**Reduction Formula :-** Integration by reduction formula helps to solve the powers of elementary functions, polynomials of arbitrary degree, products of transcendental functions and the functions that can not be integrated easily, thus, easing the process of integration and its problems. It can express an integral involving a power of a function in terms of an integral that involves a lower power of that function.

**Example :-** Obtain the reduction formula for $\int \sin^n x\,dx$ and evaluate $\int \sin^6 x\,dx$.

**Solution :-** Let,

$$I_n = \int \sin^n x\,dx$$

$$= \int \sin^{n-1} x \sin x\,dx$$

Let,

$u = \sin^{n-1} x$

$v = \sin x$

$$= \sin^{n-1} x \int \sin x\,dx - \int \left\{ \frac{d}{dx} \left( \sin^{n-1} x \right) \int \sin x\,dx \right\} dx$$

$$= -\sin^{n-1} x \cos x + \int (n-1) \sin^{n-2} x \cos^2 x\,dx$$

$$= -\sin^{n-1} x \cos x + (n-1) \int \sin^{n-2} x \left(1 - \sin^2 x\right) dx$$

$$= -\sin^{n-1}x\cos x + (n-1)\int \sin^{n-2}x\,dx - (n-1)\int \sin^n x\,dx$$

$$= -\sin^{n-1}x\cos x + (n-1)I_{n-2} - (n-1)I_n$$

Thercefore,

$$I_n = -\sin^{n-1}x\cos x + (n-1)I_{n-2} - (n-1)I_n$$

$$\Rightarrow I_n + (n-1)I_n = -\sin^{n-1}x\cos x + (n-1)I_{n-2}$$

$$\Rightarrow nI_n = -\sin^{n-1}x\cos x + (n-1)I_{n-2}$$

$$\Rightarrow I_n = -\frac{1}{n}\sin^{n-1}x\cos x + \frac{n-1}{n}I_{n-2}$$

$$\Rightarrow \int \sin^n x\,dx = -\frac{1}{n}\sin^{n-1}x\cos x + \frac{n-1}{n}I_{n-2} + c$$

Now, $\int \sin^6 x\,dx$

$$= -\frac{1}{6}\sin^5 x\cos x + \frac{5}{6}I_4$$

$$= -\frac{1}{6}\sin^5 x\cos x + \frac{5}{6}\left[-\frac{1}{4}\sin^3 x\cos x + \frac{3}{4}I_2\right]$$

$$= -\frac{1}{6}\sin^5 x\cos x - \frac{5}{24}\sin^3 x\cos x + \frac{5}{8}\left[-\frac{1}{2}\sin x\cos x + \frac{1}{2}I_0\right]$$

$$= -\frac{1}{6}\sin^5 x\cos x - \frac{5}{24}\sin^3 x\cos x - \frac{5}{16}\sin x\cos x + \frac{5}{16}\int dx$$

$$= -\frac{1}{6}\sin^5 x\cos x - \frac{5}{24}\sin^3 x\cos x - \frac{5}{16}\sin x\cos x + \frac{5}{16}x + c$$

Ans.

<u>Example:-</u> Evaluate $\int x^n \sin x \, dx$

<u>Solution:-</u> Let,

$$I_n = \int x^n \sin x dx$$

Let,

$u = x^n$

$v = \sin x$

$$\Rightarrow I_n = x^n \int \sin x dx - \int \left\{ \frac{d}{dx}(x^n) \int \sin x dx \right\} dx$$

$$\Rightarrow I_n = -x^n \cos x + \int n x^{n-1} \cos x dx$$

$$\Rightarrow I_n = -x^n \cos x + n \int x^{n-1} \cos x dx$$

$$\Rightarrow I_n = -x^n \cos x + n \left[ x^{n-1} \int \cos x dx - \int \left\{ \frac{d}{dx}(x^{n-1}) \int \cos x dx \right\} dx \right]$$

$$\Rightarrow I_n = -x^n \cos x + n x^{n-1} \sin x - n(n-1) \int x^{n-2} \sin x dx$$

$$\Rightarrow I_n = -x^n \cos x + n x^{n-1} \sin x - n(n-1) I_{n-2} + C$$

Ans.

<u>Homework:-</u> Obtain the reduction formula for the following functions:-

i) $\cos^n x dx$  ii) $\sec^n x dx$  iii) $\tan^n x dx$

iv) $\cot^n x dx$  v) $\csc^n x dx$  vi) $\int x^n dx$

# Trigonometric Integral :-

Suppose we have to evaluate $\int \sin^m x \cos^n x \, dx$ where $m$ and $n$ are integers.

