Introduction to Fourier Analysis

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Preface

This book is based on the class notes of a third year, undergraduate Fourier Analysis course offered by the Pure Math Department at the University of Waterloo, Waterloo, Ontario, Canada. The course was taught by professor J. A. Baker during the winter term (January to April) 1994. The course (PMATH 353) is no longer available but its contents are offered under a different heading and course number.

A basic understanding of undergraduate calculus, real analysis and complex analysis are required to read this book. However, the first chapter reviews several theorems of real and complex analysis that are used extensively throughout this book.

There were nine assignments handed out by professor J. A. Baker during this course. The questions of these assignments are included as examples in this book, along with my solutions to them.

There was no official textbook for this course. However, professor Baker highly recommended the book *Fourier Analysis* by *T.W. Körner*. There is a companion text called *Exercises in Fourier Analysis* also by the same author.

Please report any errors or suggestions for improvement to me at the email address noted below. Thank you.

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Chapter 1

Theorems of Real and Complex Analysis

This chapter begins with a review of the calculus of complex-valued functions. It then goes on to present various theorems of real and complex analysis which are used throughout the book. The reader may already be familiar with some (or all) of these results and can skip the corresponding sections.

1.1 Calculus of Complex-Valued Functions

Proposition 1.1.1. Suppose $f : [a, b] \to \mathbb{C}$ for some interval $[a, b]$ in $\mathbb{R}$. Let $u(t) = \mathrm{Re} f(t)$ and $v(t) = \mathrm{Im} f(t)$, so that $f(t) = u(t) + iv(t)$. Also, suppose $c = a + ib$, for some $c \in \mathbb{C}$ and $a \leq t_0 \leq b$. Then

i) $\lim_{t\to t_0} f(t) = c$ iff $\lim_{t\to t_0} u(t) = a$ and $\lim_{t\to t_0} v(t) = b$.

ii) f is continuous at t_0 iff u and v are.

Proof. i)

$$|f(t) - c| = |(u(t) - a) + i(v(t) - b)| \leq |u(t) - a| + |v(t) - b|$$

Therefore,

$$\lim_{t\to t_0} f(t) = c \text{ if } \lim_{t\to t_0} u(t) = a \text{ and } \lim_{t\to t_0} v(t) = b$$

Conversely, we have the inequalities

$$|u(t) - a| \leq |f(t) - c|$$

and

$$|v(t) - b| \leq |f(t) - c|$$

from which it follows that $\lim_{t \to t_0} u(t) = a$ and $\lim_{t \to t_0} v(t) = b$ if $\lim_{t \to t_0} f(t) = c$.

ii) This follows immediately from i).

□

Definition 1.1.2. If f, t_0, u and v, are defined as above, we say that f is *differentiable* at t_0 if u and v are differentiable at t_0 (in the sense of real-valued functions). In this case, we define $f'(t_0) = u'(t_0) + iv'(t_0)$. This is the same as saying,

$$f'(t_0) = \lim_{t \to t_0} \frac{f(t) - f(t_0)}{t - t_0}$$

Definition 1.1.3. We say $f : [a, b] \to \mathbb{C}$ is *Riemann Integrable* on $[a, b]$ provided both $u = \mathrm{Re} f$ and $v = \mathrm{Im} f$ are (in the sense of real-valued functions). In this case, we define the integral of f to be

$$\int_a^b f = \int_a^b u + i \int_a^b v$$

Most rules from the calculus of real-valued functions carry over to that of complex-valued functions. For example, the Fundamental Theorem of Calculus, the product rule for derivatives and integration by parts all hold.

Proposition 1.1.4. Suppose $f : [a, b] \to \mathbb{C}$ is Riemann Integrable on its domain, $\lambda \in \mathbb{C}$ and $g = \lambda f$. Then g is Riemann Integrable on $[a, b]$ and

$$\int_a^b g = \lambda \int_a^b f$$

Proof. Let $u(t) = \mathrm{Re} f(t)$, $v(t) = \mathrm{Im} f(t)$, and $\lambda = \alpha + i\beta$. Then

$$g(t) = [\alpha u(t) - \beta v(t)] + i[\alpha v(t) + \beta u(t)]$$

Hence g is Riemann Integrable on $[a, b]$ and

$$\begin{aligned}\int_a^b g(t)\,\mathrm{d}t &= \int_a^b [\alpha u(t) - \beta v(t)]\,\mathrm{d}t + i\int_a^b [\alpha v(t) + \beta u(t)]\,\mathrm{d}t \\ &= (\alpha + i\beta)\left(\int_a^b u(t)\,\mathrm{d}t + i\int_a^b v(t)\,\mathrm{d}t\right) \\ &= \lambda\int_a^b f(t)\,\mathrm{d}t\end{aligned}$$

This result, along with the obvious fact that

$$\int_a^b (f+g) = \int_a^b f + \int_a^b g$$

shows that integration is a linear operator on the space of complex-valued, Riemann Integrable functions (just as it is on the space of real-valued functions). □

Proposition 1.1.5. If $a < b < c$ and $f : [a, c] \to \mathbb{C}$ is Riemann Integrable on $[a, b]$ and $[b, c]$, then f is Riemann Integrable on $[a, c]$ and

$$\int_a^c f = \int_a^b f + \int_b^c f$$

Proof. The solution follows directly from the definition of the integral of f and the corresponding result for real-valued functions. □

Theorem 1.1.6 (Fundamental Theorem of Calculus)**.** i) Suppose $f : [a, b] \to \mathbb{C}$ is continuous on its domain and let $F : [a, b] \to \mathbb{C}$ be defined by $F(x) = \int_a^x f(t)\,\mathrm{d}t$. Then F is differentiable on $[a, b]$ and $F'(x) = f(x)$ for all $a \leq x \leq b$.

ii) Suppose $f : [a,b] \to \mathbb{C}$ is $\mathcal{C}^1$ on its domain. Then $f(b) - f(a) = \int_a^b f'(t)\,\mathrm{d}t$.

Proof. i) Let $f = u + iv$. Then $F(x) = \int_a^x u(t)\,\mathrm{d}t + i\int_a^x v(t)\,\mathrm{d}t$. Since u and v are continuous on $[a,b]$, we can apply the Fundamental Theorem of Calculus for real-valued functions to obtain

$$\begin{aligned} F'(x) &= \frac{\mathrm{d}}{\mathrm{dx}}\int_a^x u(t)\,\mathrm{d}t + i\frac{\mathrm{d}}{\mathrm{dx}}\int_a^x v(t)\,\mathrm{d}t \\ &= u(x) + iv(x) \\ &= f(x) \end{aligned}$$

ii) Let $f = u + iv$. Then u and v are $\mathcal{C}^1$ on $[a,b]$ so we may apply the Fundamental Theorem of Calculus for real-valued functions to obtain

$$\begin{aligned} \int_a^b f'(t)\,\mathrm{d}t &= \int_a^b u'(t)\,\mathrm{d}t + i\int_a^b v'(t)\,\mathrm{d}t \\ &= u(b) - u(a) + i[v(b) - v(a)] \\ &= [u(b) + iv(b)] - [u(a) + iv(a)] \\ &= f(b) - f(a) \end{aligned}$$

□

Proposition 1.1.7 (Product Rule of Differentiation)**.** If $h(x) = f(x)g(x)$ for $a \le x \le b$ and if f and g are differentiable at x_0 for some $x_0 \in [a,b]$, then so is h and

$$h'(x_0) = f'(x_0)g(x_0) + f(x_0)g'(x_0)$$

Proof. Let

$$f = u_1 + iv_1 \text{ and } g = u_2 + iv_2$$

Then $h = (u_1u_2 - v_1v_2) + i(u_1v_2 + v_1u_2)$ So, by the product rule for real-valued functions, h is differentiable at x_0 since u_1, v_1, u_2 and v_2 are. Furthermore,

$$\begin{aligned} h' &= [(u_1'u_2 + u_1u_2') - (v_1'v_2 + v_1v_2')] \\ &\quad + i[(u_1'v_2 + u_1v_2') + (v_1'u_2 + v_1u_2')] \\ &= [(u_1'u_2 - v_1'v_2) + i(u_1'v_2 + v_1'v_2)] \\ &\quad + [(u_1u_2' - v_1v_2') + i(u_1v_2' + v_1u_2')] \\ &= (u_1' + iv_1')(u_2 + iv_2) + (u_1 + iv_1)(u_2' + iv_2') \\ &= f'g + fg' \end{aligned}$$

where all functions are evaluated at x_0. □

Proposition 1.1.8 (Integration by Parts)**.** Suppose $f, g : [a, b] \to \mathbb{C}$ are $\mathcal{C}^1$ on their domain. Then

$$\int_a^b f(t)g'(t)\,\mathrm{d}t = f(t)g(t)\,\Big|_{t=a}^{b} - \int_a^b f'(t)g(t)\,\mathrm{d}t$$

Proof. Let $f = u_1 + iv_1$ and $g = u_2 + iv_2$. Then

$$\begin{aligned}\int_a^b f(t)g'(t)\,\mathrm{d}t &= \int_a^b \Big([(u_1(t)u_2'(t) - v_1(t)v_2'(t)] + i[u_1(t)v_2'(t) + v_1(t)u_2'(t)]\Big)\,\mathrm{d}t \\ &= \int_a^b u_1(t)u_2'(t)\,\mathrm{d}t - \int_a^b v_1(t)v_2'(t)\,\mathrm{d}t \\ &\quad + i\left(\int_a^b u_1(t)v_2'(t)\,\mathrm{d}t + \int_a^b v_1(t)u_2'(t)\,\mathrm{d}t\right)\end{aligned}$$

Since u_1, v_1, u_2 and v_2 are $\mathcal{C}^1$ on $[a, b]$, we can apply Integration by Parts for real-valued functions on each of these integrals. After doing so and rearranging, we obtain

$$\begin{aligned}\int_a^b f(t)g'(t)\,\mathrm{d}t &= \Big([u_1(t)u_2(t) - v_1(t)v_2(t)] + i[u_1(t)v_2(t) + v_1(t)u_2(t)]\Big)\,\Big|_{t=a}^{b} \\ &\quad - \int_a^b \Big([u_1'(t)u_2(t) - v_1'(t)v_2(t)] + i[u_1'(t)v_2(t) + v_1'(t)u_2(t)]\Big)\,\mathrm{d}t \\ &= f(t)g(t)\,\Big|_{t=a}^{b} - \int_a^b f'(t)g(t)\,\mathrm{d}t\end{aligned}$$

□

Unfortunately, there is no parallel to the Intermediate Value Theorem since the complex numbers are not an ordered field. Also the Mean Value Theorem does not hold. To see this, let $f(t) = e^{it}$. Then $f(0) = f(2\pi) = 0$ but $f'(t) \neq 0$ for any $0 < t < 2\pi$ since

$$|f'(t)| = \left|ie^{it}\right| = 1 \text{ for all } t \in \mathbb{R}$$

However, we do have the following two Mean Value Inequalities.

Proposition 1.1.9. Suppose $f : [a, b] \to \mathbb{C}$ is Riemann Integrable on its domain and let

$$M = \sup\{|f(t)| : a \leq t \leq b\}$$

Then

$$\left| \int_a^b f(t)\,\mathrm{d}t \right| \leq \int_a^b |f(t)|\,\mathrm{d}t \leq M(b-a)$$

Proof. This proposition is the complex version of the corresponding result for real- valued functions. To prove it, we will compare $\int_a^b |f(t)|\,\mathrm{d}t$ to a real number whose absolute value is equal to the modulus of $\int_a^b f(t)\,\mathrm{d}t$. Let

$$re^{i\theta} = \int_a^b f(t)\,\mathrm{d}t \text{ where } r \geq 0$$

Then

$$\begin{aligned} r = e^{-i\theta} \int_a^b f(t)\,\mathrm{d}t &= \int_a^b e^{-i\theta} f(t)\,\mathrm{d}t \\ &= \mathrm{Re}\left(\int_a^b e^{-i\theta} f(t)\,\mathrm{d}t \right) \text{ (since } r \text{ is real)} \\ &= \int_a^b \mathrm{Re}(e^{-i\theta} f(t))\,\mathrm{d}t \leq \int_a^b \left| e^{-i\theta} f(t) \right| \mathrm{d}t \\ &= \int_a^b |f(t)|\,\mathrm{d}t \end{aligned}$$

But,

$$r = \left| \int_a^b f(t)\,\mathrm{d}t \right|$$

So this proves the first inequality. For the second, note that

$$\int_a^b |f(t)|\,\mathrm{d}t \leq \int_a^b M\,\mathrm{d}t = M(b-a)$$

□

Proposition 1.1.10. Suppose I is an open interval in $\mathbb{R}$, $[a, b] \subset I$, $f : I \to \mathbb{C}$ and f is $\mathcal{C}^1$ on I. Then there exists $M > 0$ such that $|f(b) - f(a)| \leq M(b-a)$.

Proof. Let $M = \max \{|f'(t)| : a \leq t \leq b\}$. Then by the Fundamental Theorem of Calculus,

$$f(b) - f(a) = \int_a^b f'(t)\,\mathrm{d}t$$

So,

$$|f(b) - f(a)| = \left|\int_a^b f'(t)\,\mathrm{d}t\right| \leq \int_a^b |f'(t)|\,\mathrm{d}t \leq \int_a^b M\,\mathrm{d}t = M(b-a)$$

□

1.2 Three Convergence Theorems

In this section, we establish three convergence theorems which will be used in later sections. The first two theorems deal with derivatives and the second with integrals.

Theorem 1.2.1. i) Suppose $\{f_k\}_{k=1}^{\infty}$ is a sequence of complex-valued, continuously differentiable functions on a set $X \subset \mathbb{R}$ which converge pointwise on their domain and whose derivatives converge uniformly on X. Then $\{f_k\}_{k=1}^{\infty}$ converges pointwise on X to a $\mathcal{C}^1$ function which satisfies

$$\frac{d}{dx} \lim_{k\to\infty} f_k(x) = \lim_{k\to\infty} f_k'(x)$$

ii) Suppose $\{f_k\}_{k=1}^{\infty}$ is a sequence of complex-valued, continuously differentiable functions on a set $X \subset \mathbb{R}$. Futhermore, suppose $\sum_{k=1}^{\infty} f_k$ converges pointwise on X and $\sum_{k=1}^{\infty} f_k'$ converges uniformly on X. Then $\sum_{k=1}^{\infty} f_k$ converges pointwise on X to a $\mathcal{C}^1$ function which satisfies

$$\frac{d}{dx} \sum_{k=1}^{\infty} f_k(x) = \sum_{k=1}^{\infty} f_k'(x)$$

Proof. i) Let $g = \lim_{k\to\infty} f_k'$. Then for $x \in X$,

$$\begin{aligned} f(x) &= \lim_{k\to\infty} f_k(x) \\ &= \lim_{k\to\infty} \left(f_k(x_0) + \int_{x_0}^{x} f_k'(t)\,\mathrm{d}t \right) \\ &= f(x_0) + \lim_{k\to\infty} \int_{x_0}^{x} f_k'(t)\,\mathrm{d}t \\ &= f(x_0) + \int_{x_0}^{x} g(t)\,\mathrm{d}t \end{aligned}$$

But g is continuous on X, so the result follows by the Fundamental Theorem of Calculus.

ii) Apply part i) to the sequence of partial sums

$$s_n(x) = \sum_{k=1}^{n} f(x)$$

□

Theorem 1.2.2. Suppose $\{f_k\}_{k=1}^{\infty}$ is a sequence of complex-valued, continuously differentiable functions on a bounded interval $[a, b]$.

i) If the sequence $\{f_k\}_{k=1}^{\infty}$ converges pointwise at some $x_0 \in [a, b]$ and $\{f_k'\}_{k=1}^{\infty}$ converges uniformly to some $g \in \mathcal{C}([a, b])$ then $\{f_k\}_{k=1}^{\infty}$ converges uniformly on $[a, b]$ to some $f \in \mathcal{C}^1([a, b])$ and $f'(x) = g(x)$ for all $x \in [a, b]$.

ii) If $\sum_{k=1}^{\infty} f_k$ converges pointwise at some $x_0 \in [a, b]$ and $\sum_{k=1}^{\infty} f_k'$ converges uniformly on $[a, b]$ then $\sum_{k=1}^{\infty} f_k$ converges uniformly on $[a, b]$ to a $\mathcal{C}^1$ function which satisfies

$$\frac{d}{dx} \sum_{k=1}^{\infty} f_k(x) = \sum_{k=1}^{\infty} f_k'(x)$$

Proof. i) Fix $\epsilon > 0$. Choose N_1 such that $|f_m(x_0) - f_n(x_0)| < \epsilon/2$ for $n, m > N_1$. Choose N_2 such that $|f_m'(t) - f_n'(t)| < \epsilon/2(b - a)$ for $n, m > N_2$ and all $t \in [a, b]$. Let $N = \max(N_1, N_2)$. Then for $n, m > N$,

$$\begin{aligned}
|f_m(x) - f_n(x)| &= \left| \left(f_m(x_0) + \int_a^b f_m'(t)\,\mathrm{d}t \right) - \left(f_n(x_0) + \int_a^b f_n'(t)\,\mathrm{d}t \right) \right| \\
&\le |f_m(x_0) - f_n(x_0)| + \left| \int_a^b f_m'(t)\,\mathrm{d}t - \int_a^b f_n'(t)\,\mathrm{d}t \right| \\
&< \epsilon/2 + \int_a^b |f_m'(t) - f_n'(t)|\,\mathrm{d}t \\
&< \epsilon/2 + \int_a^b \frac{\epsilon}{2(b-a)}\,\mathrm{d}t \\
&= \epsilon/2 + \epsilon/2 \\
&= \epsilon
\end{aligned}$$

Hence, $\{f_k\}_{k=1}^{\infty}$ satisfies the uniform Cauchy criterion and so converges uniformly to some $f : [a,b] \to \mathbb{C}$. The remaining assertion can be proven identically as in Theorem 1.2.1.

ii) Apply part i) to the sequence of partial sums

$$s_n(x) = \sum_{k=1}^{n} f(x)$$

□

Theorem 1.2.3. i) Suppose $h : [a,b] \to \mathbb{C}$ is bounded on its domain and is Riemann Integrable on $[c,b]$ for all $a < c < b$. Then h is Riemann Integrable on $[a,b]$ and $\int_a^b h = \lim_{c\to a^+} \int_c^b h$.

ii) Suppose $h : [a,b] \to \mathbb{C}$ is bounded on its domain and is Riemann Integrable on $[a,c]$ for all $a < c < b$. Then h is Riemann Integrable on $[a,b]$ and $\int_a^b h = \lim_{c\to b^-} \int_a^c h$.

Proof. We will only prove part i). The second part follows by a similar argument. First note that we may assume, without loss of generality, that h is real-valued. (For the general case, just apply the following argument to the real and imaginary parts of of h.) Fix $\epsilon > 0$. Let

$$M = \sup\{|h(x)| : a < x < b\}$$

Choose $c \in (a, b)$ such that $(c - a) < \epsilon/4M$. Since h is Riemann Integrable on $[c, b]$, we can find a partition $c = x_1 < x_2 < \ldots < x_n = b$ such that

$$\sum_{k=2}^{n} M_k(x_k - x_{k-1}) - \sum_{k=2}^{n} m_k(x_k - x_{k-1}) < \epsilon/2$$

where

$$M_k = \sup\{h(x) : x_{k-1} \leq x \leq x_k\}$$

and

$$m_k = \inf\{h(x) : x_{k-1} \leq x \leq x_k\}$$

Let $x_0 = a$,

$$M_1 = \sup\{h(x) : x_0 = a \leq x \leq x_1 = c\}$$

and

$$m_1 = \inf\{h(x) : x_0 = a \leq x \leq x_1 = c\}$$

Then

$$\begin{aligned}
&\sum_{k=1}^{n} M_k(x_k - x_{k-1}) - \sum_{k=1}^{n} m_k(x_k - x_{k-1}) \\
&= (M_1 - m_1)(c - a) + \sum_{k=2}^{n}(M_k - m_k)(x_k - x_{k-1}) \\
&< \frac{2M\epsilon}{4M} + \epsilon/2 = \epsilon
\end{aligned}$$

It follows that h is Riemann Integrable on $[a, b]$. Moreover,

$$\left|\int_a^b h - \int_c^b h\right| = \left|\int_a^c h\right| \leq \int_a^c |h| \leq M(c - a) \to 0$$

as $c \to a^+$. $\square$

Corollary 1.2.4. i) Suppose $h : [a, b] \to \mathbb{C}$ is right continous at a and Riemann Integrable on $[c, b]$ for all $c \in (a, b)$. Then h is Riemann Integrable on $[a, b]$ and $\int_a^b h = \lim_{c \to a^+} \int_c^b h$.

ii) Suppose $h : [a, b] \to \mathbb{C}$ is left continuous at b and Riemann Integrable on $[a, c]$ for all $c \in (a, b)$. Then h is Riemann Integrable on $[a, b]$ and $\int_a^b h = \lim_{c \to b^-} \int_a^c h$.

Proof. Again, we will only prove part i). The second part follows by a similar argument. Choose $c > a$ such that $|h(t) - h(a)| \le 1$ for $t \in [a, c]$. Then $|h(t)| \le |h(a)| + 1$ on for $t \in [a, c]$. So h is bounded on $[a, c]$. Also, it is bounded on $[c, b]$ because it is Riemann Integrable on this interval. Therefore, h satisfies the hypotheses of Theorem 1.2.3 and the result follows. □

1.3 Riemann-Lebesgue Lemma

The Riemann-Lebesgue Lemma is used extensively in this book and will be proven in this section.

Theorem 1.3.1 (Riemann-Lebesgue Lemma)**.** Suppose $f : [a, b] \to \mathbb{C}$ is Riemann Integrable on its domain. Then

$$\lim_{\substack{\lambda \to \infty \\ \lambda \in \mathbb{R}}} \int_a^b f(x) \sin(\lambda x + \alpha) \, \mathrm{d}x = 0$$

Proof. Case 1. $f = c$ on $[a, b]$ for some $c \in \mathbb{R}$. Then for all $\lambda \in \mathbb{R}$, $\lambda \neq 0$,

$$\begin{aligned} &\int_a^b f(x) \sin(\lambda x + \alpha) \, \mathrm{d}x \\ &= c \int_a^b \sin(\lambda x + \alpha) \, \mathrm{d}x \\ &= \frac{-c \cos(\lambda x + \alpha)}{\lambda} \bigg|_{x=a}^{b} \\ &= \frac{c}{\lambda} \left(\cos(\lambda a + \alpha) - \cos(\lambda b + \alpha) \right) \end{aligned}$$

So,

$$\left| \int_a^b f(x) \sin(\lambda x + \alpha) \, \mathrm{d}x \right| \le \frac{2\,|c|}{|\lambda|} \to 0$$

as $\lambda \to \infty$.

Case 2. f is a step function. Then there exist $x_0, x_1, \ldots, x_n$ and $c_1, c_2, \ldots, c_n$ such that $a = x_0 < x_1 < \ldots < x_n = b$ and $f(x) = c_k$ for $x_{k-1} < c_k \leq x_k$, $1 \leq k \leq n$. In this case we have

$$\int_a^b f(x)\sin(\lambda x + \alpha)\,\mathrm{d}x = \sum_{k=1}^{n} \int_{x_{k-1}}^{x_k} c_k \sin(\lambda x + \alpha)\,\mathrm{d}x$$

$\to 0$ as $\lambda \to \infty$ by case 1.

Case 3. f is real-valued. Fix $\epsilon > 0$. Since f is Riemann Integrable, there exists a step function $g : [a, b] \to \mathbb{R}$ such that

$$\int_a^b |f(x) - g(x)|\,\mathrm{d}x < \epsilon/2$$

Then

$$\begin{aligned}
&\left|\int_a^b f(x)\sin(\lambda x + \alpha)\,\mathrm{d}x - \int_a^b g(x)\sin(\lambda x + \alpha)\,\mathrm{d}x\right| \\
&= \left|\int_a^b (f(x) - g(x))\sin(\lambda x + \alpha)\,\mathrm{d}x\right| \\
&\leq \int_a^b |f(x) - g(x)|\,\mathrm{d}x \\
&< \epsilon/2
\end{aligned}$$

Therefore,

$$\begin{aligned}
&\limsup_{\lambda\to\infty} \left|\int_a^b f(x)\sin(\lambda x + \alpha)\,\mathrm{d}x\right| \\
&\leq \limsup_{\lambda\to\infty} \left|\int_a^b (f(x) - g(x))\sin(\lambda x + \alpha)\,\mathrm{d}x\right| \\
&+ \ \limsup_{\lambda\to\infty} \left|\int_a^b g(x)\sin(\lambda x + \alpha)\,\mathrm{d}x\right| \\
&< \epsilon/2 + \limsup_{\lambda\to\infty} \left|\int_a^b g(x)\sin(\lambda x + \alpha)\,\mathrm{d}x\right| \\
&< \epsilon
\end{aligned}$$

(by Case 2 and the fact that g is a step function).

Since this is true for all $\epsilon > 0$,

$$\limsup_{\lambda\to\infty} \left| \int_a^b f(x)\sin(\lambda x + \alpha)\,\mathrm{d}x \right| = 0$$

and the result follows.

Case 4. For the general case in which f is complex-valued, simply apply the previous case to the real and imaginary parts of f.

$\square$

Corollary 1.3.2. Let $f : [a, b] \to \mathbb{C}$ be Riemann Integrable on its domain. Then

i)

$$\int_a^b f(x)\sin\lambda x\,\mathrm{d}x \to 0$$

as $\lambda \to \infty$ or $\lambda \to -\infty$.

ii)

$$\int_a^b f(x)\cos\lambda x\,\mathrm{d}x \to 0$$

as $\lambda \to \infty$ or $\lambda \to -\infty$.

iii)

$$\int_a^b f(x)e^{i\lambda x}\,\mathrm{d}x \to 0$$

as $\lambda \to \infty$ or $\lambda \to -\infty$.

Proof. i) Apply the Riemann-Lebesgue Lemma with $\alpha = 0$.

ii) Apply the Riemann-Lebesgue Lemma with $\alpha = \pi/2$.

iii) This is a simple consequence of parts i) and ii) and the fact that $e^{i\lambda x} = \cos\lambda x + i\sin\lambda x$ for all $\lambda, x \in \mathbb{R}$.

$\square$

1.4 Absolute Integrability

The concept of absolute integrability will be dealt with in this section. It will be used in the next section to prove Fubini's Theorem.

Definition 1.4.1. Suppose $a \in \mathbb{R}$, $f : [a, +\infty) \to \mathbb{C}$ and f is Riemann Integrable on $[a, \lambda]$ for all $a < \lambda \in \mathbb{R}$. We say f is *absolutely integrable* on $[a, +\infty)$ provided there exists $M > 0$ such that

$$\int_a^\lambda |f(t)| \,\mathrm{d}t \leq M \text{ for all } a \leq \lambda \in \mathbb{R}$$

Or equivalently,

$$\lim_{\lambda \to +\infty} \int_a^\lambda |f(t)| \,\mathrm{d}t \text{ exists.}$$

If $b \in \mathbb{R}$, absolute integrability on $(-\infty, b]$ is defined analagously.

If f is absolutely integrable on $[a, +\infty)$ and $a \leq \lambda < \mu$ then

$$\begin{aligned} &\left| \int_a^\mu f(t) \,\mathrm{d}t - \int_a^\lambda f(t) \,\mathrm{d}t \right| \\ &\leq \int_\lambda^\mu |f(t)| \,\mathrm{d}t \\ &= \int_a^\mu |f(t)| \,\mathrm{d}t - \int_a^\lambda |f(t)| \,\mathrm{d}t \end{aligned}$$

$\to 0$ as λ, $\mu \to +\infty$. By the Cauchy criteriion,

$$\lim_{\lambda \to +\infty} \int_a^\lambda f(t) \,\mathrm{d}t$$

exists. Denote it by $\int_a^{+\infty} f(t) \,\mathrm{d}t$.

Example 1.4.2. Define $f : [0, +\infty) \to \mathbb{R}$ by

$$f(t) = \begin{cases} \dfrac{\sin t}{t} & \text{if } t > 0 \\ 1 & \text{if } t = 0 \end{cases}$$

Show that $\lim_{\lambda\to+\infty}\int_0^\lambda f(t)\,\mathrm{d}t$ exists but f is not absolutely integrable on $[0,+\infty)$.

Solution. It will be shown in Theorem 2.6.1, using techniques from complex analysis, that

$$\lim_{\lambda\to+\infty}\int_0^\lambda f(t)\,\mathrm{d}t = \pi/2$$

However, we can show that

$$\lim_{\lambda\to+\infty}\int_0^\lambda |f(t)|\,\mathrm{d}t = +\infty$$

Define $g : [\pi,+\infty)$ by

$$g(t) = \begin{cases} \dfrac{1}{\pi/2(k\pi+\pi/2)}(t-k\pi) & k\pi \le t \le k\pi+\pi/2 \\ \dfrac{1}{\pi/2(k\pi+\pi/2)}((k+1)\pi-t) & k\pi+\pi/2 \le t \le (k+1)\pi \end{cases}$$

for $k \in \mathbb{N}$. It is easy to see that $|f(t)| \ge g(t)$ for all $\pi \le t \in \mathbb{R}$.

Then for $n \in \mathbb{N}$

$$\begin{aligned}
\int_0^{(n+1)\pi} |f(t)|\,\mathrm{d}t &\ge \int_\pi^{(n+1)\pi} |f(t)|\,\mathrm{d}t \\
&= \sum_{k=1}^n \int_{k\pi}^{(k+1)\pi} \left|\frac{\sin t}{t}\right| \mathrm{d}t \\
&\ge \sum_{k=1}^n \int_{k\pi}^{(k+1)\pi} g(t)\,\mathrm{d}t \\
&= \sum_{k=1}^n \frac{1}{2}\left(\frac{\pi}{k\pi+\pi/2}\right) \\
&= \sum_{k=1}^n \frac{1}{2k+1}
\end{aligned}$$

which goes to $+\infty$ as $n\to\infty$.

Definition 1.4.3. Suppose $f : \mathbb{R} \to \mathbb{C}$ and f is Riemann Integrable on $[a,b] \subset \mathbb{R}$. We say f is *absolutely integrable on* $\mathbb{R}$ provided it is absolutely integrable on $(-\infty, 0]$ and $[0, +\infty)$. In this case we define

$$\int_{\mathbb{R}} f = \int_{-\infty}^{\infty} f(x)\,\mathrm{d}x = \int_{-\infty}^{0} f(x)\,\mathrm{d}x + \int_{0}^{+\infty} f(x)\,\mathrm{d}x$$

and conclude that

$$\left| \int_{-\infty}^{\infty} f(x)\,\mathrm{d}x \right| \le \int_{-\infty}^{\infty} |f(x)|\,\mathrm{d}x$$

as well as

$$\int_{-\infty}^{\infty} f(x)\,\mathrm{d}x = \int_{-\infty}^{a} f(x)\,\mathrm{d}x + \int_{a}^{b} f(x)\,\mathrm{d}x + \int_{b}^{+\infty} f(x)\,\mathrm{d}x$$

for all $a < b$, $a, b \in \mathbb{R}$.

Denote the set of all absolutely integrable functions on $\mathbb{R}$ by $\mathcal{R}^1$. Clearly $\mathcal{R}^1$ is a complex vector space equipped with the linear functional

$$f \mapsto \int_{\mathbb{R}} f$$

Moreover, if $f \in \mathcal{R}^1$ and $f > 0$, then $\int_{-\infty}^{\infty} f(x)\,\mathrm{d}x > 0$.

Proposition 1.4.4. Suppose $f : \mathbb{R} \to \mathbb{C}$ is Riemann Integrable on $[a,b]$ for all $[a,b] \subset \mathbb{R}$. Then the following are equivalent

i) $f \in \mathcal{R}^1$

ii) There exists $M > 0$ such that

$$\int_{a}^{b} |f(x)|\,\mathrm{d}x \le M$$

whenever $a < b$.

iii) For all $\epsilon > 0$ there exists $A > 0$ such that

$$\int_{a}^{b} |f(x)|\,\mathrm{d}x < \epsilon$$

whenever $[a,b] \cap [-A, A] = \varnothing$.

Proof. i) $\Rightarrow$ ii) If $0 \le a < b$ then, since f is absolutely integrable on $[0, +\infty)$, we have

$$\int_a^b |f(x)|\,\mathrm{d}x \le \int_0^b |f(x)|\,\mathrm{d}x \le M_1$$

for some $M_1 > 0$.

Similarly, since f is absolutely integrable on $(-\infty, 0]$, there exists $M_2 > 0$ such that

$$\int_a^b |f(x)|\,\mathrm{d}x \le M_2$$

whenever $a < b \le 0$.

Let $M = M_1 + M_2$. Then ii) is satisfied with this choice of M whenever $0 \le a < b$ or $a < b \le 0$. If $a < 0 < b$ then we also have

$$\begin{aligned}\int_a^b |f(x)|\,\mathrm{d}x &= \int_a^0 |f(x)|\,\mathrm{d}x + \int_0^b |f(x)|\,\mathrm{d}x \\ &\le M_1 + M_2 \\ &= M\end{aligned}$$

ii) $\Rightarrow$ i) There exists $M > 0$ such that

$$\int_0^\lambda |f(x)|\,\mathrm{d}x \le M$$

for all $\lambda > 0$. Thus, by definition, f is absolutely integrable on $[0, +\infty)$. Similarly, f is absolutely integrable on $(-\infty, 0]$. So $f \in \mathcal{R}^1$.

i) $\Rightarrow$ iii) If $f \in \mathcal{R}^1$ then it is absolutely integrable on $[0, +\infty)$. This means

$$\lim_{b\to\infty} \int_0^b |f(x)|\,\mathrm{d}x$$

exists. Fix $\epsilon > 0$. By the Cauchy criterion, there exists $A_1 > 0$ such that

$$\int_a^b |f(x)|\,\mathrm{d}x = \int_0^b |f(x)|\,\mathrm{d}x - \int_0^a |f(x)|\,\mathrm{d}x < \epsilon$$

whenever $A_1 < a < b \in \mathbb{R}$. A similar argument shows that if $f \in \mathcal{R}^1$ then there exists $A_2 > 0$ such that

$$\int_a^b |f(x)|\,\mathrm{d}x = \int_a^0 |f(x)|\,\mathrm{d}x - \int_b^0 |f(x)|\,\mathrm{d}x < \epsilon$$

whenever $a < b < -A_2$. Let $A = \max(A_1, A_2)$. Then $[a,b] \cap [-A, A] = \varnothing$ iff $a < b < -A \leq -A_2$ or $A_1 \leq A < a < b$. The result then follows.

iii) $\Rightarrow$ i) Fix $\epsilon > 0$. Then by assumption there exists $A > 0$ such that

$$\int_a^b |f(x)| \,\mathrm{d}x < \epsilon$$

whenever $b > a > A$. Hence

$$\lim_{a,b\to\infty} \int_a^b |f(x)| \,\mathrm{d}x = 0$$

So by the Cauchy criterion, f is absolutely integrable on $[0, +\infty)$. A similar argument shows that f is absolutely integrable on $(-\infty, 0]$. Therefore, $f \in \mathcal{R}^1$ as required.

$\square$

Sometimes, the quantity

$$\int_{-\infty}^{\infty} |f(x)| \,\mathrm{d}x$$

is denoted by $\|f\|_1$. This is called the *1-norm* of f. This terminology is indeed justified, as the next theorem shows.

Theorem 1.4.5. The function $\|\cdot\|_1 : \mathcal{R}^1 \to \mathbb{R}$ is a norm provided we identify functions which agree almost everywhere.

Proof. Clearly, if $f \in \mathcal{R}^1$ then

$$\|f\|_1 = \int_{-\infty}^{\infty} |f(x)| \,\mathrm{d}x \geq 0$$

with equality iff $f = 0$ almost everywhere.

If in addition, $\lambda \in \mathbb{C}$ then

$$\|\lambda f\|_1 = \int_{-\infty}^{\infty} |\lambda f(x)| \,\mathrm{d}x = |\lambda| \int_{-\infty}^{\infty} |f(x)| \,\mathrm{d}x = |\lambda| \, \|f\|_1$$

Finally, the triangle inequality holds because if $f, g \in \mathcal{R}^1$ then

$$\begin{aligned}\|f+g\|_1 &= \int_{-\infty}^{\infty} |f(x)+g(x)|\,\mathrm{d}x \\ &\le \int_{-\infty}^{\infty} [|f(x)|+|g(x)|]\,\mathrm{d}x \\ &= \int_{-\infty}^{\infty} |f(x)|\,\mathrm{d}x + \int_{-\infty}^{\infty} |g(x)|\,\mathrm{d}x \\ &= \|f\|_1 + \|g\|_1\end{aligned}$$

□

Proposition 1.4.6. Given $f \in \mathcal{R}^1$ define

$$f_n(x) = \begin{cases} f(x) & \text{if } |x| \le n \\ 0 & \text{if } |x| > n \end{cases}$$

for $n \in \mathbb{N}$. Then $f_n \to f$ in $\mathcal{R}^1$ with respect to the 1-norm.

Proof.

$$\begin{aligned}\|f-f_n\|_1 &= \int_{-\infty}^{\infty} |f(x)-f_n(x)|\,\mathrm{d}x \\ &= \int_{-\infty}^{-n} |f(x)|\,\mathrm{d}x + \int_{n}^{\infty} |f(x)|\,\mathrm{d}x \\ &= \int_{-\infty}^{\infty} |f(x)|\,\mathrm{d}x - \int_{-n}^{n} |f(x)|\,\mathrm{d}x\end{aligned}$$

$\to 0$ as $n \to \infty$ since $f \in \mathcal{R}^1$. □

Theorem 1.4.7. A function $\varphi : \mathbb{R} \to \mathbb{C}$ is said to have *compact support* if it vanishes outside a compact set. Let

$$C_c(\mathbb{R}) = \{\varphi : \mathbb{R} \to \mathbb{C} : \varphi \text{ is continuous and has compact support}\}$$

Then, $C_c(\mathbb{R})$ is a dense linear subspace of $\mathcal{R}^1$.

Proof. Let $f \in \mathcal{R}^1$. Then given $\epsilon > 0$ there exists $N_1 > 0$ such that

$$\int_{-\infty}^{\infty} |f(x)| \, \mathrm{d}x - \int_{-n}^{n} |f(x)| \, \mathrm{d}x = \int_{-\infty}^{-n} |f(x)| \, \mathrm{d}x + \int_{n}^{\infty} |f(x)| \, \mathrm{d}x < \epsilon/2$$

whenever $n > N_1$.

Also, given $n \in \mathbb{N}$, there exist a step function $s_n(x) : \mathbb{R} \to \mathbb{C}$ and a continuous function $\varphi_n(x) : \mathbb{R} \to \mathbb{C}$, both of which vanish outside $[-n, n]$, such that

$$\int_{-n}^{n} |f(x) - s_n(x)| \, \mathrm{d}x < 1/2n$$

and

$$\int_{-n}^{n} |s_n(x) - \varphi_n(x)| \, \mathrm{d}x < 1/2n$$

Choose $N_2 > 0$ such that $1/N_2 < \epsilon/2$ and let $N = \max(N_1, N_2)$. Then for $n > N$, we have

$$\begin{aligned}
\|f - \varphi_n\|_1 &= \int_{-\infty}^{\infty} |f(x) - \varphi_n(x)| \, \mathrm{d}x \\
&= \int_{-\infty}^{-n} |f(x)| \, \mathrm{d}x + \int_{n}^{\infty} |f(x)| \, \mathrm{d}x + \int_{-n}^{n} |f(x) - \varphi_n(x)| \, \mathrm{d}x \\
&< \epsilon/2 + \int_{-n}^{n} |f(x) - s_n(x)| \, \mathrm{d}x + \int_{-n}^{n} |s_n(x) - \varphi_n(x)| \, \mathrm{d}x \\
&< \epsilon/2 + 1/2n + 1/2n \\
&< \epsilon/2 + \epsilon/2 \\
&= \epsilon
\end{aligned}$$

and so $\|f - \varphi_n\|_1 \to 0$ as $n \to \infty$. □

Proposition 1.4.8. Suppose $f : \mathbb{R} \to \mathbb{C}$ and f is Riemann Integrable on $[a, b] \subset \mathbb{R}$ for all $a < b$. Furthermore, suppose there is a function $h \in \mathcal{R}^1$ such that $|f(x)| \leq h(x)$ for all $x \in \mathbb{R}$. Then $f \in \mathcal{R}^1$.

Proof. For $0 < \lambda \in \mathbb{R}$ we have

$$\int_{0}^{\lambda} |f(x)| \, \mathrm{d}x \leq \int_{0}^{\lambda} h(x) \, \mathrm{d}x$$

Hence,

$$\lim_{\lambda \to +\infty} \int_0^\lambda |f(x)| \, \mathrm{d}x \le \lim_{\lambda \to +\infty} \int_0^\lambda h(x) \, \mathrm{d}x$$

and this latter limit exists since $h \in \mathcal{R}^1$. Therefore, f is absolutely integrable on $[0, +\infty)$. A Similar argument shows that f is absolutely integrable on $(-\infty, 0]$. So, by the definition of absolute integrability on $\mathbb{R}$, $f \in \mathcal{R}^1$. □

1.5 Fubini's Theorem

The main result of this section is Fubini's Theorem which deals with the order of integration of an integral. One the consequences of Fubini's Theorem is a result due to Leibniz, which will be proven in the next section.

Theorem 1.5.1 (Fubini. Version 1)**.** Suppose $F : [a, b] \times [c, d] \to \mathbb{C}$ is continuous and define

$$\varphi(x) = \int_c^d F(x, y) \, \mathrm{d}y, \ a \le x \le b$$

and

$$\psi(y) = \int_a^b F(x, y) \, \mathrm{d}x, \ c \le y \le d$$

Then,

i) φ and ψ are both continuous on their domain.

ii)
$$\int_a^b \varphi(x) \, \mathrm{d}x = \int_c^d \psi(y) \, \mathrm{d}y$$

Proof. i) Since F is continuous on the compact set $[a, b] \times [c, d]$, it is uniformly continuous. So given $\epsilon > 0$ we can choose $\delta > 0$ such that $|F(x, y) - F(s, t)| < \epsilon/(d - c)$ whenever $\|(x, y) - (s, t)\| < \delta$. Thus, if

$x, s \in [a, b]$ and $|x - s| < \delta$, we have

$$\begin{aligned}
|\varphi(x) - \varphi(s)| &= \left| \int_c^d F(x, y)\,\mathrm{d}y - \int_c^d F(s, y)\,\mathrm{d}y \right| \\
&\leq \int_c^d |F(x, y) - F(s, y)|\,\mathrm{d}y \\
&\leq \int_c^d \epsilon/(d - c)\,\mathrm{d}y \text{ since } \|(x, y) - (s, y)\| < \delta \\
&= \epsilon
\end{aligned}$$

Therefore, φ is continuous on $[a, b]$. A similar argument shows that ψ is continuous on $[c, d]$.

ii) Without loss of generality, we may assume F is real-valued. The general case can be handled by applying this special case to the real and imaginary parts of F.

Choose $\delta > 0$ such that $|F(x, y) - F(s, t)| < \epsilon/(b - a)(d - c)$ whenever $\|(x, y) - (s, t)\| < \delta$. Choose partitions $a = x_0 < x_1 < \ldots < x_m = b$ of $[a, b]$ and $c = y_0 < y_1 < \ldots < y_n = d$ of $[c, d]$ such that $|x_j - x_{j-1}| < \delta/\sqrt{2}$ for $1 \leq j \leq m$ and $|y_k - y_{k-1}| < \delta/\sqrt{2}$ for $1 \leq k \leq n$. Then,

$$\begin{aligned}
\int_a^b \varphi(x)\,\mathrm{d}x &= \sum_{j=1}^m \int_{x_{j-1}}^{x_j} \varphi(x)\,\mathrm{d}x \\
&= \sum_{j=1}^m \int_{x_{j-1}}^{x_j} \int_c^d F(x, y)\,\mathrm{d}y\,\mathrm{d}x \\
&= \sum_{j=1}^m \int_{x_{j-1}}^{x_j} \sum_{k=1}^n \int_{y_{k-1}}^{y_k} F(x, y)\,\mathrm{d}y\,\mathrm{d}x \\
&= \sum_{j=1}^m \sum_{k=1}^n \int_{x_{j-1}}^{x_j} \int_{y_{k-1}}^{y_k} F(x, y)\,\mathrm{d}y\,\mathrm{d}x \\
&\leq \sum_{j=1}^m \sum_{k=1}^n M_{jk}(x_j - x_{j-1})(y_k - y_{k-1})
\end{aligned}$$

Where, $M_{jk} = \max\{F(x, y) : (x, y) \in [x_{j-1}, x_j] \times [y_{k-1}, y_k]\}$. Denote this latter sum by U.

Similarly,

$$\int_a^b \varphi(x)\,\mathrm{d}x \geq \sum_{j=1}^{m}\sum_{k=1}^{n} m_{jk}(x_j - x_{j-1})(y_k - y_{k-1}) = L$$

where $m_{jk} = \min\{F(x,y) : (x,y) \in [x_{j-1}, x_j] \times [y_{k-1}, y_k]\}$.

A similar argument shows that

$$L \leq \int_c^d \psi(y)\,\mathrm{d}y \leq U$$

But, by the choice of the x_j's and y_k's the diameter of $[x_{j-1}, x_j] \times [y_{k-1}, y_k]$ is less than δ. So $M_{jk} - m_{jk} \leq \epsilon/(b-a)(d-c)$.

