Ques 2:

a) Base Case: $n=1$

When $n=1$, A is a 1×1 matrix

$$A = [\alpha_{11}]$$

The scalar α_{11} would be stored in υ_{11}. Assuming there is some error introduced in representing it as floating point number, we get

$$\check{\upsilon}_{11} = \frac{\alpha_{11}}{1+\epsilon} \quad \langle \text{under ACM} \rangle$$

$$\check{\upsilon}_{11} = \frac{\alpha_{11}}{1+\theta_1} \quad \langle \text{Lemma 6.3.2.1} \rangle$$

$$\check{\upsilon}_{11}(1+\theta_1) = \alpha_{11} \quad \langle \text{multiply } (1+\theta_1) \text{ on both sides} \rangle$$

$$\check{\upsilon}_{11} + \check{\upsilon}_{11}\theta_1 = \alpha_{11} \quad \langle \text{distribute} \rangle$$

$$\check{\upsilon}_{11} = \alpha_{11} - \check{\upsilon}_{11}\theta_1 \quad \langle \text{subtract } \check{\upsilon}_{11}\theta_1 \text{ on both sides} \rangle$$

$$\check{\upsilon}_{11} = \alpha_{11} + \delta\alpha_{11} \quad \langle \text{Let } \delta\alpha_{11} = -\check{\upsilon}_{11}\theta_1 \rangle$$

where $|\delta\alpha_{11}| = |\check{\upsilon}_{11}\theta_1|$

Hence;

$$\check{L}\check{U} = A + \Delta A \quad \text{where } \check{L} = [1], \ \check{U} = [\check{\upsilon}_{11}]$$

and

$$
\begin{aligned}
|\delta\alpha_{11}| &= |\theta_1 \check{\upsilon}_{11}| \\
&= |\theta_1||\check{\upsilon}_{11}| \\
&\leq \gamma_1 |\check{\upsilon}_{11}| && \langle \text{since } |\theta_1| \leq \gamma_1 \rangle \\
&= \gamma_n |\check{\upsilon}_{11}| && \langle \text{since } n = 1 \rangle \\
&= [illegible] \\
&= \gamma_n |1||\check{\upsilon}_{11}| \\
&= \gamma_n |\check{L}||\check{U}| && \langle \text{since } |\check{L}| = 1,\ |\check{U}| = |\check{\upsilon}_{11}| \rangle
\end{aligned}
$$

Hence, $|\delta\alpha_{11}| \leq \gamma_n |\check{L}||\check{U}|$

equivalently $|\Delta A| \leq \gamma_n |\check{L}||\check{U}|$

Ques 2

b) Inductive Step: Assume that for $A_{00} \in \mathbb{R}^{n\times n}$ the bordered LU factorization computes (in floating point arithmetic) $\check{L}_{00}$ and $\check{U}_{00}$ where

$$A_{00} + \Delta A_{00} = \check{L}_{00}\check{U}_{00} \text{ with } |\Delta A_{00}| \le \gamma_n |\check{L}_{00}||\check{U}_{00}|$$

Then, for $A \in \mathbb{R}^{(n+1)\times(n+1)}$ where

$$A = \left(\begin{array}{c|c} A_{00} & a_{01} \\ \hline a_{10}^T & \alpha_{11} \end{array}\right)$$

it computes the following:

(i) $A_{00} + \Delta A_{00} = \check{L}_{00}\check{U}_{00}$ (given) where $|\Delta A_{00}| \le \gamma_n |\check{L}_{00}||\check{U}_{00}| \le \gamma_{n+1}|\check{L}_{00}||\check{U}_{00}|$

(ii) $\check{L}_{00} u_{01} = a_{10}^T$ (overwriting a_{10} w/ the result)

<Using Thm 6.4.1.3 for backward error analysis of triangular solve>

$(\check{L}_{00} + \Delta\check{L}_{00})\check{U}_{01} = a_{01}$ where $|\Delta\check{L}_{00}| \le \theta_n |\check{L}_{00}|$

A corollary from Thm 6.4.1.3 can be states as follows:

Thm 6.4.1.3 (result): $(L + \Delta L)\check{x} = y$ where $|\Delta L| \le \max(\gamma_2, \gamma_{n-1})|L|$ or $|\Delta L| \le \gamma_n |L|$

Corollary 1: The vector $\check{x}$ computed using Thm 6.4.1.3 satisfies $|L\check{x} - y| = |\Delta L\check{x}|$ where $|\Delta L\check{x}| \le \gamma_n |L||\check{x}|$

Proof:

