Exam2

Q1: Propose a bordered algorithm for computing the Cholesky factorization of a SPD matrix A.

Sol: a) Consider $A = LL^T$ where L is a lower triangular matrix and A and L are partitioned as follows:

$$A = \left(\begin{array}{c|c} A_{00} & a_{01} \\ \hline a_{10}^T & \alpha_{11} \end{array}\right) \text{ and } L = \left(\begin{array}{c|c} L_{00} & 0 \\ \hline l_{10}^T & \lambda_{11} \end{array}\right)$$

By substituting these partitioned matrices into $A = LL^T$, we find that:

$$\left(\begin{array}{c|c} A_{00} & a_{01} \\ \hline a_{10}^T & \alpha_{11} \end{array}\right) = \left(\begin{array}{c|c} L_{00} & 0 \\ \hline l_{10}^T & \lambda_{11} \end{array}\right)\left(\begin{array}{c|c} L_{00} & 0 \\ \hline l_{10}^T & \lambda_{11} \end{array}\right)^T$$

$$= \left(\begin{array}{c|c} L_{00} & 0 \\ \hline l_{10}^T & \lambda_{11} \end{array}\right)\left(\begin{array}{c|c} L_{00}^T & l_{10} \\ \hline 0 & \lambda_{11} \end{array}\right) \quad \text{<transpose of L>}$$

So, $$\left(\begin{array}{c|c} A_{00} & a_{01} \\ \hline a_{10}^T & \alpha_{11} \end{array}\right) = \left(\begin{array}{c|c} L_{00}L_{00}^T & L_{00}l_{10} \\ \hline l_{10}^T L_{00}^T & l_{10}^T l_{10} + \lambda_{11}\lambda_{11} \end{array}\right) \quad \text{<matrix-matrix multiplication>}$$

- Since A is SPD, we don't need to compute a_{01} (it is just a transpose of a_{10}^T), Hence;

$$\left(\begin{array}{c|c} A_{00} & \star \\ \hline a_{10}^T & \alpha_{11} \end{array}\right) = \left(\begin{array}{c|c} L_{00}L_{00}^T & \cancel{L_{00}l_{10}}\ \star \\ \hline l_{10}^T L_{00}^T & l_{10}^T l_{10} + \lambda_{11}^2 \end{array}\right)$$

- Hence
  - (i) $A_{00} = L_{00}L_{00}^T$ = Cholesky factorization of submatrix A_{00}.
  - (ii) $a_{10}^T = l_{10}^T L_{00}^T$
  - (iii) $\alpha_{11} = l_{10}^T l_{10} + \lambda_{11}^2$

(i) Assuming that L_{00} has been computed in previous iterations $A_{00} = L_{00}L_{00}^T$ = Cholesky factorization of A_{00} can be obtained by previous iterations of the algorithm

(ii) $a_{10}^T = l_{10}^T L_{00}^T$

Assuming we know what a_{10}^T ~~and l_{00}~~ (given) and L_{00}^T (computed previously) is known, we can get l_{10}^T as follows:

$$l_{10}^T = a_{10}^T L_{00}^{-T}$$ exists bc A is SPD

(iii) $\alpha_{11} = l_{10}^T l_{10} + \lambda_{11}^2$ (where $(l_{10})^T = l_{10}^T$)

Since we know α_{11} (given) and l_{10}^T (computed in step ii), we can get $l_{10}^T l_{10}$; to get λ_{11}, we can do the following:

$$\lambda_{11} = \sqrt{\alpha_{11} - l_{10}^T l_{10}}.$$

So, the algorithm looks like:

A := CHOL_FACT_BORDERED (A) [overwrites lower triangular part of A w/ its Cholesky factor L]

Partition $A \rightarrow \left(\begin{array}{c|c} A_{TL} & \star \\ \hline A_{BL} & A_{BR} \end{array}\right)$

where A_{TL} is 0X0 matrix

while $n(A_{TL}) < n(A)$ do

Repartition

$$\left(\begin{array}{c|c} A_{TL} & \star \\ \hline A_{BL} & A_{BR} \end{array}\right) \rightarrow \left(\begin{array}{c|cc} A_{00} & \star & \star \\ \hline a_{10}^T & \alpha_{11} & \star \\ A_{20} & a_{21} & A_{22} \end{array}\right)$$

where α_{11} is 1X1 matrix

$a_{10}^T := l_{10}^T := a_{10}^T A_{00}^{-T}$ < Computing w/ only lower triangular part of A >

