3. Linear Classifiers

GEV6135 Deep Learning for Visual Recognition and Applications

Kibok Lee

Assistant Professor of

Applied Statistics / Statistics and Data Science

Sep 15, 2022

How to Contact Us

- via CLASSUM
  - All questions about the course should go here.
  - Students are encouraged to ask and answer each other.
  - We will also use this to communicate with you.

LearnUs

- Email: Only for sensitive and/or confidential issues

Assignment 2

- Due **Monday 9/26, 11:59pm KST**

- K-Nearest Neighbors classification

- Please read the instruction carefully!
  - Do **not write or modify any code outside** of the designated blocks.
  - Do **not add or delete cells** from the notebook.
  - Do **not import** additional libraries.
    - + Do not use torch.nn unless instructed.
  - **Run all cells**, and do **not clear out the outputs**, before submitting.
  - Do **not zip by yourself**, run the provided code.

Last time: Image Classification

Input: image

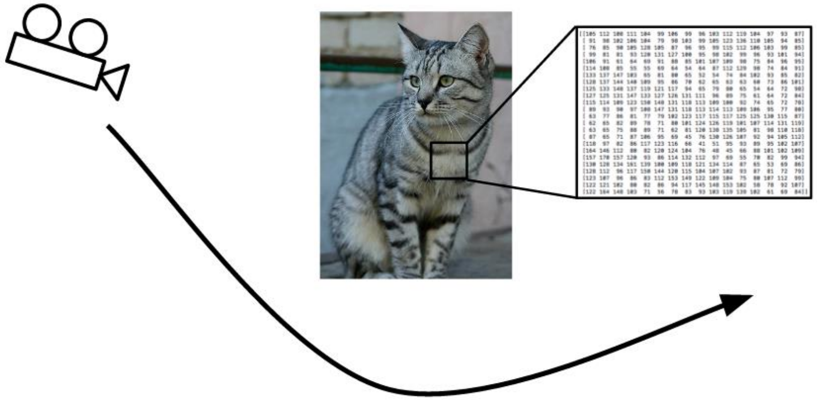
Output: Assign image to one of a fixed set of categories

→

cat
bird
deer
dog
truck

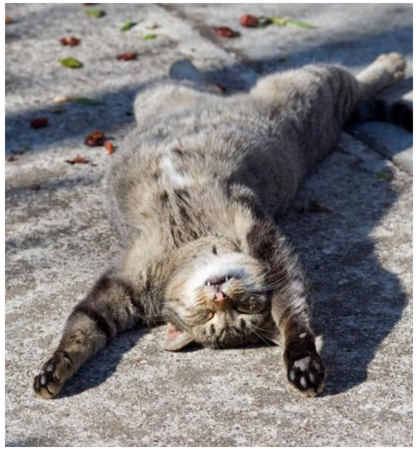
Last time: Challenges of Recognition

Viewpoint



Illumination

Deformation



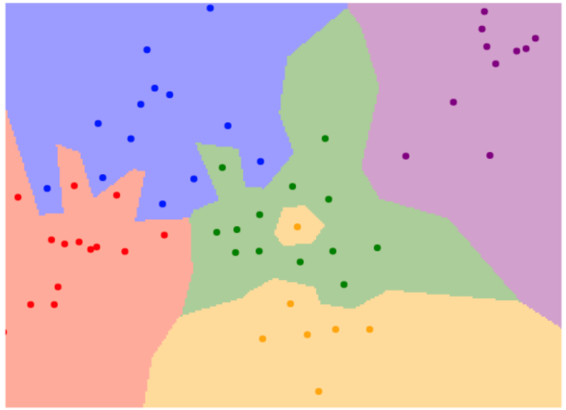
Occlusion

Clutter

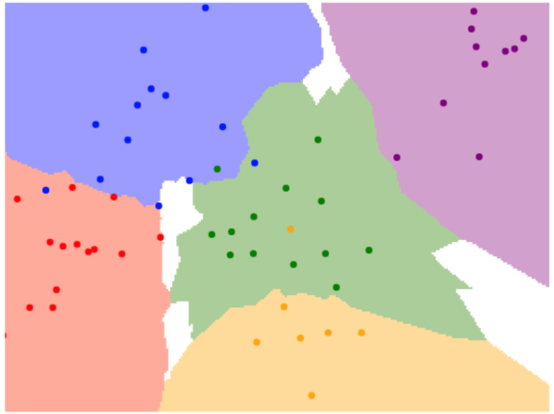
Intraclass Variation

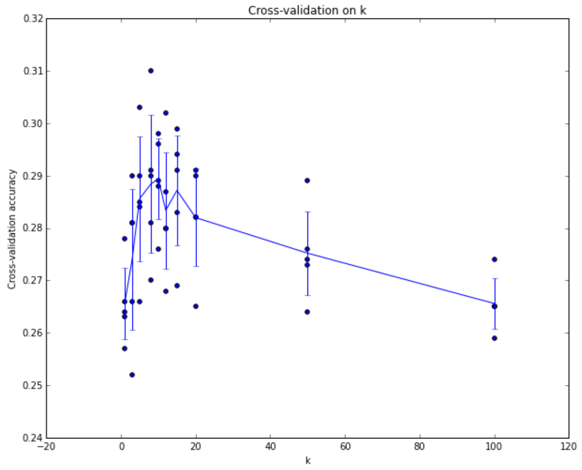
Last time: Data-Drive Approach, kNN

1-NN classifier



5-NN classifier





train	test

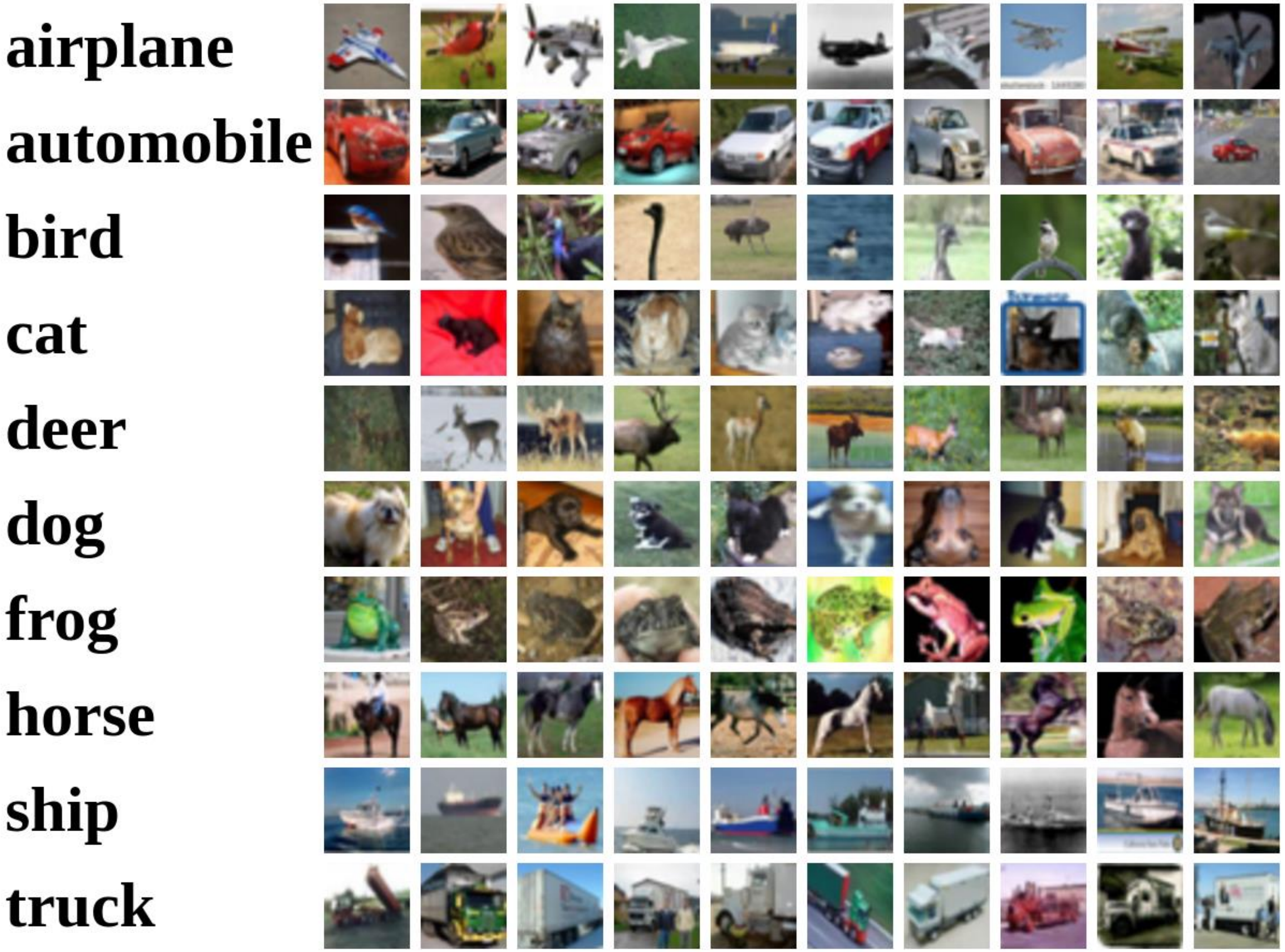
train	validation	test

Today: Linear Classifiers

Neural Network

Linear classifiers

Recall CIFAR10



50,000 training images
each image is **32x32x3**

10,000 test images.