Using Thm 6.4.1.3 $(L + \Delta L)\check{x} = y$

$= L\check{x} + \Delta L\check{x} = y$

$= |L\check{x} - y| = |\Delta L\check{x}|$

where

$|\Delta L\check{x}| = |\Delta L||\check{x}|$

$\le \gamma_n |L||\check{x}|$ (In Thm 6.4.1.3, $|\Delta L| \le \gamma_n |L|$) & corollary 6.4.1.4

Using Corollary 1 to $\check{L}_{00} U_{01} = a_{10}$,

$= \check{L}_{00}\check{U}_{01} + \Delta\check{L}_{00}\check{U}_{01} = a_{01}$

$= \check{L}_{00}\check{U}_{01} = a_{01} - \Delta\check{L}_{00}\check{U}_{01}$

Set $\delta a_{01} = -\Delta\check{L}_{00}\check{U}_{01}$

Hence,

$$|\check{L}_{00}\check{U}_{01}| = |a_{01} + \delta a_{01}|$$

where $|\delta a_{01}| = |\Delta \check{L}_{00}\check{U}_{01}|$

$= |\Delta \check{L}_{00}||\check{U}_{01}|$

$\leq \gamma_n |\check{L}_{00}||\check{U}_{01}|$ (Thm 6.4.1.3 & Corollary 6.4.1.4)

$\leq \gamma_{n+1} |\check{L}_{00}||\check{U}_{01}|$

Hence

$$|a_{01} + \delta a_{01}| = |\check{L}_{00}\check{U}_{01}|$$

where $|\delta a_{01}| \leq \gamma_{n+1} |\check{L}_{00}||\check{U}_{01}|$.

(ii) $l_{10}^T \check{U}_{00} = a_{10}^T$ (overwriting a_{10}^T w/ result)

Transpose of both sides

$(l_{10}^T \check{U}_{00})^T = (a_{10}^T)^T$

$\check{U}_{00}^T l_{10} = a_{10}$

<Using Thm 6.4.1.3 for backward error results of triangular solve>

$(\check{U}_{00}^T + \Delta \check{U}_{00}^T) \check{l}_{10} = a_{10}$

$\check{U}_{00}^T \check{l}_{10} + \Delta \check{U}_{00}^T \check{l}_{10} = a_{10}$

Transpose both sides

$(\check{U}_{00}^T \check{l}_{10} + \Delta \check{U}_{00}^T \check{l}_{10})^T = (a_{10})^T$

$= \check{l}_{10}^T \check{U}_{00} + \check{l}_{10}^T \Delta \check{U}_{00} = a_{10}^T$

Using results from Thm 6.4.1.3 and Corollary 1:

$\check{l}_{10}^T \check{U}_{00} = a_{10}^T - \check{l}_{10}^T \Delta \check{U}_{00}$

$\check{l}_{10}^T \check{U}_{00} = a_{10}^T + \delta a_{10}^T$ where $\delta a_{10}^T = -\check{l}_{10}^T \Delta \check{U}_{00}$

Hence $|\check{l}_{10}^T \check{U}_{00}| = |a_{10}^T + \delta a_{10}^T|$

where

$|\delta a_{10}^T| = |\check{l}_{10}^T \Delta \check{U}_{00}|$

$= |\check{l}_{10}^T||\Delta \check{U}_{00}|$

$\leq |\check{l}_{10}^T| \gamma_n |\check{U}_{00}|$ (Corollary 6.4.1.4)

$= \gamma_n |\check{l}_{10}^T||\check{U}_{00}|$

$\leq \gamma_{n+1} |\check{l}_{10}^T||\check{U}_{00}|$

(iii) $\check{U}_{11} = \alpha_{11} - \check{l}_{10}^T \check{u}_{01}$

Using reference from 6.4.3 (Goal Oriented & Modular stability analysis),

Using Thm 5.1 of the above reference, we have

R2-f: For given $v = y - Ax$, where $A \in \mathbb{R}^{m\times n}$; $y, v \in \mathbb{R}^m$, $x \in \mathbb{R}^n$ using floating point arithmetic

$$\check{v} = y - Ax + \delta\check{v} \quad \text{where } |\delta v| \le \gamma_n |A||x| + \gamma_1 |\check{v}|$$
$$\le \gamma_n (|A||x| + |\check{v}|)$$

In the above theorem, if we allow $\delta\check{v} = \delta\check{y}$, then we obtain Corollary 2 to obtain backward error result:

Corollary 2: $\check{v} = y - Ax + \delta y$ where $\delta y \le \gamma_n (|A||x| + |\check{v}|)$

