Assignment – 1

Subject: Mathematics-I (Linear Algebra & Calculus) **Subject Code: MA XXX**

Unit – I: Linear Algebra

1. Define a vector space.

2. Show that $R^n(R)$ is a vector space using the following operations:
(i) vector addition: $(x_1,\ x_2,\ \ldots,\ x_n)+(y_1,\ y_2,\ \ldots,\ y_n)=(x_1+y_1,\ x_2+y_2,\ \ldots,\ x_n+y_n)$
(ii) scalar multiplication: $a(x_1,\ x_2,\ \ldots,\ x_n)=(ax_1,\ ax_2,\ \ldots,\ ax_n),\ \ a\in R$

3. Show that $R(C)$ is not a vector space.

4. Show that $R^2(R)$ is not a vector space with the operations
$(x_1,y_1)+(x_2,y_2)=(x_1+x_2,y_1+y_2), \quad c(x_1,y_1)=(cx_1,y_1)$.

5. Define linear span of a set.

6. Show that each of the following set of vectors generates $R^3(R)$.
(i) $\{(1,0,0),(1,1,0),\ (1,1,1)\}$
(ii) $\{(1,2,3),(0,1,2),\ (0,0,1)\}$
(iii) $\{(1,2,1),(2,1,0),\ (1,-1,2)\}$

7. Let $v_1=(1,-2,3),\ v_2=(3,-7,10)$, and $v_3=(2,1,9)$. Examine if v is a linear combination of v_1,v_2,v_3, where $v=(4,9,19)$.

8. Let $w_1=(1,-2,1,3),\ w_2=(1,2,-1,1)$, and $w_3=(2,3,1,-1)$. Examine if w is a linear combination of w_1,w_2,w_3, where $w=(3,0,5,-1)$.

9. Let $u_1=(1,-1,0),\ u_2=(0,1,-1),\ u_3=(0,0,1)$ be elements of $\mathbb{R}^3$. Show that the set of vectors $\{u_1,u_2,u_3\}$ is linearly independent.

10. Let $v_1=(1,-1,0),\ v_2=(0,1,-1),\ v_3=(0,2,1),\ v_4=(1,0,3)$ be elements of $\mathbb{R}^3$. Show that the set of vectors $\{v_1,v_2,v_3,v_4\}$ is linearly dependent.

11. Examine whether the following set of vectors is linearly independent.
a) $\{(3,-2,0,4),\ \ (5,0,0,1),\ \ (-6,1,0,1),\ \ (2,0,0,3)\}$
b) $\{(3,4,7),\ \ (2,0,3),\ \ (8,2,3),\ \ (5,5,6)\}$
c) $\{\,(1,1,0),\ \ (1,0,0),\ \ (1,1,1)\}$

12. Define basis and dimension of a vector space.

13. Show that the set of vectors $e_1=(1,\ 0,\ \ldots,0),\ e_2=(0,\ 1,\ \ldots,0),\ldots e_n=(0,\ 0,\ \ldots,1)$ is a basis of $R^n(R)$.

14. Determine whether or not each of the following form a basis of $R^3(R)$:
(i) $\{(2,1,0),(0,1,2),(-7,2,5),(8,0,0)\}$
(ii) $\{(2,1,4),(1,-1,2),\ (3,1,-2)\}$
(iii) $\{(1,1,2),(1,2,5),\ (5,3,4)\}$

15. Reduce the following matrices to Echelon form and hence find their ranks:

(a) $A=\begin{bmatrix} 1 & 3 & 5 \\ 2 & -1 & 4 \\ -2 & 8 & 2 \end{bmatrix}$ (b) $A=\begin{pmatrix} 7 & -1 & 0 & 5 \\ 0 & 1/2 & 1 & 1/3 \\ 3 & -1 & 1/4 & 4 \\ 5 & -3 & 1 & 11 \end{pmatrix}$

16. Find the rank and nullity of the following matrices:

(a) $A=\begin{bmatrix} 8 & 2 & 5 \\ 16 & 6 & 29 \\ 4 & 0 & -7 \end{bmatrix}$ (b) $A=\begin{bmatrix} 8 & 0 & 4 \\ 0 & 2 & 0 \\ 4 & 0 & 2 \\ 0 & 4 & 0 \end{bmatrix}$ (c) $A=\begin{pmatrix} 2 & -2 & 0 & 6 \\ 4 & 2 & 0 & 2 \\ 1 & -1 & 0 & 3 \\ 1 & -2 & 1 & 2 \end{pmatrix}$

17. Find all real λ & μ for which the rank of the matrix P is 2,

(a) $P=\begin{bmatrix} 1 & 2 & 3 & 1 \\ 2 & 5 & 3 & \lambda \\ 1 & 1 & 6 & \lambda+1 \end{bmatrix}$ (b) $P=\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & -1 & \lambda \\ 5 & 7 & 1 & \lambda^2 \end{bmatrix}$ (c) $P=\begin{bmatrix} 1 & 1 & 1 & 6 \\ 1 & 2 & 3 & 10 \\ 1 & 2 & \lambda & \mu \end{bmatrix}$

18. Show that the system of equations

$\begin{bmatrix} 1 & -1 & 1 \\ 2 & 1 & -3 \\ 1 & 1 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix}=\begin{bmatrix} 4 \\ 0 \\ 2 \end{bmatrix}$ has a unique solution and find the solution.

