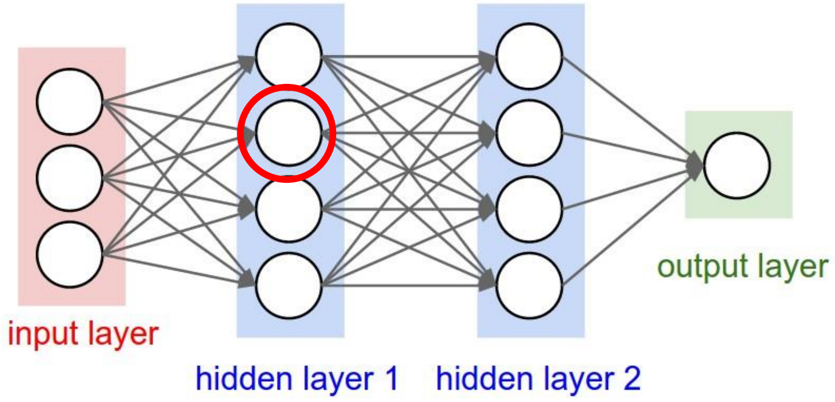


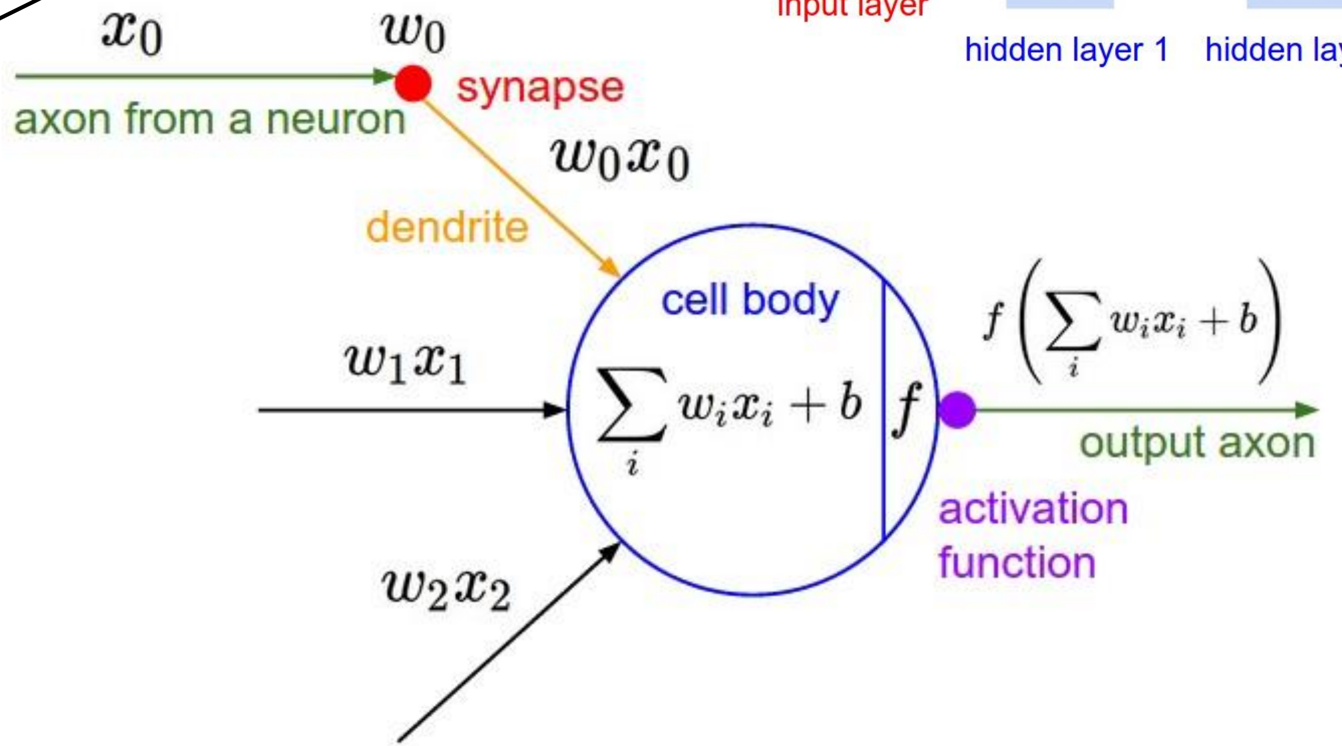
Firing rate is a nonlinear function of inputs

Biological Neuron

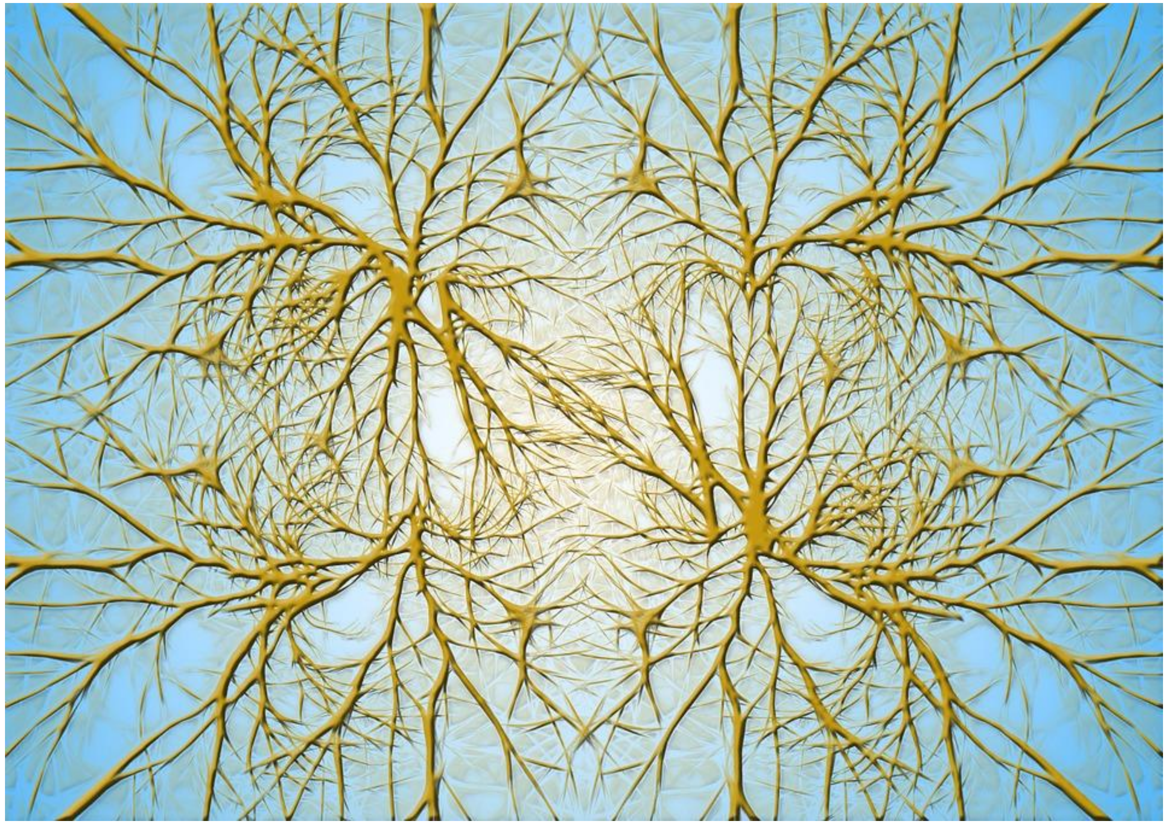


Artificial Neuron





Biological Neurons:
Complex connectivity patterns



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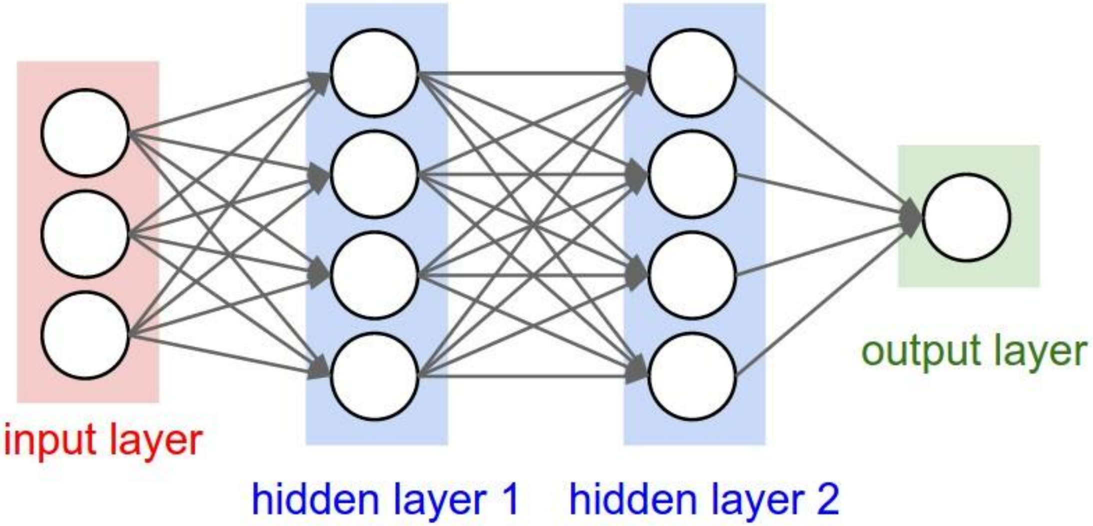
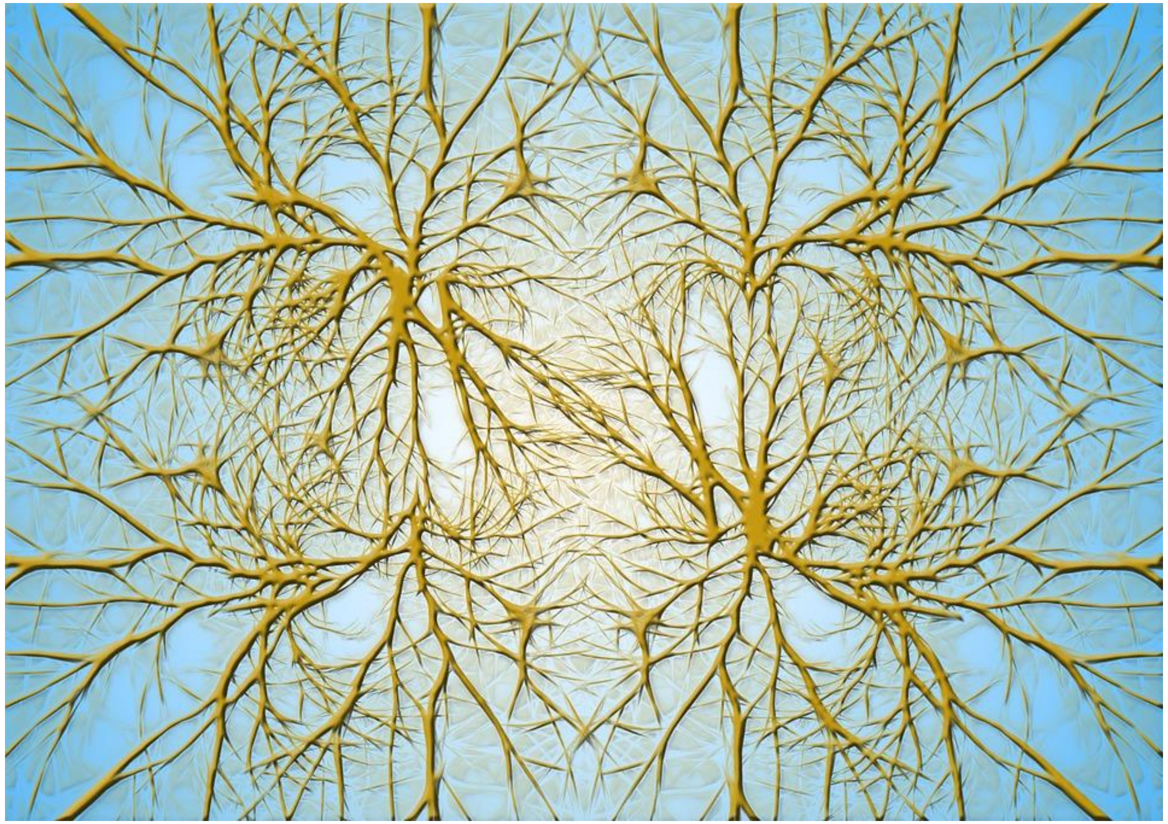
Neurons in a neural network:
Organized into regular layers for computational efficiency



Biological Neurons:
Complex connectivity patterns



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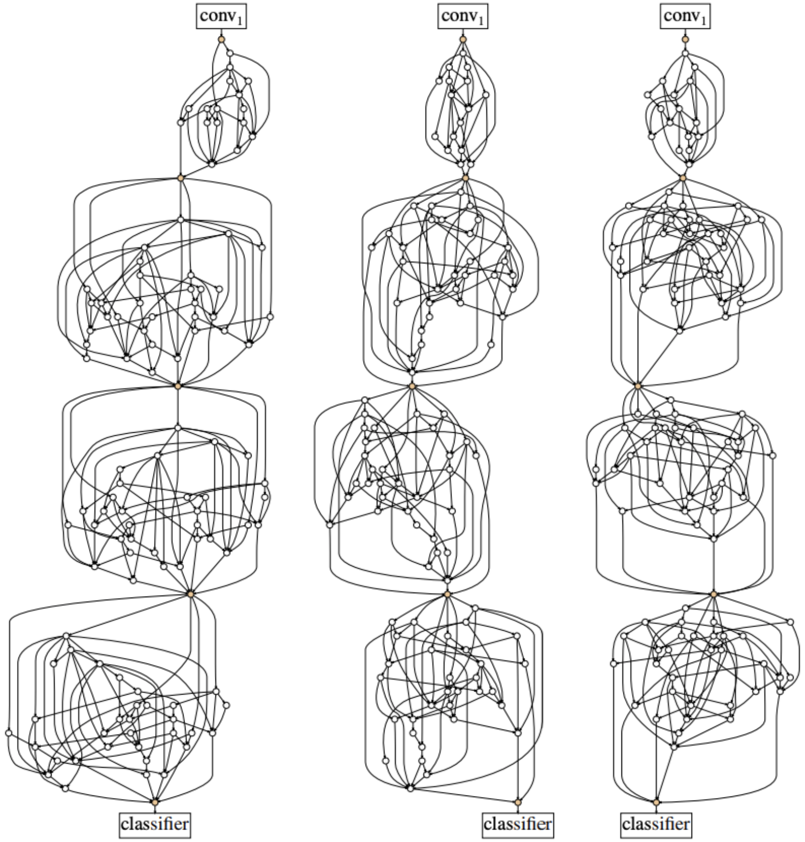
But neural networks with random connections can work too!



