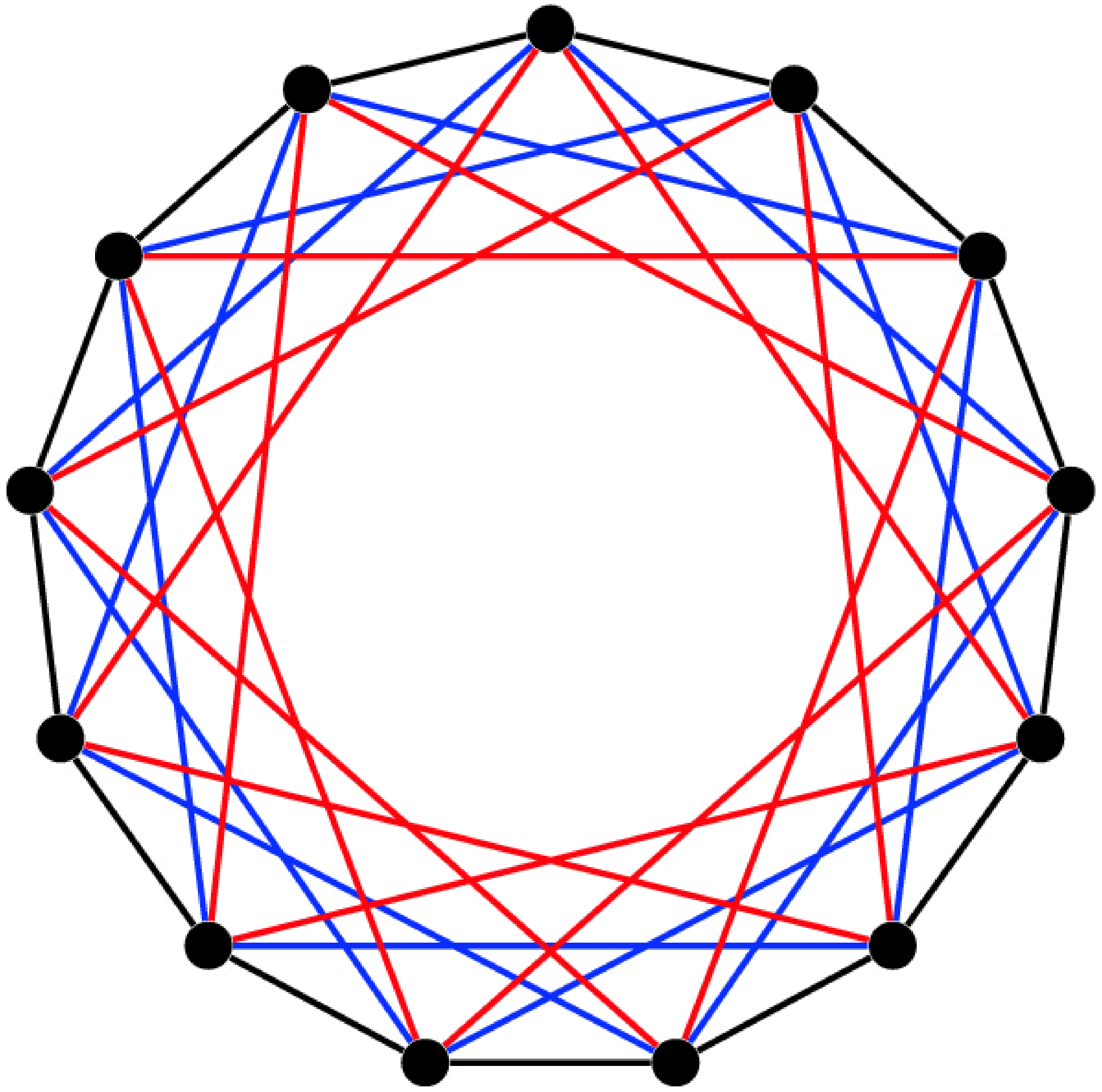
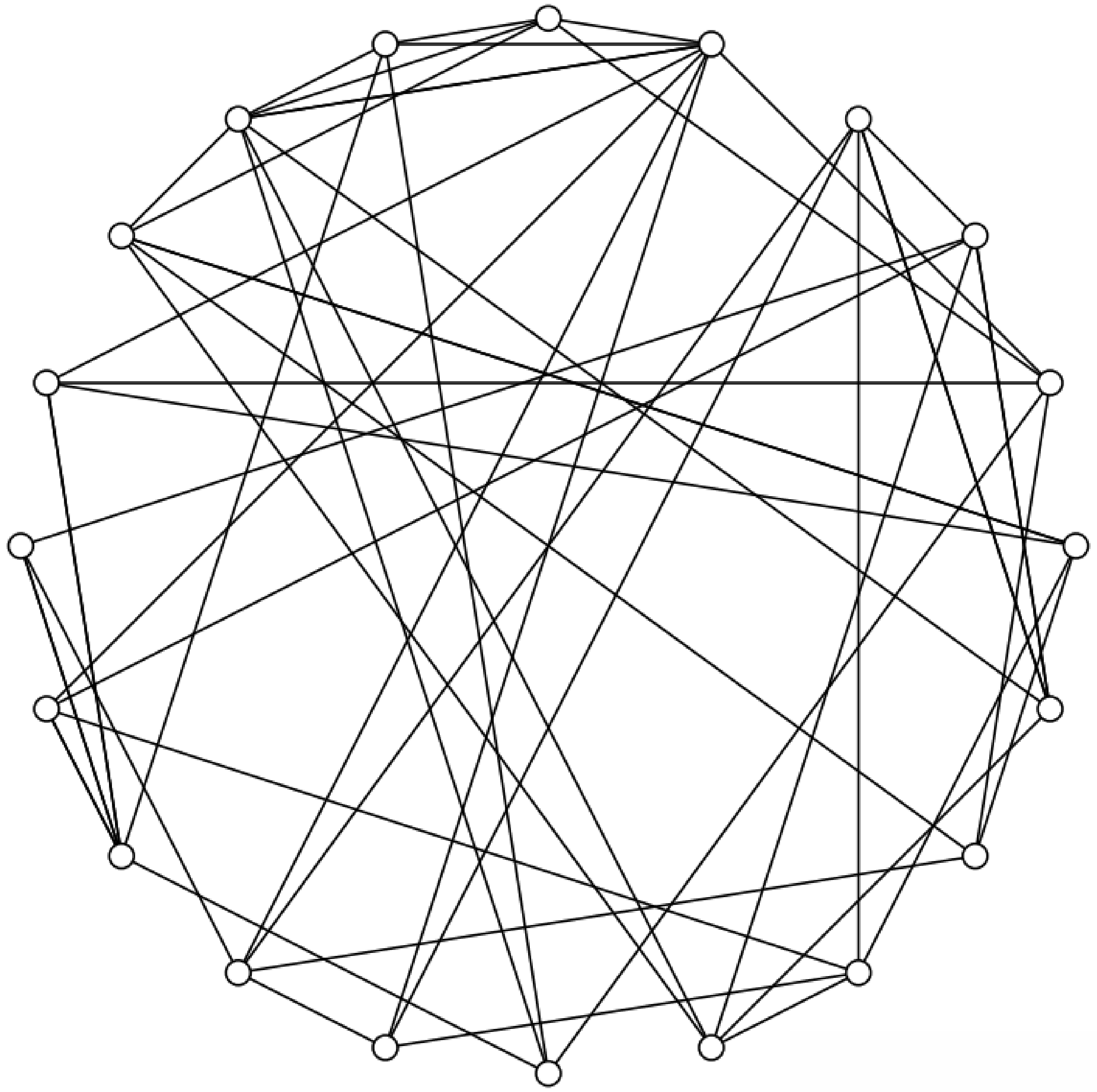
# Infinite Graphs and Randomness





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# Random Graphs

## 3.2. What is a Random Graph?

We may define a probability space on graphs of a given order $n \geq 1$ as follows. Fix a vertex set $V$ consisting of $n$ distinct elements, usually taken as $[n] = \{1, 2, \ldots, n\}$, and fix $p \in [0, 1]$. Define the space of *random graphs of order $n$ with edge probability $p$*, written $G(n, p)$, with sample space equalling the set of all $2^{\binom{n}{2}}$ (labelled) graphs with vertices $V$, and

$$\mathbb{P}(G) = p^{|E(G)|}(1 - p)^{\binom{n}{2} - |E(G)|}.$$

## Graph Property:

A property preserved by isomorphism;
for example, being planar and possessing Hamilton cycles

## A.A.S: Asymptotically almost surely

We say that G element of G(n, p) satisfies Property, P asymptotically almost surely (or a. a. s. for short) if when n tends to infinity Probability(G element of G(n, p) satisfies P) tends to 1.

