6.3 REGION OF CONVERGENCE (ROC)

The value of Z for which the equation (6.7) converges is called the Region of Convergence (ROC).

6.3.1 Relationship between Z-Transform and DTFT

The Z-Transform reduces to the discrete-time Fourier transform when the magnitude of the transform Z is unity ($r = 1$).

$$X(Z) = X(e^{j\omega})\Big|_{Z=e^{j\omega}} \tag{6.10}$$

SOLVED PROBLEMS

Problem 6.1 Determine the Z-Transform of the signal

$$x(n) = a^n u(n)$$

and find the ROC.

Solution

$$x(n) = a^n u(n)$$

By equation (6.6), the Z-Transform is given by,

$$X(Z) = \sum_{n=-\infty}^{\infty} x(n)Z^{-n}$$

$$X(Z) = \sum_{n=-\infty}^{\infty} [a^n u(n)]Z^{-n}$$

$$X(Z) = \sum_{n=0}^{\infty} a^n Z^{-n}$$

$$X(Z) = \sum_{n=0}^{\infty} (aZ^{-1})^n$$

$$X(Z) = \frac{1}{1-(aZ^{-1})}$$

Hint $\sum_{n=0}^{\infty} a^n = \frac{1}{(1-a)}$

For convergence of $X(s)$,

$$|aZ^{-1}| < 1 \Rightarrow |Z| > |a|$$

$$X(Z) = \frac{Z}{Z-a}$$

Location of zeros: $Z = 0$

Location of poles: $Z = a$

The ROC is outside the inner circle of radius a as shown in Fig. 6.1

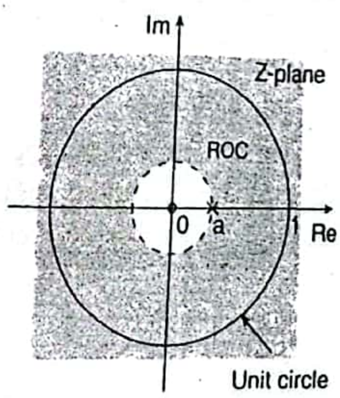


Fig. 6.1

Problem 6.2 Determine the *Z*-Transform of the signal

$$x(n) = -a^n u(-n-1)$$

and plot the ROC.

Solution

$$x(n) = -a^n u(-n-1)$$

Taking *Z*-Transform,

$$X(Z) = \sum_{n=-\infty}^{\infty} [-a^n u(-n-1)]Z^{-n}$$

$$X(Z) = \sum_{n=-\infty}^{-1} -a^n Z^{-n} = -\sum_{n=+1}^{\infty} a^{-n} Z^n$$

$$X(Z) = -\sum_{n=1}^{\infty} (a^{-1}Z)^n$$

Hint $\sum_{n=k}^{\infty} a^n = \frac{a^k}{(1-a)}$

$$X(Z) = \frac{-a^{-1}Z}{1-a^{-1}Z} \qquad (1)$$

For convergence of $X(Z)$,

$$|a^{-1}Z| < 1 \Rightarrow |Z| < |a|$$

On simplifying equation (1)

$$X(Z) = \frac{Z}{Z-a}$$

Location of zeros: $Z = 0$; Location of poles: $Z = a$

ROC : $|Z| < |a|$

The ROC is inside the inner circle of radius a as shown in Fig. 6.2.

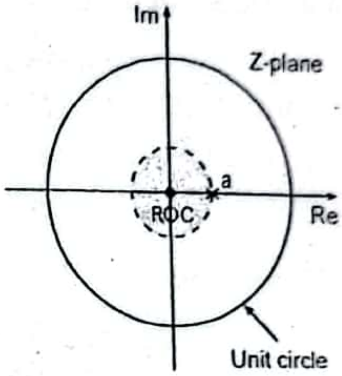


Fig. 6.2

Problem 6.3 Determine the *Z*-Transform of the signal

$$x(n) = a^n u(n) - b^n u(n), b > a$$

and plot the ROC.

Solution

$$x(n) = a^n u(n) - b^n u(n)$$

Taking *Z*-Transform,

$$X(Z) = \sum_{n=-\infty}^{\infty} x(n)Z^{-n} = \sum_{n=-\infty}^{\infty} [a^n u(n) - b^n u(n)]Z^{-n}$$

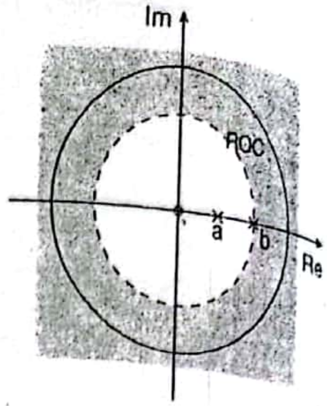
$$X(Z) = \sum_{n=0}^{\infty} a^n Z^{-n} - \sum_{n=0}^{\infty} b^n Z^{-n}$$

$$X(Z) = \sum_{n=0}^{\infty} (aZ^{-1})^n - \sum_{n=0}^{\infty} (bZ^{-1})^n \qquad (1)$$

$$X(Z) = \frac{1}{1-aZ^{-1}} - \frac{1}{1-bZ^{-1}}$$

For convergence of $X(Z)$.

$$|aZ^{-1}| < 1 \Rightarrow |Z| > |a|$$
$$|bZ^{-1}| < 1 \Rightarrow |Z| > |b|$$



$$X(Z) = \frac{Z}{Z-a} - \frac{Z}{Z-b}$$

Fig. 6.3

Location of zeros: $Z = 0$

Location of poles: $Z = a$ and $Z = b$

Note ROC should not contain any poles in it. Therefore, $|Z| > |b|$ will be considered for ROC. If $|Z| > |a|$ is considered, then pole b will be enclosed within the ROC. Hence, it is not converged.

