$$\begin{pmatrix} L_{00} & 0 & 0 \\ \lambda_{10} e_L^T & 1 & \upsilon_{12} e_F^T \\ 0 & 0 & U_{22} \end{pmatrix} \begin{pmatrix} D_{00} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & E_{22} \end{pmatrix} \begin{pmatrix} L_{00}^T & \lambda_{10} e_L & 0 \\ 0 & 1 & 0 \\ 0 & \upsilon_{12} e_F & U_{22}^T \end{pmatrix} \begin{pmatrix} x_0 \\ x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} L_{00} D_{00} & 0 & 0 \\ \lambda_{10} e_L^T D_{00} & 0 & \upsilon_{12} e_F^T E_{22} \\ 0 & 0 & U_{22} E_{22} \end{pmatrix} \begin{pmatrix} L_{00}^T & \lambda_{10} e_L & 0 \\ 0 & 1 & 0 \\ 0 & \upsilon_{12} e_F & U_{22}^T \end{pmatrix} \begin{pmatrix} x_0 \\ x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} L_{00} D_{00} L_{00}^T & L_{00} D_{00} \lambda_{10} e_L & 0 \\ \lambda_{10} e_L^T D_{00} L_{00}^T & \lambda_{10} e_L^T D_{00} \lambda_{10} e_L + \upsilon_{12} e_F^T E_{22} \upsilon_{12} e_F & \upsilon_{12} e_F^T E_{22} U_{22}^T \\ 0 & U_{22} E_{22} \upsilon_{12} e_F & U_{22} E_{22} U_{22}^T \end{pmatrix} \begin{pmatrix} x_0 \\ x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

(i) $L_{00} D_{00} L_{00}^T x_0 + L_{00} D_{00} \lambda_{10} e_L x_1 = 0$

(ii) $\lambda_{10} e_L^T D_{00} L_{00}^T x_0 + \lambda_{10} e_L^T D_{00} \lambda_{10} x_1 + \upsilon_{12} e_F^T E_{22} \upsilon_{12} e_F x_1$
$+ \upsilon_{12} e_F^T E_{22} U_{22}^T x_2 = 0$

(iii) $U_{22} E_{22} \upsilon_{12} e_F x_1 + U_{22} E_{22} U_{22}^T x_2 = 0$

let $x_1 = 1$

Then, <u>using (iii)</u>

$$U_{22} E_{22} \upsilon_{12} e_F x_1 + U_{22} E_{22} U_{22}^T x_2 = 0$$

$$U_{22} E_{22} U_{22}^T x_2 = -(U_{22} E_{22} \upsilon_{12} e_F x_1)$$

$$x_2 = U_{22}^{-T} E_{22}^{-1} U_{22}^{-1} (-U_{22} E_{22} \upsilon_{12} e_F x_1)$$

$$x_2 = U_{22}^{-T} (-\upsilon_{12} e_F x_1) \quad \langle U_{22}^{-1} U_{22} = I,\ E_{22}^{-1} E_{22} = I \rangle$$

<u>or</u> $U_{22}^T x_2 = -\upsilon_{12} e_F x_1 \rightarrow a$

$U_{22}^T x_2 = -\upsilon_{12} e_F \quad \langle x_1 = 1 \rangle$

<u>Using (ii)</u>

$$\lambda_{10} e_L^T D_{00} L_{00}^T x_0 + \lambda_{10} e_L^T D_{00} \lambda_{10} x_1 + \upsilon_{12} e_F^T E_{22} \upsilon_{12} e_F x_1 + \upsilon_{12} e_F^T E_{22} U_{22}^T x_2 = 0$$

Plugging in x_2

$= \lambda_{10}e_L^T D_{00} L_{00}^T x_0 + \lambda_{10} e_L^T D_{00} \lambda_{10} e_L x_1 + \cancel{\lambda_{00}} \upsilon_{12} e_F^T E_{22} \upsilon_{12} e_F x_1$
$+ \upsilon_{12} e_F^T E_{22} U_{22}^T U_{22}^{-T}(-\upsilon_{12} e_F x_1) = 0$

< Since $U_{22}^T U_{22}^{-T} = I$, last two terms cancel out >

$= \lambda_{10} e_L^T D_{00} L_{00}^T x_0 + \lambda_{10} e_L^T D_{00} \lambda_{10} e_L x_1 + \upsilon_{12} e_F^T E_{22} \upsilon_{12} e_F x_1 - \upsilon_{12} e_F^T E_{22} \upsilon_{12} x_1 = 0$