In this case we say any random G element of G(n, p) satisfies P with high probability (w.h.p)

## Adjacency Properties:

Let us look into a few properties which are asymptotically satisfied for a random graph.

An adjacency property is a global property of a graph asserting that
for every set S of vertices of some fixed type,
there is a vertex joined to some of the vertices of S in a prescribed way

## n-e.c Adjacency Property

A graph is n-existentially closed or n-e.c., if
for all disjoint sets of vertices A and B with |A U B| = n (one of A or B can be empty),
there is a vertex z not in A U B joined to each vertex of A and no vertex of B.

## Cartesian Product of Graphs

The Cartesian product of two graphs $G=(V_G,E_G)$ and $H=(V_H,E_H)$ denoted as $G\square H$, is a graph whose vertex set and edge set are defined as follows:

- Vertex Set:
- $V(G\square H)=V_G\times V_H$
- This means each vertex in $G\square H$ is a pair $(g,h)$ where $g\in V_G$ and $h\in V_H$.
- Edge Set:
- Two vertices $(g_1, h_1)$ and $(g_2, h_2)$ in $G\square H$ are adjacent if and only if:
    - $g_1 = g_2$ and $(h_1,h_2)\in E_H$ (i.e., the second coordinate changes according to an edge in H), or
    - $h_1=h_2$ and $(g_1,g_2)\in E_G$ (i.e., the first coordinate changes according to an edge in G).

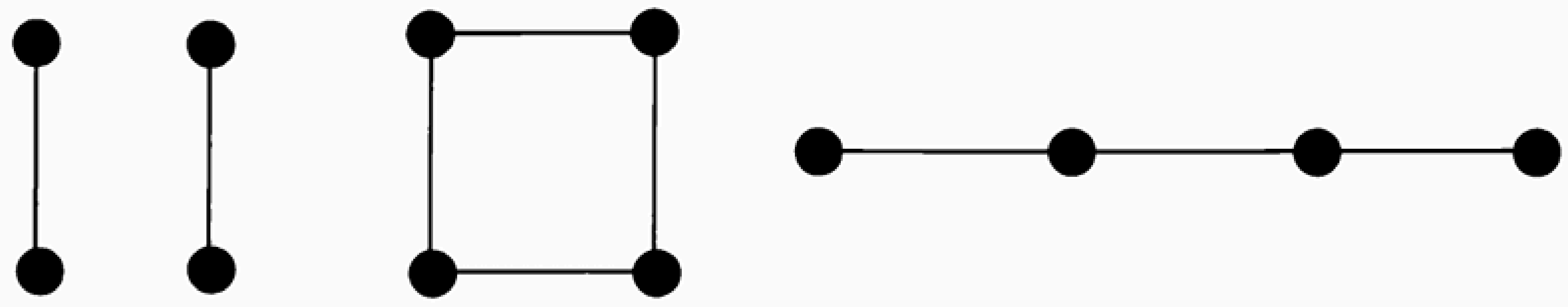


**Figure 3.2.** The 1-e.c. graphs of order 4.

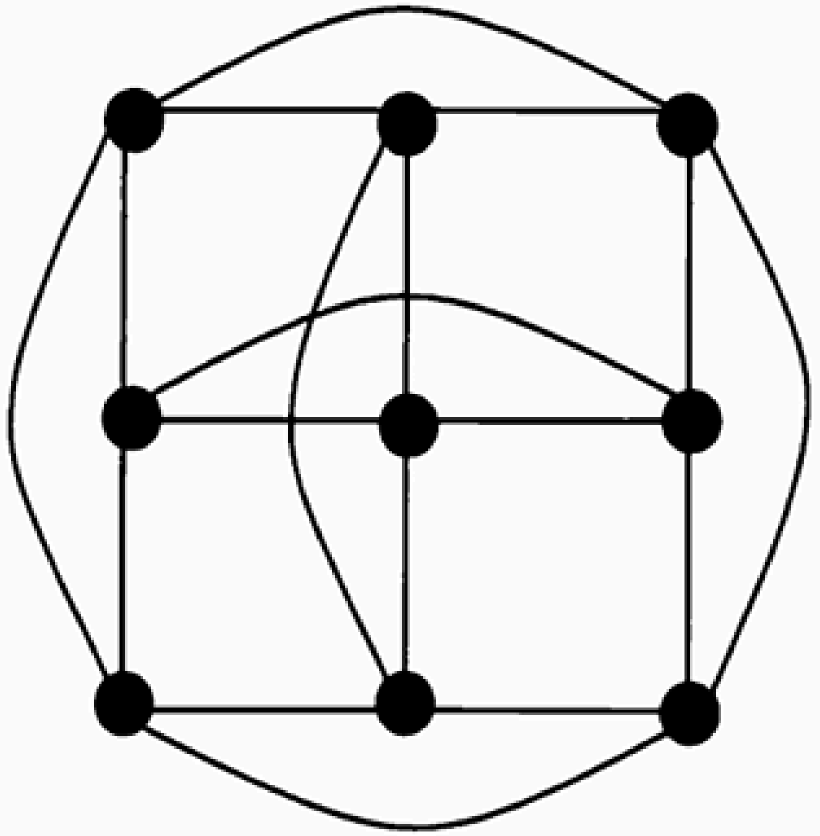
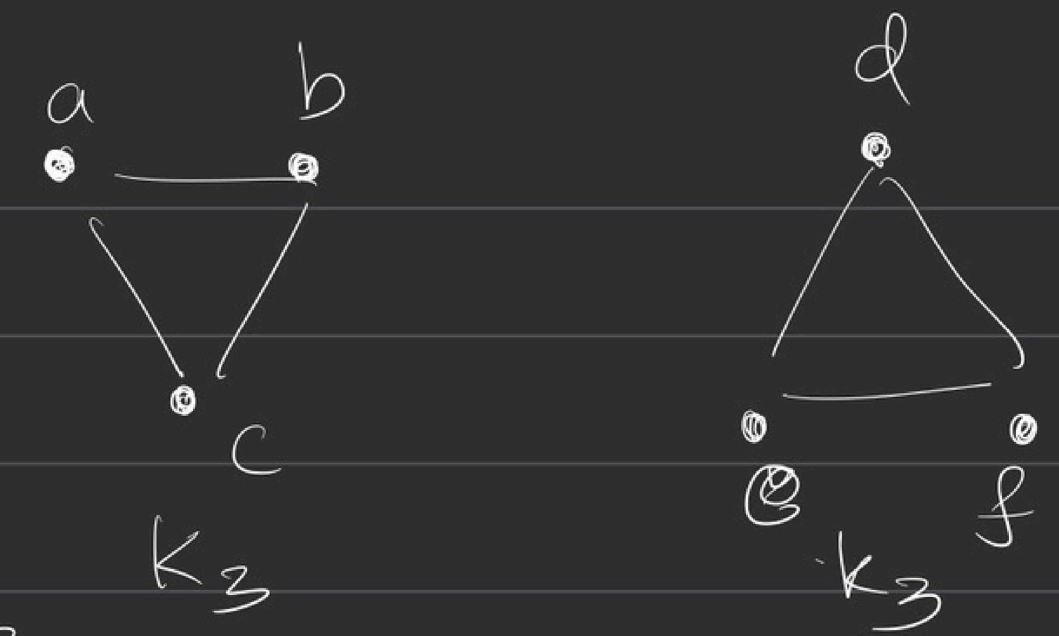


