Jane Street Puzzle Document

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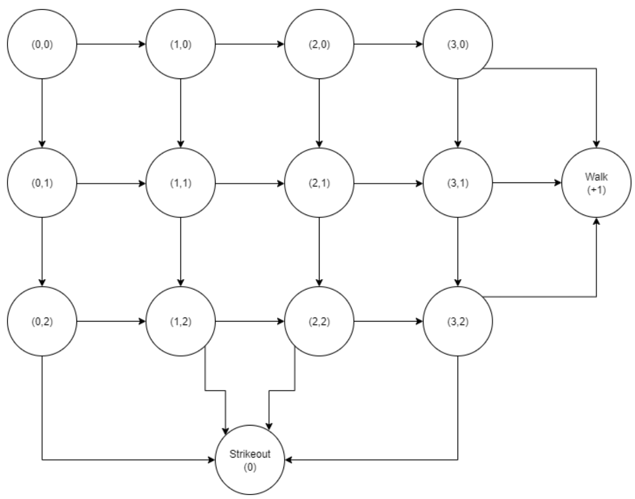
November 4, 2025

1 October 2025

Let $S = (S_B, S_S)$ denote the current at-bat state. We can draw the following state diagram:



The state contains two absorbing states 'Walk' and 'Strikeout', which yield +1 and 0 points to the batter, respectively. Note that at from any state, we can also travel to a third absorbing state 'Home Run', which yields +4 points to the batter.

Let A_S denote the probability of the pitcher throwing a ball in at-bat state S. Let B_S denote the probability of the batter swinging in at-bat state S. Under any given state S, the occurrences are as follows:

	Ball	**Strike**
Swing	$A_S B_S$	$(1 - A_S)B_S$
Wait	$A_S(1 - B_S)$	$(1 - A_S)(1 - B_S)$

Given these mixed strategies, let $E(S)$ denote the expected number of points the batter will score while currently in state S.

	Ball	**Strike**
Swing	$A_S B_S E(S_B, S_S + 1)$	$(1 - A_S)B_S[4p + (1 - p)E(S_B, S_S + 1)]$
Wait	$A_S(1 - B_S)E(S_B + 1, S_S)$	$(1 - A_S)(1 - B_S)E(S_B, S_S + 1)$

Hence,

$$
\begin{aligned}
E(S_B,S_S) &= A_S B_S E(S_B,S_S+1) + (1-A_S)B_S[4p + (1-p)E(S_B,S_S+1)] \\
&+ A_S(1-B_S)E(S_B+1,S_S) + (1-A_S)(1-B_S)E(S_B,S_S+1) \\
\\
&= A_S(1-B_S)E(S_B+1,S_S) \\
&+ (1-A_S)B_S(4p) \\
&+ (A_S B_S + (1-A_S)B_S(1-p) + (1-A_S)(1-B_S))E(S_B,S_S+1) \\
\\
&= A_S(1-B_S)E(S_B+1,S_S) \\
&+ (1-A_S)B_S(4p) \\
&+ (A_S B_S - pB_S + pA_S B_S + 1 - A_S)E(S_B,S_S+1)
\end{aligned}
$$

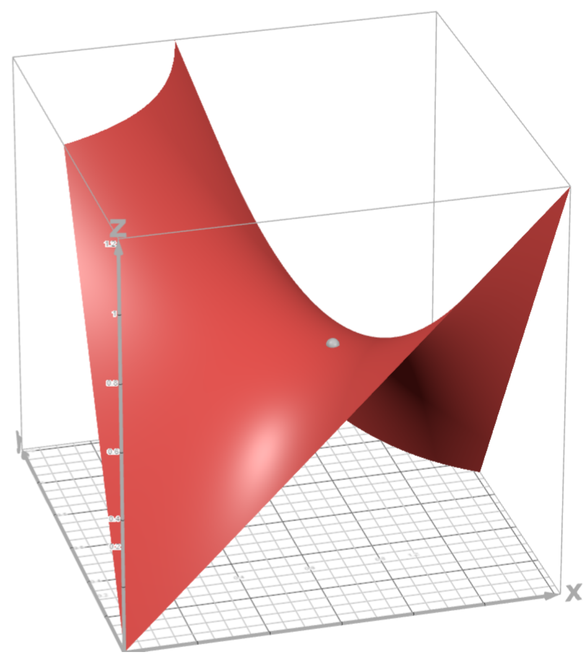


Figure 1: Plot of the expected value as a function of A_S (x-axis) and B_S (y-axis) for p = 0.5

