Z-transform:

In mathematics and signal processing, the **Z-transform** converts a discrete-time signal, which is a sequence of real or complex numbers, into a complex frequency-domain (**z-domain** or **z-plane**) representation.

Z-transform and Inverse Z transform

- The Z-transform of a discrete time signal $x(n)$ is represented by $X(Z)$

$$x(n) \xleftrightarrow{ZT} X(Z)$$

- $$\boxed{X(Z) = \sum_{n=-\infty}^{\infty} x(n)\, Z^{-n}}$$

- $x(t) \longrightarrow$ freq $\longrightarrow$ F.T. , L.T.

  $x(n) \longrightarrow$ freq $\longrightarrow$ D.F.T , Z.T

- $$\text{F.T.}[x(t)] = \int_{-\infty}^{\infty} x(t)\, e^{-j\omega t}\, dt$$

- $$\text{D.F.T.}[x(n)] = \sum_{n=-\infty}^{\infty} x(n)\, e^{-j\omega n}$$

The Z-transform can be defined as either a *one-sided* or *two-sided* transform.

Bilateral Z-transform

The *bilateral* or *two-sided* Z-transform of a discrete-time signal $x[n]$ is the formal power series $X(z)$ defined as

$$X(z) = \mathcal{Z}\{x[n]\} = \sum_{n=-\infty}^{\infty} x[n] z^{-n}$$

where n is an integer and z is, in general, a complex number:

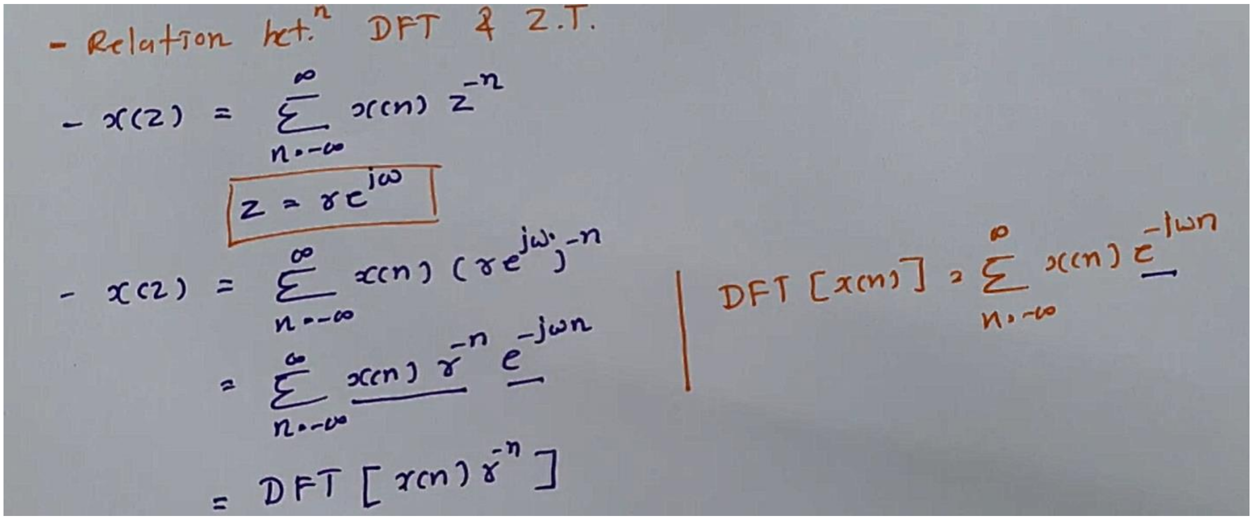
$$z = Ae^{j\phi} = A \cdot (\cos\phi + j\sin\phi)$$

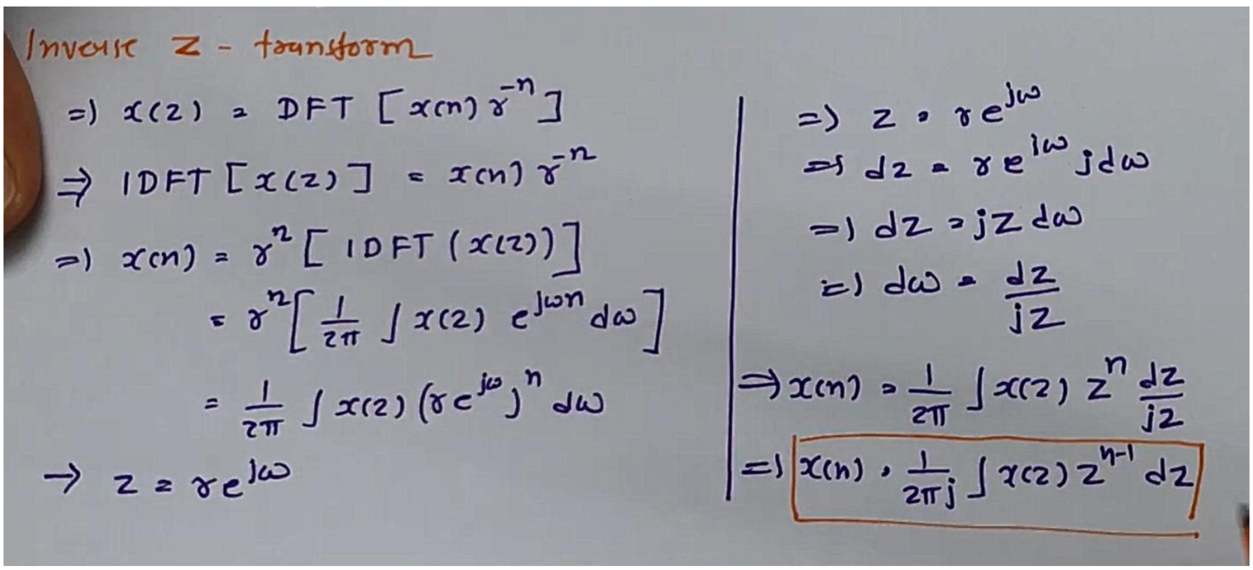
where A is the magnitude of z, j is the imaginary unit, and ϕ is the *complex argument* (also referred to as *angle* or *phase*) in radians.

Unilateral Z-transform

Alternatively, in cases where $x[n]$ is defined only for $n \geq 0$, the *single-sided* or *unilateral* Z-transform is defined as

$$X(z) = \mathcal{Z}\{x[n]\} = \sum_{n=0}^{\infty} x[n] z^{-n}.$$





Region of Convergence

Region of Convergence is the range of complex variable Z in the Z-plane. The Z-transformation of the signal is finite or convergent. So, ROC represents those set of values of Z, for which $X\,(Z)$ has a finite value.

Properties of ROC

- ROC does not include any pole.
- For right-sided signal, ROC will be outside the circle in Z-plane.
- For left sided signal, ROC will be inside the circle in Z-plane.
- For stability, ROC includes unit circle in Z-plane.
- For Both sided signal, ROC is a ring in Z-plane.
- For finite-duration signal, ROC is entire Z-plane.

Example

Let us find the Z-transform and the ROC of a signal given as $x(n)=\{7,3,4,9,5\}$, where origin of the series is at 3.

