Indefinite Integral :- If the interval of integration is not defined then we get an arbitrary constant after completing the integration. This type of integral is called indefinite integral.

$$\int f(x)\,dx = F(x) + C$$

Definite Integral :- If the interval of integration is defined then the arbitrary constant gets vanished. This type of integral is called definite integral.

$$\int_a^b f(x)\,dx = F(b) - F(a)$$

Properties of Indefinite Integral :-

1. $\int k\,dx = kx + c$

2. $\int [f(x) \pm g(x)]\,dx = \int f(x)\,dx \pm \int g(x)\,dx$

3. $\int cf(x)\,dx = c\int f(x)\,dx$

Properties of Definite Integrals:-

1. $\int_a^b [f(x) \pm g(x)]\,dx = \int_a^b f(x)\,dx \pm \int_a^b g(x)\,dx$

2. $\int_a^a f(x)\,dx = 0$

3. $\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$

4. If f is integrable on $[a,b]$, then $\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$

5. If f is integrable on $[a,b]$ and $f(x) \geqslant 0$ for all $x \in [a,b]$, then $\int_a^b f(x)\,dx \geqslant 0$.

6. If f and g are integrable on $[a,b]$ and $f(x) \geqslant g(x)$ for all $x \in [a,b]$, then

$$\int_a^b f(x)\,dx \geqslant \int_a^b g(x)\,dx$$

Practice Problems:-

Chapter - 5.2 $\rightarrow$ 15-34

Continuity Implies Integrability :-

If f is continuous on the closed interval $[a, b]$, then $\int_a^b f(x)$ exists. That means f is integrable on the interval.

Let $y = \frac{1}{x^2}$. Then $\int y\,dx = \int_{-1}^{3} \frac{1}{x^2}\,dx$. The integrand is discontinuous at $x = 0$. So the function $f(x)$ is not integrable on this interval.

Example :-

① $\int \frac{x^3 - 5x^2}{x}\,dx$

$= \int (x^2 - 5x)\,dx$

$= \frac{x^3}{3} - \frac{5x^2}{2} + c$

② $\int (x^2 - 4)(2x + 3)\,dx$

$= \int (2x^3 - 8x + 3x^2 - 12)\,dx$

$= \frac{2x^4}{4} - \frac{8x^2}{2} + \frac{3x^3}{3} - 12x + c$

$= \frac{x^4}{2} - 4x^2 + x^3 - 12x + c$

③ $\int_0^{\pi/4} \sin(2x)\,dx$

$= \left[-\frac{1}{2}\cos 2x\right]_0^{\frac{\pi}{4}}$

$= \left[-\frac{1}{2}\cos 2\left(\frac{\pi}{4}\right) + \frac{1}{2}\cos(2 \times 0)\right]$

$= \frac{1}{2}$

④ $\int_0^{\pi} \frac{\sin x}{1 - \sin^2 x}\,dx$

$= \int_0^{\pi} \frac{\sin x}{\cos^2 x}\,dx$

$= \int_0^{\pi} \sec x \tan x\,dx$

$= [\sec x]_0^{\pi}$.

$= -1 + 1 = 0$.

Ans.

Integrals with Variable Limits :-

$$\int_{t-2}^{\sin t} (x^2-2x+3)\,dx = \left[\frac{x^3}{3} - x^2 + 3x\right]_{t-2}^{\sin t}$$

$$= \left[\frac{\sin^3 t}{3} - \sin^2 t + 3\sin t\right] - \left[\frac{(t-2)^3}{3} - (t-2)^2 + 3(t-2)\right]$$

$$= \frac{\sin^3 t}{3} - \sin^2 t + 3\sin t - \frac{t^3}{3} + 3t^2 - 11t + \frac{38}{3}$$

Logarithmic Integral :-

Natural logarithm of x is denoted by $\ln(x)$ and is defined by the integral

$$\ln x = \int_1^x \frac{1}{t}\,dt, \quad x>0.$$

Example:- $\displaystyle\int_2^4 \left(\frac{6+x^2}{x^3}\right) dx$

Solution:- $$\int_2^4 \left(\frac{6+x^2}{x^3}\right) dx = \int_2^4 \left(\frac{6}{x^3} + \frac{x^2}{x^3}\right) dx$$

$$= \int_2^4 \left(\frac{6}{x^3} + \frac{1}{x}\right) dx$$

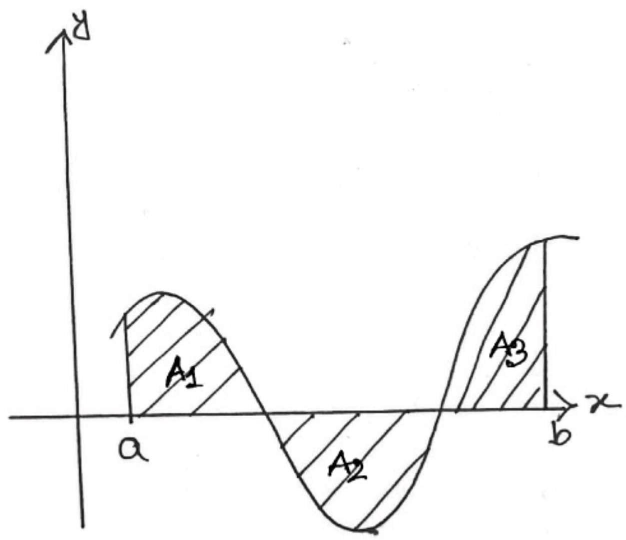
$$= \left[\frac{6x^{-2}}{-2} + \ln x\right]_2^4$$

$$= \frac{9}{16} + \ln(2)$$

Ans

Net Signed Area:-

In the previous topics, we assumed that f is continuous and nonnegative on the interval [a, b]. If f is continuous and attains both positive and negative values on [a,b], then it no longer represents the area between the curve $y = f(x)$ and the interval [a,b] on the x-axis; rather, it represents a difference of areas- the area of the region that is above the interval [a,b] and below the curve $y = f(x)$ minus the area of the region that is below the interval [a,b] and above the curve $y = f(x)$. We call this net signed area between the graph of $y = f(x)$ and the interval [a,b]