$$
\begin{aligned}
\frac{\delta E(S_B,S_S)}{\delta B_S} &= -A_S E(S_B+1,S_S) \\
&+ (1-A_S)(4p) \\
&+ (A_S - p + pA_S)E(S_B,S_S+1) = 0 \\
0 &= 4p - pE(S_B,S_S+1) + A_S((1+p)E(S_B,S_S+1) - 4pE(S_B+1,S_S)) \\
A_S &= \frac{pE(S_B,S_S+1) - 4p}{(1+p)E(S_B,S_S+1) - 4p - E(S_B+1,S_S)}
\end{aligned}
$$

$$
\begin{aligned}
\frac{\delta E(S_B, S_S)}{\delta A_S} &= (1 - B_S)E(S_B + 1, S_S) \\
&- B_S(4p) \\
&+ (B_S + pB_S - 1)E(S_B, S_S + 1) = 0 \\
0 &= (E(S_B + 1, S_S) - E(S_B, S_S + 1)) \\
&+ B_S((1 + p)E(S_B, S_S + 1) - 4p - E(S_B + 1, S_S)) \\
B_S &= \frac{E(S_B, S_S + 1) - E(S_B + 1, S_S)}{(1 + p)E(S_B, S_S + 1) - 4p - E(S_B + 1, S_S)}
\end{aligned}
$$

Hence, the mixed-strategy Nash equilibrium lies at

$$
\begin{aligned}
\dot{A}_S &= \frac{pE(S_B, S_S + 1) - 4p}{(1 + p)E(S_B, S_S + 1) - 4p - E(S_B + 1, S_S)} \\
\dot{B}_S &= \frac{E(S_B, S_S + 1) - E(S_B + 1, S_S)}{(1 + p)E(S_B, S_S + 1) - 4p - E(S_B + 1, S_S)}
\end{aligned}
$$

Note that $\dot{A}_S + \dot{B}_S = 1$. This condition doesn't always hold but does when the a pair of diagonal entries in the EV-table are equal (in this case, (Ball, Swing), and (Strike, Wait)). We can simplify the expected value equation to the following:

$$
\begin{aligned}
E(S_B, S_S) &= \dot{A}_S(1 - \dot{B}_S)E(S_B + 1, S_S) \\
&+ (1 - \dot{A}_S)\dot{B}_S(4p) \\
&+ (\dot{A}_S\dot{B}_S - p\dot{B}_S + p\dot{A}_S\dot{B}_S + 1 - \dot{A}_S)E(S_B, S_S + 1) \\
&= \dot{A}_S{}^2 E(S_B + 1, S_S) \\
&+ (1 - \dot{A}_S)^2(4p) \\
&+ (\dot{A}_S(1 - \dot{A}_S) - p(1 - \dot{A}_S) + p\dot{A}_S(1 - \dot{A}_S) + 1 - \dot{A}_S)E(S_B, S_S + 1) \\
&= \dot{A}_S{}^2 E(S_B + 1, S_S) \\
&+ (1 - \dot{A}_S)^2(4p) \\
&+ (1 - \dot{A}_S)(\dot{A}_S - p + p\dot{A}_S + 1)E(S_B, S_S + 1) \\
&= \dot{A}_S{}^2 (E(S_B + 1, S_S) + 4p - (1 + p)E(S_B, S_S + 1)) \\
&+ \dot{A}_S(-8p + (1 - p)E(S_B, S_S + 1) - (1 - p)E(S_B, S_S + 1)) \\
&+ (4p + (1 - p)E(S_B, S_S + 1)
\end{aligned}
$$