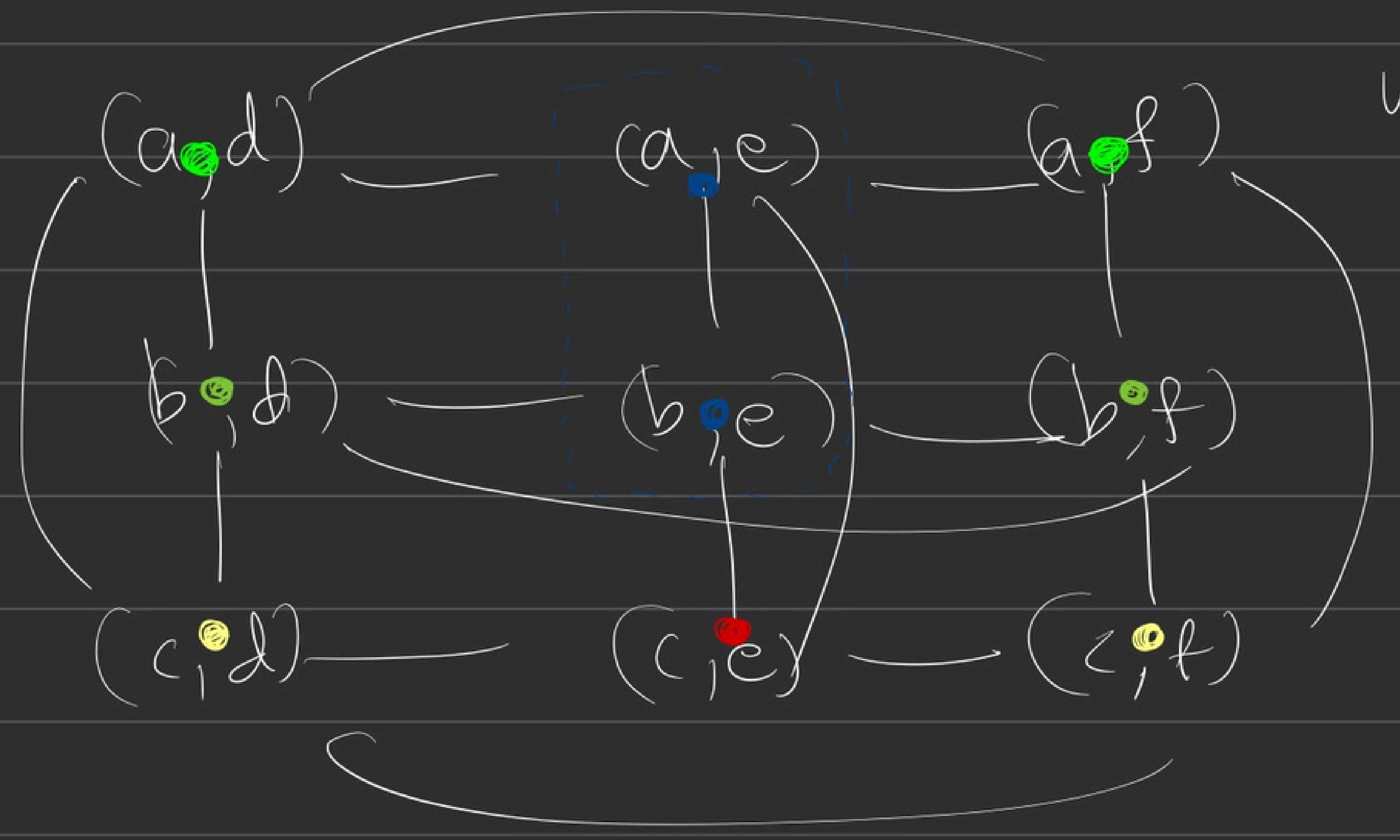
**Figure 3.3.** The graph $K_3 \square K_3$.

$$20 \leq m_{ec}(3) \leq 28.$$




Cartesian product $K_3 \square K_3$

$m_{ec}(2) = 9$




we need $2^2 = 4$ vertices for each 2-set

# Quasi Randomness

A graph is considered "quasi-random" if it exhibits many of the same properties as a truly random graph of the same size and edge density, even though it might not be generated by a truly random process.

## Notations involved-Quasi Randomness

To each graph G of order n we may associate its n x n adjacency matrix A(G).

As A(G) is a real-symmetric matrix, it has n real eigenvalues

which are ordered by absolute value: $|\lambda 1| >= |\lambda 2| \ldots >= |\lambda n|$

Ns (H, G) be the number of labelled subgraphs of G isomorphic to H,

while Nis(H, G) is the number of labelled induced subgraphs of G isomorphic to H

For X C V_G, let e(X) is the number of edges in the subgraph induced by X on G.

For vertices u and v, define s(u, v) to be the set of vertices joined to both u and v or neither u nor v.

# Quasi Randomness - Properties

Any graph that is quasi random satisfies a few properties

These properties all hold a.a.s. in G(n, 1/2)

Any deterministic graph that satisfies any of these properties satisfy all of them

(P1) For all $X \subseteq V(G_n)$

$$e(X) = \frac{1}{4}|X|^2 + o(n^2).$$

(P2) $e(G) \geq (1+o(1))\frac{n^2}{4}$, and $N_S(C_4, G_n) \leq (1+o(1))\frac{n^4}{16}$.

(P3) $e(G) \geq (1+o(1))\frac{n^2}{4}$, and for any fixed graph $H$ of order $4 \leq t \leq n$,

$$N_{IS}(H, G_n) = (1+o(1))n^t 2^{-\binom{t}{2}}.$$

# Quasi Randomness - Properties

(P4) $\sum_{u,v \in V(G_n)} \left| |N(u) \cap N(v)| - \frac{n}{4} \right| = o(n^3)$.

(P5) $\sum_{u,v \in V(G_n)} \left| s(u,v) - \frac{n}{2} \right| = o(n^3)$.

