Rank of a matrix:— Rank of a non-zero matrix A is defined as the order of the largest size non-singular square sub-matrix of A and it is denoted by rank(A) or $\rho(A)$.

The rank of a zero (null) matrix is defined to be zero.

e.g., (i) if $A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}$ then $\begin{pmatrix} 1 & 2 \\ 4 & 5 \end{pmatrix}$ is a square submatrix of A of highest order 2 and $\begin{vmatrix} 1 & 2 \\ 4 & 5 \end{vmatrix} = 5-8 = -3 \neq 0$, so that rank$(A) = 2$

(ii) if $A = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 4 & 6 & 8 \end{pmatrix}$ then there are ${}^4C_2 = 6$ square submatrices of A of highest order 2 and all are easily seen to be singular. but A is a non-zero matrix and therefore rank$(A) = 1$.

Note:— 1) rank of a matrix is a non-negative integer. Rank of a non-zero matrix is a +ve integer.

2) Rank of a non-singular square matrix of order n is n and in particular the rank of a unit matrix of order n is n.

Ex.1. Find the rank of the matrix $A = \begin{pmatrix} 1 & 2 & -1 & 3 \\ 3 & 4 & 0 & -1 \\ -1 & 0 & -2 & 7 \end{pmatrix}$

Soln. There are ${}^4C_3 = 4$ square submatrices of order 3 of A and their determinants are:

$$\begin{vmatrix} 1 & 2 & -1 \\ 3 & 4 & 0 \\ -1 & 0 & -2 \end{vmatrix} = -1(0+4)-0-2(4-6) = -4+4 = 0$$

$$\begin{vmatrix} 1 & 2 & 3 \\ 3 & 4 & -1 \\ -1 & 0 & 7 \end{vmatrix} = -1(-2-12)-0+7(4-6) = 14-14 = 0$$

$$\begin{vmatrix} 1 & -1 & 3 \\ 3 & 0 & -1 \\ -1 & -2 & 7 \end{vmatrix} = -3(-7+6)+0+1(-2-1) = 3-3 = 0$$

$$\begin{vmatrix} 2 & -1 & 3 \\ 4 & 0 & -1 \\ 0 & -2 & 7 \end{vmatrix} = 0+2(-2-12)+7(0+4) = -28+28=0$$

$\therefore$ all square submatrices of order 3 are singular. $\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$ is a square submatrix of A of order 2 and $\begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix} = 4-6 = -2 \neq 0$ $\therefore$ rank $(A) = 2$.

Elementary operation or Elementary Transformation:

The following operations when applied to a matrix are called elementary row operations:

i) interchange of any two rows : $R_i \longleftrightarrow R_j$

ii) multiplication of every element of a row by a non-zero scalar k : $R_i \longrightarrow kR_i$

iii) addition to the elements of a row, a constant multiple of corresponding elements of another row:

$$R_i \longrightarrow R_i + kR_j$$

Similarly elementary column operations are defined which are denoted by $C_i \longleftrightarrow C_j$. $C_i \longrightarrow kC_i$, $C_i \longrightarrow C_i + kC_j$.

Theorem :— Elementary operations donot alter the rank of a matrix.

Echelon form of a matrix or Echelon matrix :—

A matrix is said to be in Echelon form if the number of zeros preceeding the first non-zero entry of any row increases row by row untill only zero rows remain (if zero rows exist). The first non-zero entries of the rows of an Echelon matrix are called distinguished elements & if each of them is unity & if it is the only non-zero element in the corresponding column then the matrix is said to be a row reduced echelon matrix.

e.g., (i) $\begin{pmatrix} 1 & 2 & -3 & 0 \\ 0 & 5 & -1 & 2 \\ 0 & 0 & 0 & 7 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$ is an Echelon matrix.

(ii) $\begin{pmatrix} 0 & 2 & -3 & 4 \\ 0 & 0 & 3 & -5 \\ 0 & 0 & 0 & 1 \end{pmatrix}$ is an Echelon matrix.

(iii) $\begin{pmatrix} 1 & 0 & 4 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$ is a row reduced Echelon matrix.

but $\begin{pmatrix} 2 & 0 & 0 & 5 \\ 0 & 3 & 0 & 1 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 3 & 0 \end{pmatrix}$ is not an Echelon matrix

Theorem :— The rank of a matrix in Echelon form is equal to the number of non-zero rows in it.

Note :— i) To find the rank of a non-zero matrix we reduce it to Echelon form using elementary operations and then the rank of the matrix is equal to the number of non-zero rows in its echelon form.

ii) Every diagonal matrix, unit matrix, scalar matrix and upper triangular matrix is clearly in Echelon form.

Ex. 1. Under what condition the rank of the matrix $\begin{pmatrix} 2 & 4 & 2 \\ 2 & 1 & 2 \\ 1 & 0 & x \end{pmatrix}$ is 3 ?