Solution − Applying the formula we have -

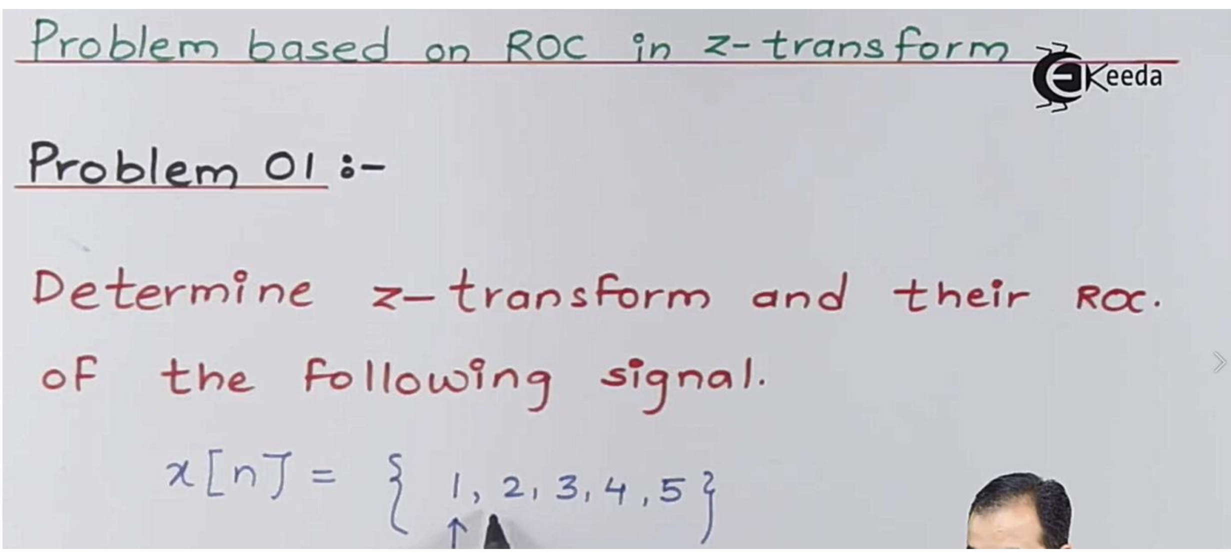
$$X(z) = \sum_{n=-\infty}^{\infty} x(n)Z^{-n}$$

$$= \sum_{n=-1}^{3} x(n)Z^{-n}$$

$$= x(-1)Z + x(0) + x(1)Z^{-1} + x(2)Z^{-2} + x(3)Z^{-3}$$

$$= 7Z + 3 + 4Z^{-1} + 9Z^{-2} + 5Z^{-3}$$

ROC is the entire Z-plane excluding Z = 0, ∞



Solution :-

$$x[n] = \{\underset{\underset{0}{\uparrow}}{1}, \underset{1}{2}, \underset{2}{3}, \underset{3}{4}, \underset{4}{5}\}$$

$$z[x[n]] = X(z) = \sum_{n=-\infty}^{\infty} x(n) \cdot z^{-n}.$$

$n = 0$ to 4

$$\therefore X(z) = \sum_{n=0}^{4} x(n) \cdot z^{-n}.$$

$$= x(0) + x(1) \cdot z^{-1} + x(2) \cdot z^{-2} + x(3)\, z^{-3} + x(4) \cdot z^{-4}.$$

$$\underline{X(z)} = 1 + 2z^{-1} + 3z^{-2} + 4z^{-3} + 5z^{-4}$$

$$X(z) = 1 + 2z^{-1} + 3z^{-2} + 4z^{-3} + 5z^{-4}.$$

$$\therefore X(z) = 1 + \frac{2}{z} + \frac{3}{z^2} + \frac{4}{z^3} + \frac{5}{z^4}$$

at $z = 0$, $X(z) = \infty$.

At $z = \infty$, $X(z) = 1 + 0 + 0 + 0 + 0 = 1$.

The ROC of $X(z)$ is available over the entire region of z-plane, except $z = 0$.

Problem 02:-

Determine z-transform and their ROC of the following discrete time signal,

$$x[n] = \{1, 2, 3, 4, \underset{\uparrow}{5}\}$$

Solution:-

$$x[n] = \{\underset{-4}{1}, \underset{-3}{2}, \underset{-2}{3}, \underset{-1}{4}, \underset{\uparrow 0}{5}\}$$

By the definition of z-transform,

$$z[x(n)] = X(z) = \sum_{n=-\infty}^{\infty} x(n).z^{-n}$$

$n = -4$ to 0.

$$\therefore X(z) = \sum_{n=-4}^{0} x(n) \cdot z^{-n}.$$

$$= x(-4) \cdot z^{4} + x(-3) \cdot z^{3} + x(-2) \cdot z^{2} + x(-1) z^{1} + x(0).$$

$$X(z) = 1 \cdot z^{4} + 2z^{3} + 3z^{2} + 4z + 5.$$

$$\therefore X(z) = z^4 + 2z^3 + 3z^2 + 4z + 5$$

R.O.C.

At $z=0$,

$$X(z) = 5$$

At $z=\infty$,

$$X(z) = \infty$$

The ROC of $X(z)$ is available over entire region of z-plane, except $z=\infty$.

A finite sequence $x(n)$ is defined as $x(n) = \{\underset{\uparrow}{\overset{x(0)}{5}}, \overset{x(1)}{3}, \overset{x(2)}{-3}, \overset{x(3)}{0}, \overset{x(4)}{4}, \overset{x(5)}{-2}\}$ find $X(z)$

$$Z.T[x(n)] = X(z) = \sum_{n=-\infty}^{\infty} x(n)\, z^{-n}$$

$$= \sum_{n=0}^{5} x(n)\, z^{-n}$$

$$= x(0)z^{0} + x(1)z^{-1} + x(2)z^{-2} + \underbrace{x(3)z^{-3}}_{0} + x(4)z^{-4} + x(5)z^{-5}$$

$$\therefore X(z) = 5 + 3z^{-1} - 3z^{-2} + 4z^{-4} - 2z^{-5}$$

Z- transform for finite sequence

A finite sequence $x(n)$ is defined as $x(n) = \{5, 3, \underset{\uparrow}{-3}, 0, 4, -2\}$ find $X(z)$ of given sequence.

$$- \; x(n) = \{\overset{x(-2)}{5}, \overset{x(-1)}{3}, \underset{\uparrow}{\overset{x(0)}{-3}}, \overset{x(1)}{0}, \overset{x(2)}{4}, \overset{x(3)}{-2}\}$$

$$- \; X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n} = \sum_{n=-2}^{3} x(n) z^{-n}$$

$$= x(-2) z^{2} + x(-1) z^{1} + x(0) z^{0} + x(1) z^{-1} + x(2) z^{-2} + x(3) z^{-3}$$

$$= 5z^{2} + 3z^{1} - 3z^{0} + 0 \times z^{-1} + 4z^{-2} + (-2) z^{-3}$$

$$\boxed{X(z) = 5z^{2} + 3z^{1} - 3 + 4z^{-2} - 2z^{-3}}$$

A Finite duration sequence $x(n) = \{\underset{\uparrow}{5}, 3, 0, 1, 2, 4\}$ Find Z- transform of $x(n)$.

$$- \; x(n) = \{\underset{\uparrow}{\overset{x(0)}{5}}, \overset{x(1)}{3}, \overset{x(2)}{0}, \overset{x(3)}{1}, \overset{x(4)}{2}, \overset{x(5)}{4}\}$$

$$- \; X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n} = \sum_{n=0}^{5} x(n) z^{-n}$$

$$= x(0) z^{0} + x(1) z^{-1} + x(2) z^{-2} + x(3) z^{-3} + x(4) z^{-4} + x(5) z^{-5}$$

$$= 5z^{0} + 3z^{-1} + 0z^{-2} + 1 \times z^{-3} + 2 \times z^{-4} + 4 \times z^{-5}$$

$$= \boxed{5 + 3z^{-1} + z^{-3} + 2z^{-4} + 4z^{-5}}$$

Linearity

It states that when two or more individual discrete signals are multiplied by constants, their respective Z-transforms will also be multiplied by the same constants.