Parametric Approach

Image

Array of **32x32x3** numbers
(3072 numbers total)

f(**x**,**W**)

10 numbers giving
class scores

W
parameters
or weights

Parametric Approach: Linear Classifier

(3072,)

$$\boxed{f(x,W)} = \boxed{W}\boxed{x}$$

(10,) (10, 3072)

Image

Array of **32x32x3** numbers
(3072 numbers total)

f(**x**,**W**)

10 numbers giving class scores

W

parameters or weights

Parametric Approach: Linear Classifier

$$\underbrace{\boxed{f(x,W)}}_{(10,)} = \underbrace{\boxed{W}}_{(10,\ 3072)}\overbrace{\boxed{x}}^{(3072,)} + \boxed{b}\ \ (10,)$$

Image

Array of **32x32x3** numbers (3072 numbers total)

$\longrightarrow$ f(**x**,**W**) $\longrightarrow$ **10** numbers giving class scores

$\uparrow$

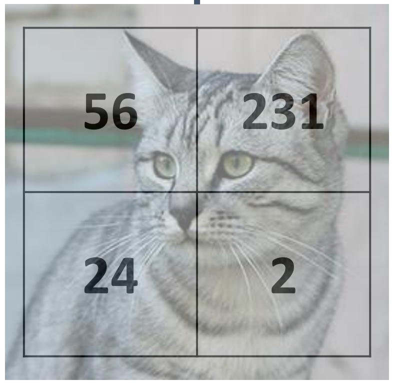
W

parameters or weights

Example for 2x2 image, 3 classes (cat/dog/ship)

Stretch pixels into column

$$f(x,W) = Wx + b$$



Input image
(2, 2)

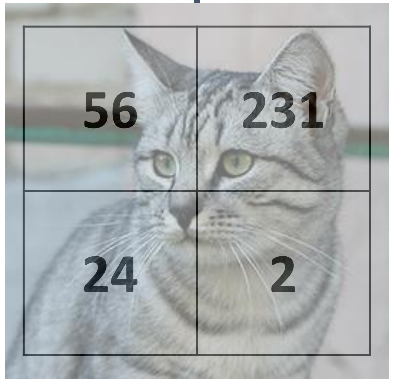
56
231
24
2

(4,)

Example for 2x2 image, 3 classes (cat/dog/ship)

Stretch pixels into column

$f(x,W) = Wx + b$



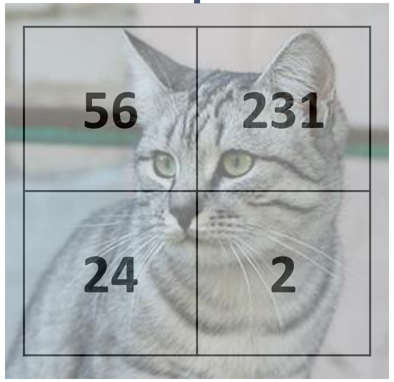
Input image
(2, 2)

$$\underbrace{\begin{bmatrix} 0.2 & -0.5 & 0.1 & 2.0 \\ 1.5 & 1.3 & 2.1 & 0.0 \\ 0 & 0.25 & 0.2 & -0.3 \end{bmatrix}}_{W\ (3,\ 4)} \underbrace{\begin{bmatrix} 56 \\ 231 \\ 24 \\ 2 \end{bmatrix}}_{(4,)} + \underbrace{\begin{bmatrix} 1.1 \\ 3.2 \\ -1.2 \end{bmatrix}}_{b\ (3,)} = \underbrace{\begin{bmatrix} -96.8 \\ 437.9 \\ 61.95 \end{bmatrix}}_{(3,)}$$

Linear Classifier: Algebraic Viewpoint

Stretch pixels into column

f(x,W) = Wx + b



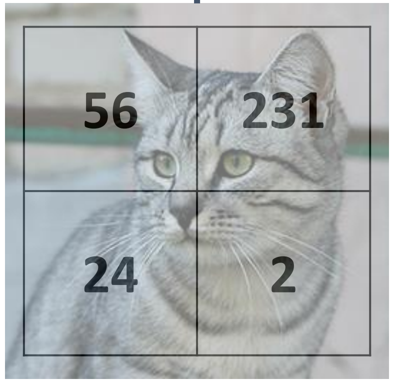
Input image
(2, 2)

$$\underbrace{\begin{bmatrix} 0.2 & -0.5 & 0.1 & 2.0 \\ 1.5 & 1.3 & 2.1 & 0.0 \\ 0 & 0.25 & 0.2 & -0.3 \end{bmatrix}}_{W\ (3,\ 4)} \underbrace{\begin{bmatrix} 56 \\ 231 \\ 24 \\ 2 \end{bmatrix}}_{(4,)} + \underbrace{\begin{bmatrix} 1.1 \\ 3.2 \\ -1.2 \end{bmatrix}}_{b\ (3,)} = \underbrace{\begin{bmatrix} -96.8 \\ 437.9 \\ 61.95 \end{bmatrix}}_{(3,)}$$

Linear Classifier: Bias Trick

Add extra one to data vector; bias is absorbed into last column of weight matrix

Stretch pixels into column



Input image
(2, 2)

0.2	-0.5	0.1	2.0	1.1
1.5	1.3	2.1	0.0	3.2
0	0.25	0.2	-0.3	-1.2

W (3, 5)

56
231
24
2
1

(5,)

=

-96.8
437.9
61.95

(3,)

Linear Classifier: Predictions are Linear!

$$f(x, W) = Wx \quad \text{(ignore bias)}$$

$$f(cx, W) = W(cx) = c * f(x, W)$$

Linear Classifier: Predictions are Linear!