So we have,

$$\begin{aligned}\left|\int_a^b \varphi(x)\,\mathrm{d}x - \int_c^d \psi(y)\,\mathrm{d}y\right| &\leq U - L \\ &= \sum_{j=1}^{m}\sum_{k=1}^{n}(M_{jk} - m_{jk})(x_j - x_{j-1})(y_k - y_{k-1}) \\ &\leq \sum_{j=1}^{m}\sum_{k=1}^{n}\frac{\epsilon}{(b-a)(d-c)}(x_j - x_{j-1})(y_k - y_{k-1}) \\ &= \epsilon\end{aligned}$$

Since this is true for all $\epsilon > 0$,

$$\int_a^b \varphi(x)\,\mathrm{d}x = \int_c^d \psi(y)\,\mathrm{d}y$$

□

A natural question to ask is whether Theorem 1.5.1 holds when one or both of the intervals $[a,b]$ and $[c,d]$ are replaced with unbounded intervals. The next two versions of Fubini's theorem deal with that question and generalize it further.

Theorem 1.5.2 (Fubini. Version 2)**.** Suppose that $X \subset \mathbb{R}^m$ and X is either compact or open. Also suppose that $F : X \times \mathbb{R} \to \mathbb{C}$ is continuous and there exists $h \in \mathcal{R}^1$ such that $|F(x,t)| \leq |h(t)|$ for all $(x,t) \in X \times \mathbb{R}$.

i) The function $\varphi : X \to \mathbb{C}$ defined by

$$\varphi(x) = \int_{-\infty}^{\infty} F(x,t)\,\mathrm{d}t$$

is continuous on its domain.

ii) If $m = 1$ and $[a,b] \subset X$ and we define $\psi : \mathbb{R} \to \mathbb{C}$ by

$$\psi(t) = \int_a^b F(x,t)\,\mathrm{d}x$$

then ψ is continuous and $\psi \in \mathcal{R}^1$.

iii) The functions φ and ψ satisfy

$$\int_a^b \varphi(x)\,\mathrm{d}x = \int_{-\infty}^{\infty} \psi(t)\,\mathrm{d}t$$

Proof. i) Let

$$\varphi_n(x) = \int_{-n}^{n} F(x,t)\,\mathrm{d}t$$

for $x \in X$ and $n \in \mathbb{N}$.

To show that φ_n is continuous, proceed in a manner similar to that in Theorem 1.5.1. Choose a compact set $A \subset X$. Then $A \times [-n,n]$ is compact. Therefore, F is uniformly continuous on $A \times [-n,n]$. So given $\epsilon > 0$ there exists $\delta > 0$ such that $|F(x,s) - F(y,t)| < \epsilon/2n$ whenever $(x,s),(y,t) \in A \times [-n,n]$ and $\|(x,s) - (y,t)\| < \delta$. Then,

$$\begin{aligned}
|\varphi_n(x) - \varphi_n(y)| &= \left| \int_{-n}^{n} F(x,t)\,\mathrm{d}t - \int_{-n}^{n} F(y,t)\,\mathrm{d}t \right| \\
&\leq \int_{-n}^{n} |F(x,t) - F(y,t)|\,\mathrm{d}t \\
&\leq \int_{-n}^{n} \epsilon/2n\,\mathrm{d}t \\
&= \epsilon
\end{aligned}$$

whenever $x,y \in A$ and $\|x - y\| < \delta$. So φ_n is continuous on A and hence on its entire domain.

To show that φ is continuous,

$$\begin{aligned}
|\varphi(x) - \varphi_n(x)| &= \left| \int_{-\infty}^{\infty} F(x,t)\,\mathrm{d}t - \int_{-n}^{n} F(x,t)\,\mathrm{d}t \right| \\
&= \left| \int_{-\infty}^{-n} F(x,t)\,\mathrm{d}t + \int_{n}^{\infty} F(x,t)\,\mathrm{d}t \right| \\
&\le \int_{-\infty}^{-n} |F(x,t)|\,\mathrm{d}t + \int_{n}^{\infty} |F(x,t)|\,\mathrm{d}t \\
&\le \int_{-\infty}^{-n} |h(t)|\,\mathrm{d}t + \int_{n}^{\infty} |h(t)|\,\mathrm{d}t \\
&= \int_{-\infty}^{\infty} |h(t)|\,\mathrm{d}t - \int_{-n}^{n} |h(t)|\,\mathrm{d}t
\end{aligned}$$

But

$$\int_{-\infty}^{\infty} |h(t)|\,\mathrm{d}t = \lim_{n\to\infty} \int_{-n}^{n} |h(t)|\,\mathrm{d}t$$

It follows that $\{\varphi_n\}$ converges uniformly to φ on X. Therefore φ is continuous since each φ_n is.

ii) By Theorem 1.5.1, ψ is continuous on bounded intervals. It follows that ψ is continuous on its entire domain.

To show that $\psi \in \mathcal{R}^1$, we have

$$\begin{aligned}
\int_{-n}^{n} |\psi(t)|\,\mathrm{d}t &= \int_{-n}^{n} \left| \int_{a}^{b} F(x,t)\,\mathrm{d}x \right| \mathrm{d}t \\
&\le \int_{-n}^{n} \left(\int_{a}^{b} |F(x,t)|\,\mathrm{d}x \right) \mathrm{d}t \\
&= \int_{a}^{b} \left(\int_{-n}^{n} |F(x,t)|\,\mathrm{d}t \right) \mathrm{d}x \\
&\le \int_{a}^{b} \left(\int_{-n}^{n} |h(t)|\,\mathrm{d}t \right) \mathrm{d}x \\
&\le (b-a) \int_{-\infty}^{\infty} |h(t)|\,\mathrm{d}t \\
&< \infty
\end{aligned}$$

since $h \in \mathcal{R}^1$.

iii) To show the equality of

$$\int_a^b \varphi(x)\,\mathrm{d}x = \int_{-\infty}^{\infty} \psi(t)\,\mathrm{d}t$$

first note that

$$\begin{aligned}
\int_{-\infty}^{\infty} \psi(t)\,\mathrm{d}t &= \lim_{n\to+\infty} \int_{-n}^{n} \psi(t)\,\mathrm{d}t \\
&= \lim_{n\to+\infty} \int_{-n}^{n} \left(\int_a^b F(x,t)\,\mathrm{d}x\right)\mathrm{d}t \\
&= \lim_{n\to+\infty} \int_a^b \left(\int_{-n}^{n} F(x,t)\,\mathrm{d}t\right)\mathrm{d}x
\end{aligned}$$

Therefore,

$$\begin{aligned}
&\left|\int_a^b \varphi(x)\,\mathrm{d}x - \int_{-n}^{n} \psi(t)\,\mathrm{d}t\right| \\
&= \left|\int_a^b \varphi(x)\,\mathrm{d}x - \int_a^b \left(\int_{-n}^{n} F(x,t)\,\mathrm{d}t\right)\mathrm{d}x\right| \\
&= \left|\int_a^b \left(\int_{-\infty}^{\infty} F(x,t)\,\mathrm{d}t\right)\mathrm{d}x - \int_a^b \left(\int_{-n}^{n} F(x,t)\,\mathrm{d}t\right)\mathrm{d}x\right| \\
&= \left|\int_a^b \left(\int_{-\infty}^{\infty} F(x,t)\,\mathrm{d}t - \int_{-n}^{n} F(x,t)\,\mathrm{d}t\right)\mathrm{d}x\right| \\
&\leq \int_a^b \left|\int_{-\infty}^{\infty} F(x,t)\,\mathrm{d}t - \int_{-n}^{n} F(x,t)\,\mathrm{d}t\right|\mathrm{d}x \\
&= \int_a^b \left|\int_{-\infty}^{-n} F(x,t)\,\mathrm{d}t + \int_{n}^{\infty} F(x,t)\,\mathrm{d}t\right|\mathrm{d}x \\
&\leq \int_a^b \left(\int_{-\infty}^{-n} |F(x,t)|\,\mathrm{d}t + \int_{n}^{\infty} |F(x,t)|\,\mathrm{d}t\right)\mathrm{d}x \\
&\leq \int_a^b \left(\int_{-\infty}^{-n} |h(t)|\,\mathrm{d}t + \int_{n}^{\infty} |h(t)|\,\mathrm{d}t\right)\mathrm{d}x \\
&= \int_a^b \left(\int_{-\infty}^{\infty} |h(t)|\,\mathrm{d}t - \int_{-n}^{n} |h(t)|\,\mathrm{d}t\right)\mathrm{d}x \\
&= (b-a)\left(\int_{-\infty}^{\infty} |h(t)|\,\mathrm{d}t - \int_{-n}^{n} |h(t)|\,\mathrm{d}t\right)
\end{aligned}$$

$\to 0$ as $n \to +\infty$.

Thus,

$$\int_a^b \varphi(x)\,\mathrm{d}x = \int_{-\infty}^{\infty} \psi(t)\,\mathrm{d}t$$

$\square$

Theorem 1.5.3 (Fubini. Version 3)**.** Suppose $F : \mathbb{R}^2 \to \mathbb{C}$ is continuous and there exist $h, k \in \mathcal{R}^1$ such that $|F(x,t)| \le h(x)k(t)$ for all $x, t \in \mathbb{R}$. Define

$$\varphi(x) = \int_{-\infty}^{\infty} F(x,t)\,\mathrm{d}t,\ x \in \mathbb{R}$$

and

$$\psi(t) = \int_{-\infty}^{\infty} F(x,t)\,\mathrm{d}x,\ t \in \mathbb{R}$$

Then,

i) φ and ψ are well-defined.

ii) φ and ψ are absolutely integrable on $\mathbb{R}$.

iii) φ and ψ are continuous.

iv)

$$\int_{-\infty}^{\infty} \varphi(x)\,\mathrm{d}x = \int_{-\infty}^{\infty} \psi(t)\,\mathrm{d}t$$

Proof. i) To see that φ is well-defined,

$$\begin{aligned}
|\varphi(x)| &= \left| \int_{-\infty}^{\infty} F(x,t)\,\mathrm{d}t \right| \\
&\le \int_{-\infty}^{\infty} |F(x,t)|\,\mathrm{d}t \\
&\le \int_{-\infty}^{\infty} |h(x)|\,|k(t)|\,\mathrm{d}t \\
&= |h(x)| \int_{-\infty}^{\infty} |k(t)|\,\mathrm{d}t \\
&= |h(x)|\,\|k\|_1
\end{aligned}$$

Similarly, $|\psi(t)| \le |k(t)|\,\|h\|_1$.

ii) The above remarks also show that φ and ψ are absolutely integrable on $\mathbb{R}$ because,

$$\left|\int_{-\infty}^{\infty} \varphi(x)\,\mathrm{d}x\right| \le \int_{-\infty}^{\infty} |\varphi(x)|\,\mathrm{d}x \le \int_{-\infty}^{\infty} |h(x)|\,\|k\|_1\,\mathrm{d}x = \|h\|_1\,\|k\|_1$$

and similarly,

$$\left|\int_{-\infty}^{\infty} \psi(t)\,\mathrm{d}t\right| \le \int_{-\infty}^{\infty} |\psi(t)|\,\mathrm{d}t \le \int_{-\infty}^{\infty} |k(t)|\,\|h\|_1\,\mathrm{d}t = \|k\|_1\,\|h\|_1$$

iii) To show that φ is continuous, let

$$\varphi_n(x) = \int_{-n}^{n} F(x,t)\,\mathrm{d}t$$

Then, φ_n is continuous by Theorem 1.5.2.

So we have,

$$\begin{aligned}
|\varphi(x) - \varphi_n(x)| &= \left|\int_{-\infty}^{\infty} F(x,t)\,\mathrm{d}t - \int_{-n}^{n} F(x,t)\,\mathrm{d}t\right| \\
&= \left|\int_{-\infty}^{-n} F(x,t)\,\mathrm{d}t + \int_{n}^{\infty} F(x,t)\,\mathrm{d}t\right| \\
&\le \int_{-\infty}^{-n} |F(x,t)|\,\mathrm{d}t + \int_{n}^{\infty} |F(x,t)|\,\mathrm{d}t \\
&\le \int_{-\infty}^{-n} |h(x)|\,|k(t)|\,\mathrm{d}t + \int_{n}^{\infty} |h(x)|\,|k(t)|\,\mathrm{d}t \\
&= |h(x)| \left(\int_{-\infty}^{\infty} |k(t)|\,\mathrm{d}t - \int_{-n}^{n} |k(t)|\,\mathrm{d}t\right)
\end{aligned}$$

$\to 0$ as $n \to +\infty$ uniformly on bounded intervals. This is because $k \in \mathcal{R}^1$ and h is bounded on bounded intervals (because $h \in \mathcal{R}^1$). Thus, the sequence of continuous functions $\{\varphi_n\}$ converges uniformly to φ on each bounded interval. Hence φ is continuous.

A similar argument shows that ψ is continuous.

iv)

$$
\begin{aligned}
&\left|\int_{-\infty}^{\infty} \varphi(x)\,\mathrm{d}x - \int_{-n}^{n}\left(\int_{-n}^{n} F(x,t)\,\mathrm{d}t\right)\mathrm{d}x\right| \\
&\leq \left|\int_{-\infty}^{\infty} \varphi(x)\,\mathrm{d}x - \int_{-n}^{n}\left(\int_{-\infty}^{\infty} F(x,t)\,\mathrm{d}t\right)\mathrm{d}x\right| \\
&+ \left|\int_{-n}^{n}\left(\int_{-\infty}^{\infty} F(x,t)\,\mathrm{d}t\right)\mathrm{d}x - \int_{-n}^{n}\left(\int_{-n}^{n} F(x,t)\,\mathrm{d}t\right)\mathrm{d}x\right|
\end{aligned}
$$

Now,

$$
\begin{aligned}
&\left|\int_{-\infty}^{\infty} \varphi(x)\,\mathrm{d}x - \int_{-n}^{n}\left(\int_{-\infty}^{\infty} F(x,t)\,\mathrm{d}t\right)\mathrm{d}x\right| \\
&= \left|\int_{-\infty}^{\infty} \varphi(x)\,\mathrm{d}x - \int_{-n}^{n} \varphi(x)\,\mathrm{d}x\right|
\end{aligned}
$$

$\to 0$ as $n \to +\infty$.

Also,

$$
\begin{aligned}
&\left|\int_{-n}^{n}\left(\int_{-\infty}^{\infty} F(x,t)\,\mathrm{d}t\right)\mathrm{d}x - \int_{-n}^{n}\left(\int_{-n}^{n} F(x,t)\,\mathrm{d}t\right)\mathrm{d}x\right| \\
&\leq \int_{-n}^{n}\left|\int_{-\infty}^{-n} F(x,t)\,\mathrm{d}t + \int_{n}^{\infty} F(x,t)\,\mathrm{d}t\right|\mathrm{d}x \\
&\leq \int_{-n}^{n}\left(\int_{-\infty}^{-n} |F(x,t)|\,\mathrm{d}t + \int_{n}^{\infty} |F(x,t)|\,\mathrm{d}t\right)\mathrm{d}x \\
&\leq \int_{-n}^{n}\left(\int_{-\infty}^{-n} |h(x)|\,|k(t)|\,\mathrm{d}t + \int_{n}^{\infty} |h(x)|\,|k(t)|\,\mathrm{d}t\right)\mathrm{d}x \\
&= \left(\int_{-n}^{n} |h(x)|\,\mathrm{d}x\right)\left(\int_{-\infty}^{\infty} |k(t)|\,\mathrm{d}t - \int_{-n}^{n} |k(t)|\,\mathrm{d}t\right) \\
&\leq \|h\|_1 \left(\int_{-\infty}^{\infty} |k(t)|\,\mathrm{d}t - \int_{-n}^{n} |k(t)|\,\mathrm{d}t\right)
\end{aligned}
$$

$\to 0$ as $n \to +\infty$.

Thus,

$$
\int_{-\infty}^{\infty} \varphi(x)\,\mathrm{d}x = \lim_{n\to+\infty} \int_{-n}^{n}\left(\int_{-n}^{n} F(x,t)\,\mathrm{d}t\right)\mathrm{d}x
$$

and similarly,

$$\int_{-\infty}^{\infty} \psi(t)\,\mathrm{d}t = \lim_{n\to+\infty} \int_{-n}^{n} \left(\int_{-n}^{n} F(x,t)\,\mathrm{d}x \right) \mathrm{d}t$$

But,

$$\int_{-n}^{n} \left(\int_{-n}^{n} F(x,t)\,\mathrm{d}t \right) \mathrm{d}x = \int_{-n}^{n} \left(\int_{-n}^{n} F(x,t)\,\mathrm{d}x \right) \mathrm{d}t$$

so the result follows.

□

1.6 Leibniz Rule

Leibniz Rule is concerned with differentiating under an integral. The proof depends on Fubini's Theorem discussed in the last section.

Theorem 1.6.1 (Leibniz Rule. Version 1)**.** Suppose $F : [a,b] \times [c,d] \to \mathbb{C}$ and $\frac{\partial F}{\partial x}(x,y)$ is defined and continuous on $[a,b] \times [c,d]$. Let

$$\varphi(x) = \int_{c}^{d} F(x,y)\,\mathrm{d}y$$

Then φ is continuously differentiable on $[a,b]$ and

$$\varphi'(x) = \int_{c}^{d} \frac{\partial F}{\partial x}(x,y)\,\mathrm{d}y$$

Proof. Let

$$\rho(x) = \int_{c}^{d} \frac{\partial F}{\partial x}(x,y)\,\mathrm{d}y$$

Then by Theorem 1.5.1 and the fact that $\frac{\partial F}{\partial x}(x,y)$ is continuous, ρ is continuous on $[a,b]$.

Now, for $x \in [a,b]$

$$\begin{aligned}\int_a^x \rho(s)\,\mathrm{d}s &= \int_a^x \int_c^d \frac{\partial F}{\partial s}(s,y)\,\mathrm{d}y\,\mathrm{d}s \\ &= \int_c^d \int_a^x \frac{\partial F}{\partial s}(s,y)\,\mathrm{d}s\,\mathrm{d}y \quad \text{(by Theorem 1.5.1)} \\ &= \int_c^d [F(x,y) - F(a,y)]\,\mathrm{d}y \quad \text{(by Theorem 1.1.6)} \\ &= \varphi(x) - \varphi(a)\end{aligned}$$

Therefore,

$$\varphi(x) = \varphi(a) + \int_a^x \rho(s)\,\mathrm{d}s$$

and because ρ is continuous on $[a,b]$, we have

$$\varphi'(x) = \rho(x)$$

by the Fundamental Theorem of Calculus. □

Example 1.6.2. Suppose $a,b,c,d \in \mathbb{R}$, $K = [a,b] \times [c,d]$, $f : K \to \mathbb{C}$ is continuous, $U = \mathbb{R}^2 \backslash K$, and for $(x,y) \in U$ we define

$$u(x,y) = \int_c^d \left(\int_a^b f(s,t) \log[(x-s)^2 + (y-t)^2]\,\mathrm{d}s \right) \mathrm{d}t$$

Prove that u is harmonic on U. That is, it is $\mathcal{C}^\infty$ and

$$\frac{\partial^2 u}{\partial x^2}(x,y) + \frac{\partial^2 u}{\partial y^2}(x,y) = 0$$

for all $(x,y) \in U$.

Solution. Since $\log[(x-s)^2 + (y-t)^2]$ is $\mathcal{C}^\infty$ with respect to x and y, for $(x,y) \in U$, we have by Theorem 1.6.1

$$\frac{\partial^k u}{\partial x^k}(x,y) = \int_c^d \left(\int_a^b f(s,t) \frac{\partial^k}{\partial x^k} \log[(x-s)^2 + (y-t)^2]\,\mathrm{d}s \right) \mathrm{d}t$$

for $k \in \mathbb{N}$. Similarly, all partial derivatives of u with respect to y exist as well. For the second assertion, note that

$$\begin{aligned}
\frac{\partial^2 u}{\partial x^2}(x,y) &= \int_c^d \left(\int_a^b f(s,t) \frac{\partial^2}{\partial x^2} \log[(x-s)^2 + (y-t)^2] \, \mathrm{d}s \right) \mathrm{d}t \\
&= \int_c^d \left(\int_a^b f(s,t) \frac{\partial}{\partial x} \left[\frac{2(x-s)}{(x-s)^2 + (y-t)^2} \right] \mathrm{d}s \right) \mathrm{d}t \\
&= \int_c^d \left(\int_a^b f(s,t) \frac{2[(x-s)^2 + (y-t)^2] - 4(x-s)^2}{[(x-s)^2 + (y-t)^2]^2} \, \mathrm{d}s \right) \mathrm{d}t
\end{aligned}$$

Similarly,

$$\frac{\partial^2 u}{\partial y^2}(x,y) = \int_c^d \left(\int_a^b f(s,t) \frac{2[(x-s)^2 + (y-t)^2] - 4(y-t)^2}{[(x-s)^2 + (y-t)^2]^2} \, \mathrm{d}s \right) \mathrm{d}t$$

Therefore,

$$\begin{aligned}
\frac{\partial^2 u}{\partial x^2}(x,y) + \frac{\partial^2 u}{\partial y^2}(x,y) &= \int_c^d \left(\int_a^b f(s,t) \frac{2[(x-s)^2 + (y-t)^2] - 4(x-s)^2}{[(x-s)^2 + (y-t)^2]^2} \, \mathrm{d}s \right) \mathrm{d}t \\
&\quad + \int_c^d \left(\int_a^b f(s,t) \frac{2[(x-s)^2 + (y-t)^2] - 4(y-t)^2}{[(x-s)^2 + (y-t)^2]^2} \, \mathrm{d}s \right) \mathrm{d}t \\
&= \int_c^d \left(\int_a^b f(s,t) \frac{4[(x-s)^2 + (y-t)^2] - 4[(x-s)^2 + (y-t)^2]}{[(x-s)^2 + (y-t)^2]^2} \, \mathrm{d}s \right) \mathrm{d}t \\
&= 0
\end{aligned}$$

as required.

Theorem 1.6.3 (Liebniz Rule. Version 2)**.** Suppose $X \subset \mathbb{R}$ is open, $F : X \times \mathbb{R} \to \mathbb{C}$ and $\frac{\partial F}{\partial x}(x,y)$ is defined and continuous on $X \times \mathbb{R}$. Furthermore, suppose there exists $h \in \mathcal{R}^1$ such that

$$\left| \frac{\partial F}{\partial x}(x,y) \right| \leq |h(y)|$$

for all $y \in \mathbb{R}$. Let

$$\varphi(x) = \int_{-\infty}^{\infty} F(x,y) \, \mathrm{d}y$$

for $x \in X$. Then φ is continuously differentiable on its domain and

$$\varphi'(x) = \int_{-\infty}^{\infty} \frac{\partial F}{\partial x}(x, y)\,\mathrm{d}y$$

Proof. The proof of this theorem is very similar to that of Theorem 1.6.1.

Let

$$\rho(x) = \int_{-\infty}^{\infty} \frac{\partial F}{\partial x}(x, y)\,\mathrm{d}y$$

Then by Theorem 1.5.2 and the fact that $\frac{\partial F}{\partial x}(x, y)$ is defined and continuous, ρ is continuous on X.

Now, for $x_0, x \in X$

$$\begin{aligned}
\int_{x_0}^{x} \rho(s)\,\mathrm{d}s &= \int_{x_0}^{x}\int_{-\infty}^{\infty} \frac{\partial F}{\partial s}(s, y)\,\mathrm{d}y\,\mathrm{d}s \\
&= \int_{-\infty}^{\infty}\int_{x_0}^{x} \frac{\partial F}{\partial s}(s, y)\,\mathrm{d}s\,\mathrm{d}y \quad \text{(by Theorem 1.5.2)} \\
&= \int_{-\infty}^{\infty} [F(x, y) - F(x_0, y)]\,\mathrm{d}y \quad \text{(by Theorem 1.1.6)} \\
&= \varphi(x) - \varphi(x_0)
\end{aligned}$$

Therefore,

$$\varphi(x) = \varphi(x_0) + \int_{x_0}^{x} \rho(s)\,\mathrm{d}s$$

and because ρ is continuous on X, we have

$$\varphi'(x) = \rho(x)$$

by the Fundamental Theorem of Calculus. □

Example 1.6.4. Let

$$f(x) = \left(\int_0^x e^{-t^2}\,\mathrm{d}t\right)^2 + \int_0^1 \frac{e^{-x^2(1+t^2)}}{1+t^2}\,\mathrm{d}t$$

Show that

i) $f'(x) = 0$ for all $x \in \mathbb{R}$.

ii) $f(x) = \pi/2$ for all $x \in \mathbb{R}$.

iii)

$$\int_0^\infty e^{-x^2}\,\mathrm{d}x = \sqrt{\frac{\pi}{2}}$$

Solution. i) Since

$$\left|\frac{e^{-x^2(1+t^2)}}{1+t^2}\right| \leq e^{-x^2}$$

for $0 \leq t \leq 1$ and clearly $e^{-x^2} \in \mathcal{R}^1$, we can apply Theorem 1.6.3 to obtain

$$\begin{aligned}
f'(x) &= 2e^{-x^2}\left(\int_0^x e^{-t^2}\,\mathrm{d}t\right) + \int_0^1 \frac{\partial}{\partial x}\left(\frac{e^{-x^2(1+t^2)}}{1+t^2}\right)\mathrm{d}t \\
&= 2e^{-x^2}\left(\int_0^x e^{-t^2}\,\mathrm{d}t\right) - 2xe^{-x^2}\int_0^1 e^{-x^2t^2}\,\mathrm{d}t \\
&= 2e^{-x^2}\left(\int_0^x e^{-t^2}\,\mathrm{d}t\right) - 2e^{-x^2}\int_0^x e^{-s^2}\,\mathrm{d}s \quad (s = xt) \\
&= 0
\end{aligned}$$

ii)

$$\begin{aligned}
f(0) &= \left(\int_0^0 e^{-t^2}\,\mathrm{d}t\right)^2 + \int_0^1 \frac{1}{1+t^2}\,\mathrm{d}t \\
&= \arctan(1) - \arctan(0) \\
&= \frac{\pi}{2}
\end{aligned}$$

Hence $f(x) = \pi/2$ for all $x \in \mathbb{R}$ since f is constant by part i).

iii) From part ii) we obtain

$$\left(\int_0^\infty e^{-t^2}\,\mathrm{d}t\right)^2 = \frac{\pi}{2} - \lim_{x\to\infty}\int_0^1 \frac{e^{-x^2(1+t^2)}}{1+t^2}\,\mathrm{d}t$$

But

$$\frac{e^{-x^2(1+t^2)}}{1+t^2} \to 0$$

uniformly, as $x \to \infty$ for $0 \le t \le 1$. Hence

$$\left(\int_0^\infty e^{-t^2}\,\mathrm{d}t \right)^2 = \frac{\pi}{2}$$

from which the result follows.

Example 1.6.5. i) Prove that for all $x \in \mathbb{R}$, the map $t \mapsto e^{-t^2/2} \cos xt$ belongs to $\mathcal{R}^1$.

ii) Let

$$g(x) = \int_{-\infty}^{\infty} e^{-t^2/2} \cos xt\,\mathrm{d}t$$

for $x \in \mathbb{R}$. Prove that $g \in \mathcal{C}^1(\mathbb{R})$ and $g'(x) = -xg(x)$.

iii) Deduce that $g(x) = 2\sqrt{\pi} e^{-x^2/2}$ for all $x \in \mathbb{R}$.

Solution. i) This follows immediately from the inequality

$$\left| e^{-t^2/2} \cos xt \right| \le e^{-t^2/2}$$

ii) From part i), the expression under the integral sign belongs to $\mathcal{R}^1$ and hence by Theorem 1.6.3, g belongs to $\mathcal{C}^1(\mathbb{R})$ and

$$\begin{aligned} g'(x) &= \int_{-\infty}^{\infty} \frac{\partial}{\partial x} \left[e^{-t^2/2} \cos xt \right] \mathrm{d}t \\ &= \int_{-\infty}^{\infty} -te^{-t^2/2} \sin xt\,\mathrm{d}t \\ &= \int_{-\infty}^{\infty} \frac{d}{dt}(e^{-t^2/2}) \sin xt\,\mathrm{d}t \\ &= e^{-t^2/2} \sin xt \,\Big|_{t=-\infty}^{\infty} - x \int_{-\infty}^{\infty} e^{-t^2/2} \cos xt\,\mathrm{d}t \\ &= -xg(x) \end{aligned}$$

iii) Let

$$h(x) = \frac{g(x)}{e^{-x^2/2}}$$

Then by part ii) $h \in \mathcal{C}^1(\mathbb{R})$ and

$$\begin{aligned} h'(x) &= \frac{1}{e^{-x^2}}\left[g'(x)e^{-x^2/2} + xg(x)e^{-x^2/2}\right] \\ &= \frac{1}{e^{-x^2}}\left[-xg(x)e^{-x^2/2} + xg(x)e^{-x^2/2}\right] \\ &= 0 \end{aligned}$$

Hence

$$g(x) = Ce^{-x^2/2}$$

But

$$\begin{aligned} g(0) &= \int_{-\infty}^{\infty} e^{-t^2/2}\cos 0\,\mathrm{d}t \\ &= \sqrt{2}\int_{-\infty}^{\infty} e^{-s^2}\,\mathrm{d}s \quad (t = \sqrt{2}s) \\ &= 2\sqrt{2}\int_{0}^{\infty} e^{-s^2}\,\mathrm{d}s \\ &= 2\sqrt{2}\sqrt{\frac{\pi}{2}} \quad \text{(by Example 1.6.4)} \\ &= 2\sqrt{\pi} \end{aligned}$$

Therefore,

$$g(x) = 2\sqrt{\pi}e^{-x^2/2}$$

Theorem 1.6.6 (Leibniz Rule. Version 3)**.** Suppose $X \subset \mathbb{R}^n$ is open, $F : X \times \mathbb{R} \to \mathbb{C}$ and

$$\frac{\partial F}{\partial x_k}(x_1, x_2, \ldots, x_n, y)$$

is defined and continuous for $1 \le k \le n$. Furthermore, suppose there exist functions $h_k \in \mathcal{R}^1, 1 \le k \le n$ such that

$$\left|\frac{\partial F}{\partial x_k}(x_1, x_2, \ldots, x_n, y)\right| \le |h_k(y)|$$

for $1 \leq k \leq n$. Let

$$\varphi(x) = \int_{-\infty}^{\infty} F(x, y)\,\mathrm{d}y$$

for $x \in X$. Then φ is continuously differentiable on its domain and

$$\frac{\partial \varphi}{\partial x_k}(x_1, x_2, \ldots, x_n, y) = \int_{-\infty}^{\infty} \frac{\partial F}{\partial x_k}(x_1, x_2, \ldots, x_n, y)\,\mathrm{d}y$$

Proof. The proof follows directly from Theorem 1.6.3. □

Chapter 2

Fourier Series

2.1 The Heat Problem

The one dimensional heat problem is the following: Given a continuous function $f : [0, \pi] \to \mathbb{R}$ which satisfies $f(0) = f(\pi) = 0$ and represents the temperature at time zero of a wire of length π, is it possible to find a continuous function $u : [0, \pi] \times [0, \infty) \to \mathbb{R}$ which represents the temperature of the wire at time $t \geq 0$?

It can be shown (although we will not do it here) using methods from physics, that u must have partial derivatives on $(0, \pi) \times (0, \infty)$ and furthermore, it must satisfy the following conditions

$$u(x, 0) = f(x) \text{ for all } x \in [0, \pi] \tag{2.1}$$

$$u(0, t) = u(\pi, t) = 0 \text{ for all } t \geq 0 \tag{2.2}$$

$$\frac{\partial u}{\partial t}(x, t) = c^2 \frac{\partial^2 u}{\partial x^2}(x, t) \text{ for all } (x, t) \in (0, \pi) \times (0, \infty) \tag{2.3}$$

where c is a real constant.

To find a possible solution for u, let us make two simplying assumptions, namely, that $c = 1$ and $u(x, t) = X(x)T(t)$ for some functions $X : [0, \pi] \to \mathbb{R}$ and $T : [0, \infty) \to \mathbb{R}$.

By condition 2.3, we must have $X(x)T'(t) = X''(x)T(t)$. If we omit the trivial case where $X = T = 0$, then there exist $0 < x_0 < \pi$ and $t_0 > 0$ such that $X(x_0) \neq 0$ and $T(t_0) \neq 0$.

Let

$$\lambda = \frac{T'(t_0)}{T(t_0)} = \frac{X''(x_0)}{X(x_0)}$$

Then $X(x_0)T'(t) = X''(x_0)T(t)$ and $X(x)T'(t_0) = X''(x)T(t_0)$. So we have

$$T'(t) = \lambda T(t) \text{ and} \tag{2.4}$$

$$X''(x) = \lambda X(x) \tag{2.5}$$

There are three possibilities for λ. Either $\lambda > 0, \lambda = 0$ or $\lambda < 0$. If $\lambda > 0$ then by equation 2.5 we must have

$$X(x) = \alpha e^{\sqrt{\lambda}x} + \beta e^{-\sqrt{\lambda}x}$$

But by condition 2.2, this implies $\alpha = \beta = 0$ which is impossible.

If $\lambda = 0$ then $X''(x) = 0$ so $X(x) = ax$ for some constant a. Again, condition 2.2 implies $a = 0$ so this is impossible.

Therefore $\lambda < 0$. Say $\lambda = -k^2$ for some $k \in \mathbb{R}, k > 0$. Equation 2.5 implies

$$X(x) = \alpha \cos kx + \beta \sin kx$$

But condition 2.2 implies $\alpha = 0$ and $k \in \mathbb{N}$.

Substituting λ in equation 2.4, we obtain

$$T(t) = \gamma e^{-k^2 t} \text{ for } t \geq 0$$

Therefore, u is of the form

$$u(x,t) = be^{-k^2 t} \sin kx \text{ for } (x,t) \in [0,\pi] \times [0,\infty)$$

where b is a constant. It follows that for any choice of constants $b_1, b_2, \ldots, b_n$ the function

$$u(x,t) = \sum_{k=1}^{n} b_k e^{-k^2 t} \sin kx \tag{2.6}$$

satisfies conditions 2.2 and 2.3 above. For condition 2.2 to hold, we must be able to write f in the form

$$f(x) = \sum_{k=1}^{n} b_k \sin kx \tag{2.7}$$

for some constants $b_1, b_2, \ldots, b_n$.

Another possibility for f and u is that there exist real scalars $b_1, b_2, \ldots$ such that

$$f(x) = \sum_{k=1}^{\infty} b_k \sin kx \tag{2.8}$$

and

$$u(x,t) = \sum_{k=1}^{\infty} b_k e^{-k^2 t} \sin kx \tag{2.9}$$

Of course, the series in 2.9 must be $\mathcal{C}^2$ and we must be able to differentiate it term by term on $(0, \pi) \times (0, \infty)$.

We will show in this chapter the conditions that f must satisfy for equation 2.8 to hold and hence, for the function u in 2.9 to represent a solution to the heat problem.

In the next section, we will also show that the solution to the heat problem derived here is unique.

Before concluding this section, we will show that the function u, defined by 2.9 is $\mathcal{C}^\infty$.

Lemma 2.1.1. Suppose $\{b_k\}_{k=1}^{\infty}$ is a bounded real sequence.

i) Let $\tau > 0$ and $x \in \mathbb{R}$ and define $\varphi_k(t) : [\tau, \infty) \to \mathbb{R}$ by $\varphi_k(t) = b_k e^{-k^2 t} \sin kx$. Then

$$\sum_{k=1}^{\infty} k^n \varphi_k(t)$$

converges uniformly on $[\tau, \infty)$ for fixed $0 \leq n \in \mathbb{Z}$.

ii) Let $t > 0$ and define $\psi_k(x) : \mathbb{R} \to \mathbb{R}$ by $\psi_k(x) = b_k e^{-k^2 t} \sin kx$. Then

$$\sum_{k=1}^{\infty} k^n \psi_k(x)$$

converges uniformly on $\mathbb{R}$ for fixed $0 \leq n \in \mathbb{Z}$.

Proof. Let M be an upper bound for the set $\{|b_k| : k \geq 1\}$.

i) For $t \geq \tau > 0$ we have

$$\begin{aligned} |k^n \varphi_k(t)| &= \left| k^n b_k e^{-k^2 t} \sin kx \right| \\ &\leq M k^n e^{-k^2 t} \\ &\leq M k^n e^{-k^2 \tau} \\ &\leq M k^n e^{-k\tau} \\ &= M k^n a^k \text{ where } a = e^{-\tau} < 1 \end{aligned}$$

By the ratio test, $\sum_{k=1}^{\infty} M k^n a^k$ converges and so, by the Weierstrass M-test, $\sum_{k=1}^{\infty} k^n \varphi_k(t)$ converges uniformly on $[\tau, \infty)$.

ii) The proof is similar to that in part i).

□

Theorem 2.1.2. Let $u : \mathbb{R} \times (0, \infty)$ be defined by

$$u(x,t) = \sum_{k=1}^{\infty} b_k e^{-k^2 t} \sin kx$$

where $\{b_k\}_{k=1}^{\infty}$ is a bounded real sequence. Then u is $\mathcal{C}^{\infty}$ on its domain.

Proof. Fix $x \in \mathbb{R}$ and $\tau > 0$ and let $\varphi_k(t) : [\tau, \infty) \to \mathbb{R}$ be defined by $\varphi_k(t) = b_k e^{-k^2 t} \sin kx$. Then by Lemma 2.1.1, the function $\varphi : [\tau, \infty) \to \mathbb{R}$ defined by

$$\varphi(t) = \sum_{k=1}^{\infty} \varphi_k(t)$$

converges uniformly on its domain. Also by Lemma 2.1.1,

$$\sum_{k=1}^{\infty} \varphi_k'(t) = \sum_{k=1}^{\infty} -k^2 b_k e^{-k^2 t} \sin kx$$

converges uniformly on its domain.

Therefore, by Theorem 1.2.1, φ is $\mathcal{C}^1$ on $[\tau, \infty)$ and $\varphi'(t) = \sum_{k=1}^{\infty} \varphi_k'(t)$.

Assume φ is $\mathcal{C}^n$ for $n > 1$ and that $\varphi^{(n)}(t) = \sum_{k=1}^{\infty} \varphi_k^{(n)}(t)$. Then, noting that $\varphi_k^{(n+1)}(t) = (-1)^{(n+1)} k^{2(n+1)} b_k e^{-k^2 t} \sin kx$ and using Lemma 2.1.1 combined with Theorem 1.2.1 again, it is easy to see that φ is $\mathcal{C}^{n+1}$ on $[\tau, \infty)$.

Therefore, by induction, φ is $\mathcal{C}^\infty$ on $[\tau, \infty)$ for all $\tau > 0$. It follows that φ is $\mathcal{C}^\infty$ on $(0, \infty)$.

Now let $\psi_k(x) = b_k e^{-k^2 t} \sin kx$ for $x \in \mathbb{R}$ and fixed $t > 0$. An argument identical to that used above shows that the function $\psi(x) = \sum_{k=1}^\infty \psi_k(x)$ is $\mathcal{C}^\infty$ on $\mathbb{R}$.

Thus, u is $\mathcal{C}^\infty$ on $\mathbb{R} \times (0, \infty)$. □

2.2 Maximum Minimum Priniple

Roughly stated, the Maximum Minimum Principle says the maximum (minimum) value of a function on a bounded region, under certain conditions, occurs along the boundary of the region. This result will be used to prove the uniqueness of the solution to the heat problem. But we begin with a lemma that will be used to prove the principle.

Lemma 2.2.1. Suppose $v : [a, b] \times [0, \infty) \to \mathbb{R}$ is continuous on its domain and $\mathcal{C}^2$ on $(a, b) \times (0, \infty)$. Futhermore, suppose

$$\frac{\partial v}{\partial t}(x,t) - \frac{\partial^2 v}{\partial x^2}(x,t) < 0$$

for $(x, t) \in (a, b) \times (0, \infty)$.

Fix $T > 0$ and define

$$B_T = \{(a,t) : 0 \le t \le T\} \cup \{(b,t) : 0 \le t \le T\} \cup \{(x,0) : a \le x \le b\}$$

Then $\max\{v(x,t) : a \le x \le b, 0 \le t \le T\} = \max\{v(x,t) : (x,t) \in B_T\}$.

Proof. Since $[a,b]\times[0,T]$ is compact and v is continuous, v attains an absolute maximum on this region. Say this absolute maximum occurs at the point (x_0, t_0). If $a < x_0 < b$ or $0 < t_0 \le T$ then by the first derivative test we must have

$$\frac{\partial v}{\partial t}(x_0,t_0) = \frac{\partial v}{\partial x}(x_0,t_0) = 0$$

By the second derivative test we must have

$$\frac{\partial^2 v}{\partial x^2}(x_0,t_0) < 0$$

But then

$$\frac{\partial v}{\partial t}(x,t) - \frac{\partial^2 v}{\partial x^2}(x,t) > 0$$

which is a contradiction. □

Theorem 2.2.2 (Maximum Minimum Principle)**.** Suppose $u : [a,b]\times[0,\infty) \to \mathbb{R}$ is continous on its domain, $\mathcal{C}^2$ on $(a,b)\times(0,\infty)$ and satisfies

$$\frac{\partial u}{\partial t}(x,t) = \frac{\partial^2 u}{\partial x^2}(x,t)$$

for $(x,t) \in (a,b)\times(0,\infty)$.

Then for all $T > 0$,

i)

$$\max\{u(x,t) : a \le x \le b, 0 \le t \le T\} = \max\{u(x,t) : (x,t) \in B_T\}$$

ii)

$$\min\{u(x,t) : a \le x \le b, 0 \le t \le T\} = \min\{u(x,t) : (x,t) \in B_T\}$$

where B_T is defined as in Lemma 2.2.1.

Proof. i) Given $T > 0$, let $M_T = \max\{u(x,t) : (x,t) \in B_T\}$. Fix $\epsilon > 0$ and define

$$v(x,t) = u(x,t) + \epsilon x^2$$

for $(x,t) \in [a,b]\times[0,\infty)$. Then v is continuous on its domain, $\mathcal{C}^2$ on $(a,b)\times(0,\infty)$ and satisfies

$$\begin{aligned}
&\frac{\partial v}{\partial t}(x,t) - \frac{\partial^2 v}{\partial x^2}(x,t) \\
&= \frac{\partial u}{\partial t}(x,t) - \frac{\partial^2 u}{\partial x^2}(x,t) - 2\epsilon \\
&= -2\epsilon \\
&< 0
\end{aligned}$$

for all $(x,t) \in (a,b)\times(0,\infty)$.

By Lemma 2.2.1 $v(x,t) \le M_T$ when $a \le x \le b$ and $0 \le t \le T$. So

$$u(x,t) + \epsilon x^2 \le M_T$$

on this same region. Since this is true for all $\epsilon > 0$ and $T > 0$, the result follows.

ii) Apply part i) with u replaced by $-u$.

□

Definition 2.2.3. Let $C_0[0,\pi]$ be defined by the set of all continuous, real-valued functions on $[0,\pi]$ which satisfy $f(0) = f(\pi) = 0$.

Corollary 2.2.4. Suppose $f_1, f_2 \in C_0[0,\pi]$ and u_1, u_2 are the solutions to the heat problem corresponding to f_1, f_2, respectively. Furthermore, suppose $f_1(x) \le f_2(x)$ for all $x \in [0,\pi]$. Then $u_1(x,t) \le u_2(x,t)$ for all $(x,t) \in [0,\pi] \times [0,\infty)$.

Proof. Let $f = f_1 - f_2$. Then $u = u_1 - u_2$ is the solution to the heat problem corresponding to f. Now u satisfies the hypotheses of Theorem 2.2.2 so $u(x,t) \le M_T$ for all $(x,t) \in [0,\pi] \times [0,\infty)$ (where M_T is defined as in the proof of Theorem 2.2.2). But $M_T \le 0$. So the result follows. □

Corollary 2.2.5. Suppose $f \in C_0[0,\pi]$ and u is the solution to the heat problem corresponding to f. If $|f(x)| \le M$ for some $M > 0$ and all $x \in [0,\pi]$ then $|u(x,t)| \le M$ for all $(x,t) \in [0,\pi] \times [0,\infty)$.

Proof. Since $-M \le f(x) \le M$ for all $x \in [0,\pi]$ the value of M_T in the proof of Theorem 2.2.2 satisfies $-M \le M_T \le M$ as well. Therefore, by Theorem 2.2.2, $-M \le u(x,t) \le M$ for all $(x,t) \in [0,\pi] \times [0,\infty)$. □

Corollary 2.2.6. Suppose $f_1, f_2 \in C_0[0,\pi]$, u_1, u_2 are the solutions to the heat problem corresponding to f_1, f_2 respectively and for some $\epsilon > 0$ we have $|f_1(x) - f_2(x)| \le \epsilon$. Then $|u_1(x,t) - u_2(x,t)| \le \epsilon$ for all $(x,t) \in [0,\pi] \times [0,\infty)$.

Proof. Let $f = f_1 - f_2$ and $u = u_1 - u_2$. Then u is the solution to the heat problem corresponding to f. So the result is an immediate consequence of Corollary 2.2.5. □

Theorem 2.2.7 (Uniqueness of the Heat Equation)**.** If $f \in C_0[0, \pi]$ then there is at most one solution to the heat problem corresponding to f.

Proof. Suppose u_1 and u_2 are two solutions of the heat problem corresponding to f. Apply Corollary 2.2.6 with $f_1 = f_2 = f$. □

2.3 The Fourier Problem

Recall from section 1 of this chapter that the heat problem has a solution corresponding to $f \in C_0[0, \pi]$ if we can find real constants $b_1, b_2, \ldots, b_n$ such that

$$f(x) = \sum_{k=1}^{n} b_k \sin kx$$

for $x \in [0, \pi]$.

In this section, we ask a slightly different question, called the *Fourier Problem*: Given $f : \mathbb{R} \to \mathbb{R}$ is it possible to find real numbers $a_0, a_1, b_1, a_2, b_2, \ldots$ such that

$$f(x) = \frac{a_0}{2} + \sum_{k=1}^{\infty} (a_k \cos kx + b_k \sin kx)$$

for all $x \in \mathbb{R}$. If this is possible, then we must have $f(x + 2\pi) = f(x)$, which leads to the following definition and results.

