Method of Trigonometric Substitution

To start with we will be concerned with integrals that contains expression of the form $\sqrt{a^2-x^2}$, $\sqrt{x^2+a^2}$, $\sqrt{x^2-a^2}$ in which 'a' is a positive constant. The basic idea for evaluating such integrals is to make a substitution for x that will eliminate the radical. We make the substitution as

	Substitution	Angle restriction
$\sqrt{a^2-x^2}$	$x=a\sin\theta$	$-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$
$\sqrt{a^2+x^2}$	$x=a\tan\theta$	$-\frac{\pi}{2} < \theta < \frac{\pi}{2}$
$\sqrt{x^2-a^2}$	$x=a\sec\theta$	$\begin{cases} 0 \leq \theta < \frac{\pi}{2} \ (x \geq a) \\ \pi/2 < \theta \leq \pi \ (x \leq -a) \end{cases}$

Example:- Evaluate $\displaystyle\int \frac{dx}{x^2\sqrt{4-x^2}}$

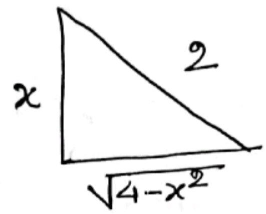
Solution:- $\displaystyle\int \frac{dx}{x^2\sqrt{4-x^2}}$

Let
$x = 2\sin\theta$
$\Rightarrow dx = 2\cos\theta\, d\theta$

$$= \int \frac{2\cos\theta\, d\theta}{4\sin^2\theta\sqrt{4-4\sin^2\theta}}$$

$$= \int \frac{2\cos\theta}{8\sin^2\theta\sqrt{1-\sin^2\theta}}\,d\theta$$

$$= \int \frac{\cos\theta}{4\sin^2\theta\cos\theta}\,d\theta$$



$\sin\theta = \frac{x}{2} = \frac{\text{Perpendicular}}{\text{Hypotenuse}}$

$\cot\theta = \frac{\sqrt{4-x^2}}{x} = \frac{\text{Adjacent}}{\text{Perpendicular}}$

$$= \frac{1}{4}\int \frac{1}{\sin^2\theta}\,d\theta$$

$$= \frac{1}{4}\int \operatorname{cosec}^2\theta\,d\theta$$

$$= \frac{-1}{4}\cot\theta + c$$

$$= \frac{-1}{4}\,\frac{\sqrt{4-x^2}}{x} + c$$

Ans.

Homework: $\int \frac{\sqrt{x^2-25}}{x}\,dx\,;\ x \geq 5.$

Integrals Involving ax^2+bx+c

Integrals that involve a quadratic expression ax^2+bx+c, where $a \neq 0$, and $b \neq 0$ can often be evaluated by first completing the square, then making an appropriate substitution.

Example: Evaluate $\int \sqrt{4x^2 - 16x + 52}\, dx$

Solution: $\int \sqrt{4x^2 - 16x + 52}\, dx$

Let,

$x - 2 = 3\tan\theta$

$\Rightarrow x = 2 + 3\tan\theta$

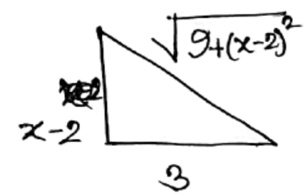
$\Rightarrow dx = 3\sec^2\theta\, d\theta$

$$= \int 2\sqrt{x^2 - 4x + 13}\, dx$$

$$= 2\int \sqrt{x^2 - 2x\cdot 2 + 4 + 9}\, dx$$

$$= 2\int \sqrt{(x-2)^2 + 3^2}\, dx$$

$$= 2\int \sqrt{9\tan^2\theta + 9}\; 3\sec^2\theta\, d\theta$$



$$= 6\int 3\sqrt{1 + \tan^2\theta}\; \sec^2\theta\, d\theta$$

$\sec\theta = \frac{\sqrt{9+(x-2)^2}}{3}$

$\tan\theta = \frac{x-2}{3}$

$$= 18\int \sec^3\theta\, d\theta.$$

$$= 18\left(\frac{1}{2}\sec\theta\tan\theta + \frac{1}{2}\ln\left|\sec\theta + \tan\theta\right|\right) + c$$

[By applying reduction formula)

$$= 18\left(\frac{1}{2}\frac{\sqrt{9+(x-2)^2}}{3}\frac{x-2}{3} + \frac{1}{2}\ln\left|\frac{\sqrt{9+(x-2)^2}}{3} + \frac{(x-2)}{3}\right|\right) + c$$

Practice Problem

Chapter 7.4 → 1–26

Ans.