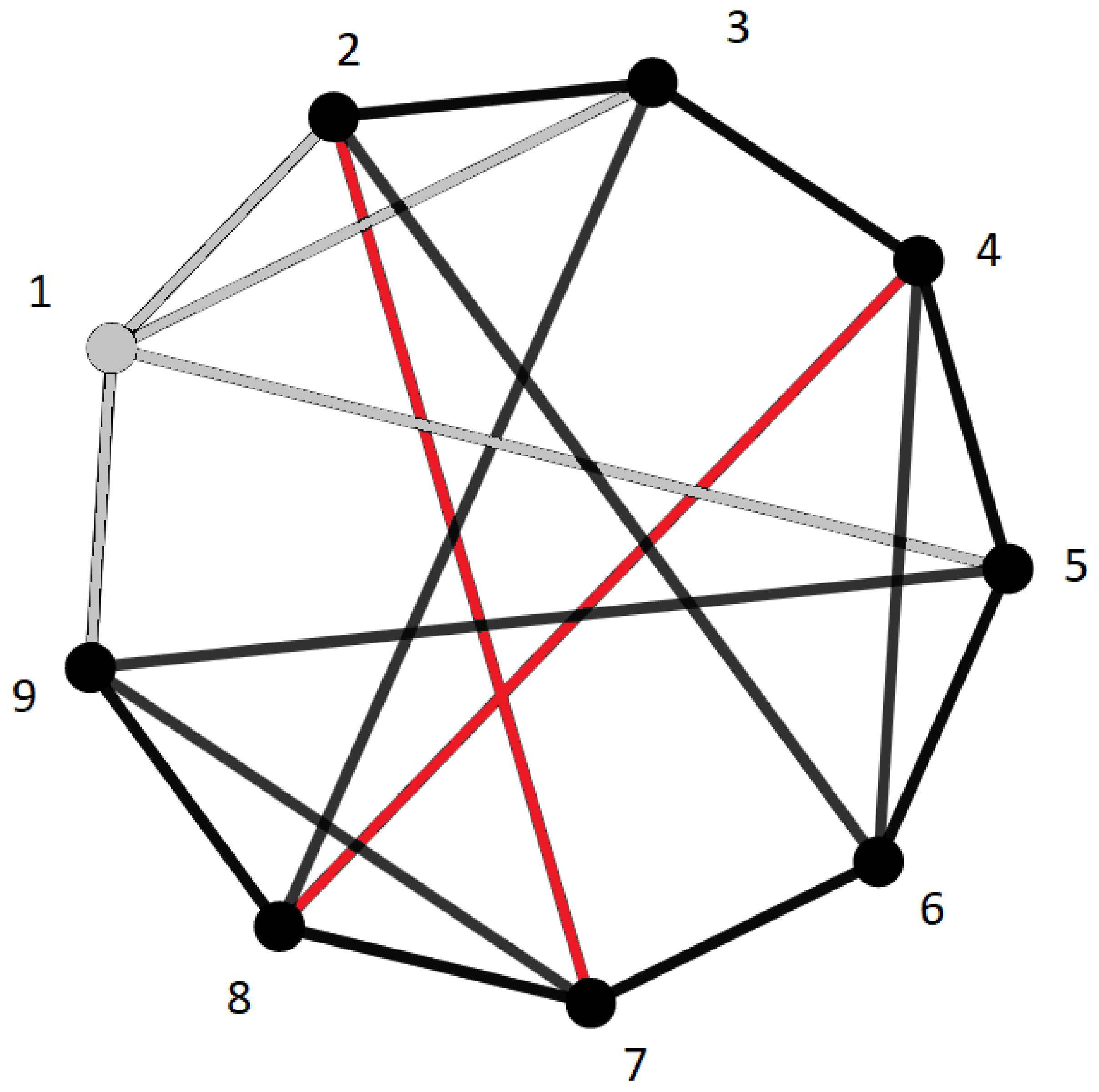
(P6) $e(G) \geq (1 + o(1))\frac{n^2}{4}$ and for $2 \leq i \leq n$

$$\lambda_1 = (1 + o(1))\frac{n}{2}, \ \lambda_n = o(n).$$

THEOREM 3.4 ([70]). *If $G_n$ satisfies any one of the six properties above, then it satisfies all of them.*

# Strongly Regular Graphs(SRG)

A k-regular graph G with v vertices, such that each pair of joined vertices has exactly A common neighbors, and each pair of non-joined vertices has exactly p common neighbors is called a strongly regular graph; we say that G is an SRG (v, k, A, p)




Example - Paley graphs

# Paley Graphs

## Quadratic Residue

A quadratic residue modulo an integer n (typically a prime p) is an integer a such that there exists a non-zero integer x satisfying

x^2≡a(modn).

In other words, a is a quadratic residue modulo n if it is congruent to a perfect square modulo n. If no such x exists, then a is called a quadratic non-residue.

For example consider n = 7. The set of quadratic residues is {1,2,4}, while the set of quadratic non-residues is {3,5,6}

QUADRATIC RESIDUES

$$6^2 = 36 \equiv 1 \ (mod\ 7)$$

$$5^2 = 25 \equiv 4 \ (mod\ 7)$$

$$3^2 = 9 \equiv 2 \ (mod\ 7)$$

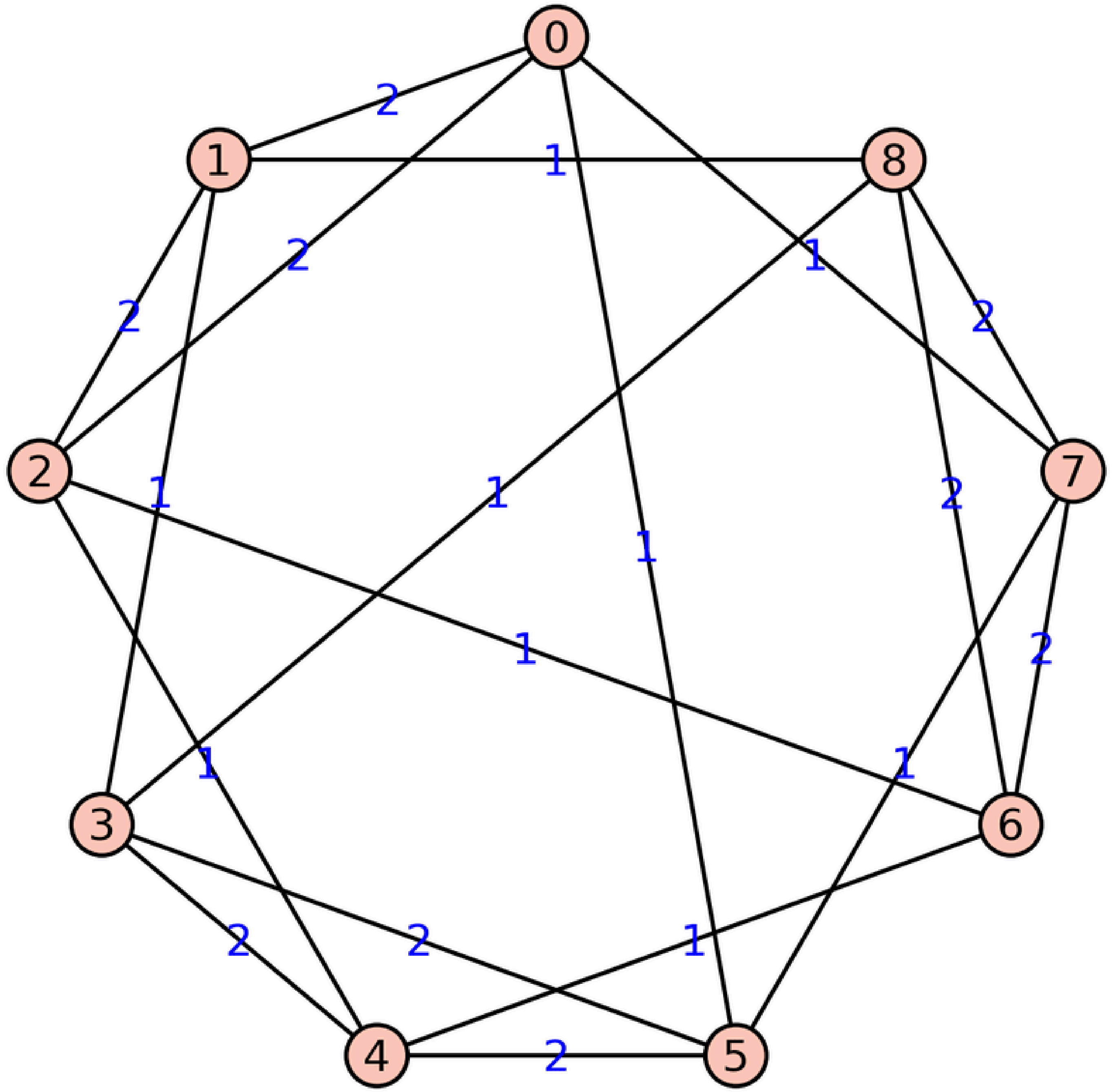
# Paley Graphs

**Vertex Set:**

Choose a positive integer q (often a prime or a prime power) such that
$q \equiv 1 \pmod{4}$.
Label the vertices with the numbers
$0,1,2,\ldots,q-1$.

