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Intuitionistic Fuzzy Block Matrix and its Some Properties

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Abstract. If in an intuitionistic fuzzy matrix each element is again a smaller intuitionistic fuzzy matrix then the intuitionistic fuzzy matrix is called intuitionistic fuzzy block matrix (IBFMs). In this paper, the concept of intuitionistic fuzzy block matrices (IBFMs) are introduced and defined different types of intuitionistic fuzzy block matrices (IBFMs). The operations direct sum, Kronecker sum, Kronecker product of intuitionistic fuzzy matrices are presented and shown that their resultant matrices are intuitionistic fuzzy block matrices (IBFMs). Also, some relational operations are presented and prove some properties of intuitionistic fuzzy block matrices (IBFMs).

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1. Introduction

Atanassov [4] introduced the concept of intuitionistic fuzzy sets (IFSs), which is a generalization of fuzzy subsets. Later on much research works have done with this concept by Atanasov and others. The term fuzzy matrix has important role in fuzzy algebra. For definition of fuzzy matrix we follow the definition of Dubois and Prade [7], i.e. a matrix with fuzzy member as its element. This class of fuzzy matrices consist of applicable matrices which can model uncertain aspects and the works on them are limited. Some of the most interesting works on these matrices can be seen in [5, 10, 18, 19]. Thomson [16] defined convergence of a square fuzzy matrix. Pal and Shyamal [15] introduced two new operators on fuzzy matrices and shown several properties of them. By the concept of IFSs, first time Pal [12] introduced intuitionistic fuzzy determinant. Latter on Pal and Shyamal [13, 14] introduced intuitionistic fuzzy matrices (IFMs) and distance between intuitionistic fuzzy matrices. Bhowmik and Pal [5, 6] presented some results on intuitionistic fuzzy matrices, intuitionistic circulant fuzzy matrices and generalized intuitionistic fuzzy matrices.

The general rectangular or square array of the numbers are known as matrix and if the elements are intuitionistic fuzzy then the matrix is called intuitionistic fuzzy matrix. If we delete some rows or some columns or both or neither then the intuitionistic fuzzy matrix is called intuitionistic fuzzy submatrix. The concept of non-empty subset in set theory and the principle of combination are combinedly used for the construction and calculation of the number intuitionistic fuzzy submatrices of a given intuitionistic fuzzy matrix.

Again, if an intuitionistic fuzzy matrix is divided or partitioned into smaller intuitionistic fuzzy matrices called cells or blocks with consecutive rows and columns by drawing dotted horizontal lines of full width between rows and vertical lines of full height between columns, then the intuitionistic fuzzy matrix is called intuitionistic fuzzy block matrix. There are lots of advantages noted in partitioning an intuitionistic fuzzy matrix A into blocks or cells. It simplifies the writing or printing of an IFM A in compact form and thus save space. It exhibits some smaller structure of A. It also simplifies computation.

The structure of this paper is organized as follows. In Section 2, the preliminaries and some definitions are given. In Section 3, different kinds of intuitionistic submatrix and intuitionistic fuzzy block matrix are given. Section 4 deals with direct sum, Kronecker sum and Kronecker product of intuitionistic fuzzy block matrix. In Section 5, some relational operations on intuitionistic fuzzy block matrices are gained. Finally, at the end of this paper a conclusion is given.

2. Preliminaries

Here some preliminaries, definitions of IFSs and IFMs are recalled and presented some algebric operations of IFMs and different types of IFMs.

2.1 Fuzzy set and intuitionistic fuzzy set

Definition 2.1 ***(Fuzzy set (FS))*** *A fuzzy set* A *in a universal set* X *is defined as* $A=\{\langle x,\mu_A(x)\rangle \mid x\in X\}$ *where* $\mu_A: X\to[0,1]$ *is a mapping called the membership function of the fuzzy set* A.

Definition 2.2 ***(Intuitionistic fuzzy set (IFS))*** *An intuitionistic fuzzy set (IFS)* A *over* X *is an object having the form* $A=\{x,\langle\mu_A(x),\nu_A(x)\rangle : x\in X\}$ *; where* $\mu_A: X\to[0,1]$ *and* $\nu_A: X\to[0,1]$*, where* $\mu_A(x)$ *and* $\nu_A(x)$ *are called the* *membership and non-membership values of* x *in* A *satisfying the condition* $0 \le \mu_A(x) + \nu_A(x) \le 1$.

Some operations on IFSs

In the following we define some relational operations on IFSs. Let A and B be two IFSs on X, where

$$A = \{x, \langle \mu_A(x), \nu_A(x) \rangle : x \in X\} \text{ and}$$
$$B = \{x, \langle \mu_B(x), \nu_B(x) \rangle : x \in X\}.$$

Then, (1) $A = B \Leftrightarrow \mu_A(x) = \mu_B(x)$ *and* $\nu_A(x) = \nu_B(x)$, *for all* $x \in X$.

(2) $A \subseteq B$ *iff* $\mu_A(x) \le \mu_B(x)$ *and* $\nu_A(x) \ge \nu_B(x)$ *for all* $x \in X$.

(3) $\overline{A} = \{x, \langle \nu_A(x), \mu_A(x) \rangle : x \in X\}$.

(4) $A \cap B = \{\langle \min\{\mu_A(x), \mu_B(x)\}, \max\{\nu_A(x), \nu_B(x)\} \rangle : x \in X\}$.

(5) $A \cup B = \{\langle \max\{\mu_A(x), \mu_B(x)\}, \min\{\nu_A(x), \nu_B(x)\} \rangle : x \in X\}$.

2.2 Fuzzy matrix and intuitionistic fuzzy matrix

Definition 2.3 ***(Fuzzy matrix (FM))*** A fuzzy matrix (FM) of order $m \times n$ is defined as $A = [a_{ij\mu}]$ where $a_{ij\mu}$ is the membership value of the ij-th element in A.

Definition 2.4 ***(Intuitionistic fuzzy matrix (IFM))*** An intuitionistic fuzzy matrix (IFM) of order $m \times n$ is defined as $A = [\langle a_{ij\mu}, a_{ij\nu} \rangle]$ where $a_{ij\mu}$ and $a_{ij\nu}$ are the membership and non-membership values of the ij-th element in A satisfying the condition $0 \le a_{ij\mu} + a_{ij\nu} \le 1$ for all i, j.

Let $F_{m\times n}$ denotes the set of all IFMs of order $m \times n$. In particular F_n denotes the set of all IFMs of order $n \times n$.

Some algebric operations of IFMs

Let A and B be two IFMs, such that $A = [\langle a_{ij\mu}, a_{ij\nu} \rangle]$ and $B = [\langle b_{ij\mu}, b_{ij\nu} \rangle] \in F_{m\times n}$,

1. Matrix addition and subtraction are given by
$A + B = [\langle \max\{a_{ij\mu}, b_{ij\mu}\}, \min\{a_{ij\nu}, b_{ij\nu}\} \rangle]$ *and* $A - B = [\langle a_{ij\mu} - b_{ij\mu}, a_{ij\nu} - b_{ij\nu} \rangle]$,
where $a_{ij\mu} - b_{ij\mu} = \begin{cases} a_{ij\mu}, & a_{ij\mu} \ge b_{ij\mu} \\ 0, & elsewhere \end{cases}$ and $a_{ij\nu} - b_{ij\nu} = \begin{cases} a_{ij\nu}, & a_{ij\nu} < b_{ij\nu} \\ 0, & otherwise \end{cases}$.

2. Componentwise matrix multiplication is given by
$$A \circ B = [\langle \min\{a_{ij\mu}, b_{ij\mu}\}, \max\{a_{ij\nu}, b_{ij\nu}\} \rangle].$$

3. Let A, B be two IFMs of order $m \times n$ and $n \times p$. Then the matrix product AB is given by $AB = [\langle \sum_k \min\{a_{ik\mu}, b_{kj\mu}\}, \prod_k \max\{a_{ik\nu}, b_{kj\nu}\} \rangle] \in F_{m\times p}$.