Mathematically,

$$a_1x_1(n) + a_2x_2(n) = a_1X_1(z) + a_2X_2(z)$$

Proof − We know that,

$$X(Z) = \sum_{n=-\infty}^{\infty} x(n)Z^{-n}$$

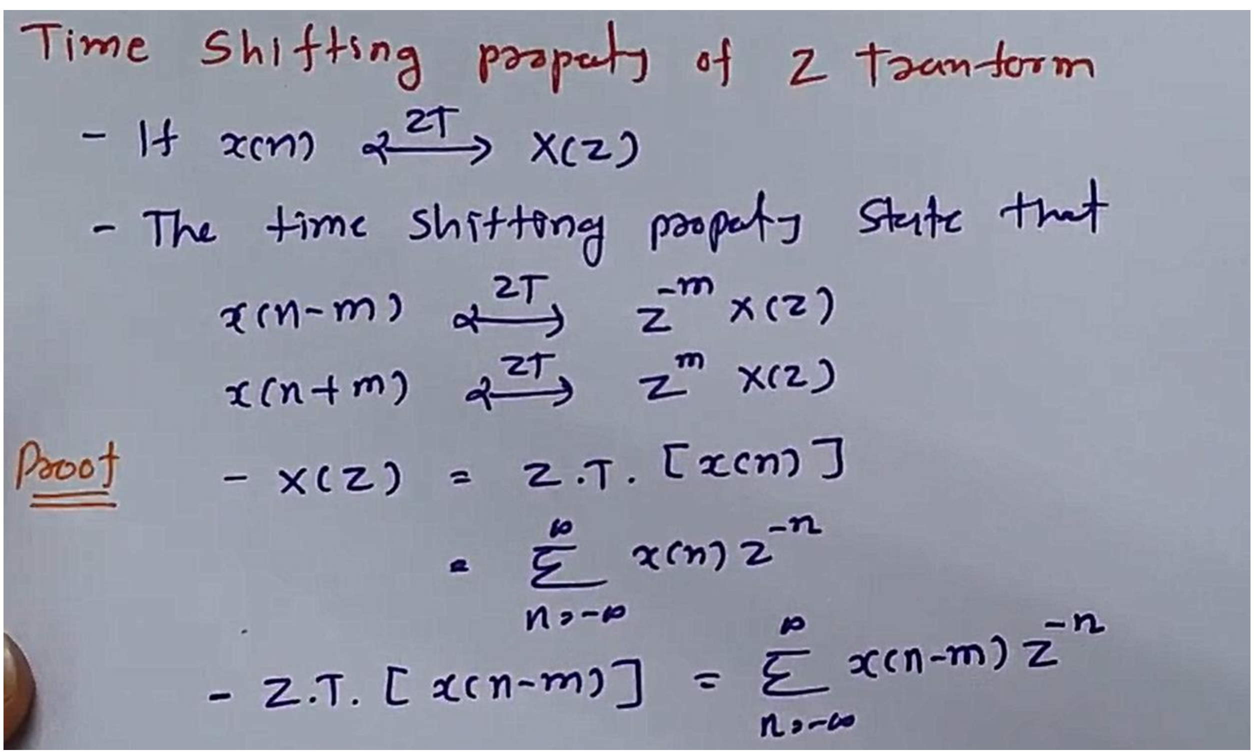
$$= \sum_{n=-\infty}^{\infty}(a_1x_1(n) + a_2x_2(n))Z^{-n}$$

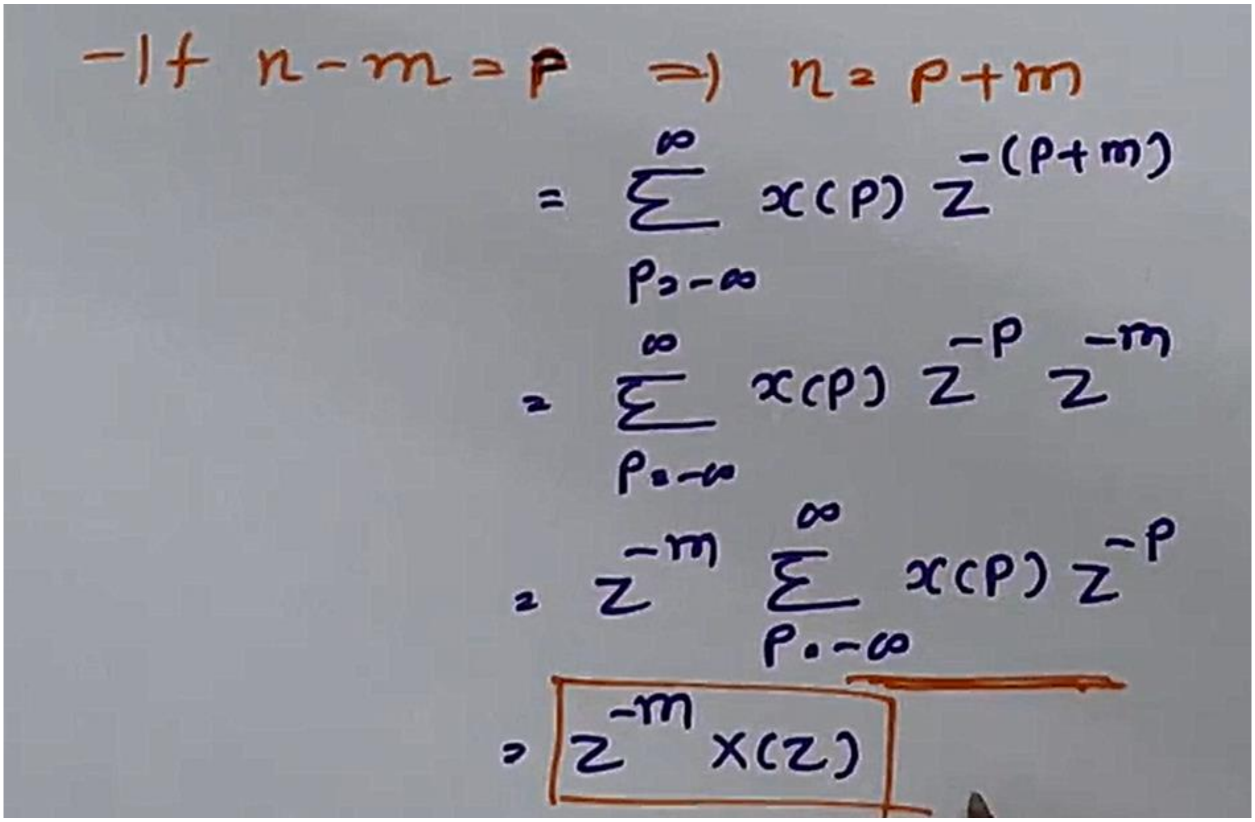
$$= a_1\sum_{n=-\infty}^{\infty} x_1(n)Z^{-n} + a_2\sum_{n=-\infty}^{\infty} x_2(n)Z^{-n}$$