$$f(x, W) = Wx \quad \text{(ignore bias)}$$

$$f(cx, W) = W(cx) = c * f(x, W)$$

Image

Scores

-96.8
437.8
62.0

0.5 * Image

0.5 * Scores

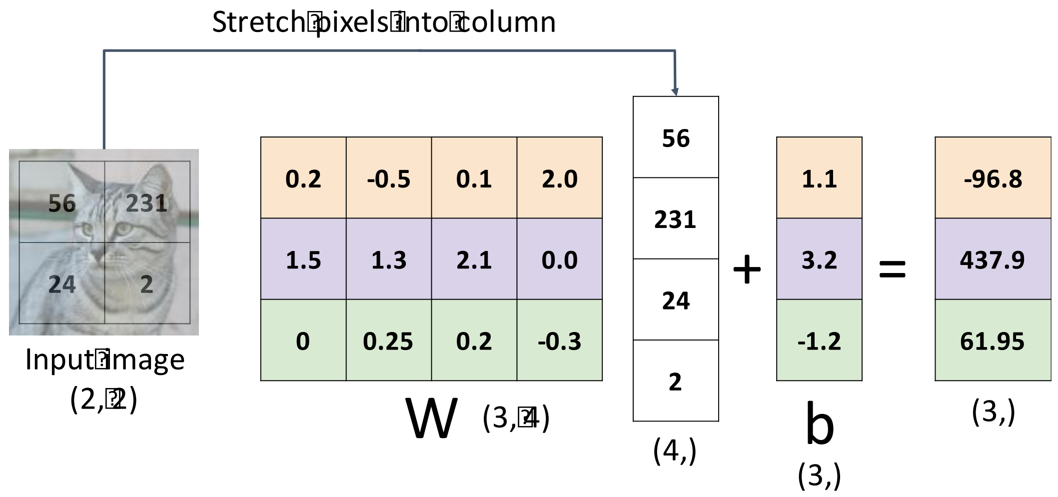
-48.4
218.9
31.0

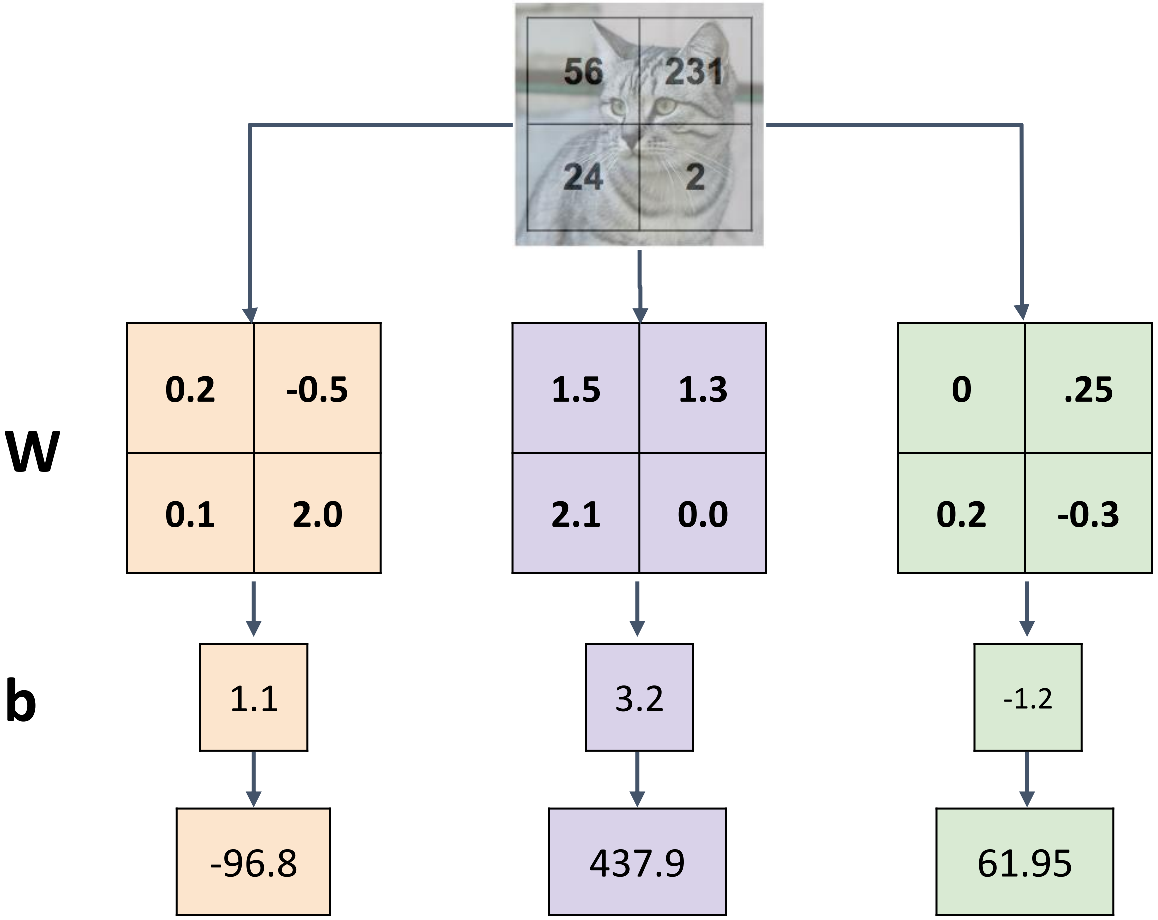
Interpreting a Linear Classifier

Instead of stretching pixels into columns, we can equivalently stretch rows of W into images!

