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Published in IET Control Theory and Applications
Received on 22nd May 2010
Revised on 9th July 2012
doi: 10.1049/iet-cta.2010.0278

ISSN 1751-8644

State estimation with delayed measurements incorporating time-delay uncertainty

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Abstract: This study is concerned with the problem of state estimation using delayed measurements, especially when the delay is uncertain. In real-world applications, measurements are often randomly delayed, and hence not immediately available to the filter when taken by a sensor. By modelling uncertain delay as a probabilistic density function, the effect of uncertain delayed measurements is accounted for by the proposed estimator, combined with the augmented state Kalman filter. Use of the uncertain delay model resolves the randomness of the delay, and the augmentation is used to handle the delayed measurements. Consequently, the proposed algorithm is able to provide consistent estimates, regardless of the uncertainty of the delay. Monte Carlo simulations using one-dimensional particle were conducted with three kinds of representative uncertainty models, based on Gaussian, gamma and uniform distributions. The simulations verified the reliability and consistency of the proposed estimator.

1 Introduction

State estimation in dynamical systems is crucial in real-world applications, because the true state is unknown and sensors provide only a sequence of noisy measurements. Filtering methods, such as the extended Kalman filter (EKF) and the particle filter, are commonly used to acquire an estimate of the true state from noisy measurements. In general, it is assumed that the measurements are transmitted to the filter without any delay. In other words, a currently available measurement reflects the current state. Thus, the current state can be corrected by the current measurement. In practice, however, this assumption may be unjustified. For instance, when a filtering processor is connected to a sensor through a network, there is a fundamental communication time. Moreover, if raw sensor data require post-processing (such as image processing) in order to update the state of the dynamical system, processing time is needed, resulting in a delay between the acquisition of a measurement and its availability to the filter. Let us suppose that a vehicle is controlled by an operator through a communication network in a remote site. The operator should know the vehicle pose to control the vehicle cleverly. If, however, the fundamental communication time delay is not handled in the filtering method to estimate the vehicle pose, the operator may cause an accident, crashing an obstacle, because the vehicle pose which the operator is recognising is wrong. In the remote site, the delay should be compensated to estimate the vehicle pose correctly. This kind of state estimation problem is referred to as the out-of-sequence measurement (OOSM) problem [1], the time-delayed measurement problem [2] or the time-varying measurement problem [3].

When measurements are delayed, the current state cannot be directly corrected using the current measurement, since a delayed measurement is actually carrying information about a past state. The past state corresponding to a delayed measurement must be determined before using the delayed measurement in state estimation. The current state must also be corrected after correcting the appropriate past state. In a number of studies, a given delay is assumed in order to simplify the problem. Used as a stamping time on every measurement, this known delay can be implemented by a well-synchronised system. In other research, the delay is assumed to be unknown, and algorithms for utilising unknown delayed measurements are investigated.

In the known delay problem, backward prediction (or retrodiction) of the corresponding past state from the current state is used as a means of incorporating delayed measurements. Based on the delayed measurement and the corresponding backward-predicted state, the current state can be updated using a filtering process. Bar and Shalom [1] proposed an optimal solution and sub-optimal solutions for one-step delayed measurements. In [4], extended versions of [1] were also proposed for multiple-step delayed measurements. The difference between the optimal and sub-optimal algorithms lies in whether or not process noise in the backward prediction is assumed to have a non-zero conditional mean. Zhang *et al.* [5] suggested a sub-optimal scheme to reduce the computational effort and optimal schemes for specific cases.

As another solution to the known delay problem, a delayed measurement is directly used to estimate a preserved past state, and a new prediction of the current state is then obtained from the corrected past state. This recalculation technique can be elaborately implemented in terms of state augmentation. The augmented state vector is formed by piling up states in several steps. Although a delayed measurement is not directly related to the current state, the augmented state vector containing the corresponding past state can be associated with the delayed measurement in a measurement model. Challa *et al.* [6] derived a Bayesian solution to the OOSM problem, using the augmented state Kalman filter (ASKF). Merwe *et al.* [7] applied the sigma point Kalman filter to the augmentation technique, instead of the standard EKF. Leonard and Rikoski [8] used augmentation for the stochastic mapping of a mobile robot equipped with ultrasonic sensors. By utilising measurements calculated from several steps of raw ultrasonic data, the related robot poses in the augmented vector were updated automatically according to the measurement model.

Instead of obtaining an estimate of the current state from the corrected past state, Larsen *et al.* [2] suggested extrapolation of a delayed measurement. By means of this extrapolation, a measurement for the current state was calculated and used in conjunction with the Kalman filter to correct the current state.

As was previously noted, in the above methods, it is assumed that the amount of delay is known. However, in many practical applications, it is difficult to determine the time delay precisely. Therefore delay uncertainty should be considered in the filtering technique. For example, communication time through a network might not be constant in a real-world situation. The processing time required to obtain meaningful measurements from raw sensor data is also variable. Challa *et al.* [6] applied an augmented state probabilistic data association filter (AS-PDAF) to the OOSM problem in cluttered measurements, as an extension of [9, 10]. The probabilistic data association method was used to calculate the probability that a delayed measurement was induced from each past state in the augmented vector. These probabilities were used to correct the augmented vector. Julier and Uhlmann [11] suggested a covariance union algorithm for handling unknown time delays. They assumed the delay to be uniformly distributed, with given maximum and minimum time delays. Both the maximum and minimum trackable past states were updated via the delayed measurement, and the means and covariances of both updated states were then unified by covariance union.

In this paper, a state estimation algorithm is proposed that incorporates uncertainty of measurement delay. Uncertain delay means that the delay must be represented statistically, using a probabilistic density function (PDF), even though the distribution is not exactly known. In [12], internet traffic delay was modelled by a shifted gamma distribution. Chen *et al.* [13] used such a model to predict round-trip time delay. According to [11], uncertain delay can be represented by the uniform distribution, and the Gaussian distribution is also a strong candidate for modelling uncertain delay [14].

We adopt augmentation of the state vector in the proposed estimator for handling delayed measurement. The process model and measurement model are transformed into suitable forms for the augmented vector, so that the current state can be updated systematically using a delayed measurement. When the delay is given, we can determine the corresponding past state in the augmented vector. The past state is updated using the delayed measurement, and the current state is simultaneously corrected in the augmented vector. However, when the delay is not known precisely, a probability is obtained that the measurement was generated by one of the past states in the augmented vector, based on the PDF of the delay. In other words, the measurement has a certain probability of belonging to each past state. The basic framework for the proposed technique is as follows. First of all, each estimate of the augmented vector is preserved by assuming that the measurement comes from every past state in the augmented vector. A weighted mean of the preserved estimates is then calculated to obtain the actual estimate of the augmented vector. In this updating process, the derived probabilities are the weights, and the EKF is used to obtain the individual estimates. Similarly, the covariance of the augmented vector can also be obtained. The estimate of the current state is found at the top of the estimate of the augmented vector.

