4.10.21 21BAI1830

PRACTICE SET 1 CALCULUS

1. i) $f(x) = x^2\sqrt{8-x^3}$ $u = x^2$ $u' = 2x$ $v = (8-x^3)^{1/2}$

$f'(x) = 2x(8-x^3)^{1/2} - \frac{3x^4}{2\sqrt{8-x^3}}$ $u_1 = 8-x^3$ $u_1' = -3x^2$ $v_1 = u^{1/2}$ $v_1' = \frac{1}{2}u^{-1/2}$

$= \frac{-x(7x^3-32)}{2\sqrt{8-x^3}}$ $v' = \frac{-3x^2}{2(8-x^3)^{1/2}}$

$\frac{-x(7x^3-32)}{2\sqrt{8-x^3}} = 0$

$-x = 0$, $x = 0$ $7x^3 - 32 = 0$, $x = 1.6597$ $2(-x^3+8)^{1/2} = 0$, $x = 2$.

$\therefore$ critical points @ $x=0$, $x=1.6597$, $x=2$

interval:	$-\infty < x < 0$	$0 < x < 1.6597$	$1.6597 < x < 2$
	$f'(-1) = -6.5$	$f'(1) = 4.72$	$f'(1.7) = -1.156$
sign:	$-$	$+$	$-$
behaviour:	decreasing	increasing	decreasing.

$f'(x) = \frac{-x(7x^3-32)}{2\sqrt{8-x^3}}$ $u = x(7x^3-32)$ $u_1 = x$ $u_1' = 1$ $v_1 = (7x^3-32)$ $v_1' = 21x^2$ $v = 2(8-x^3)^{1/2}$ $v' = \frac{-3x^2}{2(8-x^3)^{1/2}}$

$f''(x) = \frac{(8-x^3)^{1/2}(28x^3-32) + \frac{3x^3(7x^3-32)}{2(8-x^3)^{1/2}}}{2(8-x^3)}$ $u' = 21x^3 + 7x^3 - 32 = 28x^3 - 32$

$0 = -\frac{1}{2}\left(\frac{(8-x^3)^{1/2}(28x^3-32) + \frac{3x^3(7x^3-32)}{2(8-x^3)^{1/2}}}{(8-x^3)}\right)$

~~$(8-x^3)^{1/2}(28x^3-32) = \frac{3x^3(7x^3-32)}{2(8-x^3)^{1/2}}$~~ ~~$x(8-x^3)^{1/2} = 0$~~

~~$0 = \frac{1}{2}(35x^6 -$~~ $(8-x^3) = 0$, $x = 2$

$(8-x^3)^{1/2}(28x^3-32) = -\frac{3x^3(7x^3-32)}{2(8-x^3)^{1/2}}$

$2(8-x^3)(28x^3-32) = -3x^3(7x^3-32)$

$(16-2x^3)(28x^3-32) = -2(x^6+96x^3)$

$448x^3 - 512 - 56x^6 + 64x^3 = -2(x^6+96x^3)$

$-35x^6 + 416x^3 + 512 = 0$

$x^3(-35x^3+416) = -512.$

[redacted]

unable to calculate roots ∴ found roots graphically.

$x = 1.117$ ← inflection point

~~∴ f''(x) = 0 is positive~~

$-\infty < x < 1.117$	$1.117 < x < \infty$
$f''(1) = 1.457$	$f''(1.5) = 7.442$
+	−
~~increasing~~ concave up	~~decreasing~~ concave down

∴ curve is concave up on interval $(-\infty, 1.117)$ and concave down on $(1.117, \infty)$

ii) $g(x) = \dfrac{x^3}{3x^2+1}$

$u = x^3$ $\quad v = 3x^2+1$

$u' = 3x^2$ $\quad v' = 6x$

$g'(x) = \dfrac{6x^4 - 3x^2(3x^2+1)}{(3x^2+1)^2}$

$\dfrac{6x^4 - 3x^2(3x^2+1)}{(3x^2+1)^2} = 0$

$x = 0$ ← critical point

$-\infty < x < 0$	$0 < x < \infty$
$f'(-1) = 0.375$	$f'(1) = 0.375$
+	+
increasing	increasing.