Problem 6.4 Determine the Z-Transform of the signal

$$x(n) = a^n u(n) - b^n u(-n-1) \quad (a \text{ and } b) < 1, b > a$$

and plot the ROC.

Solution

$$x(n) = a^n u(n) - b^n u(-n-1)$$

Taking Z-Transform,

$$X(Z) = \sum_{n=-\infty}^{\infty} x(n) Z^{-n} = \sum_{n=-\infty}^{\infty} \left[a^n u(n) - b^n u(-n-1)\right] Z^{-n}$$

$$X(Z) = \sum_{n=0}^{\infty} a^n Z^{-n} - \sum_{n=-\infty}^{-1} b^n Z^{-n}$$

$$X(Z) = \sum_{n=0}^{\infty} a^n Z^{-n} - \sum_{n=1}^{\infty} b^{-n} Z^{n}$$

$$X(Z) = \sum_{n=0}^{\infty} (aZ^{-1})^n - \sum_{n=1}^{\infty} (b^{-1}Z)^n$$

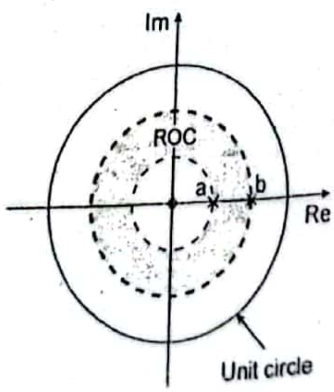
$$X(Z) = \frac{1}{1-aZ^{-1}} - \frac{b^{-1}Z}{1-b^{-1}Z} \quad (1)$$

For convergence of $X(Z)$,

$$|aZ^{-1}| < 1 \Rightarrow |Z| > |a|$$

Similarly,

$$|b\,Z^{-1}| < 1 \Rightarrow |Z| < |b|$$



On simplifying equation (1), we obtain

$$X(Z) = \frac{Z[(Z-b)+(Z-a)]}{(Z-a)(Z-b)}$$

Location of poles: $Z = a$ and $Z = b$

In this problem, the ROC lies between a and b.

Problem 6.5 Determine the Z-Transform of the signal

$$x(n) = -a^n u(-n-1) - b^n u(-n-1),\ b > a$$

and plot the ROC.

Solution

$$x(n) = -a^n u(-n-1) - b^n u(-n-1)$$

Taking Z-Transform,

$$X(Z) = \sum_{n=-\infty}^{\infty} x(n)Z^{-n} = \sum_{n=-\infty}^{\infty} [-a^n u(-n-1) - b^n u(-n-1)]Z^{-n}$$

$$X(Z) = \sum_{n=-\infty}^{-1} -a^n Z^{-n} - \sum_{n=-\infty}^{-1} b^n Z^{-n}$$

$$X(Z) = -\sum_{n=1}^{\infty} a^{-n} Z^n - \sum_{n=1}^{\infty} b^{-n} Z^n$$

$$X(Z) = -\sum_{n=1}^{\infty} (a^{-1}Z)^n - \sum_{n=1}^{\infty} (b^{-1}Z)^n$$

$$X(Z) = -\left[\frac{a^{-1}Z}{1-a^{-1}Z}\right] - \left[\frac{b^{-1}Z}{1-b^{-1}Z}\right] \quad (1)$$

For convergence of $X(Z)$,

$$|a^{-1}Z| < 1 \Rightarrow |Z| < |a|$$

Similarly,

$$|b^{-1}Z| < 1 \Rightarrow |Z| < |b|$$

On simplifying equation (1), we obtain

$$X(Z) = \frac{Z}{Z-a} + \frac{Z}{Z-b}$$

$$X(Z) = \frac{Z(Z-b) + Z(Z-a)}{(Z-a)(Z-b)}$$

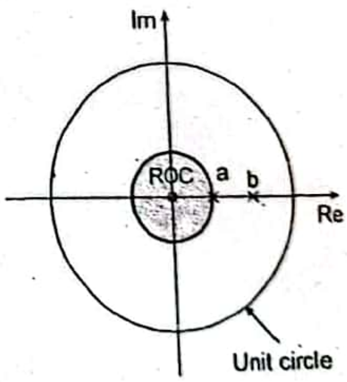


Fig. 6.5

Location of poles: $Z = a$ and $Z = b$

Since ROC should not enclose any poles in it, the ROC is for $|Z| < |a|$. If $|Z| < |b|$ is selected, then ROC encloses $|Z| < |a|$.

