Vectors in $\mathbb{R}^n$:- Let $\mathbb{R}$ be the set of all real numbers and n is a fixed +ve integer (and we assume $n \geq 2$) then $\mathbb{R}^n = \{(x_1, x_2, ---x_n) \mid x_i \in \mathbb{R}, i=1,2,---,n\}$ is the set of all ordered n-tuples of real numbers.

If $x = (x_1, x_2, ---, x_n) \in \mathbb{R}^n$, then x is called a point or a row-vector or simply a vector in $\mathbb{R}^n$-space, which is also denoted by $x = (x_1\ x_2\ ---x_n)$ or $x = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$ and in later case it is called a column-vector.

Each component in a vector $x = (x_1, x_2, ---, x_n)$ is a real number and is called a scalar.

Two vectors $x = (x_1, x_2, ---, x_n)$ & $y = (y_1, y_2, ---, y_n)$ in $\mathbb{R}^n$ are said to be equal written as $x = y$ if $x_i = y_i$ $\forall\ i = 1, 2, ---, n$. The vector $(0, 0, ---, 0)$ in $\mathbb{R}^n$ is called the zero-vector and is denoted by 0.

The sum of two vectors $x = (x_1, x_2, ---, x_n)$ & $y = (y_1, y_2, ---, y_n)$ in $\mathbb{R}^n$ is denoted & defined as $x + y = (x_1 + y_1, x_2 + y_2, ---, x_n + y_n)$ which is also a unique vector in $\mathbb{R}^n$, i.e., $\mathbb{R}^n$ is closed under vector addition.

The product of a scalar k and a vector $x = (x_1, x_2, ---, x_n)$ in $\mathbb{R}^n$ is denoted & defined as $kx = (kx_1, kx_2, ---, kx_n)$ which is also a unique vector in $\mathbb{R}^n$, i.e., $\mathbb{R}^n$ is closed under scalar multiplication.

For any vectors $x, y, z \in \mathbb{R}^n$ and scalars $\alpha, \beta \in \mathbb{R}$ it is easy to see that the following conditions hold:

(i) $(x+y)+z = x+(y+z)$ (ii) $x + 0 = x = 0 + x$

(iii) $-x + x = 0 = x + (-x)$ where $-x = (-x_1, -x_2, ---, -x_n)$

(iv) $x+y = y+x$, (v) $\alpha(x+y) = \alpha x + \alpha y$, (vi) $(\alpha+\beta)x = \alpha x + \beta x$

(vii) $(\alpha\beta)x = \alpha(\beta x)$, (viii) $1x = x$.

The set $\mathbb{R}^n$ which is closed under vector addition & scalar multiplication and also all the above conditions from (i) to (viii) hold $\forall\, x, y, z \in \mathbb{R}^n$ & $\forall\, \alpha, \beta \in \mathbb{R}$, is called a vector space (or Linear space) of dimension n. $\mathbb{R}^n$ is also called Euclidean-n-space.

Note:- 1) In particular $\mathbb{R}^2 = \{(x_1, x_2) \mid x_1, x_2 \in \mathbb{R}\}$ is two-dimensional Euclidean space or Euclidean plane, $\mathbb{R}^3 = \{(x_1, x_2, x_3) \mid x_1, x_2, x_3 \in \mathbb{R}\}$ is three-dimensional Euclidean space, which are both vector (Linear) spaces.

2) $-y = (-1)y = (-y_1, -y_2, ---, -y_n)$

$x - y = x + (-y) = (x_1 - y_1, x_2 - y_2, ---, x_n - y_n)$

$0x = (0x_1, 0x_2, ---, 0x_n) = (0, 0, ---, 0) = 0$, zero vector.

Linear (or Vector) subspace:- A non-empty subset W of $\mathbb{R}^n$ is called a linear subspace or simply a subspace of $\mathbb{R}^n$ if W itself a linear space under same vector addition & scalar multiplication as in $\mathbb{R}^n$.