3. Different kinds of intuitionistic submatrix and intuitionistic fuzzy block matrix

In this section the concept of intuitionistic submatrix and intuitionistic fuzzy block matrices are introduced and presented some properties of them. We begin this section with some definitions:

3.1 Intuitionistic fuzzy submatrix

Definition 3.1 ***(Intuitionistic fuzzy submatrix)*** *An intuitionistic fuzzy submatrix of an intuitionistic fuzzy matrix of order ≥ 1 is obtained by deleting some rows or some columns or both (not necessarily consecutive) or neither.*

The intuitionistic fuzzy matrix itself is its intuitionistic fuzzy submatrix.

The maximum number of intuitionistic fuzzy submatrices of an $n \times m$ intuitionistic fuzzy matrix is $(2^n - 1)(2^m - 1)$.

Definition 3.2 ***(Intuitionistic fuzzy principal submatrix)*** *The intuitionistic fuzzy submatrix of order $(n-r)$ obtained by deleting r rows and columns of an n square intuitionistic fuzzy matrix is called intuitionistic fuzzy principal submatrix.*

The first order principal intuitionistic fuzzy submatrices obtained from the following third order IFM $\begin{bmatrix} \langle a_{11\mu}, a_{11\nu} \rangle & \langle a_{12\mu}, a_{12\nu} \rangle & \langle a_{13\mu}, a_{13\nu} \rangle \\ \langle a_{21\mu}, a_{21\nu} \rangle & \langle a_{22\mu}, a_{22\nu} \rangle & \langle a_{23\mu}, a_{23\nu} \rangle \\ \langle a_{31\mu}, a_{31\nu} \rangle & \langle a_{32\mu}, a_{32\nu} \rangle & \langle a_{33\mu}, a_{33\nu} \rangle \end{bmatrix}$ are $[\langle a_{11\mu}, a_{11\nu} \rangle]$, $[\langle a_{12\mu}, a_{12\nu} \rangle]$ and $[\langle a_{13\mu}, a_{13\nu} \rangle]$.

Second order intuitionistic fuzzy submatrices are $\begin{bmatrix} \langle a_{11\mu}, a_{11\nu} \rangle & \langle a_{12\mu}, a_{12\nu} \rangle \\ \langle a_{21\mu}, a_{21\nu} \rangle & \langle a_{22\mu}, a_{22\nu} \rangle \end{bmatrix}$,

$$\begin{bmatrix} \langle a_{22\mu}, a_{22\nu} \rangle & \langle a_{23\mu}, a_{23\nu} \rangle \\ \langle a_{32\mu}, a_{32\nu} \rangle & \langle a_{33\mu}, a_{33\nu} \rangle \end{bmatrix}, \begin{bmatrix} \langle a_{11\mu}, a_{11\nu} \rangle & \langle a_{13\mu}, a_{13\nu} \rangle \\ \langle a_{31\mu}, a_{31\nu} \rangle & \langle a_{33\mu}, a_{33\nu} \rangle \end{bmatrix}.$$

Third order intuitionistic fuzzy submatrix is the matrix itself.

Definition 3.3 ***(Leading principal intuitionistic fuzzy submatrix).*** *The intuitionistic fuzzy submatrix of order $(n-r)$ obtained by deleting last r rows and columns of an n square intuitionistic fuzzy matrix A is called leading principal intuitionistic fuzzy submatrix.*

The first order leading principal intuitionistic fuzzy submatrix is $[\langle a_{11\mu}, a_{11\nu} \rangle]$.

The second order leading principal intuitionistic fuzzy submatrix of the above intuitionistic fuzzy matrix is $\begin{bmatrix} \langle a_{11\mu}, a_{11\nu} \rangle & \langle a_{12\mu}, a_{12\nu} \rangle \\ \langle a_{21\mu}, a_{21\nu} \rangle & \langle a_{22\mu}, a_{22\nu} \rangle \end{bmatrix}$

Definition 3.4 ***(Intuitionistic fuzzy partition matrix).*** *If an IFM is divided or partitioned into smaller IFMs called blocks or cells with consecutive rows and columns by separated by dotted horizontal lines of full width between rows and vertical lines of full height between columns, then the IFM is called intuitionistic fuzzy partition matrix.*

The elements of intuitionistic fuzzy partition matrix are smaller IFMs.

It is also called the intuitionistic fuzzy block matrix.

Advantage of intuitionistic fuzzy partitioning: The following advantages may be noted in partitioning an IFM A into blocks or cells:

1. The partitioning may simplicity the writing or printing in compact form thus saves space.
2. It exhibits some smaller structures of A which is of great interest.

Definition 3.5 ***(Conformal partition).*** *Two IFMs of same order are said to be conformally or identically partitioned if*

1. both the IFMs are partitioned in such way that the number of columns of two intuitionistic fuzzy partition matrices are same,
2. the corresponding blocks are of same order.

3.2 Intuitionistic fuzzy block matrix

Definition 3.6 ***(Intuitionistic fuzzy block matrix).*** *The intuitionistic fuzzy matrix whose elements are blocks obtained by partitioning is called intuitionistic fuzzy block matrix.*

Thus $$A=\begin{bmatrix} \langle a_{11\mu},a_{11\nu}\rangle & \langle a_{12\mu},a_{12\nu}\rangle & \vdots & \langle a_{13\mu},a_{13\nu}\rangle & \langle a_{13\mu},a_{13\nu}\rangle \\ \cdots & \cdots & \vdots & \cdots & \cdots \\ \langle a_{21\mu},a_{21\nu}\rangle & \langle a_{22\mu},a_{22\nu}\rangle & \vdots & \langle a_{23\mu},a_{23\nu}\rangle & \langle a_{24\mu},a_{24\nu}\rangle \\ \langle a_{31\mu},a_{31\nu}\rangle & \langle a_{32\mu},a_{32\nu}\rangle & \vdots & \langle a_{33\mu},a_{33\nu}\rangle & \langle a_{34\mu},a_{34\nu}\rangle \end{bmatrix}=\begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix}$$

where $P_{11}=\lfloor\langle a_{11\mu},a_{11\nu}\rangle \quad \langle a_{12\mu},a_{12\nu}\rangle\rfloor$, $P_{12}=\lfloor\langle a_{13\mu},a_{13\nu}\rangle \quad \langle a_{14\mu},a_{14\nu}\rangle\rfloor$,

$$P_{21}=\begin{bmatrix} \langle a_{21\mu},a_{21\nu}\rangle & \langle a_{22\mu},a_{22\nu}\rangle \\ \langle a_{31\mu},a_{31\nu}\rangle & \langle a_{32\mu},a_{32\nu}\rangle \end{bmatrix} and P_{22}=\begin{bmatrix} \langle a_{23\mu},a_{23\nu}\rangle & \langle a_{24\mu},a_{24\nu}\rangle \\ \langle a_{33\mu},a_{33\nu}\rangle & \langle a_{34\mu},a_{34\nu}\rangle \end{bmatrix}$$

The IFM $A=\begin{bmatrix} P_{11} & \vdots & P_{12} \\ \cdots & \cdots & \cdots \\ P_{21} & \vdots & P_{22} \end{bmatrix}$ *is an example of intuitioni stic fuzzy block matrix .*

Definition 3.7 ***(Transpose of intuitionistic fuzzy block matrix).*** *The transpose of intuitionistic fuzzy block matrix is the transpose of both blocks and constituents blocks.* $A^T=\begin{bmatrix} P_{11}^T & P_{21}^T \\ P_{12}^T & P_{22}^T \end{bmatrix}$.