Homework :-

1. $\int \sqrt{1-4x^2}\,dx$

2. $\int \dfrac{dx}{(4+x^2)^2}$

3. $\int \dfrac{dx}{1+2x^2+x^4}$

4. $\int \dfrac{3x^3}{\sqrt{1-x^2}}\,dx$

5. $\int_{\sqrt{2}}^{2} \dfrac{\sqrt{2x^2-4}}{x}\,dx$

6. $\int \dfrac{\sqrt{4y^2-16y+19}}{(y-2)^6}\,dy$

7. $\int \dfrac{2}{(x-3)^5\sqrt{-x^2+6x-5}}\,dx$

Integration of Rational Functions By Partial fractions

Let us consider a rational function $f(x) = \frac{P(x)}{Q(x)}$

where P and Q are polynomials. It is possible to express f as a sum of simpler fractions provided that the degree of P is less than the degree of Q. Such a rational function is called proper. Recall that if

$$P(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$$

where $a_n \neq 0$, then the degree of P is n. If f is improper, then $\deg(P) \leq \deg(Q)$, then we must take the preliminary step of dividing Q into P (by long division) until a remainder $R(x)$ is obtained such that $\deg(R) < \deg(Q)$. The result is,

$$f(x) = \frac{P(x)}{Q(x)} = S(x) + \frac{R(x)}{Q(x)}$$

where S and R are also polynomials.

Case I:- The denominator $Q(x)$ is a product of distinct linear factors (Linear Factor Rule)

This means that we can write

$$Q(x) = (a_1 x + b_1)(a_2 x + b_2) \cdots (a_k x + b_k)$$

Thereeforce, ① implies, $\dfrac{5x-10}{(x-4)(x+1)} = \dfrac{2}{x-4} + \dfrac{3}{x+1}$

So, $\displaystyle\int \frac{5x-4}{x^2-3x-4}\,dx = \int \frac{5x-10}{(x-4)(x+1)}\,dx$

$$= \int \left(\frac{2}{x-4} + \frac{3}{x+1}\right) dx$$

$$= \int \frac{2}{x-4}\,dx + \int \frac{3}{x+1}\,dx$$

$$= 2 \ln|x-4|\,dx + 3\ln|x+1| + c$$

Ans.

Case II:- $Q(x)$ is a product of lineare factors, some of which are repeated

suppose that the first lineare factore (a_1x+b_1) is repeated r times; that is, $(a_1x+b_1)^r$ occurs in the factorcization of $Q(x)$. Then, instead of the single term $\dfrac{A_1}{a_1x+b_1}$ in equation ①, we would use

$$\frac{A_1}{a_1x+b_1} + \frac{A_2}{(a_1x+b_1)^2} + \cdots + \frac{A_r}{(a_1x+b_1)^r} \quad \ldots (ii)$$

Example:- Evaluate $\int \frac{4x}{x^3-x^2-x+1}\,dx$.

Solution:-

Let,

$$\frac{4x}{x^3-x^2-x+1} = \frac{4x}{(x-1)^2(x+1)} = \frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{C}{x+1} \quad \cdots (1)$$

Multiplying (1) by $(x-1)^2(x+1)$ we get,

$$4x = A(x-1)(x+1) + B(x+1) + C(x-1)^2 \quad (2)$$

When $x=1$, (2) implies, $4 = 2B$

$\Rightarrow B = 2$

When $x=-1$, (2) implies, $-4 = 4C$

$\Rightarrow C = -1$

From (2),

$$4x = A(x^2-1) + B(x+1) + C(x^2-2x+1)$$

$$\Rightarrow 4x = (A+C)x^2 + (-A+B+C) + (B-2C)x \quad \cdots (2)$$

Equating the coefficients from (2),

$$A+C = 0$$

$$B-2C = 4$$

$$-A+B+C = 0$$

Solving $A=1$, $B=2$, $C=-1$.