**Edge Set:**

For any two distinct vertices i and j (considered modulo q), connect i and j with an edge if and only if the difference $(i-j)$ (computed modulo q) is a quadratic residue modulo q.



## Paley Graph - Quasi Random

Paley Graphs are Quasi Random because they satisfy P5

$$(P5) \quad \sum_{u,v \in V(G_n)} \left| s(u,v) - \frac{n}{2} \right| = o(n^3).$$

**Proof**:

Let us consider an element z of the field for it to be vertex contributing to s(u,v) it has to be satisfy:
either
1)z-u is a quadratic residue and z-v is a quadratic residue
or
2)z-u is not a quadratic residue and z-v is not a quadratic residue

# Paley Graph - Quasi Random

**Proof**:

Let us consider an element z of the field for it to be vertex contributing to s(u,v) it has to be satisfy:
either
1)z-u is a quadratic residue and z-v is a quadratic residue
or
2)z-u is not a quadratic residue and z-v is not a quadratic residue
We can include both these conditions in one expression, following term must be a quadratic residue

$$\frac{z-u}{z-v} = a \qquad \frac{z-u}{z-v} = 1 + \frac{v-u}{z-v} = a,$$

Here there exists a 1 to 1 relation between a and z and a is a square
No of squares in a the field is (q-1)/2 and we need to subtract 1 because a can not take value 1

## Paley Graph - Quasi Random

**Proof**:

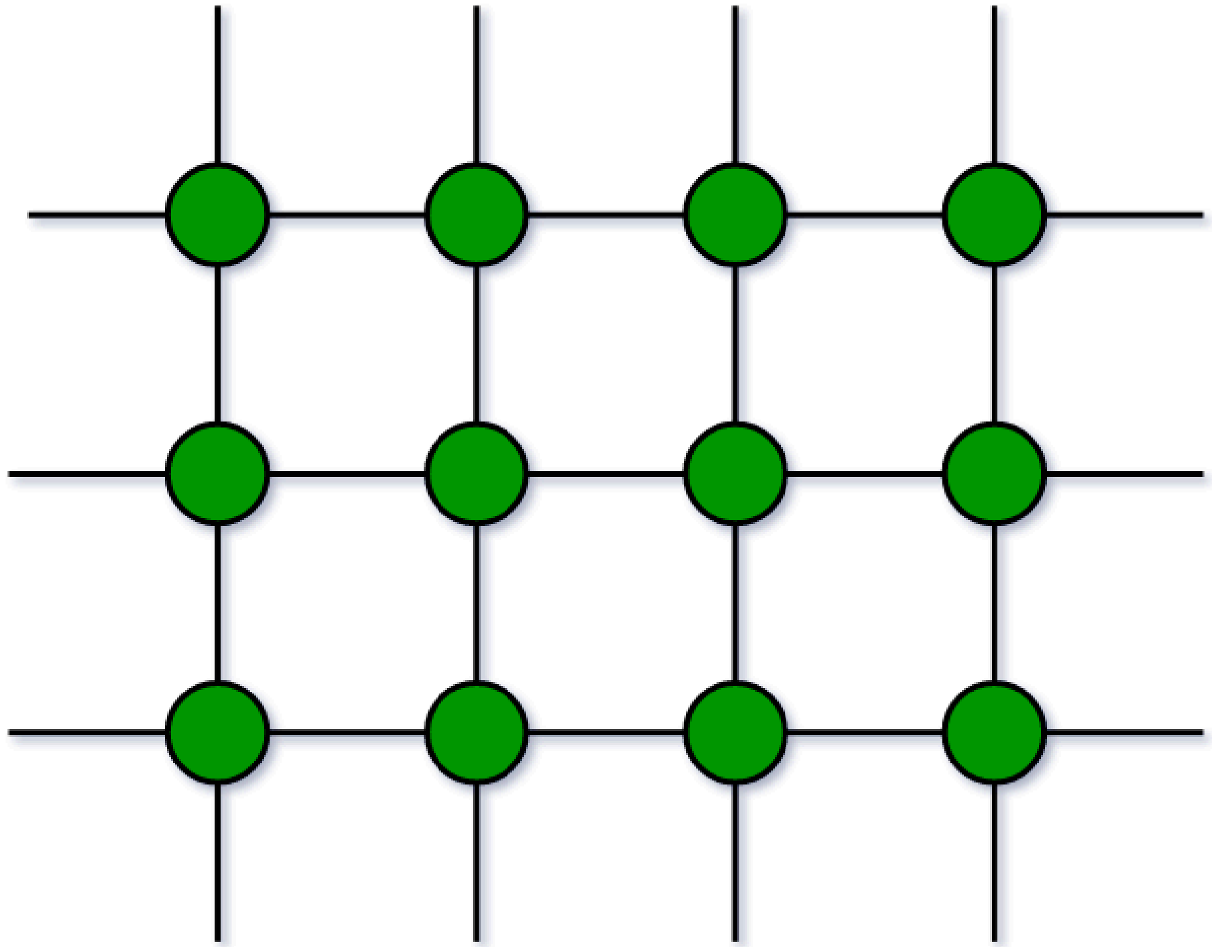
it follows that $s(u,v) = \frac{1}{2}(q-3)$. Hence,

$$\begin{aligned}
\sum_{u,v\in V(G_n)} \left| s(u,v) - \frac{q}{2} \right| &= \sum_{u,v\in V(G_n)} \left| \frac{1}{2}(q-3) - \frac{q}{2} \right| \\
&= \binom{q}{2}\frac{3}{2} = \frac{3\left(q^2-q\right)}{4} \\
&= o(q^3). \quad \square
\end{aligned}$$

THEOREM 3.7 ([**29, 38**]). *If $q > n^2 2^{2n-2}$, then $P_q$ is $n$-e.c.*

# THE INFINITE WEB

The **static web** consists of a finite number of pages (~54 billion), while the **dynamic web** can generate infinitely many pages based on user input or parameters. Thus, the web graph appears infinite in practice, though bounded by physical constraints.