The proposed technique offers several advantages. It provides a general framework for the uncertain delayed measurement problem, when the uncertain delay is modelled by any type of PDF. The uncertainty of the delay is itself resolved, since the PDF of the delay is sexplicitly incorporated into the proposed estimator. By directly considering the statistical properties of the delay, a consistent estimate is obtained, regardless of other conditions, including measurement noise and process noise. By adopting augmentation, the correlation between a past state and the current state can be calculated autonomously, without complicated considerations. In addition, it is not necessary to define an inverse process model for backward predictions. (Inverse process models are not easily obtained, especially in non-linear systems.) Finally, instead of the EKF, other kinds of filters can also be applied to the proposed framework without changing the concept of the algorithm.

This paper is organised as follows. In Section 2, the uncertain delayed measurement problem is defined. Section 3 introduces the augmented state Kalman filter to handle the delayed measurements. Section 4 presents the proposed filtering algorithm, in which uncertain delay is explicitly incorporated. In Section 5, simulation results are analysed, and are used to evaluate the proposed method. A summary of our conclusions is given in Section 6.

2 Uncertain delayed measurement

The delayed measurement problem is defined in Section 2.1, and uncertain delay models are described in Section 2.2.

2.1 Delayed measurement

The system dynamics can be described by a discrete-time equation known as the process model

$$x_{k+1} = f(k, x_k) + v_k \tag{1}$$

where x_k is the state vector at time step k, and v_k is the process noise. For the sake of simplicity, no control input is assumed. v_k is assumed to be additive and have a white Gaussian distribution with mean zero and covariance Q_k.

The measurement model is

$$z_k = h(k, x_k) + w_k \tag{2}$$

where z_k is a measurement taken by a sensor and w_k is the measurement noise. w_k is assumed to have a white Gaussian distribution with mean zero and covariance R_k.

If the moment when a sensor takes a measurement coincides with the moment when the measurement is available to the filter, (1), and (2) can be applied to general filtering methods such as the EKF and the particle filter. This situation can be represented as in Fig. 1*a* with respect to the time sequence. However, in the delayed measurement case, the two moments disagree, as shown in Fig. 1*b*. If each delay is denoted by τ_k, the measurement model should be reformulated as

$$z_k = h(k - \tau_k, x_{k-\tau_k}) + w_{k-\tau_k} \tag{3}$$

where τ_k is a function of time. Although z_k is available to the filter at time step k, x_k should not be directly corrected using the measurement z_k, since this measurement is related to a past state vector $x_{k-\tau_k}$. The filter should take these delays into consideration to obtain a proper estimate.

2.2 Uncertainty of delay

If the amount of delay is constant, or known to the filter, it can easily be accounted for in the correction step. Unfortunately, the amount of delay is a random variable, since it results from network communications and/or the processing time required for raw sensor data. Of course, if the uncertainty is small relative to the time interval, it is not critical. On the other hand, if the level of the uncertainty is significant, it should be incorporated into the filtering algorithm. This case is shown in Fig. 2. The measurements taken at t_i may normally arrive in the filter at t_{i+3}, but they will sometimes arrive at different time steps, such as t_{i+2}, t_{i+4}, or t_{i+5}.

Even though the delay cannot be specified exactly, we can assume that the PDF of the delay is given. State estimation is then performed in terms of the PDF of the delay. In practice, the shape of the PDF should be modelled on a case-by-case basis, according to a system. In [15], the delay distribution of wireless local area network (WLAN) was analysed with variable packet length. In [13], the statistical properties of the internet-based tele-operation was analyzed using the round trip time delay. The PDF may be a Gaussian distribution (as in general noise modelling), a gamma distribution (as in the internet traffic model), or a uniform distribution when the statistics cannot be specified, so that only the maximum and minimum delays are given. These PDFs are described as follows.

- Gaussian distribution

$$p(t) = \frac{1}{\sqrt{2\sigma^2}} \exp\left(-\frac{(t-\mu)^2}{2\sigma^2}\right) \tag{4}$$

where μ and σ^2 are the mean and variance of the delay.

- Gamma distribution

$$p(t) = \frac{t^{\alpha-1}\exp(-t/\beta)}{\beta^\alpha \Gamma(\alpha)} \tag{5}$$

where α and β are parameters for the shape and scale of the PDF. These parameters are related to the mean and variance of the delay by $\alpha = \mu^2/\sigma^2$ and $\beta = \sigma^2/\mu$.

- Uniform distribution

$$p(t) = \begin{cases} \frac{1}{b-a} & \text{for } t \in [a, b] \\ 0 & \text{otherwise} \end{cases} \tag{6}$$

where $b > a > 0$.

Although other PDFs could be used to represent delay uncertainty, these three are considered here because they are representative cases of uncertain delay.

We can predict the probability that a measurement is available to the filter within a given delayed interval by integrating the chosen PDF over that interval. This probability is then combined with the ASKF to solve the uncertain delayed measurement problem.

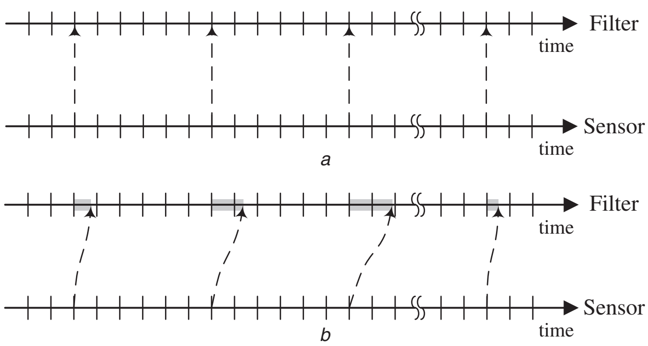

Fig. 1 *Measurement sequences in the sensor and the filter*
a No delayed measurements
b Uncertain delayed measurements

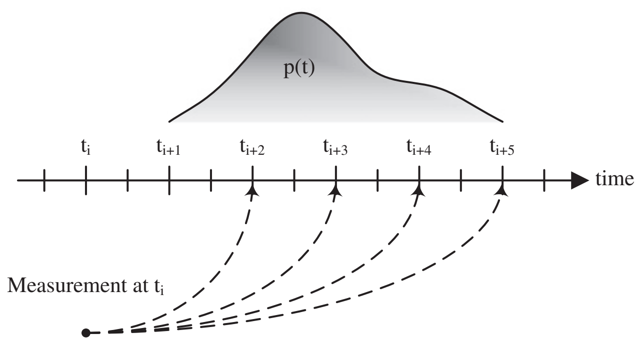

Fig. 2 *Uncertainty of delay; $p(t)$ denotes a probability density function for the delay*

3 Augmented state Kalman filter

Augmentation is a systematic and straightforward method for handling delayed measurements. Hence, the ASKF is used as a basic framework for the proposed estimator of state estimation with uncertain delayed measurements. The augmentation for delayed measurements is described in Section 3.1. The ASKF algorithm is then described in Section 3.2.

3.1 Augmented state vector for delayed measurement

Augmentation basically involves binding several state vectors to a single augmented state vector. By augmenting the current and past states, the current measurement (related to the past state) can be used to directly correct the augmented state vector. Assuming that there is a one-step delay, the process model is modified as

$$\begin{bmatrix} x_{k+1} \\ x_k \end{bmatrix} = \begin{bmatrix} f(k, x_k) + v_k \\ x_k \end{bmatrix}$$

where $\begin{bmatrix} x_{k+1}^{\mathrm{T}} & x_k^{\mathrm{T}} \end{bmatrix}^{\mathrm{T}}$ is the augmented state vector. Rewriting the measurement equation as

$$z_k = h\left(k, \begin{bmatrix} \mathbf{0} & \mathbf{I} \end{bmatrix} \begin{bmatrix} x_{k+1} \\ x_k \end{bmatrix}\right) + w_k$$

where $\mathbf{I}$ is the identity matrix, the current measurement z_k can be used to update $\begin{bmatrix} x_{k+1}^{\mathrm{T}} & x_k^{\mathrm{T}} \end{bmatrix}^{\mathrm{T}}$.