Therefore,

$$\int \frac{4x}{x^3-x^2-x+1}\,dx = \int \frac{4x}{(x-1)^2(x+1)}\,dx$$

$$= \int \frac{1}{x-1}\,dx + \int \frac{2}{(x-1)^2}\,dx - \int \frac{1}{x+1}\,dx$$

$$= \ln|x-1| - \frac{2}{x-1} - \ln|x+1| + c$$

Ans.

Case III :- $Q(x)$ contains irreducible quadratic, none of which is repeated (Quadratic factor Rule)

If $Q(x)$ has the factor ax^2+bx+c, where $b^2-4ac<0$, in addition to the partial fractions in Equations (i) and (ii), the expression for $\frac{R(x)}{Q(x)}$ will have a term of the form.

$$\frac{Ax+B}{ax^2+bx+c} \quad \ldots \text{(iii)}$$

Example :- Evaluate $\int \frac{x^2+x-2}{3x^3-x^2+3x-1}\,dx$

Solution :- To evaluate the integral we first split the rational function into partial fractions.

~~Sol~~, Here,

$$\frac{x^2+x-2}{3x^3-x^2+3x-1} = \frac{x^2+x-2}{x^2(3x-1)+1(3x-1)} = \frac{x^2+x-2}{(3x-1)(x^2+1)}$$

Let,

$$\frac{x^2+x-2}{(3x-1)(x^2+1)} = \frac{A}{3x-1} + \frac{Bx+C}{x^2+1} \quad \cdots (1)$$

Multiplying (1) by $(3x-1)(x^2+1)$ we get,

$$x^2+x-2 = A(x^2+1) + (Bx+C)(3x-1)$$

$$\Rightarrow x^2+x-2 = (A+3B)x^2 + (3C-B)x + (A-C) \quad \cdots (2)$$

Equating the coefficients from (2) we get

$$A+3B = 1$$

$$3C-B = 1$$

$$A-C = 2$$

Solving, $A = \frac{-7}{5}$, $B = \frac{4}{5}$, $C = \frac{3}{5}$.

Therefore,

$$\int \frac{x^2+x-2}{(3x-1)(x^2+1)}\,dx = \int \frac{-7/5}{3x-1}\,dx + \int \frac{\frac{4}{5}x+\frac{3}{5}}{x^2+1}\,dx$$

$$= \frac{-7}{15}\ln|3x-1| + \frac{2}{5}\ln|x^2+1| + \frac{3}{5}\tan^{-1}x + c$$

Ans

Case IV :- $Q(x)$ contains a repeated irreducible factor

If $Q(x)$ has the factor $(ax^2+bx+c)^r$ where $b^2-4ac<0$, then instead of the single partial fraction (iii), the sum

$$\frac{A_1x+B_1}{ax^2+bx+c}+\frac{A_2x+B_2}{(ax^2+bx+c)^2}+\cdots+\frac{A_rx+B_r}{(ax^2+bx+c)^r} \quad \cdots \text{(iv)}$$

occurs in the partial fraction decomposition of $\frac{R(x)}{Q(x)}$. Each of the terms in (iv) can be integrated by using a substitution of by first completing the square if necessary.

Example :- Evaluate the integral $\int \frac{3x^4+4x^3+16x^2+20x+9}{(x+2)(x^2+3)^2}dx$.

Solution :- From the given integrand, the denominator is $(x+2)(x+3)^2$. By the linear factor rule, the factor $x+2$ introduces the single term $\frac{A}{x+2}$. By the quadratic factor rule, the factor $(x^2+3)^2$ introduces two terms: $\frac{Bx+C}{x^2+3}+\frac{Dx+E}{(x^2+3)^2}$.

Let,

$$\frac{3x^4+4x^3+16x^2+20x+9}{(x+2)(x^2+3)^2}=\frac{A}{x+2}+\frac{Bx+C}{x^2+3}+\frac{Dx+E}{(x^2+3)^2} \quad \cdots (1)$$

Multiplying ① by $(x+2)(x+3)^2$ we get,

$$3x^4+4x^3+16x^2+20x+9 = A(x^2+3)^2+(Bx+C)(x+2)(x^2+3) + (Dx+E)(x+2)$$

$$\Rightarrow 3x^4+4x^3+16x^2+20x+9 = A(x^4+6x^2+9) + (Bx^2+Cx+2Bx+2C)(x^2+3) + (Dx^2+Ex+Dx+2E)$$

$$\Rightarrow 3x^4+4x^3+16x^2+20x+9 = (A+B)x^4+(2B+C)x^3 + (6A+3B+2C+D)x^2+(6B+3C+2D+E)x+9A+6C+2E \quad \cdots ②$$