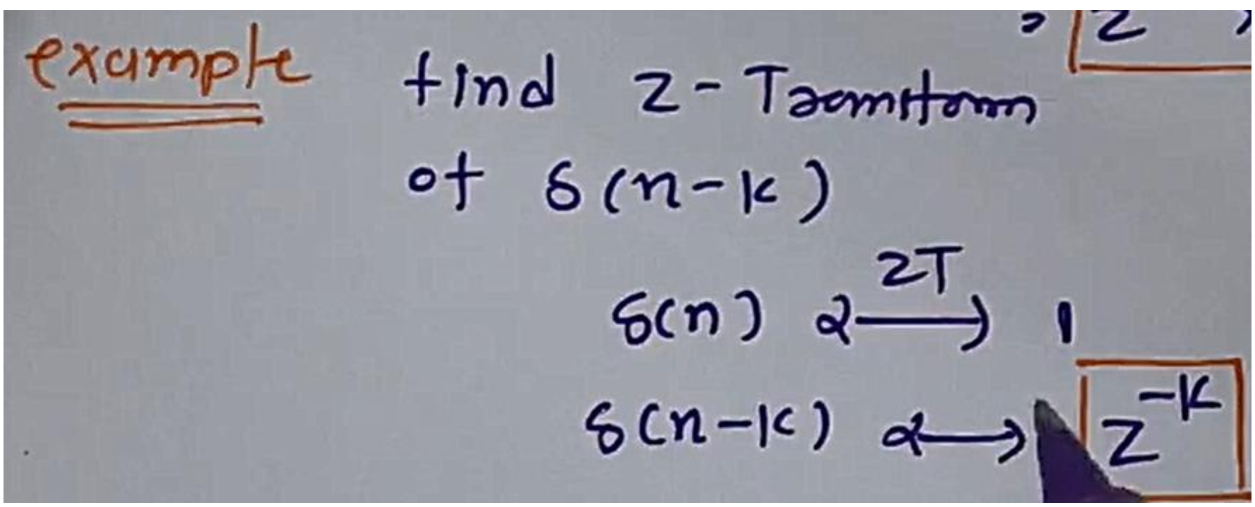
$$= a_1X_1(z) + a_2X_2(z) \qquad\qquad HenceProved$$

Here, the ROC is $ROC_1 \bigcap ROC_2$.

Time Shifting







Time shifting property depicts how the change in the time domain in the discrete signal will affect the Z-domain, which can be written as;

$$x(n - n_0) \longleftrightarrow X(Z)Z^{-n}$$

Or $$x(n-1) \longleftrightarrow Z^{-1}X(Z)$$

Proof –

Let $y(P) = X(P - K)$

$$Y(z) = \sum_{p=-\infty}^{\infty} y(p)Z^{-p}$$

$$= \sum_{p=-\infty}^{\infty} (x(p-k))Z^{-p}$$

Let s = p-k

$$= \sum_{s=-\infty}^{\infty} x(s)Z^{-(s+k)}$$

$$= \sum_{s=-\infty}^{\infty} x(s)Z^{-s}Z^{-k}$$

$$= Z^{-k}[\sum_{s=-\infty}^{\infty} x(m)Z^{-s}]$$

$$= Z^{-k}X(Z) \qquad\qquad HenceProved$$

Time Scaling

Scaling property of Z Transform

- $x(n) \xleftrightarrow{ZT} X(z)$

Then scaling property of Z Transform states that

$a^n x(n) \xleftrightarrow{ZT} X[z/a]$

Proof

$$X(z) = Z.T.[x(n)]$$

$$= \sum_{n=-\infty}^{\infty} x(n) z^{-n}$$

$$Z.T.[a^n (x(n))] = \sum_{n=-\infty}^{\infty} a^n x(n) z^{-n}$$

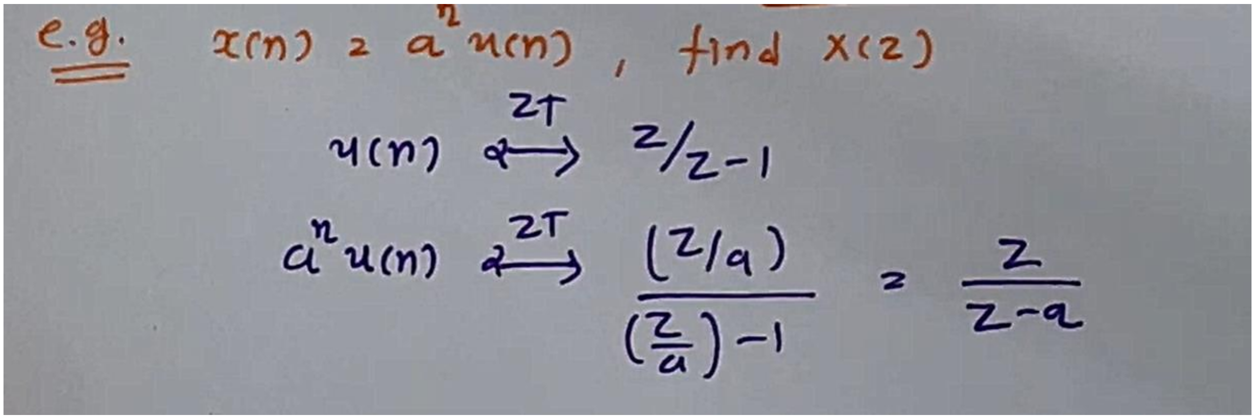
$$Z.T.[a^n (x(n))] = \sum_{n=-\infty}^{\infty} a^n x(n) z^{-n}$$

$$= \sum_{n=-\infty}^{\infty} x(n) (az^{-1})^n$$

$$= \sum_{n=-\infty}^{\infty} x(n) \left(\frac{1}{az^{-1}}\right)^{-n}$$

$$= \sum_{n=-\infty}^{\infty} x(n) (a^{-1}z)^{-n}$$

$$= \boxed{X(z/a)}$$



Time Scaling property tells us, what will be the Z-domain of the signal when the time is scaled in its discrete form, which can be written as;

$$a^n x(n) \longleftrightarrow X(a^{-1}Z)$$

Proof −

Let $y(p) = a^p x(p)$

$$Y(P) = \sum_{p=-\infty}^{\infty} y(p) Z^{-p}$$

$$= \sum_{p=-\infty}^{\infty} a^p x(p) Z^{-p}$$

$$= \sum_{p=-\infty}^{\infty} x(p) [a^{-1}Z]^{-p}$$

$$= X(a^{-1}Z) \qquad Hence proved$$

Convolution

This depicts the change in Z-domain of the system when a convolution takes place in the discrete signal form, which can be written as −

$$x_1(n) * x_2(n) \longleftrightarrow X_1(Z).\, X_2(Z)$$

Proof −

$$X(Z) = \sum_{n=-\infty}^{\infty} x(n) Z^{-n}$$

$$= \sum_{n=-\infty}^{\infty} [\sum_{k=-\infty}^{\infty} x_1(k) x_2(n-k)] Z^{-n}$$

$$= \sum_{k=-\infty}^{\infty} x_1(k) [\sum_{n}^{\infty} x_2(n-k) Z^{-n}]$$

$$= \sum_{k=-\infty}^{\infty} x_1(k) [\sum_{n=-\infty}^{\infty} x_2(n-k) Z^{-(n-k)} Z^{-k}]$$