Definition 3.8 ***(Diagonal blocks).*** *The blocks along the diagonal of the intuitionistic fuzzy block matrix are called diagonal blocks. The blocks P_{ij} for which $i = j$ are diagonal blocks. Thus P_{11} and P_{22} diagonal blocks of the intuitionistic fuzzy block matrix* $A = \begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix}$.

Definition 3.9 ***(Square intuitionistic fuzzy block matrix).*** *If the numbers of rows and the numbers columns of blocks are equal then the matrix is called square intuitionistic fuzzy block matrix.*

Thus the partitioned IFM

$$A = \begin{bmatrix} \langle a_{11\mu}, a_{11\nu} \rangle & \langle a_{12\mu}, a_{12\nu} \rangle & \vdots & \langle a_{13\mu}, a_{13\nu} \rangle & \langle a_{14\mu}, a_{14\nu} \rangle & \vdots & \langle a_{15\mu}, a_{15\nu} \rangle & \langle a_{16\mu}, a_{16\nu} \rangle \\ \langle a_{21\mu}, a_{21\nu} \rangle & \langle a_{22\mu}, a_{22\nu} \rangle & \vdots & \langle a_{23\mu}, a_{23\nu} \rangle & \langle a_{24\mu}, a_{24\nu} \rangle & \vdots & \langle a_{25\mu}, a_{25\nu} \rangle & \langle a_{26\mu}, a_{26\nu} \rangle \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \langle a_{31\mu}, a_{31\nu} \rangle & \langle a_{32\mu}, a_{32\nu} \rangle & \vdots & \langle a_{33\mu}, a_{33\nu} \rangle & \langle a_{34\mu}, a_{34\nu} \rangle & \vdots & \langle a_{35\mu}, a_{35\nu} \rangle & \langle a_{36\mu}, a_{36\nu} \rangle \\ \langle a_{41\mu}, a_{41\nu} \rangle & \langle a_{42\mu}, a_{42\nu} \rangle & \vdots & \langle a_{43\mu}, a_{43\nu} \rangle & \langle a_{44\mu}, a_{44\nu} \rangle & \vdots & \langle a_{45\mu}, a_{45\nu} \rangle & \langle a_{46\mu}, a_{46\nu} \rangle \end{bmatrix}$$

$= . \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \end{bmatrix}$. is a square intuitionistic fuzzy block matrix, since all A_{ij} are square blocks.

Definition 3.10 ***(Regularly partition intuitionistic fuzzy matrix).*** *If the diagonal blocks of a square intuitionistic fuzzy block matrix are square IFMs then the intuitionistic fuzzy block matrix is said to be regularly fuzzy partition or regular intuitionistic fuzzy block matrix.*

Definition 3.11 ***(Diagonal intuitionistic fuzzy block matrix).*** *If a square intuitionistic fuzzy block matrix is such that the blocks $A_{ij} = [\langle 0,1 \rangle]$ for all $i \neq j$, the intuitionistic fuzzy matrix A is said to be a diagonal intuitionistic fuzzy block matrix.*

Note that IFM A needs not be square but it must be partitioned as a square intuitionistic fuzzy block matrix.

Thus, $\begin{pmatrix} A_{11} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & A_{22} & \mathbf{0} \end{pmatrix}$, *where* $\mathbf{0} = [\langle 0,1 \rangle]$ is a diagonal intuitionistic fuzzy block matrix.

Definition 3.12 ***(Triangular intuitionistic fuzzy block matrix).*** *If the square or rectangular blocks above (or below) the square diagonal blocks of a square* *intuitionistic fuzzy block matrix are all zero, then the intuitionistic fuzzy matrix is said to be the lower (or upper) triangular intuitionistic fuzzy block matrix.*

Definition 3.13 ***(Quasidiagonal intuitionistic fuzzy block matrix).*** *It is a intuitionistic fuzzy block matrix whose diagonal blocks are square intuitionistic fuzzy matrices of different order and off diagonal blocks are zero intuitionistic fuzzy matrices.*

Thus, $$A = \begin{bmatrix} D_1 & \mathbf{0} & \cdots & \mathbf{0} \\ & D_2 & \cdots & \vdots \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{0} & \mathbf{0} & \cdots & D_r \end{bmatrix}$$

is a quasidiagonal matrix whose diagonal blocks D_i , $i = 1,2,\cdots,s$ are square intuitionistic fuzzy matrices of different orders.

The IFM A is also called decomposable intuitionistic fuzzy block matrix or pseudo-diagonal intuitionistic fuzzy block matrix or direct sum of intuitionistic fuzzy matrices D_1, $D_2,\cdots,D_s$ taken in this order.

Theorem 3.1 *If* $A = [a_{ij}]_{m\times n}$ *and* $B = [b_{ij}]_{n\times p}$ *are two intuitionistic fuzzy matrices such that* $AB = C = [c_{ij}]_{m\times p}$ *then the* j *-th column of* C *is* $AB_{j\cdot}$*, where*

$$B_{j\cdot} = \begin{bmatrix} \langle b_{1j\mu}, b_{1j\nu}\rangle \\ \langle b_{2j\mu}, b_{2j\nu}\rangle \\ \vdots \\ \langle b_{nj\mu}, b_{1j\nu}\rangle \end{bmatrix}$$

are the column partition of IFM B .

Proof: Let IFM B of order $n \times p$ be partition into p column vectors ($n\times 1$) IFMs as

$$B = [B_{1\cdot} \quad B_{2\cdot} \quad \cdots \quad B_{j\cdot} \quad \cdots \quad B_{p\cdot}] \text{ where } j = 1,2,\cdots,p$$

to find a column of the product AB . From the product rule of the IFMs, the elements of the product is

$$c_{ij} = [\langle \sum_k \min\{a_{ik\mu}, b_{kj\mu}\}, \prod_k \max\{a_{ik\nu}, b_{kj\nu}\}\rangle], i = 1,2,\cdots,m \text{ and } j = 1,2,\cdots,n,$$

where $A = [\langle a_{ik\mu}, a_{ik\nu}\rangle]_{m\times n}$, $B = [\langle b_{kj}, b_{ik\nu}\rangle]_{n\times p}$, $C = [\langle c_{ik\mu}, c_{ik\nu}\rangle]_{m\times n}$ and $C = AB.$

Therefore j -th column of C is obtained by giving the values $1,2,\cdots,m$ to i and it is

$$C_{j\cdot} = \begin{bmatrix} \langle \sum_k \min\{a_{1k\mu}, b_{kj\mu}\}, \prod_k \max\{a_{1k\nu}, b_{kj\nu}\}\rangle \\ \langle \sum_k \min\{a_{2k\mu}, b_{kj\mu}\}, \prod_k \max\{a_{2k\nu}, b_{kj\nu}\}\rangle \\ \vdots \\ \langle \sum_k \min\{a_{mk\mu}, b_{kj\mu}\}, \prod_k \max\{a_{mk\nu}, b_{kj\nu}\}\rangle \end{bmatrix}$$

$$= \begin{bmatrix} \langle a_{11\mu}, a_{11\nu}\rangle & \langle a_{12\mu}, a_{12\nu}\rangle & \cdots & \langle a_{1n\mu}, a_{11\nu}\rangle \\ \langle a_{21\mu}, a_{21\nu}\rangle & \langle a_{22\mu}, a_{22\nu}\rangle & \cdots & \langle a_{2n\mu}, a_{2n\nu}\rangle \\ \cdots & \cdots & \vdots & \cdots \\ \langle a_{m1\mu}, a_{m1\nu}\rangle & \langle a_{m2\mu}, a_{m2\nu}\rangle & \cdots & \langle a_{mn\mu}, a_{mn\nu}\rangle \end{bmatrix} \begin{bmatrix} \langle b_{1j\mu}, b_{1j\nu}\rangle \\ \langle b_{2j\mu}, b_{2j\nu}\rangle \\ \vdots \\ \langle b_{nj\mu}, b_{1j\nu}\rangle \end{bmatrix}, j = 1,2,\cdots,p$$

$= AB_{j\cdot}, j = 1,2,\cdots,p.$

Hence the theorem.