Soln Let A be the given matrix, then $|A| = \begin{vmatrix} 2 & 4 & 2 \\ 2 & 1 & 2 \\ 1 & 0 & x \end{vmatrix} = 1(8-2) - 0 + x(2-8)$
$= 6(1-x)$

rank$(A) = 3$ if $|A| \neq 0$

i.e., if $6(1-x) \neq 0$ i.e., if $x \neq 1$.

$\therefore$ rank of given matrix will be 3 under the condition $x \neq 1$.

Ex. 2. Find the rank of the matrix:

(i) $\begin{pmatrix} 4 & -2 & 6 & 8 \\ 1 & 1 & -3 & -1 \\ 15 & -3 & 9 & 21 \end{pmatrix}$ (ii) $A = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \end{pmatrix}$

Soln (i) Let $A = \begin{pmatrix} 4 & -2 & 6 & 8 \\ 1 & 1 & -3 & -1 \\ 15 & -3 & 9 & 21 \end{pmatrix} \underset{R_1 \leftrightarrow R_2}{\sim} \begin{pmatrix} 1 & 1 & -3 & -1 \\ 4 & -2 & 6 & 8 \\ 15 & -3 & 9 & 21 \end{pmatrix}$

$\sim \begin{pmatrix} 1 & 1 & -3 & -1 \\ 0 & -6 & 18 & 12 \\ 0 & -18 & 54 & 36 \end{pmatrix}$, $R_2 \to R_2 - 4R_1$, $R_3 \to R_3 - 15R_1$

$\sim \begin{pmatrix} 1 & 1 & -3 & -1 \\ 0 & -6 & 18 & 12 \\ 0 & 0 & 0 & 0 \end{pmatrix}$ $R_3 \to R_3 - 3R_2$

which is in Echelon form having two non-zero rows.

$\therefore$ rank $(A) = 2$.

(ii) $A = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \end{pmatrix} \sim \begin{pmatrix} 1 & 2 & 3 \\ 0 & 0 & 0 \end{pmatrix}$, $R_2 \to R_2 - 2R_1$ which is in Echelon form with only one non-zero row and therefore rank $(A) = 1$.

Ex. 3. Find the rank of the matrix $\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \\ p & p & p \end{pmatrix}$ where $p \in \mathbb{R}$, $p \neq 0$

Soln Applying $R_2 \to R_2 - R_1$, $R_3 \to R_3 - R_1$, $R_4 \to R_4 - pR_1$ to the given matrix we obtain $\begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ which is in Echelon form with one non-zero row.

$\therefore$ rank of given matrix is 1.

Ex. 4. If $U = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$, find rank of U & of U^2

Soln. U is in Echelon form with two non-zero rows & therefore rank$(U) = 2$.

Now $U^2 = UU = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

which is in Echelon form with only one non-zero row and therefore rank$(U^2) = 1$.

Ex. 5. Find the value of x so that rank $(A) < 3$ where

$A = \begin{pmatrix} 3x-8 & 3 & 3 \\ 3 & 3x-8 & 3 \\ 3 & 3 & 3x-8 \end{pmatrix}$

Soln. $A = \begin{pmatrix} 3x-8 & 3 & 3 \\ 3 & 3x-8 & 3 \\ 3 & 3 & 3x-8 \end{pmatrix}$ is a square matrix of order 3 so rank(A) will be less than 3 if $|A| = 0$.

or $\begin{vmatrix} 3x-8 & 3 & 3 \\ 3 & 3x-8 & 3 \\ 3 & 3 & 3x-8 \end{vmatrix} = 0$

$C_1 \to C_1 + C_2 + C_3$

or $\begin{vmatrix} 3x-2 & 3 & 3 \\ 3x-2 & 3x-8 & 3 \\ 3x-2 & 3 & 3x-8 \end{vmatrix} = 0$

$R_2 \to R_2 - R_1,\ R_3 \to R_3 - R_1$

or $\begin{vmatrix} 3x-2 & 3 & 3 \\ 0 & 3x-11 & 0 \\ 0 & 0 & 3x-11 \end{vmatrix} = 0$

or $(3x-2)(3x-11)^2 = 0$

$\therefore\ 3x-2 = 0$ or $3x-11 = 0$

$\therefore\ x = \frac{2}{3}$ or $x = \frac{11}{3}$

$\therefore$ rank(A) < 3 if $x = \frac{2}{3}$ or $\frac{11}{3}$.

Note:-1. Rank of a matrix A and its transpose A′ are same i.e; rank(A) = rank(A′).

This is because the value of a determinant remains the same if rows are written as columns and for finding rank of a matrix we search largest size square submatrix of the given matrix whose determinant is non.zero.

Note:-2. Finding rank using determinantal approach is cumbersome.