Definition 2.3.1. If f is a complex-valued function on $\mathbb{R}$ and p is a non-zero real number, we say that f *has period* p or f *is p-periodic* provided $f(x + p) = f(x)$ for all $x \in \mathbb{R}$. We denote the set of all p-periodic functions by P_p.

Proposition 2.3.2. i) If $f \in P_p$ then $f(x + kp) = f(x)$ for all $k \in \mathbb{Z}$ and $x \in \mathbb{R}$.

ii) Let $T = \{z \in \mathbb{C} : |z| = 1\}$ and suppose $g : T \to \mathbb{C}$ is given. Then $f : \mathbb{R} \to \mathbb{C}$ defined by $f(x) = g(e^{ix})$ is 2π-periodic. Conversely, given $f \in P_{2\pi}$, there exists a unique $g : T \to \mathbb{C}$ such that $f(x) = g(e^{ix})$ for all $x \in \mathbb{R}$. The map $f \mapsto g$ is an algebra isomorphism between $P_{2\pi}$ and the set of functions from T to $\mathbb{C}$.

iii) Given $a \in \mathbb{R}$ and $f_0 : [a, a + 2\pi) \to \mathbb{C}$ there exists a unique 2π-periodic function f such that $f_0(x) = f(x)$ for all $x \in [a, a + 2\pi)$.

iv) If $f \in P_p$ and we let $g(x) = f(px)$ then $g \in P_1$ and the map $f \mapsto g$ is and algebra isomorphism between P_p and P_1.

v) If $f \in P_{2\pi}$ then f is Riemann Integrable on $[a, b]$ whenever $a < b$ iff f is Riemann Integrable on $[c, c + 2\pi]$ for some $c \in \mathbb{R}$, in which case

$$\int_c^{c+2\pi} f = \int_0^{2\pi} f$$

Proof. i) Trivial.

ii) Trivial.

iii) Trivial.

iv) It suffices to note that $g(x+1) = f(p(x+1)) = f(px+p) = f(px) = g(x)$.

v) The first statement is trivial. To prove the equality of the integrals, choose $n \in \mathbb{Z}$ such that $2\pi n < c \leq 2\pi(n + 1)$. Then

$$\begin{aligned}
\int_c^{c+2\pi} f(x)\,\mathrm{d}x &= \int_c^{2\pi(n+1)} f(x)\,\mathrm{d}x + \int_{2\pi(n+1)}^{c+2\pi} f(x)\,\mathrm{d}x \\
&= \int_{c-2\pi n}^{2\pi} f(x + 2\pi n)\,\mathrm{d}x + \int_0^{c-2\pi n} f(x + 2\pi(n+1))\,\mathrm{d}x \\
&= \int_{c-2\pi n}^{2\pi} f(x)\,\mathrm{d}x + \int_0^{c-2\pi n} f(x)\,\mathrm{d}x \quad \text{by part i)} \\
&= \int_0^{2\pi} f(x)\,\mathrm{d}x
\end{aligned}$$

$\square$

In the remainder of this book we will only consider functions which are 2π-periodic.

In order to find real numbers $a_0, a_1, b_1, a_2, b_2, \ldots$ such that

$$f(x) = \frac{a_0}{2} + \sum_{k=1}^{\infty}(a_k \cos kx + b_k \sin kx)$$

for all $x \in \mathbb{R}$, we need the following lemma.

Lemma 2.3.3. If $m, n \in \mathbb{Z}$ then

i)

$$\int_{-\pi}^{\pi} \cos mx \,\mathrm{d}x = \int_{-\pi}^{\pi} \sin mx \,\mathrm{d}x = 0$$

provided $m \neq 0$.

ii)

$$\int_{-\pi}^{\pi} \cos mx \sin nx \,\mathrm{d}x = 0$$

iii)

$$\begin{aligned} &\int_{-\pi}^{\pi} \cos mx \cos nx \,\mathrm{d}x \\ &= \int_{-\pi}^{\pi} \sin mx \sin nx \,\mathrm{d}x = 0 \text{ if } m \neq \pm n \end{aligned}$$

iv)

$$\int_{-\pi}^{\pi} \cos^2 mx \,\mathrm{d}x = \int_{-\pi}^{\pi} \sin^2 mx \,\mathrm{d}x = \pi$$

v)

$$\int_{-\pi}^{\pi} e^{imx} e^{-inx} \,\mathrm{d}x = \begin{cases} 0 & \text{if } m \neq n \\ 2\pi & \text{if } m = n \end{cases}$$

Proof. i) Trivial.

ii) It suffices to see that $\cos mx \sin nx$ is odd.

iii) Integrating by parts twice, we obtain

$$\begin{aligned}
&\int_{-\pi}^{\pi} \cos mx \cos nx \,\mathrm{d}x \\
&= \frac{\sin mx \cos nx}{m} \Big|_{x=-\pi}^{\pi} + \frac{n}{m} \int_{-\pi}^{\pi} \sin mx \sin nx \,\mathrm{d}x \\
&= \frac{-n \cos mx \sin nx}{m^2} \Big|_{x=-\pi}^{\pi} + \frac{n^2}{m^2} \int_{-\pi}^{\pi} \cos mx \cos nx \,\mathrm{d}x \\
&= \frac{n^2}{m^2} \int_{-\pi}^{\pi} \cos mx \cos nx \,\mathrm{d}x
\end{aligned}$$

Since $m \neq \pm n$, we must have

$$\int_{-\pi}^{\pi} \cos mx \cos nx \,\mathrm{d}x = 0$$

The other integral is handled similarly.

iv) Using the identity $\cos 2x = 2\cos^2 x - 1$ we obtain,

$$\begin{aligned}
\int_{-\pi}^{\pi} \cos^2 mx \,\mathrm{d}x &= \int_{-\pi}^{\pi} \frac{1 + \cos 2mx}{2} \,\mathrm{d}x \\
&= \frac{x}{2} \Big|_{x=-\pi}^{\pi} + \frac{\sin 2mx}{4m} \Big|_{x=-\pi}^{\pi} \\
&= \pi
\end{aligned}$$

To obtain the other result, use the identity $\cos 2x = 1 - 2\sin^2 x$.

v)

$$\begin{aligned}
&\int_{-\pi}^{\pi} e^{imx} e^{-inx} \,\mathrm{d}x = \int_{-\pi}^{\pi} e^{i(m-n)x} \,\mathrm{d}x \\
&= \int_{-\pi}^{\pi} \cos(m-n)x \,\mathrm{d}x + i \int_{-\pi}^{\pi} \sin(m-n)x \,\mathrm{d}x
\end{aligned}$$

If $m \neq n$ then by part i) this expression is zero. If $m = n$ then $\sin(m - n)x = 0$ and $\cos(m - n)x = 1$, so the result follows.

□

Theorem 2.3.4. Suppose for some real numbers $a_0, a_1, b_1, a_2, b_2, \ldots$ the series

$$\frac{a_0}{2} + \sum_{k=1}^{\infty}(a_k \cos kx + b_k \sin kx)$$

converges uniformly to a given function $f : \mathbb{R} \to \mathbb{R}$. Then

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx \, \mathrm{d}x \text{ for } 0 \leq n \in \mathbb{Z}$$

and

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx \, \mathrm{d}x \text{ for } 0 < n \in \mathbb{Z}$$

Proof. Since the given series converges uniformly, f is continuous on $\mathbb{R}$ and hence Riemann Integrable on $\mathbb{R}$. Then, integrating f on the interval $[-\pi, \pi]$, we obtain

$$\begin{aligned}\int_{-\pi}^{\pi} f(x)\,\mathrm{d}x &= \int_{-\pi}^{\pi}\frac{a_0}{2}\,\mathrm{d}x + \sum_{k=1}^{\infty}\left(a_k\int_{-\pi}^{\pi}\cos kx\,\mathrm{d}x + b_k\int_{-\pi}^{\pi}\sin kx\,\mathrm{d}x\right)\\ &= \int_{-\pi}^{\pi}\frac{a_0}{2}\,\mathrm{d}x\\ &= a_0\pi\end{aligned}$$

by using Lemma 2.3.3. (Note that we can integrate the series term by term because of uniform convergence.)

Similarly,

$$\begin{aligned}\int_{-\pi}^{\pi} f(x)\cos nx\,\mathrm{d}x &= \int_{-\pi}^{\pi}\frac{a_0}{2}\cos nx\,\mathrm{d}x\\ &\quad + \sum_{k=1}^{\infty}\left(a_k\int_{-\pi}^{\pi}\cos kx\cos nx\,\mathrm{d}x + b_k\int_{-\pi}^{\pi}\sin kx\cos nx\,\mathrm{d}x\right)\\ &= \int_{-\pi}^{\pi} a_n\cos^2 nx\,\mathrm{d}x\\ &= a_n\pi\end{aligned}$$

again by using Lemma 2.3.3. So the result is true for the sequence $\{a_n : 0 \leq n \in \mathbb{Z}\}$. A similar argument can be used for the sequence $\{b_n : 0 < n \in \mathbb{Z}\}$. $\square$

Corollary 2.3.5. If $f : \mathbb{R} \to \mathbb{R}$ is odd then $a_n = 0$ for all $n \geq 0$. If f is even then $b_n = 0$ for all $n > 0$.

Proof. If f is odd then $f(x) \cos nx$ is odd for all $n \geq 0$. So the result follows from the formula for a_n given in Theorem 2.3.4. Similarly, if f is even then $f(x) \sin nx$ is odd and so b_n is zero for all $n \in \mathbb{N}$. □

We now deal with a reformulation of the Fourier Problem: Given a 2π-periodic function $f : \mathbb{R} \to \mathbb{C}$, do there exist complex scalars $\ldots, c_{-2}, c_{-1}, c_0, c_1, c_2, \ldots$ such that

$$f(x) = c_0 + \sum_{k=1}^{\infty} (c_k e^{ikx} + c_{-k} e^{-ikx})$$

for $x \in \mathbb{R}$?

The next theorem deals with this question.

Theorem 2.3.6. Suppose $f : \mathbb{R} \to \mathbb{C}$ is 2π-periodic and $f(x) = F(e^{ix})$ for some complex-valued function F analytic on the annulus $A = \{z \in \mathbb{C} : r < |z| < R\}$, where $r < 1 < R$. Then the reformulation of the Fourier problem, given above, has a solution. Moreover, the scalars $\{c_k : k \in \mathbb{Z}\}$ are given by

$$c_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) e^{-inx} \, \mathrm{d}x$$

Proof. Since F is analytic on A, it has a Laurent expansion in that region, namely

$$F(z) = c_0 + \sum_{k=1}^{\infty} (c_k z^k + c_{-k} z^{-k})$$

Since $f(x) = F(e^{ix})$, we have

$$f(x) = c_0 + \sum_{k=1}^{\infty} (c_k e^{ikx} + c_{-k} e^{-ikx})$$

for $x \in \mathbb{R}$.

Now the series given above converges uniformly on the annulus A, hence we can integrate term by term to obtain

$$\begin{aligned} \int_{-\pi}^{\pi} f(x) e^{-inx} \, \mathrm{d}x &= \sum_{k=-\infty}^{\infty} c_k \int_{-\pi}^{\pi} e^{ikx} e^{-inx} \, \mathrm{d}x \\ &= 2\pi c_n \end{aligned}$$

by Lemma 2.3.3. □

Definition 2.3.7. The real scalars $a_0, a_1, b_1, a_2, b_2, \ldots$ are called the *trigonometric Fourier coefficients of f*. The complex scalars $\ldots, c_{-2}, c_{-1}, c_0, c_1, c_2, \ldots$ are called the *exponential Fourier coefficients of f*.

Proposition 2.3.8. The trigonometric and exponential fourier coefficients of a function f are related by

$$\begin{aligned}
a_0 &= 2c_0 \\
a_n &= c_n + c_{-n} \text{ for } 0 < n \in \mathbb{Z} \\
b_n &= i(c_n - c_{-n}) \text{ for } 0 < n \in \mathbb{Z}
\end{aligned}$$

Proof. Clearly $a_0 = 2c_0$. For $a_n, n > 0$,

$$\begin{aligned}
c_n + c_{-n} &= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{inx}\,\mathrm{d}x + \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{-inx}\,\mathrm{d}x \\
&= \frac{1}{2\pi}\left(\int_{-\pi}^{\pi} f(x)\cos nx\,\mathrm{d}x + i\int_{-\pi}^{\pi} f(x)\sin nx\,\mathrm{d}x\right) \\
&\quad + \frac{1}{2\pi}\left(\int_{-\pi}^{\pi} f(x)\cos(-nx)\,\mathrm{d}x + i\int_{-\pi}^{\pi} f(x)\sin(-nx)\,\mathrm{d}x\right) \\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi} 2f(x)\cos nx\,\mathrm{d}x \\
&= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,\mathrm{d}x \\
&= a_n
\end{aligned}$$

The expression for b_n is handled similarly. □

Theorem 2.3.9. Suppose f is a 2π-periodic, $\mathcal{C}^p$ function for some $p \in \mathbb{N}$. Then there exists $M_p > 0$ such that

i) $|a_k| \leq M_p/|k|^p$ for all $k \in \mathbb{N}$.

ii) $|b_k| \leq M_p/|k|^p$ for all $k \in \mathbb{N}$.

iii) $|c_k| \le M_p / |k|^p$ for all $0 \neq k \in \mathbb{Z}$.

Proof. i) Using the formula for a_k and integrating by parts, we obtain

$$\begin{aligned} a_k &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos kx \, \mathrm{d}x \\ &= \frac{f(x)\sin kx}{k\pi}\bigg|_{x=-\pi}^{\pi} + \frac{1}{k\pi}\int_{-\pi}^{\pi} f'(x)\sin kx \, \mathrm{d}x \\ &= \frac{1}{k\pi}\int_{-\pi}^{\pi} f'(x)\sin kx \, \mathrm{d}x \end{aligned}$$

Now integrating by parts $p-1$ more times

$$a_k = \frac{1}{k^p\pi}\int_{-\pi}^{\pi} f^{(p)}(x)\sin kx \, \mathrm{d}x$$

Since $f \in \mathcal{C}^p$, $f^{(p)}\sin kx$ is continuous and hence bounded on $[-\pi, \pi]$. Therefore,

$$\begin{aligned} |a_k| &= \frac{1}{|k|^p\pi}\left|\int_{-\pi}^{\pi} f^{(p)}(x)\sin kx \, \mathrm{d}x\right| \\ &\le \frac{1}{|k|^p\pi}\int_{-\pi}^{\pi} \left|f^{(p)}(x)\sin kx\right| \mathrm{d}x \\ &\le \frac{1}{|k|^p\pi}\int_{-\pi}^{\pi} M \, \mathrm{d}x \\ &\le \frac{M_p}{|k|^p} \end{aligned}$$

for some $M_p > 0$.

ii) This is handled similarly to part i).

iii) This is handled similarly to part i).

□

Corollary 2.3.10. If $f \in \mathcal{C}^2$ is 2π-periodic then the Fourier series of f converges uniformly on $\mathbb{R}$.

Proof. By Theorem 2.3.9 $|c_k| \le M/|k|^2$ for some $M > 0$ and all $0 \ne k \in \mathbb{Z}$. Therefore $\left|c_k e^{ikx}\right| \le M/|k|^2$. Now the series $\sum_{k=1}^{\infty} \frac{M}{k^2}$ and $\sum_{k=-\infty}^{-1} \frac{M}{k^2}$ both converge. So by the Weierstrass M-test, the Fourier series of f

$$s_n(x) = \sum_{k=-\infty}^{\infty} c_k e^{ikx}$$

converges uniformly (and absolutely) on $\mathbb{R}$. $\square$

Although Corollary 2.3.10 says the Fourier series of a function f converges uniformly, it doesn't necessarily have to converge to the function f itself. In fact it does, and this will be shown in Theorem 2.5.1.

2.4 Dirichlet's Kernel

Suppose $a_0, a_1, b_1, \ldots$ are the trigonometric Fourier coefficients of a given function f and $\ldots, c_{-1}, c_0, c_1, \ldots$ its exponential Fourier coefficients. The *Fourier Series of f* is the series with partial sums

$$\begin{aligned} s_n(x) &= \frac{a_0}{2} + \sum_{k=1}^{n} a_k \cos kx + b_k \sin kx \\ &= c_0 + \sum_{k=1}^{n} (c_k e^{ikx} + c_{-k} e^{-ikx}) \end{aligned}$$

In this section, we will derive some properties of this series.

Let s_n be the nth partial sum of the Fourier series of f. Then

$$\begin{aligned} s_n(x) &= \sum_{k=-n}^{n} c_k e^{ikx} \\ &= \sum_{k=-n}^{n} \left(\frac{1}{2\pi} \int_{-\pi}^{\pi} f(t) e^{-ikt}\, \mathrm{d}t \right) e^{ikx} \\ &= \frac{1}{2\pi} \int_{-\pi}^{\pi} \left(\sum_{k=-n}^{n} f(t) e^{-ikt} e^{ikx} \right) \mathrm{d}t \\ &= \frac{1}{2\pi} \int_{-\pi}^{\pi} \left(f(t) \sum_{k=-n}^{n} e^{ik(x-t)} \right) \mathrm{d}t \\ &= \frac{1}{2\pi} \int_{-\pi}^{\pi} f(t) D_n(x-t)\, \mathrm{d}t \end{aligned}$$

where $D_n(u) = \sum_{k=-n}^{n} e^{iku}$ for all $u \in \mathbb{R}$. We call $\{D_n\}_{n=0}^{\infty}$ the *Dirichlet Kernel.*

Proposition 2.4.1. For each $n \in \mathbb{Z}$, $n \geq 0$, we have

$$D_n(x) = \begin{cases} 2n+1 & \text{if } x = 2\pi l, l \in \mathbb{Z} \\ \dfrac{\sin(n+1/2)x}{\sin x/2} & \text{otherwise} \end{cases}$$

Proof. Clearly, if $x = 2\pi l$ for $l \in \mathbb{Z}$ then $e^{ikx} = 1$ so $\sum_{k=-n}^{n} e^{ikx} = 2n+1$. Otherwise, we have

$$\begin{aligned} D_n(x) &= \sum_{k=-n}^{n} e^{ikx} \\ &= \sum_{k=-n}^{n} (e^{ix})^k \\ &= e^{-inx} \sum_{k=0}^{2n} (e^{ix})^k \\ &= e^{-inx} \left(\frac{1-(e^{ix})^{2n+1}}{1-e^{ix}} \right) \\ &= \frac{e^{-inx}}{e^{ix/2}} \left(\frac{1-(e^{ix})^{2n+1}}{e^{-ix/2}-e^{ix/2}} \right) \\ &= e^{-i(n+1/2)x} \left(\frac{1-(e^{ix})^{2n+1}}{-2i\sin x/2} \right) \\ &= \frac{e^{-i(n+1/2)x} - e^{i(n+1/2)x}}{-2i\sin x/2} \\ &= \frac{-2i\sin(n+1/2)x}{-2i\sin x/2} \\ &= \frac{\sin(n+1/2)x}{\sin x/2} \end{aligned}$$

□

Theorem 2.4.2. For all $n \in \mathbb{N}$,

i) D_n is real-valued, $\mathcal{C}^\infty$, 2π-periodic, even and satisfies $|D_n(x)| \leq 2n+1$.

ii) $\int_{-\pi}^{\pi} D_n(t)\,\mathrm{d}t = 2\pi$

iii) For all $\delta \in (0,\pi)$,

$$\lim_{n\to\infty} \int_{\delta}^{\pi} D_n(t)\,\mathrm{d}t = \int_{-\pi}^{-\delta} D_n(t)\,\mathrm{d}t = 0$$

Proof. i) That D_n is real-valued, $\mathcal{C}^\infty$, 2π-periodic, even and satisfies $|D_n(x)| \leq 2n+1$ follows from its definition.

ii)

$$\begin{aligned}
\int_{-\pi}^{\pi} D_n(t)\,\mathrm{d}t &= \int_{-\pi}^{\pi} \sum_{k=-n}^{n} e^{ikt}\,\mathrm{d}t \\
&= \int_{-\pi}^{\pi} 1\,\mathrm{d}t + \sum_{\substack{k=-n \\ k\neq 0}}^{n} \int_{-\pi}^{\pi} e^{ikt}\,\mathrm{d}t \\
&= 2\pi
\end{aligned}$$

iii) If $0 < \delta < \pi$ then

$$\int_{\delta}^{\pi} D_n(t)\,\mathrm{d}t = \int_{\delta}^{\pi} g(t)\sin(n+1/2)t\,\mathrm{d}t$$

where $g(t) = 1/(\sin t/2)$. Since g is continuous on $[\delta,\pi]$, it is Riemann Integrable on this interval and hence this integral goes to zero as $n \to \infty$ by Theorem 1.3.1. The other integral can be deduced by noting that D_n is even.

$\square$

Proposition 2.4.3. Let f be a 2π-periodic, Riemann Integrable function and s_n the nth partial sum of its Fourier series. Then

i)

$$s_n(x) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x-t) D_n(t)\,\mathrm{d}t$$

ii)
$$s_n(x) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x+t)D_n(t)\,\mathrm{d}t$$

Proof. i)
$$\begin{aligned}
s_n(x) &= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)D_n(x-t)\,\mathrm{d}t \\
&= -\frac{1}{2\pi}\int_{x+\pi}^{x-\pi} f(x-s)D_n(s)\,\mathrm{d}s \text{ where } s = x-t \\
&= \frac{1}{2\pi}\int_{x-\pi}^{x+\pi} f(x-s)D_n(s)\,\mathrm{d}s \\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x-s)D_n(s)\,\mathrm{d}s
\end{aligned}$$

by Proposition 2.3.2 and the fact that f is 2π-periodic.

ii) By part i) we have
$$\begin{aligned}
s_n(x) &= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x-t)D_n(t)\,\mathrm{d}t \\
&= -\frac{1}{2\pi}\int_{\pi}^{-\pi} f(x+s)D_n(-s)\,\mathrm{d}s \text{ where } s = -t \\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x+s)D_n(s)\,\mathrm{d}s \text{ since } D_n \text{ is even}
\end{aligned}$$

□

2.5 The Representation Problem

The *Representation Problem* asks if the Fourier series of a given function f converges pointwise to f. In this section, we will deduce two results which answer this question in certain cases. Also, the Riemann Localization Theorem will be presented.

Theorem 2.5.1. Suppose f is a 2π-periodic, Riemann Integrable function and s_n is the nth partial sum of its Fourier series. If f is differentiable at some $x \in \mathbb{R}$ then s_n converges pointwise to f at x.

Before presenting the proof, we will give an intuitive justification of this result. If $n \in \mathbb{N}$ is large and $0 < \delta < \pi$ then

$$\int_{\delta}^{\pi} f(x+t) D_n(t)\,\mathrm{d}t \simeq 0$$

and

$$\int_{-\pi}^{-\delta} f(x+t) D_n(t)\,\mathrm{d}t \simeq 0$$

by Theorem 2.4.2. Therefore, if $0 < \delta < \pi$ is small and $n \in \mathbb{N}$ is large, we can use the continuity of f at x to deduce that

$$\begin{aligned}
f(x) &= \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) D_n(t)\,\mathrm{d}t \\
&\simeq \frac{1}{2\pi} \int_{-\pi}^{-\delta} f(x+t) D_n(t)\,\mathrm{d}t + \frac{1}{2\pi} \int_{-\delta}^{\delta} f(x+t) D_n(t)\,\mathrm{d}t \\
&\quad + \frac{1}{2\pi} \int_{\delta}^{\pi} f(x+t) D_n(t)\,\mathrm{d}t \\
&= \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x+t) D_n(t)\,\mathrm{d}t \\
&= s_n(x)
\end{aligned}$$

Proof.

$$\begin{aligned}
s_n(x) - f(x) &= \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x+t) D_n(t)\,\mathrm{d}t - \frac{f(x)}{2\pi} \int_{-\pi}^{\pi} D_n(t)\,\mathrm{d}t \\
&= \frac{1}{2\pi} \int_{-\pi}^{\pi} [f(x+t) - f(x)] D_n(t)\,\mathrm{d}t \\
&= \frac{1}{2\pi} \int_{-\pi}^{\pi} g(t) \sin(n+1/2)t\,\mathrm{d}t
\end{aligned}$$

where $g : [-\pi, \pi] \to \mathbb{C}$ is defined by

$$g(t) = \begin{cases} \dfrac{f(x+t) - f(x)}{\sin t/2} & \text{if } 0 < |t| \leq \pi \\ 2f'(x) & \text{if } t = 0 \end{cases}$$

Note that g is continous at zero because

$$g(t) = 2\left(\frac{f(x+t) - f(x)}{t}\right) \frac{t/2}{\sin x/2}$$

$\to 2f'(x) = g(0)$ as $t \to 0$. Now g is Riemann Integrable on $[\delta, \pi]$ and $[-\pi, -\delta]$ for all $0 < \delta < \pi$. So by Corollary 1.2.4 g is Riemann Integrable on $[-\pi, \pi]$. Therefore by Theorem 1.3.1,

$$s_n(x) - f(x) = \frac{1}{2\pi}\int_{-\pi}^{\pi} g(t)\sin(n + 1/2)t\,\mathrm{d}t$$

$\to 0$ as $n \to \infty$. □

Corollary 2.5.2. If $f \in \mathcal{C}^2$ is 2π-periodic then the Fourier series of f converges uniformly on $\mathbb{R}$ to f.

Proof. This follows immediately from Theorem 2.5.1 and Corollary 2.3.10. □

Theorem 2.5.3 (Jordan)**.** Suppose f is a 2π-periodic, Riemann Integrable function and s_n the nth partial sum of its Fourier series. Furthermore, suppose there exists $x_0 \in \mathbb{R}$ and $\delta > 0$ such that f is differentiable on $(x_0, x_0 + \pi)$ and $(x_0 - \pi, x_0)$. If $\lambda_+ = \lim_{x \to x_0^+} f'(x)$ and $\lambda_- = \lim_{x \to x_0^-} f'(x)$ both exist then

i) $f(x_0+) = \lim_{x \to x_0^+} f(x)$ exists.

ii) $f(x_0-) = \lim_{x \to x_0^-} f(x)$ exists.

iii)
$$\lim_{x \to x_0^+} \frac{f(x) - f(x_0+)}{x - x_0} = \lambda_+$$

iv)
$$\lim_{x \to x_0^-} \frac{f(x) - f(x_0-)}{x - x_0} = \lambda_-$$

v)
$$\lim_{n \to \infty} s_n(x) = \frac{f(x_0+) + f(x_0-)}{2}$$

Proof. i) Since $\lim_{x\to x_0^+} f'(x)$ exists, there are $M > 0$ and $\delta_0 > 0$ such that $|f'(x)| < M$ for $x_0 < x \leq x_0 + \delta_0$. Fix $\epsilon > 0$ and let $\delta' = \min(\delta_0, \epsilon/2M)$. Then, by the Mean Value Theorem, if $x, y \in (x_0, x_0 + \delta']$ with $x < y$,

$$\begin{aligned} |f(x) - f(y)| &= |f'(t)|\,|x - y| \text{ where } t \in (x, y) \subset (x_0, x_0 + \delta') \\ &< M\frac{\epsilon}{2M} \\ &= \epsilon/2 \end{aligned}$$

Therefore, if we let $z = x_0 + \delta'$, then

$$f(z) - \epsilon/2 \leq \limsup_{x\to x_0^+} f(x) \leq f(z) + \epsilon/2$$

Similarly,

$$f(z) - \epsilon/2 \leq \liminf_{x\to x_0^+} f(x) \leq f(z) + \epsilon/2$$

So we have $\limsup_{x\to x_0^+} f(x) - \liminf_{x\to x_0^+} f(x) \leq \epsilon$ and the result follows.

ii) This limit is handled similarly as that in part i).

iii) Let $g(x) = f(x)$ for $x \in (x_0, x_0 + \delta)$ and $g(x_0) = f(x_0+)$. Fix $\epsilon > 0$ and choose $0 < \delta_0 < \delta$ so that $|f'(x) - \lambda_+| < \epsilon$ if $x_0 < x < x_0 + \delta_0$. Now by part i), g is continuous on $[x_0, x_0 + \delta_0]$ and differentiable on $(x_0, x_0 + \delta_0)$. So by the Mean Value Theorem, if $x \in (x_0, x_0 + \delta_0)$

$$\frac{f(x) - f(x_0+)}{x - x_0} = \frac{g(x) - g(x_0)}{x - x_0} = g'(t)$$

for some $t \in (x_0, x) \subset (x_0, x_0 + \delta_0)$. So $g'(t) = f'(t)$ and hence,

$$\left|\frac{f(x) - f(x_0+)}{x - x_0} - \lambda_+\right| = |f'(t) - \lambda_+| < \epsilon$$

if $x_0 < x < x_0 + \delta_0$.

iv) This result is handled similarly to part iii).

v) Since D_n is even, Theorem 2.4.2 implies $\int_0^\pi D_n(t)\,\mathrm{d}t = \pi$. Therefore,

$$\begin{aligned}\frac{1}{\pi}\int_0^\pi f(x_0+t)D_n(t)\,\mathrm{d}t - f(x_0+) &= \frac{1}{\pi}\int_0^\pi f(x_0+t)D_n(t)\,\mathrm{d}t \\ &\quad - \frac{1}{\pi}\int_0^\pi f(x_0+)D_n(t)\,\mathrm{d}t \\ &= \frac{1}{\pi}\int_0^\pi [f(x_0+t) - f(x_0+)]D_n(t)\,\mathrm{d}t \\ &= \frac{1}{\pi}\int_0^\pi g(t)D_n(t)\,\mathrm{d}t\end{aligned}$$

where

$$g(t) = \begin{cases} \dfrac{f(x_0+t) - f(x_0+)}{\sin t/2} & \text{if } 0 < t < \pi \\ 2\lambda_+ & \text{if } t = 0 \end{cases}$$

As was argued in the proof of Theorem 2.5.1, g is Riemann Integrable on $[0,\pi]$ and

$$\lim_{n\to\infty}\frac{1}{\pi}\int_0^\pi g(t)D_n(t)\,\mathrm{d}t = 0$$

Therefore,

$$\lim_{n\to\infty}\frac{1}{\pi}\int_0^\pi f(x_0+t)D_n(t)\,\mathrm{d}t = f(x_0+)$$

A similar argument shows that

$$\lim_{n\to\infty}\frac{1}{\pi}\int_{-\pi}^0 f(x_0+t)D_n(t)\,\mathrm{d}t = f(x_0-)$$

To complete the proof, note that

$$\begin{aligned}s_n(x_0) &= \frac{1}{2\pi}\int_{-\pi}^\pi f(x+t)D_n(t)\,\mathrm{d}t \\ &= \frac{1}{2}\left(\frac{1}{\pi}\int_{-\pi}^0 f(x+t)D_n(t)\,\mathrm{d}t + \frac{1}{\pi}\int_0^\pi f(x+t)D_n(t)\,\mathrm{d}t\right)\end{aligned}$$

$\to (f(x_0+) + f(x_0-))/2$ as $n \to \infty$.

$\square$

Corollary 2.5.4. Suppose f is 2π-periodic, Riemann Integrable and f is $\mathcal{C}^1$ on (x_{k-1}, x_k) where $-\pi = x_0 < x_1 < \ldots < x_n = \pi$. If $\lim_{x \to x_k^+} f'(x)$ exists for $0 \leq k \leq n-1$ and $\lim_{x \to x_k^-} f'(x)$ exists for $1 \leq k \leq n$ then $s_n(x) \to (f(x+) + f(x-))/2$ for all $x \in \mathbb{R}$. (Here, s_n denotes the nth partial sum of the Fourier series of f.)

Proof. This follows immediately from Theorem 2.5.3. □

We now present several examples of Fourier series of various functions.

Example 2.5.5. Prove that

$$x^2 = \frac{\pi^2}{3} + 4 \sum_{k=1}^{\infty} \frac{(-1)^k}{k^2} \cos kx$$

for $-\pi \leq x \leq \pi$.

Solution. By Proposition 2.3.2, the function $f(x) = x^2, -\pi \leq x \leq \pi$ has a unique 2π-periodic extension to $\mathbb{R}$. Moreover, this extension is even so by Corollary 2.3.5 $b_n = 0$ for all $n \in \mathbb{N}$. Now this extension of f satisfies the hypothesis of Theorem 2.5.3 and is also continuous on $\mathbb{R}$ so the Fourier series of f converges pointwise to f. The a_n's are computed as follows:

$$\begin{aligned} a_0 &= \frac{1}{\pi} \int_{-\pi}^{\pi} x^2 \, \mathrm{d}x \\ &= \frac{x^3}{3\pi} \bigg|_{x=-\pi}^{\pi} \\ &= \frac{2\pi^2}{3} \end{aligned}$$

and for $0 < n \in \mathbb{Z}$ we have (using integration by parts twice)

$$\begin{aligned} a_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} x^2 \cos nx \,\mathrm{d}x \\ &= \frac{1}{\pi}\left(\frac{x^2 \sin nx}{n}\Big|_{x=-\pi}^{\pi} - \frac{2}{n}\int_{-\pi}^{\pi} x \sin nx \,\mathrm{d}x\right) \\ &= \frac{1}{\pi}\left(\frac{2}{n^2} x \cos nx \Big|_{x=-\pi}^{\pi} + \int_{-\pi}^{\pi} \cos nx \,\mathrm{d}x\right) \\ &= \frac{1}{\pi}\left(\frac{(-1)^n 4\pi}{n^2}\right) \\ &= \frac{(-1)^n 4}{n^2} \end{aligned}$$

as required.

Example 2.5.6. Prove that

$$|x| = \frac{\pi}{2} - \frac{4}{\pi}\sum_{k=1}^{\infty} \frac{\cos(2k-1)x}{(2k-1)^2}$$

Solution. As in Example 2.5.5, we consider the unique (even) 2π-periodic extension of $f(x) = |x|, -\pi \le x \le \pi$ to $\mathbb{R}$. Then $b_n = 0$ for all $n \in \mathbb{N}$ and we compute the a_n's as follows:

$$\begin{aligned} a_0 &= \frac{1}{\pi}\int_{-\pi}^{\pi} |x| \,\mathrm{d}x \\ &= \frac{2}{\pi}\int_{0}^{\pi} x \,\mathrm{d}x \\ &= \frac{2}{\pi}\left(\frac{\pi^2}{2}\right) \\ &= \pi \end{aligned}$$

and for $a_n, n > 0$ we have (again, using integration by parts)

$$\begin{aligned}
a_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} |x| \cos nx \,\mathrm{d}x \\
&= \frac{2}{\pi}\int_{0}^{\pi} x \cos nx \,\mathrm{d}x \\
&= \frac{2}{\pi}\left(\frac{x \sin nx}{n}\bigg|_{x=0}^{\pi} - \frac{1}{n}\int_0^{\pi} \sin nx \,\mathrm{d}x \right) \\
&= \frac{2}{\pi}\left(\frac{\cos nx}{n^2}\bigg|_{x=0}^{\pi} \right) \quad (*)
\end{aligned}$$

Now

$$\frac{\cos nx}{n^2}\bigg|_{x=0}^{\pi} = \begin{cases} 0 & \text{if } n \text{ is even} \\ \dfrac{-2}{n^2} & \text{if } n \text{ is odd} \end{cases}$$

Therefore, from $(*)$ we get $a_{2n} = 0$ and

$$a_{2n-1} = -\frac{4}{\pi(2n-1)^2}$$

Since f is continuous on its domain, the Fourier series of f converges pointwise to f by Theorem 2.5.3.

Example 2.5.7. Define $f : [-\pi, \pi] \to \mathbb{R}$ by

$$f(x) = \begin{cases} 1 & 0 < x < \pi \\ -1 & -\pi < x < 0 \\ 0 & x = -\pi, 0, \pi \end{cases}$$

Find the Fourier series of f and show that it converges pointwise to f.

Solution. Consider the 2π-periodic extension of f to $\mathbb{R}$. This extension is continuous for all $x \in \mathbb{R}, x \neq \pi l, l \in \mathbb{Z}$. Also, for $x = \pi l, l \in \mathbb{Z}$ we have $f(x+) + f(x-) = 0 = f(x)$. Now f satisfies the hypothesis of Theorem 2.5.3

and so the Fourier series of f converges pointwise to f. Noting that f is odd we get $a_n = 0$ for all $0 \leq n \in \mathbb{Z}$ and

$$\begin{aligned} b_n &= \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx \, \mathrm{d}x \\ &= \frac{2}{\pi} \int_{0}^{\pi} f(x) \sin nx \, \mathrm{d}x \\ &= \frac{2}{\pi} \int_{0}^{\pi} \sin nx \, \mathrm{d}x \\ &= -\frac{2 \cos nx}{\pi n} \bigg|_{x=0}^{\pi} \\ &= \frac{2}{\pi n} (1 - \cos n\pi) \end{aligned}$$

Now

$$\frac{2}{\pi n} (1 - \cos n\pi) = \begin{cases} \dfrac{4}{\pi n} & \text{if } n \text{ is odd} \\ 0 & \text{if } n \text{ is even} \end{cases}$$

Therefore,

$$f(x) = \frac{4}{\pi} \sum_{k=1}^{\infty} \frac{\sin(2k-1)x}{2k-1}$$

Example 2.5.8. Prove that

$$\cos ax = \frac{\sin \pi a}{\pi a} + \sum_{k=1}^{\infty} (-1)^k \frac{2a \sin \pi a}{\pi(a^2 - k^2)} \cos kx$$

where $a \in \mathbb{R}, a \notin \mathbb{N}$ and $-\pi \leq x \leq \pi$.

Solution. As in Example 2.5.5, we consider the unique (even) 2π-periodic extension of $f(x) = \cos ax, -\pi \leq x \leq \pi$ to $\mathbb{R}$. Then $b_n = 0$ for all $n \in \mathbb{N}$ and we compute the a_n's as follows:

$$\begin{aligned} a_0 &= \frac{1}{\pi}\int_{-\pi}^{\pi} \cos ax \, \mathrm{d}x \\ &= \frac{2}{\pi}\int_{0}^{\pi} \cos ax \, \mathrm{d}x \\ &= \frac{2\sin ax}{\pi a}\Big|_{x=0}^{\pi} \\ &= \frac{2\sin \pi a}{\pi a} \end{aligned}$$

and for $n > 0$ we have (using integration by parts)

$$\begin{aligned} a_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} \cos ax \cos nx \, \mathrm{d}x \\ &= \frac{2}{\pi}\int_{0}^{\pi} \cos ax \cos nx \, \mathrm{d}x \\ &= \frac{2}{\pi}\left(\frac{\cos ax \sin nx}{n}\Big|_{x=0}^{\pi} + \frac{a}{n}\int_0^{\pi} \sin ax \sin nx \, \mathrm{d}x\right) \\ &= \frac{2}{\pi}\left(-\frac{a \sin ax \cos nx}{n^2}\Big|_{x=0}^{\pi} + \frac{a^2}{n^2}\int_0^{\pi} \cos ax \cos nx \, \mathrm{d}x\right) \\ &= (-1)^{n+1}\frac{2a\sin \pi a}{\pi n^2} + \frac{2a^2}{\pi n^2}\int_0^{\pi} \cos ax \cos nx \, \mathrm{d}x \end{aligned}$$

Rearranging this equation we get

$$\frac{2}{\pi}\left(1 - \frac{a^2}{n^2}\right)\int_0^{\pi} \cos ax \cos nx \, \mathrm{d}x = (-1)^{n+1}\frac{2a\sin \pi a}{\pi n^2}$$

and so

$$\begin{aligned} a_n &= \frac{2}{\pi}\int_0^{\pi} \cos ax \cos nx \, \mathrm{d}x \\ &= (-1)^{n+1}\frac{2an^2 \sin \pi a}{\pi n^2(n^2 - a^2)} \\ &= (-1)^{n}\frac{2a \sin \pi a}{\pi(a^2 - n^2)} \end{aligned}$$

as required. (Note that by Theorem 2.5.3 this Fourier series converges to f since f is continuous on $\mathbb{R}$.)

Example 2.5.9. Find real scalars $a_0, a_1, \dots$ such that

$$e^{-|x|} = \frac{a_0}{2} + \sum_{k=1}^{\infty} a_k \cos kx$$

for $-\pi \leq x \leq \pi$.

Solution. As in Example 2.5.5, we consider the unique (even) 2π-periodic extension of $f(x) = e^{-|x|}, -\pi \leq x \leq \pi$ to $\mathbb{R}$. Then $b_n = 0$ for all $n \in \mathbb{N}$ and we compute the a_n's as follows:

$$\begin{aligned} a_0 &= \frac{1}{\pi}\int_{-\pi}^{\pi} e^{-|x|}\,\mathrm{d}x \\ &= \frac{2}{\pi}\int_0^{\pi} e^{-x}\,\mathrm{d}x \\ &= \frac{2(1-e^{-\pi})}{\pi} \end{aligned}$$

and for $n > 0$ we have (using integration by parts)

$$\begin{aligned} a_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} e^{-|x|}\cos nx\,\mathrm{d}x \\ &= \frac{2}{\pi}\int_0^{\pi} e^{-x}\cos nx\,\mathrm{d}x \\ &= \frac{2}{\pi}\left(\frac{e^{-x}\sin nx}{n}\bigg|_{x=0}^{\pi} + \frac{1}{n}\int_0^{\pi} e^{-x}\sin nx\,\mathrm{d}x\right) \\ &= \frac{2}{\pi}\left(-\frac{e^{-x}\cos nx}{n^2}\bigg|_{x=0}^{\pi} - \frac{1}{n^2}\int_0^{\pi} e^{-x}\cos nx\,\mathrm{d}x\right) \\ &= \frac{2(1-(-1)^n e^{-\pi})}{\pi n^2} - \frac{2}{\pi n^2}\int_0^{\pi} e^{-x}\cos nx\,\mathrm{d}x \end{aligned}$$

Rearranging this equation we get

$$\frac{2}{\pi}\left(1+\frac{1}{n^2}\right)\int_0^{\pi} e^{-x}\cos nx\,\mathrm{d}x = \frac{2(1-(-1)^n e^{-\pi})}{\pi n^2}$$

and so

$$\begin{aligned} a_n &= \frac{2}{\pi}\int_0^\pi e^{-x}\cos nx \,\mathrm{d}x \\ &= \frac{2n^2(1-(-1)^n e^{-\pi})}{\pi n^2(n^2+1)} \\ &= \frac{2(1-(-1)^n e^{-\pi})}{\pi(n^2+1)} \end{aligned}$$

Therefore because of continuity and Theorem 2.5.3, we have

$$e^{-|x|} = \frac{(1-e^{-\pi})}{\pi} + 2\sum_{k=1}^{\infty}\left(\frac{1-(-1)^k e^{-\pi}}{\pi(k^2+1)}\right)\cos kx$$

for $-\pi \le x \le \pi$. In fact, this series converges uniformly because

$$\left|\left(2\frac{(-1)^k e^{-\pi}-1}{\pi(k^2+1)}\right)\cos kx\right| \le \frac{4}{\pi k^2}$$

and the series

$$\sum_{k=1}^{\infty}\frac{4}{\pi k^2}$$

converges.

Theorem 2.5.10 (Riemann's Localization Theorem)**.** Suppose f and g are 2π-periodic Riemann Integrable functions which agree on $(x_0-\delta, x_0+\delta)$ for some $x_0 \in \mathbb{R}$ and $\delta > 0$. Let s_n be the nth partial sum of the Fourier series of f and t_n the nth partial sum of the Fourier series of g. If $s_n(x_0)$ converges to some l then so does $t_n(x_0)$.

Proof. Let $h = f - g$ and $u_n = s_n - t_n$. Then u_n is the nth partial sum of the Fourier series of h and we have

$$\begin{aligned} u_n(x_0) &= \frac{1}{2\pi}\int_{-\pi}^{\pi} h(x_0+t)D_n(t)\,\mathrm{d}t \\ &= \frac{1}{2\pi}\int_{-\pi}^{-\delta} h(x_0+t)D_n(t)\,\mathrm{d}t + \frac{1}{2\pi}\int_{-\delta}^{\delta} h(x_0+t)D_n(t)\,\mathrm{d}t \\ &\quad + \frac{1}{2\pi}\int_{\delta}^{\pi} h(x_0+t)D_n(t)\,\mathrm{d}t \\ &= \frac{1}{2\pi}\int_{-\pi}^{-\delta} h(x_0+t)D_n(t)\,\mathrm{d}t + \frac{1}{2\pi}\int_{\delta}^{\pi} h(x_0+t)D_n(t)\,\mathrm{d}t \end{aligned}$$

since $h(x)=0$ on $(x_0-\delta, x_0+\delta)$. Now

$$\begin{aligned}
&\frac{1}{2\pi}\int_{-\pi}^{-\delta} h(x_0+t)D_n(t)\,\mathrm{d}t+\frac{1}{2\pi}\int_{\delta}^{\pi} h(x_0+t)D_n(t)\,\mathrm{d}t\\
&=\frac{1}{2\pi}\int_{-\pi}^{-\delta}\frac{h(x_0+t)}{\sin t/2}\sin(n+1/2)t\,\mathrm{d}t\\
&+\frac{1}{2\pi}\int_{\delta}^{\pi}\frac{h(x_0+t)}{\sin t/2}\sin(n+1/2)t\,\mathrm{d}t
\end{aligned}$$

$\to 0$ as $n\to\infty$ by Theorem 1.3.1. $\square$

2.6 Dini's Theorem

Dini's Theorem gives another set of conditions for a function f to satisfy under which its Fourier series converges. But before proving this theorem, we need some preliminary results.

Theorem 2.6.1. If

$$g(t)=\begin{cases}\dfrac{\sin t}{t} & t>0\\ 1 & t=0\end{cases}$$

then

$$\int_0^\infty g(t)\,\mathrm{d}t=\lim_{R\to\infty}\int_0^R\frac{\sin t}{t}\,\mathrm{d}t=\pi/2$$

Proof. Let $f(z)=e^{iz}/z$ for $z\in\mathbb{C}\diagdown\{0\}$. For $r>0$ let $\gamma_r(t)=re^{it}, 0\le t\le\pi$. We will now divide the proof into several steps.