Theorem 3.2 *Let A be an $m\times n$ IFM and B be an $n\times p$ IFM. Let B $(or A)$ be partitioned into two blocks by column partitioning only. Then the product AB is also partitioned into two blocks of same column (row) partitioning.*

Proof: Let $B = [B_1 \quad B_2]$ where B_1 is of order $n\times t$ and B_2 is of order $n\times(p-t)$ IFM. Then

$$AB = A\lfloor b_{1\cdot} \quad b_{2\cdot} \quad \cdots \quad b_{t\cdot} \quad \vdots \quad b_{(t+1)\cdot} \quad \cdots \quad b_{p\cdot} \rfloor$$

$$where\, b_{j\cdot} = \begin{bmatrix} \langle b_{1j\mu}, b_{1j\nu}\rangle \\ \langle b_{2j\mu}, b_{2j\nu}\rangle \\ \vdots \\ \langle b_{nj\mu}, b_{nj\nu}\rangle \end{bmatrix},$$

$$= \lfloor Ab_{1\cdot} \quad Ab_{2\cdot} \quad \cdots \quad Ab_{t\cdot} \quad \vdots \quad Ab_{(t+1)\cdot} \quad \cdots \quad \cdots \quad Ab_{p\cdot} \rfloor = [AB_1 \quad \vdots \quad AB_2]$$

Hence the theorem. □

3.3 Operations on intuitionistic fuzzy block matrix

Addition: The conformal intuitionistic fuzzy matrices can be added by block as addition of two intuitionistic fuzzy matrices of the same dimensions.

$$A + B = \begin{bmatrix} A_{11} + B_{11} & A_{12} + B_{12} & \cdots & A_{1q} + B_{1q} \\ A_{21} + B_{21} & A_{22} + B_{22} & \cdots & A_{2q} + B_{2q} \\ \cdots & \cdots & \cdots & \cdots \\ A_{p1} + B_{p1} & A_{p2} + B_{p2} & \cdots & A_{pq} + B_{pq} \end{bmatrix}.$$

Scalar multiplication: As in scalar multiplication of an intuitionistic fuzzy matrix by a scalar, each block of partition intuitionistic fuzzy matrix is multiplied by scalar.

Thus, $$\alpha A = \begin{bmatrix} \alpha A_{11} & \alpha A_{12} & \cdots & \alpha A_{1q} \\ \alpha A_{21} & \alpha A_{22} & \cdots & \alpha A_{2q} \\ \cdots & \cdots & \cdots & \cdots \\ \alpha A_{p1} & \alpha A_{p2} & \cdots & \alpha A_{pq} \end{bmatrix}.$$

Multiplication of intuitionistic fuzzy partition matrices: Let $A = [\langle a_{ij\mu}, a_{ij\nu} \rangle]_{m\times n}$ and $B = [\langle b_{jk\mu}, b_{jk\nu} \rangle]_{n\times p}$ be two IFMs conformable for multiplication. Let A_{ij} and B_{ij} denote blocks of A and B.

For IFM multiplication AB by partition to be conformable, the number of column partitioning of A must be equal to that of row partitioning of B and in addition blocks must be conformable for multiplication of IFM and

$$AB = [\langle c_{ik\mu}, c_{ik\nu} \rangle]_{r\times t} = [\sum_{j=1}^{s} A_{ij} B_{jk}]$$

which is again a partition IFM with r horizontal and t vertical dotted lines.

Theorem 3.3 *If $AB = C$ the intuitionistic fuzzy submatrix containing rows i_1, $i_2, \cdots, i_r$ and columns j_1, $j_2, \cdots, j_s$ of C is equal to the product of the intuitionistic fuzzy submatrix with these rows of A and the intuitionistic fuzzy submatrix with these columns of B.*

Proof: Let $A = [\langle a_{ij\mu}, a_{ij\nu} \rangle]_{m\times n}$ and $B = [\langle b_{jk\mu}, b_{jk\nu} \rangle]_{n\times p}$ be two IFMs. Then

$$C = [\langle c_{ik\mu}, c_{ik\nu} \rangle] = [\langle \sum_{j=1}^{n} \min\{a_{ij\mu}, b_{jk\mu}\}, \prod_{j=1}^{n} \max\{a_{ij\mu}, b_{jk\mu}\} \rangle],$$

where $i = 1,2,\cdots,m$ and $k = 1,2,\cdots,p$. Now the intuitionistic fuzzy submatrix C_{ik} of IFM C with rows $i_1, i_2, \cdots, i_r$ and columns $j_1, j_2, \cdots, j_s$ is obtained by replacing row i and column k of C by these rows and columns. It is

$$C_{ik} = [\langle \sum_{j=1}^{n} \min\{a_{ij\mu}, b_{jk\mu}\}, \prod_{j=1}^{n} \max\{a_{ij\mu}, b_{jk\mu}\} \rangle], \tag{1}$$

where $i = i_1, i_2, \cdots, i_r$ and columns $k = j_1, j_2, \cdots, j_s$. Again the product of the given intuitionistic fuzzy submatrices A_i and B_k of IFM A and B respectively is

$A_i B_k$

$$=\begin{bmatrix} \langle a_{i_1 1\mu}, a_{i_1 1\nu}\rangle & \langle a_{i_1 2\mu}, a_{i_1 2\nu}\rangle & \cdots & \langle a_{i_1 n\mu}, a_{i_1 n\nu}\rangle \\ \langle a_{i_2 1\mu}, a_{i_2 1\nu}\rangle & \langle a_{i_2 2\mu}, a_{i_2 2\nu}\rangle & \cdots & \langle a_{i_2 n\mu}, a_{i_2 n\nu}\rangle \\ \vdots & \vdots & \vdots & \vdots \\ \langle a_{i_r 1\mu}, a_{i_r 1\nu}\rangle & \langle a_{i_r 2\mu}, a_{i_r 2\nu}\rangle & \cdots & \langle a_{i_r n\mu}, a_{i_r n\nu}\rangle \end{bmatrix}_{r\times n} = \begin{bmatrix} \langle b_{1 j_1\mu}, b_{1 j_1\nu}\rangle & \langle b_{1 j_2\mu}, b_{1 j_2\nu}\rangle & \cdots & \langle b_{1 j_n\mu}, b_{1 j_n\nu}\rangle \\ \langle b_{2 j_1\mu}, b_{2 j_1\nu} & \langle b_{2 j_2\mu}, b_{2 j_2\nu}\rangle & \cdots & \langle b_{2 j_2\mu}, b_{2 j_2\nu}\rangle \\ \vdots & \vdots & \ddots & \vdots \\ \langle b_{n j_1\mu}, b_{n j_1\nu}\rangle & \langle b_{n j_2\mu}, b_{n j_2\nu}\rangle & \cdots & \langle b_{n j_2\mu}, b_{n j_2\nu}\rangle \end{bmatrix}_{n\times s}$$

$$=\begin{bmatrix} \langle \sum_{j=1}^{n} \min\{a_{ij\mu}, b_{jk\mu}\}, \prod_{j=1}^{n} \max\{a_{ij\nu}, b_{jk\nu}\}\rangle & \cdots & \langle \sum_{j=1}^{n} \min\{a_{ij\mu}, b_{jk\mu}\}, \prod_{j=1}^{n} \max\{a_{ij\nu}, b_{jk\nu}\}\rangle \\ \langle \sum_{j=1}^{n} \min\{a_{ij\mu}, b_{jk\mu}\}, \prod_{j=1}^{n} \max\{a_{ij\nu}, b_{jk\nu}\}\rangle & \cdots & \langle \sum_{j=1}^{n} \min\{a_{ij\nu}, b_{jk\nu}\}, \prod_{j=1}^{n} \max\{a_{ij\nu}, b_{jk\nu}\}\rangle \\ \vdots & \vdots & \vdots \\ \langle \sum_{j=1}^{n} \min\{a_{ij\mu}, b_{jk\mu}\}, \prod_{j=1}^{n} \max\{a_{ij\nu}, b_{jk\nu}\}\rangle & \cdots & \langle \sum_{j=1}^{n} \min\{a_{ij\mu}, b_{jk\mu}\}, \prod_{j=1}^{n} \max\{a_{ij\nu}, b_{jk\nu}\}\rangle \end{bmatrix}$$

$$=\left[\langle \sum_{j=1}^{n} \min\{a_{ij\mu}, b_{jk\mu}\}, \prod_{j=1}^{n} \max\{a_{ij\nu}, b_{jk\nu}\}\rangle\right]_{r\times s} \tag{2}$$

where $i = i_1, i_2, \cdots, i_r$ and columns $k = j_1, j_2, \cdots, j_s$.