For multiple-step delays, the process model can be extended to

$$\mathbf{x}_{k+1} = \begin{bmatrix} \begin{bmatrix} f(k, x_k) \\ \begin{bmatrix} \mathbf{I} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \ddots & \mathbf{0} & \vdots \\ \mathbf{0} & \mathbf{0} & \mathbf{I} & \mathbf{0} \end{bmatrix} \mathbf{x}_k \end{bmatrix} \end{bmatrix} + \begin{bmatrix} v_k \\ \mathbf{0} \\ \vdots \\ \mathbf{0} \end{bmatrix} \equiv \mathbf{f}(k, \mathbf{x}_k) + \mathbf{v}_k \tag{7}$$

where $\mathbf{x}_k$ is the augmented state vector defined by $\begin{bmatrix} x_k^{\mathrm{T}} & x_{k-1}^{\mathrm{T}} & \ldots & x_{k-n}^{\mathrm{T}} \end{bmatrix}^{\mathrm{T}}$ and n is the maximum number of delay steps.

The measurement equation is rewritten as

$$\mathbf{z}_k = h\left(k - \tau_k, \begin{bmatrix} \mathbf{0} \\ \vdots \\ \mathbf{I} \\ \vdots \\ \mathbf{0} \end{bmatrix}^{\mathrm{T}} \begin{bmatrix} x_k \\ \vdots \\ x_{k-\tau_k} \\ \vdots \\ x_{k-n} \end{bmatrix}\right) + \begin{bmatrix} \mathbf{0} \\ \vdots \\ w_{k-\tau_k} \\ \vdots \\ \mathbf{0} \end{bmatrix} \equiv \mathbf{h}(k, \tau_k, \mathbf{x}_k) + \mathbf{w}_k \tag{8}$$

where τ_k represents the delay, which should be less than n, and $\mathbf{I}$ is placed at the corresponding time step $k - \tau_k$.

As (7) and (8) indicate, the process model and measurement equation can be revised for state estimation using the augmented state vector. These models can be directly applied to general filtering algorithms such as EKF.

3.2 Extended Kalman filter

If the number of delays τ_k is given, the augmented state vector can be estimated recursively via the EKF algorithm, which consists of prediction and update stages. If $\mathbf{x}_k$, the state vector, represents a real value, $\hat{\mathbf{x}}_k$ denotes the estimate for the state vector, and the estimate for the error covariance is defined by

$$\mathbf{P}_k = \mathbf{E}\left\{(\mathbf{x}_k - \hat{\mathbf{x}}_k)(\mathbf{x}_k - \hat{\mathbf{x}}_k)^{\mathrm{T}} | \mathbf{z}_k\right\}$$

where $\mathbf{E}\{\cdot\}$ denotes expectation.

In the prediction stage, state prediction is carried out by the process model. The error covariance is propagated by the Jacobian of the process model and the process noise covariance, $\mathbf{Q}$.

procedure PREDICTION($\hat{\mathbf{x}}_{k-1}, \mathbf{P}_{k-1}, \mathbf{Q}_{k-1}$)
  $\hat{\mathbf{x}}_k^- \leftarrow \mathbf{f}(k-1, \hat{\mathbf{x}}_{k-1})$
  $\mathbf{F}_{k-1} \leftarrow \nabla_{\mathbf{x}} \mathbf{f}(k, \mathbf{x})|_{\mathbf{x}=\hat{\mathbf{x}}_{k-1}}$
  $\mathbf{P}_k^- \leftarrow \mathbf{F}_{k-1}\mathbf{P}_{k-1}\mathbf{F}_{k-1}^{\mathrm{T}} + \mathbf{Q}_{k-1}$
  return $\hat{\mathbf{x}}_k^-, \mathbf{P}_k^-$
end procedure

The update stage is executed subsequently, based on the predicted state and the error covariance. The measurement model is required to obtain the measurement prediction induced by the predicted state. In addition, the Jacobian of the measurement model and the measurement noise, $\mathbf{R}$, are needed to obtain the Kalman gain, $\mathbf{K}$. Finally, the Kalman gain and the measurement are used to correct both the state vector and the error covariance via the following procedure. Unlike the prediction stage, the amounts of delay τ_k are necessary to determine the measurement model for the delayed measurements.

procedure UPDATE($\hat{\mathbf{x}}_k^-, \mathbf{P}_k^-, \mathbf{z}_k, \mathbf{R}_k, \tau_k$)
  $\hat{\mathbf{z}}_k \leftarrow \mathbf{h}(k, \tau_k, \hat{\mathbf{x}}_k^-)$
  $\mathbf{H}_k \leftarrow \nabla_{\mathbf{x}} \mathbf{h}(k, \tau_k, \mathbf{x})|_{\mathbf{x}=\hat{\mathbf{x}}_k^-}$
  $\mathbf{K}_k \leftarrow \mathbf{P}_k^- \mathbf{H}_k^{\mathrm{T}} (\mathbf{H}_k \mathbf{P}_k^- \mathbf{H}_k^{\mathrm{T}} + \mathbf{R}_k)^{-1}$
  $\hat{\mathbf{x}}_k \leftarrow \hat{\mathbf{x}}_k^- + \mathbf{K}_k(\mathbf{z}_k - \hat{\mathbf{z}}_k)$
  $\mathbf{P}_k \leftarrow \mathbf{P}_k^- - \mathbf{K}_k(\mathbf{H}_k \mathbf{P}_k^- \mathbf{H}_k^{\mathrm{T}} + \mathbf{R}_k)\mathbf{K}_k^{\mathrm{T}}$
  return $\hat{\mathbf{x}}_k, \mathbf{P}_k$
end procedure

The returned values of $\hat{\mathbf{x}}_k$ and $\mathbf{P}_k$ are used in the prediction stage of the next time step.

The ASKF is implemented by applying the above prediction and update stages to the augmented state vector. It provides consistent estimates for the state vector and the error covariance in the delayed measurement problem. However, as was previously noted, the number of delayed steps τ_k must be known in order to obtain reliable estimates from the ASKF. When the τ_k are not known precisely, the uncertain delay is modelled by a PDF to obtain a consistent estimator.