1. $n$ odd $\rightarrow$ a) Split off ~~the term~~ a factor of $\cos x$

   b) Apply the identity $\cos^2 x = 1 - \sin^2 x$

   c) Make the substitution $u = \sin x$.

**Example :-** Evaluate $\int \sin^4 x \cos^5 x \, dx$

**Solution :-** $\int \sin^4 x \cos^5 x \, dx$

Let,

$u = \sin x$

$\Rightarrow du = \cos x \, dx$

$$= \int \sin^4 x \cos^4 x \cdot \cos x \, dx$$

$$= \int \sin^4 x (\cos^2 x)^2 \cos x \, dx$$

$$= \int \sin^4 x (1 - \sin^2 x)^2 \cos x \, dx$$

$$= \int u^4 (1 - u^2)^2 \, du$$

$$= \int u^4 (1 - 2u^2 + u^4) \, du$$

$$= \int (u^4 - 2u^6 + u^8) \, du$$

$$= \frac{u^5}{5} - \frac{2u^7}{7} + \frac{u^9}{9} + c = \frac{\sin^5 x}{5} - \frac{2\sin^7 x}{7} + \frac{\sin^9 x}{9} + c$$

Ans.

2. $m$ odd $\rightarrow$ a) Split off a factor of $\sin x$

b) Apply the identity $\sin^2 x = 1 - \cos^2 x$

c) Make the substitution $u = \cos x$

Example: Evaluate $\int \sin^5 x \cos^4 x \, dx$.

Solution: $\int \sin^5 x \cos^4 x \, dx$

Let,

$u = \cos x$

$\Rightarrow du = -\sin x \, dx$

$\Rightarrow -du = \sin x \, dx$

$= \int \sin^4 x \cos^4 x \sin x \, dx$

$= \int (\sin^2 x)^2 \cos^4 x \sin x \, dx$

$= \int (1 - \cos^2 x)^2 \cos^4 x \sin x \, dx$

$= \int (1 - u^2)^2 u^4 (-du)$

$= -\int (1 - 2u^2 + u^4) u^4 \, du$

$= -\int (u^4 - 2u^6 + u^8) \, du$

$= -\dfrac{u^9}{9} + \dfrac{2u^7}{7} - \dfrac{u^5}{5} + c$

$= \dfrac{-\cos^9 x}{9} + \dfrac{2\cos^7 x}{7} - \dfrac{\cos^5 x}{5} + c$

Ans.

3. $m$ even, $n$ even $\rightarrow$ a) Use the relevant identities

$$\sin^2 x = \frac{1}{2}(1-\cos 2x)$$

$$\cos^2 x = \frac{1}{2}(1+\cos 2x)$$

to reduce the power of $\sin x$ and $\cos x$

b) Then use the reduction formula.

Example: Evaluate $\int \sin^4 x \cos^4 x \, dx$

Solution: $\int \sin^4 x \cos^4 x \, dx$

$$= \int (\sin^2 x)^2 (\cos^2 x)^2 \, dx$$

$$= \int \left\{\frac{1}{2}(1-\cos 2x)\right\}^2 \left\{\frac{1}{2}(1+\cos 2x)\right\}^2 dx$$

$$= \frac{1}{16}\int \left\{(1-\cos 2x)(1+\cos 2x)\right\}^2 dx$$

$$= \frac{1}{16}\int (1-\cos^2 2x)^2 \, dx$$

Let $u = 2x$

$\Rightarrow \frac{du}{2} = dx$

$$= \frac{1}{16}\int (\sin^2 2x)^2 \, dx$$

$$= \frac{1}{16}\int \sin^4 2x \, dx$$

$$= \frac{1}{32}\int \sin^4 u \, du$$ = Apply Reduction Formula.

Ans

# Integrating Products of Tangents and Secants

If $m$ and $n$ are positive integers then the integral $\int \tan^m x \sec^n x\, dx$ can be evaluated in the following way :-

1. $n$ even $\rightarrow$ 
   a) Split off a factor of $\sec^2 x$
   b) Apply the identity $\sec^2 x = 1 + \tan^2 x$.
   c) Make the substitution $u = \tan x$.

