Change to Polar Coordinates in Double Integral

If f is continuous on a polar rectangle R given by $0 \leq a \leq r \leq b$, $\alpha \leq \theta \leq \beta$, where $0 \leq \beta - \alpha \leq 2\pi$, then

$$\iint_R f(x,y)\,dA = \int_\alpha^\beta \int_a^b f(r\cos\theta, r\sin\theta)\, r\,dr\,d\theta$$

In polar coordinates,

$$x = r\cos\theta,$$

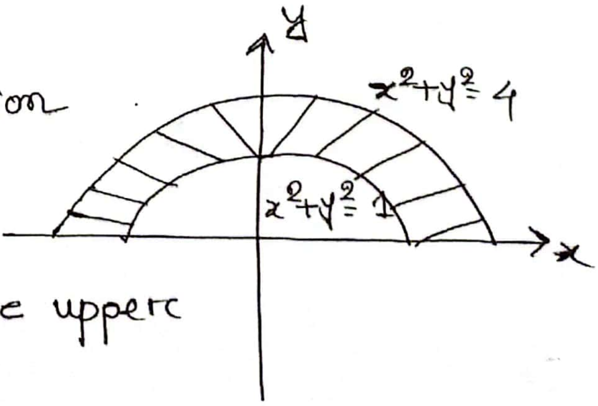
$$y = r\sin\theta$$

$$x^2 + y^2 = r^2 \quad \text{and} \quad \frac{\partial(x,y)}{\partial(r,\theta)} = r$$

Be careful not to forget the additional factor r on the right side of the formula.

Example:- Evaluate $\iint_R (3x+4y^2)\,dA$ where R is the region in the upper half plane bounded by the circles $x^2+y^2=1$ and $x^2+y^2=4$.

Solution:- The given region is in between the circles $x^2+y^2=1$ and $x^2+y^2=4$ in the upper half plane.



Since, $x^2+y^2=r^2$ in the polar coordinates.

therefore,

$$x^2+y^2=1 \qquad \text{and} \qquad x^2+y^2=4$$

$$\Rightarrow r^2=1 \qquad\qquad \Rightarrow r^2=4$$

$$\Rightarrow r=1 \qquad\qquad \Rightarrow r=2$$

Since the region is in the upper half plane then the angle changes from 0 to π.

Therefore, $$\iint_R (3x+4y^2)\,dA = \int_0^{\pi}\int_1^2 (3r\cos\theta + 4r^2\sin^2\theta)\,r\,dr\,d\theta$$

$$= \int_0^{\pi}\int_1^2 (3r^2\cos\theta + 4r^3\sin^2\theta)\,dr\,d\theta$$

$$= \int_0^{\pi} \Big[r^3\cos\theta + r^4\sin^2\theta\Big]_1^2 d\theta$$

$$= \int_0^{\pi} (8\cos\theta + 16\sin^2\theta - \cos\theta - \sin^2\theta)\,d\theta$$

$$= \int_0^{\pi} (7\cos\theta + 15\sin^2\theta)\,d\theta$$

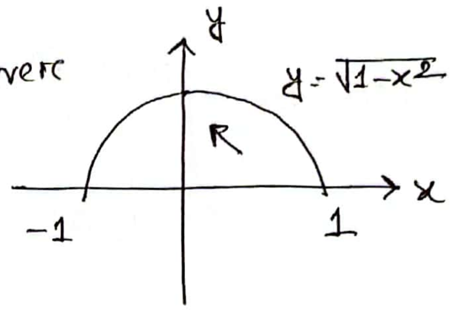
$$= \int_0^{\pi} \left(7\cos\theta + \frac{15}{2} - \frac{15}{2}\cos 2\theta\right)d\theta$$

$$= \frac{15\pi}{2}.$$ Ans.