$= \lambda_{10} e_L^T D_{00} L_{00}^T x_0 + \lambda_{10} e_L^T D_{00} \lambda_{10} e_L x_1 = 0$

$= \lambda_{10} e_L^T D_{00} L_{00}^T x_0 = -\lambda_{10} e_L^T D_{00} \lambda_{10} e_L x_1$

$$L_{00}^T x_0 = \frac{1}{\lambda_{10} e_L^T} D_{00}^{-1} (-\lambda_{10} e_L^T D_{00} \lambda_{10} e_L x_1)$$

$L_{00}^T x_0 = -\lambda_{10} e_L x_1$ < $D_{00}^{-1} D_{00} = I$ > b

(or $x_0 = L_{00}^{-T}(-\lambda_{10} e_L x_1)$)

(or $x_0 = L_{00}^{-T}(-\lambda_{10} e_L)$)

Check hint:

$$\begin{pmatrix} L_{00}^T & \lambda_{10} e_L & 0 \\ 0 & 1 & 0 \\ 0 & \upsilon_{12} e_F^T & U_{22}^T \end{pmatrix} \begin{pmatrix} x_0 \\ x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

- $L_{00}^T x_0 + \lambda_{10} e_L x_1 = 0$
  Using b, we know $L_{00}^T x_0 = -\lambda_{10} e_L x_1$
  $-\lambda_{10} e_L x_1 + \lambda_{10} e_L x_1 = 0$
  $0 = 0$

- $1 x_1 = 1$
  $x_1 = 1$

- $\upsilon_{12} e_F^T x_1 + U_{22}^T x_2 = 0$
  Using a, we know $U_{22}^T x_2 = -\upsilon_{12} e_F^T x_1$
  $\upsilon_{12} e_F^T x_1 - \upsilon_{12} e_F^T x_1 = 0$
  $0 = 0$

Solving w/ a bidiagonal matrix

Let A be $n \times n$ bidiagonal matrix, $x \in \mathbb{C}^n$, $y \in \mathbb{C}^n$

where $A \rightarrow \begin{pmatrix} \alpha_{0,0} & & & \\ \alpha_{1,0} & \alpha_{1,1} & & \\ & \alpha_{2,1} & \alpha_{2,2} & \\ & & \ddots & \ddots \end{pmatrix}$ $\quad x \rightarrow \begin{pmatrix} \chi_0 \\ \chi_1 \\ \vdots \\ \chi_{n-1} \end{pmatrix}$ $\quad y \rightarrow \begin{pmatrix} \psi_0 \\ \vdots \\ \psi_{n-1} \end{pmatrix}$

Then $\chi_0 = \psi_0 / \alpha_{0,0}$

$\chi_1 = (\psi_1 - \alpha_{1,0}(\chi_0)) / \alpha_{1,1}$

$\chi_2 = (\psi_2 - \alpha_{2,1}(\chi_1)) / \alpha_{2,2}$ and so on

So, for an $n \times n$ bidiagonal matrix, it requires n divides, $(n-1)$ subtractions and $(n-1)$ ~~additions~~ multiplications

So, $\approx n + n-1 + n-1 \approx 3n-2 \approx O(n)$.

Using the fact that we use unit lower/upper triangular bidiagonal matrices, we can eliminate the divides resulting in $(n-1)$ subtractions & $(n-1)$ multiplications

So, $\approx 2n-2 \approx O(n)$

Putting it together

- $L_{00}^T x_0 = -\lambda_{10} e_L \chi_1$
- $\chi_1 = 1$
- $U_{22}^T x_2 = -U_{12} e_F \chi_1$

Cost of Computation

- $L_{00}^T x_0 = -\lambda_{10} e_L \chi_1$

  Solving w/ a unit lower bidiagonal system $\approx O(n)$

  Transposing a matrix $\approx O(n^2)$

- $U_{22}^T x_2 = -U_{12} e_F \chi_1$

  Solving w/ a unit upper bidiagonal system $\approx O(n)$

  Transposing a matrix $\approx O(n^2)$

Total: $\approx O(n+n) = O(2n) \approx O(n)$ <without cost of transposing>

$\approx O(n+n+n^2+n^2) = O(2n+2n^2) \approx O(n^2)$ <with cost of transposing>