Theorem:- A necessary and sufficient condition for a non-empty subset W of $\mathbb{R}^n$ to be a subspace is that $\alpha x + \beta y \in W$ $\forall\, x, y \in W$ & $\forall\, \alpha, \beta \in \mathbb{R}$.

e.g., i) $W_1 = \{(x, 0) \mid x \in \mathbb{R}\}$ is a subspace of $\mathbb{R}^2$

ii) $W_2 = \{(x, 0, z) \mid x, z \in \mathbb{R}\}$ is a subspace of $\mathbb{R}^3$

In particular $\{0\}$ & $\mathbb{R}^n$ are always subspaces of $\mathbb{R}^n$

Linear Combination, Linear dependence and linear independence of vectors in $\mathbb{R}^n$:—

If $v_1, v_2, ---, v_m$ are m vectors in $\mathbb{R}^n$ and $\alpha_1, \alpha_2, ---, \alpha_m$ are scalars in $\mathbb{R}$ then a vector of the form $\alpha_1 v_1 + \alpha_2 v_2 + --- + \alpha_m v_m$ is called a linear combination of vectors $v_1, v_2, ---, v_m$.

A finite number of vectors $v_1, v_2, ---, v_m$ in $\mathbb{R}^n$ are said to be linearly dependent (L.D.) if $\exists$ scalars $\alpha_1, \alpha_2, ---, \alpha_m \in \mathbb{R}$, not all zero i.e., at least one $\alpha_i \neq 0$, such that $\alpha_1 v_1 + \alpha_2 v_2 + --- + \alpha_m v_m = 0$, zero vector.

Also $v_1, v_2, ---, v_m$ are said to be linearly independent (L.I.) if they are not L.D. i.e., if

$$\alpha_1 v_1 + \alpha_2 v_2 + --- + \alpha_m v_m = 0 \quad \text{where } \alpha_i \in \mathbb{R}$$

$$\Rightarrow \alpha_1 = 0, \alpha_2 = 0, ---, \alpha_m = 0.$$

e.g., i) In $\mathbb{R}^3$, $x = (1, 0, 0)$, $y = (0, 1, 0)$, $z = (0, 0, 1)$ are L.I. because —

$$\alpha_1 x + \alpha_2 y + \alpha_3 z = 0 \quad \text{where } \alpha_1, \alpha_2, \alpha_3 \in \mathbb{R}$$

$$\Rightarrow \alpha_1 (1,0,0) + \alpha_2 (0,1,0) + \alpha_3 (0,0,1) = (0,0,0)$$

$$\Rightarrow (\alpha_1, 0, 0) + (0, \alpha_2, 0) + (0, 0, \alpha_3) = (0,0,0)$$

$$\Rightarrow (\alpha_1, \alpha_2, \alpha_3) = (0,0,0) \Rightarrow \alpha_1 = 0, \alpha_2 = 0, \alpha_3 = 0.$$

ii) In $\mathbb{R}^3$, $u = (1, 0, -2)$, $v = (2, 1, 0)$, $w = (-1, -1, -2)$ are L.D. because —

$$-u + v + w = -(1, 0, -2) + (2, 1, 0) + (-1, -1, -2)$$

$$\text{or } (-1)u + 1v + 1w = (-1+2-1,\ 0+1-1,\ 2+0-2)$$

$$= (0,0,0)$$

$$= 0, \text{ zero vector}$$

Linear span:– If $S=\{v_1, v_2, ---, v_m\} \subseteq \mathbb{R}^n$, then the set of all linear combinations of vectors in S is denoted by $L(S)$ and is called linear span of S which is a subspace of $\mathbb{R}^n$

If $L(S)=\mathbb{R}^n$ then S is called a generating system of $\mathbb{R}^n$, i.e., if every vector in $\mathbb{R}^n$ can be expressed as a linear combination of vectors in S then S will be called a generating system of $\mathbb{R}^n$. In this case we also say that $\mathbb{R}^n$ is spanned by S.

e.g., i) $S=\{e_1=(1,0,0,---,0),\ e_2=(0,1,0,---,0),\ ---,\ e_n=(0,0,0,---,0,1)\}$ is a generating system of $\mathbb{R}^n$.

ii) $S=\{e_1=(1,0,0),\ e_2=(0,1,0),\ e_3=(0,0,1)\}$ is a " " " $\mathbb{R}^3$

iii) $S=\{e_1=(1,0),\ e_2=(0,1)\}$ is a g.s. of $\mathbb{R}^2$.