Xie et al, "Exploring Randomly Wired Neural Networks for Image Recognition", ICCV 2019

Be very careful with brain analogies!

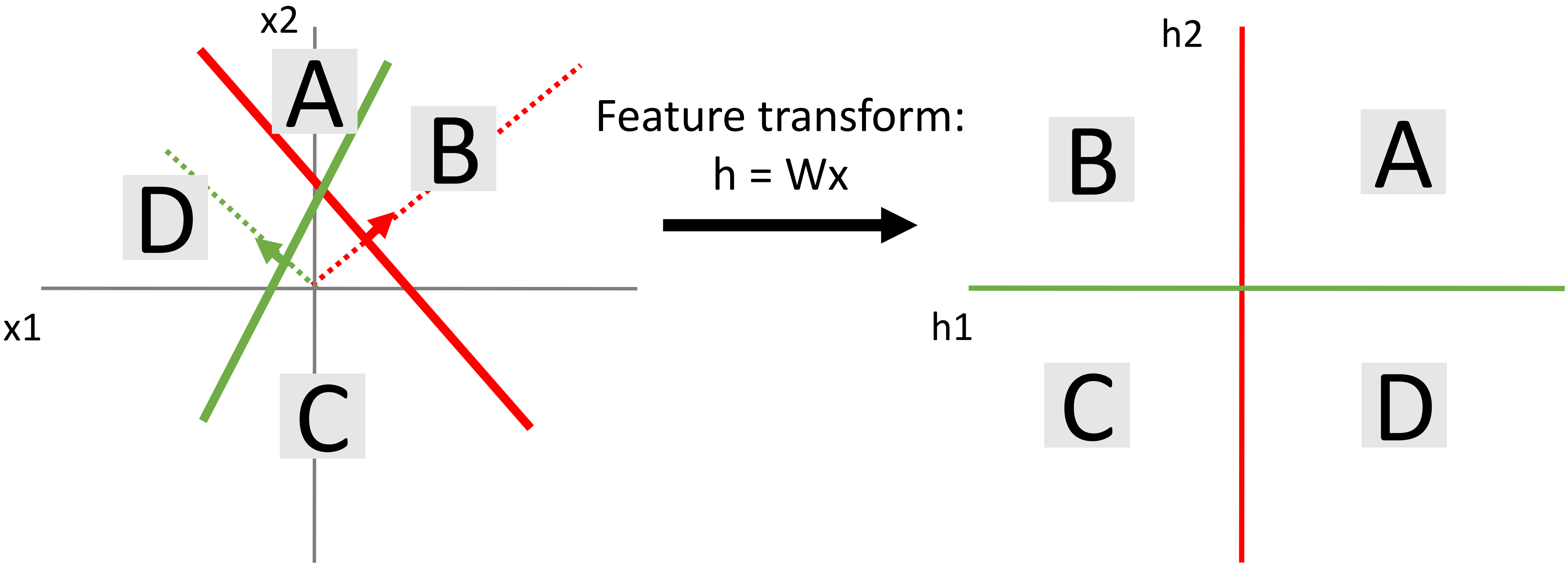
Biological Neurons:

- Many different types
- Dendrites can perform complex non-linear computations
- Synapses are not a single weight but a complex non-linear dynamical system
- Abstracting a neuron by “firing rate” isn’t enough; temporal sequences of activations matter too (spiking neural networks)

[Dendritic Computation. London and Hausser]

Space Warping

Consider a linear transform: h = Wx
Where x, h are both 2-dimensional



Space Warping

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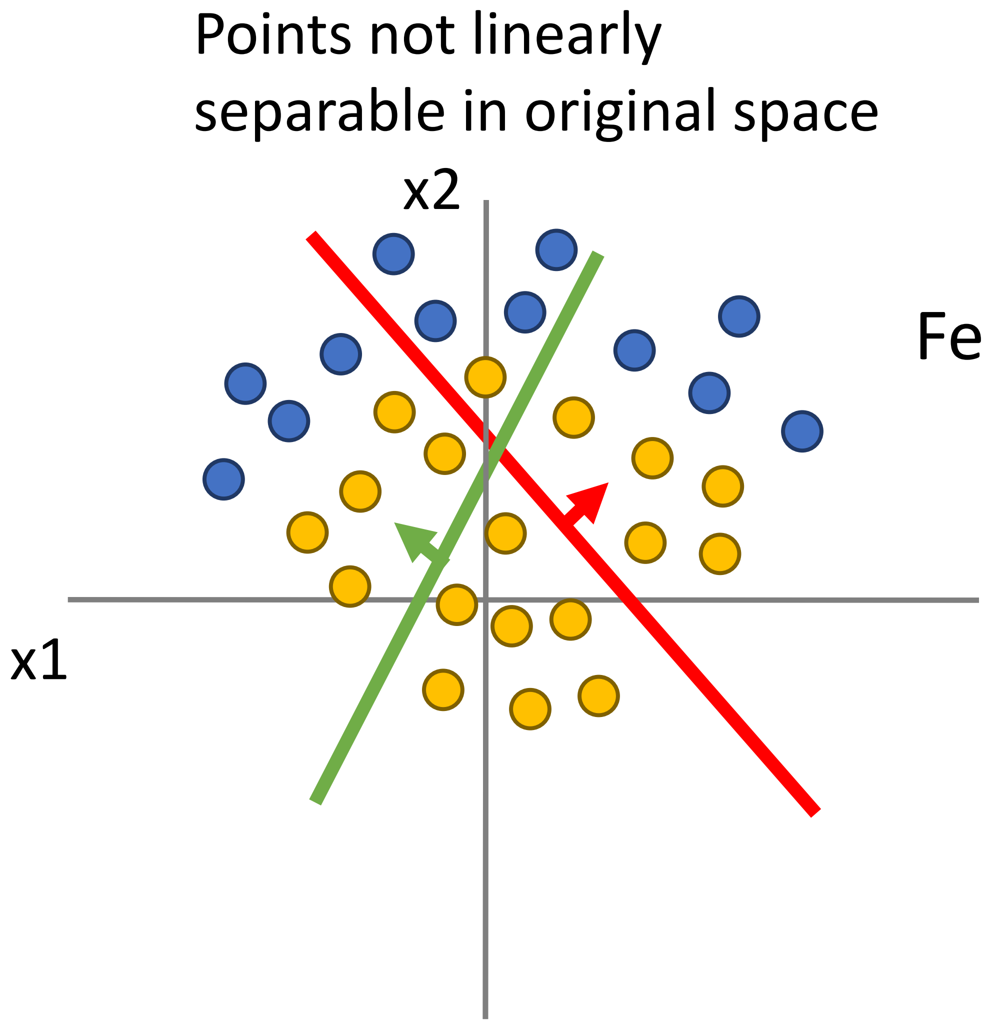
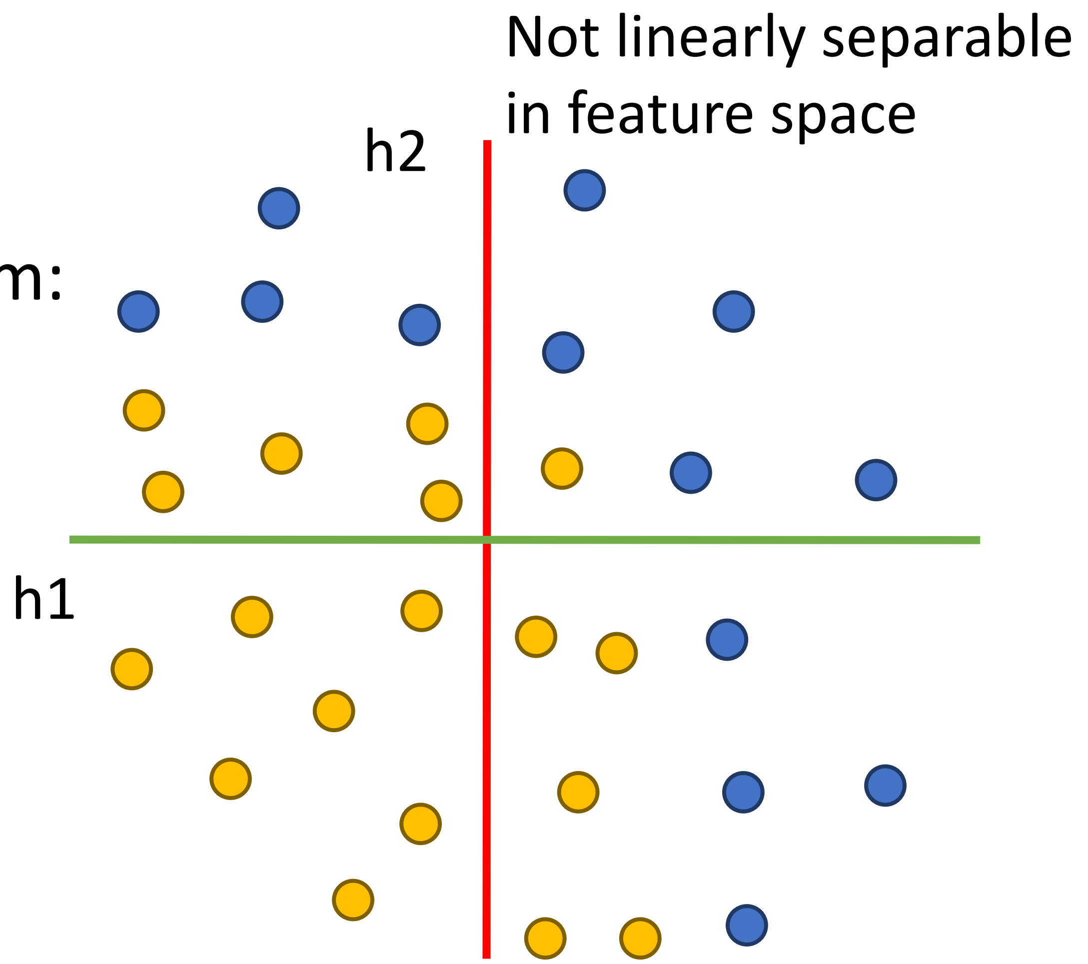


Feature transform:
h = Wx

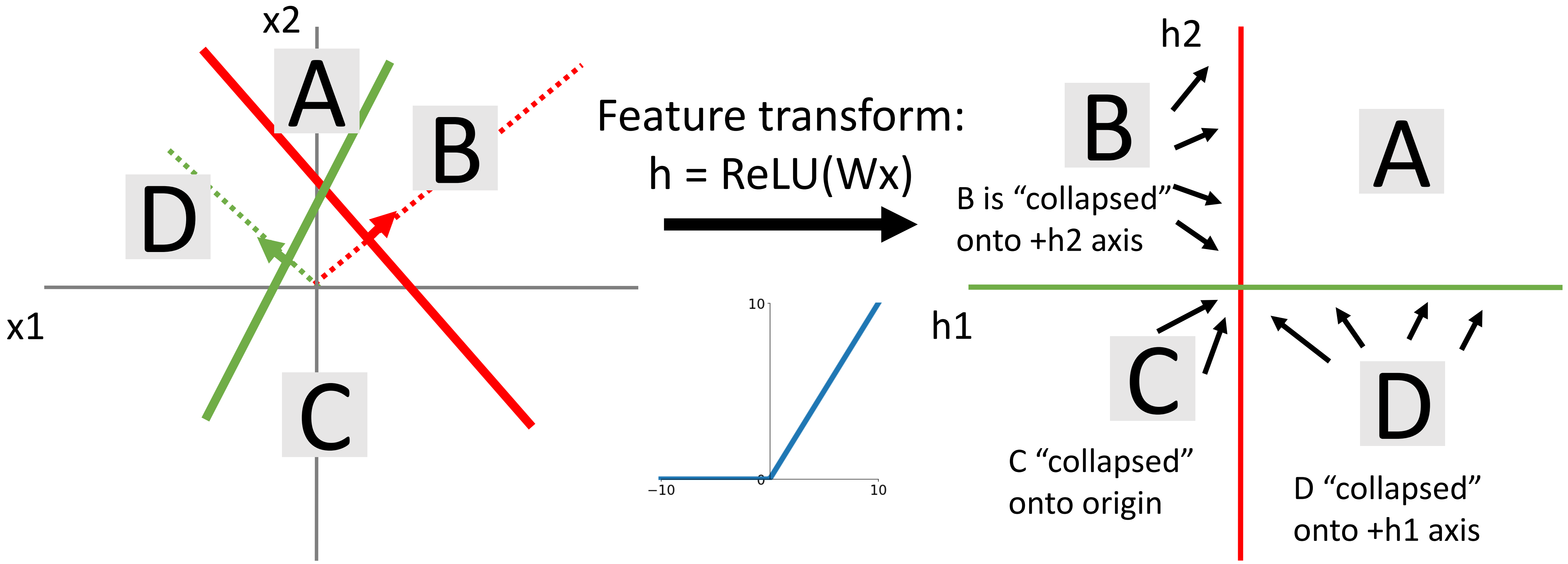


Space Warping

Consider a neural net hidden layer:

h = ReLU(Wx) = max(0, Wx)