Equating the power-like coefficients in ② we get,

$$\left.\begin{aligned} A+B &= 3 \\ 2B+C &= 4 \\ 6A+3B+2C+D &= 16 \\ 6B+3C+2D+E &= 20 \\ 9A+6C+2E &= 9 \end{aligned}\right\} \cdots ③$$

By solving ③, we get

$A=1,\ B=2,\ C=0,\ D=4,\ E=0$

Thus, $$\frac{3x^4+4x^3+16x^2+20x+9}{(x+2)(x^2+3)^2} = \frac{1}{x+2}+\frac{2x}{x^2+3}+\frac{4x}{(x^2+3)^2}$$

Thereforce,

$$\cancel{\int \frac{3x^4+3x^3-5x^2+x-1}{x^2+x-2}\,dx}$$

$$\int \frac{3x^4+4x^3+16x^2+20x+9}{(x+2)(x^2+3)^2}\,dx$$

$$= \int \frac{1}{x+2}\,dx + \int \frac{2x}{x^2+3}\,dx + \int \frac{4x}{(x^2+3)^2}\,dx$$

$$= \ln|x+2| + \ln|x^2+3| + 2\int \frac{du}{u^2}$$

Let,
$u = x^2+3$
$\Rightarrow du = 2x\,dx$

$$= \ln|x+2| + \ln|x^2+3| - \frac{2}{u} + c$$

$$= \ln|x+2| + \ln|x^2+3| - \frac{2}{x^2+3} + c$$

Ans.

Example :- Evaluate $\displaystyle\int \frac{3x^4+3x^3-5x^2+x-1}{x^2+x-2}\,dx$

Solution :- Since the intagrand is an improper rational function, we perform the long division

$$
\begin{array}{r|l|l}
x^2+x-2 & 3x^4+3x^3-5x^2+x-1 & 3x^2+1 \\
 & \underline{3x^4+3x^3-6x^2} & \\
 & \qquad x^2+x-1 & \\
 & \qquad \underline{x^2+x-2} & \\
 & \qquad\qquad 1 &
\end{array}
$$

It follows that.

$$\frac{3x^4+3x^3-5x^2+x-1}{x^2+x-2} = 3x^2+1+\frac{1}{x^2+x-2}$$

$$= 3x^2+1+\frac{1}{(x+2)(x-1)}$$

Let,

$$\frac{1}{(x+2)(x-1)} = \frac{A}{x+2} + \frac{B}{x-1} \quad \cdots (1).$$

Multiplying (1) by $(x+2)(x-1)$ we get,

$$1 = A(x-1) + B(x+2) \quad \cdots (2)$$

When $x=1$, (2) implies $1 = 3B \Rightarrow B = \frac{1}{3}$

When $x=-2$, (2) implies $1 = -3A \Rightarrow A = \frac{-1}{3}$.

$$\frac{1}{(x+2)(x-1)} = \frac{-1/3}{x+2} + \frac{1/3}{x-1}$$

Therefore,

$$\int \frac{3x^4+3x^3-5x^2+x-1}{x^2+x-2}\,dx$$

$$= \int \left(3x^2+1 - \frac{1/3}{x+2} + \frac{1/3}{x-1}\right) dx$$

$$= x^3+x - \frac{1}{3}\ln|x+2| + \frac{1}{3}\ln|x-1| + c$$

Ans.

Practice Problem

Chapter 7·5 → 9–34

Homework:

1. $$\int \frac{6y-7}{(2y+1)(4y^2+1)}\,dy$$

2. $$\int \frac{7x+2x^2}{(x-4)(2x+3)(2x+1)}\,dx$$

3. $$\int \frac{4x^3-x}{x^2-x-30}\,dx$$

4. $$\int \frac{x^6-6x^5+3x^4-10x^3-9x^2+12x-27}{x^4+3x^2}\,dx$$

5. $$\int \frac{x^2+x+1}{(x^2+1)^2}\,dx$$

6. $$\int \frac{8-t^3}{(t-3)(t+1)^2}\,dt$$