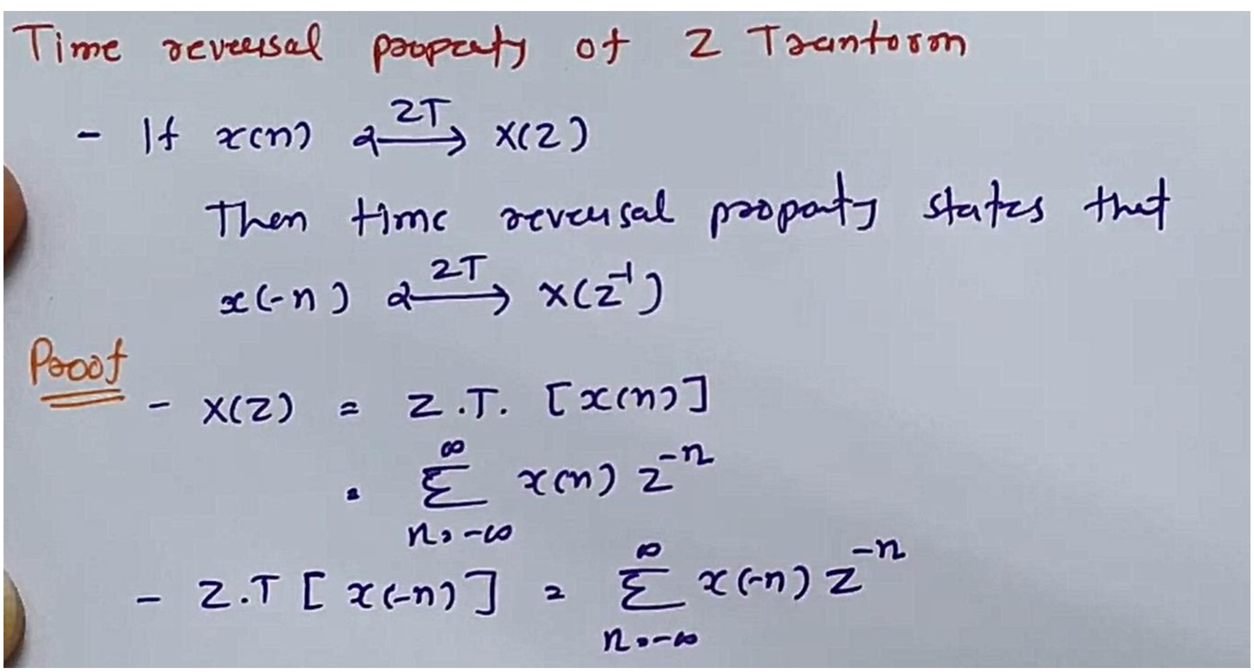
Let n-k = l, then the above equation cab be written as −

$$X(Z) = \sum_{k=-\infty}^{\infty} x_1(k) [Z^{-k} \sum_{l=-\infty}^{\infty} x_2(l) Z^{-l}]$$

$$= \sum_{k=-\infty}^{\infty} x_1(k) X_2(Z) Z^{-k}$$

$$= X_2(Z) \sum_{k=-\infty}^{\infty} x_1(Z) Z^{-k}$$

$$= X_1(Z).\, X_2(Z) \qquad\qquad HenceProved$$



$$- \; Z.T[x(-n)] = \sum_{n=-\infty}^{\infty} x(-n) Z^{-n}$$

If $p = -n$

$$= \sum_{p=-\infty}^{\infty} x(p) \, Z^{p}$$

$$= \sum_{p=-\infty}^{\infty} x(p) \, (Z^{-1})^{-p}$$

$$= X(Z^{-1})$$

e.g. $x(n) = a^{n} u(-n)$. find $X(Z)$

$$- \; x(n) = a^{n} u(-n)$$

$$= (1/a)^{-n} u(-n)$$

$$x(n) = (1/a)^{n} u(n) \xleftrightarrow{ZT} \frac{Z}{Z - 1/a}$$

$$(1/a)^{-n} u(-n) \xleftrightarrow{ZT} \frac{Z^{-1}}{Z^{-1} - 1/a} = \frac{1/Z}{1/Z - 1/a} = \boxed{\frac{a}{a - Z}}$$

Multiplication & Convolution property of Z-Transform

If $x_1(n) \xleftrightarrow{ZT} X_1(Z)$

$x_2(n) \xleftrightarrow{ZT} X_2(Z)$

- Multiplication property states that

$$x_1(n) \, x_2(n) \xleftrightarrow{ZT} X_1(Z) * X_2(Z)$$

- convolution property states that

$$x_1(n) * x_2(n) \xleftrightarrow{ZT} X_1(Z) \, X_2(Z).$$

e.g. $u(n-1) * \delta(n)$, find Z-Transform

- $x_1(n) = u(n-1) \xleftrightarrow{ZT} X_1(z) = \frac{1}{z-1}$
- $x_2(n) = \delta(n) \xleftrightarrow{ZT} X_2(z) = 1$
- $X(z) = ZT[x_1(n) * x_2(n)]$

$$= X_1(z) X_2(z)$$

$$= \frac{1}{z-1}$$

e.g. $u(n-2) * \delta(n-3)$, find Z-Transform.

- $x_1(n) = u(n-2) \xleftrightarrow{ZT} X_1(z) = z^{-2}\left(\frac{z}{z-1}\right) = \frac{1}{z}\left(\frac{1}{z-1}\right)$
- $x_2(n) = \delta(n-1) \xleftrightarrow{ZT} X_2(z) = z^{-1} = 1/z$
- $X(z) = ZT[x_1(n) * x_2(n)]$

$$= X_1(z) X_2(z)$$

$$= \frac{1}{z}\left(\frac{1}{z-1}\right)\left(\frac{1}{z}\right)$$

$$= \frac{1}{z^2(z-1)}$$

Successive Differentiation

Successive Differentiation property shows that Z-transform will take place when we differentiate the discrete signal in time domain, with respect to time. This is shown as below.

$$\frac{dx(n)}{dn} = (1 - Z^{-1})X(Z)$$

Proof −

Consider the LHS of the equation − $\frac{dx(n)}{dn}$

$$= \frac{[x(n) - x(n-1)]}{[n - (n-1)]}$$

$$= x(n) - X(n-1)$$

$$= x(Z) - Z^{-1}x(Z)$$

$$= (1 - Z^{-1})x(Z) \qquad\qquad HenceProved$$

Multiplication in Time

It gives the change in Z-domain of the signal when multiplication takes place at discrete signal level.

$$x_1(n).\,x_2(n) \longleftrightarrow (\frac{1}{2\Pi j})[X1(Z) * X2(Z)]$$

Conjugation in Time

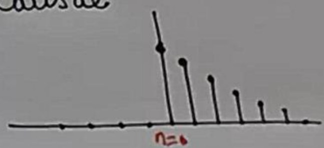
This depicts the representation of conjugated discrete signal in Z-domain.