Where x, h are both 2-dimensional



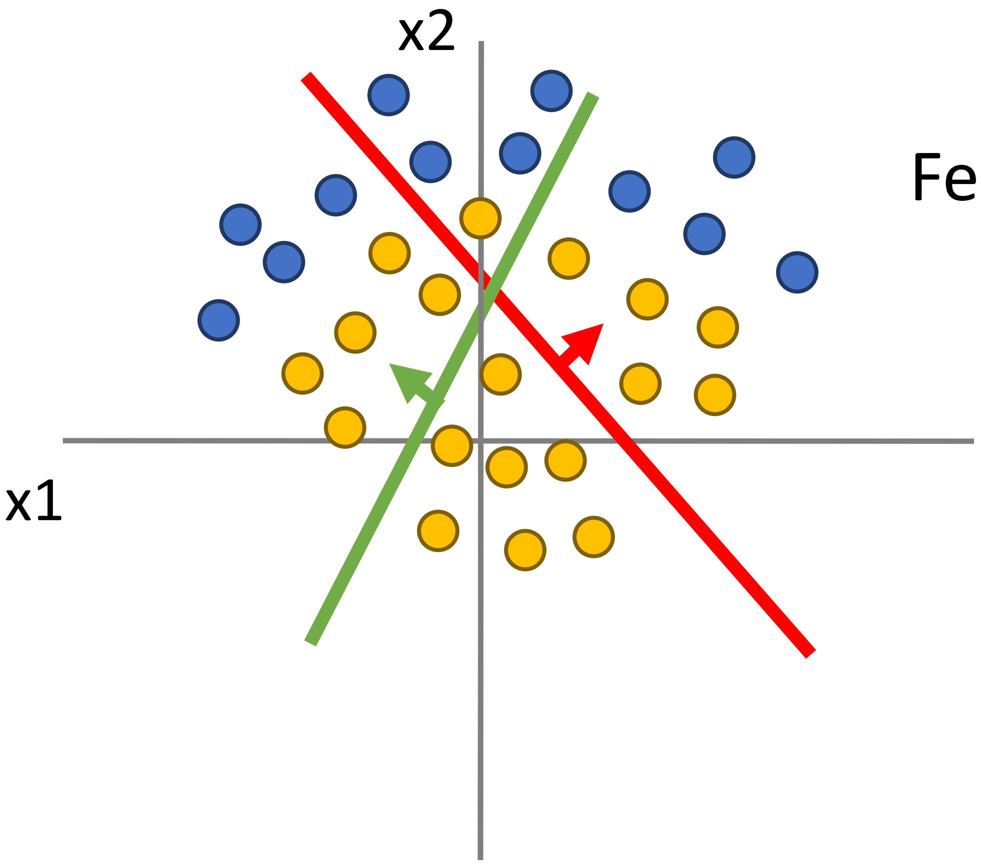
Space Warping

Consider a neural net hidden layer:

$h = \text{ReLU}(Wx) = \max(0, Wx)$

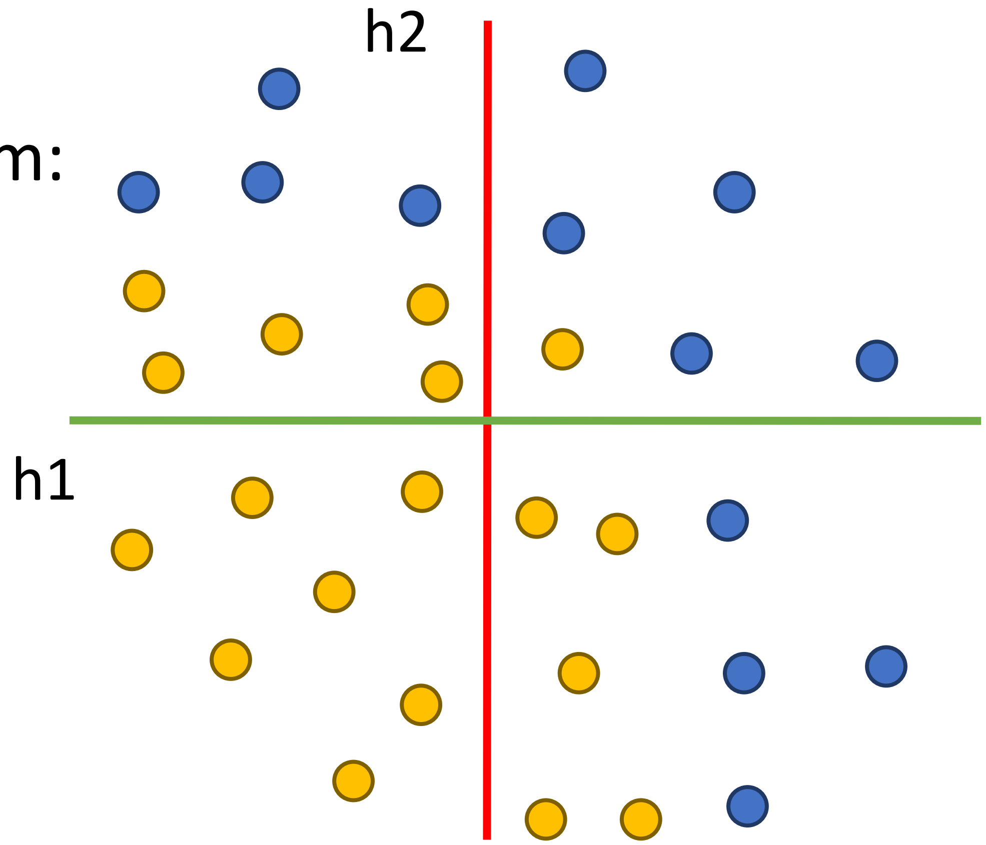
Where x, h are both 2-dimensional

Points not linearly separable in original space



Feature transform:

$h = Wx$



Space Warping

Consider a neural net hidden layer:
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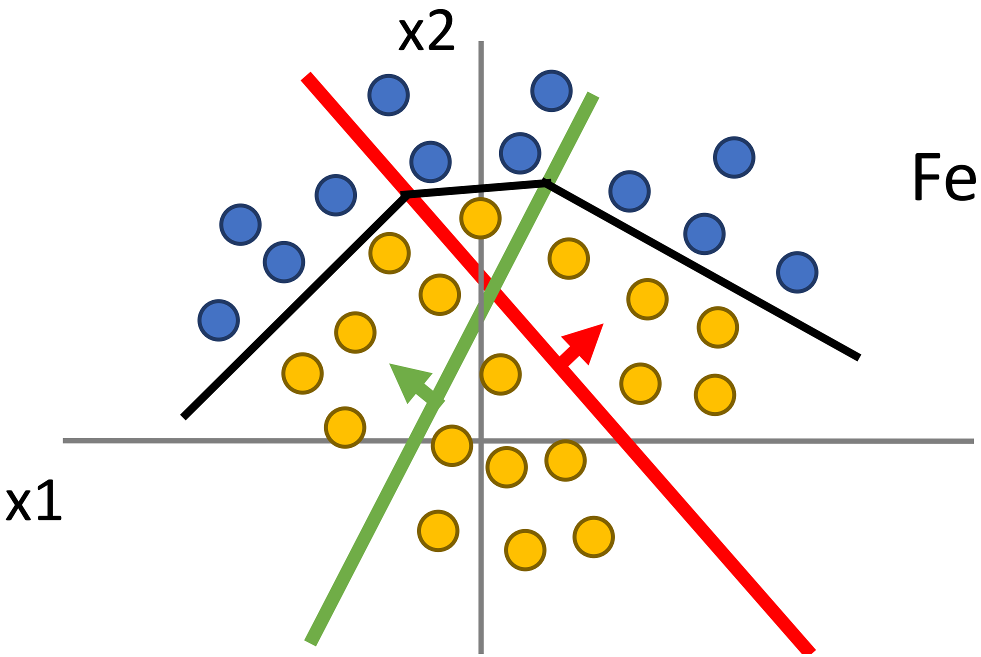
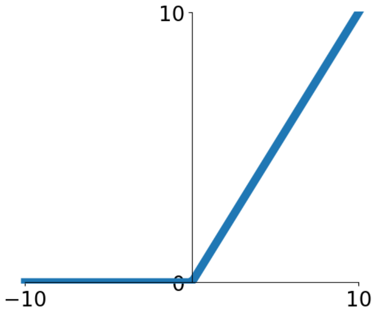
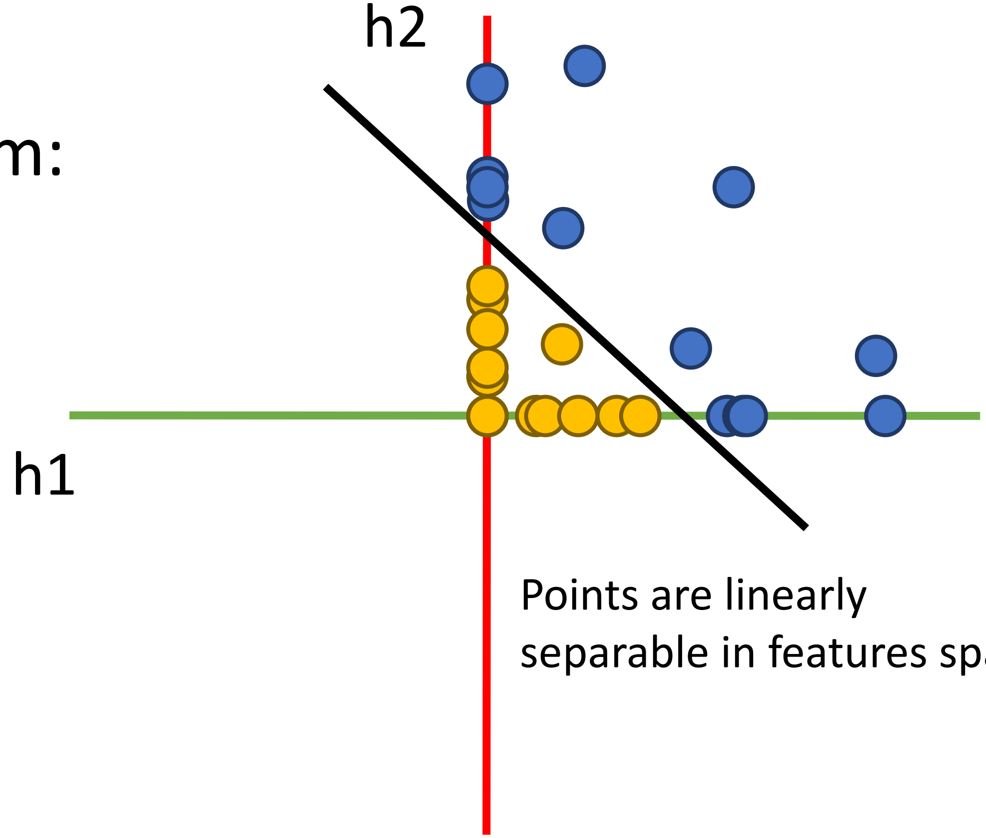