Step 1. First of all, we have

$$\begin{aligned}
\int_\epsilon^R f(z)\,\mathrm{d}z+\int_{-R}^{-\epsilon} f(z)\,\mathrm{d}z&=\int_\epsilon^R\frac{e^{it}}{t}\,\mathrm{d}t+\int_{-R}^{-\epsilon}\frac{e^{it}}{t}\,\mathrm{d}t\\
&=\int_\epsilon^R\frac{e^{it}}{t}\,\mathrm{d}t+\int_R^\epsilon-\frac{e^{-is}}{-s}\,\mathrm{d}s\\
&=\int_\epsilon^R\frac{e^{it}}{t}\,\mathrm{d}t-\int_\epsilon^R\frac{e^{-is}}{s}\,\mathrm{d}s\\
&=2i\int_\epsilon^R\frac{\sin t}{t}\,\mathrm{d}t
\end{aligned}$$

Step 2. Now note that for $\epsilon > 0$,

$$\begin{aligned}
&\left|\int_{\gamma_\epsilon} \frac{e^{iz}}{z}\,\mathrm{d}z - \int_{\gamma_\epsilon} \frac{1}{z}\,\mathrm{d}z\right| \quad (*)\\
&= \left|\int_{\gamma_\epsilon} \frac{e^{iz}-1}{z}\,\mathrm{d}z\right|\\
&\leq M\pi\epsilon
\end{aligned}$$

where

$$M = \max\left\{\left|\frac{e^{iz}-1}{z}\right| : |z| = \epsilon\right\}$$

But

$$\lim_{z\to 0} \frac{e^{iz}-1}{z} = i$$

so the expression in $(*)$ goes to zero as $\epsilon \to 0$. Noting that

$$\int_{\gamma_\epsilon} \frac{1}{z}\,\mathrm{d}z = \int_0^\pi \frac{\epsilon i e^{it}}{\epsilon e^{it}}\,\mathrm{d}t = \pi i$$

we get

$$\lim_{\epsilon\to 0+} \int_{\gamma_\epsilon} \frac{e^{iz}}{z}\,\mathrm{d}z = \pi i$$

Step 3. The final integral we need to evaluate is

$$\begin{aligned}
\left|\int_{\gamma_R} f(z)\,\mathrm{d}z\right| &= \left|\int_0^\pi \frac{e^{i(R\cos t + iR\sin t)}}{Re^{it}} iRe^{it}\,\mathrm{d}t\right|\\
&\leq \int_0^\pi e^{-R\sin t}\,\mathrm{d}t\\
&= 2\int_0^{\pi/2} e^{-R\sin t}\,\mathrm{d}t
\end{aligned}$$

But $2/\pi \leq (\sin t)/2 \leq 1$ for $0 < t \leq \pi/2$. So $-R\sin t \leq -2Rt/\pi$ for $0 \leq t \leq \pi/2$. Hence

$$\begin{aligned}
2\int_0^{\pi/2} e^{-R\sin t}\,\mathrm{d}t &\leq 2\int_0^{\pi/2} e^{-2Rt/\pi}\,\mathrm{d}t\\
&= \frac{\pi}{R}(1 - e^{-R})
\end{aligned}$$

$\to 0$ as $R \to \infty$.

Step 4. Since f is analytic on the starlike region $\mathbb{C}\diagdown\{iy : y \leq 0\}$, Cauchy's Theorem says that

$$0 = \int_{\epsilon}^{R} f(z)\,\mathrm{d}z + \int_{\gamma_R} f(z)\,\mathrm{d}z + \int_{-R}^{-\epsilon} f(z)\,\mathrm{d}z + \int_{\gamma_\epsilon} f(z)\,\mathrm{d}z$$

So putting the above results together we get

$$\begin{aligned} 2i\int_{\epsilon}^{R} \frac{\sin t}{t}\,\mathrm{d}t &= \int_{\gamma_\epsilon} f(z)\,\mathrm{d}z - \int_{\gamma_R} f(z)\,\mathrm{d}z \\ &= \pi i - \int_{\gamma_R} f(z)\,\mathrm{d}z \end{aligned}$$

Letting $R \to \infty$ we see that

$$2i\int_{\epsilon}^{\infty} \frac{\sin t}{t}\,\mathrm{d}t = \pi i$$

But from Corollary 1.2.3,

$$\int_{0}^{\infty} \frac{\sin t}{t}\,\mathrm{d}t = \lim_{\epsilon\to 0^+}\int_{\epsilon}^{\infty} \frac{\sin t}{t}\,\mathrm{d}t$$

and so

$$\int_{0}^{\infty} \frac{\sin t}{t}\,\mathrm{d}t = \pi/2$$

as required.

□

Corollary 2.6.2. i) For any $a > 0$

$$\lim_{\lambda\to\infty}\int_{0}^{a} \frac{\sin \lambda t}{t}\,\mathrm{d}t = \pi/2$$

ii) For any $b > 0$

$$\int_{0}^{\infty} \frac{\sin bt}{t}\,\mathrm{d}t = \pi/2$$

Proof. i)

$$\begin{aligned}\int_0^a \frac{\sin \lambda t}{t}\,\mathrm{d}t &= \int_0^{\lambda a} \frac{\sin x}{x/\lambda}\frac{1}{\lambda}\,\mathrm{d}x \text{ where } x = \lambda t \\ &= \int_0^{\lambda a} \frac{\sin x}{x}\,\mathrm{d}x\end{aligned}$$

$\to \pi/2$ as $\lambda \to \infty$ by Theorem 2.6.1.

ii) This proven similarly to that in part i).

□

Corollary 2.6.3. Suppose $\delta > 0$ and $g : [0, \delta] \to \mathbb{C}$ is Riemann Integrable on its domain. Furthermore, suppose $g(0+) = \lim_{t \to 0^+} g(t)$ exists and

$$\int_0^\delta \frac{g(t) - g(0+)}{t}\,\mathrm{d}t$$

exists. Then

$$\lim_{\substack{\lambda \to \infty \\ \lambda \in \mathbb{R}}} \int_0^\delta \frac{g(t)}{t} \sin \lambda t\,\mathrm{d}t = \frac{\pi}{2} g(0+)$$

Proof. By Theorem 1.3.1

$$\begin{aligned}0 &= \lim_{\substack{\lambda \to \infty \\ \lambda \in \mathbb{R}}} \int_0^\delta \frac{g(t) - g(0+)}{t} \sin \lambda t\,\mathrm{d}t \\ &= \lim_{\substack{\lambda \to \infty \\ \lambda \in \mathbb{R}}} \int_0^\delta \frac{g(t)}{t} \sin \lambda t\,\mathrm{d}t - \lim_{\substack{\lambda \to \infty \\ \lambda \in \mathbb{R}}} g(0+) \int_0^\delta \frac{\sin \lambda t}{t}\,\mathrm{d}t\end{aligned}$$

But, by Corollary 2.6.2,

$$\lim_{\substack{\lambda \to \infty \\ \lambda \in \mathbb{R}}} \int_0^\delta \frac{\sin \lambda t}{t}\,\mathrm{d}t = \pi/2$$

and so the result follows. □

Theorem 2.6.4 (Dini)**.** Suppose f is a 2π-periodic, Riemann Integrable function on $\mathbb{R}$. Furthermore suppose for some $x_0 \in \mathbb{R}$ and $\delta > 0$ the following all exist:

i) $f(x_0+) = \lim_{x\to x_0^+} f(x)$

ii) $f(x_0-) = \lim_{x\to x_0^-} f(x)$

iii)
$$\int_0^\delta \frac{f(x_0+t)-f(x_0+)}{t}\,\mathrm{d}t$$

iv)
$$\int_0^\delta \frac{f(x_0-t)-f(x_0-)}{t}\,\mathrm{d}t$$

Then
$$\lim_{n\to\infty} s_n(x_0) = \frac{f(x_0+)+f(x_0-)}{2}$$
where s_n is the nth partial sum of the Fourier series of f.

Proof. Since D_n is even, Theorem 2.4.2 implies $\int_0^\pi D_n(t)\,\mathrm{d}t = \pi$. Therefore,

$$\begin{aligned}
&\frac{1}{\pi}\int_0^\pi f(x_0+t)D_n(t)\,\mathrm{d}t - f(x_0+)\\
&= \frac{1}{\pi}\int_0^\pi f(x_0+t)D_n(t)\,\mathrm{d}t - \frac{f(x_0+)}{\pi}\int_0^\pi D_n(t)\,\mathrm{d}t\\
&= \frac{1}{\pi}\int_0^\pi [f(x_0+t)-f(x_0+)]D_n(t)\,\mathrm{d}t\\
&= \frac{1}{\pi}\int_0^\pi g(t)\sin(n+1/2)t\,\mathrm{d}t
\end{aligned}$$

where
$$g(t) = 2\left(\frac{f(x_0+t)-f(x_0+)}{t}\right)\frac{t/2}{\sin t/2}$$

Now the hypothesis implies g is Riemann Integrable on $[0,\pi]$ so by Theorem 1.3.1
$$\lim_{n\to\infty}\frac{1}{\pi}\int_0^\pi g(t)\sin(n+1/2)t\,\mathrm{d}t = 0$$

Therefore,
$$\lim_{n\to\infty}\frac{1}{\pi}\int_0^\pi f(x_0+t)D_n(t)\,\mathrm{d}t = f(x_0+)$$

A similar argument shows that

$$\lim_{n\to\infty} \frac{1}{\pi}\int_0^\pi f(x_0 - t)D_n(t)\,\mathrm{d}t = f(x_0-)$$

To complete the proof,

$$\begin{aligned} s_n(x_0) &= \frac{1}{2\pi}\left(\int_0^\pi f(x_0+t)D_n(t)\,\mathrm{d}t + \int_0^\pi f(x_0-t)D_n(t)\,\mathrm{d}t\right) \\ &= \frac{1}{2}\left(\frac{1}{\pi}\int_0^\pi f(x_0+t)D_n(t)\,\mathrm{d}t + \frac{1}{\pi}\int_0^\pi f(x_0-t)D_n(t)\,\mathrm{d}t\right) \end{aligned}$$

$\to (f(x_0+) + f(x_0-))/2$ as $n \to \infty$. □

Corollary 2.6.5. i) Suppose $\delta > 0$ and $g : [0, \delta] \to \mathbb{C}$ is Riemann Integrable on its domain. Furthermore suppose $g(0+) = \lim_{x\to 0^+} g(x)$ exists and $|g(t) - g(0+)| \le M\,|t|$ for some $M > 0$. Then the conditions of Corollary 2.6.3 are satisfied.

ii) Suppose that f is a 2π-periodic, Riemann Integrable function on $\mathbb{R}$ and for some $x_0 \in \mathbb{R}$ and $M, \delta > 0$ we have $|f(x_0 + t) - f(x_0)| \le M\,|t|\,, -\delta \le t \le \delta$. Then the conditions of Theorem 2.6.4 are satisfied.

iii) If f satisfies the hypothesis of Theorem 2.5.3 then it also satisfies the hypothesis of Theorem 2.6.4.

Proof. i) Let $h(t) = (g(t) - g(0+))/t$. Then $|h(t)| \le M$ for $t \in [0, \delta]$. Also, h is Riemann Integrable on $[\epsilon, \delta]$ for all $0 < \epsilon < \delta$. Therefore, by Theorem 1.2.3 h is Riemann Integrable on $[0, \delta]$ and we have satsified all the conditions of Corollary 2.6.3.

ii) Since

$$\lim_{t\to 0} |f(x_0 + t) - f(x_0)| \le \lim_{t\to 0} M\,|t| = 0$$

we have $f(x_0+) = f(x_0-) = f(x_0)$. Futhermore, the function $g : (0, \delta] \to \mathbb{C}$ defined by $g(t) = (f(x_0 + t) - f(x_0+))/t$ is bounded on its domain and Riemann Integrable on $[\epsilon, \delta]$ and $[-\delta, -\epsilon]$, for all $0 < \epsilon < \delta$. Therefore, by Theorem 1.2.3 g is Riemann Integrable on $[0, \delta]$ and and $[-\delta, 0]$ and we have satisfied all the conditions of Theorem 2.6.4.

iii) This follows from Theorem 2.5.3 and Corollary 1.2.4.

□

2.7 The Heat Problem (Revisited)

Recall the Heat Problem stated in the first section of this chapter: Given a continuous function $f : [0, \pi] \to \mathbb{R}$ which satisfies $f(0) = f(\pi) = 0$ and represents the temperature at time zero of a wire of length π, is it possible to find a continuous function $u : [0, \pi] \times [0, \infty) \to \mathbb{R}$ which represents the temperature of the wire at time $t \geq 0$?

We stated that u must have partial derivatives on $(0, \pi) \times (0, \infty)$ and furthermore, it must satisfy the following conditions

$$\begin{aligned}
&u(x,0) = f(x) \text{ for all } x \in [0,\pi] \ (*) \\
&u(0,t) = u(\pi,t) = 0 \text{ for all } t \geq 0 \ (\dagger) \\
&\frac{\partial u}{\partial t}(x,t) = c^2 \frac{\partial^2 u}{\partial x^2}(x,t) \text{ for all } (x,t) \in (0,\pi) \times (0,\infty) \ (\ddagger)
\end{aligned}$$

where c is a real constant.

We showed that the funtion u defined by

$$u(x,t) = \sum_{k=1}^{\infty} b_k e^{-k^2 t} \sin kx \ (*)$$

for some real scalars $b_k, k \in \mathbb{N}$ satisfies $(\dagger)$ and $(\ddagger)$ and that this u is $\mathcal{C}^\infty$ on $\mathbb{R} \times (0, \infty)$.

Therefore, to show that u represents the solution to the Heat Problem corresponding to f, we only have to show that u is continuous on $[0, \pi] \times \{0\}$ and that f can be written in the form

$$f(x) = \sum_{k=1}^{\infty} b_k \sin kx$$

In this section we present one condition, which, if satisfied, allows f to have a solution u for the Heat Problem.

Theorem 2.7.1. Suppose f belongs to $C_0[0, \pi]$ and that $f \in \mathcal{C}^2$ on $[0, \pi]$. Then the Heat Problem has a (unique) solution (namely, the function u stated above.)

Proof. By Proposition 2.3.2, there exists a unique, odd, 2π-periodic extension of f to $\mathbb{R}$. Call this extension g. Then by Corollary 2.3.5 the Fourier series of g has the form

$$\sum_{k=1}^{\infty} b_k \sin kx$$

and this series converges uniformly to g on $\mathbb{R}$ by Corollary 2.5.2. Therefore, the function u given in $(*)$ represents a solution to the Heat Problem for f provided we can show it is continuous at $(x_0, 0)$ for $x_0 \in [0, \pi]$. In fact, we will show that u is continuous on $\mathbb{R} \times \{0\}$.

Let $M > 0$ be an upper bound for the b_k's. Fix $\epsilon > 0$. Given $x_0 \in \mathbb{R}$ choose $\delta' > 0$ such that $|g(x) - g(x_0)| < \epsilon/2$ whenever $|x - x_0| < \delta'$. Next choose $N \in \mathbb{N}$ such that

$$M \sum_{k=N+1}^{\infty} \frac{2}{k^2} < \epsilon/4$$

Finally, choose $\tau > 0$ such that

$$M \sum_{k=1}^{N} \frac{1 - e^{-k^2\tau}}{k^2} < \epsilon/4$$

Let $\delta = \min(\delta', \tau)$. Then if $\|(x,t) - (x_0, 0)\| < \delta$ we have

$$\begin{aligned}
|u(x,t) - u(x_0,0)| &\leq |u(x,t) - u(x,0)| + |u(x,0) - u(x_0,0)| \\
&= \left| \sum_{k=1}^{\infty} b_k e^{-k^2 t} \sin kx - \sum_{k=1}^{\infty} b_k \sin kx \right| + |g(x) - g(x_0)| \\
&\leq \sum_{k=1}^{\infty} |b_k| \left(1 - e^{-k^2 t}\right) + \epsilon/2 \\
&\leq M \sum_{k=1}^{\infty} \frac{1 - e^{-k^2 t}}{k^2} + \epsilon/2
\end{aligned}$$

By Theorem 2.3.9. Therefore,

$$\begin{aligned}
|u(x,t) - u(x_0,0)| &\le M \sum_{k=1}^{N} \frac{1-e^{-k^2 t}}{k^2} + M \sum_{k=N+1}^{\infty} \frac{1-e^{-k^2 t}}{k^2} + \epsilon/2 \\
&\le M \sum_{k=1}^{N} \frac{1-e^{-k^2 \tau}}{k^2} + M \sum_{k=N+1}^{\infty} \frac{2}{k^2} + \epsilon/2 \\
&\le \epsilon/4 + \epsilon/4 + \epsilon/2 \\
&= \epsilon
\end{aligned}$$

And so u is continuous on $\mathbb{R} \times [0,\infty)$. Thus the restriction of this u to $[0,\pi] \times [0,\infty)$ represents the solution to the Heat Problem for f. (This is solution is unique by Theorem 2.2.7.) $\square$

Chapter 3

Féjer's Theorem

3.1 Theorems of Abel and Dirichlet

In this section we present some results due to Abel and Dirichlet concerning infinite series.

Theorem 3.1.1 (Abel's Partial Summation Formula)**.** Suppose $\{a_k\}_{k=0}^{\infty}$ and $\{b_k\}_{k=0}^{\infty}$ are complex sequences and we define $A_n = \sum_{k=0}^{n} a_k$ for $n \geq 0$, $A_{-1} = 0$. Then, if $0 \leq p < q$, we have

$$\sum_{k=p}^{q} a_k b_k = A_q b_q - A_{p-1} b_p + \sum_{k=p}^{q-1} A_k (b_k - b_{k+1})$$

Proof. For $0 \leq p < q$, we have

$$\begin{aligned}
\sum_{k=p}^{q} a_k b_k &= \sum_{k=p}^{q} (A_k - A_{k-1}) b_k \\
&= \sum_{k=p}^{q} A_k b_k - \sum_{k=p}^{q} A_{k-1} b_k \\
&= \sum_{k=p}^{q} A_k b_k - \sum_{k=p-1}^{q-1} A_k b_{k+1} \\
&= A_q b_q - A_{p-1} b_p + \sum_{k=p}^{q-1} A_k (b_k - b_{k+1})
\end{aligned}$$

□

Theorem 3.1.2 (Abel's Lemma)**.** Suppose, in addition to the hypothesis of Theorem 3.1.1, there exists $M > 0$ such that $|A_n| \leq M$ for all $n \geq 0$ and $b_k \geq b_{k+1} \geq 0$ for all $k \geq 0$. Then

$$\left| \sum_{k=p}^{q} a_k b_k \right| \leq 2Mb_p$$

whenever $0 \leq p < q$.

Proof. By Theorem 3.1.1,

$$\begin{aligned} \left| \sum_{k=p}^{q} a_k b_k \right| &= \left| A_q b_q - A_{p-1} b_p + \sum_{k=p}^{q-1} A_k (b_k - b_{k+1}) \right| \\ &\leq |A_q|\, b_q + |A_{p-1}|\, b_p + \sum_{k=p}^{q-1} |A_k|\, (b_k - b_{k+1}) \\ &\leq Mb_q + Mb_p + \sum_{k=p}^{q-1} M(b_k - b_{k+1}) \\ &= 2Mb_p \end{aligned}$$

(provided $0 \leq p < q$). □

Theorem 3.1.3 (Dirichlet's Test)**.** Suppose $\{a_k\}_{k=0}^{\infty}$ is a complex sequence such that for some $M > 0$ and all $n \geq 0$,

$$\left| \sum_{k=0}^{\infty} a_k \right| \leq M$$

Also suppose that $\{b_k\}_{k=0}^{\infty}$ is a real sequence which decreases to zero as $k \to \infty$. Then the series $\sum_{k=0}^{\infty} a_k b_k$ is convergent.

Proof. Given $\epsilon > 0$ choose $N > 0$ such that $b_p < \epsilon/2M$ if $p > N$. Let $S_n = \sum_{k=0}^{n} a_k b_k$ for $n \geq 0$. Then by Theorem 3.1.2, if $q > p > N$,

$$|S_q - S_{p-1}| \leq 2Mb_p \leq 2M\frac{\epsilon}{2M} = \epsilon$$

Therefore the series $\sum_{k=0}^{\infty} a_k b_k$ is Cauchy and hence converges. $\square$

Theorem 3.1.4 (Dirichlet's Test for Uniform Convergence)**.** Suppose X is a non-empty set, $u_k : X \to \mathbb{C}$ for $k \geq 0$ and

$$\left|\sum_{k=0}^{n} u_k(x)\right| \leq M$$

for some $M > 0$ and all $n \geq 0, x \in X$. Also suppose that $\{b_k\}_{k=0}^{\infty}$ is a real sequence which decreases to zero as $k \to \infty$. Then the series $\sum_{k=0}^{\infty} b_k u_k(x)$ is uniformly convergent on X.

Proof. This result follows immediately from Theorem 3.1.3 by noting that the given series is uniformly Cauchy. $\square$

Corollary 3.1.5. Let $\{b_k\}_{k=0}^{\infty}$ be a real sequence which decreases to zero as $k \to \infty$. Then the series $\sum_{k=0}^{\infty} b_k z^k$ converges if $z \in \mathbb{C}$ and $|z| \leq 1$. In fact, if $0 < \delta < 2$, then $\sum_{k=0}^{\infty} b_k z^k$ converges uniformly on $\{z \in \mathbb{C} : |z| \leq 1 \text{ and } |z - 1| \geq \delta\}$.

Proof. Suppose the hypotheses of the theorem hold. Then for all $n \geq 0$,

$$\left|\sum_{k=0}^{n} z^k\right| = \left|\frac{1 - z^{n+1}}{1 - z}\right| \leq \frac{2}{\delta}$$

Hence the hypotheses of Theorem 3.1.4 hold and the result follows. $\square$

Theorem 3.1.6 (Abel's Theorem)**.** If $\sum_{k=0}^{\infty} c_k$ is a convergent complex series, so that $\sum_{k=0}^{\infty} c_k z^k$ has radius of convergence at least 1, then $\sum_{k=0}^{\infty} c_k x^k$ converges uniformly for $0 \leq x \leq 1$.

Proof. Since $\sum_{k=0}^{\infty} c_k$ converges it is Cauchy. So given $\epsilon > 0$ there exists $N > 0$ such that for $q > p > N$ we have

$$\left| \sum_{k=p+1}^{q} c_k \right| \leq \frac{\epsilon}{3}$$

Now, if $0 \leq x \leq 1$ then $1 \geq x^k \geq x^{k+1} \geq 0$ for all $k \geq 0$. So by a slight modification of Theorem 3.1.2, we have for $q > p > N$,

$$\left| \sum_{k=p+1}^{q} c_k x^k \right| \leq 2(\frac{\epsilon}{3}) x^{p+1} \leq \frac{2\epsilon}{3} < \epsilon$$

Therefore the series $\sum_{k=0}^{\infty} c_k x^k$ is uniformly Cauchy and hence converges uniformly on $0 \leq x \leq 1$. □

Corollary 3.1.7.

Proof. □

Example 3.1.8. For $\alpha \in \mathbb{R}$ and $0 \leq k \in \mathbb{Z}$, define

$$\binom{\alpha}{k} = \begin{cases} 1 & k = 0 \\ \dfrac{\alpha(\alpha-1)\cdots(\alpha-k+1)}{k!} & k > 0 \end{cases}$$

Notice that this definition coincides with the usual one when $0 \leq \alpha \in \mathbb{Z}$.

i) Show that the radius of convergence of the power series

$$\sum_{k=1}^{\infty} \binom{\alpha}{k} x^k$$

is at least 1.

ii) If

$$f(x) = 1 + \sum_{k=1}^{\infty} \binom{\alpha}{k} x^k$$

for $-1 < x < 1$, use the theory of power series to deduce that $(1 + x)f'(x) = \alpha f(x)$ and therefore that

$$f(x) = (1 + x)^{\alpha}$$

iii) If $\alpha > -1$, show that

$$\sum_{k=1}^{\infty} \binom{\alpha}{k}$$

is convergent and that

$$1 + \sum_{k=1}^{\infty} \binom{\alpha}{k} x^k$$

converges uniformly on $[0, 1]$ to $(1 + x)^{\alpha}$.

iv) Show that for all $\epsilon > 0$, there exist real scalars $c_0, c_1, \ldots, c_n$ such that

$$\left| |x| - \sum_{k=0}^{n} c_k x^k \right| < \epsilon$$

for $-1 \leq x \leq 1$.

Solution. i) The radius of convergence of the given power series is

$$\begin{aligned} \lim_{k\to\infty} \left| \binom{\alpha}{k} \Big/ \binom{\alpha}{k+1} \right| &= \lim_{k\to\infty} \left| \frac{k+1}{\alpha - k} \right| \\ &= 1 \end{aligned}$$

ii) If we let $f_k(x) = \binom{\alpha}{k} x^k$ for $k > 0$ then $f_k'(x) = (\alpha - k)\binom{\alpha}{k} x^k$. By a similar argument as that used in part i), the radius of convergence of

$$\sum_{k=1}^{\infty} f_k'(x)$$

is also 1. Hence by Theorem 1.2.2, we can differentiate f term-by-term to obtain

$$\begin{aligned}(1+x)f'(x) &= f'(x) + xf'(x) \\ &= \alpha + \sum_{k=1}^{\infty}(\alpha - k)\binom{\alpha}{k}x^k + \sum_{k=1}^{\infty} k\binom{\alpha}{k}x^k \\ &= \alpha\left(1 + \sum_{k=1}^{\infty}\binom{\alpha}{k}x^k\right) \\ &= \alpha f(x)\end{aligned}$$

Now, to see that $f(x) = (1+x)^\alpha$, let

$$g(x) = \frac{f(x)}{(1+x)^\alpha}$$

Then

$$\begin{aligned}g'(x) &= \frac{f'(x)(1+x)^\alpha - \alpha(1+x)^{\alpha-1}f(x)}{(1+x)^{2\alpha}} \\ &= \frac{f'(x)(1+x)^\alpha - (1+x)^{\alpha-1}(1+x)f'(x)}{(1+x)^{2\alpha}} \\ &= 0\end{aligned}$$

So g is a constant. But $g(0) = 1$ so the result is proved.

iii) If we can show that the series

$$\sum_{k=1}^{\infty}\binom{\alpha}{k}$$

converges, then by Theorem 3.1.6, the power series

$$1 + \sum_{k=1}^{\infty}\binom{\alpha}{k}x^k$$

converges uniformly on $[0,1]$. We will divide the solution into three cases.

Case 1. If $0 \leq \alpha \in \mathbb{Z}$ then the sum in the infinite series is just a finite sum so it converges.

Case 2. If $\alpha > 0$ and $\alpha \notin \mathbb{Z}$, then let $N = \lceil \alpha \rceil$. If $k > N$, we have

$$\begin{aligned}
\binom{\alpha}{k} &= \frac{\alpha}{k}\frac{\alpha-1}{1}\frac{\alpha-2}{2}\cdots\frac{\alpha-(k-2)}{k-2}\frac{\alpha-(k-1)}{k-1} \\
&= \frac{\alpha}{k}\left(\frac{\alpha}{1}-1\right)\left(\frac{\alpha}{2}-1\right)\cdots\left(\frac{\alpha}{k-1}-1\right) \\
&= (-1)^{k-1}\frac{\alpha}{k}\prod_{j=1}^{k-1}\left(1-\frac{\alpha}{j}\right) \\
&= (-1)^{k-1}\beta\beta_k
\end{aligned}$$

where

$$\beta = \begin{cases} \alpha\left(1-\frac{\alpha}{1}\right)\left(1-\frac{\alpha}{2}\right)\cdots\left(1-\frac{\alpha}{N-1}\right) & \text{if } N > 1 \\ 1 & \text{if } N = 1 \end{cases}$$

and

$$\beta_k = \left(1-\frac{\alpha}{N}\right)\left(1-\frac{\alpha}{N+1}\right)\cdots\left(1-\frac{\alpha}{k-1}\right)$$

for $k > N$. Now notice that for $k > N$,

$$\begin{aligned}
\left|\binom{\alpha}{k+1}\Big/\binom{\alpha}{k}\right| &= \frac{\beta\beta_{k+1}}{\beta\beta_k} \\
&= \frac{k}{k+1}\left(1-\frac{\alpha}{k}\right) \\
&< 1
\end{aligned}$$

This shows that the terms of the series

$$\sum_{k=N+1}^{\infty}\binom{\alpha}{k}$$

are decreasing, and that

$$\log\left(1-\frac{\alpha}{k}\right) < 0$$

Therefore,

$$\lim_{k\to\infty}\log\left|\binom{\alpha}{k}\right| = \log\alpha - \log k + \sum_{j=1}^{k-1}\log\left|1-\frac{\alpha}{j}\right|$$

$\to -\infty$. So the terms of the given series converge to zero. Recall Leibniz Theorem for infinite series which states that if $\{a_k\}_{k=1}^{\infty}$ is a sequence of positive reals decreasing to zero, then the series

$$\sum_{k=1}^{\infty}(-1)^k a_k$$

converges. Therefore,

$$\begin{aligned}\sum_{k=1}^{\infty}\binom{\alpha}{k} &= \sum_{k=1}^{N}\binom{\alpha}{k} + \sum_{k=N+1}^{\infty}\binom{\alpha}{k} \\ &= \sum_{k=1}^{N}\binom{\alpha}{k} + \beta \sum_{k=N+1}^{\infty}(-1)^{k-1}\beta_k\end{aligned}$$

converges if $\alpha > 0$.

Case 3. If $-1 < \alpha < 0$ we will also show that $\binom{\alpha}{k}$ is decreasing and, in fact, that these terms converge to zero. For all $k > 0$, we have

$$\begin{aligned}\left|\binom{\alpha}{k+1}\Big/\binom{\alpha}{k}\right| &= \frac{k}{k+1}\left|1-\frac{\alpha}{k}\right| \\ &= \frac{k}{k+1}\frac{k+|\alpha|}{k} \\ &= \frac{k+|\alpha|}{k+1} \\ &< 1\end{aligned}$$

since $|\alpha| < 1$. Now

$$\begin{aligned}\log\left|\binom{\alpha}{k}\right| &= \log|\alpha| - \log k + \sum_{j=1}^{k-1}\log\frac{|\alpha-j|}{j} \\ &= \log|\alpha| - \sum_{j=1}^{k-1}[\log(j+1)-\log j] + \sum_{j=1}^{k-1}[\log(j+|\alpha|)-\log j] \\ &= \log|\alpha| - \sum_{j=1}^{k-1}[\log(j+1)-\log(j+|\alpha|)] \\ &= \log|\alpha| - \sum_{j=1}^{k-1}\frac{(j+1)-(j+|\alpha|)}{x_j}\end{aligned}$$

where $\{x_j\}_{j=1}^{\infty}$ is a real sequence satisfying $j + |\alpha| < x_j < j + 1$ (here we have used the Mean Value Theorem). Therefore,

$$\begin{aligned}
\log\left|\binom{\alpha}{k}\right| &= \log|\alpha| - \sum_{j=1}^{k-1} \frac{1-|\alpha|}{x_j} \\
&< \log|\alpha| - \sum_{j=1}^{k-1} \frac{1-|\alpha|}{\lceil x_j \rceil} \\
&= \log|\alpha| - \sum_{j=1}^{k-1} \frac{1-|\alpha|}{j+1}
\end{aligned}$$

$\to -\infty$ as $k \to \infty$ since $1 - |\alpha| > 0$. This implies that $\binom{\alpha}{k} \to 0$ as $k \to \infty$, as stated and so, by Leibniz Theorem again, the series

$$\sum_{k=1}^{\infty} \binom{\alpha}{k}$$

converges in this case.

iv) From part iii), we know the power series

$$f(x) = 1 + \sum_{k=1}^{\infty} \binom{\alpha}{k} x^k$$

converges uniformly on $[0, 1]$ to $(1 + x)^{\alpha}$. Therefore, it must converge uniformly on $(-1, 1]$ to $(1+x)^{\alpha}$ because the radius of convergence is at least 1. Also, this series converges at -1 because the terms decrease to zero and are alternating in sign (here we are using Leibniz Theorem as in part iii)). Thus, the series converges uniformly on $[-1, 1]$. So given $\epsilon > 0$, choose $N > 0$ such that

$$\left| f(x) - \sum_{k=1}^{N} \binom{\frac{1}{2}}{k} x^k \right| < \epsilon$$

If $-1 \le x \le 1$ then $-1 \le x^2 - 1 \le 0$ so

$$|x| = \sqrt{1 + (x^2 - 1)} = f(x^2 - 1)$$

(if we let $\alpha = 1/2$). Therefore,

$$\begin{aligned}\left| f(x^2-1) - \sum_{k=0}^{N} \binom{\frac{1}{2}}{k} (x^2-1)^k \right| &= \left| |x| - \sum_{k=0}^{N} \binom{\frac{1}{2}}{k} (x^2-1)^k \right| \\ &= \left| |x| - \sum_{k=0}^{n} c_k x^k \right| \\ &< \epsilon\end{aligned}$$

for some real scalars $c_0, c_1, \ldots, c_n$ (where $n = 2N$).

3.2 Abel and Cesáro Sums

Given a complex sequence $\{s_k\}_{k=0}^{\infty}$ there are other ways to define the sum $\sum_{k=0}^{\infty} s_k$. For example, although $\sum_{k=0}^{\infty}(-1)^k$ doesn't exist when applying the usual definition of infinite sums, Euler would have said it converges to $1/2$ because

$$\begin{aligned}\sum_{k=0}^{\infty}(-1)^k &= 1 + \sum_{k=1}^{\infty}(-1)^k \\ &= 1 + \sum_{k=0}^{\infty}(-1)^{k+1} \\ &= 1 - \sum_{k=0}^{\infty}(-1)^k\end{aligned}$$

and so

$$2\sum_{k=0}^{\infty}(-1)^k = 1$$

from which the assertion follows.

In this section we explore two alternative definitions of the sum of a series and then derive some related results.

Definition 3.2.1. Given a complex sequence $\{s_k\}_{k=0}^{\infty}$, its *Cesáro limit* is defined as

$$\lim_{n\to\infty} \frac{1}{n+1} \sum_{k=0}^{n} s_k$$

and this limit is denoted by

$$\mathrm{C} \lim_{n\to\infty} s_n$$

If the Cesáro limit of a sequence exists and equals $L \in \mathbb{C}$, then we say the sequence *converges in the sense of Cesáro to* L.

Example 3.2.2. Find the Cesáro limit of the sequence $\{(-1)^k\}_{k=1}^{\infty}$.

Solution. For $n \geq 0$, let $s_n = \sum_{k=0}^{n}(-1)^k$. Then $s_{2n} = 1$ and $s_{2n+1} = 0$. Therefore

$$\lim_{n\to\infty} \frac{s_{2n}}{2n} = \lim_{n\to\infty} \frac{s_{2n+1}}{2n+1} = 0$$

and hence $\mathrm{C} \lim_{k\to\infty}(-1)^k = 0$.

Theorem 3.2.3. Suppose $\lim_{k\to\infty} s_k = L$ for some sequence $\{s_k\}_{k=0}^{\infty}$. Then the Cesáro limit of this sequence is also L.

Proof. Without loss of generality, we may assume $L = 0$. Given $\epsilon > 0$, choose N_1 such that $|s_k| < \epsilon/2$ when $k > N_1$. Now choose N_2 such that

$$\left| \frac{1}{n+1} \sum_{k=0}^{N_1} s_k \right| < \epsilon/2$$

whenever $n > N_2$. Let $N = \max(N_1, N_2)$. Then if $n > N$, we have

$$\begin{aligned} \left| \frac{1}{n+1} \sum_{k=0}^{n} s_k \right| &\leq \left| \frac{1}{n+1} \sum_{k=0}^{N_1} s_k \right| + \left| \frac{1}{n+1} \sum_{k=N_1+1}^{n} s_k \right| \\ &\leq \epsilon/2 + \epsilon/2 \\ &= \epsilon \end{aligned}$$

and so

$$\mathrm{C} \lim_{k\to\infty} s_k = \lim_{n\to\infty} \frac{1}{n+1} \sum_{k=0}^{n} s_k = 0$$

as required. □

Definition 3.2.4. Given a complex series $\sum_{k=0}^{\infty} s_k$, its *Cesáro sum* is defined as the Cesáro limit of its sequence of partial sums. If this Cesáro sum is equal to $L \in \mathbb{C}$, we write

$$(\mathrm{C}) \sum_{k=0}^{\infty} s_k = L$$

Theorem 3.2.5. Suppose the Cesáro sum of a complex series $\sum_{k=0}^{\infty} s_k$ is L. Then

$$\lim_{n\to\infty} \sum_{k=0}^{n} (1 - \frac{k}{n+1}) s_k = L$$

Proof. We have

$$\begin{aligned}
\sum_{k=0}^{n} (1 - \frac{k}{n+1}) s_k &= \frac{1}{n+1} \sum_{k=0}^{n} (n+1-k) s_k \\
&= \frac{1}{n+1} \left((n+1)s_0 + ns_1 + \ldots + 2s_{n-1} + s_n \right) \\
&= \frac{1}{n+1} \left(s_0 + (s_0 + s_1) + (s_0 + s_1 + s_2) + \ldots + (s_0 + s_1 + \ldots + s_n) \right) \\
&= \frac{1}{n+1} \sum_{k=0}^{n} S_k
\end{aligned}$$

where $\{S_k\}_{k=0}^{\infty}$ is the sequence of partial sums of the series $\sum_{k=0}^{\infty} s_k$. Hence,

$$\lim_{n\to\infty} \sum_{k=0}^{n} (1 - \frac{k}{n+1}) s_k = (\mathrm{C}) \sum_{k=0}^{\infty} s_k = L$$

□

Definition 3.2.6. A complex series $\sum_{k=0}^{\infty} s_k$ is said to be *Abel summable* to $L \in \mathbb{C}$ provided the power series $\sum_{k=0}^{\infty} s_k x^k$ has radius of convergence at least 1 and

$$\lim_{x\to 1^-} \sum_{k=0}^{\infty} s_k x^k = L$$

In this case we write

$$(\mathrm{A})\sum_{k=0}^{\infty} s_k = L$$

Example 3.2.7. Show that the series $\sum_{k=0}^{\infty}(-1)^k$ is divergent but Cesáro summable and Abel summable to $1/2$.

Solution. Let $\{s_n\}_{n=0}^{\infty}$ be the sequence of partial sums of the given series. Then, for $n \geq 0$, $s_{2n} = 1$ and $s_{2n+1} = 0$. Therefore, the series diverges. However, if $\{S_n\}_{n=0}^{\infty}$ is the sequence of partial sums of the series $\sum_{k=0}^{n} s_k$, then $S_{2n} = n+1$ and $S_{2n+1} = n+1$. Therefore,

$$\lim_{n\to\infty} \frac{1}{2n} S_{2n} = \lim_{n\to\infty} \frac{n+1}{2n} = \frac{1}{2}$$

and also

$$\lim_{n\to\infty} \frac{1}{2n+1} S_{2n+1} = \lim_{n\to\infty} \frac{n+1}{2n+1} = \frac{1}{2}$$

So the Cesáro sum of this series is $1/2$. To compute the Abel sum of the series, we have

$$\lim_{x\to 1^-} \sum_{k=0}^{\infty} (-1)^k x^k = \lim_{x\to 1^-} \frac{1}{1+x} = \frac{1}{2}$$

as required.

Example 3.2.8. Show that the series $\sum_{k=0}^{\infty}(-1)^k k$ is Abel summable but not Cesáro summable.

Solution. The partial sums $\{s_n\}_{n=1}^{\infty}$ of the series are given by $s_{2n} = n$ and $s_{2n+1} = -n-1$ for $n \geq 0$. Therefore the partial sums $\{S_n\}_{n=0}^{\infty}$ of the series

$\sum_{k=0}^{\infty} s_k$ are given by

$$\begin{aligned}
S_{2n} &= \sum_{k=0}^{2n} s_k \\
&= \sum_{k=0}^{n} s_{2k} + \sum_{k=0}^{n-1} s_{2k+1} \\
&= \sum_{k=0}^{n} k + \sum_{k=0}^{n-1}(-k-1) \\
&= n + \sum_{k=0}^{n-1} k - \sum_{k=0}^{n-1} k - \sum_{k=0}^{n-1} 1 \\
&= n - n \\
&= 0
\end{aligned}$$

and

$$\begin{aligned}
S_{2n+1} &= \sum_{k=0}^{2n+1} s_k \\
&= \sum_{k=0}^{n} s_{2k} + \sum_{k=0}^{n} s_{2k+1} \\
&= \sum_{k=0}^{n} k + \sum_{k=0}^{n}(-k-1) \\
&= \sum_{k=0}^{n} k - \sum_{k=0}^{n} k - \sum_{k=0}^{n} 1 \\
&= -n - 1
\end{aligned}$$

Hence the given series is not Cesáro summable because

$$\lim_{n\to\infty} \frac{1}{2n+1} S_{2n} = 0$$

but

$$\lim_{n\to\infty} \frac{1}{2n+2} S_{2n+1} = -\frac{1}{2}$$

However, from the theory of power series, we know that the radius of convergence of $\sum_{k=0}^{\infty}(-1)^k k x^k$ is

$$\lim_{k\to\infty} \frac{\left|(-1)^k k\right|}{\left|(-1)^{k+1}(k+1)\right|} = 1$$

Also, letting $S = \sum_{k=0}^{\infty}(-1)^k k x^k$, we have

$$\begin{aligned} S + xS &= \sum_{k=1}^{\infty}(-1)^k x^k \\ &= \frac{-x}{1+x} \end{aligned}$$

which implies

$$\lim_{x\to 1^-}\sum_{k=0}^{\infty}(-1)^k k x^k = \lim_{x\to 1^-}\frac{-x}{(1+x)^2} = \frac{-1}{4}$$

and so the given series is Abel summable to $-1/4$.

Example 3.2.9. Suppose $z \in \mathbb{C}$, $|z| = 1$ but $z \neq 1$. Show that the series $\sum_{k=0}^{\infty} z^k$ is Cesáro summable and Abel summable to $1/(1-z)$.

Solution. The partial sums, $s_n, n \geq 0$ of the given series are given by

$$s_n = \frac{1 - z^{n+1}}{1 - z}$$

Therefore,

$$\begin{aligned} \lim_{n\to\infty}\frac{1}{n+1}\sum_{k=0}^{n} s_n &= \lim_{n\to\infty}\frac{1}{n+1}\sum_{k=0}^{n}\frac{1}{1-z} - \lim_{n\to\infty}\frac{1}{n+1}\sum_{k=0}^{n}\frac{z^{n+1}}{1-z} \\ &= \frac{1}{1-z} - \frac{z}{1-z}\lim_{n\to\infty}\frac{1}{n+1}\sum_{k=0}^{n} z^n \\ &= \frac{1}{1-z} - \frac{z}{1-z}\lim_{n\to\infty}\frac{1}{n+1}\frac{1}{1-z} \\ &= \frac{1}{1-z} \end{aligned}$$

So the Cesáro sum of the series is $1/(1-z)$ as required.

To compute the Abel sum, note that the radius of convergence of the series $\sum_{k=0}^{\infty} z^k x^k$ is

$$\lim_{k\to\infty}\frac{|z^k|}{|z^{k+1}|} = \lim_{k\to\infty}\frac{1}{|z|} = 1$$

Therefore, the Abel sum of the series is given by

$$\begin{aligned}\lim_{x\to 1^-}\sum_{k=0}^{\infty} z^k x^k &= \lim_{x\to 1^-}\frac{1}{1-zx}\\ &= \frac{1}{1-z}\end{aligned}$$

3.3 Féjer's Kernel

In this section we will derive an expression called Féjer's Kernel and then deduce some of its properties.