Therefore the relation (1) and (2) together give the results. □

Theorem 3.4 *If the $m\times n$ IFM A is partitioned by consecutive groups of rows into blocks A_i, $i = 1,2,\cdots,r$ and $n\times p$ IFM B is partitioned by consecutive groups of columns into blocks B_j, $j = 1,2,\cdots,s$. Then the product $AB = C$ of order $m\times p$ is partitioned into blocks by row groups exactly as A and column groups exactly as B. The ik-th block C_{ik} of C is given by $C_{ik} = A_i B_k$.*

Proof: Let $i = i_1,\ i_2,\ i_3 \cdots i_r$ be the consecutive rows of A and $k = j_1,\ j_2,\ j_3 \cdots j_s$ be the consecutive columns of B. Then $C_{ik} = \left[\sum_{j=1}^{n} a_{ij} b_{jk}\right]_{r\times s}$ (3)

where $i = i_1,\ i_2,\ i_3 \cdots i_r$ and $k = j_1,\ j_2,\ j_3 \cdots j_s$

$$A_i B_k = \left[\sum_{j=1}^{n} a_{ij} b_{jk}\right]_{r\times s} \tag{4}$$

where $i = 1,2,\cdots,r$ and $k = 1,2,\cdots,s$.

Hence from (3) and (4) *ik-th block C_{ik} of C is given by $C_{ik} = A_i B_k$* □

4. Some algebric operations of intuitionistic fuzzy matrices

In this section the direct sum, Kronecker product and Kronecker sum of the intuitionistic fuzzy matrices are defined and studied some useful properties.

4.1 Direct sum

Definition 4.1 *Let A_1, $A_2, \cdots, A_r$ be square IFMs of orders m_1, $m_2, \cdots, m_r$ respectively. The diagonal IFM*

$$diag\ (A_1, A_2, \cdots, A_r) = \begin{bmatrix} A_1 & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{0} & A_2 & \cdots & \mathbf{0} \\ \vdots & \vdots & \ddots & \cdots \\ \mathbf{0} & \mathbf{0} & \cdots & A_r \end{bmatrix}_{(m_1+m_2+\cdots+m_r)}$$

is called the direct sum of the square IFMs A_1, $A_2, \cdots, A_r$ and is expressed by $A_1 \oplus A_2 \oplus \cdots \oplus A_r$ of order $(m_1 + m_2 + \cdots + m_r)$. It is also called the block diagonalize form.

Properties of direct sum

Direct sum of IFMs possesses the following algebric properties :

1. Commutative property: Commutative property does not hold of the square intuitionistic fuzzy matrices.

Let A and B be two square intuitionistic fuzzy matrices. Then the direct sum of A and B are $A \oplus B = \begin{bmatrix} A & \mathbf{0} \\ \mathbf{0} & B \end{bmatrix}$ *and* $B \oplus A = \begin{bmatrix} B & \mathbf{0} \\ \mathbf{0} & A \end{bmatrix}$.

It is obvious that, $A \oplus B \neq B \oplus A.$

2. Associative property: Let A, B and C be three square intuitionistic fuzzy matrices. Then $A \oplus B = \begin{bmatrix} A & \mathbf{0} \\ \mathbf{0} & B \end{bmatrix} = D.$

Now, $(A \oplus B) \oplus C = D \oplus C = \begin{bmatrix} D & \mathbf{0} \\ \mathbf{0} & C \end{bmatrix} = \begin{bmatrix} A & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & B & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & C \end{bmatrix}$

Similarly, $B \oplus C = \begin{bmatrix} B & \mathbf{0} \\ \mathbf{0} & C \end{bmatrix} = E$.

Now, $A \oplus (B \oplus C) = A \oplus E = \begin{bmatrix} A & \mathbf{0} \\ \mathbf{0} & E \end{bmatrix} = \begin{bmatrix} A & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & B & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & C \end{bmatrix}$.

Therefore, $(A \oplus B) \oplus C = A \oplus (B \oplus C).$

Hence, the associative law holds for direct sum of the intuitionistic fuzzy block matrix.

3. Mixed sum: $(A + B) \oplus (C + D) = (A \oplus B) + (C \oplus D)$, if the addition are conformable corresponding intuitionistic fuzzy block matrices.

By the definition of direct sum of the intuitionistic fuzzy block matrix and intuitionistic fuzzy matrix addition,

$$(A + B) \oplus (C + D) = \begin{bmatrix} (A + B) & \mathbf{0} \\ \mathbf{0} & (C + D) \end{bmatrix} = \begin{bmatrix} A & \mathbf{0} \\ \mathbf{0} & C \end{bmatrix} + \begin{bmatrix} B & \mathbf{0} \\ \mathbf{0} & D \end{bmatrix}$$

$= (A \oplus C) + (B \oplus D).$

4. Intuitionistic fuzzy matrix multiplication of direct sum: $(A \oplus B)(C \oplus D) = (AC) \oplus (BD)$ if the multiplication is conformable for intuitionistic fuzzy matrix .

$$(A \oplus B)(C \oplus D) = \begin{bmatrix} A & \mathbf{0} \\ \mathbf{0} & B \end{bmatrix}\begin{bmatrix} C & \mathbf{0} \\ \mathbf{0} & D \end{bmatrix} = \begin{bmatrix} AC & \mathbf{0} \\ \mathbf{0} & BD \end{bmatrix} = (AC) \oplus (BD).$$

since intuitionistic fuzzy matrix multilication is conformable

5. Transposition: $(A \oplus B)^T = A^T \oplus B^T$.

Since, $A \oplus B = \begin{bmatrix} A & \mathbf{0} \\ \mathbf{0} & B \end{bmatrix}$ then $(A \oplus B)^T = \begin{bmatrix} A & \mathbf{0} \\ \mathbf{0} & B \end{bmatrix}^T = \begin{bmatrix} A^T & \mathbf{0} \\ \mathbf{0} & B^T \end{bmatrix} = A^T \oplus B^T$.

4.2 Kronecker product of intuitionistic fuzzy matrices

Let $A = [\langle a_{ij\mu}, a_{ij\nu} \rangle]_{m\times n}$ and $B = [\langle b_{ij\mu}, b_{ij\nu} \rangle]_{p\times q}$ be two rectangular IFMs. Then the Kronecker product of A and B , denoted by $A \otimes B$ is defined as the partitioned intuitionistic fuzzy matrix $A \otimes B = \begin{bmatrix} a_{11}B & a_{12}B & \cdots & a_{1n}B \\ a_{21}B & a_{22}B & \cdots & a_{2n}B \\ \vdots & \vdots & \vdots & \vdots \\ a_{m1}B & a_{m2}B & \cdots & a_{mn}B \end{bmatrix}_{mp\times nq}$

where $a_{ij} = \langle a_{ij\mu}, a_{ij\nu} \rangle$ for $i = 1,2,\cdots,m$ and $j = 1,2,\cdots,n.$ It has mn blocks. The ij th blocks $a_{ij}B$ of order $p \times q$.