**Example:-** Evaluate $\int \tan^5 x \sec^4 x\, dx$

**Solution:-** $\int \tan^5 x \sec^4 x\, dx$

Let, $u = \tan x$

$\Rightarrow du = \sec^2 x\, dx$

$$= \int \tan^5 x \sec^2 x \sec^2 x\, dx$$

$$= \int \tan^5 x (1 + \tan^2 x) \sec^2 x\, dx$$

$$= \int \tan^5 x \sec^2 x\, dx + \int \tan^7 x \sec^2 x\, dx$$

$$= \int u^5 du + \int u^7 du$$

$$= \frac{u^6}{6} + \frac{u^8}{8} + c$$

$$= \frac{\tan^6 x}{6} + \frac{\tan^8 x}{8} + c \quad \text{Ans.}$$

2. m odd → 
a) Split off a factor of $\sec x \tan x$
b) Apply the identity $\tan^2 x = \sec^2 x - 1$
c) Make the substitution $u = \sec x$.

Example :- Evaluate $\int \tan^5 x \sec^3 x \, dx$

Solution :- $\int \tan^5 x \sec^3 x \, dx$

$= \int \tan^4 x \sec^2 x \sec x \tan x \, dx$

Let $u = \sec x$

$\Rightarrow du = \sec x \tan x \, dx$

$= \int (\tan^2 x)^2 \sec^2 x \sec x \tan x \, dx$

$= \int (\sec^2 x - 1)^2 \sec^2 x \sec x \tan x \, dx$

$= \int (u^2 - 1)^2 u^2 \, du$

$= \int (u^4 - 2u^2 + 1) u^2 \, du$

$= \int (u^6 - 2u^4 + u^2) \, du$

$= \frac{u^7}{7} - \frac{2u^5}{5} + \frac{u^3}{3} + c$

$= \frac{\sec^7 x}{7} - \frac{2\sec^5 x}{5} + \frac{\sec^3 x}{3} + c$

Ans.

3. $m$ even $\rightarrow$ 1. Use $\tan^2 x = \sec^2 x - 1$ to reduce the integrand to the powers of $\sec x$ alone.
$n$ odd

2. Then use the reduction formula for the power of $\sec x$.

Example :- Evaluate $\int \tan^6 x \sec^3 x \, dx$

Solution :- $\int \tan^6 x \sec^3 x \, dx$

$$= \int (\tan^2 x)^3 \sec^3 x \, dx$$

$$= \int (\sec^2 x - 1)^3 \sec^3 x \, dx$$

$$= \int (\sec^6 x - 3\sec^4 x + 3\sec^2 x - 1) \sec^3 x \, dx$$

$$= \int \sec^9 x \, dx - \int 3\sec^7 x \, dx + 3\int \sec^5 x - \int \sec^3 x \, dx$$

= Apply the reduction formula.

Ans.

4) Integrals of the form $\int \sin mx \cos nx \, dx$, $\int \sin mx \sin nx \, dx$, $\int \cos mx \cos nx \, dx$ can be found by using the trigonometric identities.

$$\sin\alpha \cos\beta = \frac{1}{2}\left[\sin(\alpha-\beta) + \sin(\alpha+\beta)\right]$$

$$\sin\alpha \sin\beta = \frac{1}{2}\left[\cos(\alpha-\beta) - \cos(\alpha+\beta)\right]$$

$$\cos\alpha \cos\beta = \frac{1}{2}\left[\cos(\alpha-\beta) + \cos(\alpha+\beta)\right]$$

Example: Evaluate $\int \sin 7x \cos 3x \, dx$.

Solution: $\int \sin 7x \cos 3x \, dx$

$$= \frac{1}{2}\int \left[\sin(7x+3x) + \sin(7x-3x)\right] dx$$

$$= \frac{1}{2}\int (\sin 10x + \sin 4x) \, dx$$

$$= -\frac{1}{20}\cos 10x - \frac{1}{8}\cos 4x + c$$

Practice Problem:

Chapter $7.3 \rightarrow 1-52$.

Homework :-

1. $\int \sin^3 a\theta \, d\theta$

2. $\int \operatorname{cosec}^4 x \, dx$

3. $\int \cot^3 x \, dx$

4. $\int \cos^{1/3} x \sin x \, dx$

5. $\int \sqrt{\tan x} \sec^4 x \, dx$

6. $\int \sec^{3/2} x \tan x \, dx$

7. $\int \cot^2 3t \sec 3t \, dt$