$g''(x) = \dfrac{6x(x^2-1)}{(3x^2+1)^3}$ $\qquad \dfrac{6x(x^2-1)}{(3x^2+1)^3} = 0$

$6x^3 - 6x = 0$ $\qquad (3x^2+1)^3 = 0$

inflection points. → $x = 1$ $x = -1$ $x = 0$

$-\infty < x < -1$	$-1 < x < 0$	$0 < x < 1$	$1 < x < \infty$
$f''(-2) = -0.016$	$f''(-0.5) = 0.4198$	$f''(0.5) = -4198.$	$f''(2) = 0.016$
−	+	−	+
concave down	concave up	concave down	concave up.

2. i) $f(x) = x^{1/3}(8+x^2)$

$f'(x) = \dfrac{7x^2+8}{3x^{2/3}}$ $\qquad \dfrac{7x^2+8}{3x^{2/3}} = 0$

$x = 0$ → ~~non~~ critical points.

$-\infty < x < 0$ $\qquad$ $0 < x < \infty$

$f'(-1) = 5$ $\qquad$ $f'(1) = 5$

$+$ $\qquad$ $+$

increasing $\qquad$ increasing.

ii. $f''(x) = \cancel{28x^2-16} \quad \frac{14\sqrt[3]{x}}{3} - \frac{2(7x^2+8)}{9x^{5/3}}$

$$0 = \frac{28x^2-16}{9x^{5/3}}$$

$x_1 = 0.7559 \qquad x_2 = -0.7559.$

$-\infty < x < -0.7559$ $\qquad$ $-0.7559 < x < 0.77559$ $\qquad$ $0.77559 < x < \infty$

$f''(-1) = -1.33$ $\qquad$ $f(0) = -16$ $\qquad$ $f'(1) = 1.33$

$-$ $\qquad$ $-$ $\qquad$ $+$

concave down $\qquad$ concave down $\qquad$ concave up.

ii) $g(x) = \frac{x^2-3}{(x+1)^2}$

$g'(x) = \frac{2x}{(x+1)^2} - \frac{2(x^2-3)}{(x+1)^3}$ $\qquad$ $\frac{2(x+3)}{(x+1)^3} = 0$

$x_1 = -3 \leftarrow$ critical point

$-\infty < x < 3$ $\qquad$ $3 < x < \infty$

$f'(1) = 1$ $\qquad$ $f'(4) = 0.112$

$+$ $\qquad$ $+$

increasing $\qquad$ increasing.

$f''(x) = \frac{2}{(x+1)^3} - \frac{6(x+3)}{(x+1)^4}$ $\qquad$ $\frac{-4(x+4)}{(x+1)^4} = 0$

$x = -4 \leftarrow$ inflection point

$-\infty < x < -4$ $\qquad$ $-4 < x < \infty$

$f''(-5) = 0.0781$ $\qquad$ $f''(0) = -16$

$+$ $\qquad$ $-$

concave up $\qquad$ concave ~~up~~ down

3. $f(x) = x^{2/3}(x^2-3)$

$f'(x) = \frac{2(x^2-3)}{3\sqrt[3]{x}} + 2x^{5/3}$ $\qquad$ $\frac{8x^2-6}{3\sqrt[3]{x}} = 0$

critical points. $\rightarrow$ $x_1 = 0.8660$ $\qquad$ $x_2 = -0.8660$

$x_3 = 0$

~~$-\infty < x < -0.8660$~~ $\qquad$ ~~$-0.8660 < x < 0.8660$~~ $\qquad$ ~~$0.8660 < x < \infty$~~

~~$f'(-1) = -0.667$~~ $\qquad$ ~~$f(0) =$~~ $\qquad$ ~~$f'(1) = 0.667$~~

$-\infty < x < -0.8660$ $\quad f'(-1) = -0.667$ $\quad -$ $\quad$ ~~concave down~~ decreasing