Note: The difference between the product of intuitionistic fuzzy matrices and Kronecker product of intuitionistic fuzzy matrices is that in product of IFM product AB requires equality of the number of columns in A and the number of rows in B while in Kronecker product it is free from such restriction.

Kronecker product of two intuitionistic fuzzy column vectors

Let $x = [\langle x_{1\mu}, x_{1\nu} \rangle \;\; \langle x_{2\mu}, x_{2\nu} \rangle \;\; \cdots \;\; \langle x_{n\mu}, x_{n\nu} \rangle]^T$

and $y = \left[\langle y_{1\mu}, y_{1\nu} \rangle \;\; \langle y_{2\mu}, y_{2\nu} \rangle \;\; \cdots \;\; \langle y_{m\mu}, y_{m\nu} \rangle\right]^T$ be two column intuitionistic fuzzy vectors. Then by definition of Kronecker product, we have

$$x \otimes y == \begin{bmatrix} \langle x_{1\mu}, x_{1\nu} \rangle y \\ \langle x_{2\mu}, x_{2\nu} \rangle y \\ \vdots \\ \langle x_{n\mu}, x_{n\nu} \rangle y \end{bmatrix}_{nm\times 1} = \begin{bmatrix} \langle x_{1\mu}, x_{1\nu} \rangle\langle y_{1\mu}, y_{1\nu} \rangle \\ \vdots \\ \langle x_{1\mu}, x_{1\nu} \rangle\langle y_{m\mu}, y_{m\nu} \rangle \\ \langle x_{2\mu}, x_{2\nu} \rangle\langle y_{1\mu}, y_{1\nu} \rangle \\ \vdots \\ \langle x_{2\mu}, x_{2\nu} \rangle\langle y_{m\mu}, y_{m\nu} \rangle \\ \vdots \\ \langle x_{n\mu}, x_{n\nu} \rangle\langle y_{1\mu}, y_{1\nu} \rangle \\ \vdots \\ \langle x_{n\mu}, x_{n\nu} \rangle\langle y_{m\mu}, y_{m\nu} \rangle \end{bmatrix}_{nm\times 1}$$

4.2 Properties of Kronecker product

1. Homogeneity: If $\alpha = \langle \alpha_1, \alpha_2 \rangle$ and $0 \le \alpha_1 \le \alpha_2 \le 1$ then $A \otimes (\alpha B) = \alpha(A \otimes B) = (\alpha A) \otimes B.$

Proof : The ij th block of $A \otimes (\alpha B) = [a_{ij}(\alpha B)]$

$= \alpha[a_{ij}B] = \alpha[(i, j)$ th block of $A \otimes B]$

$= A \otimes (\alpha B) = \alpha(A \otimes B).$

2. Commutative: The Kronecker product is not commutative, $A \otimes B \neq B \otimes A.$
3. Distributive: If B and C are conformable for addition, then

   $A \otimes (B + C) = A \otimes B + A \otimes C,\ [left\ distribution]$

   $(B + C) \otimes A = B \otimes A + C \otimes A,\ [right\ distribution]$
4. Associative: $A \otimes (B \otimes C) = (A \otimes B) \otimes C.$
5. Transposition: $(A \otimes B)^T = A^T \otimes B^T.$
6. Trace: $Tr(A \otimes B) = (TrA)(TrB).$
7. Two column vectors α and β, not necessarily of the same order:

   $\alpha^T \otimes \beta = \beta\alpha^T = \beta \otimes \alpha^T$
8. $det(A_{m\times m} \otimes B_{n\times n}) = (detA)^m (detB)^n.$

4.3 Kronecker sum

The Kronecker sum of two square intuitionistic fuzzy matrices $A_{n\times n}$ and $B_{m\times m}$ is defined by $A \dagger B = A \otimes I_m + I_n \otimes B,$ which is an $nm \times nm$ intuitionistic fuzzy matrix.

Example 1 *Let* $A = \begin{bmatrix} \langle 1,0 \rangle & \langle 0.5,0.4 \rangle \\ \langle 0.6,0.3 \rangle & \langle 1,0 \rangle \end{bmatrix}$, $B = \begin{bmatrix} \langle 1,0 \rangle & \langle 0.4,0.6 \rangle & \langle 0.7,0.3 \rangle \\ \langle 0.3,0.6 \rangle & \langle 1,0 \rangle & \langle 0.6,0.2 \rangle \\ \langle 0.1,0.8 \rangle & \langle 0.4,0.6 \rangle & \langle 1,0 \rangle \end{bmatrix}$

be two intuitionistic fuzzy matrices. Then $A \dagger B = A \otimes I_3 + I_2 \otimes B$

$$= \left[\begin{array}{ccc:ccc} \langle 1,0 \rangle & \langle 0.4,0.6 \rangle & \langle 0.7,0.3 \rangle & \langle 0.5,0.4 \rangle & \langle 0,0 \rangle & \langle 0,0 \rangle \\ \langle 0.3,0.6 \rangle & \langle 1,0 \rangle & \langle 0.6,0.2 \rangle & \langle 0,0 \rangle & \langle 0.5,0.4 \rangle & \langle 0,0 \rangle \\ \langle 0.1,0.8 \rangle & \langle 0.4,0.6 \rangle & \langle 1,0 \rangle & \langle 0,0 \rangle & \langle 0,0 \rangle & \langle 0.5,0.4 \rangle \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \langle 0.6,0.3 \rangle & \langle 0,0 \rangle & \langle 0,0 \rangle & \langle 1,0 \rangle & \langle 0.4,0.6 \rangle & \langle 0.7,0.3 \rangle \\ \langle 0,0 \rangle & \langle 0.6,0.3 \rangle & \langle 0,0 \rangle & \langle 0.6,0.3 \rangle & \langle 1,0 \rangle & \langle 0.6,0.2 \rangle \\ \langle 0,0 \rangle & \langle 0,0 \rangle & \langle 0.6,0.3 \rangle & \langle 0.1,0.8 \rangle & \langle 0.4,0.6 \rangle & \langle 1,0 \rangle \end{array}\right]$$

5. Some relational operations on intuitionistic fuzzy block matrices

Here define four special types of reflexivity and irreflexivity of a IFM.

Definition 5.1 *Let A be an IFM of any order then*

1. T_1: A is a reflexive of type-1 if $a_{ii\mu} = 1$ and $a_{ii\nu} = 0$, for all $i = 1,2,\cdots,n$.
2. T_2: A is a reflexive of type-2 if $(a_{ii\nu} \vee a_{jj\nu}) \le a_{ij\nu}$, for each i, $j = 1,2,\cdots,n$.
3. T_3: A is a reflexive of type-3 if $(a_{ii\mu} \wedge a_{jj\mu}) \ge a_{ij\mu}$, all i, $j = 1,2,\cdots,n$.
4. T_4: A is a reflexive of type-4 if $(a_{ii\nu} \vee a_{jj\nu}) \le a_{ij\nu}$ and $(a_{ii\mu} \wedge a_{jj\mu}) \ge a_{ij\mu}$, where $i = 1,2,\cdots,n$.

For irreflexivity

1. $T_1^{'}$: A is a reflexive of type-1 if $a_{ii\mu} = 0$ and $a_{ii\nu} = 1$, for all $i = 1,2,\cdots,n$.
2. $T_2^{'}$: A is a reflexive of type-2 if $(a_{ii\nu} \wedge a_{jj\nu}) \ge a_{ij\nu}$, for all i, $j = 1,2,\cdots,n$.
3. $T_3^{'}$: A is a reflexive of type-3 if $(a_{ii\mu} \vee a_{jj\mu}) \le a_{ij\mu}$, for all i, $j = 1,2,\cdots,n$.
4. $T_4^{'}$: A is a reflexive of type-4 if $(a_{ii\nu} \wedge a_{jj\nu}) \ge a_{ij\nu}$ and $(a_{ii\mu} \vee a_{jj\mu}) \le a_{ij\mu}$, for $i = 1,2,\cdots,n$.