Suppose f is a 2π-periodic, Riemann integrable function and let $s_k, k \geq 0$ be the kth partial sum of its Fourier series. Define $\sigma_n(x) : \mathbb{R} \to \mathbb{C}$ by

$$\sigma_n(x) = \frac{1}{n+1}\sum_{k=0}^{n} s_k(x)$$

Then, we have

$$\begin{aligned}\sigma_n(x) &= \frac{1}{n+1}\sum_{k=0}^{n}\frac{1}{2\pi}\int_{-\pi}^{\pi} f(x-t)D_k(t)\,\mathrm{d}t\\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x-t)\left(\sum_{k=0}^{n}\frac{1}{n+1}D_k(t)\right)\mathrm{d}t\\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x-t)F_n(t)\,\mathrm{d}t\end{aligned}$$

where

$$F_n(t) = \frac{1}{n+1}\sum_{k=0}^{n} D_k(t)$$

for $t \in \mathbb{R}$. We call $\{F_n\}_{n=0}^{\infty}$ the *Féjer's Kernel.*

Theorem 3.3.1. i) For all $n \geq 0$, Féjer's Kernel is a real-valued, even, trigonometric polynomial of "degree n".

ii) For all $n \geq 0$,

$$F_n(x) = \begin{cases} \dfrac{1}{n+1} \dfrac{\sin^2(\frac{n+1}{2}x)}{\sin^2 x/2} & \text{if } x \in \mathbb{R}, x \neq 2\pi k, k \in \mathbb{Z} \\ n+1 & \text{if } x = 2\pi k, k \in \mathbb{Z} \end{cases}$$

iii) For all $n \geq 0$ and all $x \in \mathbb{R}$, $0 \leq F_n(x) \leq F_n(0) = n+1$.

iv) For all $n \geq 0$,

$$\int_{-\pi}^{\pi} F_n(t)\,\mathrm{d}t = 2\pi$$

v) For $0 < \delta < \pi$ and $n \in \mathbb{N}$,

$$0 \leq F_n(x) \leq \frac{2}{n+1}\left(\frac{1}{1-\cos\delta}\right)$$

if $x \in \mathbb{R}$ and $\delta \leq x \leq \pi$.

vi) For $0 < \delta < \pi$,

$$\lim_{n\to\infty} \int_{\delta}^{\pi} F_n(t)\,\mathrm{d}t = 0 = \lim_{n\to\infty} \int_{-\pi}^{-\delta} F_n(t)\,\mathrm{d}t$$

vii) For $0 < \delta < \pi$,

$$\lim_{n\to\infty} \int_{-\delta}^{\delta} F_n(t)\,\mathrm{d}t = 2\pi$$

Proof. i) This follows from the properties of the Dirichlet kernel.

ii) For all $n \geq 0$, we have

$$\begin{aligned} (\sin^2 x/2)F_n(x) &= \frac{\sin^2 x/2}{2} \sum_{k=0}^{n} D_k(x) \\ &= \frac{\sin^2 x/2}{2} \sum_{k=0}^{n} \frac{\sin(k+1/2)x}{\sin x/2} \\ &= \frac{1}{n+1} \sum_{k=0}^{n} (\sin(k+1/2)x)(\sin x/2)\ (*) \end{aligned}$$

Recall that

$$\begin{aligned}\cos(a-b)-\cos(a+b) &= [\cos a\cos(-b)-\sin a\sin(-b)]\\ &\quad -[\cos a\cos b-\sin a\sin b]\\ &= [\cos a\cos b+\sin a\sin b]\\ &\quad -[\cos a\cos b-\sin a\sin b]\\ &= 2\sin a\sin b\end{aligned}$$

Substituting into $(*)$, we obtain

$$\begin{aligned}(\sin^2 x/2)F_n(x) &= \frac{1}{n+1}\sum_{k=0}^{n}\frac{\cos kx-\cos(k+1)x}{2}\\ &= \frac{1-\cos(n+1)x}{2(n+1)}\ (\dagger)\end{aligned}$$

Substituting the following identity into $(\dagger)$,

$$1-\cos 2x = 2\sin^2 x$$

we obtain

$$(\sin^2 x/2)F_n(x) = \frac{\sin^2 \frac{n+1}{2}x}{n+1}$$

which proves the assertion when $x \neq 2\pi k, k \in \mathbb{N}$. When x is a multiple of 2π, $D_k(x) = 2k+1$, hence

$$\begin{aligned}F_n(x) &= \frac{1}{n+1}\sum_{k=0}^{n} D_k(x)\\ &= \frac{1}{n+1}\sum_{k=0}^{n}(2k+1)\\ &= \frac{1}{n+1}\left(2\frac{n(n+1)}{2}+(n+1)\right)\\ &= n+1\end{aligned}$$

as required.

iii) By part ii), we clearly have $0 \leq F_n(x)$ for all $x \in \mathbb{R}$. Also, since $|D_k(x)| \leq 2k+1$, we have

$$\begin{aligned} F_n(x) &= \frac{1}{n+1} \sum_{k=0}^{n} D_k(x) \\ &\leq \frac{1}{n+1} \sum_{k=0}^{n} (2k+1) \\ &= n+1 \\ &= F_n(0) \end{aligned}$$

again by part ii).

iv) This follows from the fact that

$$\int_{-\pi}^{\pi} D_n(t)\,\mathrm{d}t = 2\pi$$

v) By part ii) above, we have

$$\begin{aligned} 0 \leq F_n(x) &= \frac{1}{n+1} \frac{\sin^2 \frac{n+1}{2} x}{\sin^2 x/2} \\ &\leq \frac{1}{n+1} \frac{1}{\sin^2 \delta/2} \\ &= \frac{1}{n+1} \frac{1}{1-\cos\delta} \end{aligned}$$

provided $0 < \delta \leq x \leq \pi$.

vi) If $0 < \delta < \pi$, then

$$\begin{aligned} 0 \leq \int_{\delta}^{\pi} F_n(t)\,\mathrm{d}t &\leq \int_{\delta}^{\pi} \frac{1}{n+1} \frac{1}{1-\cos\delta}\,\mathrm{d}t \\ &= \frac{1}{n+1} \frac{\pi-\delta}{1-\cos\delta} \end{aligned}$$

$\to 0$ as $n \to \infty$. Since $F_n(x)$ is even,

$$\lim_{n\to\infty} \int_{-\pi}^{-\delta} F_n(t)\,\mathrm{d}t = \lim_{n\to\infty} \int_{\delta}^{\pi} F_n(t)\,\mathrm{d}t = 0$$

vii) This follows from parts iv) and vi).

$\square$

3.4 Féjer's Theorem

In this section we present Féjer's Theorem which is concerned with the Cesáro sum of the Fourier series of a given function.

Definition 3.4.1. Suppose that $\{K_n\}_{n=0}^{\infty}$ is a sequence of 2π-periodic, Riemann integrable functions with the following properties:

i)
$$0 \leq K_n(x) \in \mathbb{R}$$

ii) For every $n \geq 0$,
$$\int_{-\pi}^{\pi} K_n(t)\,\mathrm{d}t = 1$$

iii) For every $0 < \delta < \pi$,
$$\lim_{n\to\infty} \int_{-\delta}^{\delta} K_n(t)\,\mathrm{d}t = 1$$

or equivalently,
$$\lim_{n\to\infty} \int_{-\pi}^{-\delta} K_n(t)\,\mathrm{d}t = \lim_{n\to\infty} \int_{\delta}^{\pi} K_n(t)\,\mathrm{d}t = 0$$

Such a sequence is called an *approximate identity in the space of 2π-periodic, Riemann integrable functions.*

Theorem 3.4.2. Suppose that f is a 2π-periodic, Riemann integrable function and that $\{K_n\}_{n=0}^{\infty}$ is an approximate identity. Define $f_n : \mathbb{R} \to \mathbb{C}$ by

$$f_n(x) = \int_{-\pi}^{\pi} f(t) K_n(x-t)\,\mathrm{d}t$$

Then

i) If f is continuous at some $x_0 \in \mathbb{R}$, then $f_n(x_0)$ converges to $f(x_0)$ as $n \to \infty$.

ii) If f is 2π-periodic and continuous then f_n converges to f uniformly on $\mathbb{R}$ as $n \to \infty$.

The intuitive idea for the proof is the following,

$$\begin{aligned}
f_n(x_0) &= \int_{-\pi}^{\pi} f(t)K_n(x_0 - t)\,\mathrm{d}t \\
&= \int_{-\pi}^{\pi} f(x_0 - t)K_n(t)\,\mathrm{d}t \\
&= \int_{-\pi}^{-\delta} f(x_0 - t)K_n(t)\,\mathrm{d}t + \int_{\delta}^{\pi} f(x_0 - t)K_n(t)\,\mathrm{d}t \\
&\quad + \int_{-\delta}^{\delta} f(x_0 - t)K_n(t)\,\mathrm{d}t \\
&\approx 0 + 0 + \int_{-\delta}^{\delta} f(x_0)K_n(t)\,\mathrm{d}t \\
&\approx f(x_0)
\end{aligned}$$

if n is large and δ is small.

Proof. i) Letting $s = x - t$, we have

$$\begin{aligned}
f_n(x) &= -\int_{x+\pi}^{x-\pi} f(x - s)K_n(s)\,\mathrm{d}s \\
&= \int_{-\pi}^{\pi} f(x - s)K_n(s)\,\mathrm{d}s
\end{aligned}$$

Now, given $\epsilon > 0$, choose $\delta > 0$ such that $|f(x) - f(x_0)| < \epsilon/2$ whenever $|x - x_0| < \delta$. Letting $M = \sup\{|f(t)| : x \in \mathbb{R}t\}$, choose $N > 0$ such that

$$\int_{-\pi}^{-\delta} K_n(t)\,\mathrm{d}t < \frac{\epsilon}{8M}$$

and

$$\int_{\delta}^{\pi} K_n(t)\,\mathrm{d}t < \frac{\epsilon}{8M}$$

whenever $n > N$. Then, we have

$$
\begin{aligned}
|f_n(x_0) - f(x_0)| &= \left| \int_{-\pi}^{\pi} f(x_0 - t) K_n(t)\,\mathrm{d}t - f(x_0) \int_{-\pi}^{\pi} K_n(t)\,\mathrm{d}t \right| \\
&\le \int_{-\pi}^{\pi} |f(x_0 - t) - f(x_0)|\, K_n(t)\,\mathrm{d}t \\
&= \int_{-\pi}^{-\delta} |f(x_0 - t) - f(x_0)|\, K_n(t)\,\mathrm{d}t + \int_{\delta}^{\pi} |f(x_0 - t) - f(x_0)|\, K_n(t)\,\mathrm{d}t \\
&\quad + \int_{-\delta}^{\delta} |f(x_0 - t) - f(x_0)|\, K_n(t)\,\mathrm{d}t \\
&< \int_{-\pi}^{-\delta} 2M K_n(t)\,\mathrm{d}t + \int_{\delta}^{\pi} 2M K_n(t)\,\mathrm{d}t + + \int_{-\delta}^{\delta} \frac{\epsilon}{2} K_n(t)\,\mathrm{d}t \\
&< 2M \frac{\epsilon}{8M} + 2M \frac{\epsilon}{8M} + \frac{\epsilon}{2} \\
&= \epsilon
\end{aligned}
$$

if $n > N$. This proves pointwise convergence at x_0.

ii) If f is 2π-periodic and continuous on $\mathbb{R}$ then it is uniformly continuous and hence the choice of δ in part i) can be made independently of x_0. The result then follows.

□

Theorem 3.4.3 (Féjer's Theorem)**.** Suppose f is a 2π-periodic, Riemann integrable function and let s_k be the kth partial sum of the Fourier series of f. Define $\sigma_n : \mathbb{R} \to \mathbb{C}$ by

$$
\sigma_n(x) = \frac{1}{n+1} \sum_{k=0}^{n} s_k
$$

Then,

i) If f is continuous at some $x_0 \in \mathbb{R}$ then $\sigma_n(x_0)$ converges to $f(x_0)$ as $n \to \infty$.

ii) If f is continuous on $\mathbb{R}$, then σ_n converges uniformly to f on $\mathbb{R}$.

Proof. Notice that

$$\begin{aligned}\sigma_n(x) &= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x-t)F_n(t)\,\mathrm{d}t\\ &= \int_{-\pi}^{\pi} f(x-t)K_n(t)\,\mathrm{d}t\end{aligned}$$

where

$$K_n(x) = \frac{1}{2\pi}F_n(x)$$

is an approximate identity by Theorem 3.3.1. Therefore, both parts of the theorem follow by Theorem 3.4.2. □

Theorem 3.4.4. Let f, s_k and σ_n be as in Theorem 3.4.3. If both $f(x_0+)$ and $f(x_0-)$ exist for some $x_0 \in \mathbb{R}$, then $\sigma_n(x_0)$ converges to $(f(x_0+)+f(x_0-))/2$ as $n \to \infty$.

Proof. The proof of this theorem is similar to that of Theorem 3.4.2. Given $\epsilon > 0$, choose $\delta > 0$ such that $|f(x) - f(x_0+)| < \epsilon/2$ whenever $0 < x - x_0 < \delta$. Letting $M = \sup\{|f(t)| : t \in \mathbb{R}\}$, choose $N_1 > 0$ such that

$$\frac{1}{\pi}\int_{\delta}^{\pi} F_n(t)\,\mathrm{d}t < \frac{\epsilon}{4M}$$

whenever $n > N_1$. Then, noting that Féjer's kernel is even, we have

$$\begin{aligned}\left|\frac{1}{\pi}\int_0^{\pi} f(x_0+t)F_n(t)\,\mathrm{d}t - f(x_0+)\right| &= \left|\frac{1}{\pi}\left(\int_0^{\pi} f(x_0+t)F_n(t)\,\mathrm{d}t - f(x_0+)\int_0^{\pi} F_n(t)\,\mathrm{d}t\right)\right|\\ &\le \frac{1}{\pi}\int_0^{\pi} |f(x_0+t) - f(x_0+)|\,F_n(t)\,\mathrm{d}t\\ &= \frac{1}{\pi}\int_0^{\delta} |f(x_0+t) - f(x_0+)|\,F_n(t)\,\mathrm{d}t\\ &\quad + \frac{1}{\pi}\int_{\delta}^{\pi} |f(x_0+t) - f(x_0+)|\,F_n(t)\,\mathrm{d}t\\ &< \frac{1}{\pi}\int_0^{\delta} \frac{\epsilon}{2}F_n(t)\,\mathrm{d}t + \frac{1}{\pi}\int_{\delta}^{\pi} 2MF_n(t)\,\mathrm{d}t\\ &< \frac{\epsilon}{2} + \frac{2M\epsilon}{4M}\\ &= \epsilon\end{aligned}$$

if $n > N_1$. Similarly, we can find N_2 such that for $n > N_2$, we have

$$\left| \frac{1}{\pi} \int_{-\pi}^{0} f(x_0 + t) F_n(t) \, \mathrm{d}t - f(x_0-) \right| < \epsilon$$

Let $N = \max\{N_1, N_2\}$. Then for $n > N$,

$$\begin{aligned}
\left| \sigma_n(x_0) - \frac{f(x_0+) + f(x_0-)}{2} \right| &\leq \frac{1}{2} \left| \frac{1}{\pi} \int_{0}^{\pi} f(x_0 + t) F_n(t) \, \mathrm{d}t - f(x_0+) \right| \\
&\quad + \frac{1}{2} \left| \frac{1}{\pi} \int_{-\pi}^{0} f(x_0 + t) F_n(t) \, \mathrm{d}t - f(x_0-) \right| \\
&< \frac{1}{2} (\epsilon + \epsilon) \\
&= \epsilon
\end{aligned}$$

This proves the theorem. □

Corollary 3.4.5. Given a function f which is continuous and 2π-periodic and $\epsilon > 0$, there exist $n > 0$ and scalars $\gamma_k \in \mathbb{C}, -n \leq k \leq n$ such that

$$\left| f(x) - \sum_{k=-n}^{n} \gamma_k e^{ikx} \right| < \epsilon$$

Proof. By Theorem 3.4.3, we can choose $n > 0$ such that $|f(x) - \sigma_n(x)| < \epsilon$

(where σ_n is defined as in the proof of Theorem 3.4.3). Then

$$\begin{aligned}
\sigma_n(x) &= \frac{1}{n+1}\sum_{j=0}^{n} s_j \\
&= \frac{1}{n+1}\sum_{j=0}^{n} \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x-t)D_j(t)\,\mathrm{d}t \\
&= \frac{1}{n+1}\sum_{j=0}^{n} \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)D_j(x-t)\,\mathrm{d}t \\
&= \frac{1}{n+1}\sum_{j=0}^{n} \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)\sum_{k=-j}^{j} e^{ik(x-t)}\,\mathrm{d}t \\
&= \frac{1}{n+1}\sum_{j=0}^{n} \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)\sum_{k=-j}^{j} e^{-ikt}e^{ikx}\,\mathrm{d}t \\
&= \sum_{k=-n}^{n} \gamma_k e^{ikx}
\end{aligned}$$

for some scalars γ_k. □

Corollary 3.4.6. Suppose $T = \{z \in \mathbb{C} : |z| = 1\}$ and $g : T \to \mathbb{C}$ is continuous. Then given $\epsilon > 0$ there exist $n > 0$ and scalars $\gamma_k \in \mathbb{C}, -n \leq k \leq n$ such that

$$\left| g(z) - \sum_{k=-n}^{n} \gamma_k z^k \right| < \epsilon$$

Proof. Let $f(x) = g(e^{ix}), x \in \mathbb{R}$. Then f satisfies the hypotheses of Corollary 3.4.5. Let $n > 0$ and γ_k be as in that Corollary. Then

$$\left| g(z) - \sum_{k=-n}^{n} \gamma_k z^k \right| = \left| g(e^{ix}) - \sum_{k=-n}^{n} \gamma_k e^{ikx} \right| = \left| f(x) - \sum_{k=-n}^{n} \gamma_k e^{ikx} \right| < \epsilon$$

□

Corollary 3.4.7. Given $\phi : [0,1] \to \mathbb{C}$ and $\epsilon > 0$ there exist $n > 0$ and scalars $a_0, a_1, \ldots, a_n \in \mathbb{C}$ such that

$$\left|\phi(t) - \sum_{k=0}^{n} a_k t^k\right| < \epsilon$$

for $0 \leq t \leq 1$.

Proof. Choose a 2π-periodic, continuous function f which agrees with ϕ on $[0,1]$. By Corollary 3.4.5 there exist $m > 0$ and scalars $\gamma_k \in \mathbb{C}, -m \leq k \leq m$ such that

$$\left|f(t) - \sum_{k=-m}^{m} \gamma_k e^{ikt}\right| < \epsilon/2$$

Let

$$g(z) = \sum_{k=-m}^{m} \gamma_k e^{ikz}$$

for $z \in \mathbb{C}$. Then the Maclaurin series of g converges uniformly to g on bounded sets (since g is entire). Hence there exists $n > 0$ such that

$$\left|g(z) - \sum_{k=0}^{n} \frac{g^{(k)}(0)}{k!} z^k\right| < \epsilon/2$$

provided $z \in \mathbb{C}$ and $|z| \leq 1$. Let

$$a_k = \frac{g^{(k)}(0)}{k!}$$

Then for $0 \leq t \leq 1$, we have

$$\begin{aligned}
\left|\phi(t) - \sum_{k=0}^{n} a_k t^k\right| &= \left|f(t) - \sum_{k=0}^{n} a_k t^k\right| \\
&\leq |f(t) - g(t)| + \left|g(t) - \sum_{k=0}^{n} a_k t^k\right| \\
&\leq \epsilon/2 + \epsilon/2 \\
&= \epsilon
\end{aligned}$$

which proves the assertion. $\square$

Theorem 3.4.8. Suppose f is a continuous, complex-valued function which has period 1. Then given $\epsilon > 0$ there exist complex scalars $c_k, -N \le k \le N$ such that

$$\left| f(x) - \sum_{k=-N}^{N} c_k e^{2\pi kxi} \right| < \epsilon$$

Proof. Let $g(x) = f(x/2\pi), x \in \mathbb{R}$. Then g is a continuous, 2π-periodic function so by Corollary 3.4.5,

$$\left| g(t) - \sum_{k=-N}^{N} c_k e^{ikt} \right| < \epsilon$$

for some complex scalars $c_k, -N \le k \le N$ and all $t \in \mathbb{R}$. Therefore,

$$\left| f(x) - \sum_{k=-N}^{N} c_k e^{2\pi kxi} \right| = \left| g(2\pi x) - \sum_{k=-N}^{N} c_k e^{2\pi kxi} \right| < \epsilon$$

$\square$

Theorem 3.4.9 (Weierstrass Approximation Theorem)**.** Given $f : [a, b] \to \mathbb{C}$ and $\epsilon > 0$ there exists a polynomial function P such that $|f(x) - P(x)| < \epsilon$ for all $a \le x \le b$.

Proof. Let

$$\phi(t) = f(a + t(b - a))$$

for $0 \le t \le 1$. By Corollary 3.4.7, there is a polynomial function Q such that $|\phi(t) - Q(t)| < \epsilon$. But

$$f(x) = \phi\left(\frac{x-a}{b-a}\right)$$

so $|f(x) - P(x)| < \epsilon$, where

$$P(x) = Q\left(\frac{x-a}{b-a}\right)$$

$\square$

Example 3.4.10. Suppose f, g are continuous and 2π-periodic functions which have the same Fourier coefficients. Show that $f = g$.

Solution. Let $h = f - g$. Then the Fourier coefficients of h are all zero. Hence $s_k(x) = 0$ for $k \geq 0$ and $x \in \mathbb{R}$, where s_k is the kth partial sum of the Fourier series of h. Therefore $\sigma_n(x) = 0, n \geq 0, x \in \mathbb{R}$, where σ_n is defined as in Theorem 3.4.3. By Theorem 3.4.3, σ_n converges uniformly to h on $\mathbb{R}$. Thus $h = 0$ and so $f = g$.

Example 3.4.11. Suppose $f : [a, b] \to \mathbb{R}$ is continuous and

$$\int_a^b x^k f(x)\,\mathrm{d}x = 0$$

for all $0 \leq k \in \mathbb{Z}$. Show that $f(x) = 0, a \leq x \leq b$.

Solution. The hypothesis implies

$$\int_a^b g(x) f(x)\,\mathrm{d}x = 0$$

for all polynomial functions g. So by Theorem 3.4.9, we can find a sequence of polynomial functions $f_n, n \geq 0$ which converge uniformly to f on $[a, b]$. Then

$$\begin{aligned} 0 &= \lim_{n\to\infty} \int_a^b f_n(x) f(x)\,\mathrm{d}x \\ &= \int_a^b f^2(x)\,\mathrm{d}x \end{aligned}$$

This implies $f(x) = 0$ for all $a \leq x \leq b$.

Example 3.4.12. Suppose $f : [a, b] \to \mathbb{R}$ is continuous and

$$\int_a^b f(x) g'(x)\,\mathrm{d}x = 0$$

for every $g \in \mathcal{C}^1[a,b]$ such that $g(a) = g(b) = 0$. Show that f is constant.

Solution. First suppose f is a polynomial which satisfies the hypothesis. Let $g(x) = (x-a)(x-b)f'(x)$. Then g also satisfies the conditions stated and hence, using integration by parts, we obtain

$$\begin{aligned} 0 &= \int_a^b f(x)g'(x)\,\mathrm{d}x \\ &= f(x)g(x)\,\Big|_{x=a}^{b} - \int_a^b f'(x)g(x)\,\mathrm{d}x \\ &= -\int_a^b (x-a)(x-b)(f'(x))^2\,\mathrm{d}x \end{aligned}$$

Since $(x-a)(x-b)(f'(x))^2 \le 0$ for all $a \le x \le b$ we must have $f'(x) = 0, a \le x \le b$ which means f is constant.

For the general case, choose a sequence of polynomials $f_n, n \ge 1$ which satisfy $|f(x) - f_n(x)| < 1/n, a \le x \le b$. This can be done by Theorem 3.4.9. Then (for arbitrary g which satisfies the hypothesis),

$$\begin{aligned} \left| \int_a^b f(x)g'(x)\,\mathrm{d}x - \int_a^b f_n(x)g'(x)\,\mathrm{d}x \right| &\le \int_a^b |f(x) - f_n(x)|\, g'(x)\,\mathrm{d}x \\ &\le \frac{1}{n}\int_a^b g'(x)\,\mathrm{d}x \\ &= \frac{1}{n}(g(b) - g(a)) \\ &= 0 \end{aligned}$$

So

$$\int_a^b f_n(x)g'(x)\,\mathrm{d}x = 0$$

for all $n \ge 1$ and hence by the remarks above, each f_n is constant. But f_n converges uniformly to f on $[a,b]$ so this means f must be constant as well.

Example 3.4.13. Suppose $f \in \mathcal{C}^1[a,b]$ and $\epsilon > 0$. Show that there is a polynomial p such that $|f(x) - p(x)| < \epsilon$ and $|f'(x) - p'(x)| < \epsilon$.

Solution. By Theorem 3.4.9, we can find a polynomial p such that $|f'(x) - p'(x)| < \min(\epsilon, \epsilon/(b-a))$. Then

$$\begin{aligned}
|f(x) - p(x)| &= \left| \int_a^x f'(t)\,\mathrm{d}t - \int_a^x p'(t)\,\mathrm{d}t \right| \\
&\leq \int_a^x |f'(t) - p'(t)|\,\mathrm{d}t \\
&\leq \int_a^x \frac{\epsilon}{b-a}\,\mathrm{d}t \\
&\leq \epsilon
\end{aligned}$$

3.5 Applications of Féjer's Theorem

Here we revisit the heat problem and also present the H. Weyl Equidistribution Theorem, both of which depend on Féjer's Theorem which was proved in the last section.

Theorem 3.5.1 (H. Weyl)**.** Suppose f is a complex-valued, Riemann integrable function on $[0,1]$ and γ is an irrational real number. Let $[\gamma]$ be the largest integer less than or equal to γ and $<\gamma>= \gamma - [\gamma]$ (so that $<\gamma>$ is the fractional part of γ). Then

$$\int_0^1 f(x)\,\mathrm{d}x = \lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} f(<k\gamma>)$$

Proof. Case 1. If f is constant the result is trivially true.

Case 2. If $f(x) = e^{2\pi m x i}$ for some $0 \neq m \in \mathbb{Z}$ then, letting $t = 2\pi x$, we get

$$\int_0^1 f(x)\,\mathrm{d}x = \frac{1}{2\pi} \int_0^{2\pi} e^{mti}\,\mathrm{d}t = 0$$

Also, for $0 < n \in \mathbb{Z}$,

$$\begin{aligned}
\frac{1}{n}\sum_{k=1}^{n} f(< k\gamma >) &= \frac{1}{n}\sum_{k=1}^{n} e^{2\pi m < k\gamma > i} \\
&= \frac{1}{n}\sum_{k=1}^{n} e^{(2\pi m < k\gamma > i) + (2\pi m [k\gamma] i)} \\
&= \frac{1}{n}\sum_{k=1}^{n} e^{2\pi m k\gamma i} \\
&= \frac{1}{n}\sum_{k=1}^{n} (e^{2\pi m\gamma i})^k \\
&= \frac{e^{2\pi m\gamma i}}{n}\left(\frac{1 - e^{2\pi m n\gamma i}}{1 - e^{2\pi m\gamma i}}\right)
\end{aligned}$$

So

$$\left|\frac{1}{n}\sum_{k=1}^{n} f(< k\gamma >)\right| \leq \frac{2}{n(1 - e^{2\pi m\gamma i})}$$

$\to 0$ as $n \to \infty$. This proves the result.

Case 3. Suppose there exist scalars c_k, $-N \leq k \leq N$ such that

$$f(x) = \sum_{k=-N}^{N} c_k e^{2\pi k\gamma x i}$$

(ie. f is a trigonometric polynomial). Then the result follows from cases 1 and 2.

Case 4. Suppose f is continuous and $f(0) = f(1)$. By Corollary 3.4.8, given $\epsilon > 0$, there exist scalars c_k, $-N \leq k \leq N$ such that $|f(x) - g(x)| < \epsilon/2$, where

$$g(x) = \sum_{k=-N}^{N} c_k e^{2\pi k x i}$$

This implies

$$\begin{aligned}
&\left|\int_0^1 f(x)\,\mathrm{d}x - \int_0^1 g(x)\,\mathrm{d}x\right| \\
&\leq \int_0^1 |f(x) - g(x)|\,\mathrm{d}x \\
&\leq \epsilon/2
\end{aligned}$$

and for all $0 < n \in \mathbb{Z}$,

$$\left|\frac{1}{n}\sum_{k=1}^{n} f(< k\gamma >) - \frac{1}{n}\sum_{k=1}^{n} g(< k\gamma >)\right| \leq \frac{1}{n}\sum_{k=1}^{n} |f(< k\gamma >) - g(< k\gamma >)| \leq \epsilon/2$$

Therefore,

$$\begin{aligned}
\left|\frac{1}{n}\sum_{k=1}^{n} f(< k\gamma >) - \int_0^1 f(x)\,\mathrm{d}x\right| &\leq \left|\frac{1}{n}\sum_{k=1}^{n} f(< k\gamma >) - \frac{1}{n}\sum_{k=1}^{n} g(< k\gamma >)\right| \\
&\quad + \left|\frac{1}{n}\sum_{k=1}^{n} g(< k\gamma >) - \int_0^1 g(x)\,\mathrm{d}x\right| \\
&\quad + \left|\int_0^1 f(x)\,\mathrm{d}x - \int_0^1 g(x)\,\mathrm{d}x\right| \\
&\leq \epsilon/2 + \epsilon/2 \\
&\quad + \left|\frac{1}{n}\sum_{k=1}^{n} g(< k\gamma >) - \int_0^1 g(x)\,\mathrm{d}x\right| \\
&= \epsilon + \left|\frac{1}{n}\sum_{k=1}^{n} g(< k\gamma >) - \int_0^1 g(x)\,\mathrm{d}x\right|
\end{aligned}$$

But, by case 3,

$$\int_0^1 g(x)\,\mathrm{d}x = \lim_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n} g(< k\gamma >)$$

Therefore,

$$\limsup_{n\to\infty} \left|\frac{1}{n}\sum_{k=1}^{n} f(< k\gamma >) - \int_0^1 f(x)\,\mathrm{d}x\right| \leq 2\epsilon$$

from which the result follows.

Case 5. Suppose f is a real-valued, Riemann integrable function on $[0,1]$. Then given $\epsilon > 0$ we can find real-valued functions f_- and f_+ in $\mathcal{C}_1$ such that $f_-(x) \le f(x) \le f_+(x)$ for all $0 \le x \le 1$ and

$$\int_0^1 f_+(x)\,\mathrm{d}x - \int_0^1 f_-(x)\,\mathrm{d}x < \epsilon \ (*)$$

Then

$$\frac{1}{n}\sum_{k=1}^{n} f_-(< k\gamma >) \le \frac{1}{n}\sum_{k=1}^{n} f(< k\gamma >) \le \frac{1}{n}\sum_{k=1}^{n} f_+(< k\gamma >) \ (\dagger)$$

But by case 4,

$$\int_0^1 f_-(x)\,\mathrm{d}x = \lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n} f_-(< k\gamma >)$$

and

$$\int_0^1 f_+(x)\,\mathrm{d}x = \lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n} f_+(< k\gamma >)$$

Substituting into $(\dagger)$ we obtain

$$\begin{aligned}\int_0^1 f_-(x)\,\mathrm{d}x &\le \liminf_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n} f(< k\gamma >)\\ &\le \limsup_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n} f(< k\gamma >)\\ &\le \int_0^1 f_+(x)\,\mathrm{d}x\end{aligned}$$

Now using $(*)$, we see that

$$\begin{aligned}&\limsup_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n} f(< k\gamma >) - \liminf_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n} f(< k\gamma >)\\ &\le \int_0^1 f_+(x)\,\mathrm{d}x - \int_0^1 f_-(x)\,\mathrm{d}x\\ &< \epsilon\end{aligned}$$

Therefore,

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} f(< k\gamma >)$$

exists and satisfies

$$\int_0^1 f_-(x)\,\mathrm{d}x \le \lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} f(< k\gamma >) \le \int_0^1 f_+(x)\,\mathrm{d}x$$

But since $f_-(x) \le f(x) \le f_+(x)$, we also have

$$\int_0^1 f_-(x)\,\mathrm{d}x \le \int_0^1 f(x)\,\mathrm{d}x \le \int_0^1 f_+(x)\,\mathrm{d}x$$

Therefore, by $(*)$

$$\left| \int_0^1 f(x)\,\mathrm{d}x - \lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} f(< k\gamma >) \right| < \epsilon$$

which proves the result.

Case 6. For the general case, when f is complex-valued, simply apply case 5 to the real and imaginary parts of f.

□

Theorem 3.5.2 (H. Weyl's Equidistribution Theorem)**.** If γ is an irrational real number and $0 \le a \le b \le 1$ then

$$\lim_{n\to\infty} \frac{1}{n} \mathrm{card}(\{< k\gamma > \in [a, b] : 1 \le k \le n\}) = b - a$$

Proof. Define $f : [0, 1] \to \mathbb{R}$ by

$$f(x) = \begin{cases} 1 & a \le x \le b \\ 0 & \text{otherwise} \end{cases}$$

Then f satisfies the hypotheses of Theorem 3.5.1 and so

$$\begin{aligned} b - a &= \int_0^1 f(x)\,\mathrm{d}x \\ &= \lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} f(< k\gamma >) \\ &= \lim_{n\to\infty} \frac{1}{n} \mathrm{card}(\{< k\gamma > \in [a, b] : 1 \le k \le n\}) \end{aligned}$$

$\square$

Theorem 3.5.3. Given a continuous function f on $[0, \pi]$ which satisfies $f(0) = f(\pi) = 0$, there exists a unique solution to the heat problem.

Proof. Extend f to an odd, continuous 2π-periodic function on $\mathbb{R}$ and, for $x \in \mathbb{R}, 0 \leq t \in \mathbb{R}$, make the following definitions.

i)

$$b_j = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin jx \, \mathrm{d}x$$

ii)

$$s_k(x) = \sum_{j=1}^{k} b_j \sin jx$$

iii)

$$u_k(x,t) = \sum_{j=1}^{k} b_j e^{-j^2 t} \sin jx$$

iv)

$$\sigma_n(x) = \frac{1}{n}(s_1(x) + s_2(x) + \ldots + s_n(x))$$

v)

$$v_n(x,t) = \frac{1}{n}(u_1(x,t) + u_2(x,t) + \ldots + u_n(x,t))$$

By Theorem 3.4.3,

$$\lim_{n \to \infty} \sigma_n(x) = \lim_{n \to \infty} s_n(x) = f(x)$$

and, moreover, this convergence is uniform on $\mathbb{R}$. Therefore, given $\epsilon > 0$, there exists $N > 0$ such that

$$|\sigma_n(x) - \sigma_m(x)| < \epsilon$$

whenever $m, n > N$. But $v_n(x,t)$ is the solution to the heat problem corresponding to $\sigma_n(x)$. Hence, by Theorem 2.2.2,

$$|v_n(x,t) - v_m(x,t)| < \epsilon$$

whenever $m, n > N$. So $\{v_n(x,t)\}_{n=1}^{\infty}$ satisfies the uniform Cauchy criterion and thus converges uniformly to some $u(x,t) : \mathbb{R} \times [0,\infty) \to \mathbb{R}$. We claim that $u(x,t)$ is the solution to the heat problem corresponding to f. We will divide our argument into steps.

Step 1. First, we have,

$$u(x,0) = \lim_{n\to\infty} v_n(x,0) = \lim_{n\to\infty} \sigma_n(x) = f(x)$$

by Theorem 3.4.3.

Step 2. Again, because each v_n is the solution to the heat problem corresponding the σ_n,

$$u(0,t) = \lim_{n\to\infty} v_n(0,t) = 0$$

and

$$u(\pi,t) = \lim_{n\to\infty} v_n(\pi,t) = 0$$

Step 3. To see that u is continous on its domain, notice that it is the uniform limit of the v_n's, each of which are continuous.

Step 4. By Theorem 2.1.2, u is $\mathcal{C}^\infty$ on its domain.

Step 5. Finally, because of uniform convergence, we have

$$\begin{aligned}
\frac{\partial u}{\partial t}(x,t) &= \frac{\partial}{\partial t} \lim_{n\to\infty} v_n(x,t) \\
&= \lim_{n\to\infty} \frac{\partial v_n}{\partial t}(x,t) \\
&= \lim_{n\to\infty} \frac{\partial^2 v_n}{\partial x^2}(x,t) \\
&= \frac{\partial^2}{\partial x^2} \lim_{n\to\infty} v_n(x,t) \\
&= \frac{\partial^2 u}{\partial x^2}(x,t)
\end{aligned}$$

□

3.6 Féjer's Example

In this section, we present an example of a continuous, 2π-periodic function whose Fourier series diveges at zero. But we first need to develop some preliminary lemmas.

Lemma 3.6.1. Suppose $x \in \mathbb{R}, x \neq 2\pi k, k \in \mathbb{Z}$ so that $e^{ix} \neq 1$ and $\sin x/2 \neq 0$. Then for all $n \in \mathbb{N}$,

i)
$$\sum_{k=1}^{n} e^{ikx} = \left(\frac{\sin nx/2}{\sin x/2}\right) e^{i(n+1)x/2}$$

ii)
$$\left|\sum_{k=1}^{n} e^{ikx}\right| \leq \frac{1}{\sin x/2}$$

iii)
$$\sum_{k=1}^{n} \sin kx = \frac{(\sin nx/2)(\sin(n+1)x/2)}{\sin x/2}$$

iv)
$$\left|\sum_{k=1}^{n} \sin kx\right| \leq \frac{1}{|\sin x/2|}$$

Proof. i) If x satisfies the hypothesis of the lemma and $n \in \mathbb{N}$, then

$$\begin{aligned}
\sum_{k=1}^{n} e^{ikx} &= e^{ix}\left(\frac{1-e^{inx}}{1-e^{ix}}\right) \\
&= \frac{e^{ix}e^{inx/2}}{e^{ix/2}}\left(\frac{e^{-inx/2}-e^{inx/2}}{e^{-ix/2}-e^{ix/2}}\right) \\
&= e^{i(n+1)x/2}\left(\frac{-2\sin nx/2}{-2\sin x/2}\right) \\
&= e^{i(n+1)x/2}\frac{\sin nx/2}{\sin x/2}
\end{aligned}$$

ii) This follows from part i).

iii) This follows from part i) by taking imaginary parts of both sides of the identity.

iv) This follows from part iii).

□

Example 3.6.2. i) Suppose $\{b_k\}_{k=1}^{\infty}$ is a real sequence decreasing to zero. Show that

$$\sum_{k=1}^{\infty} b_k \sin kx$$

converges for all $x \in \mathbb{R}$.

ii) Show that the given series converges uniformly on $0 < a \leq x \leq b < \pi$.

Solution. i) Clearly, the given series converges if $x = 2\pi k$ for some $k \in \mathbb{Z}$. So assume x is not a multiple of 2π. Then by part iv) of Lemma 3.6.1,

$$\left| \sum_{k=1}^{n} \sin kx \right| < \frac{1}{|\sin x/2|}$$

Now given $\epsilon > 0$, choose $p \in \mathbb{N}$ such that $b_p < (\epsilon\,|\sin x/2|)/2$. Then by Theorem 3.1.2, for $q > p$,

$$\begin{aligned} \left| \sum_{k=p}^{q} b_k \sin kx \right| &< 2\left(\frac{1}{|\sin x/2|} \right) b_p \\ &< \epsilon \end{aligned}$$

So the given series is Cauchy and hence converges.

ii) By part iv) of Lemma 3.6.1,

$$\begin{aligned} \left| \sum_{k=1}^{n} \sin kx \right| &< \frac{1}{|\sin x/2|} \\ &\leq \frac{1}{\sin a/2} \end{aligned}$$

So by Theorem 3.1.4, the convergence of the given series is uniform.

Example 3.6.3. Show that the series

$$\sum_{k=1}^{\infty} \frac{(-1)^k}{\sqrt{k}} \sin(1 + x/k)$$

converges uniformly on $\mathbb{R}$.

Solution.

Lemma 3.6.4. Suppose $0 < \delta < \pi$. Then

$$\sum_{k=1}^{\infty} \frac{1}{k} \sin kx$$

converges uniformly for $\delta \leq x \leq 2\pi - \delta$.

Proof. By part iv) of Lemma 3.6.1,

$$\left| \sum_{k=1}^{n} \sin kx \right| \leq \frac{1}{\sin x/2} \leq \frac{1}{\sin \delta/2}$$

if $\delta \leq x \leq 2\pi - \delta$. Therefore the result follows by Theorem 3.1.4. $\square$

Lemma 3.6.5. If $x \in \mathbb{R}, n \in \mathbb{N}$, then

$$\left| \sum_{k=1}^{n} \frac{\sin kx}{k} \right| \leq 9\sqrt{\pi}$$

Proof. We may assume, without loss of generality, that $0 < x < 2\pi$. Choose $p \in \mathbb{N}$ such that $p \leq \sqrt{\pi}/x < p+1$. Suppose $p < n$. Then

$$\begin{aligned}
\left|\sum_{k=1}^{n} \frac{\sin kx}{k}\right| &\leq \sum_{k=1}^{p}\left|\frac{\sin kx}{k}\right| + \left|\sum_{k=p+1}^{n} \frac{\sin kx}{k}\right| \\
&= \sum_{k=1}^{p}\left|\frac{\sin kx}{kx}\right| x + \left|\sum_{k=p+1}^{n} \frac{\sin kx}{k}\right| \\
&\leq px + \left|\sum_{k=p+1}^{n} \frac{\sin kx}{k}\right|
\end{aligned}$$

It follows from part iii) of Lemma 3.6.1 that

$$\left|\sum_{k=p+1}^{n} \frac{\sin kx}{k}\right| \leq \frac{2}{\sin x/2}$$

and hence, by a slight modification of Theorem 3.1.2 that

$$\left|\sum_{k=p+1}^{n} \frac{\sin kx}{k}\right| \leq 2\left(\frac{2}{\sin x/2}\right)\frac{1}{p+1}$$

Therefore,

$$\begin{aligned}
\left|\sum_{k=1}^{n} \frac{\sin kx}{k}\right| &\leq px + \frac{4}{(p+1)\sin x/2} \\
&\leq \sqrt{\pi} + \frac{4}{p+1}\frac{x/2}{\sin x/2}\frac{2}{x} \\
&\leq \sqrt{\pi} + \frac{8}{(p+1)x}\left(\frac{x/2}{\sin x/2}\right) \\
&\leq \sqrt{\pi} + \frac{8}{\sqrt{\pi}}\pi \\
&= 9\sqrt{\pi}
\end{aligned}$$

$\square$

Theorem 3.6.6 (Féjer's Example)**.** Given $\mu, n \in \mathbb{N}$ and $x \in \mathbb{R}$, define

$$\begin{aligned} Q(x,\mu,n) = &\frac{\cos \mu x}{n} + \frac{\cos(\mu+1)x}{n-1} + \ldots + \frac{\cos(\mu+n-1)x}{1} \\ &- \frac{\cos(\mu+n+1)x}{1} - \frac{\cos(\mu+n+2)x}{2} - \ldots - \frac{\cos(\mu+2n)x}{n} \end{aligned}$$

Now suppose $\{\mu_k\}_{k=1}^{\infty}$ and $\{n_k\}_{k=1}^{\infty}$ are two increasing sequences in $\mathbb{N}$ satisfying $\mu_k + 2n_k < \mu_{k+1}$ and also that $\{a_k\}_{k=1}^{\infty}$ is a sequence of positive reals such that

$$\sum_{k=1}^{\infty} a_k$$

converges. Then the series

$$\sum_{k=1}^{\infty} a_k Q(x, \mu_k, n_k)$$

converges uniformly and absolutely to a continuous, 2π-periodic function f. Moreover, the μ_k, n_k and a_k can be chosen so that the Fourier series of f diverges at 0.

Proof. We will divide the proof into multiple steps.

Step 1. Using the identity $\cos(a-b) - \cos(a+b) = 2\sin a \sin b$, we obtain

$$\begin{aligned} Q(x,\mu_k,n_k) = &\frac{\cos \mu_k x}{n_k} + \frac{\cos(\mu_k+1)x}{n_k-1} + \ldots + \frac{\cos(\mu_k+n_k-1)x}{1} \\ &- \frac{\cos(\mu_k+n_k+1)x}{1} - \frac{\cos(\mu_k+n_k+2)x}{2} - \ldots - \frac{\cos(\mu_k+2n_k)x}{n_k} \\ = &\sum_{k=1}^{n_k} \frac{1}{k} \left(\cos(\mu_k+n_k-k)x - \cos(\mu_k+n_k+k)x\right) \\ = &\, 2\sin(\mu_k+n_k) \sum_{k=1}^{n_k} \frac{\sin kx}{k} \end{aligned}$$

But by Lemma 3.6.5

$$\left| \sum_{k=1}^{n_k} \frac{\sin kx}{k} \right| \leq 9\sqrt{\pi}$$

and so, $|Q(x, \mu_k, n_k)| < 18\sqrt{\pi}$. Now, given $\epsilon > 0$ choose $N > 0$ such that

$$\sum_{k=p}^{q} a_k < \frac{\epsilon}{18\sqrt{\pi}}$$

for $q > p > N$. Then for this choice of p and q and all $x \in \mathbb{R}$, we have

$$\begin{aligned} \left| \sum_{k=p}^{q} a_k Q(x, \mu_k, n_k) \right| &< 18\sqrt{\pi} \sum_{k=p}^{q} a_k \\ &< 18\sqrt{\pi} \frac{\epsilon}{18\sqrt{\pi}} \\ &= \epsilon \end{aligned}$$

Therefore, the series

$$\sum_{k=1}^{\infty} a_k Q(x, \mu_k, n_k)$$

satisfies the uniform Cauchy criterion on $\mathbb{R}$, and hence converges uniformly and absolutely to a continuous, 2π-periodic function f.

Step 2. Notice that because $\mu_k + 2n_k < \mu_{k+1}$, the terms of $Q(x, \mu_k, n_k)$ and $Q(x, \mu_j, n_j)$ do not "overlap" when $j \neq k$. Also notice that

$$f(x) = \sum_{k=1}^{\infty} a_k Q(x, \mu_k, n_k)$$

is obtained by "bracketing" a trigonometric series of the form

$$\frac{\alpha_0}{2} + \sum_{\nu=1}^{\infty} \alpha_\nu \cos \nu x \ (*)$$

Let $s_n, n \in \mathbb{N}$ be the nth partial sum of this series. Then $s_{\mu_k - 1}(x)$ converges uniformly on $\mathbb{R}$ to f. We will show that the series $(*)$ is indeed the Fourier series of f. To see this, first suppose $\mu_j \leq \nu \leq$

$\mu_j + 2n_j$. Then

$$\begin{aligned}\frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos\nu x\,\mathrm{d}x &= \frac{1}{\pi}\int_{-\pi}^{\pi}\sum_{k=1}^{\infty} a_k Q(x,\mu_k,n_k)\cos\nu x\,\mathrm{d}x \\ &= \frac{1}{\pi}\sum_{k=1}^{\infty} a_k \int_{-\pi}^{\pi} Q(x,\mu_k,n_k)\cos\nu x\,\mathrm{d}x \\ &= \frac{a_j}{\pi}\int_{-\pi}^{\pi} Q(x,\mu_j,n_j)\cos\nu x\,\mathrm{d}x \\ &= \alpha_\nu\end{aligned}$$

Next, suppose

$$\nu \notin \bigcup_{j=1}^{\infty}\{\mu_j,\ldots,\mu_j+n_j-1,\mu_j+n_j+1,\ldots,\mu_j+2n_j\}$$

Then

$$\begin{aligned}\frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos\nu x\,\mathrm{d}x &= \frac{1}{\pi}\int_{-\pi}^{\pi}\sum_{k=1}^{\infty} a_k Q(x,\mu_k,n_k)\cos\nu x\,\mathrm{d}x \\ &= 0\end{aligned}$$

which proves the assertion.