Problem 6.6 Find the Z-Transform of the signal

$$x(n) = \sin \omega n u(n)$$

Solution

$$x(n) = \sin \omega n u(n)$$

Taking Z-Transform,

$$X(Z)=\sum_{n=-\infty}^{\infty}x(n)Z^{-n}=\sum_{n=-\infty}^{\infty}[\sin\omega n\,u(n)]Z^{-n}$$

$$X(Z)=\sum_{n=0}^{\infty}\sin\omega n\,Z^{-n}$$

$$X(Z)=\sum_{n=0}^{\infty}\left[\frac{e^{j\omega n}-e^{-j\omega n}}{2j}\right]Z^{-n}$$

$$X(Z)=\frac{1}{2j}\left[\sum_{n=0}^{\infty}e^{j\omega n}Z^{-n}-\sum_{n=0}^{\infty}e^{-j\omega n}Z^{-n}\right]$$

$$X(Z)=\frac{1}{2j}\left[\sum_{n=0}^{\infty}(e^{j\omega}Z^{-1})^{n}-\sum_{n=0}^{\infty}(e^{-j\omega}Z^{-1})^{n}\right]$$

$$X(Z)=\frac{1}{2j}\left[\frac{1}{1-e^{j\omega}Z^{-1}}-\frac{1}{1-e^{-j\omega}Z^{-1}}\right]$$

The region of convergence of $X(Z)$,

$$\left|e^{j\omega}Z^{-1}\right|<1\Rightarrow|Z|>\left|e^{j\omega}\right|$$

$$\left|e^{j\omega}Z^{-1}\right|<1\Rightarrow|Z|>\left|e^{-j\omega}\right|$$

$$\text{ROC}:|Z|>1$$

$$X(Z)=\frac{1}{2j}\left[\frac{1-e^{-j\omega}Z^{-1}-1+e^{j\omega}Z^{-1}}{(1-e^{j\omega}Z^{-1})(1-e^{-j\omega}Z^{-1})}\right]$$

$$X(Z)=\frac{1}{2j}\left[\frac{(e^{j\omega}-e^{-j\omega})Z^{-1}}{1-e^{-j\omega}Z^{-1}-e^{j\omega}Z^{-1}+Z^{-2}}\right]$$

$$X(Z)=\frac{\left(\dfrac{e^{j\omega}-e^{-j\omega}}{2j}\right)Z^{-1}}{1-2\left(\dfrac{e^{j\omega}+e^{-j\omega}}{2}\right)Z^{-1}+Z^{-2}}=\frac{\sin\omega Z^{-1}}{1-2\cos\omega Z^{-1}+Z^{-2}}$$

Problem 6.7 Find the Z-Transform of the signal $x(n)=\cos\omega n\,u(n)$

Solution The given signal is $x(n)=\cos\omega n\,u(n)$

Taking Z-Transform,

$$X(Z)=\sum_{n=-\infty}^{\infty}x(n)Z^{-n}$$

$$X(Z)=\sum_{n=-\infty}^{\infty}[\cos\omega n\,u(n)]Z^{-n}$$

$$X(Z)=\sum_{n=0}^{\infty}\cos\omega n\,Z^{-n}$$

$$X(Z)=\sum_{n=0}^{\infty}\left[\frac{e^{+j\omega n}+e^{-j\omega n}}{2}\right]Z^{-n}$$

$$X(Z)=\frac{1}{2}\sum_{n=0}^{\infty}e^{+j\omega n}Z^{-n}+\frac{1}{2}\sum_{n=0}^{\infty}e^{-j\omega n}Z^{-n}$$

$$X(Z)=\frac{1}{2}\sum_{n=0}^{\infty}(e^{j\omega}Z^{-1})^{n}+\frac{1}{2}\sum_{n=0}^{\infty}(e^{-j\omega}Z^{-1})^{n}$$

$$X(Z)=\frac{1}{2}\left[\frac{1}{1-e^{j\omega}Z^{-1}}+\frac{1}{1-e^{-j\omega}Z^{-1}}\right]$$

The region of convergence of $X(Z)$,

$$\left|e^{j\omega}Z^{-1}\right|<1\Rightarrow|Z|>\left|e^{j\omega}\right|$$

$$\left|e^{j\omega}Z^{-1}\right|<1\Rightarrow|Z|>\left|e^{+j\omega}\right|$$

ROC: $|Z|>1$

$$X(Z)=\frac{1}{2}\left[\frac{1-e^{-j\omega}Z^{-1}+1-e^{j\omega}Z^{-1}}{(1-e^{j\omega}Z^{-1})(1-e^{j\omega}Z^{-1})}\right]$$

$$X(Z)=\frac{1}{2}\left[\frac{2-(e^{+j\omega}+e^{-j\omega})Z^{-1}}{1-e^{-j\omega}Z^{-1}-e^{+j\omega}Z^{-1}+Z^{-2}}\right]$$

$$X(Z)=\frac{1}{2}\left[\frac{2-2\cos\omega Z^{-1}}{1-2\left(\frac{e^{j\omega}+e^{-j\omega}}{2}\right)Z^{-1}+Z^{-2}}\right]=\frac{1-\cos\omega Z^{-1}}{1-2\cos\omega Z^{-1}+Z^{-2}},\ |Z|>1$$