4 State estimation incorporating uncertain delay

To incorporate the uncertainty of the delay, the PDF of the delay is assumed be (4), (5) or (6), according to the particular system. If the delay is modelled by a PDF, the possibility that the current measurement belongs to one of the delayed steps can be described in terms of the PDF. A cumulative value taken over an interval is applied to the discrete system, since time is a continuous variable in the PDF. In other words, when the measurement arrives at the filter, the probability that the measurement belongs to a given time step is calculated by integrating the PDF over the time interval, as shown in Fig. 3. The probability that

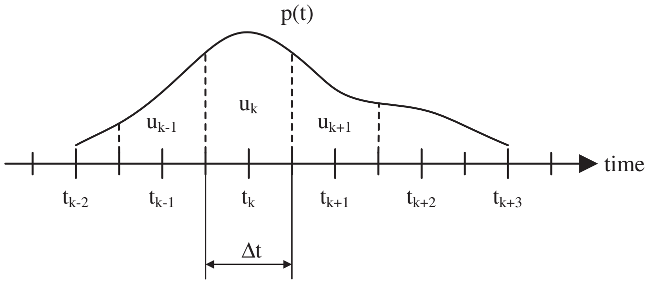

Fig. 3 *Probability that the measurement belongs to each time interval; p(t) denotes a probability density function for the delay*

IET Control Theory Appl., 2012, Vol. 6, Iss. 15, pp. 2351–2361
doi: 10.1049/iet-cta.2010.0278

the measurement belongs to the kth time step is given by

$$u_k = P\left(t_k - \tfrac{\Delta t}{2} \le t < t_k + \tfrac{\Delta t}{2}\right) = \int_{t_k - \frac{\Delta t}{2}}^{t_k + \frac{\Delta t}{2}} p(t)\mathrm{d}t \tag{9}$$

where $P(\cdot)$ denotes probability and $p(\cdot)$ denotes the PDF. After all the u_k for each time step within the maximum delay range have been calculated, these values should be normalised so that they sum to one, since the Gaussian and gamma distributions extend over an infinite-time domain, whereas the maximum delay is actually finite.

$$u_i = \eta u_k, \quad \text{where } \eta = \sum_k u_k \tag{10}$$

When the PDF of the delay is given, the maximum delay can be determined based on the corresponding cumulative distribution function (CDF) of the PDF. Practically, the maximum delay can be set as a value that makes the CDF larger than a threshold P_t(i.e. 99.6%). Then, most of the delay falls inside the maximum delay, with a probability corresponding to the threshold.

$$\tau_{\max} = \arg\min_{\tau} \left\{ \tau \,\middle|\, \int_0^{\tau} p(t)\mathrm{d}t \ge P_t \right\} \tag{11}$$

If a measurement $\mathbf{z}_k$ is given, and a correspondence c_i indicates that $\mathbf{z}_k$ is induced from the $(k-i)$th time step of the state, the optimal state estimator in the sense of the minimum mean-square error is derived by considering the correspondence c_i

$$\begin{aligned}\hat{\mathbf{x}}_k &= \mathbf{E}\left\{\mathbf{x}_k | \mathbf{z}_k\right\} \\ &= \int \mathbf{x}_k p\left(\mathbf{x}_k | \mathbf{z}_k\right) \mathrm{d}\mathbf{x} \\ &= \int \mathbf{x}_k \sum_i P\left(c_i | \mathbf{z}_k\right) p\left(\mathbf{x}_k | \mathbf{z}_k, c_i\right) \mathrm{d}\mathbf{x} \\ &= \sum_i P\left(c_i | \mathbf{z}_k\right) \mathbf{E}\left\{\mathbf{x}_k | \mathbf{z}_k, c_i\right\} \\ &= \sum_i P\left(c_i | \mathbf{z}_k\right) \hat{\mathbf{x}}_{k,i}\end{aligned} \tag{12}$$

where $\hat{\mathbf{x}}_{k,i}$ is the estimate for $\mathbf{x}_k$ when $\mathbf{z}_k$ is induced from the ith delayed step. Given that the measurement $\mathbf{z}_k$ of the state $\mathbf{x}_k$ is not directly related to the correspondence c_i, $\mathbf{z}_k$ and c_i can be assumed independent. Then, $P(c_i|\mathbf{z}_k)$ can be obtained.

$$P(c_i|\mathbf{z}_k) = P(c_i) = u_i \tag{13}$$

The final form of the optimal state estimator is a weighted sum of the estimates $\hat{\mathbf{x}}_{k,i}$.

$$\hat{\mathbf{x}}_k = \sum_i u_i \hat{\mathbf{x}}_{k,i} \tag{14}$$

The optimal error covariance estimator is derived from

$$\begin{aligned}\mathbf{P}_k &= \mathbf{E}\{(\mathbf{x}_k - \hat{\mathbf{x}}_k)(\mathbf{x}_k - \hat{\mathbf{x}}_k)^{\mathrm{T}} | \mathbf{z}_k\} \\ &= \int \mathbf{x}_k \mathbf{x}_k^{\mathrm{T}} p(\mathbf{x}_k|\mathbf{z}_k)\mathrm{d}\mathbf{x} \\ &\quad - \int (\mathbf{x}_k \hat{\mathbf{x}}_k^{\mathrm{T}} + \hat{\mathbf{x}}_k \mathbf{x}_k^{\mathrm{T}}) p(\mathbf{x}_k|\mathbf{z}_k) d\mathbf{x} + \hat{\mathbf{x}}_k \hat{\mathbf{x}}_k^{\mathrm{T}} \\ &= \sum_i u_i \mathbf{E}\{\mathbf{x}_k \mathbf{x}_k^{\mathrm{T}} | \mathbf{z}_k, c_i\} - \hat{\mathbf{x}}_k \hat{\mathbf{x}}_k^{\mathrm{T}} \\ &= \sum_i u_i [\mathbf{E}\{(\mathbf{x}_k - \hat{\mathbf{x}}_{k,i})(\mathbf{x}_k - \hat{\mathbf{x}}_{k,i})^{\mathrm{T}} | \mathbf{z}_k, c_i\} \\ &\quad + \hat{\mathbf{x}}_{k,i} \hat{\mathbf{x}}_{k,i}^{\mathrm{T}}] - \hat{\mathbf{x}}_k \hat{\mathbf{x}}_k^{\mathrm{T}} \\ &= \sum_i u_i [\mathbf{P}_{k,i} + \hat{\mathbf{x}}_{k,i} \hat{\mathbf{x}}_{k,i}^{\mathrm{T}}] - \hat{\mathbf{x}}_k \hat{\mathbf{x}}_k^{\mathrm{T}}\end{aligned} \tag{15}$$

Combining the derived equations (14) and (15) with the EKF, the state vector can be estimated consistently using uncertain delayed measurements.

Require: u_i, i is integer such that $0 \le i \le n$

```
procedure ESTIMATOR(x̂_{k−1}, P_{k−1}, Q_{k−1}, z_k, R_k)
    x̂_k^−, P_k^− ← PREDICTIONx̂_{k−1}, P_{k−1}, Q_{k−1}
    for i ← 0, n do
        x̂_{k,i}, P_{k,i} ← UPDATEx̂_k^−, P_k^−, z_k, R_k, i
    end for
    x̂_k ← Σ_i u_i x̂_{k,i}
    P_k ← Σ_i u_i[P_{k,i} + x̂_{k,i} x̂_{k,i}^T] − x̂_k x̂_k^T
    return x̂_k, P_k
end procedure
```

The returned values of $\hat{\mathbf{x}}_k$ and $\mathbf{P}_k$ are recursively input to the estimator for the next step. Note that the probability that the measurement is induced from the ith delayed step must be calculated in advance. In other words, each u_i is calculated from the given PDF before starting the algorithm.