Step 3. Finally, we will show that we can choose the μ_k, n_k and a_k so that the Fourier series of f diverges at zero. First notice that

$$\begin{aligned}|s_{\mu_k+n_k-1}(0) - s_{\mu_k-1}(0)| &= \sum_{\nu=\mu_k}^{\mu_k+n_k-1} \alpha_\nu \cos\nu 0 \\ &= a_k\left(\frac{1}{n_k}+\frac{1}{n_k-1}+\ldots+1\right) \\ &> a_k \log n_k\end{aligned}$$

Hence $\{s_n(0)\}_{n=1}^{\infty}$ will not be Cauchy, and hence divergent, if we choose $\{a_k\}_{k=1}^{\infty}$ and $\{n_k\}_{k=1}^{\infty}$ so that $\{a_k \log n_k\}_{k=1}^{\infty}$ does not converge to zero. It is easy to see that if we let $a_k = 1/k^2$ and $\mu_k = n_k = 2^{k^3}$ then this requirement is met as well as the condition $\mu_k + 2n_k < \mu_{k+1}$.

□

We conclude this section with the statement of two theorems closely related to the previous one. However, we will not prove these theorems here.

Theorem 3.6.7 (Carlson)**.** If f is a continuous, 2π-periodic function then its Fourier series converges pointwise to f almost everywhere.

Theorem 3.6.8 (Katznelson and Kahame)**.** If $E \subset \mathbb{R}$ is a null set then there is a function f such that for all $t \in E$,

$$\limsup_{n\to\infty} |s_n(t)| = \infty$$

(where s_n is the nth partial sum of the Fourier series of f).

Chapter 4

The Fourier Transform

4.1 Inner Product Spaces

In this section we define inner product spaces and normed spaces as well as present some relevant results and examples.

Definition 4.1.1. Let V be a real or complex vector space. An *inner product* $(\cdot \mid \cdot)$ on V is a function from $V \times V$ into $\mathbb{R}$ or $\mathbb{C}$ which satisfies,

i) For all $x \in V$, $(x \mid x) \geq 0$, with equality when $x = 0$.

ii) For all $x, y \in V$, $(x \mid y) = \overline{(y \mid x)}$.

iii) For all $x, y, z \in V$ and $\lambda \in \mathbb{R}$ or $\lambda \in \mathbb{C}$, $(\lambda x + y \mid z) = \lambda\,(x + y \mid z)$.

A vector space V equipped with an inner product $(\cdot \mid \cdot)$ is called an *inner product space*.

Definition 4.1.2. Let V be a real or complex vector space. A *norm* on V is a function from V into $\mathbb{R}$ or $\mathbb{C}$ which satisfies

i) For all $x \in V$, $\|x\| \geq 0$ with equality iff $x = 0$.

ii) For all $x \in V$ and $\lambda \in \mathbb{R}$ or $\lambda \in \mathbb{C}$, $\|\lambda x\| = |\lambda|\,\|x\|$.

iii) For all $x, y \in V$, $\|x+y\| \leq \|x\| + \|y\|$. This inequality is called the *Triangle Inequality.*

A vector space V equipped with a norm is called a *normed space.*

Definition 4.1.3. If V is an inner product space with inner product $(\cdot \mid \cdot)$ and $x \in V$, we define the *norm* of x as $\|x\| = \sqrt{(x \mid x)}$. This norm is well defined since $(x \mid x) \geq 0$ and is called the *norm induced from the inner product* $(\cdot \mid \cdot)$.

We will prove shortly that a norm induced from an inner product does indeed satisfy the properties of a norm stated in Definition 4.1.2. However, we first need the following result.

Proposition 4.1.4 (Cauchy Schwarz Inequality)**.** If V is an inner product space with inner product $(\cdot \mid \cdot)$, $\|\cdot\|$ the norm induced from this inner product and $x, y \in V$, then $|(x \mid y)| \leq \|x\| \, \|y\|$.

Proof. First assume $\|y\| = 1$. Then

$$\begin{aligned} 0 &\leq \|x - (x \mid y)\, y\| \\ &= (x - (x \mid y)\, y \mid x - (x \mid y)\, y) \\ &= \|x\|^2 - |(x \mid y)|^2 - |(x \mid y)|^2 + |(x \mid y)|^2 \\ &= \|x\|^2 - |(x \mid y)|^2 \end{aligned}$$

which implies $|(x \mid y)| \leq \|x\| = \|x\| \, \|y\|$. Now, if $y = 0$ then the result is clearly true so assume $y \neq 0$. Then replacing y with $y/\|y\|$ in the above argument, we see that $|(x \mid y/\|y\|)| \leq \|x\|$ which implies

$$|(x \mid y)| = \|y\| \, |(x \mid y/\|y\|)| \leq \|x\| \, \|y\|$$

$\square$

Proposition 4.1.5. Let V be an inner product space with inner product $(\cdot \mid \cdot)$ and $\|\cdot\|$ the norm induced from this inner product. Then this norm satisfies all the properties of a norm stated in Defintion 4.1.2.

Proof. Properties i) and ii) are trivial. So we will only prove the Triangle Inequality. For all $x, y \in V$,

$$\begin{aligned}
\|x+y\|^2 &= (x+y \mid x+y) \\
&= \|x\|^2 + (x \mid y) + (y \mid x) + \|y\|^2 \\
&= \|x\|^2 + (x \mid y) + \overline{(x \mid y)} + \|y\|^2 \\
&= \|x\|^2 + 2\mathrm{Re}\,(x \mid y) + \|y\|^2 \\
&\leq \|x\|^2 + 2\,|(x \mid y)| + \|y\|^2 \\
&\leq \|x\|^2 + 2\,\|x\|\,\|y\| + \|y\|^2 \quad \text{by Proposition 4.1.4} \\
&= (\|x\| + \|y\|)^2
\end{aligned}$$

from which the result follows. □

Proposition 4.1.6. Let V be a normed space.

i) For all $x, y \in V$, $|\|x\| - \|y\|| \leq \|x - y\|$.

ii) The norm function $\|\cdot\|$ is uniformly continuous from V to $[0, \infty)$.

iii) If V is a real inner product space with inner product $(\cdot \mid \cdot)$ and $\|\cdot\|$ is the norm induced from this innner product then

$$\|x+y\|^2 + \|x-y\|^2 = 2\,\|x\|^2 + 2\,\|y\|^2$$

and

$$(x \mid y) = \frac{1}{4}\left(\|x+y\|^2 - \|x-y\|^2\right)$$

iv) Suppose $\{x_k\}_{k=1}^{\infty}$ and $\{y_k\}_{k=1}^{\infty}$ are convergent sequences in V with limits a and b, respectively. Also suppose $\{\lambda_k\}_{k=1}^{\infty}$ and $\{\mu_k\}_{k=1}^{\infty}$ are real or complex sequences converging to α and β, respectively. Let $z_k = \lambda_k x_k + \mu_k y_k, k \geq 1$. Then

$$\lim_{k\to\infty} z_k = \alpha a + \beta b$$

Proof. i) By the Triangle Inequality,

$$\|x\| = \|x - y + y\| \leq \|x - y\| + \|y\|$$

which implies

$$\|x\| - \|y\| \leq \|x - y\|$$

A similar argument shows that

$$\|y\| - \|x\| \leq \|y - x\| = \|x - y\|$$

Therefore,

$$-\|x - y\| \leq \|x\| - \|y\| \leq \|x - y\|$$

from which the result follows.

ii) Given $\epsilon > 0$ let $\delta = \epsilon$. By part i) if $\|x - y\| < \delta$ then $|\|x\| - \|y\|| \leq \|x - y\| < \delta = \epsilon$.

iii) For all $x, y \in V$ we have

$$\begin{aligned} \|x + y\|^2 + \|x - y\|^2 &= (x + y \mid x + y) + (x - y \mid x - y) \\ &= \|x\|^2 + (x \mid y) + (y \mid x) + \|y\|^2 \\ &\quad + \|x\|^2 + (x \mid -y) + (-y \mid x) + \|y\|^2 \\ &= 2\|x\|^2 + 2\|y\|^2 \end{aligned}$$

and also

$$\begin{aligned} \|x + y\|^2 - \|x - y\|^2 &= \|x\|^2 + (x \mid y) + (y \mid x) + \|y\|^2 \\ &\quad - \left(\|x\|^2 + (x \mid -y) + (-y \mid x) + \|y\|^2\right) \\ &= 2(x \mid y) + 2(y \mid x) \\ &= 4(x \mid y) \end{aligned}$$

from which the second result follows.

iv) Given $\epsilon > 0$, choose $N_1 > 0$ such that if $k > N_1$,

$$|\lambda_k - \alpha| < \min\left(\frac{\epsilon}{4(\|a\| + 1)}, 1\right)$$

Choose $N_2 > 0$ such that if $k > N_2$,

$$\|x_k - a\| < \frac{\epsilon}{4(|\alpha| + 1)}$$

Choose $N_3 > 0$ such that if $k > N_3$,

$$|\mu_k - \beta| < \min\left(\frac{\epsilon}{4(\|b\|+1)}, 1\right)$$

Finally, choose $N_4 > 0$ such that if $k > N_4$,

$$\|y_k - b\| < \frac{\epsilon}{4(|\beta|+1)}$$

Let $N = \max(N_1, N_2, N_3, N_4)$. Then if $k > N$ we have

$$\begin{aligned}
\|z_k - (\alpha a + \beta b)\| &\leq \|\lambda_k x_k - \alpha a\| + \|\mu_k y_k - \beta b\| \\
&= \|\lambda_k x_k - \lambda_k a + \lambda_k a - \alpha a\| + \|\mu_k y_k - \mu_k b + \mu_k b - \beta b\| \\
&\leq \|\lambda_k x_k - \lambda_k a\| + \|\lambda_k a - \alpha a\| + \|\mu_k y_k - \mu_k b\| + \|\mu_k b - \beta b\| \\
&= |\lambda_k| \, \|x_k - a\| + |\lambda_k - \alpha| \, \|a\| + |\mu_k| \, \|y_k - b\| + |\mu_k - \beta| \, \|b\| \\
&< (|\alpha|+1)\left(\frac{\epsilon}{4(|\alpha|+1)}\right) + \left(\frac{\epsilon}{4(\|a\|+1)}\right)\|a\| \\
&\quad + (|\beta|+1)\left(\frac{\epsilon}{4(|\beta|+1)}\right) + \left(\frac{\epsilon}{4(\|b\|+1)}\right)\|b\| \\
&< \frac{\epsilon}{4} + \frac{\epsilon}{4} + \frac{\epsilon}{4} + \frac{\epsilon}{4} \\
&= \epsilon
\end{aligned}$$

From which it follows that

$$\lim_{k\to\infty} z_k = \alpha a + \beta b$$

□

Example 4.1.7. Suppose $f, g \in \mathcal{C}([a,b])$. Define

$$\langle f \mid g \rangle = \int_a^b f(t)\overline{g(t)}\,\mathrm{d}t$$

Show that this function is an inner product on $\mathcal{C}([a,b])$. Also, let $\|f\|_2 = \sqrt{\langle f \mid f \rangle}$ be the norm induced from this inner product and define the *maximum norm* on $\mathcal{C}([a,b])$ as

$$\|f\|_\infty = \max\{|f(t)| : a \leq t \leq b\}$$

Show that the maximum norm is indeed a norm on $\mathcal{C}([a,b])$ and that there exists $M > 0$ such that $\|f\|_2 \leq M\|f\|_\infty$ for all $f \in \mathcal{C}([a,b])$ but there is no $\mu > 0$ such that $\|f\|_\infty \leq \mu\|f\|_2$ for all $f \in \mathcal{C}([a,b])$. Find a sequence $\{f_k\}_{k=1}^\infty$ in $\mathcal{C}([a,b])$ such that

$$\lim_{k\to\infty} \|f_k\|_2 = 0$$

and

$$\lim_{k\to\infty} \|f_k\|_\infty = \infty$$

Finally, show that the maximum norm is not induced from any inner product on $\mathcal{C}([a,b])$.

Solution. We will divide the solution into parts.

i) If $f \in \mathcal{C}([a,b])$ then $|f|^2$ is zero almost everywhere iff f is identically zero. So

$$\langle f \mid f\rangle = \int_a^b |f(t)|^2 \,\mathrm{d}t$$

is non-negative and zero iff f is zero on $[a,b]$. Next, for $f, g \in \mathcal{C}([a,b])$,

$$\begin{aligned}
\overline{\langle f \mid g\rangle} &= \overline{\int_a^b f(t)\overline{g(t)}\,\mathrm{d}t} \\
&= \int_a^b \overline{f(t)\overline{g(t)}}\,\mathrm{d}t \\
&= \int_a^b g(t)\overline{f(t)}\,\mathrm{d}t \\
&= \langle g \mid f\rangle
\end{aligned}$$

Finally, if $f, g, h \in \mathcal{C}([a,b])$ and $\lambda \in \mathbb{R}$ then

$$\begin{aligned}
\langle \lambda f + g \mid h\rangle &= \int_a^b (\lambda f(t) + g(t))\overline{h(t)}\,\mathrm{d}t \\
&= \lambda \int_a^b f(t)\overline{h(t)}\,\mathrm{d}t + \int_a^b g(t)\overline{h(t)}\,\mathrm{d}t \\
&= \lambda\langle f \mid h\rangle + \langle g \mid h\rangle
\end{aligned}$$

ii) If $f \in \mathcal{C}([a,b])$ then $\|f\|_\infty \geq 0$ with equality iff f is identically zero. Also, if $\lambda \in \mathbb{R}$ then

$$\begin{aligned}\|\lambda f\|_\infty &= \max\{|\lambda f(t)| : a \leq t \leq b\} \\ &= \max\{|\lambda|\,|f(t)| : a \leq t \leq b\} \\ &= |\lambda| \max\{|f(t)| : a \leq t \leq b\} \\ &= |\lambda|\,\|f\|_\infty\end{aligned}$$

Finally, for the Triangle Inequality, if $f, g \in \mathcal{C}([a,b])$ then $|f(t) + g(t)| \leq |f(t)| + |g(t)|$ for all $t \in [a,b]$. Hence,

$$\begin{aligned}\|f+g\|_\infty &= \max\{|f(t)+g(t)| : a \leq t \leq b\} \\ &\leq \max\{|f(t)| + |g(t)| : a \leq t \leq b\} \\ &\leq \max\{|f(t)| : a \leq t \leq b\} + \max\{|g(t)| : a \leq t \leq b\} \\ &= \|f\|_\infty + \|g\|_\infty\end{aligned}$$

iii) For all $f \in \mathcal{C}([a,b])$ and $t \in [a,b]$ we have $|f(t)| \leq \|f\|_\infty$. Therefore

$$\begin{aligned}\|f\|_2^2 &= \int_a^b |f(t)|^2 \,\mathrm{d}t \\ &\leq \int_a^b \|f\|_\infty^2 \,\mathrm{d}t \\ &= (b-a)\,\|f\|_\infty^2\end{aligned}$$

which implies $\|f\|_2 \leq M\,\|f\|_\infty$ where $M = \sqrt{b-a}$. In the next part of this solution, we will present a sequence $\{f_k\}_{k=1}^\infty$ in $\mathcal{C}([a,b])$ such that

$$\lim_{k\to\infty} \|f_k\|_2 = 0$$

and

$$\lim_{k\to\infty} \|f_k\|_\infty = \infty$$

from which it will follow that there is no $\mu > 0$ such that $\|f\|_\infty \leq \mu\,\|f\|_2$ for all $f \in \mathcal{C}([a,b])$.

iv) Let

$$f_k(t) = k\left(\frac{t-a}{b-a}\right)^{k^3}$$

for $t \in [a,b]$. Then $\|f_k\|_\infty = k$ so $\lim_{k\to\infty} \|f_k\|_\infty = \infty$ and

$$\begin{aligned}
\|f_k\|_2^2 &= \int_a^b |f_k(t)|^2 \, \mathrm{d}t \\
&= \frac{k^2}{(b-a)^{2k^3}} \int_a^b (t-a)^{2k^3} \, \mathrm{d}t \\
&= \frac{k^2}{(b-a)^{2k^3}} \frac{(t-a)^{2k^3+1}}{2k^3+1} \Bigg|_{t=a}^{b} \\
&= \frac{k^2(b-a)}{2k^3+1}
\end{aligned}$$

$\to 0$ as $k \to \infty$.

v) Let $f(t) = (t-a)/(b-a)$ and $g(t) = 1 - f(t)$. Then $\|f\|_\infty = \|g\|_\infty = \|f+g\|_\infty = \|f-g\|_\infty = 1$. So $\|f+g\|_\infty^2 + \|f-g\|_\infty^2 \neq 2\|f\|_\infty^2 + 2\|g\|_\infty^2$. This contradicts property iii) of Proposition 4.1.6. Therefore the maximum norm is not induced from any inner product on $\mathcal{C}([a,b])$.

Example 4.1.8. Suppose $f, g \in \mathcal{R}_{2\pi}$. If we define

$$(f \mid g) = \int_{-\pi}^{\pi} f(t)\overline{g(t)} \, \mathrm{d}t$$

then properties ii) and iii) of Definition 4.1.1 hold. Also, if we define

$$\|f\|_2 = \sqrt{\int_{-\pi}^{\pi} |f(t)|^2 \, \mathrm{d}t}$$

then conditions ii) and iii) of an Definition 4.1.2 are satisfied as well. However, if f is non-zero on a set of measure zero, then $\|f\|_2 = 0$ even though $f \neq 0$.

To remedy this situation, let $\eta = \{f \in \mathcal{R}_{2\pi} | \, \|f\|_2 = 0\}$. Then η is a subspace of $\mathcal{R}_{2\pi}$.

i) If $\|f\|_2 = \|g\|_2 = 0$ then $0 \leq \|f+g\|_2 \leq \|f\|_2 + \|g\|_2 = 0$ so $\|f+g\|_2 = 0$ and hence $f + g \in \eta$.

ii) If $\lambda \in \mathbb{C}$ then $\|\lambda f\|_2 = |\lambda| \, \|f\|_2 = 0$ so $\lambda f \in \eta$.

Now let $\tilde{\mathcal{R}}_{2\pi} = \mathcal{R}_{2\pi}/\eta$ and denote its elements by $[f]$ where $f \in \mathcal{R}_{2\pi}$. Then for $[f], [g] \in \tilde{\mathcal{R}}_{2\pi}$, $[f] = [g]$ iff $f - g \in \eta$ iff $\|f - g\|_2 = 0$ iff $f - g$ is zero almost everywhere. Notice the following.

i) If $h \in \eta$ and $f \in \mathcal{R}_{2\pi}$ then $|(h \mid f)| \leq \|h\|_2 \|f\|_2 = 0$ which implies $(h \mid f) = 0$.

ii) If $h_1, h_2 \in \eta$ and $f_1, f_2 \in \mathcal{R}_{2\pi}$ then

$$(f_1 + h_1 \mid f_2 + h_2) = (f_1 \mid f_2) + (f_1 \mid h_2) + (h_1 \mid f_2) + (h_1 \mid h_2) = (f_1 \mid f_2)$$

iii) If $h \in \eta$ and $f \in \mathcal{R}_{2\pi}$ then by part ii),

$$\|f + h\|_2^2 = (f + h \mid f + h) = (f \mid f) = \|f\|_2^2$$

It follows that for $[f], [g] \in \tilde{\mathcal{R}}_{2\pi}$, the functions

$$\langle [f] \mid [g] \rangle = (f \mid g)$$

and

$$\|[f]\| = \|f\|_2$$

are well defined and define an inner product and norm, respectively. In the remainder of this text, we will simply write f for $[f]$ and $\mathcal{R}_{2\pi}$ for $\tilde{\mathcal{R}}_{2\pi}$.

4.2 Metric Spaces

In this brief section we define metric spaces and Banach spaces and prove some related results.

Definition 4.2.1. A set X together with a function $d(x, y) : X \to [0, \infty)$, $x, y \in X$ is called a *metric space* provided the function d satisfies

i) $d(x, x) = 0$

ii) $d(x, y) = d(y, x)$

iii) $d(x, y) + d(y, z) \geq d(x, z)$

This function d is called a *metric* on X. If, in addition, X is complete with respect to d, then X is called a *Banach space*.

Example 4.2.2. Let X be a normed space and define $d(x,y) = \|x-y\|$ for $x, y \in X$. Show that d is a metric on X (it is called the *metric induced by the norm* $\|\cdot\|$). Furthermore, if X is complete with respect to d (ie. it is a Banach space), and $\{x_k\}_{k=1}^{\infty}$ is a sequence in X for which $\sum_{k=1}^{\infty} \|x_k\|$ converges, then $\sum_{k=1}^{\infty} x_k$ also converges.

Solution. If $x \in X$ then $d(x,x) = \|x-x\| = \|0\| = 0$. Also, if $x, y \in X$, then $d(x,y) = \|x-y\| = \|y-x\| = d(y,x)$. Finally, by the Triangle Inequality, if $x, y, z \in X$,

$$\begin{aligned} d(x,y) + d(y,z) &= \|x-y\| + \|y-z\| \\ &\geq \|(x-y)+(y-z)\| \\ &= \|x-z\| \\ &= d(x,z) \end{aligned}$$

Thus d is indeed a metric. For the last part of the solution, let

$$S_n = \sum_{k=1}^{n} x_k$$

for $n \geq 1$. Now, given $\epsilon > 0$ choose $N > 0$ such that if $q > p > N$ we have,

$$\sum_{k=p+1}^{q} \|x_k\| < \epsilon$$

This can be done since $\sum_{k=1}^{\infty} \|x_k\|$ converges and hence is Cauchy. Then, by the Triangle Inequality,

$$d(S_q, S_p) = \|S_q - S_p\| = \left\| \sum_{k=p+1}^{q} x_k \right\| \leq \sum_{k=p+1}^{q} \|x_k\| < \epsilon$$

So the series $\sum_{k=1}^{\infty} x_k$ is Cauchy in X. Since X is complete, this implies this series converges.

Definition 4.2.3. A metric d on a set X is called a *bounded metric* provided there exists $M > 0$ such that $d(x, y) < M$ for all $x, y \in X$.

Proposition 4.2.4. Let g be a metric on a set X and define

$$d(x, y) = \frac{g(x, y)}{1 + g(x, y)}$$

for all $x, y \in X$. Then d is a bounded metric on X.

Proof. First we will show that d is indeed a metric.

i) For all $x \in X$,

$$d(x, x) = \frac{g(x, x)}{1 + g(x, x)} = 0$$

ii) For all $x, y \in X$,

$$d(x, y) = \frac{g(x, y)}{1 + g(x, y)} = \frac{g(y, x)}{1 + g(y, x)} = d(y, x)$$

iii) For all $x, y, z \in X$,

$$\begin{aligned} d(x, z) &= \frac{g(x, z)}{1 + g(x, z)} \\ &\leq \frac{g(x, y) + g(y, z)}{1 + g(x, y) + g(y, z)} \\ &= \frac{g(x, y)}{1 + g(x, y) + g(y, z)} + \frac{g(y, z)}{1 + g(x, y) + g(y, z)} \\ &\leq \frac{g(x, y)}{1 + g(x, y)} + \frac{g(y, z)}{1 + g(y, z)} \\ &= d(x, y) + d(y, z) \end{aligned}$$

Finally it is easily seen that d is bounded above by 1. $\square$

Definition 4.2.5. Two metrics d and g on a set X are called *equivalent* provided there exist positive constants C_1 and C_2 such that $C_1 g(x, y) \leq d(x, y) \leq C_2 g(x, y)$ for all $x, y \in X$.

Readers familiar with general topology will know that equivalent metrics generate identical topologies on X.

Proposition 4.2.6. Let g be a metric on a set X and define

$$d(x,y) = \frac{g(x,y)}{1+g(x,y)}$$

for all $x, y \in X$. Then d and g are equivalent metrics on X.

Proof. It is clear that $d(x,y) \leq g(x,y)$ for all $x, y \in X$. We will show that $g(x,y) \leq 2d(x,y)$ to complete the proof. This will be done in steps.

Step 1 For $x \in \mathbb{R}, x \geq 0$, let

$$f(x) = \frac{x}{1+x} - 2x$$

Then

$$f'(x) = \frac{1}{(1+x)^2} - 2$$

and

$$f''(x) = -\frac{2}{(1-x)^3}$$

Hence and

$$f'''(x) = \frac{6}{(1-x)^4}$$

□

Theorem 4.2.7. Let X be a non-empty set and $\{d_k\}_{k=1}^{\infty}$ a sequence of metrics on X. Define

$$d(x,y) = \sum_{k=1}^{\infty} \frac{1}{2^k} \frac{d_k(x,y)}{1+d_k(x,y)}$$

for all $x, y \in X$. Then

i) d is a metric on X.

ii) A sequence $\{x_n\}_{n=1}^{\infty}$ in X converges to some $x_0 \in X$ with respect to d iff that sequence also converges to x_0 with respect to each d_k.

iii) X is complete with respect to d iff it is complete with respect to each d_k.

Proof. i) All the properties of a metric can be proven for d in a similar fashion as they were in Proposition 4.2.4.

ii) Suppose $\{x_n\}_{n=1}^{\infty}$ does not converge to x_0 with respect to some d_K. This implies for some $\epsilon > 0$ there is a subsequence $\{x_{n_j}\}_{j=1}^{\infty}$ such that $d_K(x_{n_j}, x_0) > \epsilon$ for all $j \geq 1$. Hence

$$d(x_{n_j}, x_0) = \sum_{k=1}^{\infty} \frac{1}{2^k} \frac{d_k(x_{n_j}, x_0)}{1 + d_k(x_{n_j}, x_0)} \geq \frac{1}{2^K} d_K(x_{n_j}, x_0) > \frac{\epsilon}{2^K}$$

which means $\{x_n\}_{n=1}^{\infty}$ doesn't converge to x_0 with respect to d.

Conversely, suppose $\{x_n\}_{n=1}^{\infty}$ converges to x_0 with respect to each d_k. Given $\epsilon > 0$ choose $N' > 0$ such that

$$\sum_{k=N'+1}^{\infty} \frac{1}{2^k} < \epsilon/2$$

Next choose $N > 0$ such that for all $1 \leq k \leq N'$,

$$d_k(x_n, x_0) < \frac{\epsilon 2^{k-1}}{N'}$$

if $n > N$. This can be done because $\{x_n\}_{n=1}^{\infty}$ converges to x_0 with respect to each d_k. Then if $n > N$,

$$\begin{aligned}
d(x_n, x_0) &= \sum_{k=1}^{\infty} \frac{1}{2^k} \frac{d_k(x_n, x_0)}{1 + d_k(x_n, x_0)} \\
&= \sum_{k=1}^{N'} \frac{1}{2^k} \frac{d_k(x_n, x_0)}{1 + d_k(x_n, x_0)} + \sum_{k=N'+1}^{\infty} \frac{1}{2^k} \frac{d_k(x_n, x_0)}{1 + d_k(x_n, x_0)} \\
&< \sum_{k=1}^{N'} \frac{1}{2^k} \frac{\epsilon 2^{k-1}}{N'} + \sum_{k=N'+1}^{\infty} \frac{1}{2^k} \\
&< \frac{\epsilon}{2} + \frac{\epsilon}{2} \\
&= \epsilon
\end{aligned}$$

Hence $\{x_n\}_{n=1}^{\infty}$ converges to x_0 with respect to x_0.

iii) An argument similar to that in ii) can be used to show that sequences are Cauchy with respect to d iff they are Cauchy with respect to each d_k. Therefore if d is complete and $\{x_n\}_{n=1}^{\infty}$ is Cauchy with respect to d then this sequence converges to some $x_0 \in X$ with respect to d. But by part ii), this implies it converges to x_0 with respect to each d_k.

□

4.3 Parseval's Theorem

In this section we define orthogonal sets, orthonormal sets and orthonormal bases. We also present Bessel's Inequality and Parseval's Theorem.

Definition 4.3.1. Let V be an inner product space and $\{v_1, v_2, \ldots, v_n\}$ a subset of V. We call this set of vectors *orthogonal* if $(v_j \mid v_k) = 0$ whenever $j \neq k$. If in addition, $(v_j \mid v_k) = \delta_{jk}$ then we call this set of vectors *othornormal*.

Proposition 4.3.2. If $\{v_1, v_2, \ldots, v_n\}$ is an orthogonal set of vectors in a complex vector space and $\lambda_1, \lambda_2, \ldots, \lambda_n$ are complex scalars, then

$$\left\| \sum_{k=1}^{n} \lambda_k v_k \right\|^2 = \sum_{k=1}^{n} |\lambda_k|^2 \|v_k\|^2$$

Proof. Let v_k and λ_k be as in the hypothesis. Then

$$\begin{aligned}
\left\| \sum_{k=1}^{n} \lambda_k v_k \right\|^2 &= \left(\sum_{j=1}^{n} \lambda_j v_j \mid \sum_{k=1}^{n} \lambda_k v_k \right) \\
&= \sum_{j=1}^{n} \sum_{k=1}^{n} (\lambda_j v_j \mid \lambda_k v_k) \\
&= \sum_{j=1}^{n} \sum_{k=1}^{n} \lambda_j \overline{\lambda_k} (v_j \mid v_k) \\
&= \sum_{k=1}^{n} |\lambda_k|^2 \|v_k\|^2
\end{aligned}$$

□

Theorem 4.3.3 (Bessel's Inequality)**.** Suppose that $\{b_1, \ldots, b_n\}$ is an othonormal subset of an inner product space V and $x \in V$. Then

i) For all $\lambda_1, \ldots, \lambda_n \in \mathbb{C}$,

$$\left\| x - \sum_{k=1}^{n} (x \mid b_k)\, b_k \right\| \leq \left\| x - \sum_{k=1}^{n} \lambda_k b_k \right\|$$

and equality holds iff $\lambda_k = (x \mid b_k)$ for all k.

ii) (Bessel's Inequality)

$$\sum_{k=1}^{n} |(x \mid b_k)|^2 \leq \|x\|^2$$

Proof. Let x, b_k and λ_k be as in the hypothesis and define

$$\varphi(\lambda_1, \ldots, \lambda_n) = \left\| x - \sum_{k=1}^{n} \lambda_k b_k \right\|^2$$

Then

$$\begin{aligned}
\varphi(\lambda_1, \ldots, \lambda_n) &= \left(x - \sum_{j=1}^{n} \lambda_j b_j \mid x - \sum_{k=1}^{n} \lambda_k b_k \right) \\
&= \|x\|^2 - \left(x \mid \sum_{k=1}^{n} \lambda_k b_k \right) - \left(\sum_{j=1}^{n} \lambda_j b_j \mid x \right) + \left(\sum_{j=1}^{n} \lambda_j b_j \mid \sum_{k=1}^{n} \lambda_k b_k \right) \\
&= \|x\|^2 - \sum_{k=1}^{n} \overline{\lambda_k}\, (x \mid b_k) - \sum_{j=1}^{n} \lambda_j\, (b_j \mid x) + \sum_{k=1}^{n} |\lambda_k|^2 \\
&= \|x\|^2 + \sum_{k=1}^{n} \left(|(x \mid b_k)|^2 - \overline{\lambda_k}\, (x \mid b_k) - \lambda_k \overline{(x \mid b_k)} + |\lambda_k|^2 \right) - \sum_{k=1}^{n} |(x \mid b_k)|^2 \\
&= \|x\|^2 + \sum_{k=1}^{n} |(x \mid b_k) - \lambda_k|^2 - \sum_{k=1}^{n} |(x \mid b_k)|^2
\end{aligned}$$

Obviously, this quantity is minimized by the unique choice

$$\lambda_k = (x \mid b_k)$$

This proves the first assertion. For the second, simply note that

$$0 \leq \varphi((x \mid b_1), \ldots, (x \mid b_n)) = \|x\|^2 - \sum_{k=1}^{n} |(x \mid b_k)|^2$$

□

Proposition 4.3.4. In the inner product space $\mathcal{R}_{2\pi}$ (with the inner product defined in the previous section), let

$$e_k(t) = \frac{e^{ikt}}{\sqrt{2\pi}}$$

for $k \in \mathbb{Z}$. Also, for $f \in \mathcal{R}_{2\pi}$ define

$$\hat{f}(k) = \langle f \mid e_k \rangle$$

(this is sometimes called the *Fourier Transform of f* but we will reserve this terminology for a different function). Then

i) The set $\{e_k\}_{k=-\infty}^{\infty}$ is orthonormal.

ii) $\hat{f}(k) = \sqrt{2\pi}c_k$ where $c_k, k \in \mathbb{Z}$ are the exponential Fourier coefficients of f.

iii) If $s_n, n \geq 1$ is the nth partial sum of the Fourier series of f, then

$$s_n = \sum_{k=-n}^{n} \hat{f}(k)e_k$$

iv) The series $\sum_{k=0}^{\infty} \left|\hat{f}(k)\right|^2$ and $\sum_{k=1}^{\infty} \left|\hat{f}(-k)\right|^2$ both converge. Also, if we define

$$\sum_{k=-\infty}^{\infty} \left|\hat{f}(k)\right|^2 = \sum_{k=0}^{\infty} \left|\hat{f}(k)\right|^2 + \sum_{k=1}^{\infty} \left|\hat{f}(-k)\right|^2$$

then this series converges as well.

v) If $f \in \mathcal{R}_{2\pi}$ then

$$\sum_{k=1}^{\infty} \left| \frac{\hat{f}(k)}{k} \right|$$

and

$$\sum_{k=1}^{\infty} \left| \frac{\hat{f}(-k)}{k} \right|$$

both converge.

Proof. Let f, $\hat{f}$ and the e_k be as in the hypothesis.

i) By Lemma 2.3.3 we have

$$\begin{aligned} \langle e_j(t) \mid e_k(t) \rangle &= \int_{-\pi}^{\pi} \frac{e^{ijt}}{\sqrt{2\pi}} \overline{\frac{e^{ikt}}{\sqrt{2\pi}}} \, \mathrm{d}t \\ &= \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{i(j-k)t} \, \mathrm{d}t \\ &= \delta_{jk} \end{aligned}$$

ii)

$$\hat{f}(k) = \langle f \mid e_k \rangle = \frac{1}{\sqrt{2\pi}} \int_{-\pi}^{\pi} f(t) e^{-ikt} \, \mathrm{d}t = \sqrt{2\pi} c_k$$

iii) For $n \in \mathbb{N}$ and $x \in \mathbb{R}$,

$$\begin{aligned} s_n(x) &= \sum_{k=-n}^{n} \left(\frac{1}{2\pi} \int_{-\pi}^{\pi} f(t) e^{-ikt} \, \mathrm{d}t \right) e^{ikx} \\ &= \sum_{k=-n}^{n} \left(\int_{-\pi}^{\pi} f(t) \frac{e^{-ikt}}{\sqrt{2\pi}} \, \mathrm{d}t \right) \frac{e^{ikx}}{\sqrt{2\pi}} \\ &= \sum_{k=-n}^{n} \langle f \mid e_k \rangle \, e_k(x) \\ &= \sum_{k=-n}^{n} \hat{f}(k) e_k(x) \end{aligned}$$

iv) By Theorem 4.3.3, for all $n \in \mathbb{N}$,

$$\sum_{k=-n}^{n} \left|\hat{f}(k)\right|^2 = \sum_{k=-n}^{n} |\langle f \mid e_k \rangle|^2 \leq \|f\|^2 = \int_{-\pi}^{\pi} |f(t)|^2 \, \mathrm{d}t < \infty$$

since $f \in \mathcal{R}_{2\pi}$. Therefore $\sum_{k=0}^{\infty} \left|\hat{f}(k)\right|^2$ and $\sum_{k=1}^{\infty} \left|\hat{f}(-k)\right|^2$ are both convergent and

$$\sum_{k=-\infty}^{\infty} \left|\hat{f}(k)\right|^2 = \sum_{k=0}^{\infty} \left|\hat{f}(k)\right|^2 + \sum_{k=1}^{\infty} \left|\hat{f}(-k)\right|^2 \leq \int_{-\pi}^{\pi} |f(t)|^2 \, \mathrm{d}t < \infty$$

v) Let

$$S = \{k \in \mathbb{N} \mid \left|\hat{f}(k)\right| \leq \frac{1}{k}\}$$

and

$$T = \{k \in \mathbb{N} \mid \left|\hat{f}(k)\right| > \frac{1}{k}\}$$

Then

$$\begin{aligned} \sum_{k=1}^{\infty} \left|\frac{\hat{f}(k)}{k}\right| &= \sum_{k \in S} \left|\frac{\hat{f}(k)}{k}\right| + \sum_{k \in T} \left|\frac{\hat{f}(k)}{k}\right| \\ &< \sum_{k \in S} \frac{1}{k^2} + \sum_{k \in T} \left|\hat{f}(k)\right|^2 \\ &< \sum_{k=1}^{\infty} \frac{1}{k^2} + \sum_{k=-\infty}^{\infty} |\langle f \mid e_k \rangle|^2 \end{aligned}$$

Now the first sum in the above expression converges and by Theorem 4.3.3,

$$\sum_{k=-\infty}^{\infty} |\langle f \mid e_k \rangle|^2 \leq \|f\|^2 < \infty$$

since $f \in \mathcal{R}_{2\pi}$. Therefore

$$\sum_{k=1}^{\infty} \left|\frac{\hat{f}(k)}{k}\right| < \infty$$

A similar argument shows that

$$\sum_{k=1}^{\infty} \left| \frac{\hat{f}(-k)}{k} \right| < \infty$$

as well.

□

Theorem 4.3.5 (Parseval's Theorem)**.** If $f \in \mathcal{R}_{2\pi}$ then

$$\lim_{n\to\infty} \left\| f - \sum_{k=-n}^{n} \hat{f}(k) e_k \right\| = 0$$

and

$$\sum_{k=-\infty}^{\infty} \left| \hat{f}(k) \right|^2 = \|f\|^2 = \int_{-\pi}^{\pi} |f(t)|^2 \, \mathrm{d}t$$

Proof. Case i) First suppose $f \in \mathcal{C}_{2\pi}$. For $n \geq 0$, let s_n be the nth partial sum of the Fourier series of f and define

$$\sigma_n = \frac{1}{n+1} \sum_{k=0}^{n} s_k$$

By Theorem 3.4.3, given $\epsilon > 0$ there exists $N > 0$ such that for $n > N$, $|f(t) - \sigma_n(t)| < \sqrt{\epsilon/2\pi}$. Hence for this choice of n,

$$\|f - \sigma_n\|^2 = \int_{-\pi}^{\pi} |f(t) - \sigma_n(t)|^2 \, \mathrm{d}t < \int_{-\pi}^{\pi} \frac{\epsilon}{2\pi} \, \mathrm{d}t = \epsilon$$

But by Proposition 4.3.4,

$$s_n = \sum_{k=-n}^{n} \hat{f}(k) e_k$$

which means σ_n is a linear combination of the e_k's. Therefore, by Theorem 4.3.3,

$$\begin{aligned} \|f - s_n\|^2 &= \left\| f - \sum_{k=-n}^{n} \langle f \mid e_k \rangle \, e_k \right\|^2 \\ &\leq \|f - \sigma_n\|^2 \\ &< \epsilon \end{aligned}$$

if $n > N$. Thus $\lim_{n\to\infty} \|f - s_n\| = 0$. The second assertion follows by noting that $|\|f\| - \|s_n\|| \leq \|f - s_n\| \to 0$ as $n \to \infty$.

Case ii) Suppose $f \in \mathcal{R}_{2\pi}$ and $\epsilon > 0$. Choose $g \in \mathcal{C}_{2\pi}$ such that $\|f - g\| < \epsilon/2$. Also, by the first case, we can choose $N > 0$ such that

$$\left\| g - \sum_{k=-N}^{N} \hat{g}(k) e_k \right\| < \epsilon/2$$

Then if $n > N$, by Theorem 4.3.3, we have

$$\begin{aligned} \left\| f - \sum_{k=-n}^{n} \hat{f}(k) e_k \right\| &\leq \left\| f - \sum_{k=-N}^{N} \hat{g}(k) e_k \right\| \\ &\leq \|f - g\| + \left\| g - \sum_{k=-N}^{N} \hat{g}(k) e_k \right\| \\ &< \epsilon \end{aligned}$$

Thus

$$\lim_{n\to\infty} \left\| f - \sum_{k=-n}^{n} \hat{f}(k) e_k \right\| = 0$$

The second assertion follows as in Case i).

□

Definition 4.3.6. An orthonormal sequence $\{b_k\}_{k=1}^{\infty}$ in an inner product space V is said to be an *orthonormal basis for* V provided that for all $x \in V$,

$$x = \lim_{n\to\infty} \sum_{k=1}^{n} (x \mid b_k)\, b_k$$

In other words,

$$\lim_{n\to\infty} \left\| x - \sum_{k=1}^{n} (x \mid b_k)\, b_k \right\| = 0$$

Example 4.3.7. By Theorem 4.3.5, $\{e_0, e_{-1}, e_1, \ldots\}$ is an orthonormal basis for $\mathcal{R}_{2\pi}$.

Theorem 4.3.8. If $\{b_k\}_{k=1}^{\infty}$ is an orthonormal basis for an inner product space V, then for all $x, y \in V$ and $\lambda \in \mathbb{C}$

i)

$$\lambda x + y = \sum_{k=1}^{\infty} \left(\lambda \left(x \mid b_k\right) + \left(y \mid b_k\right)\right) b_k$$

ii)

$$\left(x \mid y\right) = \sum_{k=1}^{\infty} \left(x \mid b_k\right) \overline{\left(y \mid b_k\right)}$$

iii)

$$\|x\|^2 = \sum_{k=1}^{\infty} \left|\left(x \mid b_k\right)\right|^2$$

Proof. i) Trivial.

ii) Given $\epsilon > 0$ choose $N > 0$ such that if $n > N$ then

$$\left\| x - \sum_{k=1}^{n} \left(x \mid b_k\right) b_k \right\| < \frac{\epsilon}{2(\|y\| + 1)}$$

and also

$$\left\| y - \sum_{k=1}^{n} \left(y \mid b_k\right) b_k \right\| < \frac{\epsilon}{2(\|x\| + 1)}$$

(This can be done because $\{b_k\}_{k=1}^{\infty}$ is a basis for V.) Then for this choice of n,

$$\begin{aligned}
\left|(x \mid y) - \sum_{k=1}^{n} (x \mid b_k)\,\overline{(y \mid b_k)}\right| &= \left|(x \mid y) - \left(\sum_{k=1}^{n} (x \mid b_k)\, b_k \mid \sum_{k=1}^{n} (y \mid b_k)\, b_k\right)\right| \\
&\leq \left|(x \mid y) - \left(\sum_{k=1}^{n} (x \mid b_k)\, b_k \mid y\right)\right| \\
&\quad + \left|\left(\sum_{k=1}^{n} (x \mid b_k)\, b_k \mid y\right) - \left(\sum_{k=1}^{n} (x \mid b_k)\, b_k \mid \sum_{k=1}^{n} (y \mid b_k)\, b_k\right)\right| \\
&= \left|\left(x - \sum_{k=1}^{n} (x \mid b_k)\, b_k \mid y\right)\right| \\
&\quad + \left|\left(\sum_{k=1}^{n} (x \mid b_k)\, b_k \mid y - \sum_{k=1}^{n} (y \mid b_k)\, b_k\right)\right| \\
&\leq \left\|x - \sum_{k=1}^{n} (x \mid b_k)\, b_k\right\| \|y\| \\
&\quad + \left\|\sum_{k=1}^{n} (x \mid b_k)\, b_k\right\| \left\|y - \sum_{k=1}^{n} (y \mid b_k)\, b_k\right\| \quad \text{by Proposition 4.} \\
&< \frac{\epsilon}{2(\|y\| + 1)} \|y\| + \|x\| \frac{\epsilon}{2(\|x\| + 1)} \quad \text{by Theorem 4.3.3} \\
&< \frac{\epsilon}{2} + \frac{\epsilon}{2} \\
&= \epsilon
\end{aligned}$$

which proves the result.

iii) This follows immediately from part ii).

□

Corollary 4.3.9. If $f, g \in \mathcal{R}_{2\pi}$ and $\lambda \in \mathbb{C}$ then

i) $(\widehat{\lambda f + g})(k) = \lambda(\hat{f}(k) + \hat{g}(k)) = (\lambda \hat{f} + \hat{g})(k)$ for all $k \in \mathbb{Z}$.

ii)

$$\int_{-\pi}^{\pi} f(t)\overline{g(t)}\,\mathrm{d}t = \sum_{k=-\infty}^{\infty} \hat{f}(k)\overline{\hat{g}(k)}$$

iii)

$$\int_{-\pi}^{\pi} |f(t)|^2 \,\mathrm{d}t = \sum_{k=-\infty}^{\infty} \left|\hat{f}(k)\right|^2$$

Proof. All parts of this corollary follow from Theorem 4.3.8. □

4.4 Inner Product Space Isomorphisms

This section deals with unitary transformations and inner product space isomorphisms. It concludes with a discussion of l^2 spaces.

Definition 4.4.1. Let V and W be real or complex inner product spaces. A linear map $T : V \to W$ is called an inner product space *isomorphism* from V to W if it is one-to-one, onto and preserves inner products (that is, $(x \mid y) = (Tx \mid Ty)$ for all $x, y \in V$). Here it is understood that the inner product on left is the one in V and the one on the right is the one in W. A map $T : V \to W$ which satisfies all the properties of an inner product space isomorphism except for being onto is called an *embedding* of V into W.

Definition 4.4.2. Let V and W be real or complex inner product spaces. A linear map $T : V \to W$ is called a *unitary transformation* from V to W provided it preserves norms (that is, $\|x\| = \|Tx\|$ for all $x \in V$). Here the norm on the left is the one induced from the inner product in V and the one on the right is the one induced from the inner product in W.