Problem 6.8 Find the Z-Transform of $x(n)=a^{n}\cos\omega n u(n)$

Solution The given signal $x(n)=a^{n}\cos\omega n u(n)$

Taking Z-Transform,

$$X(Z)=\sum_{n=-\infty}^{\infty}x(n)Z^{-n}=\sum_{n=-\infty}^{\infty}a^{n}\cos\omega n u(n)Z^{-n}$$

$$X(Z)=\sum_{n=0}^{\infty}a^{n}\left[\frac{e^{j\omega n}+e^{-j\omega n}}{2}\right]Z^{-n}$$

$$X(Z)=\frac{1}{2}\sum_{n=0}^{\infty}a^{n}e^{j\omega n}Z^{-n}+\frac{1}{2}\sum_{n=0}^{\infty}a^{n}e^{-j\omega n}Z^{-n}$$

$$X(Z)=\frac{1}{2}\left[\sum_{n=0}^{\infty}(ae^{j\omega}Z^{-1})^{n}+\sum_{n=0}^{\infty}(ae^{-j\omega}Z^{-1})^{n}\right]$$

$$X(Z)=\frac{1}{2}\left[\frac{1}{1-ae^{j\omega}Z^{-1}}+\frac{1}{1-ae^{-j\omega}Z^{-1}}\right]$$

$$X(Z)=\frac{1}{2}\left[\frac{1-ae^{-j\omega}Z^{-1}+1-ae^{+j\omega}Z^{-1}}{(1-ae^{j\omega}Z^{-1})(1-ae^{-j\omega}Z^{-1})}\right]$$

$$X(Z)=\frac{1}{2}\left[\frac{2-aZ^{-1}(e^{j\omega}+e^{-j\omega})}{1-ae^{-j\omega}Z^{-1}-ae^{j\omega}Z^{-1}+a^2Z^{-2}}\right]$$

$$X(Z)=\frac{1}{2}\left\{\frac{2\left[1-aZ^{-1}\left(\frac{e^{j\omega}+e^{-j\omega}}{2}\right)\right]}{1-2aZ^{-1}\left(\frac{e^{j\omega}+e^{-j\omega}}{2}\right)+a^2Z^{-2}}\right\}$$

$$X(Z)=\frac{1-aZ^{-1}\cos\omega}{1-2aZ^{-1}\cos\omega+a^2Z^{-2}}$$

The region of convergence of $X(Z)$,

$$\left|ae^{-j\omega}Z^{-1}\right|<1$$

$$|Z|>a$$

Problem 6.9 Find the Z-Transform of

$$x(n)=a^n\sin\omega n\,u(n)$$

Solution The given signal,

$$x(n)=a^n\sin\omega n\,u(n)$$

Taking Z-Transform,

$$X(Z)=\sum_{n=-\infty}^{\infty}x(n)Z^{-n}=\sum_{n=-\infty}^{\infty}a^n\sin\omega n\,u(n)Z^{-n}$$

$$X(Z)=\sum_{n=0}^{\infty}a^n\left[\frac{e^{+j\omega n}-e^{-j\omega n}}{2j}\right]Z^{-n}$$

$$X(Z)=\frac{1}{2j}\left[\sum_{n=0}^{\infty}a^ne^{j\omega n}Z^{-n}-\sum_{n=0}^{\infty}a^ne^{-j\omega n}Z^{-n}\right]$$

$$X(Z)=\frac{1}{2j}\left[\sum_{n=0}^{\infty}(ae^{j\omega}Z^{-1})^n-\sum_{n=0}^{\infty}(ae^{-j\omega}Z^{-1})^n\right]$$

$$X(Z)=\frac{1}{2j}\left[\frac{1}{1-(ae^{j\omega}Z^{-1})}-\frac{1}{1-(ae^{-j\omega}Z^{-1})}\right]$$

$$X(Z)=\frac{1}{2j}\left[\frac{1-ae^{-j\omega}Z^{-1}-1+ae^{j\omega}Z^{-1}}{(1-ae^{j\omega}Z^{-1})(1-ae^{-j\omega}Z^{-1})}\right]$$

6.4.3 Two-sided Finite Sequence

A two-sided finite sequence has data value on both the left and right. The ROC of such two-sided finite sequence is the entire Z-plane except at $Z = 0$ and $Z = \infty$.

The Z-Transform of two-sided finite sequence is

$$X(Z) = \sum_{n=-m_0}^{+n_0} x(n) Z^{-n} \tag{6.13}$$

SOLVED PROBLEM

Problem 6.16 Find the Z-Transform of the given signal

$$x(n) = \{-6, 7, 3, \underset{\uparrow}{5}, 0, 2, -8, -2\}$$

Solution

$$x(n) = \{-6, 7, 3, \underset{\uparrow}{5}, 0, 2, -8, -2\}$$

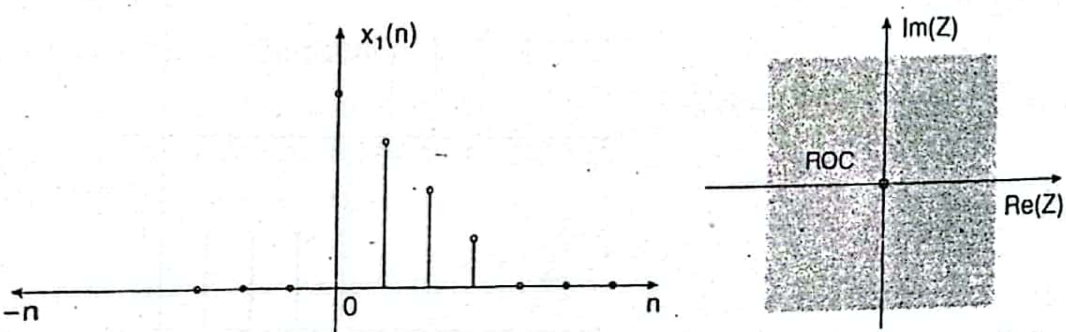
n	–3	–2	–1	0	1	2	3	4
$x(n)$	–6	7	3	5	0	2	–8	–2

$$X(Z) = \sum_{n=-3}^{4} x(n) Z^{-n} = -6Z^3 + 7Z^2 + 3Z + 5 + 2Z^{-2} - 8Z^{-3} - 2Z^{-4}$$

The ROC is the entire Z-plane except at $Z = 0$ and $Z = \infty$.