$\alpha_{11} := \lambda_{11} := \sqrt{\alpha_{11} - a_{10}^T a_{10}}$

Continue with:

$$\left(\begin{array}{c|c} A_{TL} & \star \\ \hline A_{BL} & A_{BR} \end{array}\right) \leftarrow \left(\begin{array}{cc|c} A_{00} & \star & \star \\ a_{10}^T & \alpha_{11} & \star \\ \hline A_{20} & a_{21} & A_{22} \end{array}\right)$$

endwhile

Q1 b) Prove the Cholesky factorization Theorem by showing that the bordered Cholesky Factorization algorithm is well defined for a matrix A that is SPD.

Sol: We will employ a proof by induction on n, the size of the matrix

(i) Base Case: $n=1$

Then A is 1×1 matrix ~~containing~~ containing α_{11}

$A = [\alpha_{11}]$, where α_{11} is real and positive since A is SPD [Lemma 5.4.4.1]

and the Cholesky factor of A is given by

$$\lambda_{11}^2 = \alpha_{11} \Rightarrow \lambda_{11} = \pm\sqrt{\alpha_{11}}$$

Since α_{11} is real and positive [Lemma 5.4.4.1], the Cholesky factor of A, λ_{11} is ~~well defined~~ real valued. (well defined)

If λ_{11} is restricted to be positive then $\lambda_{11} = \sqrt{\alpha_{11}}$ is well defined and unique.

(ii) Inductive Hypothesis: Assume the algorithm is well defined for SPD $A \in \mathbb{R}^{n \times n}$, we will show it is well defined for SPD $A \in \mathbb{R}^{(n+1) \times (n+1)}$

Let A be SPD where $A \in \mathbb{R}^{(n+1)\times(n+1)}$. Partition

$$A = \left(\begin{array}{c|c} A_{00} & a_{01} \\ \hline a_{10}^T & \alpha_{11} \end{array}\right) \quad \text{and} \quad L = \left(\begin{array}{c|c} L_{00} & 0 \\ \hline l_{10}^T & \lambda_{11} \end{array}\right)$$

Then:

(i) $A_{00} = L_{00} L_{00}^T$ where $A_{00} \in \mathbb{R}^{n \times n}$ is computed by previous iterations & well defined due to Inductive Hypothesis

(ii) $l_{10}^T = a_{10}^T L_{00}^{-T}$

If the diagonal elements of L_{00} were restricted to be positive, then L_{00} is nonsingular (given that A is SPD, the diagonal entries in $L_{00} \in \mathbb{R}$)

then its transpose L_{00}^T is also non-singular & its inverse L_{00}^{-T} exist.

- $l_{10}^T = a_{10}^T L_{00}^{-T}$

Transposing both sides

$(l_{10}^T)^T = (a_{10}^T L_{00}^{-T})^T$

$l_{10} = L_{00}^{-1} a^{10}$

which can be rewritten as

$L_{00} l_{10} = a_{10}$

- Since L_{00} is non singular, then $L_{00} l_{10} = a_{10}$ has a well defined solution l_{10} and it is unique.

Therefore: $l_{10}^T = a_{10}^T L_{00}^{-T}$ is well defined and unique.

(ii) $\lambda_{11} = \sqrt{\alpha_{11} - l_{10}^T l_{10}}$ (since λ_{11} is restricted to be positive)

Since α_{11} is real and positive and
l_{10}^T is well defined and unique, so is l_{10}
then $\alpha_{11} - l_{10}^T l_{10}$ exists and is unique.
Hence, $\lambda_{11} = \sqrt{\alpha_{11} - l_{10}^T l_{10}}$ exists and uniquely determined; when $\alpha_{11} - l_{10}^T l_{10}$ is real & positive.
Thus, $A \in \mathbb{R}^{(n+1) \times (n+1)}$ has a unique Chlosky factorization.

- By the Principle of Mathematical Induction, the result holds.