19. Prove that the system of equations:

$2x+5y+6z=13$, $3x+y-4z=0$, $x-3y-8z=-10$

has infinite number of solutions and find the solutions.

20. Show that the system of equations:

$2x+6y+1=0$, $6x+20y-6z-3=0$, $6y-18z+1=0$

is inconsistent(i.e., has no solution).

21. Find the condition to be satisfied by a, b, c for which the system of equations:

$x+3y+4z=a$, $-4x+2y-6z=b$, $-3x-2y-7z=c$ is consistent.

22. Find the value of θ for which the system of equations,

$2(\sin\theta)\,x+y-2z=0$, $3x+2(\cos 2\theta)\,y+3z=0$, $5x+3y-z=0$

has a non-trivial solution.

23. Show that the only real value of λ for which the following system of equations has a non-zero solution is 6:

$$x+2y+3z=\lambda x$$
$$3x+y+2z=\lambda y$$
$$2x+3y+z=\lambda z$$

24. Solve the following system of equations by using Gaussian-Elimination Method:

(a) $3.0x+6.2y=0.2$
$2.1x+8.5y=4.3$

(b) $x+2y-z=3$
$x+y+z=2$
$2x+3y=5.$
$y-2z=1$

(c) $3x + 2y + 2z - 5w = 8$
$0.6x + 1.5y + 1.5z - 5.4w = 2.7$
$1.2x - 0.3y - 0.3z + 2.4w = 2.1$

25. Solve the following system of equations by using Gauss-Jordan Method:

(a) $x + y + z = 9$
$2x - 3y + 4z = 13$
$3x + 4y + 5z = 40$

(b) $x + 2y + 3z = 9$
$2x - y + z = 8$
$3x - z = 3$

26. Find the eigenvalues and eigenvectors of the following matrices:

(a) A= $\begin{bmatrix} 5 & -2 \\ 9 & -6 \end{bmatrix}$ (b) A = $\begin{bmatrix} cos\theta & sin\theta \\ sin\theta & -cos\theta \end{bmatrix}$ (c) $A = \begin{bmatrix} -1 & -1 & 0 \\ 4 & 3 & 0 \\ 1 & 0 & 2 \end{bmatrix}$ (d) $A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$

27. State the properties of eigenvalues and eigenvectors.

28. Find the sum and product of the eigen values of $\begin{pmatrix} 3 & 1 & 4 \\ 0 & 2 & 6 \\ 0 & 0 & 5 \end{pmatrix}$.

29. Find the eigen values of $adj\ A\ \ and\ A^2 - 2A + I$, where $A = \begin{pmatrix} 3 & 1 & 4 \\ 0 & 2 & 6 \\ 0 & 0 & 5 \end{pmatrix}$.

30. Two eigen values of $A = \begin{pmatrix} 2 & 2 & 1 \\ 1 & 3 & 1 \\ 1 & 2 & 2 \end{pmatrix}$ are equal to 1 each. Find the eigen values of A^{-1}.

31. The eigen vectors of a 3×3 matrix A corresponding to the eigen values 1, 1, 3 are $\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$, $\begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$ respectively, find the matrix.

32. Verify Cayley-Hamilton theorem for the following matrix and hence find its inverse, if it exists:

(a) A= $\begin{bmatrix} 1 & 2 \\ 2 & -1 \end{bmatrix}$ (b) $A = \begin{bmatrix} 3 & -1 & 4 \\ 0 & 2 & 1 \\ 1 & -1 & -2 \end{bmatrix}$ (c) $A = \begin{bmatrix} 1 & -1 & 2 \\ 2 & -2 & 4 \\ 3 & -3 & 6 \end{bmatrix}$

33. Find A^4 using Cayley-Hamilton's theorem:

(a) A= $\begin{bmatrix} 5 & 3 \\ 3 & 2 \end{bmatrix}$ (b) A= $\begin{bmatrix} 3 & 2 & 4 \\ 4 & 3 & 2 \\ 2 & 4 & 3 \end{bmatrix}$ (c) A= $\begin{bmatrix} 7 & 2 & -2 \\ -6 & -1 & 2 \\ 6 & 2 & -1 \end{bmatrix}$ (d) A= $\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix}$

34. Verify Cayley-Hamilton theorem for $\begin{pmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{pmatrix}$ and hence find the matrix represented by $A^8 - 5A^7 + 7A^6 - 3A^5 + A^4 - 5A^3 + 8A^2 - 2A + I$.

35. Define Hermitian, Skew-Hermitian and Unitary matrices.

36. State properties of Hermitian, Skew-Hermitian and Unitary matrices.

37. Prove that $A = \begin{pmatrix} \frac{1}{2}(1+i) & \frac{1}{2}(-1+i) \\ \frac{1}{2}(1+i) & \frac{1}{2}(1-i) \end{pmatrix}$ is unitary matrix and *hence* find A^{-1}.

38. If $A = \begin{pmatrix} -1 & 2+i & 5-3i \\ 2-i & 7 & 5i \\ 5+3i & -5i & 2 \end{pmatrix}$, show that A is a Hermitian matrix and iA is a skew- Hermitian matrix.

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