Linear classifier in feature space gives nonlinear classifier in original space

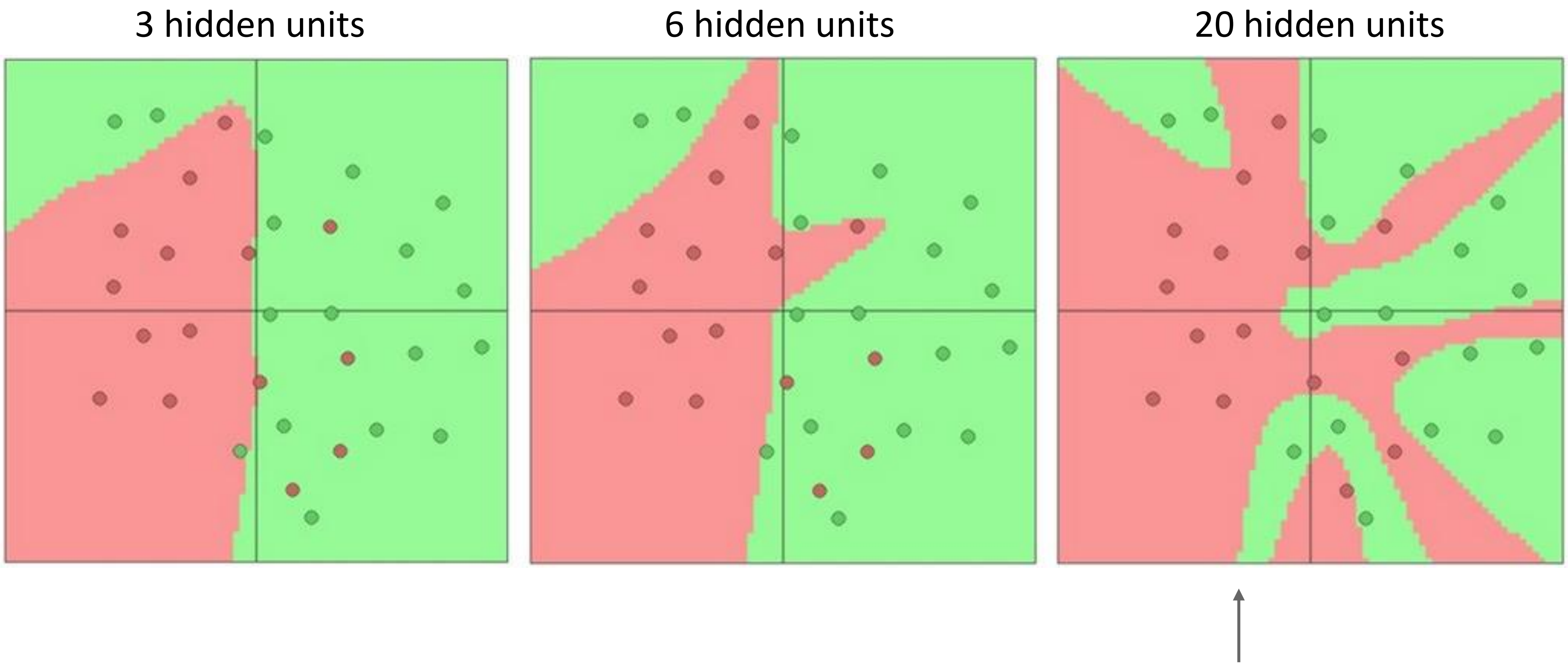
Feature transform:
$h = \text{ReLU}(Wx)$





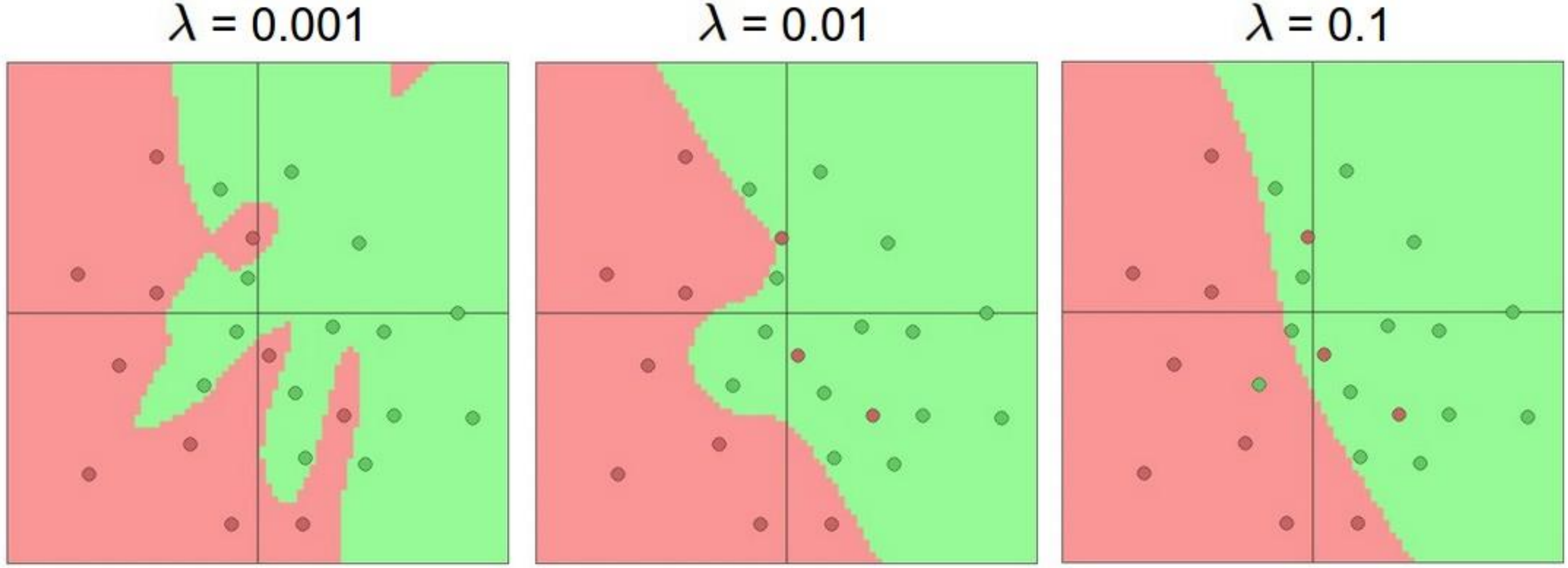
Points are linearly separable in features space!

Setting the number of layers and their sizes



More hidden units = more capacity

Don't regularize with size; instead use stronger L2



(Web demo with ConvNetJS:
http://cs.stanford.edu/people/karpathy/convnetjs/demo/classify2d.html)

Universal Approximation

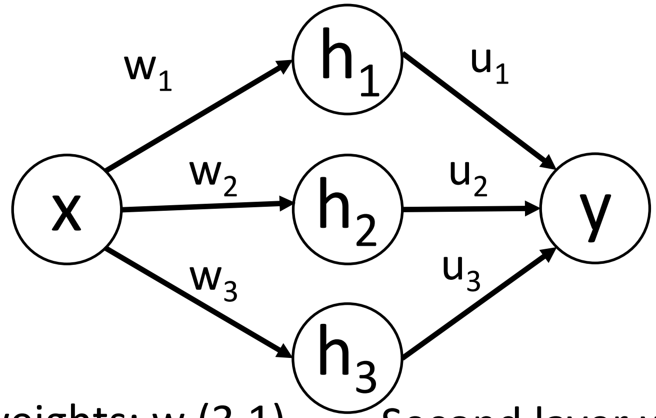
A neural network with one hidden layer can approximate any function f: R^N -> R^M with arbitrary precision*

*Many technical conditions: Only holds on compact subsets of R^N; function must be continuous; need to define "arbitrary precision"; etc

Universal Approximation

Example: Approximating a function f: R -> R with a two-layer ReLU network

Input:
x (1,)



Output:
y (1,)

First layer weights: w (3,1)
First layer bias: b (3,)

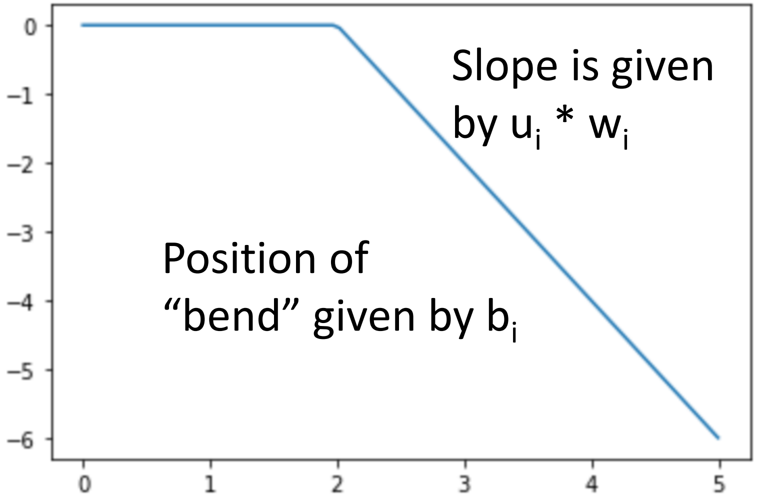
Second layer weights: u (1,3)
First layer bias: p (1,)

h1 = max(0, w1 * x + b1)
h2 = max(0, w2 * x + b2)
h3 = max(0, w3 * x + b3)
y = u1 * h1 + u2 * h2 + u3 * h3 + p

y = u1 * max(0, w1 * x + b1)
+ u2 * max(0, w2 * x + b2)
+ u3 * max(0, w3 * x + b3)
+ p

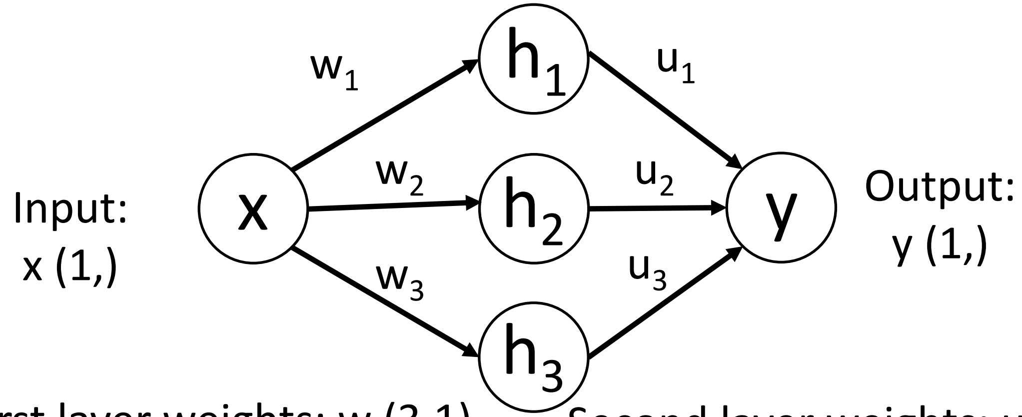
Output is a sum of shifted, scaled ReLUs:

Flip left / right based on sign of w_i



Universal Approximation

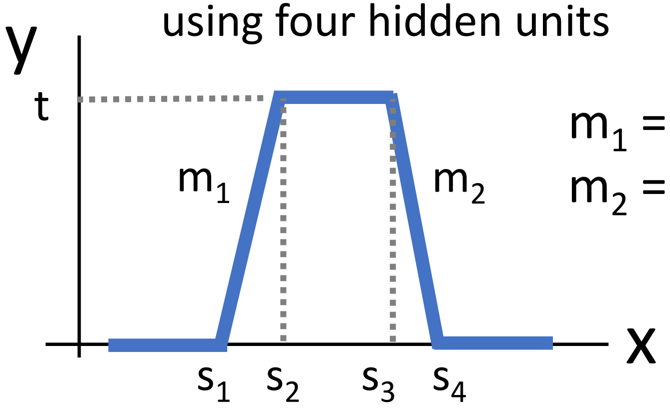
Example: Approximating a function f: R -> R with a two-layer ReLU network



First layer weights: w (3,1)
First layer bias: b (3,)

Second layer weights: u (1,3)
First layer bias: p (1,)

We can build a "bump function" using four hidden units



$m_1 = t / (s_2 - s_1)$
$m_2 = t / (s_4 - s_3)$

h1 = max(0, w1 * x + b1)
h2 = max(0, w2 * x + b2)
h3 = max(0, w3 * x + b3)
y = u1 * h1 + u2 * h2 + u3 * h3 + p

y = u1 * max(0, w1 * x + b1)
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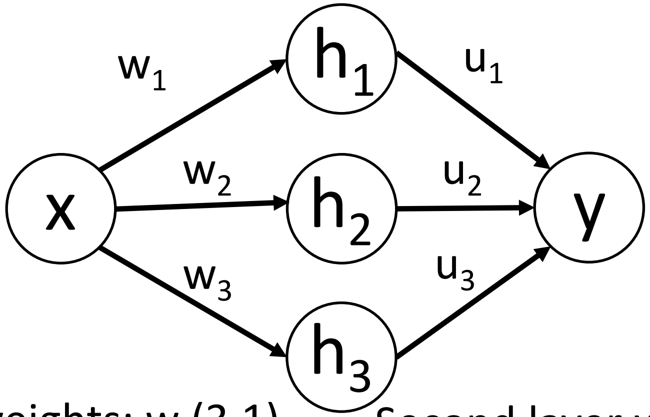

Output:
y (1,)

First layer weights: w (3,1)
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Second layer weights: u (1,3)
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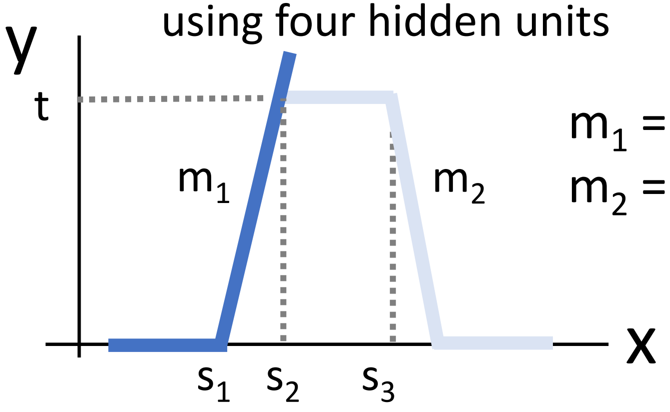
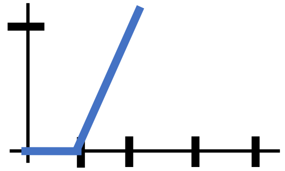
We can build a "bump function" using four hidden units