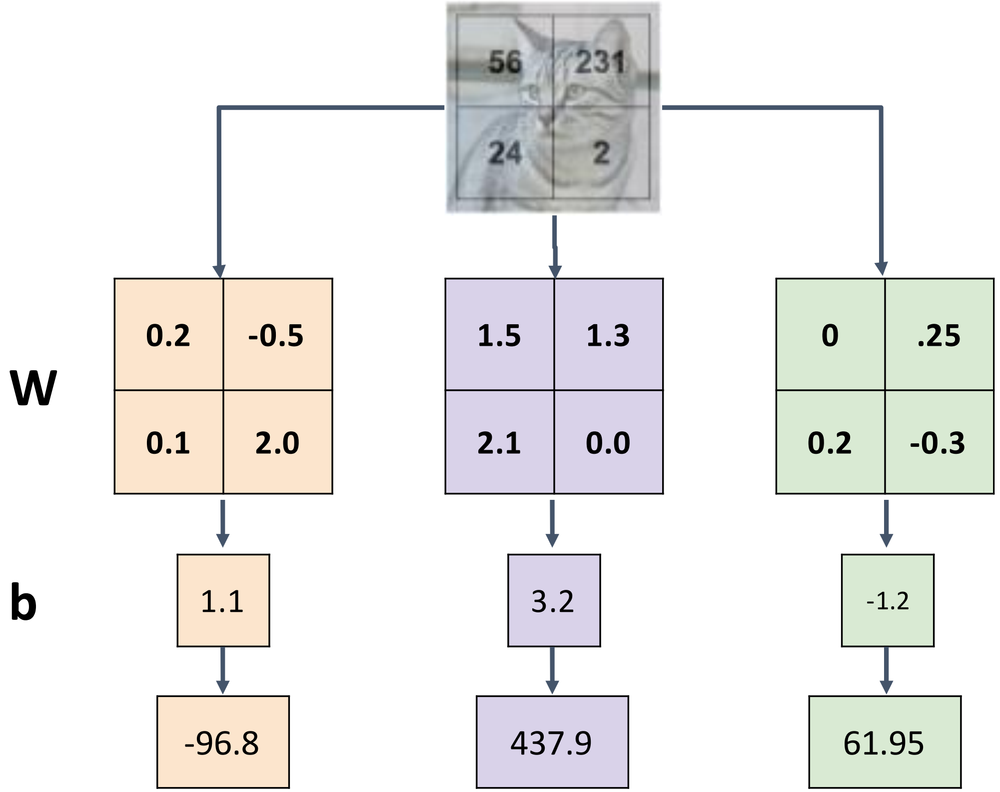
Algebraic Viewpoint

$$f(x,W) = Wx + b$$

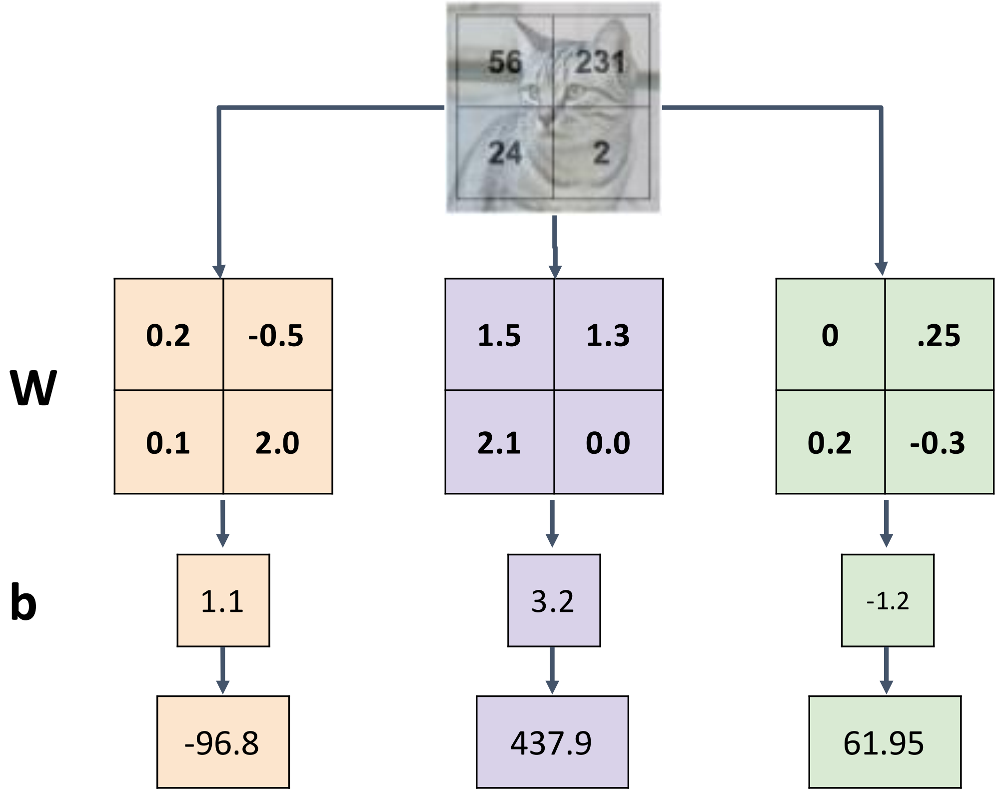




Interpreting a Linear Classifier



Interpreting a Linear Classifier: Visual Viewpoint

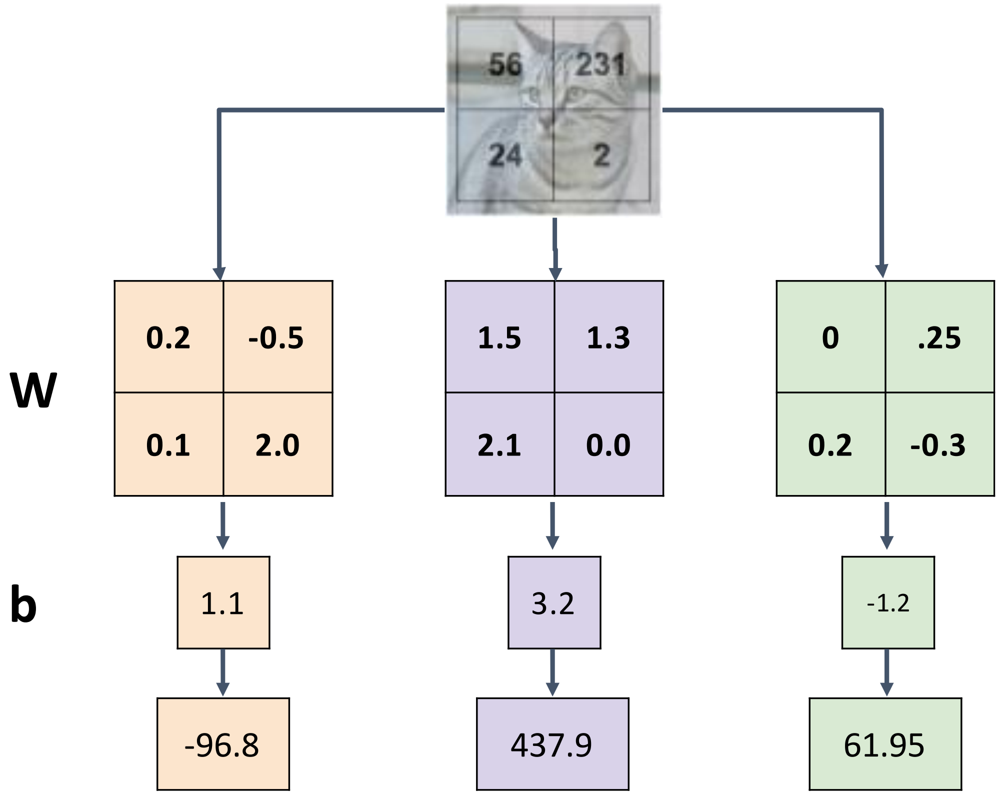


Interpreting a Linear Classifier: Visual Viewpoint

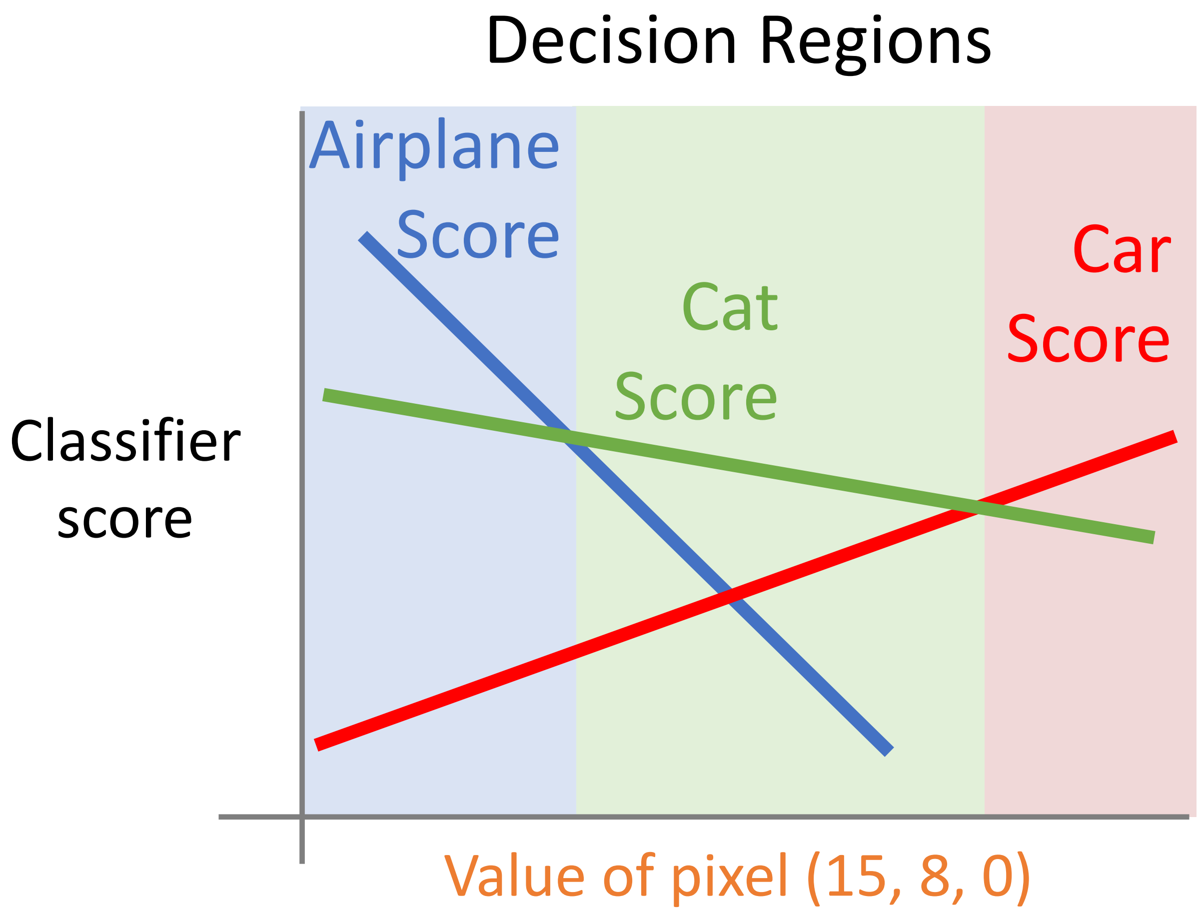
Linear classifier has one "template" per category

A single template cannot capture multiple modes of the data

e.g., horse template has 2 heads!



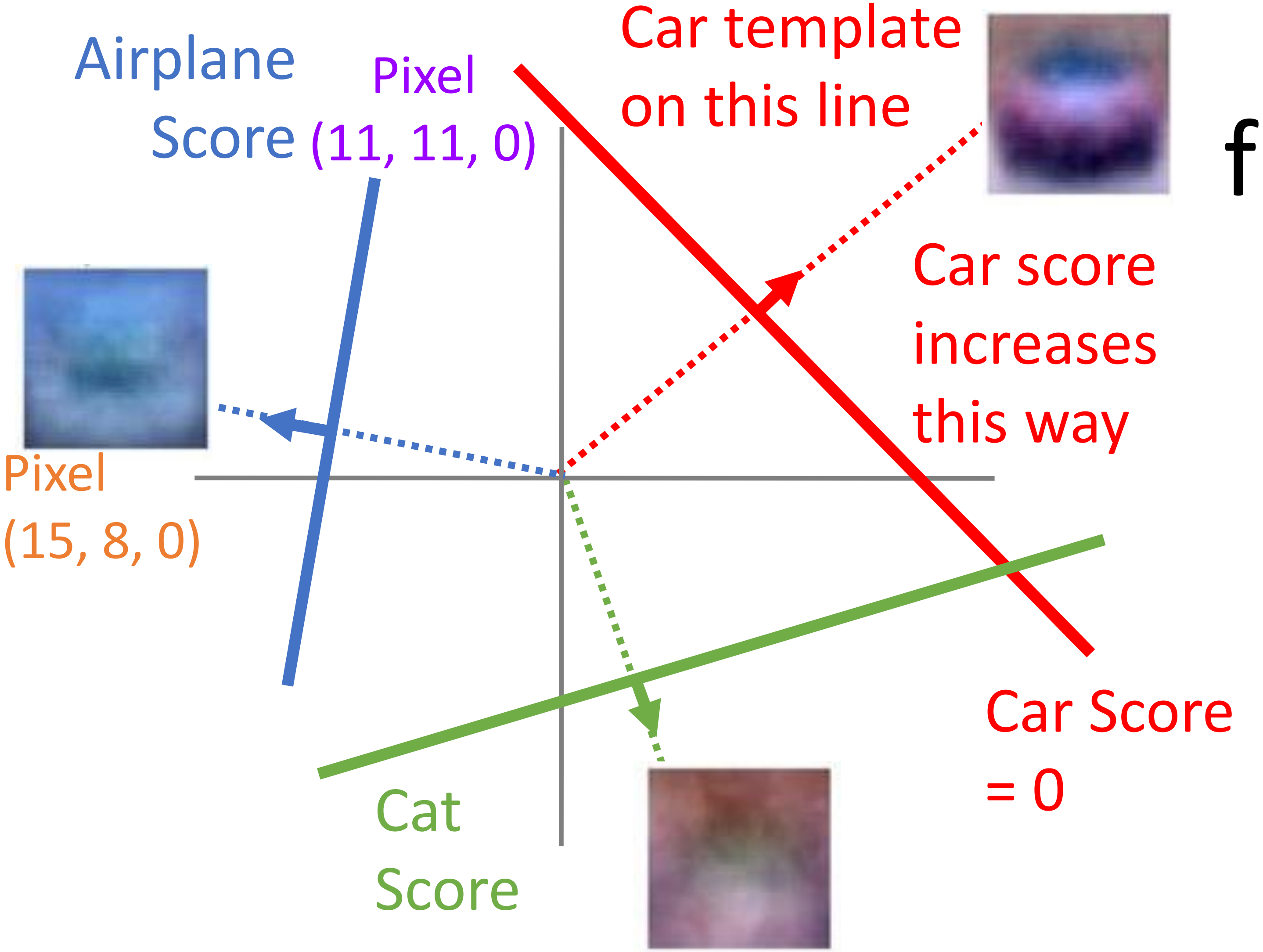
Interpreting a Linear Classifier: Geometric Viewpoint



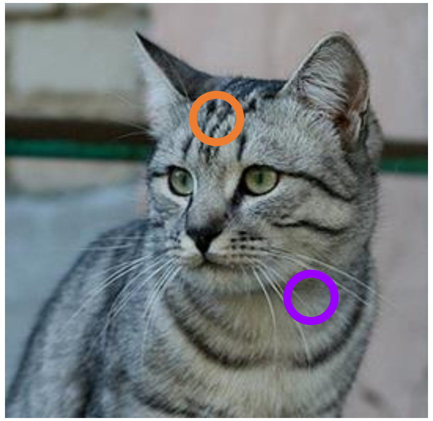
$$f(x,W) = Wx + b$$

Array of **32x32x3** numbers
(3072 numbers total)