$-0.8660 < x < 0$ $\quad f'(-0.1) = -4.25$ $\quad -$ $\quad$ concave ~~up~~ down decreasing

$0 < x < 0.8660$ $\quad f'(0.1) = 4.25$ $\quad +$ $\quad$ ~~concave up~~ increasing

$0.8660 < x < \infty$ $\quad f'(1) = 0.667$ $\quad +$ $\quad$ ~~concave up~~ increasing

$$f''(x) = \frac{16x^{2/3}}{3} - \frac{8x^2-6}{9x^{4/3}} = \frac{40x^2+6}{9x^{4/3}} = 0$$

$x = 0 \leftarrow$ inflection point

$-\infty < x < 0$ $\quad f''(-1) = 5.11$ $\quad +$ $\quad$ concave up

$0 < x < \infty$ $\quad f''(1) = 5.11$ $\quad +$ $\quad$ concave up.

ii) $g(x) = \dfrac{x^3+x}{x^2+9}$ $\qquad u = x^3+x$, $u' = 3x^2+1$ $\qquad v = x^2+9$, $v' = 2x$

$$g'(x) = \frac{3x^2+1}{x^2+9} - \frac{2x(x^3+x)}{(x^2+9)^2} = \frac{x^4+26x^2+9}{(x^2+9)^2} = 0$$

no real roots.

$$g''(x) = \frac{4x^3+52x}{(x^2+9)^2} - \frac{4x(x^4+26x^2+9)}{(x^2+9)^3}$$

$$0 = -\frac{16x(x^2-27)}{(x^2+9)}$$

$x_1 = 0$ $\quad x_2 = 5.196$ $\quad x_2 = -5.196$

$-\infty < x < -5.196$ $\quad f''(-10) = 0.983$ $\quad +$ $\quad$ concave up

$-5.196 < x < 0$ $\quad f''(-1) = -4.16$ $\quad -$ $\quad$ concave down

$0 < x < 5.196$ $\quad f''(1) = 4.16$ $\quad +$ $\quad$ concave up.

$5.196 < x < \infty$ $\quad f(10) = -9.83$ $\quad -$ $\quad$ concave down

* Q4 at the end.

5. ~~4.~~ ~~[illegible]~~

x = no. of additional ₹1000 charged.

R = ~~max.~~ revenue.

$R = (28000 + 1000x)(80 - 2x)$
$= 2240000 + 24000x - 2000x^2$

$R' = 24000 - 4000x$
$24000 - 4000x = 0$
$x = 6.$

$\therefore R = 2240000 + 24000(6) - 2000(6)^2$
$= ₹2,312,000$

6. $f(x) = x^4 - 4x^3 + 10$
$f'(x) = 4x^3 - 12x^2$

$4x^3 - 12x^2 = 0$
$4x^2(x-3) = 0$

critical values $\longrightarrow$ $x_1 = 0$ $x_2 = 3$

$-\infty < x < 0$	$0 < x < 3$	$3 < x < \infty$
$f'(-1) = -16$	$f'(1) = -8$	$f'(10) = 2800$
$-$	$-$	$+$
decreasing	decreasing	increasing.

$\therefore$ local minimum at $x = 3$ as $f'(x)$ changes from negative to positive.

$f''(x) = 4x^2 + 8(x-3)x$
$0 = 12x(x-2)$
$x_1 = 2$ $x = 0$ $\leftarrow$ inflection points.

$-\infty < x < 0$	$0 < x < 2$	$2 < x < \infty$
$f''(-1) = -36$	$f''(1) = 12$	$f''(10) = -960$
$-$	$+$	$-$
concave down	concave up	concave down.

7. $f(x) = 2x^3 - 9x^2 - 108x + 2$
$f''(x) = 12x - 18$
$12x - 18 = 0$
$x = 1.5.$

$-\infty < x < 1.5$	$1.5 < x < \infty$
$f''(1) = -6$	$f''(5) = 42$
$-$	$+$
concave down	concave up.