6.5 CHARACTERISTIC FEATURES OF SIGNALS WITH THEIR CORRESPONDING ROC

Finite Duration Signal



Causal signal

Entire Z-plane except at Z=0

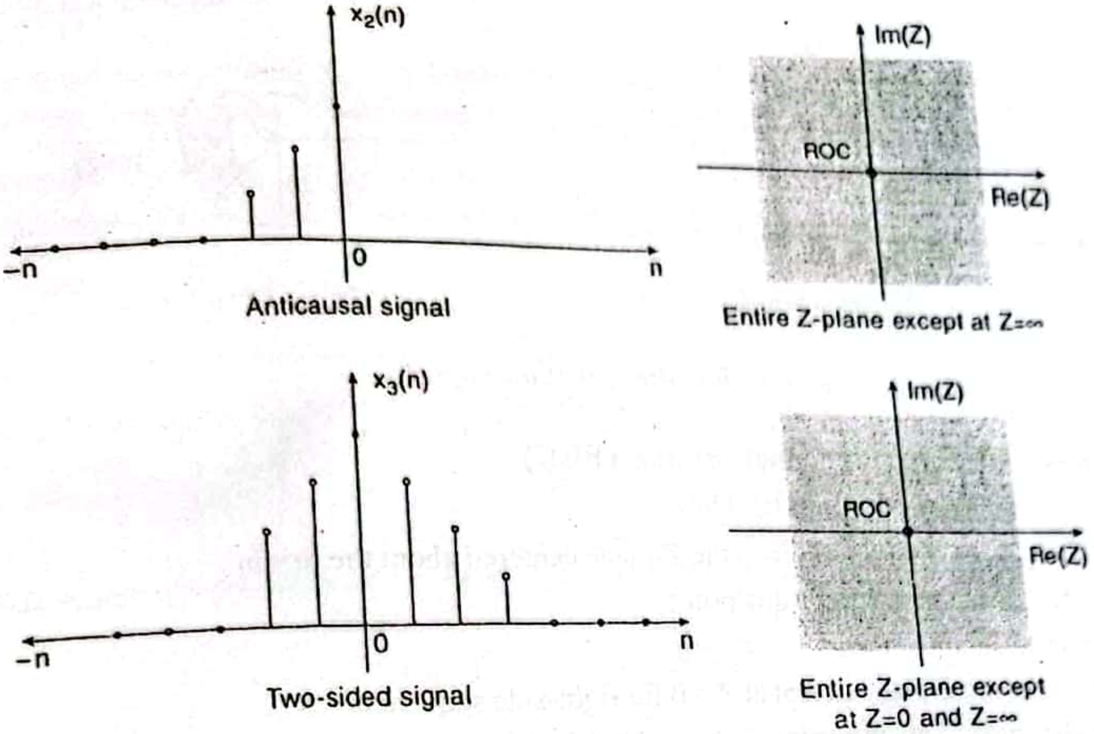
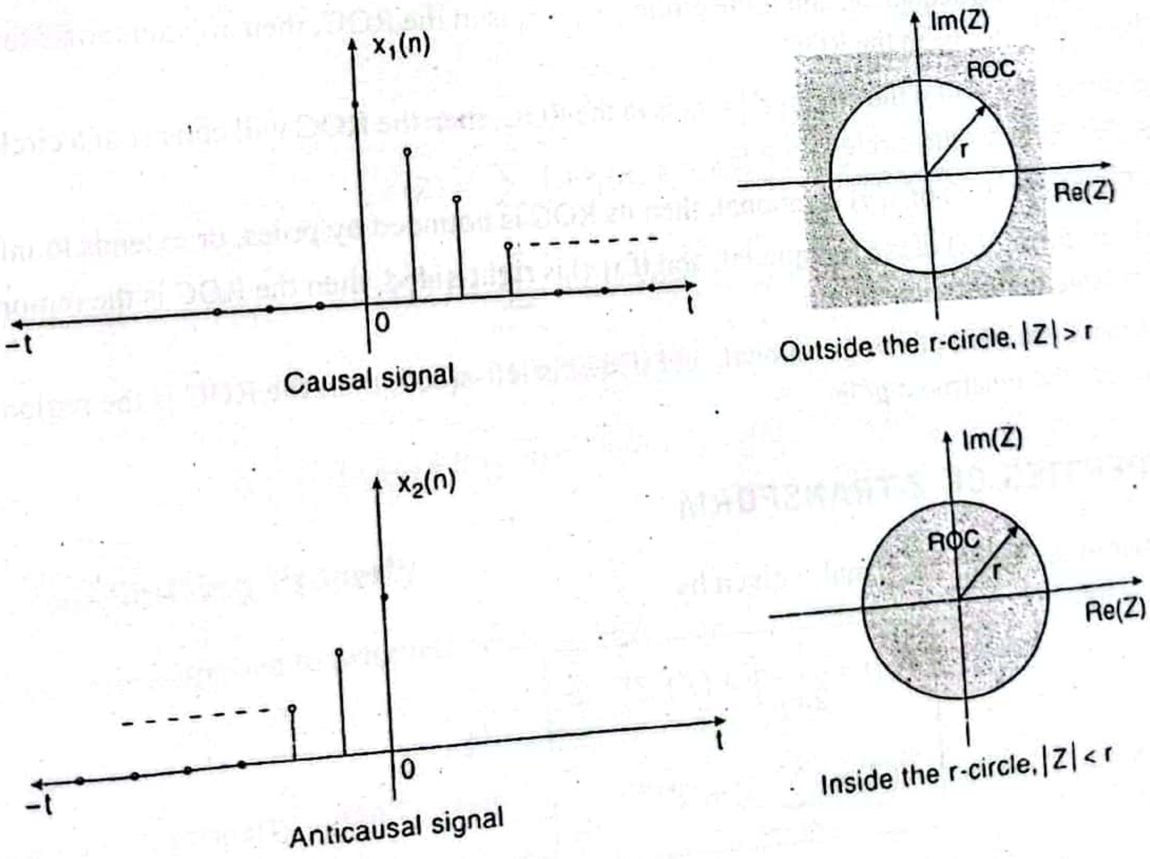