Example: Evaluate the double integral

$$\int_{-1}^{1}\int_{0}^{\sqrt{1-x^2}} (x^2+y^2)\,dy\,dx.$$

Solution: This ~~region~~ iterated integral is a double integral over the region R shown in the figure and described by



$$R = \left\{(x,y) \mid -1 \leqslant x \leqslant 1,\ 0 \leqslant y \leqslant \sqrt{1-x^2}\right\}$$

The region is a half disk, so it is more simply described in polar coordinates:

$$R = \left\{(r,\theta) \mid 0 \leqslant \theta \leqslant \pi,\ 0 \leqslant r \leqslant 1\right\}$$

Therefore, we have

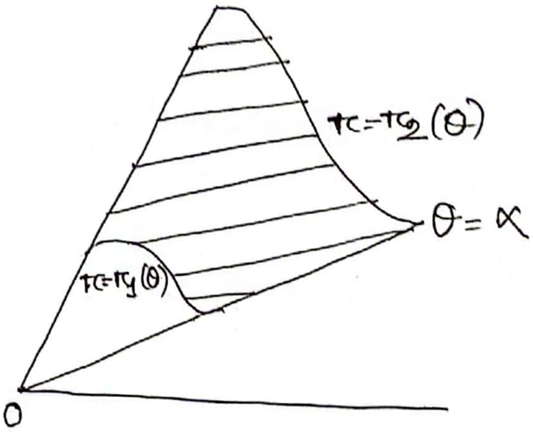
$$\int_{-1}^{1}\int_{0}^{\sqrt{1-x^2}} (x^2+y^2)\,dy\,dx = \int_{0}^{\pi}\int_{0}^{1} r^2 r\,dr\,d\theta.$$

$$= \int_{0}^{\pi}\int_{0}^{1} r^3\,dr\,d\theta$$

$$= \int_{0}^{\pi}\left[\frac{r^4}{4}\right] d\theta = \frac{\pi}{4} \quad \text{Ans.}$$

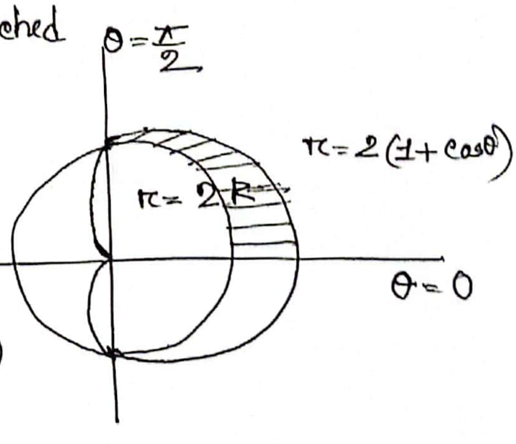
Theorem:– If R is a simple region in polar coordinates whose boundaries are the rays $\theta = \alpha$ and $\theta = \beta$ and the curves $r = r_1(\theta)$ and $r = r_2(\theta)$ shown in the figure and if $f(r,\theta)$ is continuous on R, then

$$\iint_R f(r,\theta)\, dA = \int_\alpha^\beta \int_{r_1(\theta)}^{r_2(\theta)} f(r,\theta)\, r\,dr\,d\theta .$$



Example:– Evaluate $\iint_R \sin\theta\, dA$ where R is the region in the first quadrant that is outside the circle $r=2$ and inside the Cardioid $r = 2(1+\cos\theta)$.

Solution:– The region R is sketched in the figure.



$$\iint_R \sin\theta\, dA = \int_0^{\pi/2} \int_2^{2(1+\cos\theta)} \sin(\theta)\, r\,dr\,d\theta$$

$$= \int_0^{\pi/2} \left[\tfrac{1}{2} r^2 \sin\theta \right]_2^{2(1+\cos\theta)} d\theta .$$

$$= \frac{8}{3}$$

Practice Problem: chapter 14·3 → 1-6, 7-10, 23-34

Homework :-

(1) Evaluate the following integrals in polar coordinates :-

a) $\int_0^1 \int_0^{\sqrt{1-x^2}} (x^2+y^2)\,dy\,dx$

b) $\int_0^a \int_0^{\sqrt{a^2-x^2}} \frac{dy\,dx}{(1+x^2+y^2)^{3/2}}$, $a>0$

c) $\int_0^2 \int_0^{\sqrt{2x-x^2}} \sqrt{x^2+y^2}\,dy\,dx$

d) $\int_{-a}^a \int_0^{\sqrt{a^2-x^2}} (x^2+y^2)^{1/2}\,dy\,dx$

e) $\iint_R \sin(x^2+y^2)\,dA$ where R is the region enclosed by the circle $x^2+y^2=9$.

f) $\iint_R 2y\,dA$, where R is the region in the first quadrant bounded above by the circle $(x-1)^2+y^2=1$ and below by the line $y=x$.