$m_1 = t / (s_2 - s_1)$
$m_2 = t / (s_4 - s_3)$

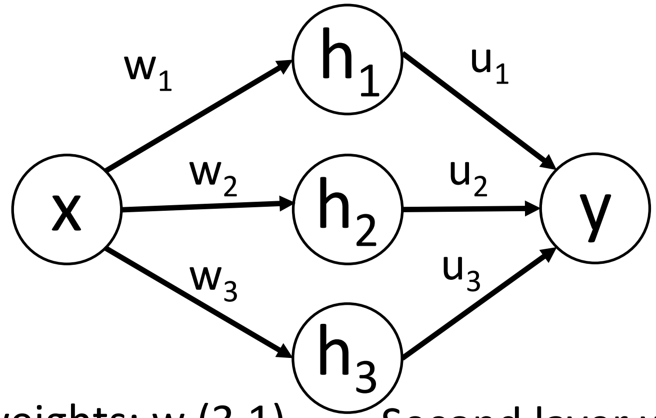


$m_1 * \max(0, x - s_1)$

Universal Approximation

Example: Approximating a function f: R -> R with a two-layer ReLU network

Input:
x (1,)



Output:
y (1,)

First layer weights: w (3,1)
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h1 = max(0, w1 * x + b1)
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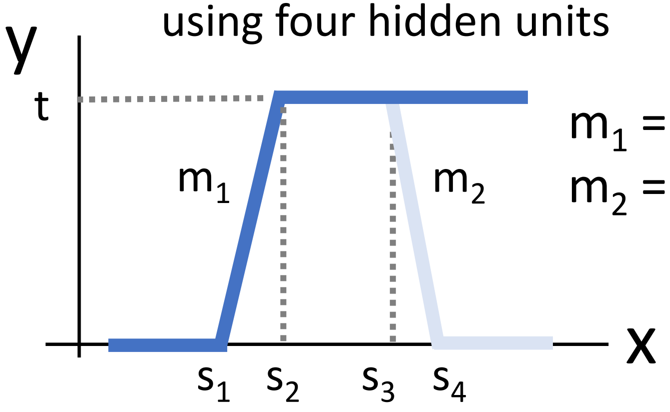
We can build a “bump function” using four hidden units



$m_1 = t / (s_2 - s_1)$
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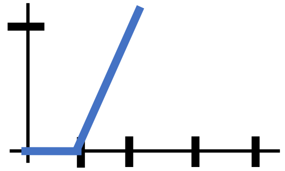
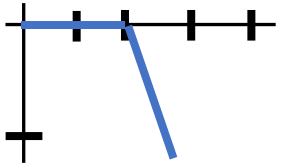


m_1 * max(0, $x - s_1$)

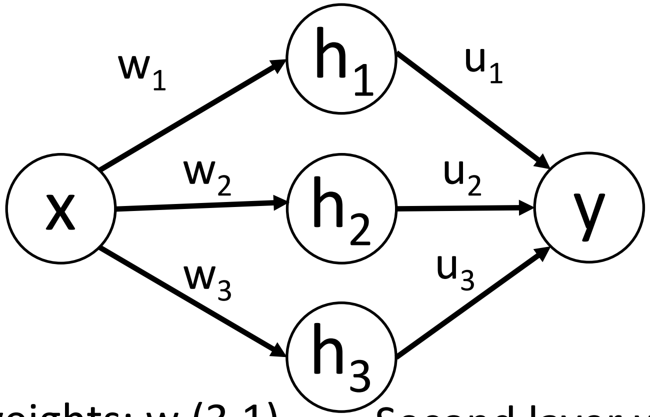


$-m_1$ * max(0, $x - s_2$)

Universal Approximation

Example: Approximating a function f: R -> R with a two-layer ReLU network

Input:
x (1,)



Output:
y (1,)

First layer weights: w (3,1)
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Second layer weights: u (1,3)
First layer bias: p (1,)

h1 = max(0, w1 * x + b1)
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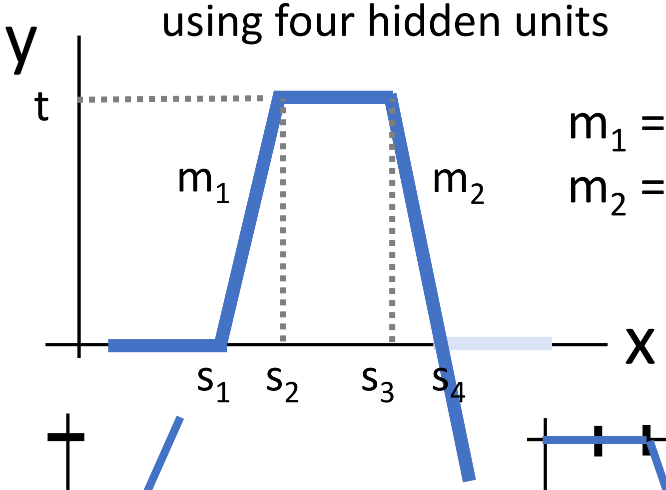
h3 = max(0, w3 * x + b3)
y = u1 * h1 + u2 * h2 + u3 * h3 + p

y = u1 * max(0, w1 * x + b1)
+ u2 * max(0, w2 * x + b2)
+ u3 * max(0, w3 * x + b3)
+ p

We can build a "bump function" using four hidden units



$m_1 = t / (s_2 - s_1)$
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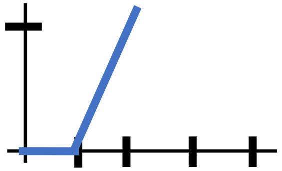
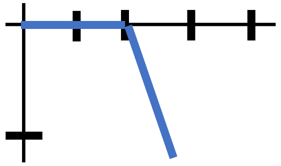
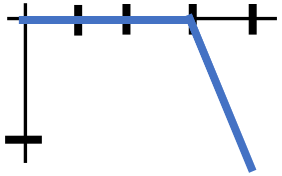


m_1 * max(0, x – s_1)



-m_1 * max(0, x – s_2)

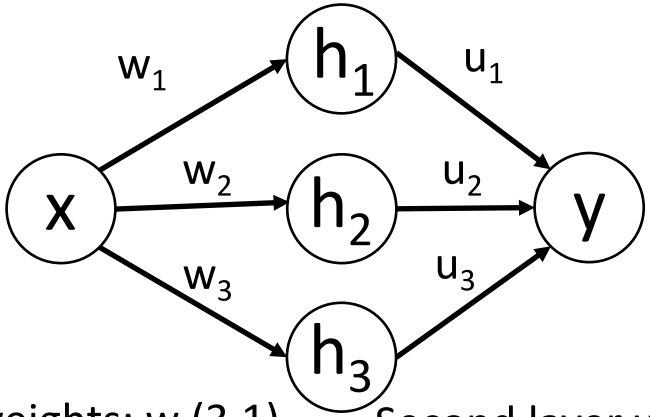


-m_2 * max(0, x – s_3)

Universal Approximation

Example: Approximating a function f: R -> R with a two-layer ReLU network

Input: x (1,)



Output: y (1,)

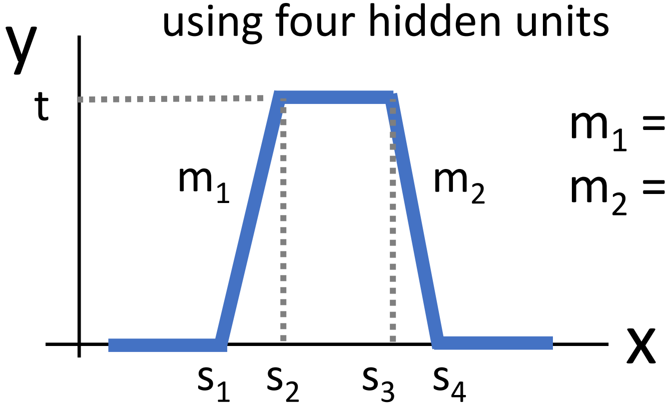
First layer weights: w (3,1)
First layer bias: b (3,)

Second layer weights: u (1,3)
First layer bias: p (1,)

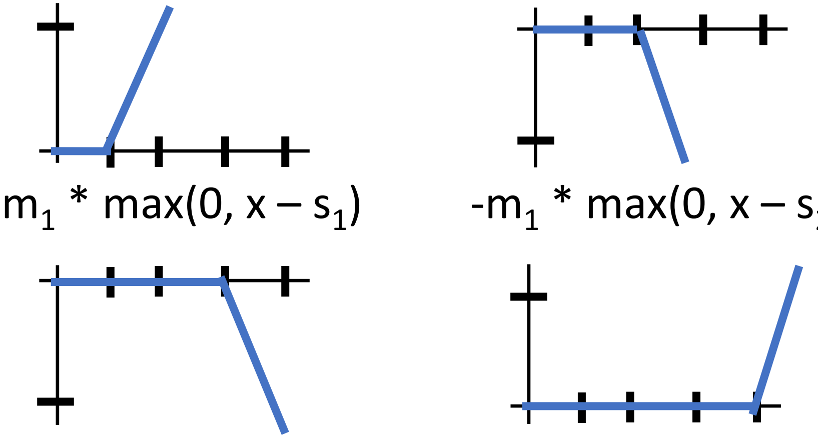
$$h1 = \max(0, w1 * x + b1)$$
$$h2 = \max(0, w2 * x + b2)$$
$$h3 = \max(0, w3 * x + b3)$$
$$y = u1 * h1 + u2 * h2 + u3 * h3 + p$$

$$y = u1 * \max(0, w1 * x + b1) + u2 * \max(0, w2 * x + b2) + u3 * \max(0, w3 * x + b3) + p$$

We can build a “bump function” using four hidden units



$m_1 = t / (s_2 – s_1)$
$m_2 = t / (s_4 – s_3)$



$m_1 * \max(0, x – s_1)$

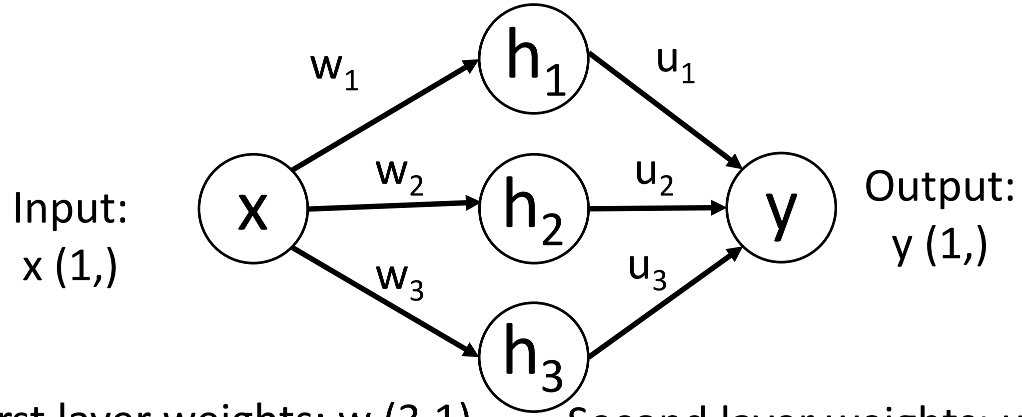
$-m_1 * \max(0, x – s_2)$

$-m_2 * \max(0, x – s_3)$

$m_2 * \max(0, x – s_4)$

Universal Approximation

Example: Approximating a function f: R -> R with a two-layer ReLU network



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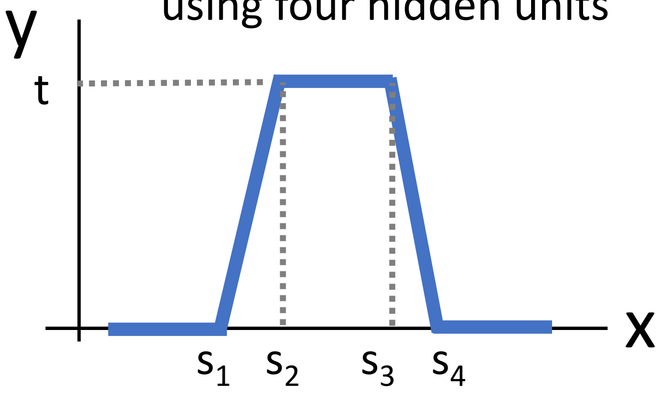
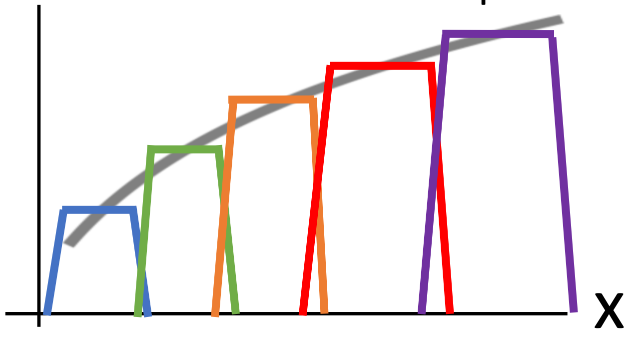
We can build a "bump function" using four hidden units



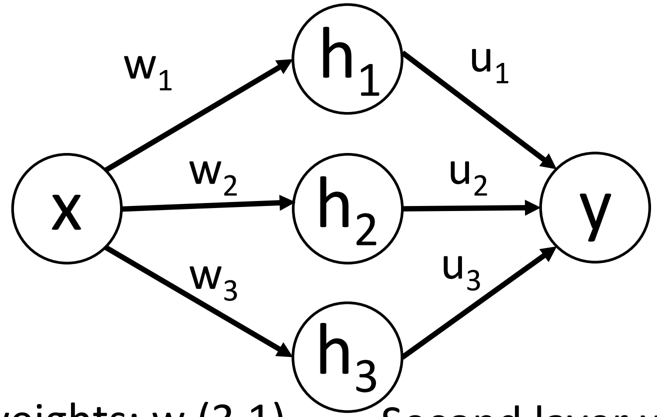
With 4K hidden units we can build a sum of K bumps



Approximate functions with bumps!