Interpreting a Linear Classifier: Geometric Viewpoint

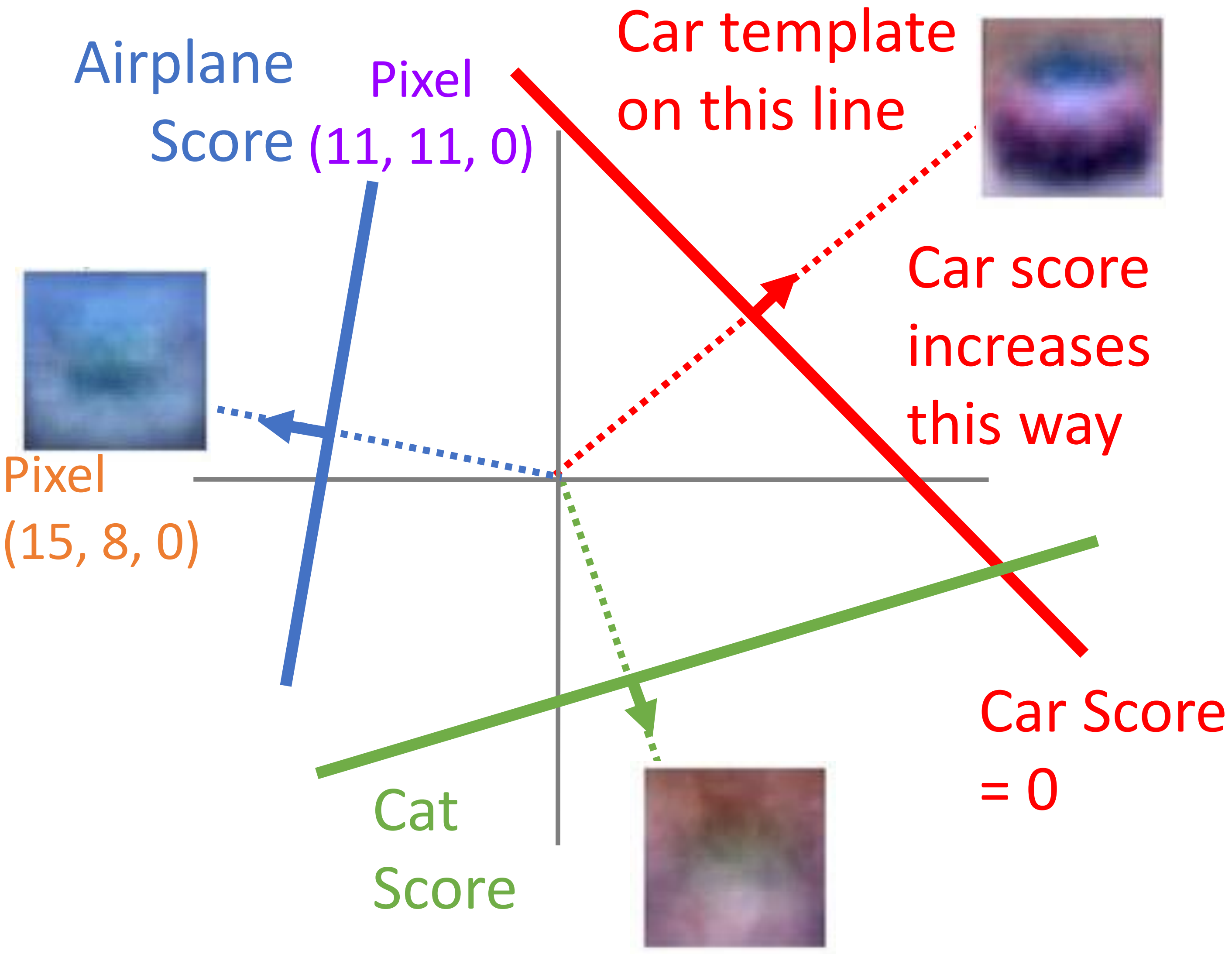


$$f(x,W) = Wx + b$$

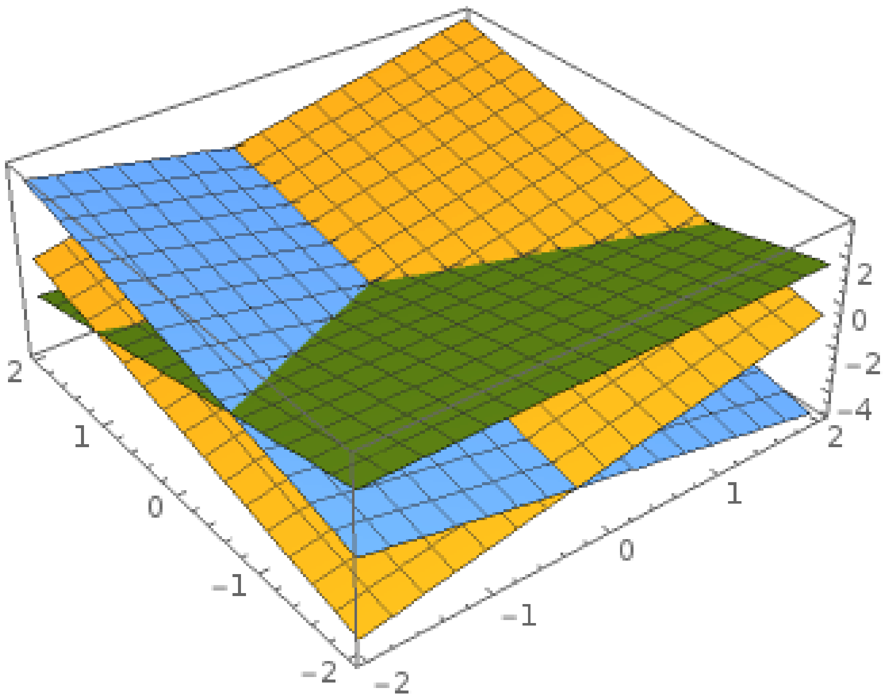


Array of **32x32x3** numbers
(3072 numbers total)

Interpreting a Linear Classifier: Geometric Viewpoint



Hyperplanes carving up a high-dimensional space

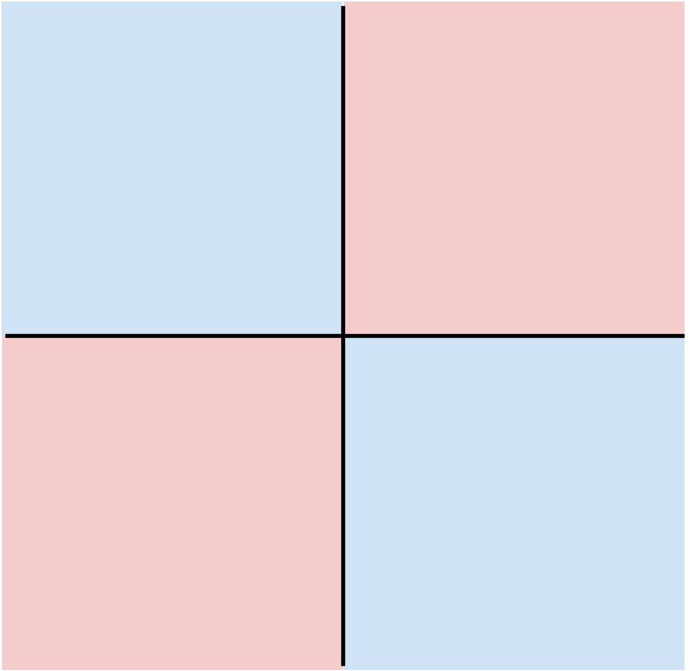


Plot created using [Wolfram Cloud]