Theorem 5.1 *If IFMs A and B be reflexive of any type then direct sum of these IFMs is also reflexive of the same type.*

Proof: (i) Let IFMs A and B be reflexive of type-1, then $\langle a_{ii\mu}, a_{ii\nu} \rangle = \langle 1,0 \rangle$ and $\langle b_{ii\mu}, b_{ii\nu} \rangle = \langle 1,0 \rangle$. Then the direct sum of these IFMs A and B be intuitionistic fuzzy block matrix $S = A \oplus B = \begin{bmatrix} A & \mathbf{0} \\ \mathbf{0} & B \end{bmatrix}$..

Now $\langle s_{ii\mu}, s_{ii\nu} \rangle = \langle 1,0 \rangle$, since diagonal elements in intuitionistic fuzzy block matrix S are IFMs A and B and diagonal elements in A and B are $\langle 1,0 \rangle$.
Hence direct sum S of the IFMs A and B is reflexive of type-1.

(ii) Let IFMs A and B be reflexive of type-2, then $a_{ii\nu} \vee a_{jj\nu} \le a_{ij\nu}$ and $b_{ii\nu} \vee b_{jj\nu} \le b_{ij\nu}$. Then the direct sum of these IFMs A and B be intuitionistic fuzzy block matrix $S = A \oplus B$.

Intuitionistic fuzzy block matrix S contains four blocks, diagonal blocks are IFMs A and B and off diagonal blocks are intuitionistic fuzzy zero matrices.

Now for A blocks we have

$$s_{ij\nu} = a_{ij\nu}\ [\ i = 1,2,\cdots,m,\ j = 1,2,\cdots,m]$$

$$\ge a_{ii\nu} \vee a_{jj\nu}\ [as\ A\ is\ reflexive\ of\ type-2] = s_{ii\nu} \vee s_{jj\nu}$$

Now for B blocks we have

$$s_{(m+p)(m+q)\nu} = b_{pq\nu}\ [\ \ p = 1,2,\cdots,n\ q = 1,2,\cdots,n]$$

$$\ge b_{pp\nu} \vee b_{qq\nu}\ [as\ B\ is\ reflexive\ of\ type-2]$$

$$= s_{(m+p)(m+p)\nu} \vee s_{(m+q)(m+q)\nu}\ [as\ off\ diagonal\ blocks\ are\ IFzero\ matrices.]$$

Therefore, $s_{kl\nu} \ge s_{kk\nu} \vee s_{ll\nu}$, $k = 1,2,\cdots,m+n, l = 1,2,\cdots,m+n.$

Hence direct sum S of the IFMs A and B is reflexive of type-2.

(iii) Let IFMs A and B be reflexive of type-3, then $a_{ii\mu} \wedge a_{jj\mu} \ge a_{ij\mu}$ and $b_{ii\mu} \wedge b_{jj\mu} \ge b_{ij\mu}$. Then the direct sum of these IFMs A and B be intuitionistic fuzzy block matrix $S = A \oplus B$.

Now for A blocks we have

$$s_{ij\mu} = a_{ij\mu}[\ i = 1,2,\cdots,m,\ j = 1,2,\cdots,m]$$

$$\le a_{ii\mu} \wedge a_{jj\mu}\ [as\ A\ is\ reflexive\ of\ type-3] = s_{ii\mu} \wedge s_{jj\mu}$$

Now for B blocks we have

$$s_{(m+p)(m+q)\mu} = b_{pq\mu}\ [\ \ p = 1,2,\cdots,n\ q = 1,2,\cdots,n]$$

$$\ge b_{pp\mu} \wedge b_{qq\mu}\ [as\ B\ is\ reflexive\ of\ type-2]$$

$$= s_{(m+p)(m+p)\mu} \wedge s_{(m+q)(m+q)\mu}\ [as\ off\ diagonal\ \ blocks\ are\ IF\ \ zero\ \ matrices.]$$

Therefore, $s_{kl\mu} \le s_{kk\mu} \wedge s_{ll\mu}$, $k = 1,2,\cdots,m+n, l = 1,2,\cdots,m+n.$

Hence direct sum S of the IFMs A and B is reflexive of type-3.

(iv) Let IFMs A and B be reflexive of type-4, then $a_{ii\nu} \vee a_{jj\nu} \le a_{ij\nu}$, $a_{ii\mu} \wedge a_{jj\mu} \ge a_{ij\mu}$ and $b_{ii\nu} \vee b_{jj\nu} \le b_{ij\nu}$ $b_{ii\mu} \wedge b_{jj\mu} \ge b_{ij\mu}$. Then the direct sum of these IFMs A and B be intuitionistic fuzzy block matrix reflexive of type-4, by using the results (ii) and (iii).

Hence direct sum of IFMs reflexive of any type is also reflexive of the same type. □

Theorem 5.2 *If IFMs A and B be reflexive of type-1 then Kronecker product of these IFMs is also reflexive of type-1.*

Proof: Let IFMs A and B be reflexive of type-1, then $\langle a_{ii\mu}, a_{ii\nu}\rangle = \langle 1,0\rangle$ and $\langle b_{ii\mu}, b_{ii\nu}\rangle = \langle 1,0\rangle$. Then the Kronecker product of these IFMs A and B be intuitionistic fuzzy block matrix

$$S = A \otimes B = \begin{bmatrix} a_{11}B & a_{12}B & \cdots & a_{1m}B \\ a_{21}B & a_{22}B & \cdots & a_{2m}B \\ \cdots & \cdots & \ddots & \cdots \\ a_{m1}B & a_{m2}B & \cdots & a_{mm}B \end{bmatrix}$$

Here $a_{ii} = \langle 1,0\rangle$ for all $i = 1,2,\cdots,m$ as A is an IFM reflexive of type-1 and diagonal elements of B are $b_{jj} = \langle 1,0\rangle$ for all $j = 1,2,\cdots,n$ as B is an IFM reflexive of type-1.

Therefore, $\langle s_{pp\mu}, s_{pp\nu}\rangle = \langle 1,0\rangle$, for $p = 1,2,\cdots,mn$, where m and n are the order of the IFMs A and B respectively.

Hence Kronecker product of IFMs reflexive of type-1 is also reflexive of type-1. □

Theorem 5.3 *If IFMs A and B be reflexive of type-2 then Kronecker product of these IFMs is also reflexive of type-2.*

Proof: (ii) Let IFMs A and B be reflexive of type-2, then $a_{ii\nu} \vee a_{jj\nu} \le a_{ij\nu}$ and $b_{ii\nu} \vee b_{jj\nu} \le b_{ij\nu}$. Then the Kronecker product of these IFMs A and B be intuitionistic fuzzy block matrix $S = A \otimes B.$

Intuitionistic fuzzy block matrix S contains mm blocks, diagonal blocks are IFMs $a_{ii}B$ and off diagonal blocks are $a_{ij}B$ when $i \ne j$.

Now for the diagonals blocks we have

$s_{pq\nu} = \max\{a_{ii\nu}, b_{pq\nu}\}$ [$i = 1,2,\cdots,m,\ p,q = 1,2,\cdots,n$]

$\ge \max\{a_{ii\nu}, b_{pp\nu} \vee b_{qq\nu}\}$ [*as B is reflexive of type* $-$ 2] $= s_{pp\nu} \vee s_{qq\nu}$.