Let

$$
d = (1 + p)E(S_B, S_S + 1) - 4p - E(S_B + 1, S_S)
$$

Then,

$$
\begin{aligned}
&= \frac{(pE(S_B,S_S+1)-4p)^2}{d^2}(-d) \\
&+ \frac{pE(S_B,S_S+1)-4p}{d}(-8p+(1-p)E(S_B,S_S+1)-(1-p)E(S_B,S_S+1)) \\
&+ (4p+(1-p)E(S_B,S_S+1) \\
&= \frac{-(pE(S_B,S_S+1)-4p)^2}{d} \\
&+ \frac{pE(S_B,S_S+1)-4p}{d}(2pE(S_B,S_S+1)-8p) \\
&+ (4p+(1-p)E(S_B,S_S+1) \\
&= \frac{-(pE(S_B,S_S+1)-4p)^2}{d} \\
&+ \frac{2(pE(S_B,S_S+1)-4p)^2}{d} \\
&+ (4p+(1-p)E(S_B,S_S+1) \\
&= \frac{(pE(S_B,S_S+1)-4p)^2}{d} + (4p+(1-p)E(S_B,S_S+1) \\
&= \frac{(pE(S_B,S_S+1)-4p)^2}{d} \\
&+ \frac{(4p+E(S_B,S_S+1)-pE(S_B,S_S+1))(E(S_B,S_S+1)+pE(S_B,S_S+1)-4p-E(S_B+1,S_S))}{d} \\
&= \frac{(pE(S_B,S_S+1)-4p)^2}{d} \\
&+ \frac{(E(S_B,S_S+1)-(E(S_B,S_S+1)-4)p)(E(S_B,S_S+1)+(E(S_B,S_S+1)-4)p-E(S_B+1,S_S))}{d} \\
&= \frac{(pE(S_B,S_S+1)-4p)^2}{d} \\
&+ \frac{E(S_B,S_S+1)^2-(E(S_B,S_S+1)-4)p)^2}{d} \\
&+ \frac{-4pE(S_B+1,S_S)-E(S_B+1,S_S)E(S_B,S_S+1)+pE(S_B+1,S_S)E(S_B,S_S+1)}{d} \\
&= \frac{(pE(S_B,S_S+1)-4p)^2}{d} \\
&+ \frac{E(S_B,S_S+1)^2}{d} - \frac{(pE(S_B,S_S+1)-4p)^2}{d} \\
&+ \frac{-4pE(S_B+1,S_S)-E(S_B+1,S_S)E(S_B,S_S+1)+pE(S_B+1,S_S)E(S_B,S_S+1)}{d} \\
&= \frac{E(S_B,S_S+1)^2}{d} \\
&+ \frac{-4pE(S_B+1,S_S)-E(S_B+1,S_S)E(S_B,S_S+1)+pE(S_B+1,S_S)E(S_B,S_S+1)}{d} \\
&= \frac{E(S_B,S_S+1)^2-4pE(S_B+1,S_S)-E(S_B+1,S_S)E(S_B,S_S+1)+pE(S_B+1,S_S)E(S_B,S_S+1)}{d} \\
&= \frac{(E(S_B,S_S+1)-E(S_B+1,S_S))^2}{d} - E(S_B+1,S_S) \\
&= \frac{(E(S_B,S_S+1)-E(S_B+1,S_S))^2}{(1+p)E(S_B,S_S+1)-4p-E(S_B+1,S_S)} - E(S_B+1,S_S)
\end{aligned}
$$

A recursive solution for $E(S_B, S_S)$! Let $P(H(S_B, S_S))$ denote the probability of hitting a certain state (S_B, S_S) from the starting position. In these terms,

$$P(H(S_B+1, S_S)|H(S_B, S_S)) = A_S(1-B_S)$$
$$P(H(S_B, S_S+1)|H(S_B, S_S)) = A_S B_S + (1-A_S)B_S(1-p) + (1-A_S)(1-B_S)$$

where,

$$A_S = \frac{pE(S_B, S_S+1) - 4p}{(1+p)E(S_B, S_S+1) - 4p - E(S_B+1, S_S)}$$
$$B_S = \frac{E(S_B, S_S+1) - E(S_B+1, S_S)}{(1+p)E(S_B, S_S+1) - 4p - E(S_B+1, S_S)}$$

By law of total probability,

$$\begin{aligned} P(H(3,2)) &= P(H(3,2)|H(2,2))P(H(2,2)) \\ &= P(H(3,2)|H(3,1))P(H(3,1)) \end{aligned}$$

To find the probability of hitting state $(3, 2)$, or $P(H(3, 2))$, we first calculate the expected values of each state and the mixed strategy probabilities of the pitcher and batter. Then, we can use dynamic methods to calculate the probabilities of hitting each state starting from state $(0, 0)$.

