

Using 'droptol' set to $1e-3$, and x defined as a random vector, the below number of iterations were observed for the different algorithms:

Method of Steepest Descent (no preconditioning)

18055

Preconditioned Method of Steepest Descent (incomplete Cholesky)

68

CG (no preconditioning)

233

PCG (incomplete Cholesky)

20

The number of iterations in the Method of Steepest being 18055 is consistent with the fact that unless the matrix A is well-conditioned, the algorithm may keep bouncing back and forth and takes a lot of iterations to converge to a minimum. Preconditioning it with a matrix M (in this case, incomplete Cholesky factorization of A) that resembles A brings it closer to resembling an Identity matrix which is an example of a well-conditioned matrix. The smaller the value of 'droptol', the more M resembles A . Using a 'droptol' of $1e-3$, the preconditioned Method of Steepest Descent reduced the number of iterations from 18055 to 68, which is less than 265 times the number of iterations taken by the Method of Steepest Descent without preconditioning.

Conjugate Gradient Descent benefits from using A -conjugate search directions such that the next search direction is the minimum in the span of all the previous search directions, instead of restricting x to be on the union of lines defined by $x_{(j)} + \alpha * p_{(j)}$ (where $j=0, \dots, k$). Hence, it takes 233 iterations which is less than 77 times the number of iterations taken by the Method of Steepest Descent. However, it doesn't outperform the preconditioned Method of Steepest Descent, which shows how the conditioning of a linear (sparse) system A has a big impact on the number of iterations to converge to a solution (bigger impact than the method in which search directions are chosen).

Preconditioning the Conjugate Gradient Method accelerates its performance as expected and brings the number of iterations to convergence to 20, less than half the number of iterations taken by the preconditioned Method of Steepest Descent which is expected since it leverages a better way to pick the next search direction (as discussed above).