Improper Integral

It is assumed ~~that~~ in the definition of definite integral $\int_a^b f(x)\,dx$ that $[a, b]$ is a finite interval and that the limit that defines the integral exists i.e. the function f is integrable. We know that continuous functions are integrable. The functions that are not bounded on the interval of integration are not integrable. We call integrals with infinite interval of integration or infinite discontinuities within the interval of integration improper integrals.

□ Improper integrals with infinite intervals of integration:-

$$\int_1^{+\infty} \frac{dx}{x^2}, \quad \int_{-\infty}^{0} e^x\,dx, \quad \int_{-\infty}^{+\infty} \frac{dx}{1+x^2}$$

□ Improper integrals with infinite discontinuities in the interval of integration

$$\int_{-3}^{3} \frac{dx}{x^2}, \quad \int_1^2 \frac{dx}{x-1}, \quad \int_0^{\pi} \tan x\,dx$$

Improper integrals with infinite discontinuities and infinite intervals of integration

$$\int_0^{+\infty} \frac{dx}{\sqrt{x}}, \quad \int_{-\infty}^{\infty} \frac{dx}{x^2-9}, \quad \int_1^{+\infty} \sec x\,dx$$

Example: Evaluate $\int_1^{+\infty} \frac{dx}{x^3}$.

Solution: $\int_1^{+\infty} \frac{dx}{x^3}$

$$= \lim_{b\to+\infty} \int_1^b \frac{dx}{x^3} = \lim_{b\to+\infty} \left[\frac{-1}{2x^2}\right]_1^b$$

$$= \lim_{b\to+\infty} \left[-\frac{1}{2b^2} + \frac{1}{2}\right]$$

$$= \frac{1}{2}.$$

Since the limit is finite the integral converges and its value is $\frac{1}{2}$.

Practice Problem:

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Example :- Evaluate $\int_{-\infty}^{\infty} \frac{dx}{1+x^2}$.

Solution :- $\int_{-\infty}^{+\infty} \frac{dx}{1+x^2}$

$$= \lim_{a \to -\infty} \int_{a}^{0} \frac{dx}{1+x^2} + \lim_{b \to +\infty} \int_{0}^{b} \frac{dx}{1+x^2}$$

$$= \lim_{a \to -\infty} \left[\tan^{-1} x \right]_{a}^{0} + \lim_{b \to +\infty} \left[\tan^{-1} x \right]_{0}^{b}$$

$$= \lim_{a \to -\infty} \left[\tan^{-1}(0) - \tan^{-1}(a) \right] + \lim_{b \to +\infty} \left[\tan^{-1}(b) - \tan^{-1}(0) \right]$$

$$= 0 + \frac{\pi}{2} + \frac{\pi}{2} - 0 = \pi .$$

Thereafore, the integral is said to converge.

Ans.

Example :- Evaluate $\int_{0}^{1} \frac{dx}{\sqrt{1-x}}$.

Solution :- $\int_{0}^{1} \frac{dx}{\sqrt{1-x}}$

$$= \lim_{k \to 1^-} \int_{0}^{k} \frac{dx}{\sqrt{1-x}} = \lim_{k \to 1^-} \left[-2\sqrt{1-x} \right]_{0}^{k}$$

$$= \lim_{k \to 1^-} \left[-2\sqrt{1-k} + 2 \right]$$

$$= 2. \quad \text{Ans.}$$

Example: Evaluate $\int_1^2 \frac{dx}{1-x}$.

Solution: $\int_1^2 \frac{dx}{1-x}$

$$= \lim_{k\to 1^+} \int_k^2 \frac{dx}{1-x} = \lim_{k\to 1^+} \Big[-\ln|1-x|\Big]_k^2$$

$$= \lim_{k\to 1^+} \Big[-\ln|-1| + \ln|1-k|\Big]$$

$$= \lim_{k\to 1^+} \ln|1-k|$$

$$= -\infty$$

Ans.

Example: Evaluate $\int_1^4 \frac{dx}{(x-2)^{2/3}}$

Solution: $\int_1^4 \frac{dx}{(x-2)^{2/3}}$

$$= \int_1^2 \frac{dx}{(x-2)^{2/3}} + \int_2^4 \frac{dx}{(x-2)^{2/3}}$$

$$= \lim_{k\to 2^-} \int_1^k \frac{dx}{(x-2)^{2/3}} + \lim_{k\to 2^+} \int_k^4 \frac{dx}{(x-2)^{2/3}}$$

$$= \lim_{k\to 2^-} \Big[3(k-2)^{1/3} - 3(1-2)^{1/3}\Big] + \lim_{k\to 2^+} \Big[3(4-2)^{1/3} - 3(k-2)^{1/3}\Big]$$

$$= 3 + 3\sqrt[3]{2}.$$

Ans.

Homework :-

1. $\displaystyle\int_0^\infty \frac{2x}{1+x^4}\,dx$

2. $\displaystyle\int_{-\infty}^{\infty} x^2 e^{-x^3}\,dx$

3. $\displaystyle\int_1^\infty \frac{\ln(x)}{x^3}\,dx$

4. $\displaystyle\int_6^\infty \frac{1}{w-1}\,dw$

5. $\displaystyle\int_0^\infty \frac{e^{1/x}}{x^2}\,dx$

6. $\displaystyle\int_{-\infty}^{\infty} \frac{y}{(y^2+1)^3}\,dy$

7. $\displaystyle\int_{-2}^{1} \frac{e^{1/x}}{x^2}\,dx$