Fig. 6.7 Finite Duration Signal

Infinite Duration Signal



Location of zeros: $Z = 1$

Location of poles: $Z = \frac{1}{2}; Z = \frac{1}{3}$

$$|Z| > \frac{1}{2} \text{ and } |Z| > \frac{1}{3}$$

Since, ROC should not enclose poles in it, $|Z| > \frac{1}{2}$ is the ROC.

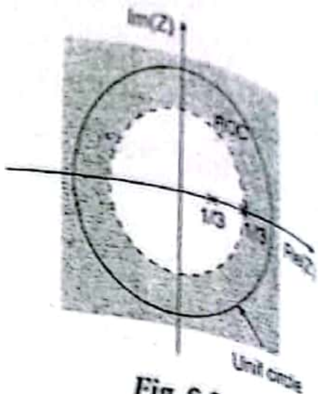


Fig. 6.9

6.8 INVERSE Z-TRANSFORM

Inverse Z-Transform can be obtained by following methods:

1. Power series method (long-division)
2. Partial fraction method
3. Residue method
4. Convolution method

6.8.1 Power Series Method (Long-division)

It is possible to express $X(Z)$ as a power series in Z^{-1} or Z of the form,

$$X(Z) = \sum_{n=-\infty}^{\infty} x(n) Z^{-n}$$

The value of the signal $x(n)$ is then given by the coefficient associated with Z^{-1}. The power series meth is limited to one-sided signals, whose ROC is either $|Z| < a$ or $|Z| > a$.

If the ROC of $X(Z)$ is $|Z| > a$, then $X(Z)$ is expressed as a power series of Z^{-1}, so that we obtain right-sid signal (inverse transform).

If the ROC of $X(Z)$ is $|Z| < a$, then $X(Z)$ is expressed as a power series of Z, so that we obtain left-sic signal (inverse transform).

In power series expansion, we use long-division. Hence, this method is also known as long-divis method. The power series method is preferred, when the given polynominal ratio is simple. The major set b of power series expansion is that, it may not result in closed form of expression for $x(n)$.

SOLVED PROBLEMS

Problem 6.25 Find the inverse Z-Transform of the given function,

$$X(Z) = \frac{1}{1 - 1.5Z^{-1} + 0.5Z^{-2}}$$

using power-series expansion method, for

(i) ROC, $|Z| > 1$

(ii) ROC, $|Z| < 1$

Solution

(i) For causal system,

$$X(Z)=\frac{1}{1-1.5Z^{-1}+0.5Z^{-2}}$$

$$
\begin{array}{r l}
 & 1+1.5Z^{-1}+1.75Z^{-2}+1.875Z^{-3}+\cdots \\
1-1.5Z^{-1}+0.5Z^{-2} \,\big) & 1 \\
 & \underline{1-1.5Z^{-1}+0.5Z^{-2}} \\
 & \quad 1.5Z^{-1}-0.5Z^{-2} \\
 & \quad \underline{1.5Z^{-1}-2.25Z^{-2}+0.75Z^{-3}} \\
 & \qquad 1.75Z^{-2}-0.75Z^{-3} \\
 & \qquad \underline{1.75Z^{-2}-2.625Z^{-3}+0.875Z^{-4}} \\
 & \qquad\quad 1.875Z^{-3}-0.875Z^{-4} \\
 & \qquad\quad \underline{1.875Z^{-3}-2.8125Z^{-4}+0.937Z^{-5}} \\
 & \qquad\qquad 1.9375Z^{-4}-0.935Z^{-5} \\
 & \qquad\qquad \vdots
\end{array}
$$