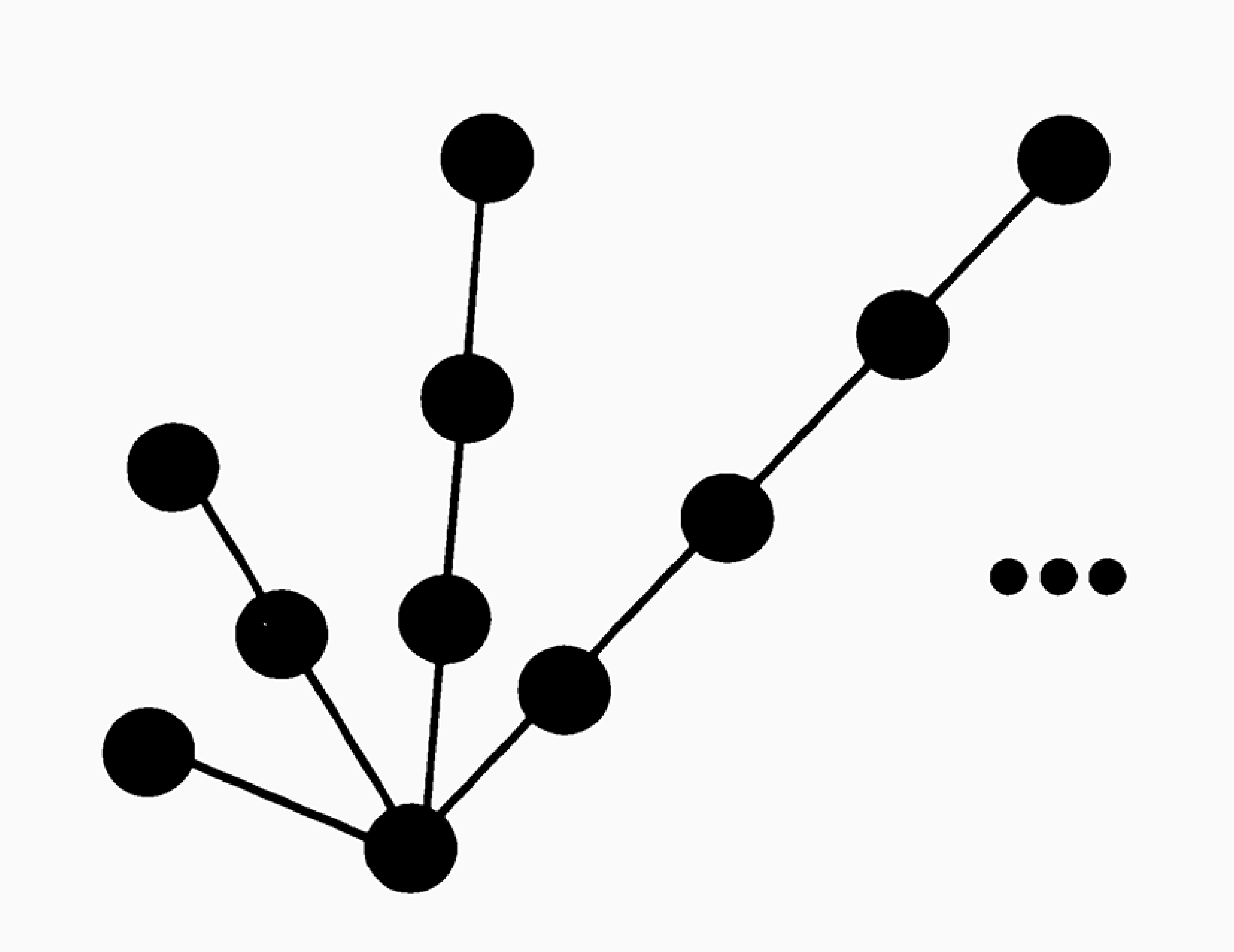


The view of G as an infinite graph presents an interesting perspective for a mathematician

Considering a graph as infinite gives rise to many interesting properties.

Consider the graph G, shown in the figure. To form G, attach a path of each finite length to a root vertex. The graph G has the property that there exists an infinite number of graphs H such that G and H are mutually embeddable ie - G <= H and H<= G.

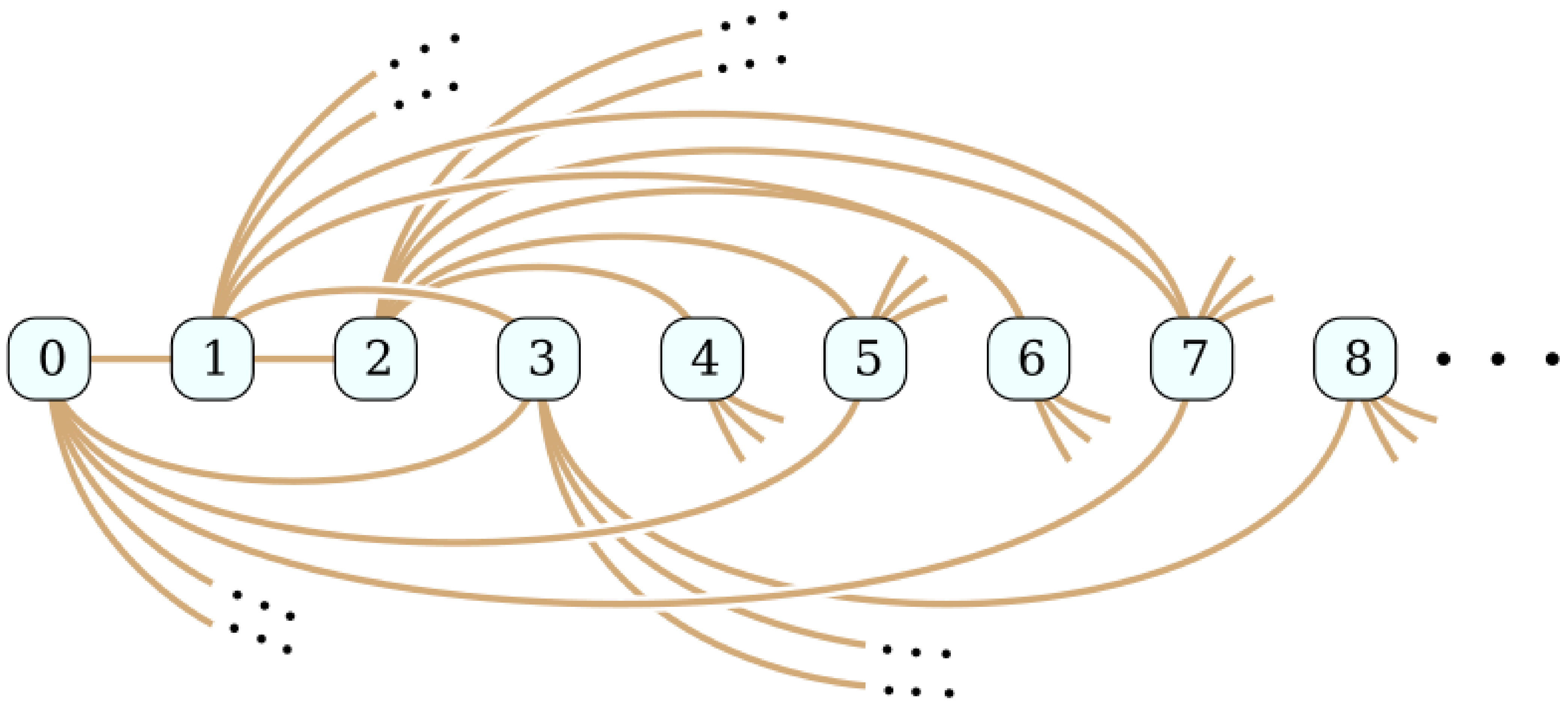
This kind of property exists only for infinite graphs, as for finite graphs, observe that the only graph that is mutually embeddable with it, is the graph itself.



# Infinite Random Graph

Consider the probability space G(N, p) consisting of graphs with V = N, the set of natural numbers in which every distinct pair of integers is joined independently with probability p.

The space G(N , p) is an infinite analogue of the finite random graph G(n, p)



## Infinite Random Graph

THEOREM 6.1. *With probability* 1, *all* $G \in G(\mathbb{N}, p)$ *are isomorphic.*

All graphs in this space are mutually isomorphic, that is, structurally apart from a relabelling of vertices all graphs can considered the same

Proof: To prove this we will consider 2 other theorems and use them. We will also use a very unique and interesting method of proving which is called back-and-forth.