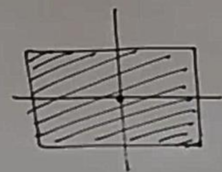
$$X^*(n) \longleftrightarrow X^*(Z^*)$$

Region of Convergence (RoC)

Finite Duration Signals

Causal

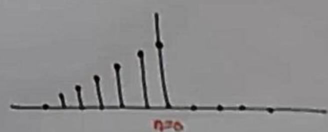


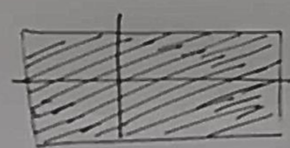


RoC

Entire z-plane except $|z| = 0$

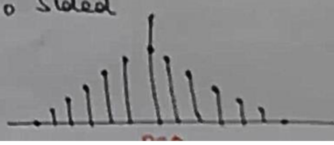
Anti Causal

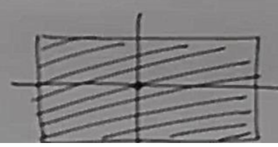




Entire z-plane except $|z| = \infty$

Two Sided

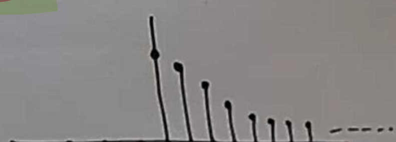


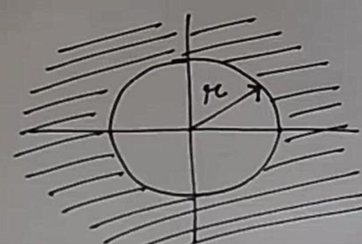


Entire z-plane except $|z| = 0$ & $|z| = \infty$

Infinite Duration Signals

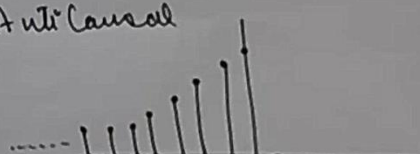
Causal

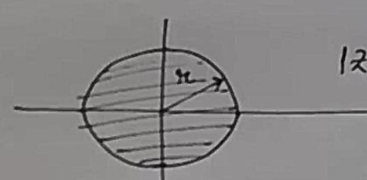




$|z| > r$

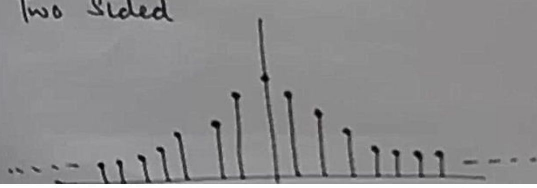
Anti Causal

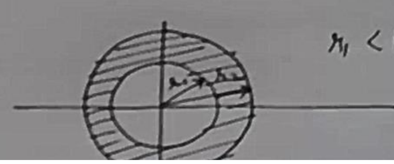




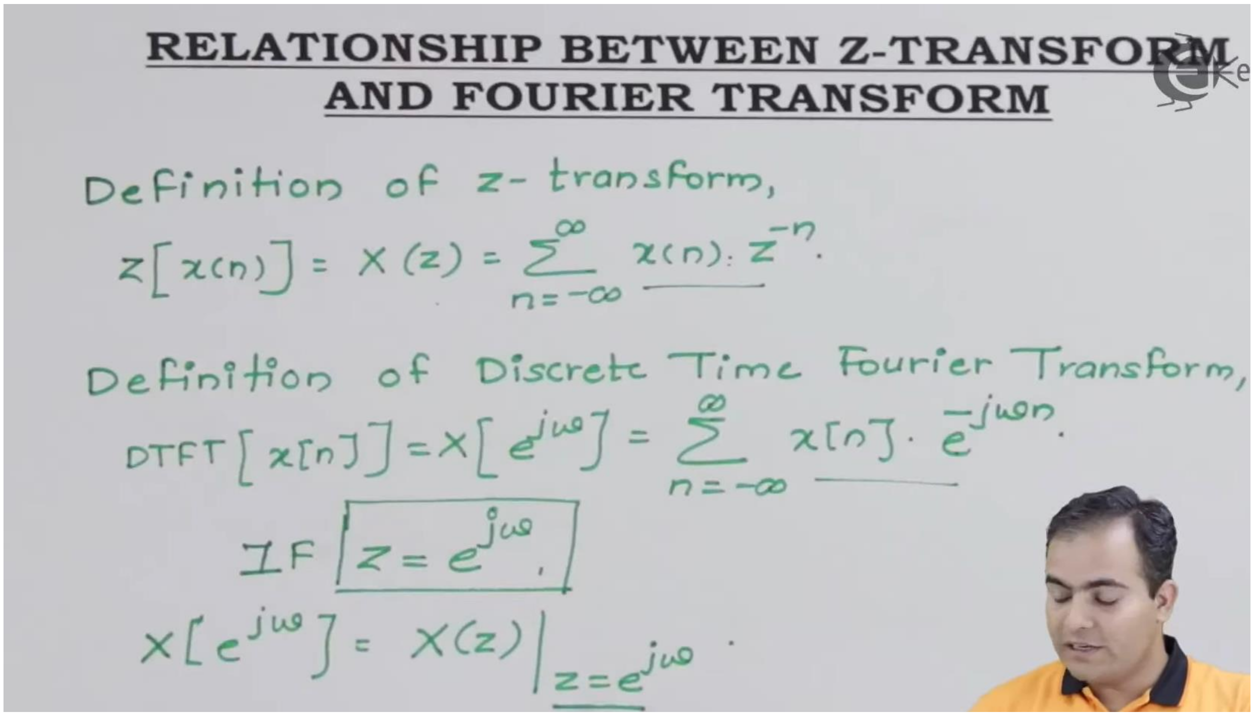
$|z| < r$

Two Sided





$r_1 < |z| < r_2$



57. Determine the z-transform of the signal $x(n) = \alpha^n u(n)$ and also the ROC and pole & zero locations of $X(z)$ in the z-plane.

Solution:

Given $x(n) = \alpha^n u(n)$

By definition $X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n}$

$$= \sum_{n=-\infty}^{\infty} \alpha^n u(n) z^{-n}$$

$$= \sum_{n=0}^{\infty} (\alpha z^{-1})^n$$

Using geometric series,

$$\sum_{n=0}^{\infty} a^n = \frac{1}{1-a}$$

$$\therefore X(z) = \frac{1}{1-\alpha z^{-1}}; |z| > |\alpha|$$

$$= \frac{z}{z-\alpha}; |z| > |\alpha|$$

There is a pole at z = α & zero at z = 0.

