Exercise 6.1

(i) Show that $\phi_1 = \delta_1 + \varepsilon_1 - \alpha_{11}$

a) $A = LDL^T$

$$= \left(\begin{array}{c|c|c} L_{00} & 0 & 0 \\ \hline \lambda_{10} e_L^T & 1 & 0 \\ \hline 0 & \lambda_{21} e_F & L_{22} \end{array}\right) \left(\begin{array}{c|c|c} D_{00} & 0 & 0 \\ \hline 0 & \delta_1 & 0 \\ \hline 0 & 0 & D_{22} \end{array}\right) \left(\begin{array}{c|c|c} L_{00}^T & \lambda_{10} e_L & 0 \\ \hline 0 & 1 & \lambda_{21} e_F^T \\ \hline 0 & 0 & L_{22}^T \end{array}\right)$$

$$= \left(\begin{array}{c|c|c} L_{00} D_{00} & 0 & 0 \\ \hline \lambda_{10} e_L^T D_{00} & \delta_1 & 0 \\ \hline 0 & \lambda_{21} e_F \delta_1 & L_{22} D_{22} \end{array}\right) \left(\begin{array}{c|c|c} L_{00}^T & \lambda_{10} e_L & 0 \\ \hline 0 & 1 & \lambda_{21} e_F^T \\ \hline 0 & 0 & L_{22}^T \end{array}\right)$$

$$\left(\begin{array}{c|c|c} A_{00} & \alpha_{10} e_L & 0 \\ \hline \alpha_{10} e_L^T & \alpha_{11} & \alpha_{21} e_F^T \\ \hline 0 & \alpha_{21} e_F & A_{22} \end{array}\right) = \left(\begin{array}{c|c|c} L_{00} D_{00} L_{00}^T & L_{00} D_{00} \lambda_{10} e_L & 0 \\ \hline \lambda_{10} e_L^T D_{00} L_{00}^T & \lambda_{10} e_L^T D_{00} \lambda_{10} e_L + \delta_1 & \delta_1 \lambda_{21} e_F^T \\ \hline 0 & \lambda_{21} e_F \delta_1 & \lambda_{21} e_F \delta_1 \lambda_{21} e_F^T + L_{22} D_{22} L_{22}^T \end{array}\right)$$

b) $A = UEU^T$

$$\left(\begin{array}{c|c|c} U_{00} & \upsilon_{01} e_L & 0 \\ \hline 0 & 1 & \upsilon_{12} e_F^T \\ \hline 0 & 0 & U_{22} \end{array}\right) \left(\begin{array}{c|c|c} E_{00} & 0 & 0 \\ \hline 0 & \varepsilon_1 & 0 \\ \hline 0 & 0 & E_{22} \end{array}\right) \left(\begin{array}{c|c|c} U_{00}^T & 0 & 0 \\ \hline \upsilon_{01} e_L^T & 1 & 0 \\ \hline 0 & \upsilon_{12} e_F & U_{22}^T \end{array}\right)$$

$$= \left(\begin{array}{c|c|c} U_{00} E_{00} & \upsilon_{01} e_L \varepsilon_1 & 0 \\ \hline 0 & \varepsilon_1 & \upsilon_{12} e_F^T E_{22} \\ \hline 0 & 0 & U_{22} E_{22} \end{array}\right) \left(\begin{array}{c|c|c} U_{00}^T & 0 & 0 \\ \hline \upsilon_{01} e_L^T & 1 & 0 \\ \hline 0 & \upsilon_{12} e_F & U_{22}^T \end{array}\right)$$

$$\left(\begin{array}{c|c|c} A_{00} & \alpha_{10} e_L & 0 \\ \hline \alpha_{10} e_L^T & \alpha_{11} & \alpha_{21} e_F^T \\ \hline 0 & \alpha_{21} e_F & A_{22} \end{array}\right) = \left(\begin{array}{c|c|c} U_{00} E_{00} U_{00}^T + \upsilon_{01} e_L \varepsilon_1 \upsilon_{01} e_L^T & \upsilon_{01} e_L \varepsilon_1 & 0 \\ \hline \varepsilon_1 \upsilon_{01} e_L^T & \varepsilon_1 + \upsilon_{12} e_F^T E_{22} \upsilon_{12} e_F & \upsilon_{12} e_F^T E_{22} U_{22}^T \\ \hline 0 & U_{22} E_{22} \upsilon_{12} e_F & U_{22} E_{22} U_{22}^T \end{array}\right)$$

c) $\left(\begin{array}{c|c|c} L_{00} & 0 & 0 \\ \hline \lambda_{10} e_L^T & 1 & \upsilon_{12} e_F^T \\ \hline 0 & 0 & U_{22} \end{array}\right) \left(\begin{array}{c|c|c} D_{00} & 0 & 0 \\ \hline 0 & \phi_1 & 0 \\ \hline 0 & 0 & E_{22} \end{array}\right) \left(\begin{array}{c|c|c} L_{00}^T & \lambda_{10} e_L & 0 \\ \hline 0 & 1 & 0 \\ \hline 0 & \upsilon_{12} e_F & U_{22}^T \end{array}\right)$