By modelling the uncertainty of the delay with an appropriate PDF, the optimal estimators for the state vector and error covariance (in the sense of the minimum mean-square error) can be derived. The proposed algorithm is capable of providing consistent estimates for the uncertain delayed measurement problem.

5 Evaluation

A linear system was selected to evaluate the proposed estimator. Since the Kalman filter functions optimally in this type of system, the choice is well suited to displaying the characteristics of the estimator. A one-dimensional (1D) particle moving at constant velocity was simulated, and the position and velocity of the particle were estimated to verify the performance of the estimator. In addition, the uncertain measurement delay was simulated using Gaussian, gamma and uniform distributions.

5.1 System description

The state variables of the 1D particle were defined by

$$x_k = \begin{bmatrix} x \\ \dot{x} \end{bmatrix} \tag{16}$$

where x is the position of the particle and $\dot{x}$ is the velocity of the particle. The particle was moving with a constant velocity, and the process noise was applied to the acceleration (i.e. the white noise acceleration model was used). If the noise is denoted by q, the process model can be written as

$$x_{k+1} = \begin{bmatrix} 1 & \Delta t \\ 0 & 1 \end{bmatrix} x_k + \begin{bmatrix} (\Delta t)^2/2 \\ \Delta t \end{bmatrix} q_k \tag{17}$$

where Δt is a constant time duration in the discrete-time system.

A sensor was used to measure the position of the particle, so that the measurement equation is given by

$$z_k = \begin{bmatrix} 1 & 0 \end{bmatrix} x_k + w_k \tag{18}$$

where w_k is the measurement noise. However, the measurement equation should be written in the form (3), since there is a measurement delay.

As described in Section 3.1, the extended process model for multiple-step delays was extended to

$$\mathbf{x}_{k+1} = \begin{bmatrix} \mathbf{\Phi} & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} \\ \mathbf{I} & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{I} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} \\ & & & \vdots & & \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{I} & \mathbf{0} \end{bmatrix} \mathbf{x}_k + \begin{bmatrix} \mathbf{\Lambda} \\ \mathbf{0} \\ \mathbf{0} \\ \vdots \\ \mathbf{0} \end{bmatrix} q_k \tag{19}$$

where $\mathbf{\Phi} = \begin{bmatrix} 1 & \Delta t \\ 0 & 1 \end{bmatrix}$, and $\mathbf{\Lambda} = \begin{bmatrix} (\Delta t)^2/2 \\ \Delta t \end{bmatrix}$. The measurement model was also rewritten as

$$\mathbf{z}_k = \begin{bmatrix} \delta(\tau_k) & 0 & \cdots & \delta(\tau_k - n) & 0 \end{bmatrix} \mathbf{x}_k + w_k \tag{20}$$

where $\delta(\cdot)$ is the Direc-delta function, and τ_k represents the delay, which should be less than n.

The uncertainty of the delay was implemented in terms of Gaussian, gamma and uniform distributions. To compare the performance of the estimator in the three cases, the means of all three distributions were assumed to be identical, and the variance of the gamma distribution was set equal to the variance of the Gaussian distribution. In the uniform distribution, the variance was not equalised, and only the maximum delay was used to model the PDF.

5.2 Simulation organisation

In the simulations, the 1D particle was moving with a piecewise constant velocity, starting from an initial condition. According to the process model, (17), the piecewise constant velocity was corrupted by the white Gaussian noise applied to the acceleration. The number of integration steps was 2000, and the sensor measured the position of the particle every ten integration steps. Thus, a total of 200 updates were carried out in the filter. The measurements were corrupted by the sensor noise, which was assumed to have a white Gaussian distribution. The parameters used in the simulations are described in Table 1.

Three kinds of uncertain delay were simulated in the measurement process. For the Gaussian and gamma distributions, the mean of the uncertain delay was five time steps, and the standard deviation was one time step. In the uniform distribution, the mean was time steps, and the maximum delay was ten time steps (twice the mean delay). Finally, the augmented state vector had 22 dimensions, since the state vector was 2D, and the maximum delay was chosen to be 10. The PDFs used in the simulations are shown in Fig. 4.

The following five cases were tested for each PDF. Most of these were used in [11] to compare performances. All algorithms were implemented in the ASKF framework to obtain a fair evaluation.

- *KnownDelay*: The time delay is known precisely. Although there is uncertainty in the delay, the filter knows the exact value of the delay. This case exhibits ideal performance.

Table 1 Parameters for the simulations

Parameter	Value
initial value	$x_{\text{initial}} = 0$, $\dot{x}_{\text{initial}} = 10$
time duration Δt	0.1
measurement frequency	1/10
integration steps	2000
acceleration noise q	$\mathcal{N}(0, 1)$
measurement noise w	$\mathcal{N}(0, 1)$
mean of time delay	$5 \times \Delta t$
standard deviation of time delay	$1 \times \Delta t$
maximum time delay	$10 \times \Delta t$

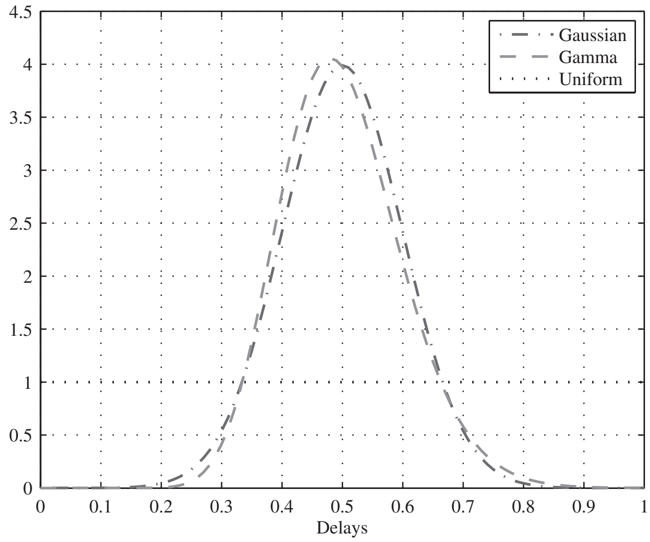

Fig. 4 *Probability density functions*

The blue dash-dotted line, red dashed line and black dotted line represent a Gaussian distribution, gamma distribution and uniform distribution, respectively