ROC of Impulse function in Z - Transform

1. $\delta(n) \xleftrightarrow{ZT} 1$

   ROC : Entire Z plane, including $z=0$ & $z=\infty$

2. $\delta(n-k) \xleftrightarrow{ZT} z^{-k}$

   ROC : Entire Z plane, including $z=\infty$ & excluding $z=0$

3. $\delta(n+k) \xleftrightarrow{ZT} z^{k}$

   ROC : Entire Z plane, Including $z=0$ & excluding $z=\infty$

$\delta(n-3)$ - Entire Z plane.
Includes $z=\infty$
Exclude $z=0$.

ROC of discrete time sequence in Z Transform

1. If sequence is purely right sided or causal, then

   ROC : Entire Z plane except $z = 0$

   $$x(n) = \{\underset{\uparrow}{1}, 2, 3, 4\} \xleftrightarrow{ZT} X(z) = 1 + 2z^{-1} + 3z^{-2} + 4$$

2. If sequence is purely Left sided or Anticausal, Then

   ROC ! Entire z plane except at $z = 0$

   Check may be ∞

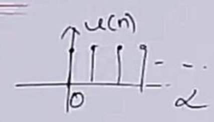
   $$x(n) = \{1, 2, 3, \underset{\uparrow}{4}\} \xleftrightarrow{ZT} X(z) = z + 2z + 3z + 4$$

3. If sequence is a two sided, Then

   ROC ! Entire z plane except at $z = 0$ & $z = \infty$

   $$x(n) = \{1, 2, \underset{\uparrow}{1}, 3, 1\} \xleftrightarrow{ZT} X(z) = 2z^{2} + 2z + 1 + 3z^{-1} + z^{-2}$$

* Z-Transform of Discrete unit step function. u(n) *



$$Z.T[u(n)]$$

$$Z.T[x(n)] = X(z) = \sum_{n=-\infty}^{\infty} x(n)\, z^{-n}$$

$$Z.T[u(n)] = \sum_{n=-\infty}^{\infty} u(n)\, z^{-n}$$

$$\boxed{\sum_{n=0}^{\infty} a^n = \frac{1}{1-a}}$$

$$= \sum_{n=0}^{\infty} z^{-n}$$

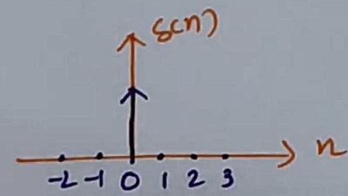
$$= \sum_{n=0}^{\infty} \left(z^{-1}\right)^n = \frac{1}{1-z^{-1}} = \frac{z}{z-1}$$

Z- Transform of Unit Impulse function

- $x(n) = \delta(n)$

- $X(z) = Z.T.[x(n)]$

$$= \sum_{n=-\infty}^{\infty} x(n)\, z^{-n}$$



$$= \sum_{n=-\infty}^{\infty} \delta(n)\, z^{-n}$$

$$= \dots + \delta(-2) z^{2} + \delta(-1) z^{1} + \underline{\delta(0)} z^{0} + \delta(1) z^{-1} + \delta(2) z^{-2} + \dots$$

$$= 0 \times z^{2} + 0 \times z^{1} + 1 \times z^{0} + 0 \times z^{-1} + 0 \times z^{-2} + \dots$$

$$= 1$$

Z- Transform of Standard basic signals

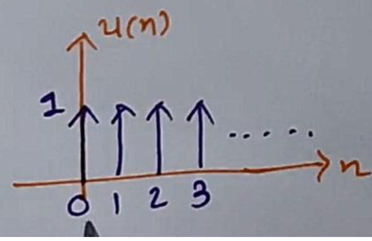
- Find The Z- transform of the signals $a^{n} u(n)$ & $a^{-n} u(n)$

- $x(n) = a^{n} u(n)$ $\qquad$ $y(n) = a^{-n} u(n)$

- $X(z) = Z.T.[x(n)]$

$$= \sum_{n=-\infty}^{\infty} x(n)\, z^{-n}$$

$$= \sum_{n=-\infty}^{\infty} a^{n} u(n)\, z^{-n}$$



0 1 2 3

$$- X(z) = \sum_{n=0}^{\infty} (a^n)(z^{-n})$$

$$= \sum_{n=0}^{\infty} (az^{-1})^n$$

We know $\sum_{n=0}^{\infty} a^n = \frac{1}{1-a}$

$$X(z) = \frac{1}{1-az^{-1}} = \frac{1}{1-(a/z)} = \frac{z}{z-a}$$

$$y(n) = a^{-n} u(n)$$

$$- Y(z) = Z.T. [y(n)]$$

$$= \sum_{n=-\infty}^{\infty} y(n) z^{-n}$$

$$= \sum_{n=-\infty}^{\infty} a^{-n} u(n) z^{-n}$$

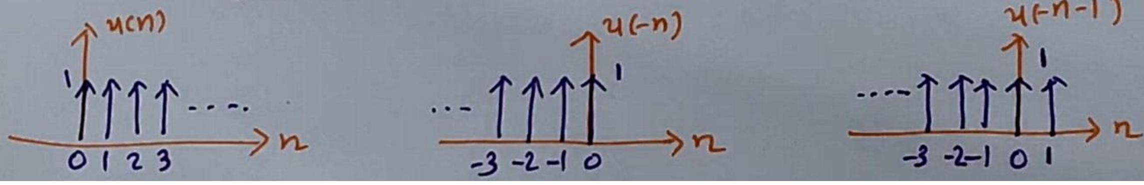
Z- Transform of standard basic signal

Find the Z- transform of the sequence $x(n) = -a^n u(-n-1)$

$$X(z) = Z.T.[x(n)]$$

$$= \sum_{n=-\infty}^{\infty} x(n) z^{-n}$$

$$= \sum_{n=-\infty}^{\infty} -a^n u(-n-1) z^{-n}$$



$$X(z) = \sum_{n=-\infty}^{1} -a^n z^{-n}$$

$$= -\sum_{n=-\infty}^{1} (az^{-1})^n$$

$$\left[\sum_{n=0}^{\infty} a^n = \frac{1}{1-a} \right]$$

- To swept limits place $n = -n$.

$$X(z) = -\sum_{n=1}^{\infty} (az^{-1})^{-n}$$

$$= -\sum_{n=1}^{\infty} (za^{-1})^n$$

$$= -\left[\sum_{n=0}^{\infty} (z a^{-1})^{n} - 1 \right]$$

$$= -\left[\frac{1}{1 - z a^{-1}} - 1 \right]$$

$$= -\left[\frac{a}{a - z} - 1 \right]$$

$$= -\left[\frac{\cancel{a} - \cancel{a} + z}{a - z} \right]$$

$$= \boxed{\frac{z}{z - a}}$$

$$x(n) = -a^{n} u(-n-1)$$

$$a^{n} u(n) \xleftrightarrow{ZT} \frac{z}{z-a}$$

$$-a^{n} u(-n-1) \xleftrightarrow{ZT} \frac{z}{z-a}$$

$$a^{-n} u(n) \xleftrightarrow{ZT} \frac{az}{az-1}$$

$$-a^{-n} u(-n-1) \xleftrightarrow{ZT} \frac{az}{az-1}$$

n

$$Y(z) = \sum_{n=0}^{\infty} a^{-n} z^{-n}$$

$$= \sum_{n=0}^{\infty} (az)^{-n}$$

we know $\sum_{n=0}^{\infty} a^{n} = \frac{1}{1-a}$

$$- Y(z) = \frac{1}{1-(az)^{-1}} = \frac{1}{1-1/az}$$

$$= \boxed{\frac{az}{az-1}}$$

$$\uparrow$$

$$y(n) = a^{-n} u(n)$$

Z- transform for finite sequence

A finite sequence $x(n)$ is defined as $x(n) = \{5, 3, \underset{\uparrow}{-3}, 0, 4, -2\}$ find $X(z)$ of given sequence.