$= \left(\begin{array}{c|c|c} L_{00} D_{00} & 0 & 0 \\ \hline \lambda_{10} e_L^T D_{00} & \phi_1 & \upsilon_{12} e_F^T E_{22} \\ \hline 0 & 0 & U_{22} E_{22} \end{array}\right) \left(\begin{array}{c|c|c} L_{00}^T & \lambda_{10} e_L & 0 \\ \hline 0 & 1 & 0 \\ \hline 0 & \upsilon_{12} e_F & U_{22}^T \end{array}\right)$

$\left(\begin{array}{c|c|c} A_{00} & \alpha_{10} e_L & 0 \\ \hline \alpha_{10} e_L^T & \alpha_{11} & \alpha_{21} e_F^T \\ \hline 0 & \alpha_{21} e_F & A_{22} \end{array}\right) = \left(\begin{array}{ccc} L_{00} D_{00} L_{00}^T & L_{00} D_{00} \lambda_{10} e_L & 0 \\ \lambda_{10} e_L^T D_{00} L_{00}^T & \lambda_{10} e_L^T D_{00} \lambda_{10} e_L + \phi_1 + \upsilon_{12} e_F^T E_{22} \upsilon_{12} e_F & \upsilon_{12} e_F^T E_{22} U_{22}^T \\ 0 & U_{22} E_{22} \upsilon_{12} e_F & U_{22} E_{22} U_{22}^T \end{array}\right)$

From $A = LDL^T$

$\alpha_{11} = \lambda_{10} e_L^T D_{00} \lambda_{10} e_L + \delta_1$; $\lambda_{10} e_L^T D_{00} \lambda_{10} e_L = \alpha_{11} - \delta_1$

From $A = UEU^T$

$\alpha_{11} = \varepsilon_1 + \upsilon_{12} e_F^T E_{22} \upsilon_{12} e_F$; $\upsilon_{12} e_F^T E_{22} \upsilon_{12} e_F = \alpha_{11} - \varepsilon_1$

From the above factorization of A (c)

$\alpha_{11} = \lambda_{10} e_L^T D_{00} \lambda_{10} e_L + \phi_1 + \upsilon_{12} e_F^T E_{22} \upsilon_{12} e_F$

$\alpha_{11} = (\alpha_{11} - \delta_1) + \phi_1 + (\alpha_{11} - \varepsilon_1)$

$\alpha_{11} = \alpha_{11} - \delta_1 + \phi_1 + \alpha_{11} - \varepsilon_1$

$0 = -\delta_1 + \phi_1 + \alpha_{11} - \varepsilon_1$

$\phi_1 = \delta_1 - \alpha_{11} + \varepsilon_1$

(ii) Cost of computing <u>one</u> twisted factorization given that you have already computed the LDL^T and UEU^T factorizations?

- Given LDL^T and UEU^T, all other entries of the twisted factorization can be computed from existing entries in L, D, U, E

  except Φ_1 which equals

$$\Phi_1 = \delta_1 + \varepsilon_1 - \alpha_{11}$$

  where δ_1, ε_1 and α_{11} can be obtained from D, E, A respectively

$\Rightarrow$ To compute $\Phi_1 = \delta_1 + \varepsilon_1 - \alpha_{11}$ takes

1 addition <u>and</u> 1 subtraction

$\Rightarrow$ $O(1)$

(iii) Cost of computing all twisted factorizations given that you have already computed the LDL^T and UEU^T factorizations?

Given the unknown is only $\phi_1 = \delta_1 + \varepsilon_1 - \alpha_{11}$

and assuming A is an $n \times n$ matrix;

and there are n possible diagonal elements

so n different possible positions to twist along the diagonal

where each twist costs 1 addition & 1 subtraction

$\Rightarrow$ as per (ii); hence $2n$

$\Rightarrow$ $O(n)$