Justification (ii):– Let $x=(a,b,c)\in\mathbb{R}^3$ be any vector in $\mathbb{R}^3$

then
$$\begin{aligned} x &= (a,0,0)+(0,b,0)+(0,0,c) \\ &= a(1,0,0)+b(0,1,0)+c(0,0,1) \\ &= ae_1+be_2+ce_3 \\ &= \text{a L.c. of vectors in } S. \end{aligned}$$

L.D. and L.I. Set:– If $S=\{v_1, v_2, ---, v_m\} \subseteq \mathbb{R}^n$ then S is said to be L.D. or L.I. according as $v_1, v_2, ---, v_m$ are L.D. or L.I. respectively.

e.g., all the sets in above three examples are L.I.

Basis:– A subset S of $\mathbb{R}^n$ is called a basis of $\mathbb{R}^n$ if S is L.I. and $L(S)=\mathbb{R}^n$. i.e., a linearly independent generating system of $\mathbb{R}^n$ is a basis of $\mathbb{R}^n$. Alternatively a basis of $\mathbb{R}^n$ can be defined as a maximal L.I. subset of $\mathbb{R}^n$ i.e., a subset of $\mathbb{R}^n$ having maximum number of L.I. vectors. and this number is called dimension of $\mathbb{R}^n$

e.g., the set $S = \{e_1 = (1,0,0,---,0),\ e_2 = (0,1,0,---,0),\ ---$
$e_n = (0,0,0,---,0,1)\}$ is a L.I. subset of $\mathbb{R}^n$ and also any vector $x = (x_1, x_2, ---, x_n)$ in $\mathbb{R}^n$ can be expressed as $x = x_1e_1 + x_2e_2 + ---+ x_ne_n$, i.e., as a linear combination of vectors in S, so that S is a basis of $\mathbb{R}^n$, which is called standard or Canonical basis of $\mathbb{R}^n$.
Also max^m number of L.I. vectors in $\mathbb{R}^n$ is n so that dimension of $\mathbb{R}^n$ is n.

Row space, Column space and rank of a matrix:—

Let A be a $m \times n$ matrix over real numbers and let $R_1, R_2, ---, R_m$ be the m rows of A and $C_1, C_2, ---, C_n$ be the n-columns of A. Each row of A consists of n-components and can be viewed as a vector in $\mathbb{R}^n$; each column of A consists of m components and can be viewed as a vector in $\mathbb{R}^m$.

The subspace of $\mathbb{R}^n$ generated by (spanned by) the rows $R_1, R_2, ---, R_m$ of A is called the row-space of A and the subspace of $\mathbb{R}^m$ generated by (spanned by) the columns $C_1, C_2, ---, C_n$ of A is called the column-space of A. The dimension of the row-space and the column space of a matrix A are same i.e., maximum number of L.I. rows and L.I. columns of A are same and this number is called the rank of the matrix A and we denote it by rank(A) or $\rho(A)$.
Some times max^m no. of L.I. rows (columns) is called row-rank (column rank) of A but row-rank(A) = column-rank(A)

Theorem:— Any set $S = \{v_1, v_2, \cdots, v_m\}$ in $\mathbb{R}^n$ containing zero vector is L.D.

Proof:— If $v_k = 0$, $1 \le k \le m$, then

$$0v_1 + 0v_2 + \cdots + 0v_{k-1} + 1v_k + 0v_{k+1} + \cdots + 0v_m = 0$$

where the coefficient $1 \neq 0$.