Theorem 4.4.3. Let V and W be real or complex inner product spaces and T a linear map from V to W. Then T is an inner product space isomorphism iff it is an onto unitary transformation from V to W.

Proof. If T preserves inner products then it clearly preserves norms because $\|x\|^2 = (x \mid x) = (Tx \mid Tx) = \|Tx\|^2$ for all $x \in V$. Also, T is onto by the very definition of isomorphism. So T is unitary and onto. Conversely, if T

preserves norms and if V and W are real inner product spaces then it is easily seen that T preserves inner products because of the identity

$$(x \mid y) = \frac{1}{4}\left(\|x+y\|^2 - \|x-y\|^2\right)$$

Now, to show that T preserves inner products when V and W are complex inner product spaces, first note that

$$\begin{aligned}\|x-iy\|^2 &= \|x\|^2 + i\,(x \mid y) - i\,(y \mid x) + \|y\|^2 \\ &= \|x\|^2 + i\,(x \mid y) - i\overline{(x \mid y)} + \|y\|^2 \\ &= \|x\|^2 + 2i\mathrm{Im}((x \mid y)) + \|y\|^2 \;\; (*)\end{aligned}$$

and similarly,

$$\|T(x-iy)\|^2 = \|Tx - iTy\|^2 = \|Tx\|^2 + 2i\mathrm{Im}((Tx \mid Ty)) + \|Ty\|^2 \;\; (\dagger)$$

Since T preserves norms, by assumption, comparing $(*)$ to $(\dagger)$, we see that $\mathrm{Im}((x \mid y)) = \mathrm{Im}((Tx \mid Ty))$. In a similar fashion,

$$\|x+y\|^2 = \|x\|^2 + (x \mid y) + (y \mid x) + \|y\|^2 = \|x\|^2 + 2\mathrm{Re}((x \mid y)) + \|y\|^2 \;\; (**)$$

and

$$\|T(x+y)\|^2 = \|Tx+Ty\|^2 = \|Tx\|^2 + 2\mathrm{Re}((Tx \mid Ty)) + \|Ty\|^2 \;\; (\ddagger)$$

Comparing $(**)$ to $(\ddagger)$ we see that $\mathrm{Re}((x \mid y))$=$\mathrm{Re}((Tx \mid Ty))$. Thus T preserves inner products. Finally note that T is onto by assumption and clearly one-to-one since $\|Tx\| = \|x\| \neq 0$ if $x \neq 0$. So T is an inner product space isomorphism from V to W. □

Proposition 4.4.4. If $T : V \to W$ is an inner product space isomorphism then it maps each basis of V to a basis of W.

Proof. Let $\{b_k\}_{k=1}^{\infty}$ be a basis for V. Then $(Tb_j \mid Tb_k) = (b_j \mid b_k) = \delta_{jk}$ so $\{Tb_k\}_{k=1}^{\infty}$ is orthonormal. To see that it is a basis, given $x \in V$ and $\epsilon > 0$ choose $N > 0$ such that if $n > N$

$$\left\| x - \sum_{k=1}^{n} (x \mid b_k)\, b_k \right\| < \epsilon$$

Then, by Theorem 4.4.3, since T preserves inner products it preserves norms, so

$$\begin{aligned}\left\|Tx - \sum_{k=1}^{n} (Tx \mid Tb_k)\, Tb_k\right\| &= \left\|T(x - \sum_{k=1}^{n} (Tx \mid Tb_k)\, b_k)\right\| \\ &= \left\|x - \sum_{k=1}^{n} (Tx \mid Tb_k)\, b_k\right\| \\ &= \left\|x - \sum_{k=1}^{n} (x \mid b_k)\, b_k\right\| \\ &< \epsilon\end{aligned}$$

So $\{Tb_k\}_{k=1}^{\infty}$ is a basis for W as required. □

Example 4.4.5. Let $T = \{z \in \mathbb{C} \mid |z| = 1\}$. Then given $f : T \to \mathbb{C}$, define $\tilde{f} : \mathbb{R} \to \mathbb{C}$ by $\tilde{f}(t) = f(e^{it})$. Let $\mathcal{R}_T = \{f : T \to \mathbb{C} \mid \tilde{f} \in \mathcal{R}_{2\pi}\}$ and define $\langle f \mid g\rangle = \left\langle \tilde{f} \mid \tilde{g} \right\rangle$ for $f, g \in \mathcal{R}_T$. It is easily verified that this defines an inner product on $\mathcal{R}_T$ and that the map $f \mapsto \tilde{f}$ is an inner product space isomorphism from $\mathcal{R}_T$ to $\mathcal{R}_{2\pi}$.

We conclude this section with a brief discussion of l^2 spaces.

Definition 4.4.6. Denote the set of all complex sequences (or equivalently complex valued functions) $\{c_k\}_{k=-\infty}^{\infty}$ which satisfy

$$\sum_{k=-\infty}^{\infty} |c_k|^2 < \infty$$

by l^2.

Proposition 4.4.7. l^2 is a vector space under pointwise addition and scalar multiplication.

Proof. Suppose $c = \{c_k\}_{k=-\infty}^{\infty}$ and $d = \{d_k\}_{k=-\infty}^{\infty}$ are sequences in l^2 and $\lambda \in \mathbb{C}$. Then, using the identity

$$2\left|c_k\overline{d_k}\right| = 2\left|c_k d_k\right| \leq \left|c_k\right|^2 + \left|d_k\right|^2$$

we get

$$\begin{aligned}\left|\sum_{k=-\infty}^{\infty}\left|\lambda c_k + d_k\right|^2\right| &\leq \left|\left|\lambda\right|^2\sum_{k=-\infty}^{\infty}\left|c_k\right|^2\right| + \left|\sum_{k=-\infty}^{\infty}\left|d_k\right|^2\right| + \left|\lambda\sum_{k=-\infty}^{\infty}c_k\overline{d_k}\right| + \left|\overline{\lambda}\sum_{k=-\infty}^{\infty}\overline{c_k}d_k\right| \\ &\leq \left|\left|\lambda\right|^2\sum_{k=-\infty}^{\infty}\left|c_k\right|^2\right| + \left|\sum_{k=-\infty}^{\infty}\left|d_k\right|^2\right| + \left|\lambda\sum_{k=-\infty}^{\infty}\left(\frac{\left|c_k\right|^2}{2} + \frac{\left|d_k\right|^2}{2}\right)\right| \\ &\quad + \left|\overline{\lambda}\sum_{k=-\infty}^{\infty}\left(\frac{\left|c_k\right|^2}{2} + \frac{\left|d_k\right|^2}{2}\right)\right| \\ &< \infty\end{aligned}$$

since $\sum_{k=-\infty}^{\infty}\left|c_k\right|^2 < \infty$ and $\sum_{k=-\infty}^{\infty}\left|d_k\right|^2 < \infty$. Thus, if $c, d \in l^2$ and $\lambda \in \mathbb{C}$, then $\lambda c + d \in l^2$. So l^2 is a complex vector space. $\square$

Proposition 4.4.8. For $c = \{c_k\}_{k=-\infty}^{\infty}$ and $d = \{d_k\}_{k=-\infty}^{\infty}$ in l^2, define

$$(c \mid d) = \sum_{k=-\infty}^{\infty} c_k\overline{d_k}$$

Then this function is well-defined and is, in fact, an inner product on l^2.

Proof. We will divide the proof into steps.

Step 1. Again, using the identity

$$2\left|c_k\overline{d_k}\right| = 2\left|c_k d_k\right| \leq \left|c_k\right|^2 + \left|d_k\right|^2$$

we see that $(c \mid d)$ converges absolutely and hence is well-defined.

Step 2. If $c \in l^2$ then

$$(c \mid c) = \sum_{k=\infty}^{\infty}\left|c_k\right|^2 \geq 0$$

with equality iff $c_k = 0$ for all $k \in \mathbb{Z}$.

Step 3. If $a, b, c \in l^2$ and $\lambda \in \mathbb{C}$ then

$$\begin{aligned}(\lambda a + b \mid c) &= \sum_{k=-\infty}^{\infty} (\lambda a_k + b_k)\overline{c_k} \\ &= \lambda \sum_{k=-\infty}^{\infty} a_k \overline{c_k} + \sum_{k=-\infty}^{\infty} b_k \overline{c_k} \\ &= \lambda\,(a \mid c) + (b \mid c)\end{aligned}$$

Step 4. Suppose $c, d \in l^2$. Then since the conjugation function is continuous, we have

$$\overline{\sum_{k=-n}^{n} c_k \overline{d_k}} \to \overline{(c \mid d)}$$

as $n \to \infty$. But

$$\overline{\sum_{k=-n}^{n} c_k \overline{d_k}} = \sum_{k=-n}^{n} \overline{c_k \overline{d_k}} = \sum_{k=-n}^{n} d_k \overline{c_k} \to (d \mid c)$$

as $n \to \infty$. Thus $\overline{(c \mid d)} = (d \mid c)$.

$\square$

As usual, we will denote $\sqrt{(c \mid c)}$ by $\|c\|$.

Proposition 4.4.9. The map $f \mapsto \hat{f}$ is an embedding from $\mathcal{R}_{2\pi}$ into l^2 and the set $\{\hat{f} \mid f \in \mathcal{R}_{2\pi}\}$ is dense in l^2.

Proof. By Corollary 4.3.9, this map is linear and preserves inner products. Moreover, if $\hat{f}(k) = 0$ for all $k \in \mathbb{Z}$ then $f = 0$. So this map is one-to-one. Therefore it is an embedding from $\mathcal{R}_{2\pi}$ into l^2.

To show that the image of this map is dense in l^2, define $b_k(j) = \delta_{jk}$ for $j, k \in \mathbb{Z}$. If $c = \{c_k\}_{k=-\infty}^{\infty}$ is in l^2 then $c = \sum_{k=-\infty}^{\infty} c_k b_k$. Hence $\{b_k\}_{k=-\infty}^{\infty}$ is a basis for l^2. Moreover, if $\{e_k\}_{k=-\infty}^{\infty}$ is the basis for $\mathcal{R}_{2\pi}$ described in the previous section, then $\hat{e}_k = b_k$ for all $k \in \mathbb{Z}$. So given $c \in l^2$, let

$f_n = \sum_{k=-n}^{n} c_k e_k$. Then $f_n \in \mathcal{R}_{2\pi}$ and

$$\begin{aligned}\left\|c - \hat{f}_n\right\|^2 &= \left\|\sum_{k=-\infty}^{\infty} c_k b_k - \sum_{k=-n}^{n} c_k \hat{e}_k\right\|^2 \\ &= \left\|\sum_{k=-\infty}^{\infty} c_k b_k - \sum_{k=-n}^{n} c_k b_k\right\|^2 \\ &= \left\|\sum_{|k|>n} c_k b_k\right\|^2 \\ &= \sum_{|k|>n} |c_k|^2\end{aligned}$$

$\to 0$ as $n \to \infty$. Hence the set $\{\hat{f} \mid f \in \mathcal{R}_{2\pi}\}$ is dense in l^2. □

The final property of l^2 we will prove is that it is complete with respect to the norm induced from its inner product. To prove this result, we will need the following inequality, which will be presented without proof.

Proposition 4.4.10 (Minkowski's Inequality)**.** Suppose $\{A_{kn}\}$ is an array of complex numbers which satisfy

$$\sum_{k=-\infty}^{\infty} |A_{kn}| < \infty$$

for all $n \in \mathbb{Z}$. Then

$$\left(\sum_{n=-\infty}^{\infty} \left|\sum_{k=-\infty}^{\infty} A_{kn}\right|^2\right)^{1/2} = \sum_{k=-\infty}^{\infty} \left(\sum_{n=-\infty}^{\infty} |A_{kn}|^2\right)^{1/2}$$

Theorem 4.4.11. l^2 is complete with respect to the norm induced from its inner product.

Proof. We will divide the proof into steps.

Step 1 Let $\{c_k\}_{k=1}^{\infty}$ be an infinite Cauchy sequence in l^2 and for each k denote the terms of c_k by $\{c_{k,n}\}_{n=-\infty}^{\infty}$. Then given $\epsilon > 0$ there exists $K > 0$ such that if $j, k > K$ and $n \in \mathbb{Z}$ is fixed we have

$$|c_{k,n} - c_{j,n}| \leq \sum_{m=-\infty}^{\infty} |c_{k,m} - c_{j,m}|^2 = \|c_k - c_j\|^2 < \epsilon$$

Hence for each $n \in \mathbb{Z}$, $\{c_{k,n}\}_{k=1}^{\infty}$ is a complex Cauchy sequence. Therefore this sequence converges to some d_n. Let $d = \{d_n\}_{n=-\infty}^{\infty}$.

Step 2 We will now show that $d \in l^2$. Since $\{c_k\}_{k=1}^{\infty}$ is Cauchy it is bounded and hence there exists $M > 0$ such that $\|c_k\| < M$ for all $k \geq 1$. Then

$$\begin{aligned}\|d\|^2 &= \lim_{N\to\infty} \sum_{|n|<N} |d_n|^2 \\ &= \lim_{k,N\to\infty} \sum_{|n|<N} |c_{k,n}|^2 \\ &= \lim_{k\to\infty} \|c_k\|^2 \\ &< M^2\end{aligned}$$

Step 3 Choose a strictly increasing sequence of positive integers $\{m_k\}_{k=1}^{\infty}$ such that if $m > m_k$ then $\|c_m - c_{m_k}\| < 2^{-k-1}$. Again, this can be done since the sequence $\{c_k\}_{k=1}^{\infty}$ is Cauchy. Then for fixed $n \in \mathbb{Z}$ and fixed $K > 0$,

$$d_n = c_{m_K,n} + \sum_{k=K}^{\infty} (c_{m_{k+1},n} - c_{m_k,n})$$

Therefore,

$$\begin{aligned}
\|d - c_{m_K}\| &= \left(\sum_{n=-\infty}^{\infty} |d_n - c_{m_K,n}|^2 \right)^{1/2} \\
&= \left(\sum_{n=-\infty}^{\infty} \left| \sum_{k=K}^{\infty} (c_{m_{k+1},n} - c_{m_k,n}) \right|^2 \right)^{1/2} \\
&= \sum_{k=K}^{\infty} \left(\sum_{n=-\infty}^{\infty} \left| c_{m_{k+1},n} - c_{m_k,n} \right|^2 \right)^{1/2} \quad \text{by Proposition 4.4.10} \\
&= \sum_{k=K}^{\infty} \left\| c_{m_{k+1}} - c_{m_k} \right\| \\
&< \sum_{k=K}^{\infty} 2^{-k-1} \text{ by the choice of } m_k \\
&= 2^{-K}
\end{aligned}$$

Step 4 For the final step, given $\epsilon > 0$ choose $K > 0$ such that $2^{-K} < \epsilon/2$. Then if $m > m_K$ we have

$$\|d - c_m\| \leq \|d - c_{m_K}\| + \|c_m - c_{m_K}\| < 2^{-K} + 2^{-K-1} < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$

□

4.5 The Fourier Transform

In this section we present the Fourier Transform of a function and its properties.

Definition 4.5.1. Suppose $f \in \mathcal{R}^1$. Then since $|f(t)e^{-ixt}| = |f|$ for all $x \in \mathbb{R}$, the map $t \mapsto f(t)e^{-ixt}$ belongs to $\mathcal{R}^1$. We define

$$\hat{f}(x) = \int_{-\infty}^{\infty} f(t)e^{-ixt} \, \mathrm{d}t$$

for all $x \in \mathbb{R}$ and call $\hat{f}$ the *Fourier Transform* of f.

Theorem 4.5.2. If $f \in \mathcal{R}^1$ then

i) $\lim_{|x|\to\infty} \hat{f}(x) = 0$

ii) $\hat{f}$ is continuous.

Proof. i) Given $\epsilon > 0$ choose $A > 0$ such that

$$\int_{-\infty}^{-A} |f(t)|\,\mathrm{d}t + \int_{A}^{\infty} |f(t)|\,\mathrm{d}t < \epsilon/2$$

Then for all $x \in \mathbb{R}$,

$$\begin{aligned}\left|\int_{-\infty}^{\infty} f(t)e^{-ixt}\,\mathrm{d}t\right| &\le \int_{-\infty}^{-A} \left|f(t)e^{-ixt}\right|\mathrm{d}t + \left|\int_{-A}^{A} f(t)e^{-ixt}\,\mathrm{d}t\right| + \int_{A}^{\infty} \left|f(t)e^{-ixt}\right|\mathrm{d}t \\ &= \int_{-\infty}^{-A} |f(t)|\,\mathrm{d}t + \left|\int_{-A}^{A} f(t)e^{-ixt}\,\mathrm{d}t\right| + \int_{A}^{\infty} |f(t)|\,\mathrm{d}t \\ &< \epsilon/2 + \left|\int_{-A}^{A} f(t)e^{-ixt}\,\mathrm{d}t\right| \quad (*)\end{aligned}$$

By Theorem 1.3.1,

$$\lim_{|x|\to\infty} \int_{-A}^{A} f(t)e^{-ixt}\,\mathrm{d}t = 0$$

Therefore, from $(*)$ we get

$$\limsup_{|x|\to\infty} \left|\hat{f}(x)\right| < \epsilon$$

Since this is true for all $\epsilon > 0$ we have

$$\lim_{|x|\to\infty} \hat{f}(x) = 0$$

ii) Fix $x \in \mathbb{R}$ and choose a real sequence h_k converging to zero as $k \to \infty$. Let $\epsilon > 0$ and choose $A > 0$ such that

$$\int_{-\infty}^{-A} |f(t)|\,\mathrm{d}t + \int_{A}^{\infty} |f(t)|\,\mathrm{d}t < \epsilon/2$$

Then for all $k \geq 1$,

$$\begin{aligned}\left|\hat{f}(x+h_k)-\hat{f}(x)\right| &= \left|\int_{-\infty}^{\infty} f(t)[e^{-i(x+h_k)t}-e^{-ixt}]\,\mathrm{d}t\right| \\ &\leq \int_{-\infty}^{-A} 2\,|f(t)|\,\mathrm{d}t + \left|\int_{-A}^{A} f(t)e^{-ixt}(e^{-ih_kt}-1)\,\mathrm{d}t\right| \\ &\quad + \int_{A}^{\infty} 2\,|f(t)|\,\mathrm{d}t \\ &< \epsilon + \int_{-A}^{A} |f(t)|\left|e^{-ih_kt}-1\right|\mathrm{d}t\end{aligned}$$

But $\left|e^{-ih_kt}-1\right| \to 0$ uniformly on $[-A, A]$ as $k \to \infty$. So

$$\lim_{k\to\infty}\left|\hat{f}(x+h_k)-\hat{f}(x)\right| \leq 2\epsilon$$

from which it follows that

$$\lim_{k\to\infty}\hat{f}(x+h_k) = f(x)$$

Thus $\hat{f}$ is continuous at x.

□

Definition 4.5.3. Let

i) $\mathcal{C}(\mathbb{R}) = \{f : \mathbb{R} \to \mathbb{C} \mid f \text{ is continuous}\}$

ii) $\mathcal{C}_0(\mathbb{R}) = \{g \in \mathcal{C}(\mathbb{R}) \mid \lim_{|x|\to\infty} g(x) = 0\}$

iii) $\mathcal{C}_c(\mathbb{R}) = \{g \in \mathcal{C}(\mathbb{R}) \mid \text{there exists } A > 0 \text{ such that } g(x) = 0 \text{ if } |x| > A\}$

The following Proposition will be stated without proof.

Proposition 4.5.4. i) $\mathcal{C}_c(\mathbb{R}) \subset \mathcal{C}_0(\mathbb{R}) \subset \mathcal{C}(\mathbb{R})$

ii) $\mathcal{C}_c(\mathbb{R}) \subset \mathcal{R}^1$

iii) $\mathcal{C}_0(\mathbb{R}) \not\subset \mathcal{R}^1$

iv) $\mathcal{R}^1 \cap \mathcal{C}(\mathbb{R}) \not\subset \mathcal{C}_0(\mathbb{R})$

v) $\mathcal{C}_0(\mathbb{R})$ is a Banach space with respect to the sup norm.

vi) $\mathcal{C}_c(\mathbb{R})$ forms a dense subspace of $\mathcal{C}_0(\mathbb{R})$.

vii) If $f \in \mathcal{R}^1$ then $\hat{f} \in \mathcal{C}_0(\mathbb{R})$.

Definition 4.5.5. Given $f \in \mathcal{R}^1$ let $\mathcal{F}f = \hat{f}$. Then $\mathcal{F}$ is a linear transformation from $\mathcal{R}^1$ to $\mathcal{C}_0(\mathbb{R})$.

Proposition 4.5.6. If $f \in \mathcal{R}^1$ then $\|\mathcal{F}f\|_\infty \leq \|f\|_1$ and $\mathcal{F}$ is uniformly continuous from $\mathcal{R}^1$ to $\mathcal{C}_0(\mathbb{R})$.

Proof. If $f \in \mathcal{R}^1$ and $x \in \mathbb{R}$ then

$$\left|\hat{f}(x)\right| = \left|\int_{-\infty}^{\infty} f(t)e^{-ixt}\,\mathrm{d}t\right| \leq \int_{-\infty}^{\infty} |f(t)|\,\mathrm{d}t = \|f\|_1$$

which means $\left\|\hat{f}\right\|_\infty \leq \|f\|_1$. This also implies for all $f, g \in \mathcal{R}^1$

$$\|\mathcal{F}f - \mathcal{F}fg\|_\infty = \|\mathcal{F}(f-g)\|_\infty \leq \|f-g\|_1$$

so that $\mathcal{F}$ is uniformly continuous from $\mathcal{R}^1$ to $\mathcal{C}_0(\mathbb{R})$. □

Example 4.5.7. Let $f(t) = e^{-|t|}$ for $t \in \mathbb{R}$. Show that $\|\mathcal{F}f\|_\infty \leq \|f\|_1$.

Solution. For any $A > 0$ we have

$$\int_0^A |f(t)|\,\mathrm{d}t = \int_0^A e^{-t}\,\mathrm{d}t = 1 - e^{-A}$$

$\to 1$ as $A \to \infty$. Similarly

$$\int_{-A}^0 |f(t)|\,\mathrm{d}t$$

$\to 1$ as $A \to \infty$. Thus $f \in \mathcal{R}^1$ and

$$\|f\|_1 = \int_{-\infty}^{\infty} |f(t)| \,\mathrm{d}t = 2$$

Now let $x \in \mathbb{R}$. Then for $A > 0$

$$\begin{aligned}
\int_0^A f(t)e^{-ixt} \,\mathrm{d}t &= \int_0^A e^{-(1+ix)t} \,\mathrm{d}t \\
&= -\frac{e^{-(1+ix)t}}{1+ix} \Big|_{t=0}^{A} \\
&= \frac{1}{1+ix} \left(1 - e^{-A}e^{-ixA}\right) \\
&\to \frac{1}{1+ix}
\end{aligned}$$

as $A \to \infty$. Similarly,

$$\int_{-\infty}^{0} f(t)e^{-ixt} \,\mathrm{d}t = \frac{1}{1-ix}$$

Thus

$$\mathcal{F}f = \frac{1}{1+ix} + \frac{1}{1-ix} = \frac{2}{1+x^2} \leq 2 = \|f\|_1$$

as required.

Example 4.5.8. Let

$$G(t) = \frac{e^{-t^2/2}}{\sqrt{2\pi}}$$

for $t \in \mathbb{R}$. (This is called the *Gaussian Distribution*.) Find $\hat{G}$.

Solution. It is easily checked that $G \in \mathcal{R}^1 \cap \mathcal{C}_0^{\infty}(\mathbb{R})$. Therefore, by Theorem

1.6.3 and using Integration by Parts, we get

$$\begin{aligned}
\hat{G}'(x) &= \frac{d}{dx}\int_{-\infty}^{\infty} G(t)e^{-ixt}\,\mathrm{d}t \\
&= \int_{-\infty}^{\infty} \frac{\partial}{\partial x}\left(\frac{e^{-t^2/2}}{\sqrt{2\pi}}e^{-ixt}\right)\mathrm{d}t \\
&= \frac{-i}{\sqrt{2\pi}}\int_{-\infty}^{\infty} te^{-t^2/2}e^{-ixt}\,\mathrm{d}t \\
&= \frac{i}{\sqrt{2\pi}}e^{-t^2/2}e^{-ixt}\,\Big|_{t=-\infty}^{\infty} - x\int_{-\infty}^{\infty}\frac{e^{-t^2/2}}{\sqrt{2\pi}}e^{-ixt}\,\mathrm{d}t \\
&= -x\hat{G}(x)
\end{aligned}$$

Thus

$$\hat{G}(x) = Ce^{-x^2/2}$$

But

$$\hat{G}(0) = \int_{-\infty}^{\infty}\frac{e^{-t^2/2}}{\sqrt{2\pi}}\,\mathrm{d}t = 1$$

So $C = 1$ and $\hat{G}(x) = \sqrt{2\pi}G(x), x \in \mathbb{R}$.

Proposition 4.5.9. Suppose $f \in \mathcal{R}^1$ and $0 < \lambda \in \mathbb{R}$. Define $f_\lambda(t) = \lambda f(\lambda t), t \in \mathbb{R}$. Then $f_\lambda \in \mathcal{R}^1$ and $\hat{f}_\lambda(x) = \hat{f}(\frac{x}{\lambda}), x \in \mathbb{R}$.

Proof. For the first part of the proof, note that

$$\begin{aligned}
\int_{-\infty}^{\infty} |f_\lambda(t)|\,\mathrm{d}t &= \lim_{A\to\infty}\int_{-A}^{A}\lambda\,|f(\lambda t)|\,\mathrm{d}t \\
&= \lim_{A\to\infty}\int_{-A/\lambda}^{A/\lambda}|f(s)|\,\mathrm{d}s \quad (s = \lambda t) \\
&= \int_{-\infty}^{\infty}|f(s)|\,\mathrm{d}s \\
&< \infty
\end{aligned}$$

So $f_\lambda \in \mathcal{R}^1$. Next,

$$\begin{aligned}
\hat{f}_\lambda(x) &= \int_{-\infty}^{\infty} f_\lambda(t) e^{-ixt}\,\mathrm{d}t \\
&= \lim_{A\to\infty} \int_{-A}^{A} \lambda f(\lambda t) e^{-ixt}\,\mathrm{d}t \\
&= \lim_{A\to\infty} \int_{-A/\lambda}^{A/\lambda} f(s) e^{-ixs/\lambda}\,\mathrm{d}s \quad (s = \lambda t) \\
&= \int_{-\infty}^{\infty} f(s) e^{-ixs/\lambda}\,\mathrm{d}s \\
&= \hat{f}(x/\lambda)
\end{aligned}$$

This completes the proof. □

Proposition 4.5.10. Suppose $f : \mathbb{R} \to \mathbb{R}$ is even. Then $\hat{f}$ is real valued and

$$\hat{f}(x) = \int_{-\infty}^{\infty} f(t) \cos xt\,\mathrm{d}t$$

for $x \in \mathbb{R}$.

Proof. By definition,

$$\begin{aligned}
\hat{f}(x) &= \int_{-\infty}^{\infty} f(t) e^{-ixt}\,\mathrm{d}t \\
&= \int_{-\infty}^{\infty} f(t) \cos(-xt)\,\mathrm{d}t + i \int_{-\infty}^{\infty} f(t) \sin(-xt)\,\mathrm{d}t \\
&= \int_{-\infty}^{\infty} f(t) \cos(xt)\,\mathrm{d}t
\end{aligned}$$

(since f is even). □

Example 4.5.11. Let

$$f(t) = \begin{cases} 1/2 & -1 \le t \le 1 \\ 0 & \text{otherwise} \end{cases}$$

Find $\hat{f}$ and show that it does not belong to $\mathcal{R}^1$.

Solution. If $x = 0$ then clearly $\hat{f}(x) = 1$. Otherwise,

$$\begin{aligned}
\hat{f}(x) &= \int_{-\infty}^{\infty} f(t)e^{-ixt}\,\mathrm{d}t \\
&= \frac{1}{2}\int_{-1}^{1} e^{-ixt}\,\mathrm{d}t \\
&= \frac{1}{-2ix}e^{-ixt}\Big|_{t=-1}^{1} \\
&= \frac{1}{2ix}\left(e^{ix} - e^{-ix}\right) \\
&= \frac{1}{2ix}\left(2i\sin x\right) \\
&= \frac{\sin x}{x}
\end{aligned}$$

This function is not in $\mathcal{R}^1$ as was shown in Example 1.4.2.

Example 4.5.12. Let

$$f(t) = te^{-|t|}$$

for $t \in \mathbb{R}$. Show that $f \in \mathcal{R}^1$ and find $\hat{f}$.

Solution. We will divide the solution into steps.

Step 1 To show that $f \in \mathcal{R}^1$ proceed as follows. First, if $t > 0$ then

$$f'(t) = \frac{e^t - te^t}{e^{2t}}$$

So f is decreasing along the positive real axis when $t > 1$. Now let $a_n = ne^{-n}, n \in \mathbb{N}$. Then

$$\lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = \frac{1}{e} < 1$$

Hence, by the ratio test,

$$\sum_{n=1}^{\infty} ne^{-n}$$

converges. But since $f(t)$ is decreasing when $t > 1$, this implies by the integral test,

$$\int_1^\infty te^{-t}\,\mathrm{d}t < \infty$$

Also,

$$\begin{aligned}\int_{-\infty}^{-1} |f(t)|\,\mathrm{d}t &= \int_{-\infty}^{-1} (-t)e^t\,\mathrm{d}t \\ &= \int_1^\infty se^{-s}\,\mathrm{d}s \quad (s = -t) \\ &< \infty\end{aligned}$$

Therefore $f \in \mathcal{R}^1$.

Step 2 Let

$$f_1(x) = \int_0^\infty f(t)e^{-ixt}\,\mathrm{d}t$$

Then using integration by parts, we get

$$\begin{aligned}f_1(x) &= \int_0^\infty te^{-t}e^{-ixt}\,\mathrm{d}t \\ &= \int_0^\infty te^{(-1-ix)t}\,\mathrm{d}t \\ &= -t\frac{e^{-(1-ix)t}}{1+ix}\Bigg|_{t=0}^{\infty} + \int_0^\infty e^{(-1-ix)t}\,\mathrm{d}t \\ &= -\frac{e^{(-1-ix)t}}{1+ix}\Bigg|_{t=0}^{\infty} \\ &= \frac{1}{1+ix}\end{aligned}$$

Similarly, if

$$f_2(x) = \int_{-\infty}^0 f(t)e^{-ixt}\,\mathrm{d}t$$

then

$$f_2(x) = -\frac{1}{1-ix}$$

Therefore,

$$\begin{aligned}\hat{f}(x) &= f_1(x) + f_2(x)\\ &= \frac{1}{1+ix} - \frac{1}{1-ix}\\ &= -\frac{2ix}{1+x^2}\end{aligned}$$

Example 4.5.13. For $k \in \mathbb{N}$ let $G_k(t) = t^k G(t), t \in \mathbb{R}$ (where $G(t) = \exp(-t^2/2)/\sqrt{2\pi}$). Show that $G_k \in \mathcal{R}^1 \cap \mathcal{C}^\infty(\mathbb{R})$ and find $\hat{G}_k$ for each $k \in \mathbb{N}$.

Solution. Step 1 For $t > 0$ it is easily checked that

$$G_k'(t) = \sqrt{2\pi}\frac{kt^{k-1}e^{t^2/2} - t^{k+1}e^{t^2/2}}{e^{t^2}}$$

Hence, along the positive real axis, G_k is decreasing when $t > \sqrt{k}$. Now for $k \in \mathbb{N}$ let $a_n = n^k G(n)$. Then

$$\begin{aligned}\lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} &= \lim_{n\to\infty} \frac{(n+1)^k \exp(n^2/2)}{n^k \exp((n+1)^2/2)}\\ &= \lim_{n\to\infty} \frac{1}{\exp((2n+1)/2)}\\ &= e^{-1/2}\\ &< 1\end{aligned}$$

Therefore, by the ratio test,

$$\sum_{n=\lceil\sqrt{k}\rceil}^{\infty} a_n = \sum_{n=\lceil\sqrt{k}\rceil}^{\infty} n^k e^{-n^2/2}/\sqrt{2\pi}$$

converges. But then, by the integral test (since $G_k(t)$ is decreasing when $t > \sqrt{k}$),

$$\int_{\sqrt{k}}^{\infty} |G_k(t)|\,\mathrm{d}t = \int_{\sqrt{k}}^{\infty} \frac{t^k e^{-t^2/2}}{\sqrt{2\pi}}\,\mathrm{d}t$$

converges as well. Thus, if k is even,

$$\begin{aligned}\int_{-\infty}^{-\sqrt{k}} |G_k(t)|\,\mathrm{d}t &= \int_{-\infty}^{-\sqrt{k}} \frac{t^k e^{-t^2/2}}{\sqrt{2\pi}}\,\mathrm{d}t \\ &= \int_{\sqrt{k}}^{\infty} \frac{s^k e^{-s^2/2}}{\sqrt{2\pi}}\,\mathrm{d}s \quad (s=-t) \\ &< \infty\end{aligned}$$

If k is odd,

$$\begin{aligned}\int_{-\infty}^{-\sqrt{k}} |G_k(t)|\,\mathrm{d}t &= \int_{-\infty}^{-\sqrt{k}} \frac{-t^k e^{-t^2/2}}{\sqrt{2\pi}}\,\mathrm{d}t \\ &= \int_{\sqrt{k}}^{\infty} \frac{s^k e^{-s^2/2}}{\sqrt{2\pi}}\,\mathrm{d}s \quad (s=-t) \\ &< \infty\end{aligned}$$

as well. Therefore, in all cases, we have $G_k \in \mathcal{R}^1$. Clearly, G_k belongs to $\mathcal{C}^\infty(\mathbb{R})$. So we have proved the first assertion.

Step 2

4.6 The Fourier Integral Theorem

In this section we present two versions of the Fourier Integral Theorem.

Lemma 4.6.1. Suppose $f \in \mathcal{R}^1$, $a \in \mathbb{R}$, $g(t) = f(t+a)$ and $h(t) = f(-t)$ for $t \in \mathbb{R}$. Then $g, h \in \mathcal{R}^1$,

$$\int_{-\infty}^{\infty} g(t)\,\mathrm{d}t = \int_{-\infty}^{\infty} h(t)\,\mathrm{d}t = \int_{-\infty}^{\infty} f(t)\,\mathrm{d}t$$

and $\|g\|_1 = \|h\|_1 = \|f\|_1$.

Proof. If $A > 0$ then

$$\begin{aligned}\int_{-A}^{A} |g(t)|\,\mathrm{d}t &= \int_{-A}^{A} |f(t+a)|\,\mathrm{d}t \\ &= \int_{a-A}^{a+A} |f(s)|\,\mathrm{d}s \quad (s = t+a) \\ &\to \int_{-\infty}^{\infty} |f(t)|\,\mathrm{d}t \\ &= \|f\|_1\end{aligned}$$

Hence $g \in \mathcal{R}^1$ and $\|g\|_1 = \|f\|_1$. A similar argument shows that

$$\int_{-A}^{A} g(t)\,\mathrm{d}t \to \int_{-\infty}^{\infty} f(t)\,\mathrm{d}t$$

as $A \to \infty$ and also that $h \in \mathcal{R}^1$, $\|h\|_1 = \|f\|_1$ and

$$\int_{-\infty}^{\infty} h(t)\,\mathrm{d}t = \int_{-\infty}^{\infty} f(t)\,\mathrm{d}t$$

□

Theorem 4.6.2 (Fourier Integral Theorem. Preliminary Version)**.** Suppose $f \in \mathcal{R}^1$ and for some $x \in \mathbb{R}$ and $\delta > 0$ the following all exist

i) $f(x+) = \lim_{y \to x^+} f(y)$

ii) $f(x-) = \lim_{y \to x^-} f(y)$

iii)
$$\int_0^{\delta} \frac{f(x+t) - f(x+)}{t}\,\mathrm{d}t$$

iv)
$$\int_0^{\delta} \frac{f(x-t) - f(x-)}{t}\,\mathrm{d}t$$

(These conditions can be met, for example, if f has left and right derivatives at x.) Then

$$\frac{f(x+)+f(x-)}{2} = \lim_{\lambda\to\infty} \frac{1}{\pi} \int_{-\infty}^{\infty} f(u) \left(\int_0^{\lambda} \cos v(u-x)\, \mathrm{d}v \right) \mathrm{d}u$$

Proof. By Theorem 1.3.1,

$$\lim_{\lambda\to\infty} \int_{-\infty}^{-\delta} f(x+t) \frac{\sin \lambda t}{t}\, \mathrm{d}t = 0$$

and

$$\lim_{\lambda\to\infty} \int_{\delta}^{\infty} f(x+t) \frac{\sin \lambda t}{t}\, \mathrm{d}t = 0$$

Also, by Corollary 2.6.3,

$$\lim_{\lambda\to\infty} \int_0^{\delta} f(x+t) \frac{\sin \lambda t}{t}\, \mathrm{d}t = \frac{\pi}{2} f(x+)$$

and

$$\lim_{\lambda\to\infty} \int_{-\delta}^{0} f(x+t) \frac{\sin \lambda t}{t}\, \mathrm{d}t = \frac{\pi}{2} f(x-)$$

Hence

$$\begin{aligned}
\lim_{\lambda\to\infty} \frac{1}{\pi} \int_{-\infty}^{\infty} f(x+t) \frac{\sin \lambda t}{t}\, \mathrm{d}t &= \lim_{\lambda\to\infty} \frac{1}{\pi} \int_{-\infty}^{-\delta} f(x+t) \frac{\sin \lambda t}{t}\, \mathrm{d}t \\
&\quad + \lim_{\lambda\to\infty} \frac{1}{\pi} \int_{-\delta}^{0} f(x+t) \frac{\sin \lambda t}{t}\, \mathrm{d}t \\
&\quad + \lim_{\lambda\to\infty} \frac{1}{\pi} \int_{0}^{\delta} f(x+t) \frac{\sin \lambda t}{t}\, \mathrm{d}t \\
&\quad + \lim_{\lambda\to\infty} \frac{1}{\pi} \int_{\delta}^{\infty} f(x+t) \frac{\sin \lambda t}{t}\, \mathrm{d}t \\
&= \frac{f(x+)+f(x-)}{2}
\end{aligned}$$

But using Lemma 4.6.1

$$\begin{aligned}
\int_{-\infty}^{\infty} f(x+t) \frac{\sin \lambda t}{t}\, \mathrm{d}t &= \int_{-\infty}^{\infty} f(u) \frac{\sin \lambda(u-x)}{u-x}\, \mathrm{d}u \quad (u = x+t) \\
&= \int_{-\infty}^{\infty} f(u) \left(\int_0^{\lambda} \cos v(u-x)\, \mathrm{d}v \right) \mathrm{d}u
\end{aligned}$$

and so

$$\frac{f(x+) + f(x-)}{2} = \lim_{\lambda \to \infty} \frac{1}{\pi} \int_{-\infty}^{\infty} f(u) \left(\int_0^{\lambda} \cos v(u - x) \, \mathrm{d}v \right) \mathrm{d}u$$

as required. $\square$

Theorem 4.6.3 (Fourier Integral Theorem)**.** Suppose $f \in \mathcal{R}^1$, f is continuous, $x \in \mathbb{R}$ and for some $\delta > 0$ the following integrals exist.

$$\int_0^{\delta} \frac{f(x+t) - f(x)}{t} \, \mathrm{d}t$$

and

$$\int_0^{\delta} \frac{f(x-t) - f(x)}{t} \, \mathrm{d}t$$

(Again, these integrals exist if, for example, f has left and right derivatives at x.) Then

$$\begin{aligned} f(x) &= \lim_{\lambda \to \infty} \int_{-\lambda}^{\lambda} \left(\int_{-\infty}^{\infty} f(u) e^{ivu} \, \mathrm{d}u \right) e^{-ixv} \, \mathrm{d}v \\ &= \lim_{\lambda \to \infty} \int_{-\lambda}^{\lambda} \hat{f}(-v) e^{-ixv} \, \mathrm{d}v \end{aligned}$$

Proof. Fix $\lambda > 0$. By Theorem 1.5.2,

$$\begin{aligned}
\frac{1}{\pi}\int_{-\infty}^{\infty} f(u)\left(\int_0^{\lambda}\cos v(u-x)\,\mathrm{d}v\right)\mathrm{d}u &= \frac{1}{\pi}\int_0^{\lambda}\left(\int_{-\infty}^{\infty} f(u)\cos v(u-x)\,\mathrm{d}u\right)\mathrm{d}v \\
&= \frac{1}{2\pi}\int_0^{\lambda}\left(\int_{-\infty}^{\infty} f(u)(e^{iv(u-x)}+e^{-iv(u-x)})\,\mathrm{d}u\right)\mathrm{d}v \\
&= \frac{1}{2\pi}\int_0^{\lambda}\left(\int_{-\infty}^{\infty} f(u)e^{ivu}\,\mathrm{d}u\right)e^{-ivx}\,\mathrm{d}v \\
&\quad + \frac{1}{2\pi}\int_0^{\lambda}\left(\int_{-\infty}^{\infty} f(u)e^{-ivu}\,\mathrm{d}u\right)e^{ivx}\,\mathrm{d}v \\
&= \frac{1}{2\pi}\int_0^{\lambda}\hat{f}(-v)e^{-ivx}\,\mathrm{d}v \\
&\quad + \frac{-1}{2\pi}\int_0^{-\lambda}\left(\int_{-\infty}^{\infty} f(u)e^{iwu}\,\mathrm{d}u\right)e^{-iwx}\,\mathrm{d}w \quad (w=-v) \\
&= \frac{1}{2\pi}\int_0^{\lambda}\hat{f}(-v)e^{-ivx}\,\mathrm{d}v \\
&\quad + \frac{1}{2\pi}\int_{-\lambda}^{0}\hat{f}(-w)e^{-iwx}\,\mathrm{d}w \\
&= \frac{1}{2\pi}\int_{-\lambda}^{\lambda}\hat{f}(-v)e^{-ivx}\,\mathrm{d}v
\end{aligned}$$

But from Theorem 4.6.2

$$f(x) = \lim_{\lambda\to\infty}\frac{1}{\pi}\int_{-\infty}^{\infty} f(u)\left(\int_0^{\lambda}\cos v(u-x)\,\mathrm{d}v\right)\mathrm{d}u$$

and so

$$f(x) = \lim_{\lambda\to\infty}\frac{1}{2\pi}\int_{-\lambda}^{\lambda}\hat{f}(-v)e^{-ivx}\,\mathrm{d}v$$

□

Definition 4.6.4. For $f \in \mathcal{R}^1$ let $\check{f}(x) = \hat{f}(-x)$ for all $x \in \mathbb{R}$ and define $\mathcal{F}^- : \mathcal{R}^1 \to \mathcal{C}_0(\mathbb{R})$ by $\mathcal{F}^- f = \check{f}$.

Corollary 4.6.5. i) $\mathcal{F}^-$ is a linear map from $\mathcal{R}^1$ to $\mathcal{C}_0(\mathbb{R})$ and $\|\mathcal{F}^- f\|_\infty \leq \|f\|_1$ for all $f \in \mathcal{R}^1$.

ii) If $f \in \mathcal{R}^1 \cap \mathcal{C}_0(\mathbb{R})$ and $\hat{f} \in \mathcal{R}^1$ then

$$f = \frac{1}{2\pi}\mathcal{F}(\mathcal{F}^- f) = \frac{1}{2\pi}\mathcal{F}^-(\mathcal{F} f)$$

Proof. i) Clearly $\mathcal{F}^-$ is linear and the second assertion follows from Proposition 4.5.6.

ii) This result follows from Theorem 4.6.3.

□

Chapter 5

Convolutions

5.1 Properties of Convolutions

In this section we define the convolution of two functions and present some relevant results.

Definition 5.1.1. If $f : \mathbb{R} \to \mathbb{C}$ and $x \in \mathbb{R}$, define $\tau_x f : \mathbb{R} \to \mathbb{C}$ by

$$(\tau_x f)(t) = f(t - x)$$

for $t \in \mathbb{R}$.

The following lemmas will be stated without proof.

Lemma 5.1.2. Suppose $x \in \mathbb{R}$, V is one of $\mathcal{C}_0(\mathbb{R}), \mathcal{C}_c(\mathbb{R})$ or $\mathcal{C}^p(\mathbb{R})$ and $f \in V$. Then $\tau_x f \in V$ and moreover the map $f \mapsto \tau_x f$ is a vector space isomorphism between V and itself.

Lemma 5.1.3. i) For each $h \in \mathbb{R}$ the restriction of τ_h to $\mathcal{R}^1$ is a norm-preserving, isomorphism of $\mathcal{R}^1$ onto itself. When we say norm-preserving, we mean that $\|f - g\|_1 = \|\tau_h f - \tau_h g\|_1$ for all $f, g \in \mathcal{R}^1$. Moreover,

$$\int_{-\infty}^{\infty} (\tau_h f)(t)\, \mathrm{d}t = \int_{-\infty}^{\infty} f(t)\, \mathrm{d}t$$

ii) For each $h \in \mathbb{R}$ the restriction of τ_h to $\mathcal{C}_0(\mathbb{R})$ is a norm-preserving, isomorphism of $\mathcal{C}_0(\mathbb{R})$ onto itself. That is,

$$\|\tau_h f - \tau_h g\|_\infty = \|f - g\|_\infty$$

In fact,

$$\widehat{\tau_h f}(x) = e^{-ihx}\hat{f}(x)$$

Lemma 5.1.4. Suppose V is one of $\mathcal{C}_0(\mathbb{R})$ or $\mathcal{R}^1$ (with their natural norms). Then the mapping

$$x \mapsto \tau_x\Big|_V$$

is a group homomorphism from $\mathbb{R}$ under addition to the group of linear operators on V under composition.