Hard Cases for a Linear Classifier

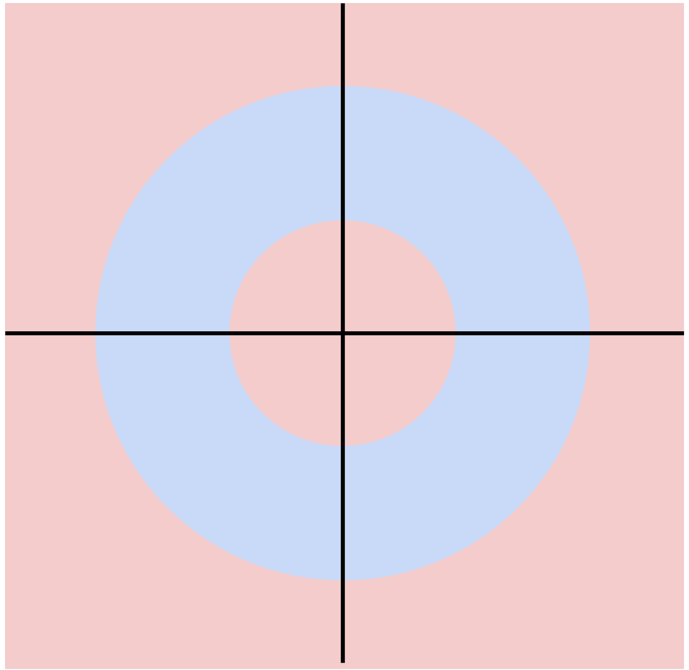
Class 1:
First and third quadrants

Class 2:
Second and fourth quadrants



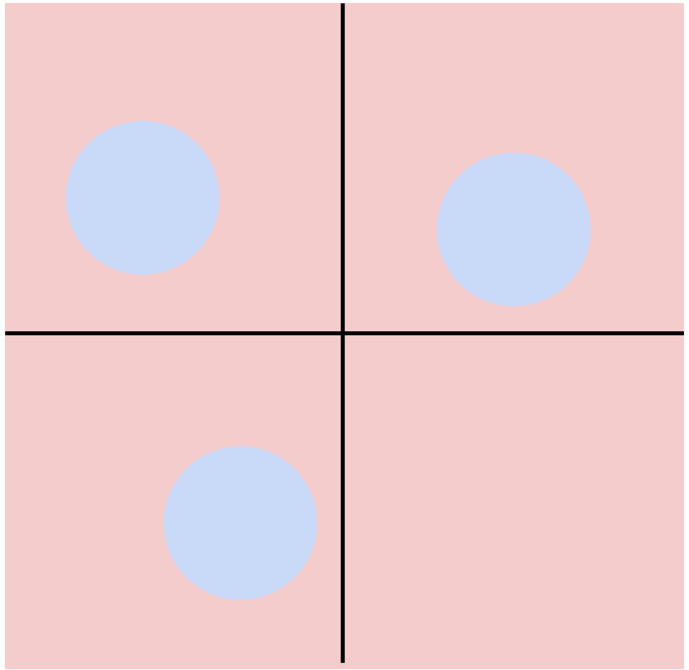
Class 1:
1 <= L2 norm <= 2

Class 2:
Everything else



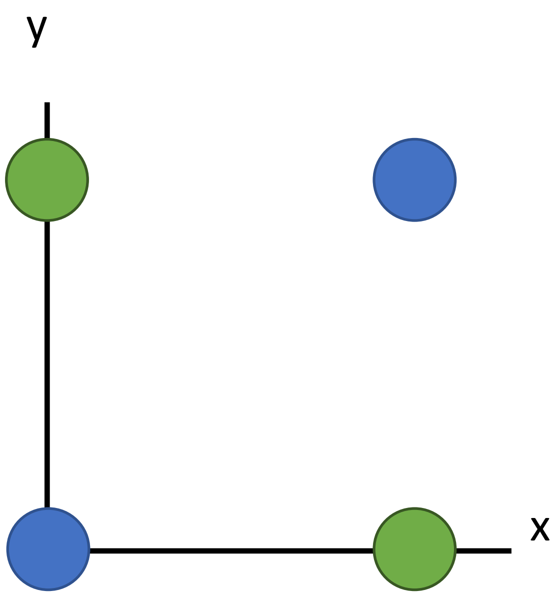
Class 1:
Three modes

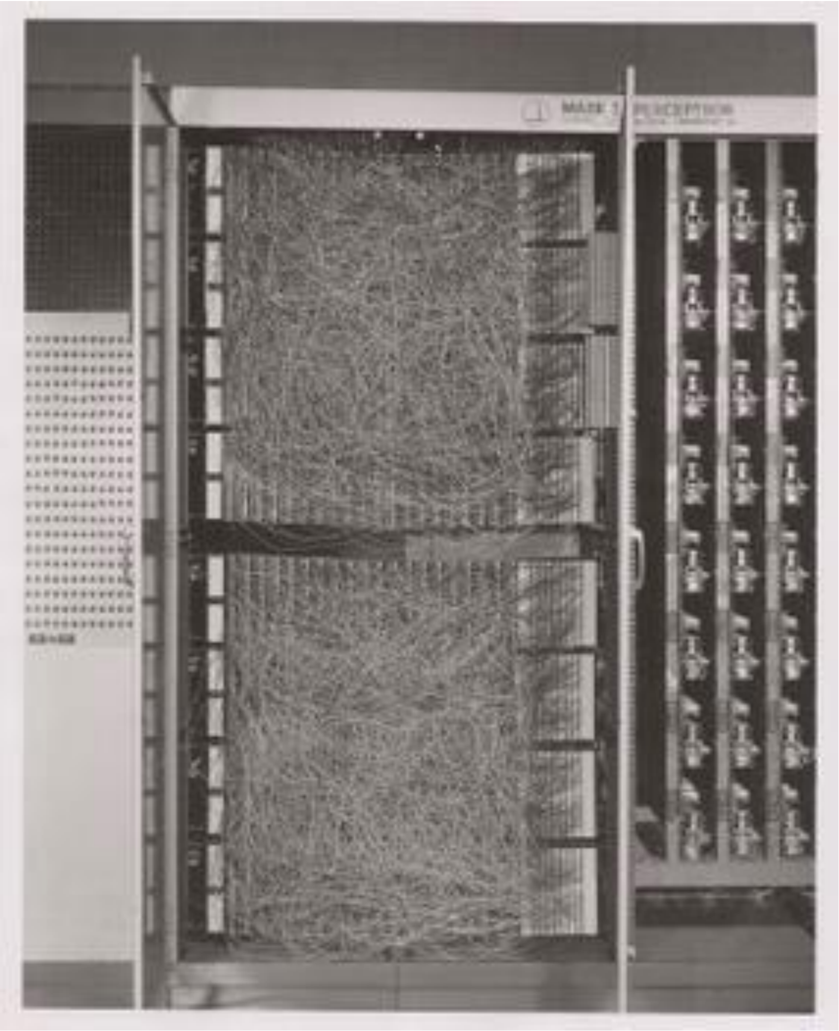
Class 2:
Everything else



Recall: Perceptron couldn't learn XOR

X	Y	F(x,y)
0	0	0
0	1	1
1	0	1
1	1	0

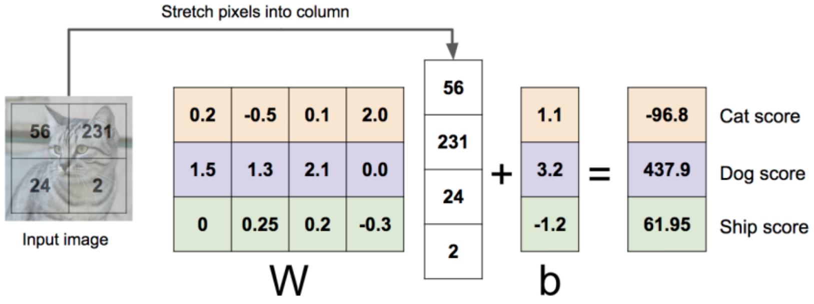




Linear Classifier: Three Viewpoints

<u>Algebraic Viewpoint</u>

f(x,W) = Wx

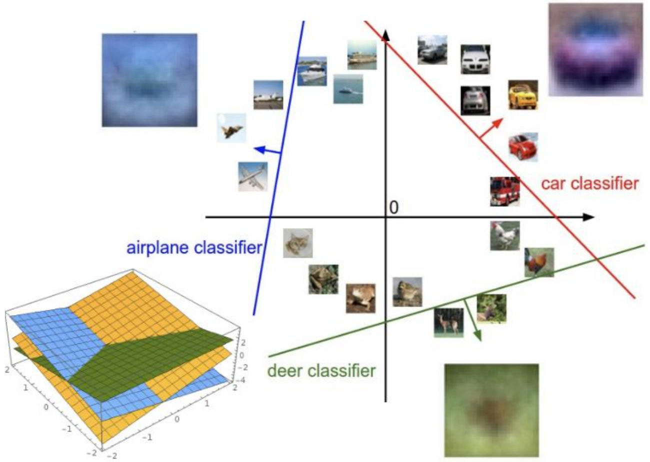


<u>Visual Viewpoint</u>

One template
per class

<u>Geometric Viewpoint</u>

Hyperplanes
cutting up space



So Far: Defined a linear **score function**

$f(x,W) = Wx + b$

airplane	-3.45	-0.51	3.42
automobile	-8.87	**6.04**	4.64
bird	0.09	5.31	2.65
cat	**2.9**	-4.22	5.1
deer	4.48	-4.19	2.64
dog	8.02	3.58	5.55
frog	3.78	4.49	**-4.34**
horse	1.06	-4.37	-1.5
ship	-0.36	-2.09	-4.79
truck	-0.72	-2.93	6.14

Given a W, we can compute class scores for an image x.

But how can we actually choose a good W?

Choosing a good W

$f(x,W) = Wx + b$



airplane	-3.45	-0.51	3.42
automobile	-8.87	**6.04**	4.64
bird	0.09	5.31	2.65
cat	**2.9**	-4.22	5.1
deer	4.48	-4.19	2.64
dog	8.02	3.58	5.55
frog	3.78	4.49	**-4.34**
horse	1.06	-4.37	-1.5
ship	-0.36	-2.09	-4.79
truck	-0.72	-2.93	6.14

TODO:

1. Use a **loss function** to quantify how good a value of W is

2. Find a W that minimizes the loss function (**optimization**)

Loss Function

A **loss function** tells how good our current classifier is

Low loss = good classifier
High loss = bad classifier

(Also called: **objective function**; **cost function**)

Negative loss function sometimes called **reward function**, **profit function**, **utility function**, **fitness function**, etc

Given a dataset of examples

$$\{(x_i, y_i)\}_{i=1}^{N}$$

Where x_i is image and
y_i is (integer) label

Loss for a single example is

$$L_i(f(x_i, W), y_i)$$

Loss for the dataset is average of per-example losses:

$$L = \frac{1}{N}\sum_i L_i(f(x_i, W), y_i)$$

Cross-Entropy Loss (Multinomial Logistic Regression)

Want to interpret raw classifier scores as **probabilities**

$$s = f(x_i; W) \qquad P(Y = k \mid X = x_i) = \frac{\exp(s_k)}{\sum_j \exp(s_j)}$$

Softmax function

cat	**3.2**
car	5.1
frog	-1.7

Cross-Entropy Loss (Multinomial Logistic Regression)

Want to interpret raw classifier scores as **probabilities**

$$\boxed{s = f(x_i; W)}$$

$$\boxed{P(Y = k \mid X = x_i) = \frac{\exp(s_k)}{\sum_j \exp(s_j)}}$$ Softmax function

Probabilities must be >= 0

Probabilities must sum to 1

cat	**3.2**	**24.5**	**0.13**
car	5.1	164.0	0.87
frog	-1.7	0.18	0.00

$\xrightarrow{\text{exp}}$ $\xrightarrow{\text{normalize}}$

Unnormalized log-probabilities / logits

unnormalized probabilities

probabilities

Cross-Entropy Loss (Multinomial Logistic Regression)

Want to interpret raw classifier scores as **probabilities**

$$s = f(x_i; W)$$

$$P(Y = k \mid X = x_i) = \frac{\exp(s_k)}{\sum_j \exp(s_j)}$$ Softmax function

$$L_i = -\log P(Y = y_i \mid X = x_i)$$

Probabilities must be >= 0

Probabilities must sum to 1

	Unnormalized log-probabilities / logits		unnormalized probabilities		probabilities
cat	**3.2**	exp →	**24.5**	normalize →	**0.13**
car	5.1		164.0		0.87
frog	-1.7		0.18		0.00