Now for off diagonal blocks we have

$$s_{pq\nu} = \max\{a_{ij\nu}, b_{pq\nu}\}\ [\ i = 1,2,\cdots,m,\ p,q = 1,2,\cdots,n]$$
$$\ge \max\{a_{ij\nu}, b_{pp\nu} \vee b_{qq\nu}\}\ [as\ B\ is\ reflexive\ of\ type-2] = s_{pp\nu} \vee s_{qq\nu}.$$

Again, $a_{ii\nu} \vee a_{jj\nu} \le a_{ij\nu}$ for all i, $j = 1,2,\cdots,m.$

Therefore, $s_{kl\nu} \ge s_{kk\nu} \vee s_{ll\nu}$, $k = 1,2,\cdots,mn, l = 1,2,\cdots,mn.$

Hence Kronecker product of two reflexive IFMs of type-2 is intuitionistic fuzzy block matrix S, which is also reflexive of type-2. □

Theorem 5.4 *If IFMs A and B be reflexive of type-3 then Kronecker product of these IFMs is also reflexive of type-3.*

Proof: Let IFMs A and B be reflexive of type-3, then $a_{ii\nu} \wedge a_{jj\nu} \geq a_{ij\nu}$ and $b_{ii\nu} \wedge b_{jj\nu} \geq b_{ij\nu}$. Then the Kronecker product of these IFMs A and B be intuitionistic fuzzy block matrix $S = A \otimes B$.
Intuitionistic fuzzy block matrix S contains mm blocks, diagonal blocks are IFMs $a_{ii}B$ and off diagonal blocks are $a_{ij}B$ when $i \neq j$.

Now for the diagonals blocks

$$s_{pq\mu} = \min\{a_{ii\mu}, b_{pq\mu}\}[where i = 1,2,\cdots,m,\ p,q = 1,2,\cdots,n]$$

$$\leq \min\{a_{ii\mu}, b_{pp\mu} \vee b_{qq\mu}\}\,[as\ B\ is\ reflexive\ of\ type-3] = s_{pp\mu} \vee s_{qq\mu}.$$

Now for off diagonal blocks

$$s_{pq\mu} = \min\{a_{ij\mu}, b_{pq\mu}\}[\ i = 1,2,\cdots,m,\ p,q = 1,2,\cdots,n]$$

$$\leq \max\{a_{ij\mu}, b_{pp\mu} \vee b_{qq\mu}\}\,[as\ B\ is\ reflexive\ of\ type-3] = s_{pp\mu} \vee s_{qq\mu}.$$

Again, $a_{ii\mu} \wedge a_{jj\mu} \geq a_{ij\mu}$ for all i, $j = 1,2,\cdots,m.$

Therefore, $s_{kl\nu} \leq s_{kk\mu} \vee s_{ll\mu}$, $k = 1,2,\cdots,mn, l = 1,2,\cdots,mn.$

Hence Kronecker product of reflexive IFMs of type-2 is intuitionistic fuzzy block matrix S, reflexive of type-3. □

Theorem 5.5 *If IFMs A and B be reflexive of type-4 then Kronecker product of these IFMs is also reflexive of type-4.*
Proof: Let IFMs A and B be reflexive of type-4, i.e., $(a_{ii\nu} \vee a_{jj\nu}) \leq a_{ij\nu}$, $(a_{ii\mu} \wedge a_{jj\mu}) \geq a_{ij\mu}$ and $(b_{ii\nu} \vee b_{jj\nu}) \leq b_{ij\nu}$ and $(b_{ii\mu} \wedge b_{jj\mu}) \geq b_{ij\mu}$.
Then from the Theorems 0.3 and 0.4, Kronecker product of these IFMs A and B be intuitionistic fuzzy block matrix which is reflexive of type-4. □

Theorem 5.6 *If IFMs A and B be reflexive of type-1 then Kronecker sum of these IFMs is also reflexive of type-1.*
Proof: Let IFMs A and B of order $m \times m$ and $n \times n$ respectively be reflexive of type-1, then $\langle a_{ii\mu}, a_{ii\nu}\rangle = \langle 1,0\rangle$ and $\langle b_{ii\mu}, b_{ii\nu}\rangle = \langle 1,0\rangle$. Then the Kronecker sum of these IFMs A and B be intuitionistic fuzzy block matrix

$$S = A \dagger B = (A \otimes I_n) + (I_m \otimes B)$$

where I_n and I_m are the intuitionistic fuzzy identity matrices.

Now intuitionistic fuzzy identity matrices I_m and I_n are reflexive IFMs of type-1.
Therefore, by the Theorem 0.1, direct sum of reflexive IFMs of type-1, is again reflexive of type-1.
Hence Kronecker sum of IFMs reflexive of type-1 is also reflexive of type-1. □

It can easily be shown by examples for the reflexivity of type-2, type-3 and type-4, the condition that IFMs A and B is reflexive imply reflexivity of Kronecker sum of these IFMs.

Example2 Let $A=\begin{bmatrix}\langle 0.6,0.3\rangle & \langle 0.5,0.4\rangle \\ \langle 0.6,0.4\rangle & \langle 0.7,0.2\rangle\end{bmatrix}$, $B=\begin{bmatrix}\langle 0.6,0.3\rangle & \langle 0.6,0.4\rangle & \langle 0.4,0.6\rangle \\ \langle 0.5,0.5\rangle & \langle 0.8,0.2\rangle & \langle 0.6,0.4\rangle \\ \langle 0.3,0.6\rangle & \langle 0.5,0.4\rangle & \langle 0.7,0.3\rangle\end{bmatrix}$

be two reflexive IFMs of order 2 and 3 respectively of type-2, type-3 or type-4.

Now the Kronecker sum of these IFMs A and B be intuitionistic fuzzy block matrix S where

$S = A \dagger B = (A \otimes I_3) + (I_2 \otimes B)$

$$=\begin{bmatrix}\langle 0.6,0.3\rangle & \langle 0.6,0.3\rangle & \langle 0.4,0.3\rangle & \langle 0.5,0.3\rangle & \langle 0.0,0.4\rangle & \langle 0.0,0.4\rangle \\ \langle 0.5,0.3\rangle & \langle 0.8,0.2\rangle & \langle 0.6,0.3\rangle & \langle 0.0,0.4\rangle & \langle 0.5,0.2\rangle & \langle 0.0,0.4\rangle \\ \langle 0.3,0.3\rangle & \langle 0.5,0.3\rangle & \langle 0.7,0.3\rangle & \langle 0.0,0.4\rangle & \langle 0.0,0.4\rangle & \langle 0.5,0.3\rangle \\ \langle 0.6,0.3\rangle & \langle 0.0,0.4\rangle & \langle 0.0,0.4\rangle & \langle 0.7,0.2\rangle & \langle 0.6,0.2\rangle & \langle 0.4,0.2\rangle \\ \langle 0.0,0.4\rangle & \langle 0.6,0.2\rangle & \langle 0.0,0.4\rangle & \langle 0.5,0.2\rangle & \langle 0.8,0.2\rangle & \langle 0.6,0.2\rangle \\ \langle 0.0,0.4\rangle & \langle 0.0,0.4\rangle & \langle 0.6,0.4\rangle & \langle 0.3,0.2\rangle & \langle 0.5,0.2\rangle & \langle 0.7,0.2\rangle\end{bmatrix}$$

Here $(s_{ii\nu} \vee s_{jj\nu}) \leq s_{ij\nu}$ and $(s_{ii\mu} \wedge s_{jj\mu}) \geq s_{ij\mu}$, for all $i = 1,2,\cdots,6$ and $j = 1,2,\cdots,6$.

Hence, intuitionistic fuzzy block matrix S is reflexive of type-2, type-3 or type-4.

6. Conclusion

In this paper, we divide or partition an intuitionistic fuzzy matrix into an intuitionistic fuzzy block matrix. Also, introduce different types of intuitionistic fuzzy submatrices and intuitionistic fuzzy block matrix. Also we derive the operations direct sum, Kronecker sum and Kronecker product of intuitionistic fuzzy matrix. Some relational operation on intuitionistic fuzzy matrix and intuitionistic fuzzy block matrix are presented. In the next paper, we should try prove regularity, symmetricity, convergency and some other related properties of intuitionistic fuzzy block matrices.

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