Definition 5.1.5. If $f : \mathbb{R} \to \mathbb{C}$, define $f^- : \mathbb{R} \to \mathbb{C}$ by $f^-(t) = f(-t)$ and write $If = f^-$.

Lemma 5.1.6. i) If V is either $\mathcal{R}^1$ or $\mathcal{C}_0(\mathbb{R})$ then the restriction of I to V is a norm-preserving isomorphism of V onto itself.

ii) If $f \in \mathcal{R}^1$ then

$$\widehat{If}(x) = \hat{f}(-x) = (I\hat{f})(x) = \check{f}(x)$$

Definition 5.1.7. Let $\mathbb{R}_{loc}$ be the set of all functions from $\mathbb{R}$ to $\mathbb{C}$ which are Riemann integrable on $[a, b]$ whenever $a, b \in \mathbb{R}$ and $a < b$. Note that $\mathbb{R}_{loc} = \mathcal{C}(\mathbb{R}) \cup \mathcal{R}^1$. Let $\mathbb{R}_c$ be the subset of functions in $\mathbb{R}_{loc}$ which vanish outside a compact set. Note that $\mathbb{R}_c \in \mathcal{R}^1$.

Definition 5.1.8. Suppose $f, g \in \mathbb{R}_{loc}$ and that, for all $x \in \mathbb{R}$, the map

$$t \mapsto f(t)g(x - t)$$

belongs to $\mathcal{R}^1$. Then we say f *is convolvable with* g and we define $f * g : \mathbb{R} \to \mathbb{C}$ by

$$(f * g)(x) = \int_{-\infty}^{\infty} f(t)g(x-t)\,\mathrm{d}t$$

for all $x \in \mathbb{R}$. We call $f * g$ the *convolution of* f *with* g.

Proposition 5.1.9. Suppose $f, g \in \mathbb{R}_{loc}$ and $f * g$ is defined. Then $g * f$ is defined and the two functions are equal.

Proof. Let $x \in \mathbb{R}$ and $A > 0$. Then

$$\begin{aligned}\int_{-A}^{A} |g(t)f(x-t)|\,\mathrm{d}t &= -\int_{x+A}^{x-A} |g(x-s)f(s)|\,\mathrm{d}s \ \ (s = x-t) \\ &= \int_{x-A}^{x+A} |f(s)g(x-s)|\,\mathrm{d}s \\ &\le \int_{-\infty}^{\infty} |f(t)g(x-t)|\,\mathrm{d}t \\ &< \infty\end{aligned}$$

Hence the map $t \mapsto g(t)f(x-t)$ belongs to $\mathcal{R}^1$ and

$$\int_{-\infty}^{\infty} |g(t)f(x-t)|\,\mathrm{d}t \le \int_{-\infty}^{\infty} |f(t)g(x-t)|\,\mathrm{d}t$$

Similarly,

$$\lim_{A\to\infty} \int_{-A}^{A} g(t)f(x-t)\,\mathrm{d}t = \int_{-\infty}^{\infty} f(t)g(x-t)\,\mathrm{d}t$$

ie. $(g * f)(x) = (f * g)(x)$ for all $x \in \mathbb{R}$. □

Lemma 5.1.10. If $f_1, f_2, g \in \mathbb{R}_{loc}, \alpha \in \mathbb{C}$ and both $f_1 * g$ and $f_2 * g$ are defined then $(\alpha f_1 + f_2) * g$ is defined and $(\alpha f_1 + f_2) * g = \alpha f_1 * g + f_2 * g$.

Proof. Trivial. □

Lemma 5.1.11. If $f, g \in \mathbb{R}_{loc}, f * g$ is defined and $h \in \mathbb{R}$ then $(\tau_h f) * g$ is defined and $(\tau_h f) * g = \tau_h(f * g)$.

Proof. For any $A > 0$,

$$\begin{aligned}
\int_{-A}^{A} (\tau_h f)(t) g(x-t)\,\mathrm{d}t &= \int_{-A}^{A} f(t-h) g(x-t)\,\mathrm{d}t \\
&= \int_{-A-h}^{A-h} f(s) g(x-s-h)\,\mathrm{d}s \quad (s = t-h) \\
&= \int_{-A-h}^{A-h} f(s) g((x-h)-s)\,\mathrm{d}s \\
&\to (f * g)(x-h) \quad \text{as } A \to \infty \\
&= \tau_h((f * g)(x))
\end{aligned}$$

□

Lemma 5.1.12. If $f \in \mathbb{R}_c$ and $g \in \mathbb{R}_{loc}$ then $f * g$ is defined. Moreover, if either $f \in \mathcal{C}(\mathbb{R})$ or $g \in \mathcal{C}(\mathbb{R})$ then $f * g \in \mathcal{C}(\mathbb{R})$ as well.

Proof. We will divide the proof into steps.

i) Choose $a > 0$ such that $f(t) = 0$ for $|t| > a$. Then for $a \leq A \in \mathbb{R}$ and $x \in \mathbb{R}$,
$$\int_{-A}^{A} f(t) g(x-t)\,\mathrm{d}t = \int_{-a}^{a} f(t) g(x-t)\,\mathrm{d}t$$
So $f * g$ is defined and
$$(f * g)(x) = \int_{-a}^{a} f(t) g(x-t)\,\mathrm{d}t$$
for all $x \in \mathbb{R}$.

ii) Now notice that for all $x \in \mathbb{R}$,
$$\begin{aligned}
|(f * g)(x)| &\leq \int_{-a}^{a} |f(t)|\,|g(x-t)|\,\mathrm{d}t \\
&\leq \|f\|_\infty \int_{x-a}^{x+a} |g(s)|\,\mathrm{d}s \quad (s = x-t)
\end{aligned}$$

iii) Suppose $f \in \mathcal{C}(\mathbb{R})$. If $h \in \mathbb{R}$ and $|h| < 1$ then for all $x \in \mathbb{R}$,

$$\begin{aligned} |(f * g)(x-h) - (f * g)(x)| &= |\tau_h(f * g)(x) - (f * g)(x)| \\ &= |((\tau_h f - f) * g)(x)| \\ &\le \|\tau_h f - f\|_\infty \int_{x-a-1}^{x+a+1} |g(t)| \, \mathrm{d}t \ \text{ by Step 2} \\ &\to 0 \ \text{ as } h \to 0 \end{aligned}$$

(for all fixed $x \in \mathbb{R}$) because f is uniformly continuous on $[x-1, x+1]$.

□

Lemma 5.1.13. Suppose $f \in \mathbb{R}_{loc} \cap B(\mathbb{R})$ (where $B(\mathbb{R})$ is the set of bounded functions on $\mathbb{R}$). If $g \in \mathcal{R}^1$ then $f * g$ is defined and belongs to $B(\mathbb{R})$. Moreover, $\|f * g\|_\infty \le \|f\|_\infty \|g\|_1$.

Proof. This follows from the proof of Lemma 5.1.12. □

Lemma 5.1.14. If $f \in \mathcal{C}(\mathbb{R})$ and $g \in \mathcal{C}_c^p(\mathbb{R})$ for some $p \in \mathbb{N}$ then $f * g \in \mathcal{C}^p(\mathbb{R})$ and

$$(f * g)^{(k)}(x) = (f * g^{(k)})x$$

for all $x \in \mathbb{R}$, $1 \le k \le p$.

Proof. Let $h(x) = (f * g)(x)$ for $x \in \mathbb{R}$. Choose $a > 0$ such that $g(t) = 0$ for $|t| \ge a$. Then

$$h(x) = \int_{x-a}^{x+a} f(t)g(x-t) \, \mathrm{d}t$$

For $b > 0, -b < x < b$ we have

$$h(x) = \int_{-b-a}^{b+a} f(t)g(x-t) \, \mathrm{d}t$$

If we let

$$F(x,t) = f(t)g(x-t)$$

for $|t| \leq a+b$ and $-b < x < b$ then we may apply Theorem 1.6.1 to conclude that $h \in \mathcal{C}^1$ on $(-b, b)$ and

$$\begin{aligned} h'(x) &= \int_{-b-a}^{b+a} f(t)g'(x-t)\,\mathrm{d}t \\ &= \int_{-\infty}^{\infty} f(t)g'(x-t)\,\mathrm{d}t \end{aligned}$$

It follows that $h \in \mathcal{C}^1(\mathbb{R})$ and by induction that $h \in \mathcal{C}^p(\mathbb{R})$ and that $h^{(k)} = f * g^{(k)}$ for $1 \leq k \leq p$. □

Lemma 5.1.15. If $f \in \mathcal{C}^p(\mathbb{R})$ for some $p \in \mathbb{N}$ and $g \in \mathcal{C}_c(\mathbb{R})$ then $f * g \in \mathcal{C}^p(\mathbb{R})$ and

$$(f * g)^{(k)} = f^{(k)} * g$$

for $1 \leq k \leq p$.

Proof. The proof is similar to that used in Lemma 5.1.14. □

5.2 Theorem of Approximation by Convolution

In this section we present the Theorem of Approximation by Convolution and two of its consequences. One of which involves the heat problem and another providing an alternate proof of the Weierstrass Approximation Theorem. But before presenting any of these results, we prove the following.

Theorem 5.2.1 (Convolution Theorem for the Fourier Transform)**.** Suppose $f, g \in \mathcal{R}^1 \cap \mathcal{C}_0(\mathbb{R})$. Then $f * g \in \mathcal{R}^1 \cap \mathcal{C}_0(\mathbb{R})$, $\widehat{f * g} = \hat{f}\hat{g}$ and $\|f * g\|_1 \leq \|f\|_1 \|g\|_1$.

Proof. We will divide the proof into steps.

i) For $k \in \mathbb{N}$ choose $\gamma_k \in \mathcal{C}_0(\mathbb{R})$ such that $0 \le \gamma_k(t) \le 1$, $\gamma_k(t) = 1$ if $-k \le t \le k$ and $\gamma_k(t) = 0$ if $|t| \ge k+1$. Now let $g_k = \gamma_k g$ for $k \in \mathbb{N}$. Then for $x \in \mathbb{R}$,

$$\begin{aligned}
|(f * g)(x) - (f * g_k)(x)| &= |(f * (g - g_k))(x)| \\
&\le \|f\|_\infty \|g - g_k\|_1 \\
&\quad \text{(by Step 2 of the proof of Lemma 5.1.12)} \\
&\le \|f\|_\infty \left(2\int_{-\infty}^{-k} |g(t)|\,\mathrm{d}t + 2\int_k^\infty |g(t)|\,\mathrm{d}t \right)
\end{aligned}$$

It follows that $\{f * g_k\}_{k=1}^\infty$ converges to $f * g$ uniformly on $\mathbb{R}$ and hence that $f * g$ lies in $\mathcal{C}_0(\mathbb{R})$.

ii) For any $A > 0$,

$$\begin{aligned}
\int_{-A}^{A} |(f * g)(x)|\,\mathrm{d}x &\le \int_{-A}^{A} \left(\int_{-\infty}^{\infty} |f(t)|\,|g(x-t)|\,\mathrm{d}t \right) \mathrm{d}x \\
&= \int_{-\infty}^{\infty} \left(\int_{-A}^{A} |f(t)|\,|g(x-t)|\,\mathrm{d}x \right) \mathrm{d}t \quad \text{(by Theorem 1.5.2)} \\
&= \int_{-\infty}^{\infty} |f(t)| \left(\int_{-A}^{A} |g(x-t)|\,\mathrm{d}x \right) \mathrm{d}t \\
&\le \int_{-\infty}^{\infty} |f(t)|\,\|g\|_1\,\mathrm{d}t \\
&= \|f\|_1\,\|g\|_1
\end{aligned}$$

Hence $f * g \in \mathcal{R}^1$ and $\|f * g\|_1 \le \|f\|_1 \|g\|_1$. A similar argument shows that

$$\int_{-\infty}^{\infty} (f * g)(x)\,\mathrm{d}x = \int_{-\infty}^{\infty} f(t)\,\mathrm{d}t \int_{-\infty}^{\infty} g(t)\,\mathrm{d}t$$

iii) For any $y \in \mathbb{R}$,

$$\begin{aligned}
\widehat{(f * g)}(y) &= \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} f(t) g(x-t)\,\mathrm{d}t \right) e^{-iyx}\,\mathrm{d}x \\
&= \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} f(t) e^{-iyt} g(x-t) e^{-iy(x-t)}\,\mathrm{d}t \right) \mathrm{d}x \\
&= \int_{-\infty}^{\infty} (f_y * g_y)(x)\,\mathrm{d}x
\end{aligned}$$

where $f_y(t) = f(t)e^{-iyt}$ and $g_y(t) = g(t)e^{-iyt}$ for all $t \in \mathbb{R}$. By the remarks in Step 1, we get

$$\begin{aligned}\widehat{(f * g)}(y) &= \int_{-\infty}^{\infty} (f_y * g_y)(x)\,\mathrm{d}x \\ &= \int_{-\infty}^{\infty} f(t)e^{-iyt}\,\mathrm{d}t \int_{-\infty}^{\infty} g(s)e^{-iys}\,\mathrm{d}s \\ &= \hat{f}(y)\hat{g}(y)\end{aligned}$$

□

Theorem 5.2.2. Suppose $f \in \mathcal{R}^1 \cap \mathcal{C}_0(\mathbb{R}) \cap \mathcal{C}^1(\mathbb{R})$ and $f' \in \mathcal{R}^1$. Then $\widehat{f'}(x) = ix\hat{f}(x)$.

Proof. For all $x \in \mathbb{R}$,

$$\begin{aligned}\widehat{f'}(x) &= \int_{-\infty}^{\infty} f'(t)e^{-ixt}\,\mathrm{d}t \\ &= f(t)e^{-ixt}\Big|_{t=-\infty}^{\infty} + ix\int_{-\infty}^{\infty} f(t)e^{-ixt}\,\mathrm{d}t \\ &= ix\hat{f}(x)\end{aligned}$$

□

Theorem 5.2.3. Suppose $g \in \mathcal{R}^1$, $0 \le g(t) \in \mathbb{R}$ for all $t \in \mathbb{R}$ and

$$\int_{-\infty}^{\infty} g(t)\,\mathrm{d}t = 1$$

For $\lambda > 0$ let $H_\lambda g(t) = \lambda g(\lambda t)$ for all $t \in \mathbb{R}$. If $f : \mathbb{R} \to \mathbb{C}$ and f is bounded and uniformly continuous on $\mathbb{R}$ then

$$\lim_{\lambda\to\infty} (f * H_\lambda g)(x) = f(x) \ \ \text{(uniformly)}$$

Proof. Choose $M > 0$ such that $|f(t)| < M$ for all $t \in \mathbb{R}$. Let $\epsilon > 0$. Choose $\delta > 0$ such that $|f(s) - f(t)| < \epsilon/2$ whenever $|s - t| \leq \delta$. Choose $\lambda_0 > 0$ such that

$$\int_{-\infty}^{\infty} g(s)\,\mathrm{d}s - \int_{-\lambda\delta}^{\lambda\delta} g(s)\,\mathrm{d}s < \frac{\epsilon}{4M}$$

whenever $0 < \lambda_0 < \lambda \in \mathbb{R}$. Note that

$$\int_{-\infty}^{\infty} H_\lambda g(t)\,\mathrm{d}t = \int_{-\infty}^{\infty} g(s)\,\mathrm{d}s$$

for all $\lambda > 0$. Then for all $x \in \mathbb{R}$ and $\lambda > \lambda_0$,

$$\begin{aligned}
|(f * H_\lambda g)(x) - f(x)| &= \left| \int_{-\infty}^{\infty} f(x-t)\lambda g(\lambda t)\,\mathrm{d}t - f(x) \int_{-\infty}^{\infty} \lambda g(\lambda t)\,\mathrm{d}t \right| \\
&= \left| \int_{-\infty}^{\infty} [f(x-t) - f(x)]\lambda g(\lambda t)\,\mathrm{d}t \right| \\
&\leq \int_{-\infty}^{-\delta} 2M\lambda g(\lambda t)\,\mathrm{d}t + \int_{-\delta}^{\delta} \frac{\epsilon}{2}\lambda g(\lambda t)\,\mathrm{d}t + \int_{\delta}^{\infty} 2M\lambda g(\lambda t)\,\mathrm{d}t \\
&= 2M\left(\int_{-\infty}^{-\delta} \lambda g(\lambda t)\,\mathrm{d}t + \int_{\delta}^{\infty} \lambda g(\lambda t)\,\mathrm{d}t \right) + \frac{\epsilon}{2} \int_{-\infty}^{\infty} \lambda g(\lambda t)\,\mathrm{d}t \\
&= 2M\left(\int_{-\infty}^{\infty} \lambda g(\lambda t)\,\mathrm{d}t - \int_{-\lambda\delta}^{\lambda\delta} \lambda g(\lambda t)\,\mathrm{d}t \right) + \frac{\epsilon}{2} \int_{-\infty}^{\infty} \lambda g(\lambda t)\,\mathrm{d}t \\
&< 2M\left(\frac{\epsilon}{4M}\right) + \frac{\epsilon}{2} \\
&= \epsilon
\end{aligned}$$

□

Theorem 5.2.4 (Heat Problem. Revisited)**.** Suppose $f : \mathbb{R} \to \mathbb{C}$ is bounded and uniformly continuous on $\mathbb{R}$. Define $u(x,t) : \mathbb{R} \times [0,\infty) \to \mathbb{C}$ by

$$u(x,t) = \begin{cases} \dfrac{1}{\sqrt{2\pi t}} \displaystyle\int_{-\infty}^{\infty} f(y) \exp\left(-\frac{(x-y)^2}{2t}\right)\mathrm{d}y & x \in \mathbb{R}, t > 0 \\ f(x) & x \in \mathbb{R}, t = 0 \end{cases}$$

Then

i) u is continuous on its domain.

ii) $u \in \mathcal{C}^\infty$ on $\mathbb{R} \times (0, \infty)$.

iii) $\dfrac{\partial u}{\partial t}(x,t) = \dfrac{\partial^2 u}{\partial x^2}(x,t)$ for all $x \in \mathbb{R}, t > 0$.

iv) $u(x,0) = f(x)$.

Proof. Statement iv) is obvious and statements ii) and iii) can be easily proven by using Theorem 1.6.3. For statement i), note that

$$u(x,t) = (f * H_{1/\sqrt{t}}G)(x)$$

for $x \in \mathbb{R}, t > 0$, where G is the Gaussian distribution and is defined by

$$G(t) = \frac{1}{\sqrt{2\pi}} e^{-t^2/2}$$

By Theorem 5.2.3,

$$\lim_{t \to 0} u(x,t) = \lim_{t \to 0} (f * H_{1/\sqrt{t}}G)(x) = f(x)$$

uniformly for all $x \in \mathbb{R}$. Hence u is continuous on its entire domain. □

Theorem 5.2.5 (Weierstrass Approximation Theorem Revisited)**.** Suppose $[a,b] \subset \mathbb{R}, \phi : [a,b] \to \mathbb{R}$ and ϕ is continuous. Then for all $\epsilon > 0$ there exists a polynomial function $P : \mathbb{R} \to \mathbb{R}$ such that $|\phi(x) - P(x)| < \epsilon$ for all $x \in [a,b]$.

Proof. Choose $A > 0$ such that $[a,b] \subset (-A, A)$ and choose a continuous function $f : \mathbb{R} \to \mathbb{R}$ which agrees with ϕ on $[a,b]$ and vanishes outside $(-A, A)$. Fix $\epsilon > 0$. By Theorem 5.2.3, we can choose $t > 0$ such that

$$\Big| f(x) - (f * H_{1/\sqrt{t}}G)(x) \Big| < \epsilon/2$$

for all $x \in \mathbb{R}$. Observe that

$$(f * H_{1/\sqrt{t}}G)(x) = \frac{1}{\sqrt{2\pi t}} \int_{-\infty}^{\infty} f(y) \exp\left(-\frac{(x-y)^2}{2t} \right) \mathrm{d}y$$

for all $x \in \mathbb{R}$. Let

$$F(z) = \frac{1}{\sqrt{2\pi t}} \int_{-\infty}^{\infty} f(y) \exp\left(-\frac{(z-y)^2}{2t} \right) \mathrm{d}y$$

for $z \in \mathbb{C}$. It follows from Theorem 1.5.1 that F is continuous. Now if $\gamma : [\alpha, \beta] \to \mathbb{C}$ is a closed, $\mathcal{C}^1$ curve then

$$\begin{aligned}
\int_\gamma F(z)\,\mathrm{d}z &= \int_\alpha^\beta F(\gamma(s))\gamma'(s)\,\mathrm{d}s \\
&= \int_\alpha^\beta \left(\frac{1}{\sqrt{2\pi t}} \int_{-A}^{A} f(y) \exp\left(-\frac{(\gamma(s)-y)^2}{2t} \right) \mathrm{d}y \right) \gamma'(s)\,\mathrm{d}s \\
&= \frac{1}{\sqrt{2\pi t}} \int_{-A}^{A} \left(\int_\alpha^\beta \exp\left(-\frac{(\gamma(s)-y)^2}{2t} \right) \gamma'(s)\,\mathrm{d}s \right) f(y)\,\mathrm{d}y \\
&= \frac{1}{\sqrt{2\pi t}} \int_{-A}^{A} \left(\int_\gamma \exp\left(-\frac{(z-y)^2}{2t} \right) \mathrm{d}z \right) f(y)\,\mathrm{d}y \\
&= 0
\end{aligned}$$

by Cauchy's Theorem. Thus

$$\int_\gamma F(z)\,\mathrm{d}z = 0$$

for every closed, $\mathcal{C}^1$ curve γ in $\mathbb{C}$. By Morera's Theorem, F is entire and so its Taylor series converges to it uniformly on the set $\{z \in \mathbb{C} \ : \ |z| \leq A\}$. It follows that there exists a polynomial function $P_1 : \mathbb{C} \to \mathbb{C}$ such that $|F(z) - P_1(z)| < \epsilon/2$ for all $z \in \mathbb{C}$, $|z| \leq A$. Since F is real valued on the real line, its derivatives at zero are real and so P_1 is real valued on the reals as well. Let $P = P_1|_{\mathbb{R}}$. Then

$$\begin{aligned}
|f(x) - P(x)| &\leq |f(x) - F(x)| + |F(x) - P(x)| \\
&< \frac{\epsilon}{2} + \frac{\epsilon}{2} \\
&= \epsilon
\end{aligned}$$

for all $x \in [-A, A]$. $\square$

5.3 The Schwarz Space

Here we present the Schwarz Space and prove several relevant results.

Definition 5.3.1. Let $\mathbb{S}$, the *Schwarz Space*, be the set of all $\mathcal{C}^\infty$ functions from $\mathbb{R}$ to $\mathbb{C}$ having the property that for every $k, m \in \mathbb{Z}, k \geq 0, m \geq 0$,

$$\lim_{|x|\to\infty} \left| x^m f^{(k)}(x) \right| = 0$$

Proposition 5.3.2. i) If G is the Gaussian Distribution then $G \in \mathbb{S}$.

ii) $\mathbb{S} \subset \mathcal{C}^\infty(\mathbb{R}) \cap \mathcal{C}_0(\mathbb{R})$.

iii) $\mathbb{S}$ is a complex, commutative algebra (lacking identity) with respect to pointwise addition and multiplication.

iv) If, for $f \in \mathbb{S}$, we define $f^- : \mathbb{R} \to \mathbb{C}$ by $f^-(x) = f(-x)$ then $f^- \in \mathbb{S}$.

v) If, for $f \in \mathbb{S}$, we define $\overline{f} : \mathbb{R} \to \mathbb{C}$ by $\overline{f}(x) = \overline{f(x)}$ then $\overline{f} \in \mathbb{S}$.

Proof. The proof of all parts is trivial. $\square$

Lemma 5.3.3. If $f \in \mathbb{S}$ then

i) $f' \in \mathbb{S}$.

ii) $Pf \in \mathbb{S}$ where $P : \mathbb{R} \to \mathbb{C}$ is any polynomial.

iii) $f \in \mathcal{R}^1$.

Proof. The proof of i) and ii) is trivial. For iii), if $f \in \mathbb{S}$ then $|(1 + x^2)f(x)| \leq |f(x)| + |x^2 f(x)|$ for all $x \in \mathbb{R}$. Hence

$$\lim_{|x|\to\infty} \left|(1 + x^2)f(x)\right| = 0$$

Choose $C > 0$ such that $|(1 + x^2)f(x)| \leq C$ for all $x \in \mathbb{R}$. Then $|f(x)| \leq C/(1 + x^2)$ which implies $f \in \mathcal{R}^1$. $\square$

Theorem 5.3.4. If $f \in \mathbb{S}$ then so is $\hat{f}$.

Proof. Let $f \in \mathbb{S}$. Since $f \in \mathcal{R}^1$ by Lemma 5.3.3, $\hat{f} \in \mathcal{C}_0(\mathbb{R})$ and by definition,

$$\hat{f}(x) = \int_{-\infty}^{\infty} f(t)e^{-ixt}\,\mathrm{d}t$$

for all $x \in \mathbb{R}$. By Theorem 1.6.3,

$$\hat{f}'(x) = -i\int_{-\infty}^{\infty} tf(t)e^{-ixt}\,\mathrm{d}t$$

for all $x \in \mathbb{R}$. Note that the map $t \mapsto tf(t)$ belongs to $\mathbb{S}$ and hence to $\mathcal{R}^1 \cap \mathcal{C}^\infty(\mathbb{R})$. By induction, $\hat{f} \in \mathcal{C}^\infty(\mathbb{R})$ and

$$\hat{f}^{(k)}(x) = (-i)^k \int_{-\infty}^{\infty} t^k f(t)e^{-ixt}\,\mathrm{d}t$$

for all $x \in \mathbb{R}, k \in \mathbb{N}$. Now suppose $k \in \mathbb{N}$. Then for each $x \in \mathbb{R}$,

$$\begin{aligned}
x\hat{f}^{(k)}(x) &= x(-i)^k \int_{-\infty}^{\infty} t^k f(t)e^{-ixt}\,\mathrm{d}t \\
&= (-i)^{k-1} \int_{-\infty}^{\infty} t^k f(t)\frac{d}{dt}(e^{-ixt})\,\mathrm{d}t \\
&= (-i)^{k-1} t^k f(t)e^{-ixt}\Big|_{t=-\infty}^{\infty} - (-i)^{k-1}\int_{-\infty}^{\infty} (kt^{k-1}f(t) + t^k f'(t))e^{-ixt}\,\mathrm{d}t \\
&= \hat{g}(x)
\end{aligned}$$

where g is that member of $\mathbb{S}$ defined by

$$g(t) = -(-i)^{k-1}(kt^{k-1}f(t) + t^k f'(t))$$

We know that $\hat{g} \in \mathcal{C}_0(\mathbb{R})$ so

$$\lim_{|x|\to\infty} x\hat{f}^{(k)}(x) = \lim_{|x|\to\infty} \hat{g} = 0$$

By induction, for all $m, k \geq 0$,

$$\lim_{|x|\to\infty} \left| x^m \hat{f}^{(k)}(x) \right| = 0$$

so that $\hat{f} \in \mathbb{S}$. $\square$

Definition 5.3.5. For $f \in \mathbb{S}$ define $Mf(x) = -ixf(x)$ for $x \in \mathbb{R}$. Note that M is a linear operator on $\mathbb{S}$ and for all $k \in \mathbb{N}$, $M^k f(x) = (-ix)^k f(x)$. Also, define $Df = f'$ and again note that this is a linear operator on $\mathbb{S}$. Finally, let

$$Ff = \frac{1}{\sqrt{2\pi}}\hat{f}$$

(which is also a linear operator on $\mathbb{S}$ and is called the Fourier Transform of f in some texts). (Note that by Example 4.5.8, $FG = G$ where G is the Gaussian Distribution.)

Lemma 5.3.6. If $f \in \mathbb{S}$ then

i) $DFf = FMf$

ii) $FD = -MFf$

Proof. i) follows from the proof of Theorem 5.3.4. For ii),

$$\begin{aligned} FDf(x) &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f'(t)e^{-ixt}\,\mathrm{d}t \\ &= \frac{1}{\sqrt{2\pi}} f(t)e^{-ixt}\Big|_{t=-\infty}^{\infty} - (-ix)\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(t)e^{-ixt}\,\mathrm{d}t \\ &= -(-ix)\frac{1}{\sqrt{2\pi}}\hat{f}(x) \\ &= -MFf(x) \end{aligned}$$

□

Corollary 5.3.7. If $f \in \mathbb{S}$ and $k \in \mathbb{N}$ then

i) $D^k Ff = FM^k f$

ii) $FD^k f = (-1)^k M^k Ff$

Proof. The proof of both assertions follows from Lemma 5.3.6 and induction.

□

Theorem 5.3.8. The operator F is a vector space automorphism of $\mathbb{S}$ onto itself and $F^{-1}f = Ff^{-}$ for all $f \in \mathbb{S}$.

Proof. This follows from Corollary 4.6.5. $\square$

Proposition 5.3.9. If $f, g \in \mathbb{S}$ then $f * g \in \mathbb{S}$ and

$$(f * g)(x) = \int_{-\infty}^{\infty} f(s)\,\mathrm{d}s \int_{-\infty}^{\infty} g(t)\,\mathrm{d}t$$

Proof. It follows from Theorem 1.5.3 that $f * g \in \mathcal{C}^{\infty}(\mathbb{R})$ and from Theorem 1.6.3 that

$$D^k(f * g) = f * (D^k g)$$

for all $k \in \mathbb{N}$. Now suppose $m \in \mathbb{N}$. Then for all $x \in \mathbb{R}, t \in \mathbb{R}$,

$$|x|^m \le (|x - t| + |t|)^m = \sum_{k=0}^{m} \binom{m}{k} |x - t|^{m-k} |t|^k$$

Therefore,

$$\begin{aligned} |x^m (f * g)(x)| &\le \sum_{k=0}^{m} \binom{m}{k} \int_{-\infty}^{\infty} |t|^k |f(t)| |x - t|^{m-k} |g(x - t)| \,\mathrm{d}t \\ &\le \sum_{k=0}^{m} \binom{m}{k} \int_{-\infty}^{\infty} |t|^k |f(t)| \left\| M^{m-k} g \right\|_{\infty} \mathrm{d}t \\ &= C_m \end{aligned}$$

So we have

$$\left| x^{m-1} (f * g)(x) \right| \le \frac{C_m}{|x|}$$

for $0 \ne x \in \mathbb{R}$ and hence,

$$\lim_{|x| \to \infty} \left| x^{m-1} (f * g)(x) \right| = 0$$

Thus for all $0 \le m, k \in \mathbb{N}$,

$$\lim_{|x| \to \infty} \left| x^m ((f * g)(x))^{(k)} \right| = \lim_{|x| \to \infty} \left| x^m (f * g^{(k)})(x) \right| = 0$$

Finally, the formula for $(f * g)(x)$ was derived in the proof of Theorem 5.2.1. $\square$

Definition 5.3.10. For the sake of elegance in certain formulae, we define

$$f \circledast g = \frac{1}{\sqrt{2\pi}}(f * g)$$

Theorem 5.3.11. If $f, g \in \mathbb{S}$ then $F(f \circledast g) = (Ff)(Fg)$.

Proof. This follows immediately from Theorem 5.2.1. □

Proposition 5.3.12. i) The operation $\circledast$ is associative on $\mathbb{S}$.

ii) $\mathbb{S}$ is isomorphic as an algebra with respect to $\circledast$ to itself as an algebra with respect to pointwise multiplication. F is such an isomorphism.

Proof. i) Suppose $f, g, h \in \mathbb{S}$. Then by Theorem 5.3.11,

$$\begin{aligned} F((f \circledast g) \circledast h) &= F(f \circledast g)F(h) \\ &= (Ff)(Fg)(Fh) \\ &= (Ff)F(g \circledast h) \\ &= F(f \circledast (g \circledast h)) \end{aligned}$$

But by Theorem 5.3.8 F is a bijection. Hence

$$(f \circledast g) \circledast h = f \circledast (g \circledast h)$$

ii) This follows immmediately from Theorem 5.3.11.

□

Proposition 5.3.13. Suppose $f, g \in \mathbb{S}$. Then

i) $FFf = f^-$

ii) $F^4 f = f$

iii) $\int_{-\infty}^{\infty} \hat{f}(x)g(x)\,\mathrm{d}x = \int_{-\infty}^{\infty} f(x)\hat{g}(x)\,\mathrm{d}x$

iv) $\overline{Ff} = F(\overline{f^-})$.

Proof. i) This follows from Theorem 5.3.8.

ii) This follows from i).

iii) Since $f, g \in \mathbb{S}$ they belong to $\mathcal{R}^1$. Hence by Theorem 1.5.3,

$$\begin{aligned}
\int_{-\infty}^{\infty} \hat{f}(x)g(x)\,\mathrm{d}x &= \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} f(t)e^{-ixt}\,\mathrm{d}t\right) g(x)\,\mathrm{d}x \\
&= \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} f(t)g(x)e^{-ixt}\,\mathrm{d}t\right) \mathrm{d}x \\
&= \int_{-\infty}^{\infty} f(t) \left(\int_{-\infty}^{\infty} g(x)e^{-ixt}\,\mathrm{d}x\right) \mathrm{d}t \\
&= \int_{-\infty}^{\infty} f(t)\hat{g}(t)\,\mathrm{d}t
\end{aligned}$$

iv) By definition,

$$\begin{aligned}
\overline{Ff(x)} &= \frac{1}{\sqrt{2\pi}}\overline{\int_{-\infty}^{\infty} f(t)e^{-ixt}\,\mathrm{d}t} \\
&= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \overline{f(t)}e^{ixt}\,\mathrm{d}t \\
&= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \overline{f(-s)}e^{-ixs}\,\mathrm{d}s \quad (s = -t) \\
&= F(\overline{f^-})
\end{aligned}$$

□

Definition 5.3.14. For $f, g \in \mathbb{S}$ define

$$\langle f \mid g \rangle = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)\overline{g(x)}\,\mathrm{d}x$$

Then $\langle \cdot \mid \cdot \rangle$ is an inner product on $\mathbb{S}$ and the norm induced from this inner product is obviously $\|f\|_2 = \langle f \mid f \rangle^{1/2}$.

Theorem 5.3.15. F is a unitary operator on $\mathbb{S}$. That is, for all $f, g \in \mathbb{S}$,

i) $\langle Ff \mid Fg \rangle = \langle f \mid g \rangle$.

ii) $\|Ff\|_2 = \|f\|_2$.

Proof. i) By Proposition 5.3.13,

$$\begin{aligned}
\langle Ff \mid Fg \rangle &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} Ff(x)\overline{Fg(x)}\,\mathrm{d}x \\
&= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)F(\overline{Fg(x)})\,\mathrm{d}x \\
&= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)(FF(\overline{g^{-}(x)}))\,\mathrm{d}x \\
&= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)\overline{g(x)}\,\mathrm{d}x \\
&= \langle f \mid g \rangle
\end{aligned}$$

ii) This follows immediately from part i).

□

The following theorem will be stated without proof.

Theorem 5.3.16. Let L^2 denote the metric completion of $\mathbb{S}$ with respect to $\|\cdot\|_2$. Then L^2 is a Hilbert space (a complete inner product space), F has a unique continuous extension, say $\tilde{F}$, to a map from L^2 onto itself and this $\tilde{F}$ is a Hilbert space automorphism of L^2. Moreover, $\mathbb{S}$ is a dense subspace of L^2.

5.4 A Topology for the Schwarz Space

In this section we introduce a norm on $\mathbb{S}$ and also a metric which give a topology on this space.

Definition 5.4.1. For $m \in \mathbb{N}$ and $f \in \mathbb{S}$ let

$$\|f\|_m = \sum_{j,k=0}^{m} \left\|M^k f^{(j)}\right\|_\infty$$

Proposition 5.4.2. The function $\|\cdot\|_m$ is a norm on $\mathbb{S}$ for all $m \in \mathbb{N}$.

Proof. This follows immediately from the linearity of the derivative, the fact that $\|\cdot\|_\infty$ is a norm and that the finite sum of norms is a norm. □

Proposition 5.4.3. Let $m \in \mathbb{N}$ and $\{f_\nu\}_{\nu=1}^\infty$ be a sequence in $\mathbb{S}$. Then this sequence converges to $f \in \mathbb{S}$ with respect to $\|\cdot\|_m$ iff, for all $0 \le j, k \le m$

$$\lim_{\nu \to \infty} x^k f_\nu^{(j)}(x) \to x^k f^{(j)}(x)$$

uniformly for $x \in \mathbb{R}$.

Proof. Fix $0 \le j, k \le m$. Then $\{M^k f_\nu^{(j)}\}_{\nu=1}^\infty$ converges to $M^k f^{(j)}$ with respect to the supnorm iff this sequence converges uniformly on $\mathbb{R}$ to $M^k f^{(j)}$. Now $\{f_\nu\}_{\nu=1}^\infty$ converges to f with respect to $\|\cdot\|_m$ iff this sequence converges to $M^k f^{(j)}$ with respect to the supnorm for each $0 \le j, k \le m$. Hence the result follows. □

Definition 5.4.4. For $f, g \in \mathbb{S}$ define

$$d(f,g) = \sum_{m=1}^\infty \frac{1}{2^m} \left(\frac{\|f-g\|_m}{1 + \|f-g\|_m} \right)$$

Proposition 5.4.5. The function d in Defintion 5.4.4 is a bounded metric and $\mathbb{S}$ is complete with respect to d. Also, for each $m \in \mathbb{N}$, $\|\cdot\|_m$ is complete as well.

Proof. By Proposition 4.2.4, for all $f, g \in \mathbb{S}$ and each $m \in \mathbb{N}$,

$$\frac{\|f-g\|_m}{1 + \|f-g\|_m}$$

is a metric bounded by 1. Therefore, d is a metric bounded above by 1 as well. Now we will prove that d is complete. The completeness of $\|\cdot\|_m$

for each $m \in \mathbb{N}$ will follow from Theorem 4.2.7. Let $\{f_\nu\}_{\nu=1}^\infty$ be a Cauchy sequence in $\mathbb{S}$ with respect to d. This means that

$$\lim_{\nu,\mu\to\infty} d(f_\nu, f_\mu) = 0$$

From the definition of d, it is seen that this means

$$\lim_{\nu,\mu\to\infty} \|f_\nu - f_\mu\|_m = 0$$

for each $m \in \mathbb{N}$. Now from the definition of $\|\cdot\|_m$, it follows that

$$\lim_{\nu,\mu\to\infty} \left\|f_\nu^{(j)} - f_\mu^{(j)}\right\|_\infty = 0$$

for each $j \geq 0$. Hence the sequence $\{f_\nu^{(j)}\}_{\nu=1}^\infty$ is Cauchy with respect to the supnorm for each $j \geq 0$. Since the supnorm is complete, this implies that $f_\nu^{(j)} \to f_j$ uniformly on $\mathbb{R}$ for some functions $f_j, j \geq 0$. Let $f = f_0$. By Theorem 1.2.1, $f^{(j)} = f_j = \lim_{\nu\to\infty} f_\nu^{(j)}$ for all $j \geq 0$. So we have proven that $\{f_\nu\}_{\nu=1}^\infty$ converges to a $\mathcal{C}^\infty$ function f. To show that $f \in \mathbb{S}$ we only have to prove that

$$\lim_{|x|\to\infty} x^m f^{(k)}(x) = 0$$

for all $m, k \geq 0$. Fix $\epsilon > 0$, $m \geq 0$ and $k \geq 0$. Then by the remarks above $x^m f_\nu^{(k)}(x)$ converges uniformly on $\mathbb{R}$ to $x^m f^{(k)}(x)$. Therefore we can choose $\nu > 0$ such that

$$\left|x^m f^{(k)}(x) - x^m f_\nu^{(k)}(x)\right| < \frac{\epsilon}{2}$$

for all $x \in \mathbb{R}$. Now since $f_\nu \in \mathbb{S}$,

$$\lim_{|x|\to\infty} x^m f_\nu^{(k)}(x) = 0$$

so we can choose $A > 0$ such that

$$\left|x^m f_\nu^{(k)}(x)\right| < \frac{\epsilon}{2}$$

whenever $|x| > A$. Hence, with this choice of A, ν and x we have

$$\begin{aligned}
\left|x^m f^{(k)}(x)\right| &\leq \left|x^m f^{(k)}(x) - x^m f_\nu^{(k)}(x)\right| + \left|x^m f_\nu^{(k)}(x)\right| \\
&< \frac{\epsilon}{2} + \frac{\epsilon}{2} \\
&= \epsilon
\end{aligned}$$

which completes the proof. □

Theorem 5.4.6. Let d be as in Defintion 5.4.4. Then D and F are continuous with respect to d. Also, if P is any polynomial then the map $f \mapsto Pf$ is continuous with respect to d as well.

Proof. Let $\{f_\nu\}_{\nu=1}^{\infty}$ be a sequence in $\mathbb{S}$ converging to some $f \in \mathbb{S}$ with respect to d. To show that D, F and the the map $f \mapsto Pf$ are continuous with respect to d, we have to show that $d(Df_\nu, Df) \to 0$, $d(Ff_\nu, Ff) \to 0$ and $d(Pf_\nu, Pf) \to 0$. We will do so now.

i) Since $d(f_\nu, f) \to 0$, this implies that

$$\left\| M^k f_\nu^{(j)} - M^k f^{(j)} \right\|_\infty \to 0$$

as $\nu \to 0$ for all $j, k \geq 0$. Hence for each $m \in \mathbb{N}$

$$\begin{aligned} \|Df_\nu - Df\|_m &= \sum_{j,k=0}^{m} \left\| M^k (Df_\nu)^{(j)} - M^k (Df)^{(j)} \right\|_\infty \\ &= \sum_{j,k=0}^{m} \left\| M^k f_\nu^{(j+1)} - M^k f^{(j+1)} \right\|_\infty \\ &\to 0 \end{aligned}$$

Thus $\{Df_\nu\}_{\nu=1}^{\infty}$ converges to Df with respect to $\|\cdot\|_m$ for each $m \in \mathbb{N}$. So by Theorem 4.2.7 $\{Df_\nu\}_{\nu=1}^{\infty}$ converges to Df with respect to d.

ii) Fix $0 \leq j, k \in \mathbb{Z}$. Then

$$(Ff_\nu)^{(j)} - (Ff)^{(j)} = D^j F f_\nu - D^j F f = F M^j f_\nu - F M^j f = F g_\nu$$

by Corollary 5.3.7 (where $g_\nu = M^j (f_\nu - f) \in \mathbb{S}$. By Theorem 5.3.4, $Fg_\nu \in \mathbb{S}$ for all $\nu \geq 1$. This means that

$$\lim_{|x| \to \infty} \left| x^k F g_\nu(x) \right| = \lim_{|x| \to \infty} \left| M^k F g_\nu \right| = 0$$

Therefore, given $\epsilon > 0$ we can find $0 < A \in \mathbb{R}$ such that

$$\left| M^k F g_\nu \right| = \left| x^k F g_\nu(x) \right| < \epsilon \ \ (*)$$

whenever $|x| > A$. Now, for $|x| \leq A$ we have

$$\begin{aligned}
\left|M^k F g_\nu\right| &= \left|x^k \int_{-\infty}^{\infty} (-it)^j [f_\nu(t) - f(t)] e^{-ixt}\,\mathrm{d}t\right| \\
&\leq A^k \int_{-\infty}^{\infty} \left|t^j [f_\nu(t) - f(t)]\right| \mathrm{d}t \\
&\leq A^k \int_{-\infty}^{\infty} \left|t^j [f_\nu(t) - f(t)]\right| \mathrm{d}t \\
&\to 0 \quad (\dagger)
\end{aligned}$$

as $\nu \to \infty$ since $t^j f_\nu(t)$ converges uniformly to $t^j f(t)$ on $\mathbb{R}$. By $(*)$ and $(\dagger)$ it follows that

$$\left\|M^k (F f_\nu)^{(j)} - M^k (F f)^{(j)}\right\|_\infty < \epsilon$$

for all $\epsilon > 0$ and hence this quantity equals zero. Thus for each $m \in \mathbb{N}$,

$$\begin{aligned}
\|F f_\nu - F f\|_m &= \sum_{j,k=0}^{m} \left\|M^k (F f_\nu)^{(j)} - M^k (F f)^{(j)}\right\|_\infty \\
&\to 0
\end{aligned}$$

as $\nu \to \infty$. Therefore $\{F f_\nu\}_{\nu=1}^{\infty}$ converges to Ff with respect to $\|\cdot\|_m$ for each $m \in \mathbb{N}$. So by Theorem 4.2.7 $\{F f_\nu\}_{\nu=1}^{\infty}$ converges to Ff with respect to d.

iii) Let

$$P(x) = \sum_{l=0}^{n} c_l x^l$$

for $x \in \mathbb{R}$. Then

$$
\begin{aligned}
\|Pf_\nu - Pf\|_m &= \sum_{j,k=0}^{m} \left\| M^k (Pf_\nu)^{(j)} - M^k (Pf)^{(j)} \right\|_\infty \\
&= \sum_{j,k=0}^{m} \left\| M^k \left(\sum_{l=j}^{n} k_l x^{l-j} f_\nu^{(j)}(x) \right) - M^k \left(\sum_{l=j}^{n} k_l x^{l-j} f^{(j)}(x) \right) \right\|_\infty \\
&\quad \text{(for some constants } k_l\text{)} \\
&\leq \sum_{j,k=0}^{m} \sum_{l=j}^{n} |k_l| \left\| M^k x^{l-j} (f_\nu^{(j)} - f^{(j)}) \right\|_\infty \\
&= \sum_{j,k=0}^{m} \sum_{l=j}^{n} |k_l| \left\| M^{k+l-j} (f_\nu^{(j)} - f^{(j)}) \right\|_\infty \\
&\to 0 \quad \text{(by the remarks above)}
\end{aligned}
$$

Again, by Theorem 4.2.7, this implies $\{Pf_\nu\}_{\nu=1}^\infty$ converges to Pf with respect to d.

$\square$