8. $s = -\frac{1}{2}gt^2 + v_0 t + s_0$

$s'(t) = \cancel{\frac{1}{2}} - gt + v_0$

$-gt + v_0 = 0$

$t = \frac{v_0}{g}$ $\leftarrow$ critical value.

$$S = -\frac{1}{2}g\left(\frac{V_0}{g}\right)^2 + V_0\left(\frac{V_0}{g}\right) + S_0$$

$$= \frac{-V_0^2}{2g} + \frac{V_0^2}{g} + S_0$$

$$S = \frac{V_0^2}{2g} + S_0$$

9. a) $f(x) = x^3 - 3x + k$ $\quad (0,1) \leftarrow$ continuous and differentiable interval ①

$f(0) = (0)^3 - 3(0) + k$ $\quad f(1) = (1)^3 - 3(1) + k$

$f(0) = k$ $\quad f(1) = k$

$\therefore$ ② $f(0) = f(1)$

$\therefore$ ③ there must exist atleast one value of c in interval $(0,1)$ such that $f'(c) = 0$

$f'(x) = 3x^3 - 3$

$0 = 3x^3 - 3$

$x = \pm 1$

$c = \pm 1$

$\therefore$ no value of k ~~doesn't~~ satisfys Rolle's theorm as c doesn't lie within interval $0 < c < 1$.

b) i) ~~$f(x) = 6(1 - \frac{t}{12})^2 = 6 - t + \frac{t^2}{24}$~~

~~$f'(x) =$~~ $\quad y = 6\left(1 - \frac{t}{12}\right)^2$

velocity of draining water $\longrightarrow$ $V(t) = \frac{dy}{dt} = -1 + \frac{t}{12}$ $\quad [0, 12]$

ii) $V(0) = -1 + \frac{(0)}{12}$ $\quad V(12) = -1 + \frac{(12)}{12}$

$= -1$ $\quad = 0$

$\therefore$ Fluid level falls fastest at $t = 12$ where $\frac{dy}{dt} = 0$

Fluid level falls slowest at $t = 0$ where $\frac{dy}{dt} = 1$.

10. a) $y = x^2 + 7x - 5$

$y' = 2x\frac{dx}{dy} + 7\frac{dx}{dy}$

$= \frac{2x}{3} + \frac{7}{3}$

$= 2x + 7/3$

when $x = 2$

$\frac{2(2) + 7}{3} = 11/3$

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b) $f(x) = \ln(x^4 + 20x^3 + 100)$

$$f'(x) = \frac{4x^3 + 60x^2}{x^4 + 20x^3 + 100}$$

$$\frac{4x^3 + 60x^2}{x^4 + 20x^3 + 100} = 0$$

$x_1 = 0 \quad x_2 = -15.$

~~$-\infty < x < -15$~~ ~~$f'(-20) = -80$~~ ~~$-$~~ ~~decreasing~~

$-15 < x < 0$ $f'(0.1) = 5.96 \times 10^{-3}$ $+$ increasing

$0 < x < 20$ $f(1) = 0.5289$ $+$ increasing.

11. a) $v(t) = t^3 - 2t^2 + 22t - 3 \quad [0, 126]$

$v'(t) = 3t^2 - 4t + 22$

$3t^2 - 4t + 22 = 0$

no real roots.

b) x = number. reciprocal = $1/x$

$f(x) = x + \frac{1}{x}$ ← minimum.

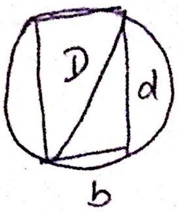
$$f'(x) = 1 - \frac{1}{x^2}$$

$$1 - \frac{1}{x^2} = 0$$

$x = \pm 1$

$x \geq 0 \therefore x = 1$

4.



$$b^2+d^2=D^2$$

$$2b\frac{db}{dd}+2d=0$$

$$\frac{db}{dd}=-\frac{d}{b}$$

$$S=bd^2$$

$$\frac{dS}{dd}=b(2d)+d^2\frac{db}{dd}=0$$

$$2bd+d^2\left(-\frac{d}{b}\right)=0$$

$$2bd=\frac{d^3}{b}$$

$$2b^2=d^2$$

$$d=b\sqrt{2}$$