Universal Approximation

Example: Approximating a function f: R -> R with a two-layer ReLU network



Input:
x (1,)

Output:
y (1,)

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First layer bias: b (3,)

Second layer weights: u (1,3)
First layer bias: p (1,)

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h3 = max(0, w3 * x + b3)
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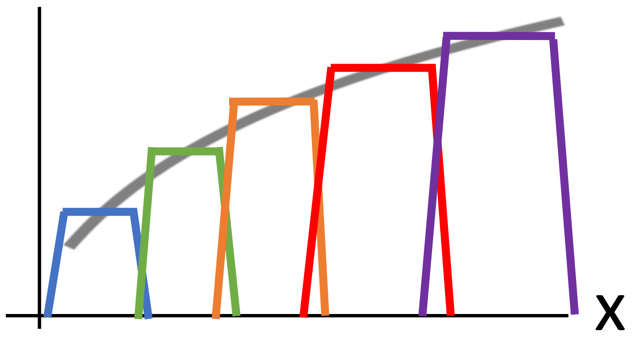
y = u1 * max(0, w1 * x + b1)
+ u2 * max(0, w2 * x + b2)
+ u3 * max(0, w3 * x + b3)
+ p

What about…
- Gaps between bumps?
- Other nonlinearities?
- Higher-dimensional functions?

See [Nielsen, Chapter 4]



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Universal Approximation

Example: Approximating a function f: R -> R with a two-layer ReLU network

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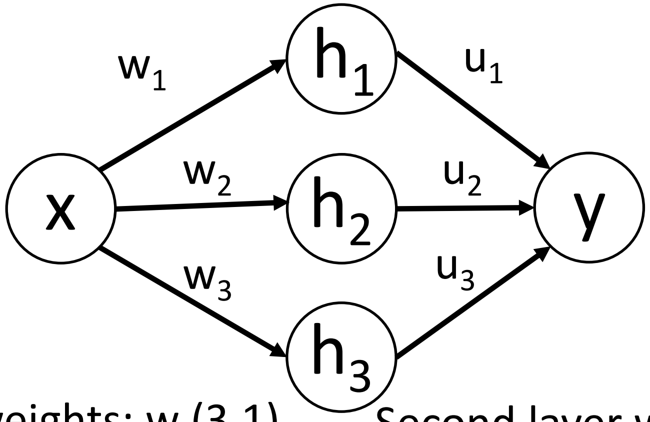

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h2 = max(0, w2 * x + b2)
h3 = max(0, w3 * x + b3)
y = u1 * h1 + u2 * h2 + u3 * h3 + p

y = u1 * max(0, w1 * x + b1)
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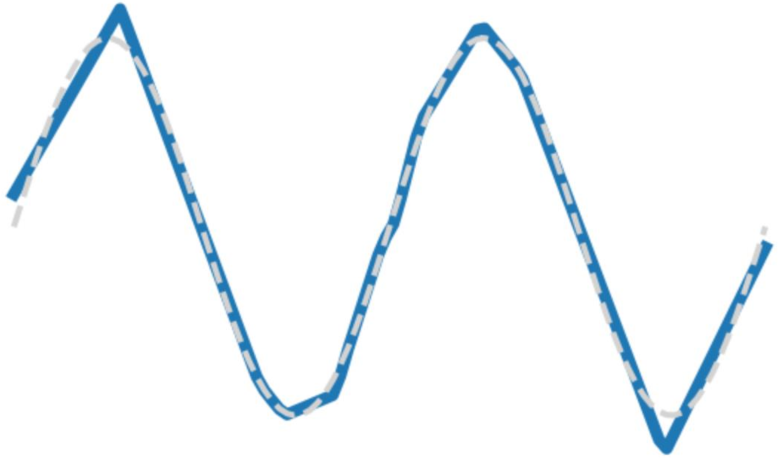
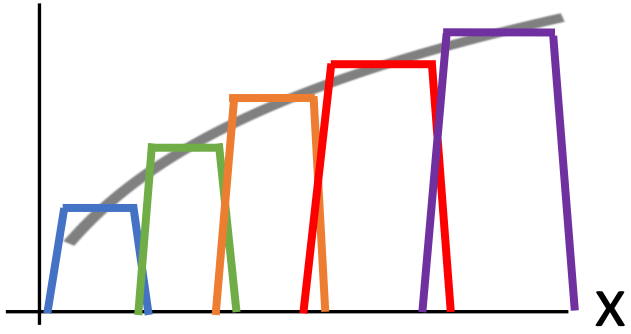
+ u3 * max(0, w3 * x + b3)
+ p

Reality check: Networks don't really learn bumps!

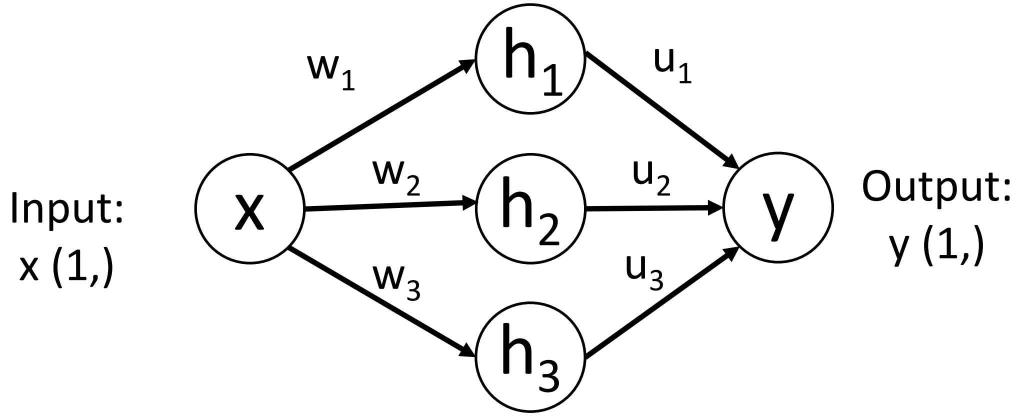




Approximate functions with bumps!

Universal Approximation

Example: Approximating a function f: R -> R with a two-layer ReLU network



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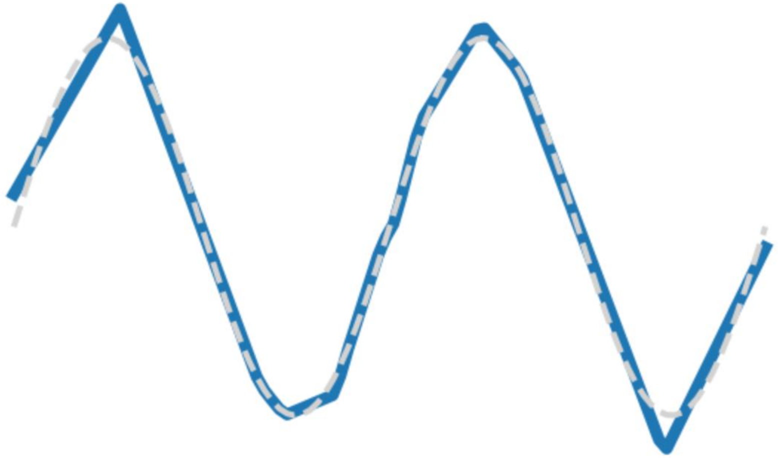
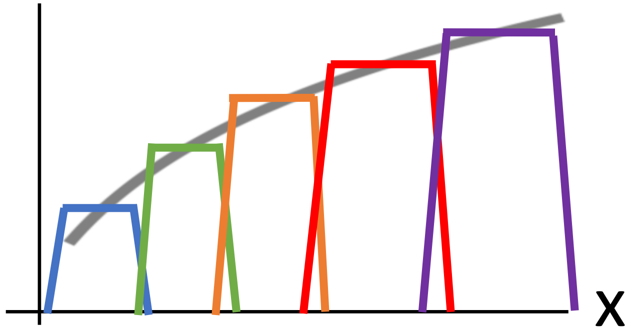


Universal approximation tells us:
- Neural nets can represent any function

Universal approximation DOES NOT tell us:
- Whether we can actually learn any function with SGD
- How much data we need to learn a function

Remember: kNN is also a universal approximator!



Approximate functions with bumps!

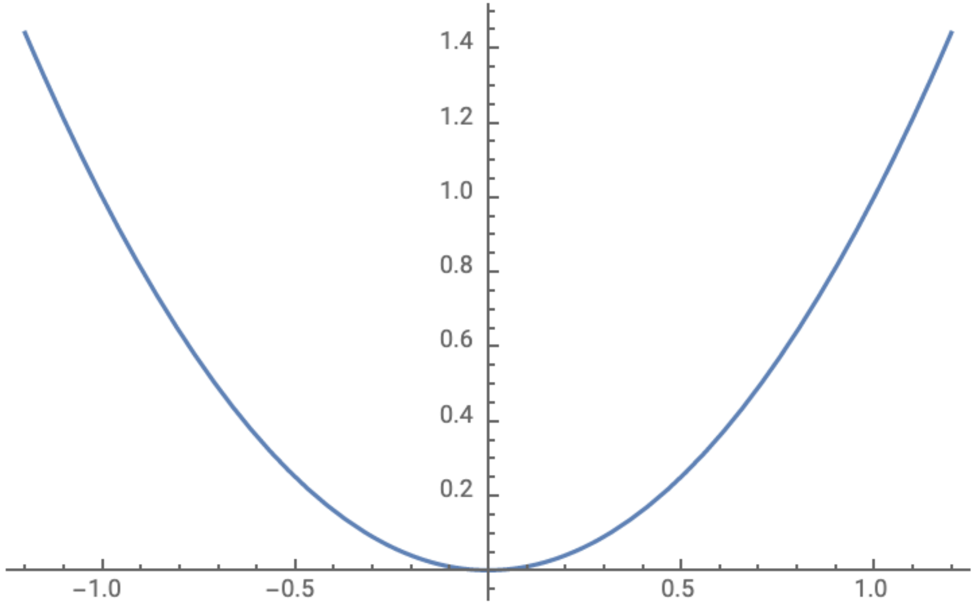
Extra topic: Convex Functions

Convex Functions

A function $f : X \subseteq \mathbb{R}^N \rightarrow \mathbb{R}$ is **convex** if for all $x_1, x_2 \in X, t \in [0, 1]$,

$$f(tx_1 + (1-t)x_2) \leq tf(x_1) + (1-t)f(x_2)$$

Example: $f(x) = x^2$ is convex:

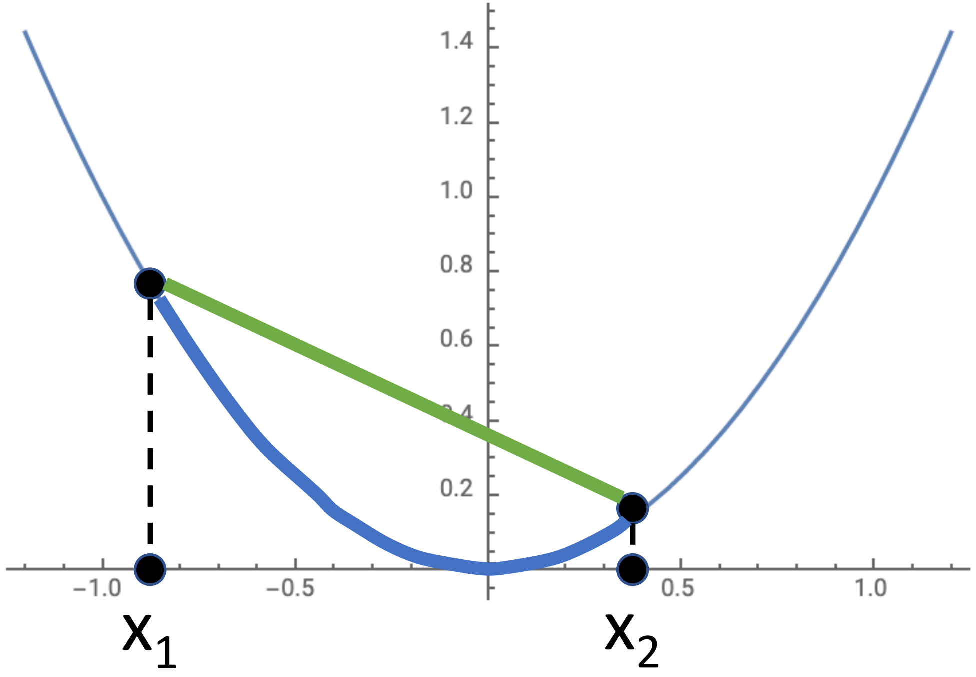


Convex Functions

A function $f : X \subseteq \mathbb{R}^N \rightarrow \mathbb{R}$ is **convex** if for all $x_1, x_2 \in X, t \in [0, 1]$,

$$\boxed{f(tx_1 + (1-t)x_2)} \leq \boxed{tf(x_1) + (1-t)f(x_2)}$$

Example: $f(x) = x^2$ is convex:

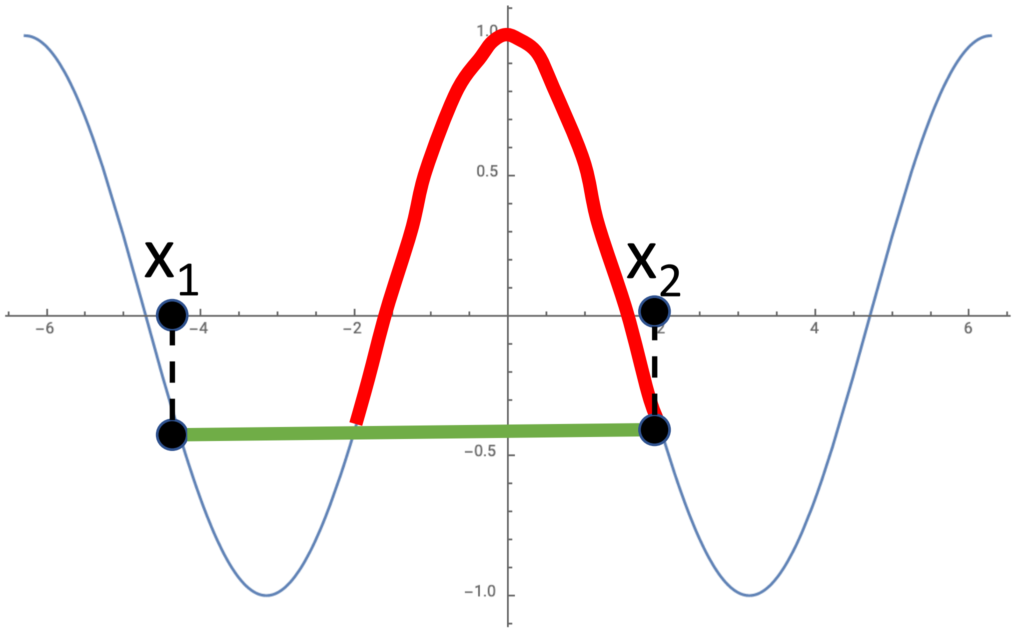


Convex Functions

A function $f : X \subseteq \mathbb{R}^N \rightarrow \mathbb{R}$ is **convex** if for all $x_1, x_2 \in X, t \in [0, 1]$,

$$\boxed{f(tx_1 + (1-t)x_2)} \leq \boxed{tf(x_1) + (1-t)f(x_2)}$$

Example: $f(x) = \cos(x)$
is <u>not</u> convex:



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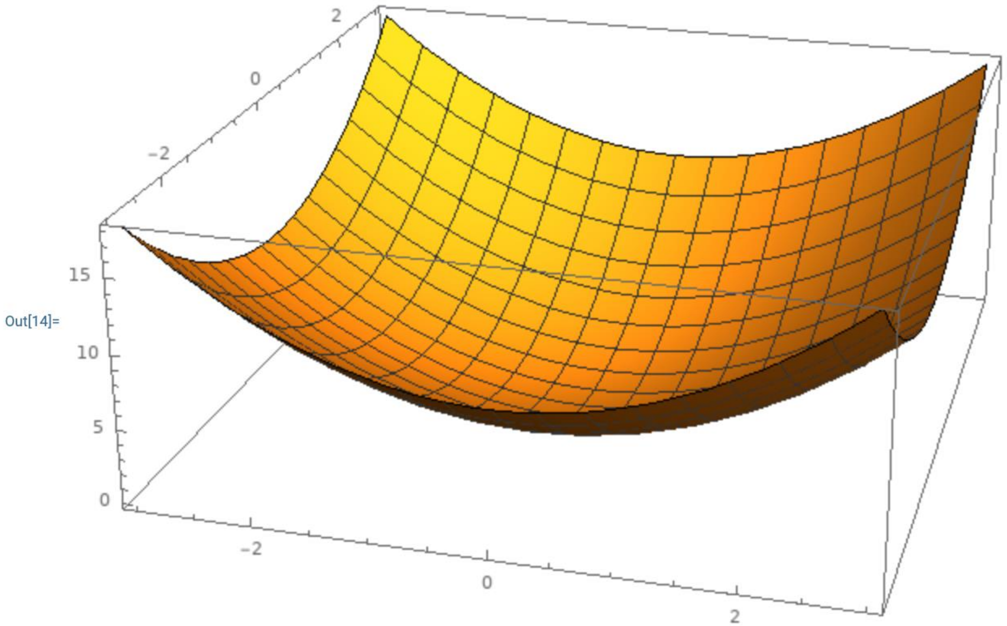
Intuition: A convex function is a (multidimensional) bowl

Generally speaking, convex functions are **easy to optimize**: can derive theoretical guarantees about **converging to global minimum***



*Many technical details inside!

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Linear classifiers optimize a **convex function!**

$$s = f(x; W) = Wx$$

$$L_i = -\log\left(\frac{e^{s_{y_i}}}{\sum_j e^{s_j}}\right)$$ Softmax

$$L_i = \sum_{j \neq y_i} \max(0, s_j - s_{y_i} + 1)$$ SVM

$$L = \frac{1}{N} \sum_{i=1}^{N} L_i + R(W)$$

R(W) = L2 or L1 regularization

*Many technical details inside!

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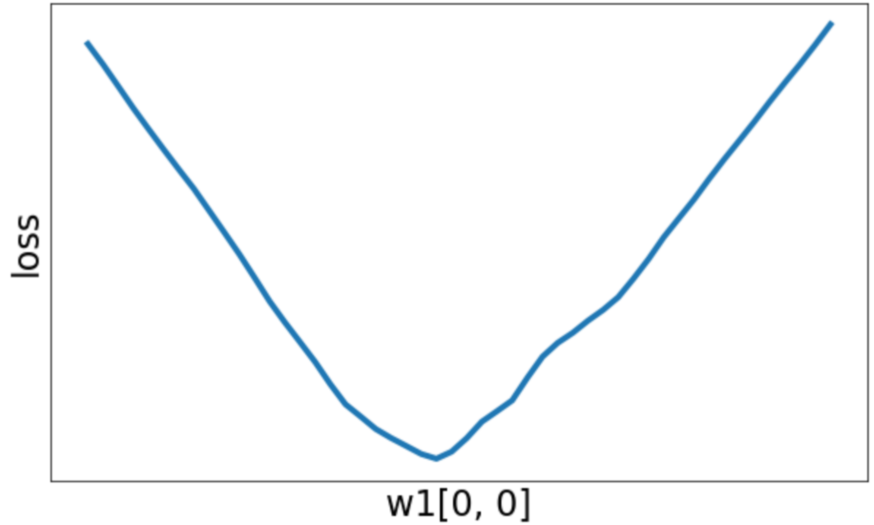
Neural net losses sometimes look convex-ish:



1D slice of loss landscape for a 4-layer ReLU network with 10 input features, 32 units per hidden layer, 10 categories, with softmax loss

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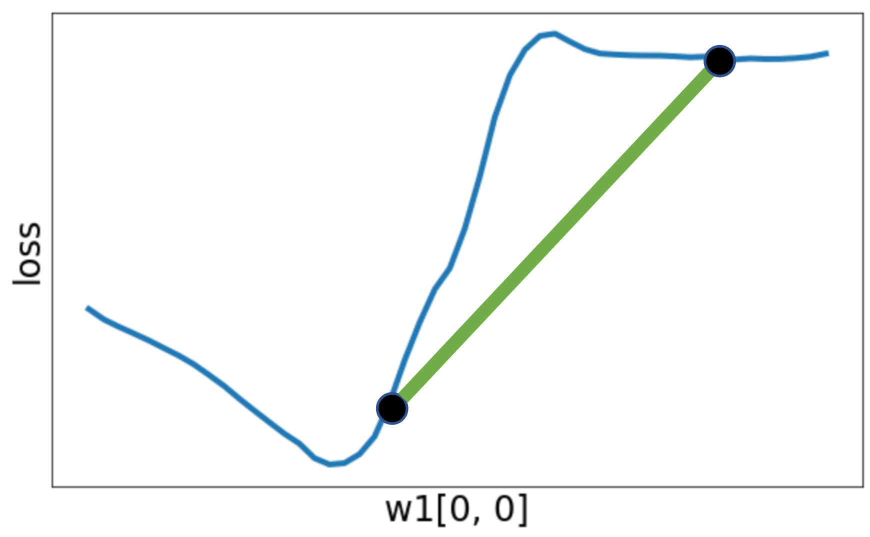
But often clearly nonconvex:



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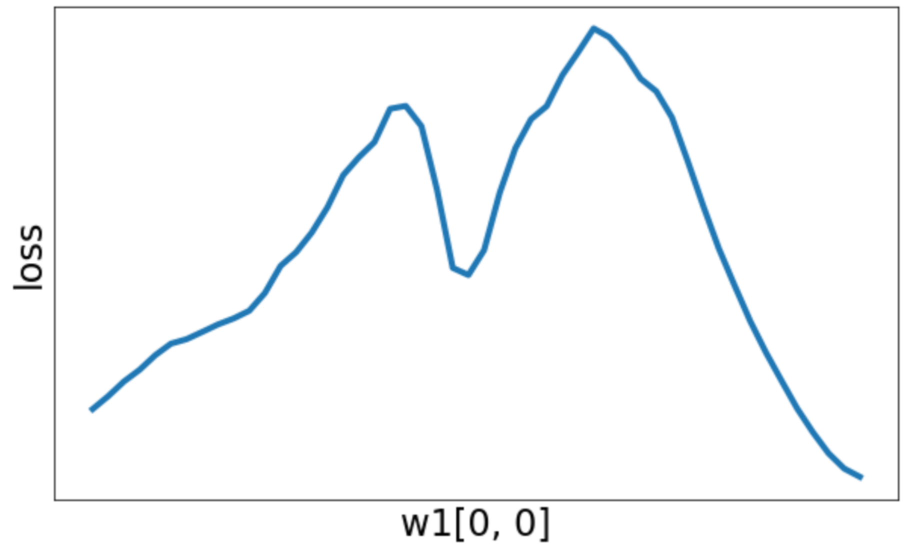
With local minima:



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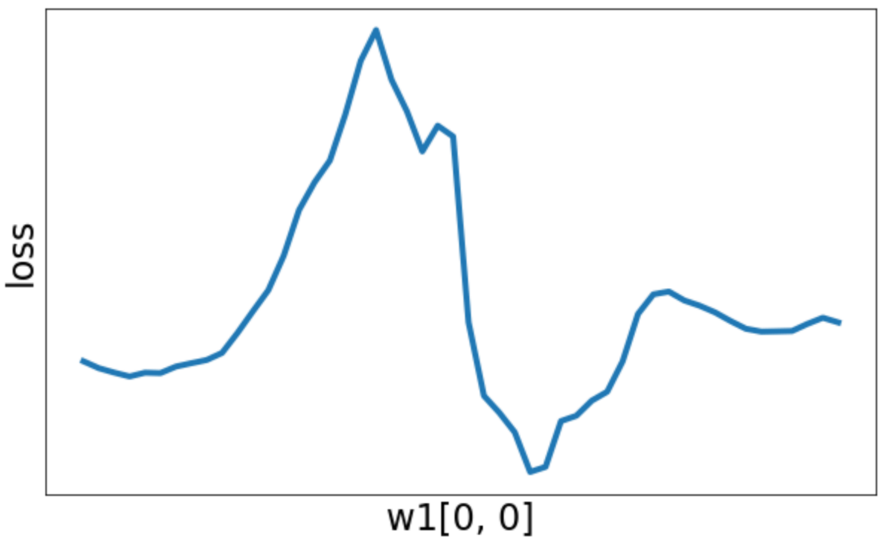
Can get very wild!



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Most neural networks need **nonconvex optimization**

- Few or no guarantees about convergence
- Empirically it seems to work anyway
- Active area of research

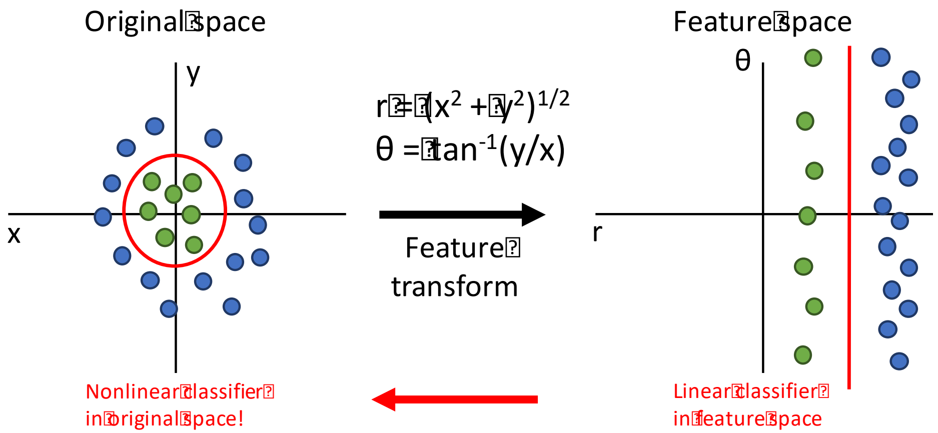
*Many technical details inside!