L_i = -log(0.13)
= **2.04**

Maximum Likelihood Estimation
Choose weights to maximize the likelihood of the observed data

Cross-Entropy Loss (Multinomial Logistic Regression)

Want to interpret raw classifier scores as **probabilities**

$$s = f(x_i; W)$$

$$P(Y = k \mid X = x_i) = \frac{\exp(s_k)}{\sum_j \exp(s_j)}$$ Softmax function

$$L_i = -\log P(Y = y_i \mid X = x_i)$$

Probabilities must be >= 0

Probabilities must sum to 1

	Unnormalized log-probabilities / logits		unnormalized probabilities		probabilities		Correct probs
cat	**3.2**	exp →	**24.5**	normalize →	**0.13**	→ Compare ←	**1.00**
car	5.1		164.0		0.87		0.00
frog	-1.7		0.18		0.00		0.00

Kullback–Leibler divergence

$$D_{KL}(\boldsymbol{Q} \,||\, \boldsymbol{P}) = \sum_y \boldsymbol{Q}(\boldsymbol{y}) \log \frac{\boldsymbol{Q}(\boldsymbol{y})}{\boldsymbol{P}(\boldsymbol{y})}$$

Cross-Entropy Loss (Multinomial Logistic Regression)

Want to interpret raw classifier scores as **probabilities**

$$s = f(x_i; W)$$

$$P(Y = k \mid X = x_i) = \frac{\exp(s_k)}{\sum_j \exp(s_j)}$$

Softmax function

Probabilities must be >= 0

Probabilities must sum to 1

$$L_i = -\log P(Y = y_i \mid X = x_i)$$

		→ exp →	→ normalize →	Compare ←	
cat	**3.2**	**24.5**	**0.13**		**1.00**
car	5.1	164.0	0.87	*Cross Entropy*	0.00
frog	-1.7	0.18	0.00		0.00

Unnormalized log-probabilities / logits

unnormalized probabilities

probabilities

$$H(P, Q) = H(Q) + D_{KL}(Q \,||\, P)$$

Correct probs

Cross-Entropy Loss (Multinomial Logistic Regression)

cat	**3.2**
car	5.1
frog	-1.7

Want to interpret raw classifier scores as **probabilities**

$$\boxed{s = f(x_i; W)}$$

$$\boxed{P(Y = k \mid X = x_i) = \frac{\exp(s_k)}{\sum_j \exp(s_j)}}$$

Softmax function

Maximize probability of correct class

$$\boxed{L_i = -\log P(Y = y_i \mid X = x_i)}$$

Putting it all together:

$$\boxed{L_i = -\log\left(\frac{\exp(s_{y_i})}{\sum_j \exp(s_j)}\right)}$$

Q: What is the min / max possible loss L_i?

A: Min 0, max +infinity

Cross-Entropy Loss (Multinomial Logistic Regression)

Want to interpret raw classifier scores as **probabilities**

cat	**3.2**
car	5.1
frog	-1.7

$$s = f(x_i; W)$$

$$P(Y = k \mid X = x_i) = \frac{\exp(s_k)}{\sum_j \exp(s_j)}$$

Softmax function

Maximize probability of correct class

$$L_i = -\log P(Y = y_i \mid X = x_i)$$

Putting it all together:

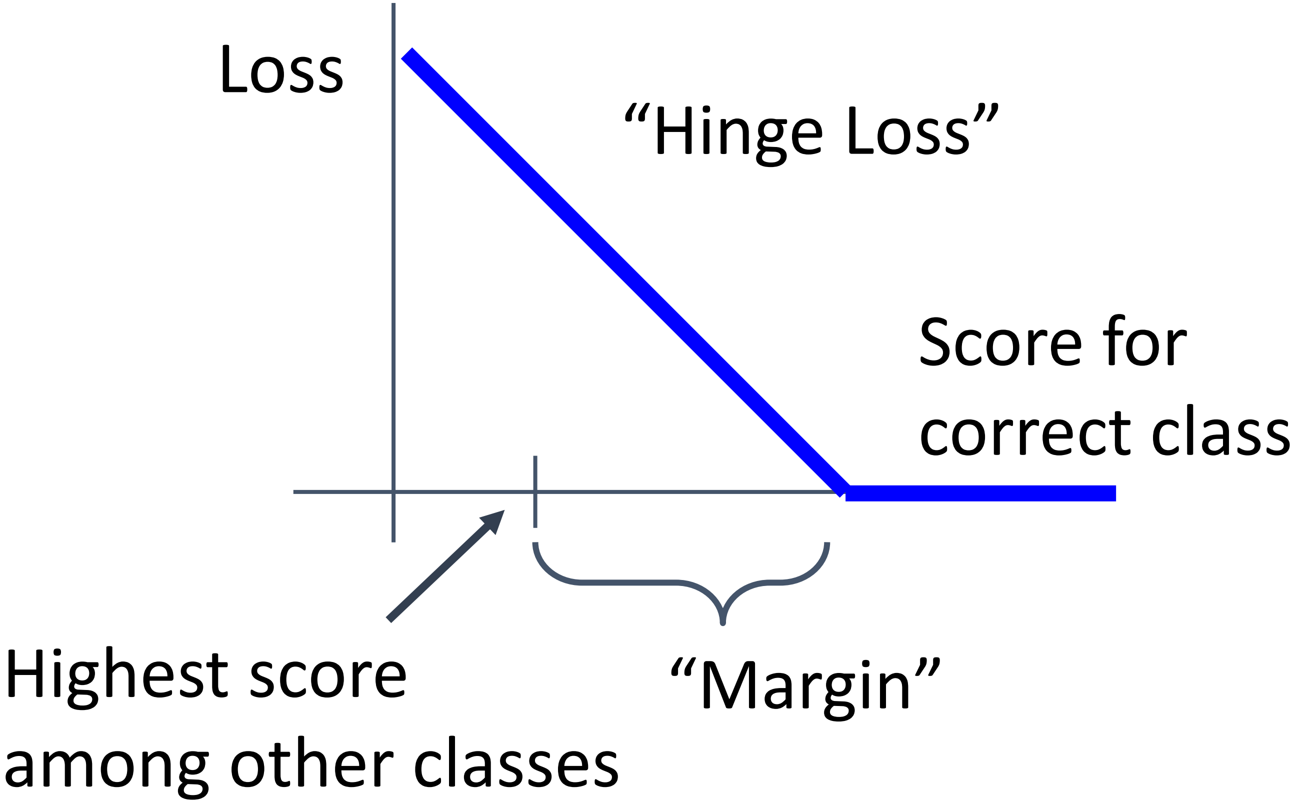
$$L_i = -\log\left(\frac{\exp(s_{y_i})}{\sum_j \exp(s_j)}\right)$$

Q: If all scores are small random values, what is the loss?

A: -log(1/C)
log(10) ≈ 2.3

Multiclass SVM Loss

"The score of the correct class should be higher than all the other scores"



Given an example (x_i, y_i)
(x_i is image, y_i is label)

Let $s = f(x_i, W)$ be scores

Then the SVM loss has the form:

$$L_i = \sum_{j \neq y_i} \max(0, s_j - s_{y_i} + 1)$$

Multiclass SVM Loss

cat	**3.2**	1.3	2.2
car	5.1	**4.9**	2.5
frog	-1.7	2.0	**-3.1**

Given an example (x_i, y_i)
(x_i is image, y_i is label)

Let $s = f(x_i, W)$ be scores

Then the SVM loss has the form:

$$L_i = \sum_{j \neq y_i} \max(0, s_j - s_{y_i} + 1)$$

Multiclass SVM Loss

			
cat	**3.2**	1.3	2.2
car	5.1	**4.9**	2.5
frog	-1.7	2.0	**-3.1**
Loss	2.9		

Given an example (x_i, y_i)
(x_i is image, y_i is label)

Let $s = f(x_i, W)$ be scores

Then the SVM loss has the form:

$$L_i = \sum_{j \neq y_i} \max(0, s_j - s_{y_i} + 1)$$

= max(0, 5.1 - 3.2 + 1)
+ max(0, -1.7 - 3.2 + 1)
= max(0, 2.9) + max(0, -3.9)
= 2.9 + 0
= 2.9

Multiclass SVM Loss

cat	**3.2**	1.3	2.2
car	5.1	**4.9**	2.5
frog	-1.7	2.0	**-3.1**
Loss	2.9	0	

Given an example (x_i, y_i)
(x_i is image, y_i is label)

Let $s = f(x_i, W)$ be scores

Then the SVM loss has the form:

$$L_i = \sum_{j \neq y_i} \max(0, s_j - s_{y_i} + 1)$$

= max(0, 1.3 - 4.9 + 1)
+max(0, 2.0 - 4.9 + 1)
= max(0, -2.6) + max(0, -1.9)
= 0 + 0
= 0

Multiclass SVM Loss

cat	**3.2**	1.3	2.2
car	5.1	**4.9**	2.5
frog	-1.7	2.0	**-3.1**
Loss	2.9	0	12.9

Given an example (x_i, y_i)
(x_i is image, y_i is label)

Let $s = f(x_i, W)$ be scores

Then the SVM loss has the form:

$$L_i = \sum_{j \neq y_i} \max(0, s_j - s_{y_i} + 1)$$

= max(0, 2.2 - (-3.1) + 1)
+max(0, 2.5 - (-3.1) + 1)
= max(0, 6.3) + max(0, 6.6)
= 6.3 + 6.6
= 12.9

Multiclass SVM Loss

cat	**3.2**	1.3	2.2
car	5.1	**4.9**	2.5
frog	-1.7	2.0	**-3.1**
Loss	2.9	0	12.9

Given an example (x_i, y_i)
(x_i is image, y_i is label)

Let $s = f(x_i, W)$ be scores

Then the SVM loss has the form:

$$L_i = \sum_{j \neq y_i} \max(0, s_j - s_{y_i} + 1)$$

Loss over the dataset is:

L = (2.9 + 0.0 + 12.9) / 3
= 5.27

Multiclass SVM Loss

			
cat	**3.2**	1.3	2.2
car	5.1	**4.9**	2.5
frog	-1.7	2.0	**-3.1**
Loss	2.9	0	12.9

Given an example (x_i, y_i)
(x_i is image, y_i is label)

Let $s = f(x_i, W)$ be scores

Then the SVM loss has the form:

$$L_i = \sum_{j \neq y_i} \max(0, s_j - s_{y_i} + 1)$$

Q: What happens to the loss if the scores for the car image change a bit?