$$X(Z)=1+1.5Z^{-1}+1.75Z^{-2}+1.875Z^{-3}+\cdots \qquad (1)$$

$$X(Z)=\sum_{n=0}^{\infty}x(n)Z^{-n} \text{ for causal system} \qquad (2)$$

On comparing the equations (1) and (2),

$x(n)=0, \quad n<0$

$x(0)=1$

$x(1)=1.5$

$x(2)=1.75$

$x(3)=1.875$

$\vdots \quad \vdots$

(ii) For anticausal system, $X(Z)=\dfrac{1}{0.5Z^{-2}-1.5Z^{-1}+1}$

$$
\begin{array}{r l}
 & 2Z^{2}+6Z^{3}+14Z^{4}+30Z^{5}+62Z^{6}+\cdots \\
0.5Z^{-2}-1.5Z^{-1}+1 \,\big) & 1 \\
 & \underline{1-3Z+2Z^{2}} \\
 & \quad 3Z-2Z^{2} \\
 & \quad \underline{3Z-9Z^{2}+6Z^{3}} \\
 & \qquad 7Z^{2}-6Z^{3} \\
 & \qquad \underline{7Z^{2}-21Z^{3}+14Z^{4}} \\
 & \qquad\quad 15Z^{3}-14Z^{4} \\
 & \qquad\quad \underline{15Z^{3}-45Z^{4}+30Z^{5}} \\
 & \qquad\qquad 31Z^{4}-30Z^{5}
\end{array}
$$

Taking inverse Z-Transform,

$$x(n)=66\left(\frac{1}{2}\right)^n-176\left(\frac{1}{3}\right)^n+111\left(\frac{1}{4}\right)^n$$

$$x(n)=66\left(\frac{1}{2}\right)^n u(n)-176\left(\frac{1}{3}\right)^n u(n)+111\left(\frac{1}{4}\right)u(n), |Z|>\frac{1}{2}$$

Problem 6.27 Find the inverse transform of

$$X(Z)=\frac{1+Z^{-1}}{1-\frac{1}{5}Z^{-1}}$$

Using long division method.

when, (i) $|Z|>\frac{1}{5}$

(ii) $|Z|<\frac{1}{5}$

Solution

(i) Let us consider, $X(Z)=\dfrac{1+Z^{-1}}{1-\frac{1}{5}Z^{-1}}, |Z|>\frac{1}{5}$

For $|Z|>\frac{1}{5}$, $X(Z)$ is a right-sided signal.

$$\begin{array}{r|l}
 & 1+\frac{6}{5}Z^{-1}+\frac{6}{25}Z^{-2}+\cdots \\
\hline
1-\frac{1}{5}Z^{-1} & 1+Z^{-1} \\
 & 1-\frac{1}{5}Z^{-1} \\
\cline{2-2}
 & \frac{6}{5}Z^{-1} \\
 & \frac{6}{5}Z^{-1}-\frac{6}{25}Z^{-2} \\
\cline{2-2}
 & \frac{6}{25}Z^{-2} \\
 & \frac{6}{25}Z^{-2}-\frac{6}{25}Z^{-3} \\
\cline{2-2}
 & \frac{6}{125}Z^{-3} \\
 & \vdots
\end{array}$$

$$X(Z)=1+\frac{6}{5}Z^{-1}+\frac{6}{25}Z^{-2}+\ldots \quad (1)$$

$$X(Z)=\sum_{n=0}x(n)Z^{-n} \quad (2)$$

6.8.2 Partial-fraction Method

If the given LTI system is a rational fraction of Z^{-1} say

$$X(Z)=\frac{B(Z)}{A(Z)}=\frac{b_0+b_1Z^{-1}+b_2Z^{-2}+\cdots+b_MZ^{-M}}{1+a_1Z^{-1}+a_2Z^{-2}+\cdots+a_NZ^{-N}} \tag{6.27}$$

For partial fraction expansion, the order of the numerator M must be less than the order of denominator N, i.e. $M<N$. If $M>N$, then perform long-division till numerator order is less than the denominator order. Therefore,

$$X(Z)=\sum_{k=0}^{M-N} f_kZ^{-k}+\frac{B'(Z)}{A(Z)} \tag{6.28}$$

The partial fraction expansion can be performed on equation (6.28).

SOLVED PROBLEM

Problem 6.28 Find the inverse Z-Transform of

$$X(Z)=\frac{Z}{(Z-1)(Z-2)(Z-3)}$$

Using partial fraction method

for (i) ROC $|Z|>3$

(ii) ROC $3>|Z|>2$

(iii) ROC $|Z|<1$

Solution

$$X(Z)=\frac{Z}{(Z-1)(Z-2)(Z-3)}$$

$$\frac{X(Z)}{Z}=\frac{A}{(Z-1)}+\frac{B}{(Z-2)}+\frac{C}{(Z-3)}$$

$$A=(Z-1)X(Z)\big|_{Z=1}=\frac{1}{(1-2)(1-3)}=\frac{1}{2}$$

$$B=(Z-2)X(Z)\big|_{Z=2}=\frac{1}{(2-1)(2-3)}=-1$$

$$C=(Z-3)X(Z)\big|_{Z=3}=\frac{1}{(3-1)(3-2)}=\frac{1}{2}$$

$$\frac{X(Z)}{Z}=\frac{1/2}{Z-1}-\frac{1}{Z-2}+\frac{1/2}{Z-3}$$

$$X(Z)=\frac{1/2}{(1-Z^{-1})}-\frac{1}{(1-2Z^{-1})}+\frac{1/2}{(1-3Z^{-1})}$$

Multiplying either side by Z^{k-1},

$$X(Z)Z^{k-1} = \sum_{n=-\infty}^{\infty} x(n)Z^{-n+k-1} \tag{6.30}$$

Integrating equation (6.30) w.r.t. Z about a closed contour C,

$$\oint_c X(Z)Z^{k-1}dz = \oint_c \sum_{n=-\infty}^{\infty} x(n)Z^{-n+k-1}dz$$

$$\oint_c X(Z)Z^{k-1}dz = \sum_{n=-\infty}^{\infty} x(n)\oint_c Z^{k-(n+1)}dz \tag{6.31}$$