# Infinite Graph R*

Start with a single-vertex graph R0, which is K1 (a graph with one vertex and no edges)

- For each non-negative integer t>0, construct the next graph Rt+1 from Rt as follows
- For every subset S of the vertex set V(Rt) (including the empty set), add a new vertex zS.

$$|V(R_{t+1})| = |V(R_t)| + 2^{|V(R_t)|}$$

## Age

Let $J = \lim_{t\to\infty} H_t$ be a limit of a chain $C = (H_t : t \in \mathbb{N})$ of graphs, where $H_t \prec H_{t+1}$ for all $t \in \mathbb{N}$. Define ages: $\mathrm{aged}(x) = t$ for $x \in V(J)$ by

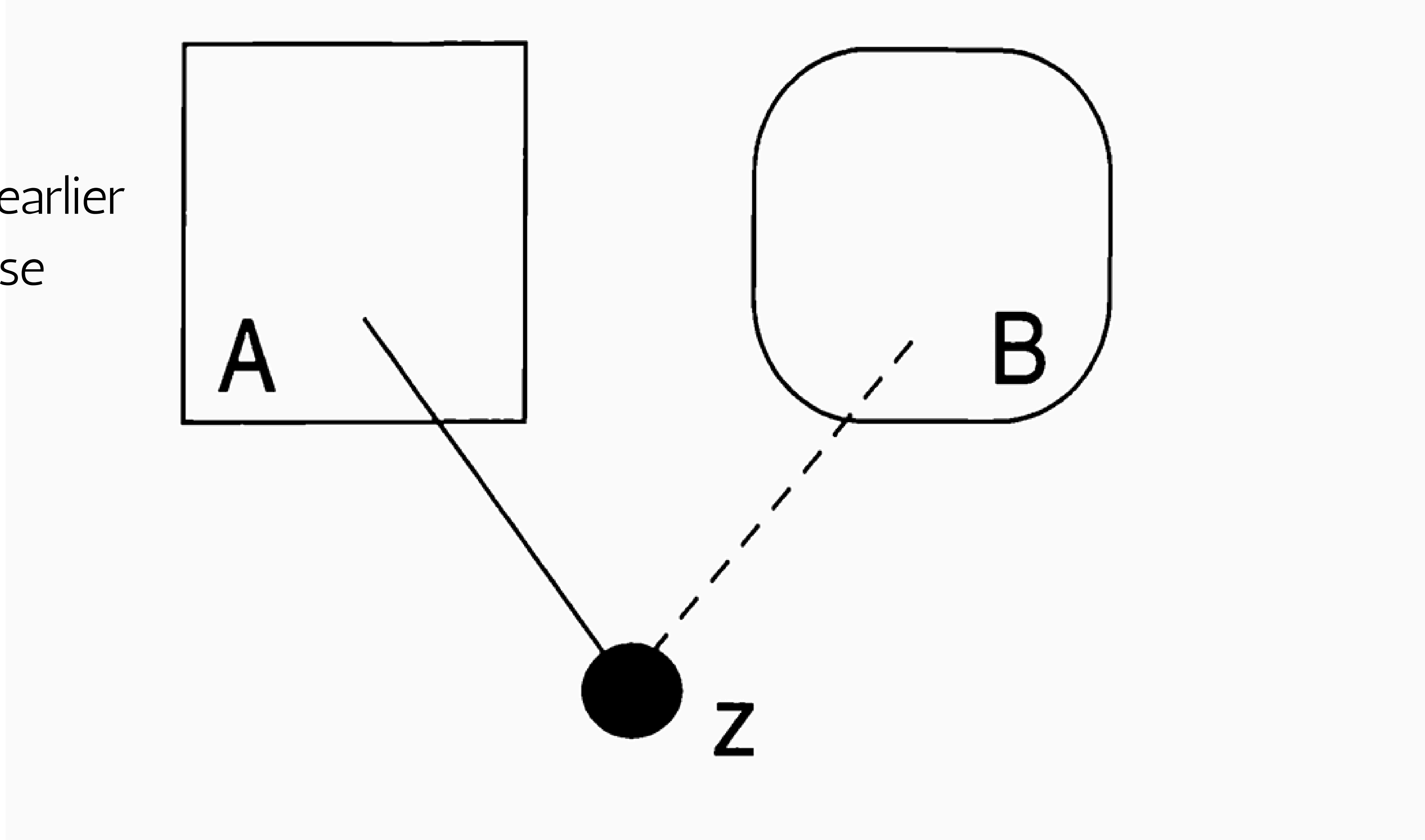
$$\mathrm{aged}(x) = \begin{cases} t & \text{if } x \in V(H_t) \setminus V(H_{t-1}) \text{ where } t > 0 \\ 0 & \text{else} \end{cases}$$

The age of a finite subset, written age(s), is max{age(x) : x E S}.

## The e.c property

A graph G is existentially closed or e. c. if for all disjoint finite sets of vertices A and B (one of which may be empty), there is a vertex z joined to all of A and to no vertex of B.

This is very similar to the n. e. c. property that was earlier defined for finite graphs, and is a conjunction of those properties.



THEOREM 6.2. *The graph $R^*$ is e.c.*

Proof: Fix finite disjoint A and B in V(R*). Let to = age(A U B). The vertex z = zA in Rto+1 is correctly joined(satisfies the ec property) to A and B.

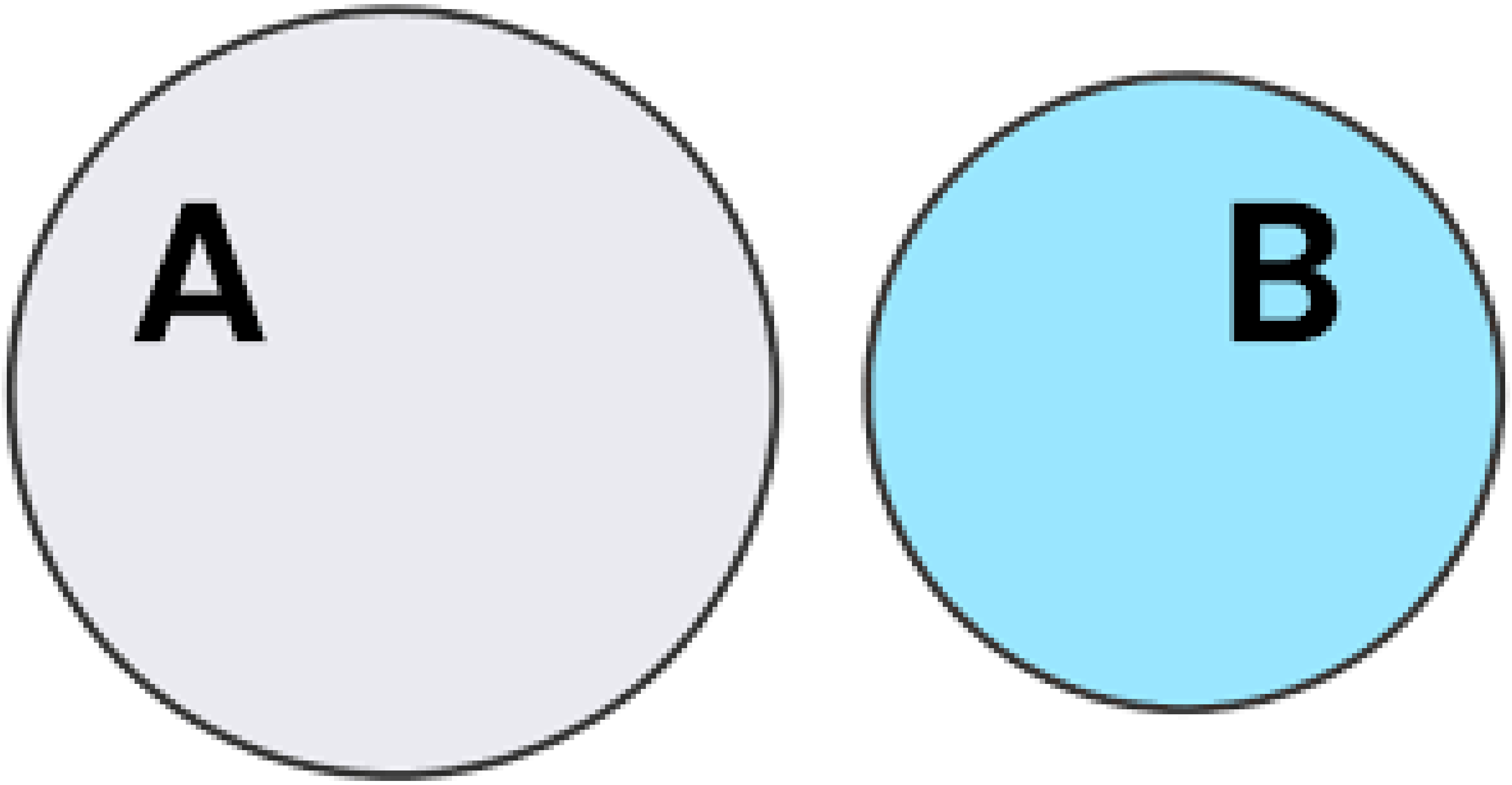
By the definition of age(x), we have that all the vertices that appear in the finite sets A and B are present in Rt0.