I implemented this in Python, which can be found on my Github. The code features a function called $find\_q(p)$, which takes in a parameter p, and calculates $q = P(H(3, 2))$. I then swept $p \in (0, 1]$ with 100 points, and obtain the following plot of q w.r.t. p.

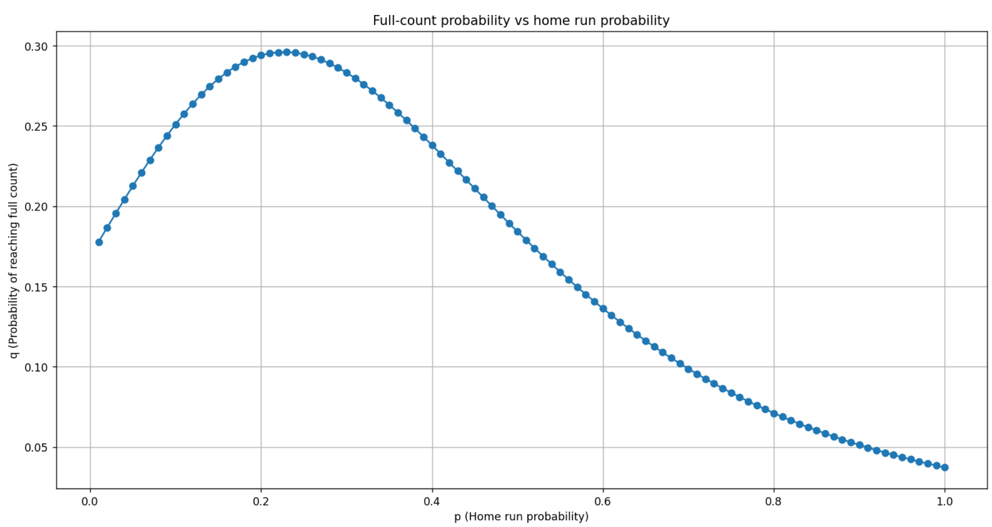


Figure 2: Probability of reaching a full count vs. home run probability

We see that q has a local maxima at some point, around 0.2, and our aim to find the $\dot{p} = argmax(find\_q(p))$. To do this, I implemented a ternary search in the search parameters. I had to consider the accuracy of the calculated numbers, since the answer must be ten-digits accurate, so at this point, I redid the calculation using mpmath to set precision to 100 digits.

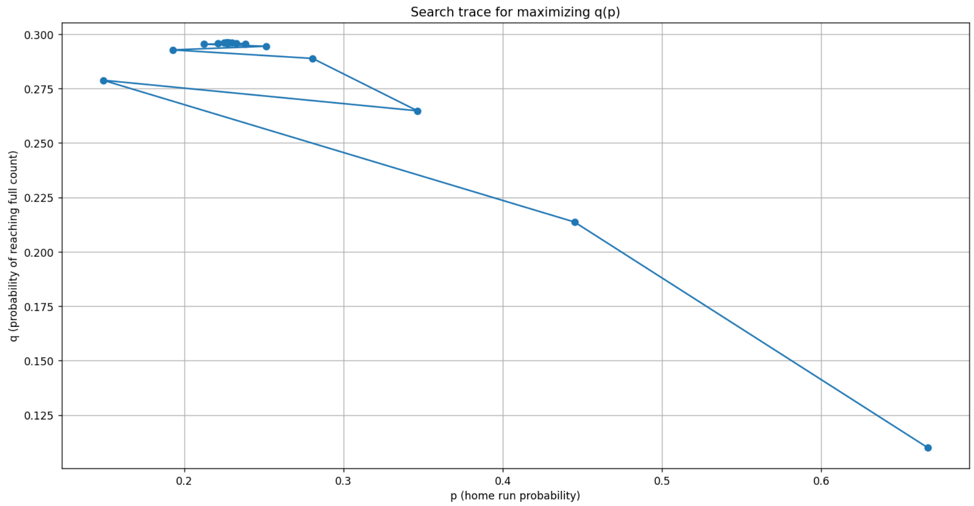


Figure 3: Progression of Ternary Search

Eventually, the search converges to a $(\dot{p}, \dot{q})$ pair of $(0.2269732325, 0.2959679933)$. Hence, our answer is $q = 0.2959679933$.