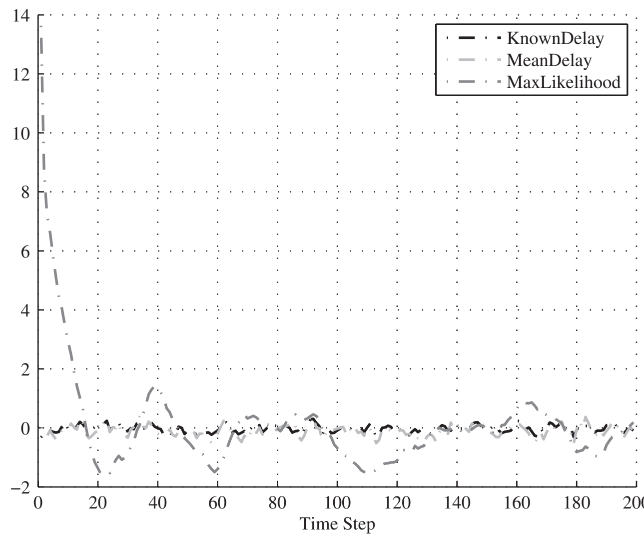

Fig. 5 *Average position errors (Gaussian distribution)*

The black, green and magenta dash-dotted lines represent *KnownDelay*, *MeanDelay* and *MaxLikelihood*, respectively

doi: 10.1049/iet-cta.2010.0278

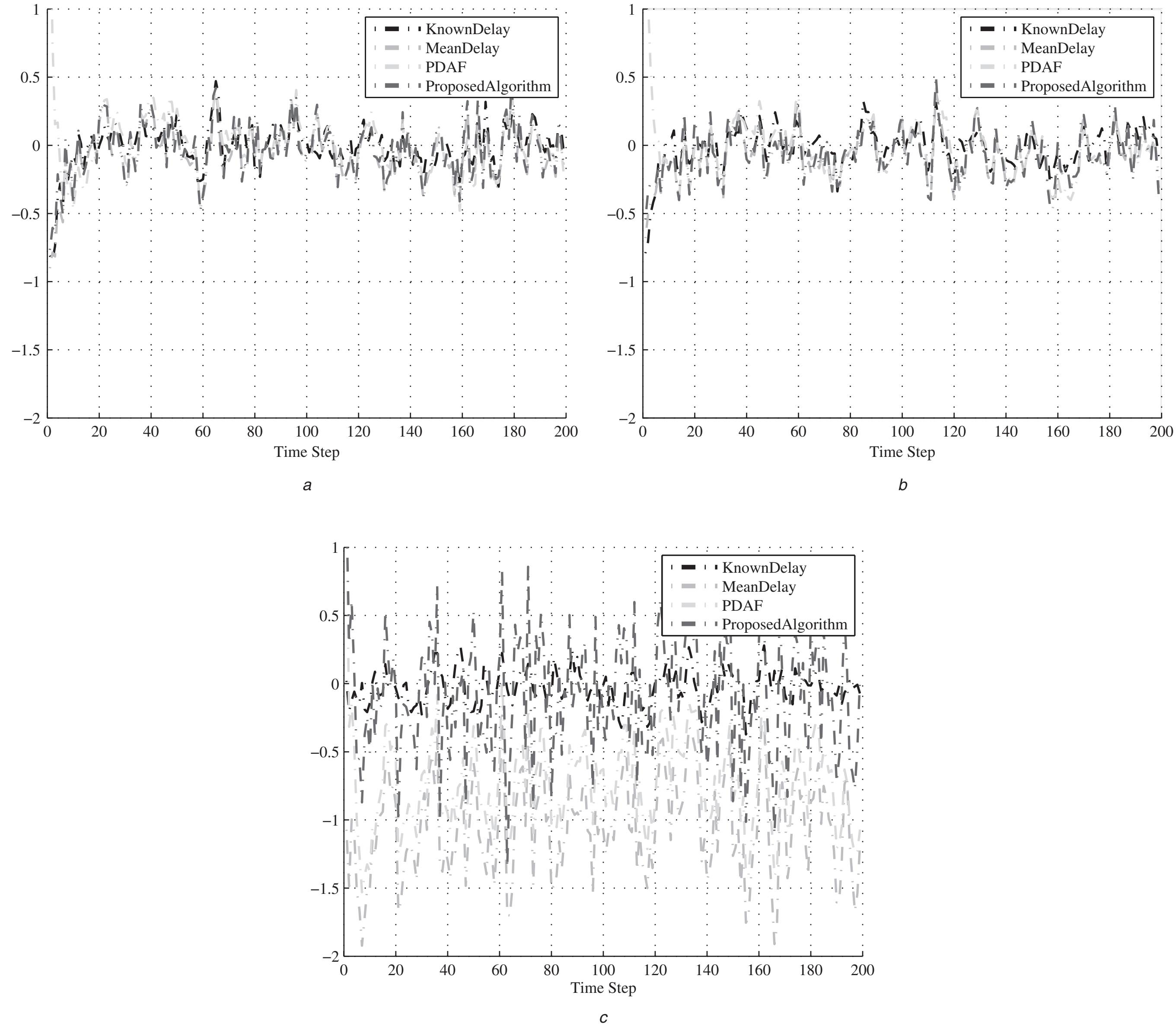

Fig. 6 *Average position errors*

The black, green, cyan and red dash-dotted lines represent *KnownDelay*, *MeanDelay*, *PDAF* and *ProposedAlgorithm*, respectively
a Gaussian distribution
b Gamma distribution
c Uniform distribution

- *MeanDelay*: The mean value of the delay is used to compensate for the delay. This case shows the effect of uncertain delay, since the uncertainty of the delay is not explicitly considered.
- *MaxLikelihood*: A time step that has the maximum-likelihood value in the augmented state vector is utilised as an appropriate delayed time step. The likelihood is obtained by comparing the states in the augmented vector with the current measurement. The filter carries out predictions and updates when the measurement is generated in the maximum-likelihood time step.
- *PDAF*: After obtaining the likelihood of each time step in the augmented state vector, the filter calculates a weighted mean for state estimation. The likelihoods are used as the weight values. The error covariance is calculated in a similar manner.
- *ProposedAlgorithm*: The proposed filtering algorithm is applied to handle the uncertain delay. The uncertainty of the delay is explicitly considered in terms of the given PDF.

Monte Carlo simulations were conducted to verify the performance of the algorithms. Since the process and measurement noise affect the results of a single trial, dozens of trials were required to guarantee the reliability of the simulation. Furthermore, the initial state was considered to be a random variable in each run. Using the two-point differencing technique in [16], the initial state and covariance were determined. By averaging the results, the performance of each algorithm could be analysed reliably. A total of 50 runs were executed in each case.