Triple Integrals in Cylindrical Coordinates

In cylindrical coordinates system, a point P in three dimensional space is represented by the ordered triple (r, θ, z). where r and θ are polar coordinates of the projection of P onto the xy-plane and z is directed distance from the xy-plane to P. To convert from cylindrical to rectangular coordinates we use

$$x = r\cos\theta, \quad y = r\sin\theta, \quad z = z$$

The Jacobian of cylindrical coordinates is defined as

$$J = \frac{\partial(x, y)}{\partial(r, \theta)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial z} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial z} \\ \frac{\partial z}{\partial r} & \frac{\partial z}{\partial \theta} & \frac{\partial z}{\partial z} \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta & 0 \\ \sin\theta & r\cos\theta & 0 \\ 0 & 0 & 1 \end{vmatrix}$$

$$= r\cos^2\theta + r\sin^2\theta$$

$$= r$$

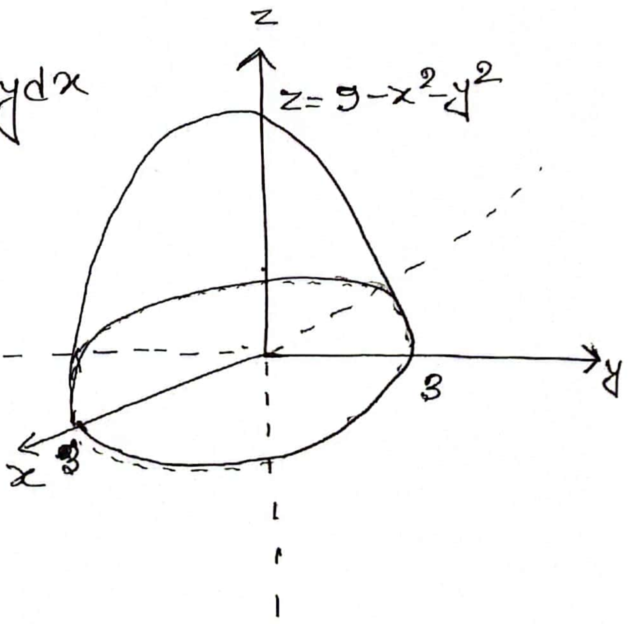
Sometimes a triple integral that is difficult to integrate in rectangular coordinates can be evaluated more easily by making the substitution $x = r\cos\theta$, $y = r\sin\theta$, $z = z$ to convert it to an integral in cylindrical coordinates. Under such a substitution, a rectangular triple integral can be expressed as an iterated integral in cylindrical coordinates as

$$\iiint_G f(x,y,z)\,dV = \underset{\substack{\text{appropriate} \\ \text{limits}}}{\iiint} f(r\cos\theta, r\sin\theta, z)\, r\,dz\,dr\,d\theta$$

It is worthwhile to use this formula when E is a solid region easily described in cylindrical coordinates and especially when the function $f(x,y,z)$ involves the expression x^2+y^2.

Example :- Evaluate $\displaystyle\int_{-3}^{3}\int_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}}\int_{0}^{9-x^2-y^2} x^2\,dz\,dy\,dx$.

Solution :- $\displaystyle\int_{-3}^{3}\int_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}}\int_{0}^{9-x^2-y^2} x^2\,dz\,dy\,dx$



$$= \int_0^{2\pi}\int_0^{3}\int_0^{9-r^2} r^2\cos^2\theta\, r\,dz\,dr\,d\theta$$

$$= \int_0^{2\pi}\int_0^{3} r^3\cos^2\theta\,\Big[z\Big]_0^{9-r^2}\,dr\,d\theta.$$

$$= \int_0^{2\pi}\int_0^{3} (9-r^2) r^3 \cos^2\theta \, dr \, d\theta$$

$$= \int_0^{2\pi}\int_0^{3} (9r^3 - r^5) \cos^2\theta \, \cancel{d\theta} \, dr \, d\theta$$

$$= \int_0^{2\pi} \left[\frac{9r^4}{4} - \frac{r^6}{6}\right]_0^3 \cos^2\theta \, d\theta$$

$$= \int_0^{2\pi} \frac{243}{4} \cos^2\theta \, d\theta$$

$$= \frac{243}{8} \int_0^{2\pi} (1 - \cos 2\theta) \, d\theta$$

$$= \frac{243}{8} \left[\theta - \frac{\sin 2\theta}{2}\right]_0^{2\pi}$$

$$= \frac{243}{8}(2\pi - 0) = \frac{243\pi}{4}$$

Ans.