$\therefore$ S is L.D.

Theorem:— The non-zero rows of a matrix in echelon form are L.I.

Proof:— Let $R_1, R_2, \cdots, R_p$ be the non-zero rows of a $m \times n$ echelon matrix A, begining from 1st row where $1 \le p \le m$

If possible let $R_1, R_2, \cdots, R_p$ are L.D. then $\exists$ scalars $\alpha_1, \alpha_2, \cdots, \alpha_p$ not all zero such that —

$$\alpha_1 R_1 + \alpha_2 R_2 + \cdots + \alpha_p R_p = 0 \quad \text{——— (1)}$$

Let k be the least suffix such that $\alpha_k \neq 0$ where $1 \le k \le p$.

$\therefore$ (1) becomes —

$$\alpha_k R_k + \alpha_{k+1} R_{k+1} + \cdots + \alpha_p R_p = 0$$

$$\Rightarrow -\alpha_k R_k = \alpha_{k+1} R_{k+1} + \cdots + \alpha_p R_p$$

$$\Rightarrow R_k = -\alpha_k^{-1}(\alpha_{k+1} R_{k+1} + \cdots + \alpha_p R_p)$$

$$\Rightarrow R_k = (-\alpha_k^{-1}\alpha_{k+1}) R_{k+1} + (-\alpha_k^{-1}\alpha_{k+2}) R_{k+2} + \cdots + (-\alpha_k^{-1}\alpha_p) R_p \quad \text{——— (2)}$$

Let the jth component of R_k is the first non-zero element then jth components of each of $R_{k+1}, R_{k+2}, \cdots, R_p$ is zero, and therefore the jth component of the vector in RHS of (2) is zero, but the jth component of the vector in LHS of (2) is not zero. Thus we meet a contradiction. Therefore our initial assumption is wrong. Therefore $R_1, R_2, \cdots, R_p$ can not be L.D. and hence they are L.I.

Note:— Since the non-zero rows of a matrix A in echelon form are L.I., so they constitute a maximal L.I. set of vectors and hence they form a basis of the row-space

of the matrix A and therefore row-rank or rank of A is equal to the number of non-zero rows of of A when it is in echelon form.

Theorem:- Elementary operations do not alter the rank of a matrix.

Note:- To find the rank of a matrix we reduce it to echelon form using elementary operations and then the rank of the matrix is equal to the number of non-zero rows in its echelon form.

Elementary matrix :- A matrix obtained from a unit matrix by an application of a single elementary operation is called an e-matrix.

Theorem:- Every elementary row (column) operation of a matrix can be obtained by pre (post) multiplication of the matrix by the corresponding e-matrix.

e.g., i) applying $R_2 \to R_2 - 3R_1$ to $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ we get an e-matrix $E_R = \begin{pmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$. Now consider $A = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 6 & 0 \\ 1 & 4 & 5 \end{pmatrix}$

then applying $R_2 \to R_2 - 3R_1$ to A we get $\begin{pmatrix} 1 & 2 & 3 \\ 0 & 0 & -9 \\ 1 & 4 & 5 \end{pmatrix}$

Also $E_R A = \begin{pmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 2 & 3 \\ 3 & 6 & 0 \\ 1 & 4 & 5 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 \\ 0 & 0 & -9 \\ 1 & 4 & 5 \end{pmatrix}$

ii) applying $C_2 \to C_2 - 2C_1$ to A we get $\begin{pmatrix} 1 & 0 & 3 \\ 3 & 0 & 0 \\ 1 & 2 & 5 \end{pmatrix}$

Also applying $C_2 \to C_2 - 2C_1$ to $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ we get

$E_c = \begin{pmatrix} 1 & -2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$, Also $AE_c = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 6 & 0 \\ 1 & 4 & 5 \end{pmatrix}\begin{pmatrix} 1 & -2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 3 \\ 3 & 0 & 0 \\ 1 & 2 & 5 \end{pmatrix}$