We may think of G(N, p) as the limit of an on-line random graph process, similar to how we defined R*. At time 0, let Xp be K1. Assuming a graph Xt at time t is defined and finite, at time t+1, add a new vertex z to form Xt+1 The vertex z is joined to each existing vertex independently with probability p. Then G(N, p) consists of limits limt-> infinity Xt.

THEOREM 6.3. *With probability* 1, $G \in G(\mathbb{N}, p)$ *is e.c.*

Proof: We, consider two disjoint sets A and B and let |A| = i and |B| = j. The probability that an edge occurs is p, and denote q = 1-p as the probability that an edge does not occur.

Now we need to link it to the e .c property, so let us consider a vertex Z correctly joined to these two sets.




Key Idea:

What will be the probability of such a vertex z (not occurring) as the number of edges t tends to infinity?

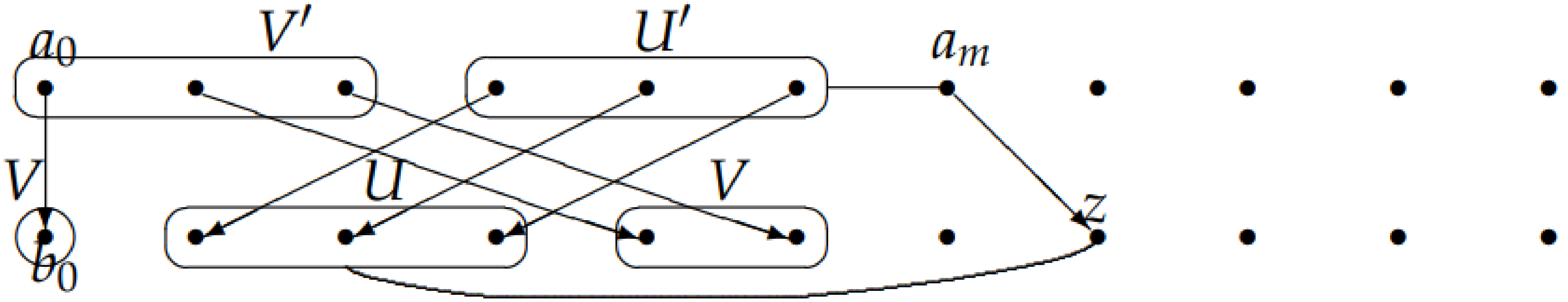
# Back and Forth method

THEOREM 6.4. *If $G$ and $H$ are e.c. graphs, then $G \cong H$.*

We use a method known to logicians as "back and forth".
Suppose that $\Gamma 1$ and $\Gamma 2$ are countable graphs satisfying $(\ast)$:
enumerate their vertex sets as $(a0, a1, \ldots)$ and $(b0, b1, \ldots)$. We build an isomorphism $\varphi$ between them in stages.

Proof:

At stage 0, map a0 to b0. At even-numbered stages, let am the first unmapped ai . Let U` and V` be its neighbours and non-neighbours among the vertices already mapped, and let U and V be their images under φ. Use (∗) in graph Γ2 to find a witness v for U and V. Then map am to z.




At odd-numbered stages, go in the other direction, using (∗) in Γ1 to choose a pre-image of the first unmapped vertex in Γ2. This approach guarantees that every vertex of Γ1 occurs in the domain, and every vertex of Γ2 in the range, of φ

## The graph R* is Universal

An existensially closed countable graph is universal in the sense every other countable graph embed in it.

Let G be any countable graph with vertices $V(G) = \{x_i : i \in \mathbb{N}\}$
Define $G_t$ as the subgraph induced by vertices $\{x_i : 0 \leqslant i \leqslant t\}$, so $G = \lim_{t\to\infty} G_t$

Base case: $f_0: G_0 \to R$

Hypothesis: $f_t$ exists

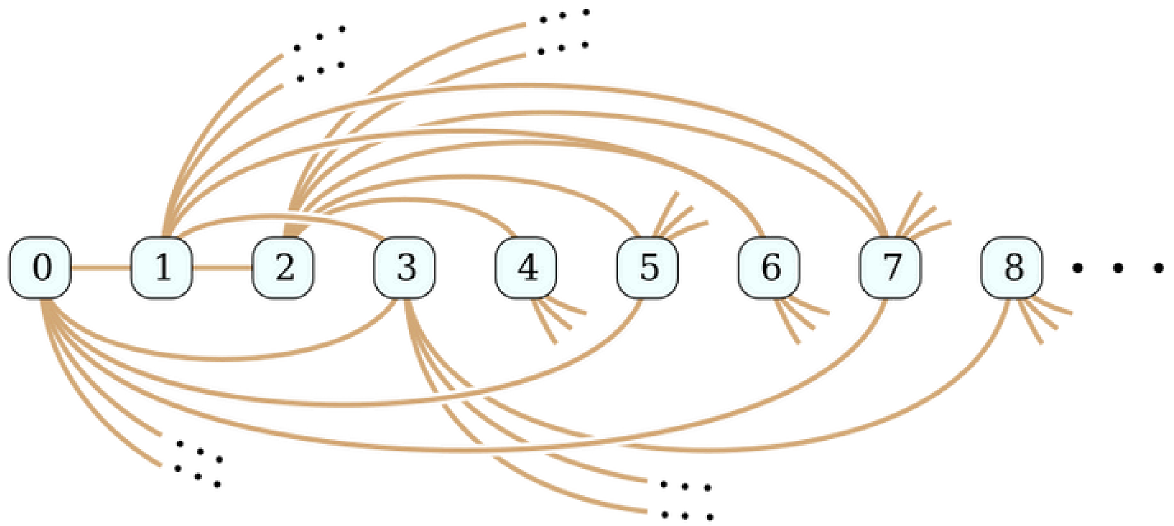
Induction:
Extend $f_t$ to $f_{t+1}$, vertex $x_{t+1}$ connects to some set S of vertices in $G_t$
By the e.c. property of R, find some z in R joined only to $f_t(S)$
Define $f_{t+1}$ by extending $f_t$ and mapping $x_{t+1}$ to z

The limit mapping $F = \lim_{t\to\infty} f_t$ gives a complete embedding of G into R1

## Countable infinite graph constructions:

### Rado Graph

A Rado graph can be constructed by connecting each pair of distinct natural numbers i<j with an edge if the i-th bit in the binary expansion of j is 1.




$$A = \{i_1, \ldots, i_m\} \quad B = \{j_1, \ldots, j_n\}$$

$$i_1 < \cdots < i_m \text{ and } j_1 < \cdots < j_n$$

$$z = 2^{i_1} + \cdots + 2^{i_m} + 2^{j_n+1}.$$

# THANK YOU !