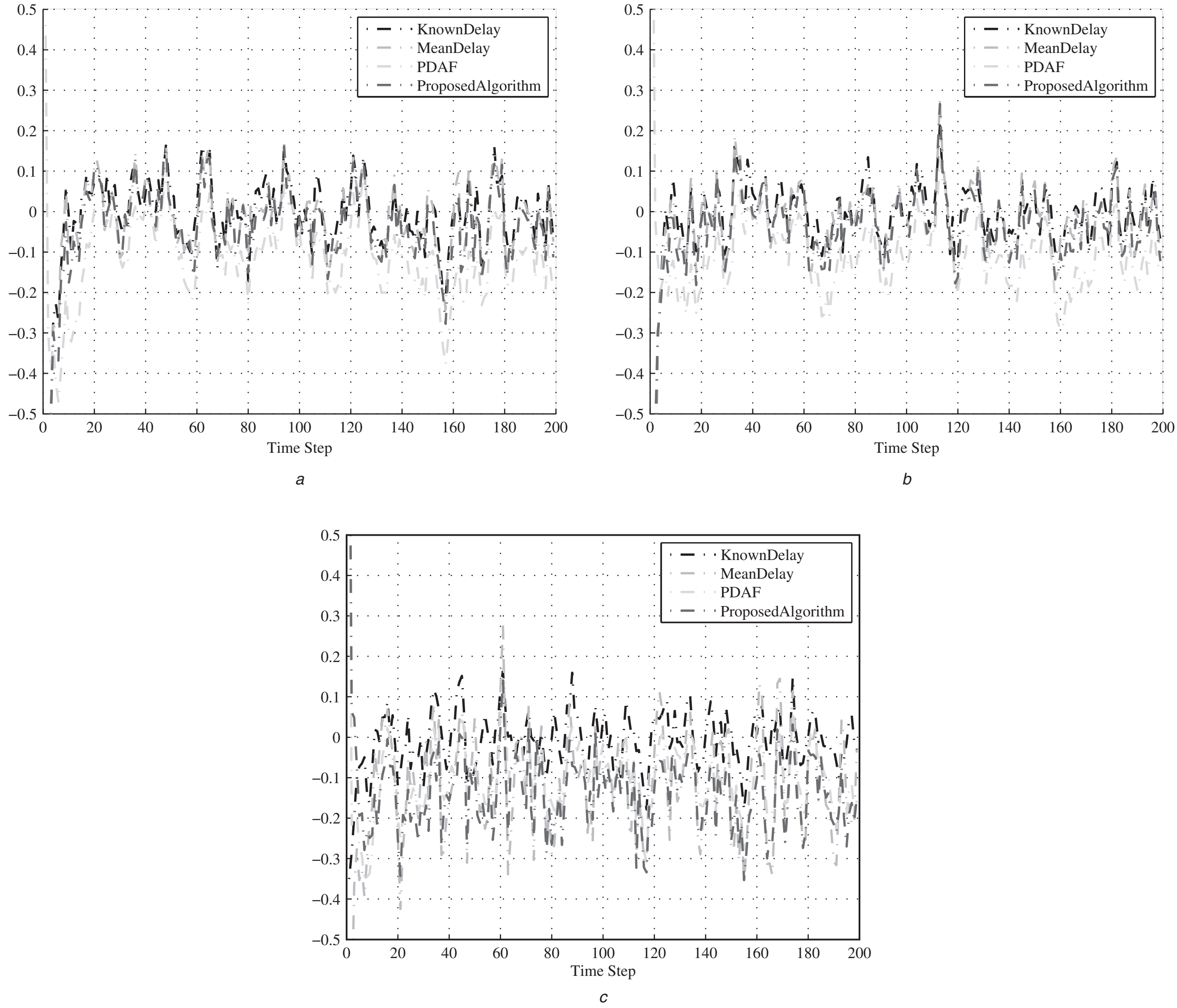

Fig. 7 *Average velocity errors*

The black, green, cyan and red dash-dotted lines represent *KnownDelay*, *MeanDelay*, *PDAF* and *ProposedAlgorithm*, respectively
a Gaussian distribution
b Gamma distribution
c Uniform distribution

5.3 Simulation results

The results for *MaxLikelihood* were abandoned because its performance was poor compared with the other cases. In Fig. 5, the average position errors show the performance of *MaxLikelihood* in relation to *KnownDelay* and *MeanDelay* for a Gaussian uncertain delay. The error produced by *MaxLikelihood* was almost four times as large as those of the other cases. This result indicates that it is inappropriate to regard the corresponding delay found by comparing likelihood values as the real delay. The reason for this is that likelihood values are affected by both sensor noise and the uncertainty of the delay. Therefore only the simulation results for the remaining four cases (excluding *MaxLikelihood*) will be shown.

Table 2 RMS of the average position error

	Gaussian	Gamma	Uniform
KnownDelay	0.16765	0.14918	0.12687
MeanDelay	0.20695	0.19064	1.10620
PDAF	0.24043	0.25425	0.80803
ProposedAlgorithm	0.20004	0.18705	0.44749

Table 3 RMS of the average velocity error

	Gaussian	Gamma	Uniform
KnownDelay	0.14812	0.12220	0.07325
MeanDelay	0.15213	0.12080	0.18231
PDAF	0.15322	0.14667	0.15217
ProposedAlgorithm	0.15042	0.11808	0.18006

IET Control Theory Appl., 2012, Vol. 6, Iss. 15, pp. 2351–2361
doi: 10.1049/iet-cta.2010.0278

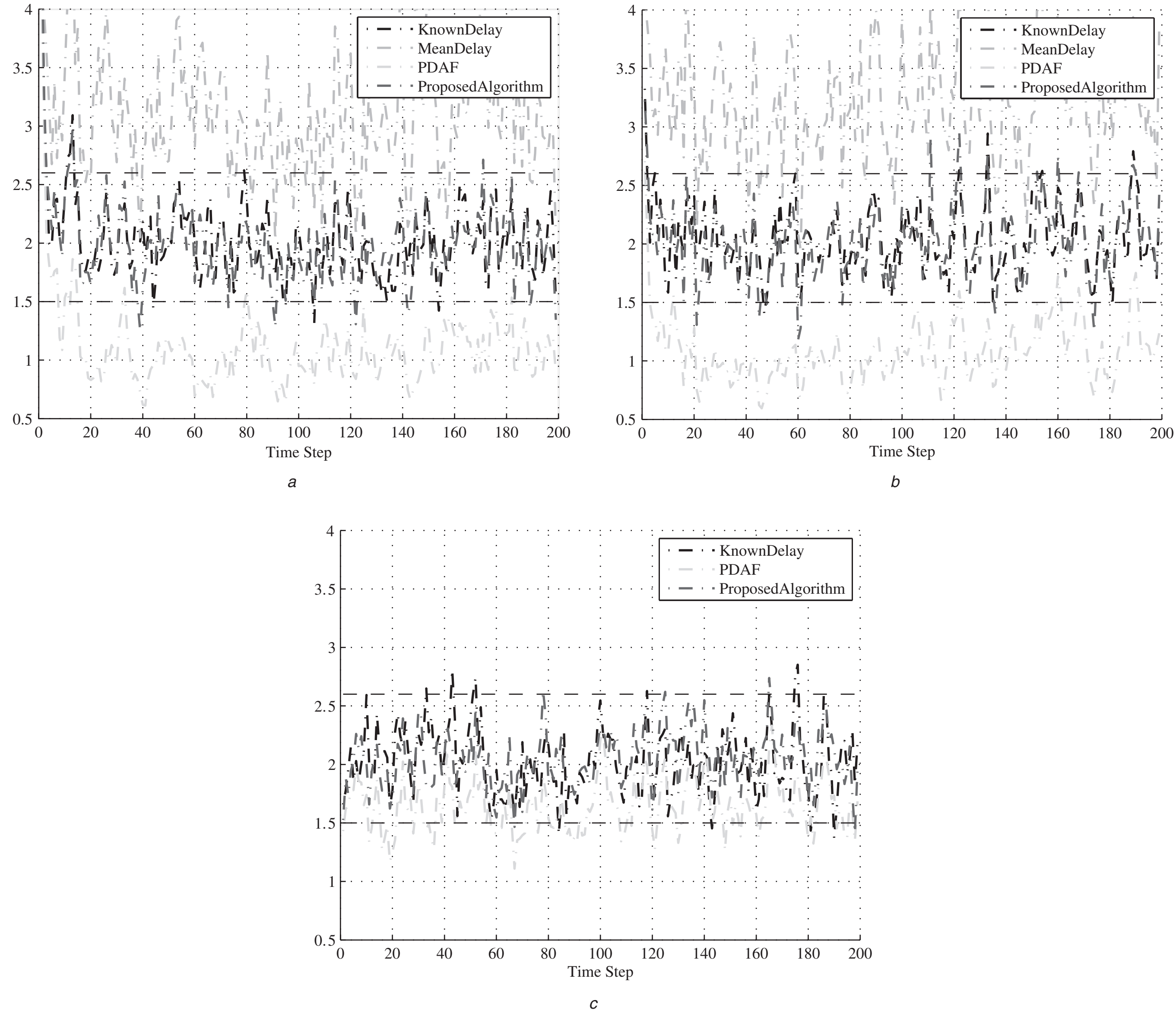

Fig. 8 *Sequences of average NEES*

The black, green, cyan and red dash-doted lines represent *KnownDelay*, *MeanDelay*, *PDAF* and *ProposedAlgorithm*, respectively. The black dashed line represents the two-sided 95% acceptance region