$$- \; x(n) = \{\overset{x(-2)}{5},\ \overset{x(-1)}{3},\ \underset{\uparrow}{\overset{x(0)}{-3}},\ \overset{x(1)}{0},\ \overset{x(2)}{4},\ \overset{x(3)}{-2}\}$$

$$- \; X(z) = \sum_{n=-\infty}^{\infty} x(n)\, z^{-n} = \sum_{n=-2}^{3} x(n)\, z^{-n}$$

$$= x(-2)\, z^{2} + x(-1)\, z^{1} + x(0)\, z^{0} + x(1)\, z^{-1} + x(2)\, z^{-2} + x(3)\, z^{-3}$$

$$= 5z^{2} + 3z^{1} - 3z^{0} + 0 \times z^{-1} + 4z^{-2} + (-2)\, z^{-3}$$

$$\boxed{X(z) = 5z^{2} + 3z^{1} - 3 + 4z^{-2} - 2z^{-3}}$$

A Finite duration sequence $x(n) = \{\underset{\uparrow}{5}, 3, 0, 1, 2, 4\}$ Find Z- transform of $x(n)$.

$$- \; x(n) = \{\underset{\uparrow}{\overset{x(0)}{5}},\ \overset{x(1)}{3},\ \overset{x(2)}{0},\ \overset{x(3)}{1},\ \overset{x(4)}{2},\ \overset{x(5)}{4}\}$$

$$- \; X(z) = \sum_{n=-\infty}^{\infty} x(n)\, z^{-n} = \sum_{n=0}^{5} x(n)\, z^{-n}$$

$$= x(0)\, z^{0} + x(1)\, z^{-1} + x(2)\, z^{-2} + x(3)\, z^{-3} + x(4)\, z^{-4} + x(5)\, z^{-5}$$

$$= 5z^{0} + 3z^{-1} + 0z^{-2} + 1 \times z^{-3} + 2 \times z^{-4} + 4 \times z^{-5}$$

$$= \boxed{5 + 3z^{-1} + z^{-3} + 2z^{-4} + 4z^{-5}}$$

Examples of Inverse Z transform

Find the signal $x(n)$ for which the z transform is

$$X(z) = 4z^{4} - z^{3} - 3z + 4z^{-1} + 3z^{-2}$$

- $x(n) = Z^{-1}[X(z)] = \boxed{\frac{1}{2\pi j} \int X(z)\, z^{n-1}\, dz}$

$$X(z) = \underset{x(-4)}{\underset{\uparrow}{4z^{4}}} - \underset{x(-3)}{\underset{\uparrow}{1z^{3}}} + \underset{x(-2)}{\underset{\uparrow}{0z^{2}}} - \underset{x(-1)}{\underset{\uparrow}{3z}} + \underset{x(0)}{\underset{\uparrow}{0z^{0}}} + \underset{x(1)}{\underset{\uparrow}{4z^{-1}}} + \underset{x(2)}{\underset{\uparrow}{3z^{-2}}}$$

- $X(z) = \sum_{n=-\infty}^{\infty} x(n)\, z^{-n}$

$$= \sum_{n=-4}^{2} x(n)\, z^{-n}$$

- $\boxed{x(n) = \{4, -1, 0, -3, \underset{\uparrow}{0}, 4, 3\}}$

If $X(z) = 2z^{2} + 3$, find $x(n)$

- $X(z) = \underset{-2}{\underline{2z^{2}}} + \underset{-1}{\underline{0 \times z^{1}}} + \underset{0}{\underline{3 \times z^{0}}}$

- $X(z) = \sum_{n=-\infty}^{\infty} x(n)\, z^{-n}$

$$= \sum_{n=-2}^{0} x(n)\, z^{-n}$$

- $x(n) = \{2, 0, \underset{\uparrow}{3}\}$

If $x(z) = 3 + 2z^{-1} + 3z^{-3}$, find $x(n)$

- $x(z) = \underbrace{3}_{x(0)} \times z^{0} + \underbrace{2}_{x(1)} \times z^{-1} + \underbrace{0}_{x(2)} \times z^{-2} + \underbrace{3}_{x(3)} \times z^{-3}$

- $x(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n} = \sum_{n=0}^{3} x(n) z^{-n}$

- $x(n) = \{\underset{\uparrow}{3}, 2, 0, 3\}$

4.18. Using the power series expansion technique, find the inverse z-transform of the following $X(z)$:

$$(a)\quad X(z)=\frac{z}{2z^2-3z+1}\qquad |z|<\frac{1}{2}$$

$$(b)\quad X(z)=\frac{z}{2z^2-3z+1}\qquad |z|>1$$

(*a*) Since the ROC is $|z|<\frac{1}{2}$, $x[n]$ is a left-sided sequence. Thus, we must divide to obtain a series in power of z. Carrying out the long division, we obtain

$$\begin{array}{r l}
 & z+3z^2+7z^3+15z^4+\cdots \\ \cline{2-2}
1-3z+2z^2 & \big)\, z \\
 & z-3z^2+2z^3 \\ \cline{2-2}
 & \quad 3z^2-2z^3 \\
 & \quad 3z^2-9z^3+6z^4 \\ \cline{2-2}
 & \qquad 7z^3-\ \ 6z^4 \\
 & \qquad 7z^3-\ 21z^4+14z^5 \\ \cline{2-2}
 & \qquad\qquad 15z^4\cdots
\end{array}$$

Thus,

$$X(z)=\cdots+15z^4+7z^3+3z^2+z$$

and so by definition (*4.3*) we obtain

$$x[n]=\{\ldots,15,7,3,1,\underset{\uparrow}{0}\}$$

(*b*) Since the ROC is $|z|>1$, $x[n]$ is a right-sided sequence. Thus, we must divide so as to obtain a series in power of z^{-1} as follows:

$$\begin{array}{r l}
 & \frac{1}{2}z^{-1}+\frac{3}{4}z^{-2}+\frac{7}{8}z^{-3}+\cdots \\ \cline{2-2}
2z^2-3z+1 & \big)\, z \\
 & z-\frac{3}{2}-\frac{1}{2}z^{-1} \\ \cline{2-2}
 & \quad \frac{3}{2}-\frac{1}{2}z^{-1} \\
 & \quad \frac{3}{2}-\frac{9}{4}z^{-1}+\frac{3}{4}z^{-2} \\ \cline{2-2}
 & \qquad \frac{7}{4}z^{-1}-\frac{3}{4}z^{-2} \\
 & \qquad\quad \vdots
\end{array}$$

Thus,

$$X(z)=\frac{1}{2}z^{-1}+\frac{3}{4}z^{-2}+\frac{7}{8}z^{-3}+\cdots$$

and so by definition (*4.3*) we obtain

$$x[n]=\left\{0,\frac{1}{2},\frac{3}{4},\frac{7}{8},\ldots\right\}$$