Convexity

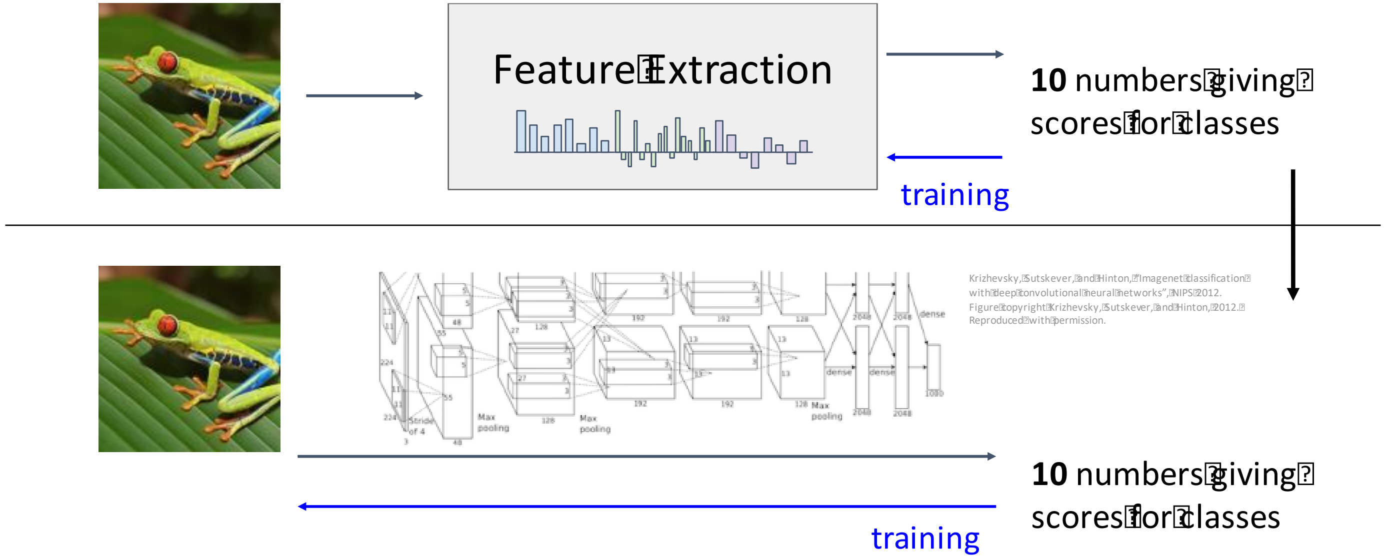
- Most linear classifiers optimize a convex function
  - Linear layer $s = f(x; W) = Wx$
  - Cross-entropy loss $L_i = -\log(\frac{e^{s_{y_i}}}{\sum_j e^{s_j}})$
  - SVM $L_i = \sum_{j \neq y_i} \max(0, s_j - s_{y_i} + 1)$
  - L1/L2 regularization $L = \frac{1}{N} \sum_{i=1}^{N} L_i + R(W)$
- Most neural networks need non-convex optimization
  - Few or no guarantees about convergence (mostly falls in a local optimum)
  - Empirically it seems to work anyway
  - Active area of research

Summary

Feature transform + Linear classifier
allows nonlinear decision boundaries



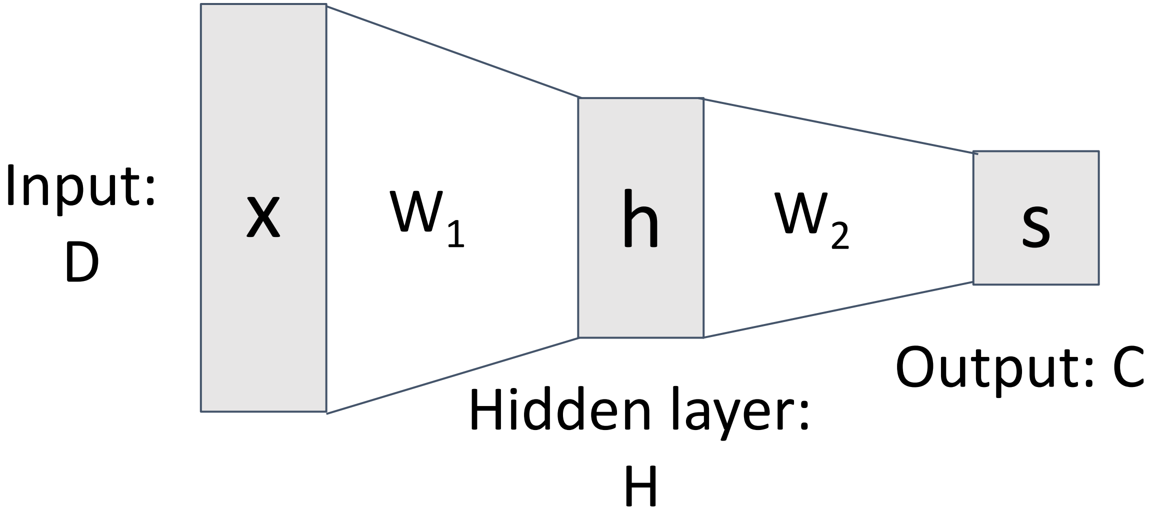
Neural Networks as learnable feature transforms



Summary

From linear classifiers to
fully-connected networks

$$s(x) = W_2\, f(W_1 x + b_1) + b_2$$



Linear classifier: One template per class

Neural networks: Many reusable templates

Summary

From linear classifiers to
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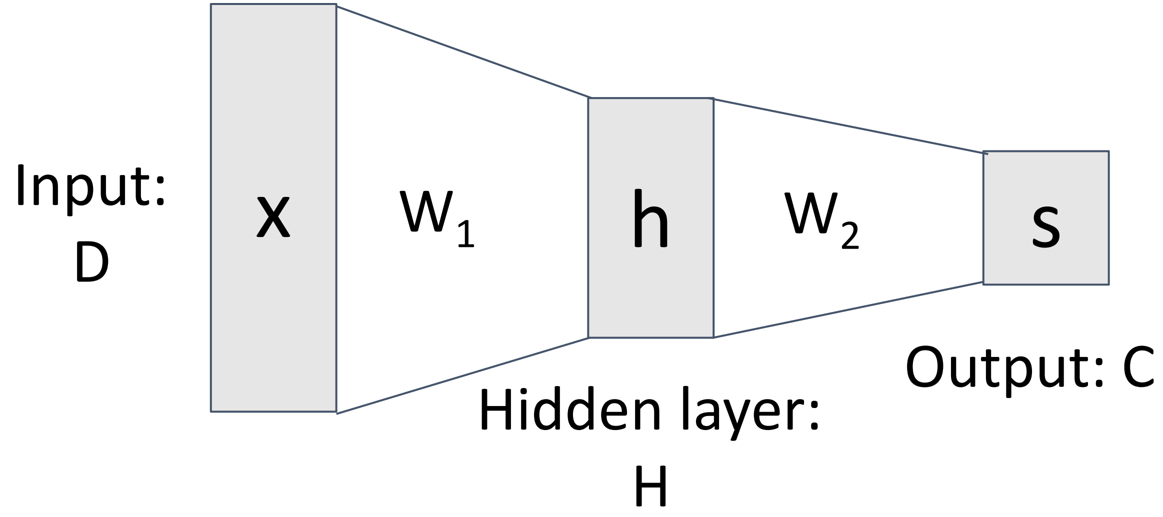
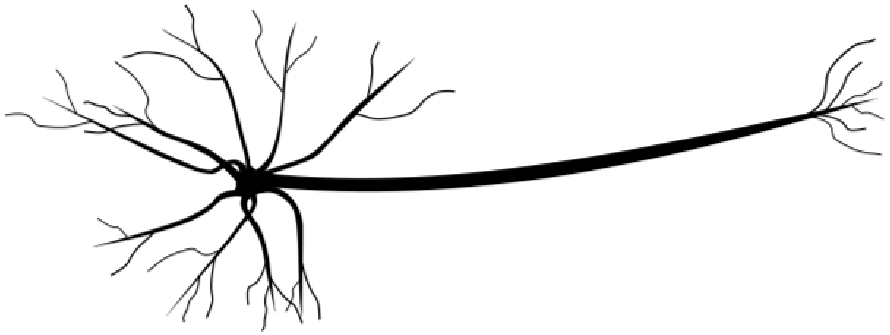


Neural networks loosely inspired by biological neurons but be careful with analogies



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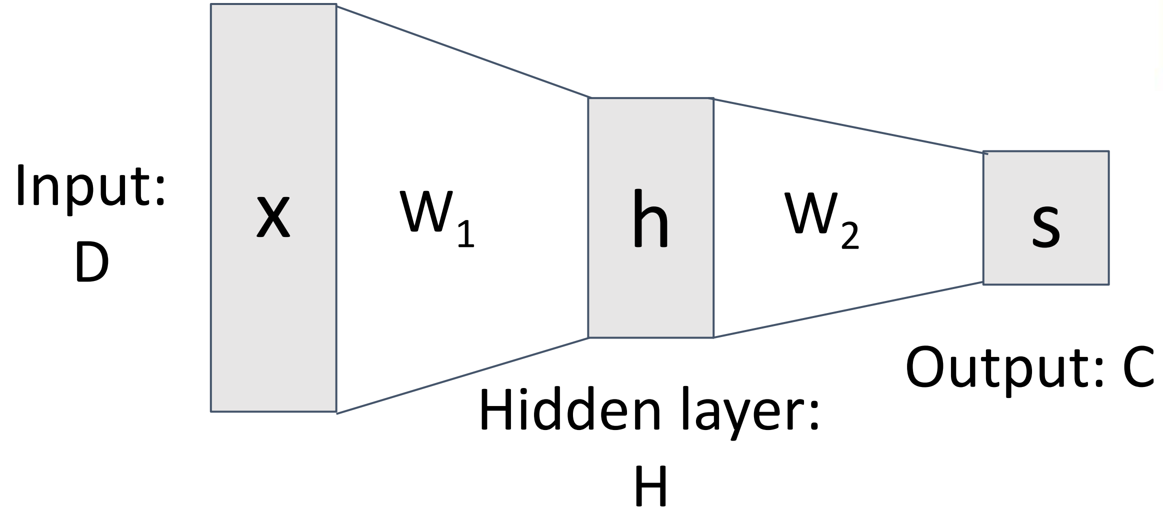
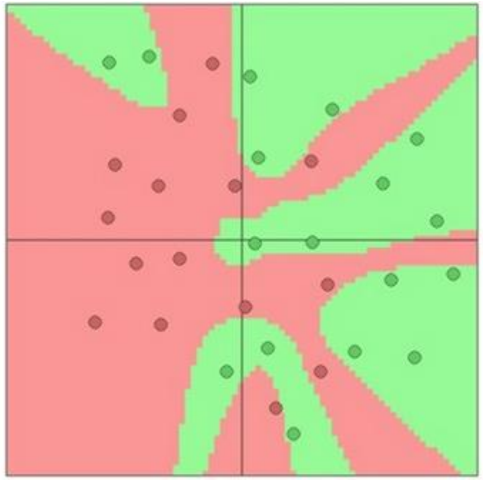
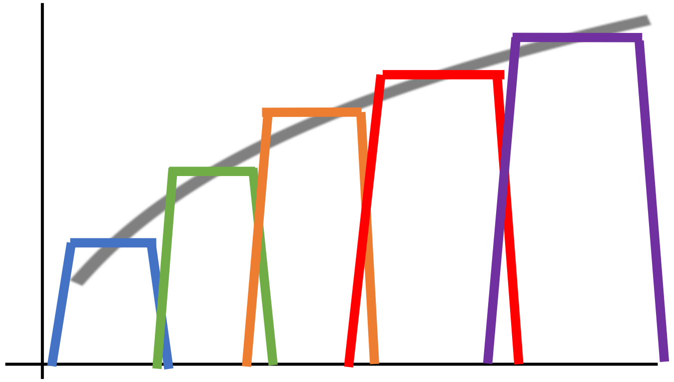


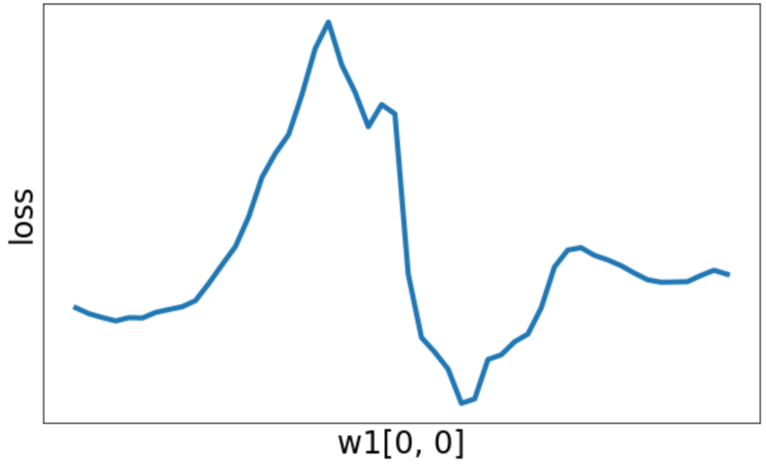
Space Warping



Universal Approximation



Nonconvex



Problem: How to compute gradients?

$s = W_2 f(W_1 x + b_1) + b_2$ Nonlinear score function

$L_i = \sum_{j \neq y_i} \max(0, s_j - s_{y_i} + 1)$ Per-element data loss

$R(W) = \sum_k W_k^2$ L2 Regularization

$L(W_1, W_2, b_1, b_2) = \frac{1}{N} \sum_{i=1}^{N} L_i + \lambda R(W_1) + \lambda R(W_2)$ Total loss

If we can compute $\frac{\partial L}{\partial W_1}, \frac{\partial L}{\partial W_2}, \frac{\partial L}{\partial b_1}, \frac{\partial L}{\partial b_2}$ then we can optimize with SGD

Next: Backpropagation