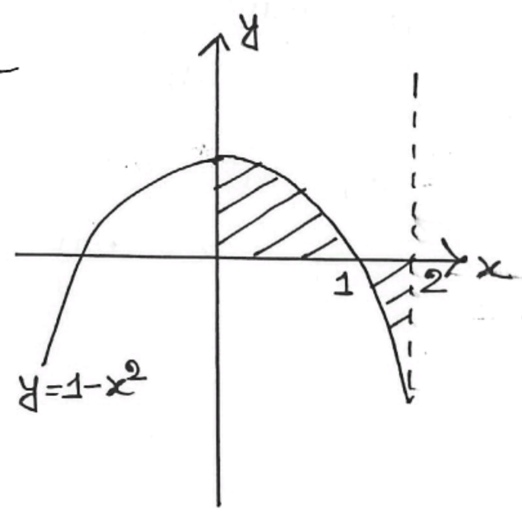
$$\text{Net signed Area} = (A_1 + A_3) - A_2$$

$$= \text{Area above } [a,b] - \text{Area below } [a,b]$$

Total Area: If f is continuous function on the interval $[a,b]$ then we define the total area between the curve $y=f(x)$ and the interval $[a,b]$ to be total area $\int_a^b |f(x)|dx$.

Example: Find the net signed area and total area between the curve $y=1-x^2$ and the x-axis over the interval $[0,2]$.

Solution: The given function is, $y=1-x^2$



$$\text{Net signed area} = \int_0^2 (1-x^2)dx$$

$$= \left[x-\frac{x^3}{3}\right]_0^2$$

$$= 2-\frac{8}{3} = -\frac{2}{3} \text{ units}^2$$

$$\text{Total area} = \int_0^2 |f(x)|\,dx \quad \text{where } |f(x)| = \begin{cases} 1-x^2, & 0 \leqslant x < 1 \\ -(1-x^2), & 1 \leqslant x < 2 \end{cases}$$

$$= \int_0^1 (1-x^2)\,dx + \int_1^2 -(1-x^2)\,dx$$

$$= \left[x-\frac{x^3}{3}\right]_0^1 + \left[-x+\frac{x^3}{3}\right]_1^2 = 2 \text{ units}^2$$

Homework :-

① $\int (7x^5 - 5x^4 + 6x^2 - 14x + 3)\, dx$

② $\int \left(w^5 + \sqrt{w^5} - \sqrt[5]{w}\right) dw$

③ $\int \left(\sec^2 u + 7 \sec u \tan u\right) du$

④ $\int \dfrac{\cos^3(v) + \sin(v)}{\cos^2(v)}\, dv$

⑤ $\int_{-123}^{123} \left[\cos^6(2x) - \sin^8(4x)\right] dx$

⑥ $\int_{1}^{6} |x-3|\, dx$

~~⑦~~ $\cancel{\int_{2}^{x^2} \sqrt{\cos t + 3}\, dt}$

⑦ $\int_{0}^{\frac{2\pi}{3}} \left[\sin(y) + \sec^2(y)\right] dy$

⑧ $\int_{4}^{1} \sqrt{x}\,(x - 2x^2 + 1)\, dx$

⑨ $\int_{0}^{1/2} \left(\dfrac{3}{\sqrt{1-x^2}} + \dfrac{7}{1+x^2}\right) dx$

⑩ $\int_{-4}^{-1} f(t)\, dt$ where $f(t) = \begin{cases} 9 + 6t^2, & t > -3 \\ 8t, & t \le -3 \end{cases}$

Homework:-

1. Estimate the net area between the function and the x-axis on the given interval using the midpoints of the subintervals for the height of the rectangles. Without looking at a graph of the function on the interval does it appear that more of the area is above or below the x-axis?

a) $h(x) = 8x - \sqrt{x+4}$ on $[-3, 2]$

b) $g(x) = 5 + x - x^2$ on $[0, 4]$

c) $f(x) = xe^{-x^2}$ on $[-1, 1]$

2. Find the net signed area and the total area under the curve $y = x - 2$ and over the interval $[0, 6]$

3. Find the net signed area and the total area under the curve $y = \sin x$ and over the interval $[0, 2\pi]$.

Fundamental Theorem of Calculus (Part I):-

If f is continuous on $[a,b]$ and F is any antiderivative of f on $[a,b]$, then

$$\int_a^b f(x)dx = F(b)-F(a) .$$

Fundamental Theorem of Calculus (Part II):-

Let f be a continuous function on an interval I and let 'a' be any point on I. If the function F is defined by

$$F(x) = \int_a^x f(t)dt$$

then F is an antiderivative of f on I i.e. $F'(x) = f(x)$ for each x in I or in alternative form

$$\frac{d}{dx}\left[\int_a^x f(t)\,dt\right] = f(x) .$$

Example:- Find $g'(x)$ using first part of the fundamental theorem of calculus where $g(x) = \int_\pi^x (1-\cos t)dt$.

Solution:- $g(x) = \int_\pi^x (1-\cos t)\,dt$

$$= \left[t - \sin t\right]_\pi^x$$

$$= x - \sin x - \pi + \sin\pi = x - \sin x - \pi .$$

Therefore, $g'(x) = 1 - \cos x$