Applying the above Corollary 2 to computation in (iii)

$$U_{11} = \alpha_{11} - \check{l}_{10}^T \check{v}_{01}$$
$$\check{v}_{11} = \alpha_{11} - \check{l}_{10}^T \check{v}_{01} + \delta\alpha_{11}$$
$$\check{v}_{11} + \check{l}_{10}^T \check{v}_{01} = \alpha_{11} + \delta\alpha_{11}$$
$$\text{where } |\delta\alpha_{11}| \le \gamma_n (|\check{l}_{10}^T||\check{v}_{01}| + |\check{v}_{11}|)$$
$$\le \gamma_{n+1} (|\check{l}_{10}^T||\check{v}_{01}| + |\check{v}_{11}|).$$

Combining results from (i), (ii), (iii), (iv), we get

$$\left(\begin{array}{c|c} A_{00} + \Delta A_{00} & a_{01} + \delta a_{01} \\ \hline a_{10}^T + \delta a_{10}^T & \alpha_{11} + \delta\alpha_{11} \end{array}\right) = \left(\begin{array}{c|c} \check{L}_{00}\check{U}_{00} & \check{L}_{00}\check{u}_{01} \\ \hline \check{l}_{10}^T\check{U}_{00} & \check{l}_{10}^T\check{u}_{01} + \check{\upsilon}_{11} \end{array}\right)$$

$$= \left(\begin{array}{c|c} \check{L}_{00} & 0 \\ \hline \check{l}_{10}^T & 1 \end{array}\right)\left(\begin{array}{c|c} \check{U}_{00} & \check{u}_{01} \\ \hline 0 & \check{\upsilon}_{11} \end{array}\right)$$

Proof of the above equality:

$$\left(\begin{array}{c|c} \check{L}_{00} & 0 \\ \hline \check{l}_{10}^T & 1 \end{array}\right)\left(\begin{array}{c|c} \check{U}_{00} & \check{u}_{01} \\ \hline 0 & \check{\upsilon}_{11} \end{array}\right) = \left(\begin{array}{c|c} \check{L}_{00}\check{U}_{00} + 0 & \check{L}_{00}\check{u}_{01} + 0 \\ \hline \check{l}_{10}^T\check{U}_{00} + 0 & \check{l}_{10}^T\check{u}_{01} + \check{\upsilon}_{11} \end{array}\right) \quad \langle \text{matrix-matrix multiply} \rangle$$

Hence

$$\left(\begin{array}{c|c} A_{00} & a_{01} \\ \hline a_{10}^T & \alpha_{11} \end{array}\right) + \left(\begin{array}{c|c} \Delta A_{00} & \delta a_{01} \\ \hline \delta a_{10}^T & \delta\alpha_{11} \end{array}\right) = \left(\begin{array}{c|c} \check{L}_{00} & 0 \\ \hline \check{l}_{10}^T & 1 \end{array}\right)\left(\begin{array}{c|c} \check{U}_{00} & \check{u}_{01} \\ \hline 0 & \check{\upsilon}_{11} \end{array}\right)$$

where

$$\left(\begin{array}{c|c} |\Delta A_{00}| & |\delta a_{01}| \\ \hline |\delta a_{10}^T| & |\delta\alpha_{11}| \end{array}\right) \leq \left(\begin{array}{c|c} \gamma_{n+1}|\check{L}_{00}||\check{U}_{00}| & \gamma_{n+1}|\check{L}_{00}||\check{u}_{01}| \\ \hline \gamma_{n+1}|\check{l}_{10}^T||\check{U}_{00}| & \gamma_{n+1}(|\check{l}_{10}^T||\check{u}_{01}| + |\check{\upsilon}_{11}| \end{array}\right)$$

$$= \left| \left(\begin{array}{c|c} \Delta A_{00} & \delta a_{01} \\ \hline \delta a_{10}^T & \delta \alpha_{11} \end{array} \right) \right| \leq \gamma_{n+1} \left| \left(\begin{array}{c|c} \check{L}_{00} \check{U}_{00} & \check{L}_{00} \check{u}_{01} \\ \hline \check{l}_{10}^T \check{U}_{00} & \check{l}_{10}^T \check{u}_{01} + \check{u}_{11} \end{array} \right) \right|$$ <Defn 6.2.6.1>

$$= \left| \left(\begin{array}{c|c} \Delta A_{00} & \delta a_{01} \\ \hline \delta a_{10}^T & \delta \alpha_{11} \end{array} \right) \right| \leq \gamma_{n+1} \left| \left(\begin{array}{c|c} \check{L}_{00} & 0 \\ \hline \check{l}_{10}^T & 1 \end{array} \right) \right| \left| \left(\begin{array}{c|c} \check{U}_{00} & \check{u}_{01} \\ \hline 0 & \check{u}_{11} \end{array} \right) \right|$$ <matrix multiplication>

- By Principle of Mathematical Induction, the result holds true for all n