By Cauchy's residue theorem,

$$\oint_c Z^{k-(n+1)}dz = 2\pi j\delta_{kn} \tag{6.32}$$

where

$$\delta_{kn} = \begin{cases} 1, & k = n \\ 0, & k \neq n \end{cases} \tag{6.33}$$

Substituting equation (6.32) in (6.31),

$$\oint_c X(Z)Z^{k-1}dz = \sum_{n=-\infty}^{\infty} x(n)(2\pi j\delta_{kn})$$

$$\oint_c X(Z)Z^{k-1}dz = 2\pi j\, x(k)$$

Since δ_{kn} exist at $k = n$ and zero otherwise.

Therefore,

$$x(k) = \frac{1}{2\pi j}\oint_c X(Z)Z^{n-1}dz \tag{6.34}$$

On replacing the dummy variable k by n results in,

$$x(n) = \frac{1}{2\pi j}\oint_c X(Z)Z^{n-1}dz \tag{6.35}$$

$x(n)$ can be obtained by finding the sum of all residues of the poles that exist inside the contour C, i.e.,

$$x(n) = \sum[\text{residues of } X(Z)Z^{n-1} \text{ at the poles inside } C]$$

$$x(n) = \sum_i (Z - Z_i)X(Z)Z^{n-1}\Big|_{Z=Z_i}$$

Note If $X(Z)\,Z^{n-1}$ does not have poles inside the contour C for any value of n, then $x(n) = 0$.

SOLVED PROBLEMS

Problem 6.29 Find the inverse Z-Transform of

$$X(Z) = \frac{Z+0.5}{(Z+0.6)(Z+0.8)}, |Z| > 0.8$$

using residue method.

Solution By definition of residue method,

$$x(n)=\frac{1}{2\pi j}\oint_c X(Z)\,Z^{n-1}\,dz$$

$$x(n)=\sum[\text{residues of } X(Z)Z^{n-1}\text{at poles within 'C'}]$$

$$x(n)=\sum\text{residues of}\left[\frac{(Z+0.5)Z^{n-1}}{(Z+0.6)(Z+0.8)}\right]\text{at poles within } C$$

The ROC $|Z|>0.8$ encloses the poles at $Z=-0.6$ and $Z=-0.8$.

For $n=0$, one more pole also exists at $Z=0$.

For $n=0$, $$x(0)=\sum\text{residues of}\frac{(Z+0.5)}{Z(Z+0.6)(Z+0.8)}$$

at poles $Z=0$; $Z=-0.6$ and $Z=-0.8$

$$x(0)=Z\left[\frac{(Z+0.5)}{Z(Z+0.6)(Z+0.8)}\right]\Bigg|_{Z=0}+(Z+0.6)\left[\frac{(Z+0.5)}{Z(Z+0.6)(Z+0.8)}\right]\Bigg|_{Z=-0.6}$$
$$+(Z+0.8)\left[\frac{(Z+0.5)}{Z(Z+0.6)(Z+0.8)}\right]\Bigg|_{Z=-0.8}$$

$$x(0)=\frac{0.5}{(0.6)(0.8)}+\frac{(-0.1)}{(-0.6)(0.2)}+\frac{(-0.3)}{(-0.8)(-0.2)}$$

$$x(0)=1.042+0.833-1.875=0$$

For $n\geq 1$,

$$x(n)=\sum\text{residues of}\left[\frac{(Z+0.5)Z^{n-1}}{(Z+0.6)(Z+0.8)}\right]\text{at poles } Z=-0.6; Z=-0.8$$

$$x(n)=(Z+0.6)\left[\frac{(Z+0.5)Z^{n-1}}{(Z+0.6)(Z+0.8)}\right]\Bigg|_{Z=-0.6}+(Z+0.8)\left[\frac{(Z+0.5)Z^{n-1}}{(Z+0.6)(Z+0.8)}\right]\Bigg|_{Z=-0.8}$$

$$x(n)=\frac{(-0.1)(-0.6)^{n-1}}{(0.2)}+\frac{(-0.3)(-0.8)^{n-1}}{(-0.2)}$$

$$x(n)=-0.5(-0.6)^{n-1}u(n-1)+1.5(-0.8)^{n-1}u(n-1)$$

Therefore, $x(n)=-0.5(-0.6)^{n-1}u(n-1)+1.5(-0.8)^{n-1}u(n-1)$

Problem 6.30 Find the inverse Z-Transform of

$$X(Z)=\frac{Z}{(Z-1)(Z^2+1)}$$

using resi[illegible]

Solution [illegible] of residue method,

$$x(n)=[illegible]\;X(Z)Z^{n-1}dz$$