Multiclass SVM Loss



cat	**3.2**	1.3	2.2
car	5.1	**4.9**	2.5
frog	-1.7	2.0	**-3.1**
Loss	2.9	0	12.9

Given an example (x_i, y_i)
(x_i is image, y_i is label)

Let $s = f(x_i, W)$ be scores

Then the SVM loss has the form:

$$L_i = \sum_{j \neq y_i} \max(0, s_j - s_{y_i} + 1)$$

Q2: What are the min and max possible loss?

Multiclass SVM Loss



cat	**3.2**	1.3	2.2
car	5.1	**4.9**	2.5
frog	-1.7	2.0	**-3.1**
Loss	2.9	0	12.9

Given an example (x_i, y_i)
(x_i is image, y_i is label)

Let $s = f(x_i, W)$ be scores

Then the SVM loss has the form:

$$L_i = \sum_{j \neq y_i} \max(0, s_j - s_{y_i} + 1)$$

Q3: If all the scores were random, what loss would we expect?

Multiclass SVM Loss



cat	**3.2**	1.3	2.2
car	5.1	**4.9**	2.5
frog	-1.7	2.0	**-3.1**
Loss	2.9	0	12.9

Given an example (x_i, y_i)
(x_i is image, y_i is label)

Let $s = f(x_i, W)$ be scores

Then the SVM loss has the form:

$$L_i = \sum_{j \neq y_i} \max(0, s_j - s_{y_i} + 1)$$

Q4: What would happen if the sum were over all classes? (including $j = y_i$)

Multiclass SVM Loss

cat	**3.2**	1.3	2.2
car	5.1	**4.9**	2.5
frog	-1.7	2.0	**-3.1**
Loss	2.9	0	12.9

Given an example (x_i, y_i)
(x_i is image, y_i is label)

Let $s = f(x_i, W)$ be scores

Then the SVM loss has the form:

$$L_i = \sum_{j \neq y_i} \max(0, s_j - s_{y_i} + 1)$$

Q5: What if the loss used a mean instead of a sum?

Multiclass SVM Loss



cat	**3.2**	1.3	2.2
car	5.1	**4.9**	2.5
frog	-1.7	2.0	**-3.1**
Loss	2.9	0	12.9

Given an example (x_i, y_i)
(x_i is image, y_i is label)

Let $s = f(x_i, W)$ be scores

Then the SVM loss has the form:

$$L_i = \sum_{j \neq y_i} \max(0, s_j - s_{y_i} + 1)$$

Q6: What if we used this loss instead?

$$L_i = \sum_{j \neq y_i} \max(0, s_j - s_{y_i} + 1)^2$$

Cross-Entropy vs SVM Loss

$$L_i = -\log\left(\frac{\exp(s_{y_i})}{\sum_j \exp(s_j)}\right)$$

$$L_i = \sum_{j \neq y_i} \max(0, s_j - s_{y_i} + 1)$$

assume scores:

[10, -2, 3]

[10, 9, 9]

[10, -100, -100]

and $y_i = 0$

Q: What is cross-entropy loss?
What is SVM loss?

A: Cross-entropy loss > 0
SVM loss = 0

Cross-Entropy vs SVM Loss

$$L_i = -\log\left(\frac{\exp(s_{y_i})}{\sum_j \exp(s_j)}\right)$$

$$L_i = \sum_{j \neq y_i} \max(0, s_j - s_{y_i} + 1)$$

assume scores:

[10, -2, 3]

[10, 9, 9]

[10, -100, -100]

and $\boxed{y_i = 0}$

Q: What happens to each loss if I slightly change the scores of the last data point?

A: Cross-entropy loss will change; SVM loss will stay the same

Cross-Entropy vs SVM Loss

$$\boxed{L_i = -\log\left(\frac{\exp(s_{y_i})}{\sum_j \exp(s_j)}\right)}$$

$$\boxed{L_i = \sum_{j \neq y_i} \max(0, s_j - s_{y_i} + 1)}$$

assume scores:

[10, -2, 3]

[10, 9, 9]

[10, -100, -100]

and $\boxed{y_i = 0}$

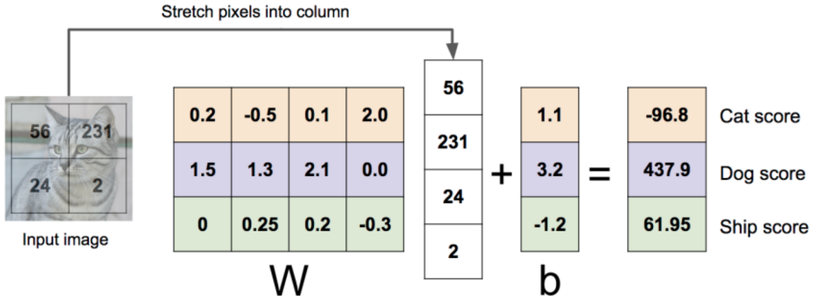
Q: What happens to each loss if I double the score of the correct class from 10 to 20?

A: Cross-entropy loss will decrease, SVM loss still 0

Recap: Three ways to think about linear classifiers

Algebraic Viewpoint

$f(x,W) = Wx$

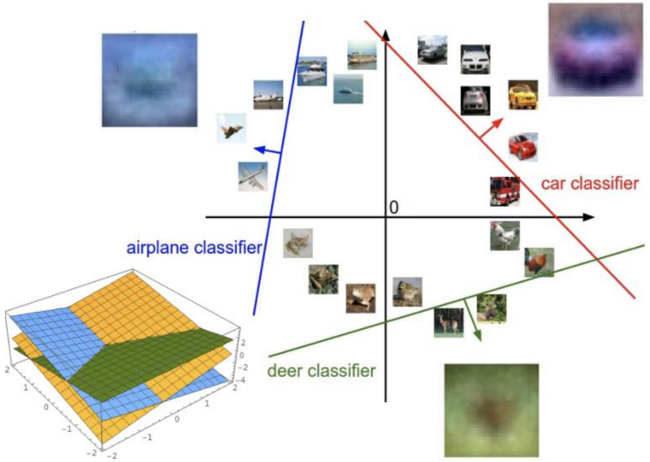


Visual Viewpoint

One template
per class

Geometric Viewpoint

Hyperplanes
cutting up space



Recap: Loss Functions quantify preferences

- We have some dataset of (x, y)
- We have a **score function:**
- We have a **loss function**:

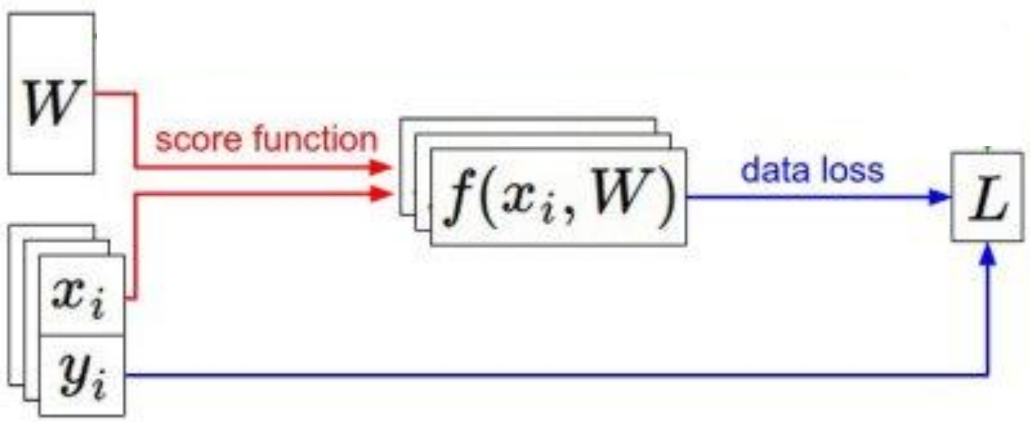
Q: How do we find the best W, b?

$$s = f(x; W, b) = Wx + b$$

Linear classifier

Softmax: $L_i = -\log\left(\frac{\exp(s_{y_i})}{\sum_j \exp(s_j)}\right)$

SVM: $L_i = \sum_{j \neq y_i} \max(0, s_j - s_{y_i} + 1)$



Next: Regularization, Optimization