a Gaussian distribution

b Gamma distribution

c Uniform distribution

The average position and velocity errors of each algorithm are plotted in Figs. 6 and 7 for each type of uncertain delay. The root-mean-square (RMS) errors for each algorithm are shown in Tables 2 and 3. *ProposedAlgorithm* produced the second-smallest estimate error according to the position and velocity RMS errors. Understandably, *KnownDelay*, in which the effect of the delay uncertainty was absent, yielded the best performance in all cases.

For a Gaussian-distributed uncertain delay, the algorithms other than *KnownDelay* had similar RMS values. Even *MeanDelay*, which was expected to exhibit the worst performance, fared reasonably well in the RMS comparison. The reason for this lies in the shape of the Gaussian distribution, in which most of the delay occurred near the mean time delay. For a gamma-distributed uncertain delay, the tendency of the RMS values was similar to that of the Gaussian-distributed uncertain delay. The RMS error of *PDAF* was larger than the others, although the differences in the RMS values were still not very meaningful. For a uniformly distributed uncertain delay, the RMS error of the position in the *MeanDelay* case was the largest, as the uncertain delay could not be captured using only a mean delay when the delay was uniformly distributed. The RMS error of the position for *ProposedAlgorithm* was slightly more than half that of *PDAF*. As a result, aside from *KnownDelay*, *ProposedAlgorithm* showed the smallest RMS error, regardless of the type of uncertain delay.

The normalsied estimated error square (NEES) can be evaluated to check the consistency of the filtering algorithm. The NEES used to check the consistency of the filter in [16, 17] is defined by

$$\epsilon_k = \{\mathbf{x}_k - \hat{\mathbf{x}}_k\}^{\mathrm{T}} \mathbf{P}_k^{-1} \{\mathbf{x}_k - \hat{\mathbf{x}}_k\} \tag{21}$$

In this simulation, the random variable NEES had a χ^2 distribution with two degrees of freedom. Through Monte Carlo simulations, the N-run average NEES is calculated as

$$\bar{\epsilon}_k = \frac{1}{N}\sum_{i=1}^{N}\epsilon_{k,i} \tag{22}$$

where k is time step and i represents each trial. When 50-run Monte Carlo tests are executed, the two-sided 95% acceptance region for $\bar{\epsilon}_k$ is $[1.5, 2.6]$. In other words, 95% of $\bar{\epsilon}_k$ is included in this acceptance region. In addition, the mean value of the average NEES should be 2.0 for two degrees of freedom. The average NEES value for each case is shown in Fig. 8, and the mean values for the average NEES are listed in Table 4.

For the Gaussian distribution, both the *KnownDelay* and *ProposedAlgorithm* sequences were in the acceptance region, as shown in Fig. 8*a*. However, the *MeanDelay* sequence was beyond the upper bound. This indicates that the covariance was underestimated, and the estimated error was outside the estimated covariance. On the other hand, the *PDAF* sequence was below the lower bound. This implies that the estimated covariance contained the estimated error, but the covariance was overestimated. These facts are also reflected in the mean value of the average NEES. *KnownDelay* and *ProposedAlgorithm* had almost the optimal value of 2.0, but *PDAF* had a smaller value and *MeanDelay* had a larger value.

For the remaining types of uncertain delay, similar results were obtained, as shown in Figs. 8*b* and *b*. The average NEES for both *ProposedAlgorithm* and *KnownDelay* were inside the acceptance region. Moreover, the mean values of the average NEES were almost equal to 2.0. The average NEES values for *MeanDelay* were very large relative to the others (see Fig. 8*c*), and thus this case is omitted. *MeanDelay* was inclined to underestimate the covariance, and the mean value of the average NEES was larger than in the other algorithms. In contrast, *PDAF* tended to overestimate the covariance. Compared with the other algorithms, the error was not too large, but the average NEES was smaller than those of the other algorithms. This indicates that the error covariance estimate is too conservative.

Hence, *KnownDelay* and *ProposedAlgorithm* provide consistent results in terms of the NEES. *KnownDelay* is expected to be consistent because it is the ideal case, and is unaffected by uncertain delay. It is noteworthy that *ProposedAlgorithm* is also consistent in spite of the effect of the uncertain delay, since the uncertainty of the delay is explicitly incorporated into the filter. *MeanDelay* is an inconsistent filter, even though the RMS errors of the position and the velocity are not too large. *PDAF* overestimates the covariance, although the estimated covariance contains the estimated error.

Table 4 Mean of the average NEES

	Gaussian	Gamma	Uniform
KnownDelay	2.0211	2.0439	2.0166
MeanDelay	3.1016	3.1945	12.9422
PDAF	1.1004	1.0821	1.6419
ProposedAlgorithm	1.9941	2.0235	2.0447

6 Conclusions

This paper presented a method of state estimation with delayed measurements when the delay itself was an uncertain variable. The proposed method provided a successful state estimation under an uncertain delay represented by a PDF. The contributions of this paper are as follows. First, an uncertain delay of PDF form was systematically incorporated into the proposed filtering algorithm. Second, using the given PDF of the uncertain delay, the optimal state estimator for the uncertain-delay measurement system was implemented as a weighted summation of the augmented state Kalman filters. Last, using a partially integrated value of the PDF as the weight, a consistent estimator for the uncertain-delay measurement system was obtained. Through Monte Carlo simulations, the proposed algorithm was verified by the 1D particle model with three types of uncertainty based on Gaussian, gamma and uniform distributions. The consistency of the proposed filter was investigated in terms of the normalised estimate error square. The proposed algorithm was shown to provide consistent estimates, regardless of the type of uncertainty, whereas other algorithms (excluding the ideal case) were inconsistent or overestimated the results. These simulation results confirm that the proposed algorithm is a consistent and effective filter for uncertain-delay measurements. The proposed algorithm can be applied to state estimation of a system in which the estimator consists of sensors connected through communication networks. A consistent estimate can be obtained regardless of the uncertain delay in the networks.

7 Acknowledgments

This work was supported in part by the Acceleration Research Program of the Ministry of Education, Science and Technology of the Republic of Korea and the National Research Foundation of Korea (Grant No. R17-2008-021-01000-0), in part by the industrial Source Technology Development Programs of the Ministry of Knowledge Economy of Korea (10038574, Development of mobile assisted robot and emotional interaction robot for the elderly), and in part by the Agency for Defence Development and by Unmanned Technology Research Center (UTRC), Korea Advanced Institute of Science and Technology.

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