Superposition Principle - Homogeneous Equations

Let $y_1, y_2, \dots, y_n$ be the solutions of the homogeneous nth order differential equation of the form

$$a_n(x)\frac{d^n y}{dx^n} + a_{n-1}(x)\frac{d^{n-1}y}{dx^{n-1}} + \dots + a_1(x)\frac{dy}{dx} + a_0(x)y = 0$$

Then the linear combination

$$y = c_1 y_1 + c_2 y_2 + \dots + c_n y_n$$

where the c_i's, $i = 1, 2, \dots, n$ are arbitrary constant is also a solution of the equation.

[Note] A set of two functions $f_1(x)$ and $f_2(x)$ is linearly independent when neither function is a constant multiple of the other on the interval.

Homogeneous Linear Equations with Constant Coefficients

We begin by considering the special case of second order equation

$$ay'' + by' + cy = 0 \quad \text{(1)}$$

The equation has constant coefficients a, b and c. Since $g(x) = 0$, it is homogeneous. There is no nonlinear term in the equation.

Let the trial solution of Equation (1) be of the form

$$y = e^{mx} \quad \text{(2)}$$

Substituting (2) in equation (1) we get

$$am^2e^{mx} + bme^{mx} + ce^{mx} = 0.$$

$$\Rightarrow am^2 + bm + c = 0 \quad \cdots \text{(3)}$$

This equation is called the auxiliary of the differential equation (1).

Solving (3) we get,

$$m = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

(1) If $b^2-4ac>0$, the roots are real and distinct.

(2) If $b^2-4ac=0$, the roots are real and equal.

(3) If $b^2-4ac<0$, the roots are complex conjugates.

The general solution of (1) is called complementary solution.

Distinct Real Roots

Under the assumption that the auxiliary equation has two unequal roots m_1 and m_2. We find the general solution as

$$y = c_1 e^{m_1 x} + c_2 e^{m_2 x}$$

Repeated Real Roots

When $m_1 = m_2$ we obtain the general solution as

$$y = c_1 e^{m_1 x} + c_2 x e^{m_1 x}$$

Conjugate Complex Roots

If m_1 and m_2 are complex, then $m_1 = \alpha + i\beta$
$m_2 = \alpha - i\beta$

Then the general solution is, $y = c_1 e^{(\alpha+i\beta)x} + c_2 e^{(\alpha-i\beta)x}$

$$= e^{\alpha x}(c_1 \cos\beta x + c_2 \sin\beta x)$$

Example:- Solve the following differential equations:-

a) $2y'' - 5y' - 3y = 0$

b) $y'' - 10y' + 25y = 0$

c) $4y'' + 4y' + 17y = 0$; $y(0) = -1$, $y'(0) = 2$

Solution:- a) Given that,

$$2y'' - 5y' - 3y = 0 \quad \cdots (1)$$

Let the trial solution be of the form

$$y = e^{mx} \quad \cdots (2)$$

Substituting (2) in equation (1) we get,

$$2m^2 e^{mx} - 5me^{mx} - 3e^{mx} = 0 .$$

$$\Rightarrow 2m^2 - 5m - 3 = 0 .$$

$$\Rightarrow m = \frac{5 \pm \sqrt{25 + 24}}{2 \cdot 2} = -\frac{1}{2}, 3 .$$

Therefore, the general solution is,

$$y = c_1 e^{-x/2} + c_2 e^{3x}$$

Ans.

b) Given that,

$$y'' - 10y' + 25y = 0 \quad \ldots (1)$$

Let the trial solution be of the form $y = e^{mx} \ldots (2)$

Substituting (2) in equation (1) we get,

$$m^2 e^{mx} - 10me^{mx} + 25e^{mx} = 0.$$

$$\Rightarrow m^2 - 10m + 25 = 0.$$

$$\Rightarrow m = \frac{10 \pm \sqrt{100 - 4.1.25}}{2.1}$$

$$\Rightarrow m = 5, 5$$

Therefore, the general solution is

$$y = c_1 e^{5x} + c_2 x e^{5x}$$

Ans.

c) Given that,

$$4y'' + 4y' + 17y = 0 \quad \ldots (1)$$

Let the trial solution be of the form $y = e^{mx} \ldots (2)$

Substituting (2) in equation (1) we get.

$$4m^2 e^{mx} + 4me^{mx} + 17e^{mx} = 0.$$

$$\Rightarrow 4m^2 + 4m + 17 = 0$$

$$\Rightarrow m = \frac{-4 \pm \sqrt{16-4\cdot4\cdot17}}{2\cdot4}$$

$$\Rightarrow m = -\frac{1}{2}+2i, \ -\frac{1}{2}-2i$$

Comparing the solution with $m = \alpha + i\beta$ we get,

$$\alpha = -\frac{1}{2}, \ \beta = 2.$$

Therefore, the general solution is

$$y = e^{-1/2x}\left(c_1 \cos 2x + c_2 \sin 2x\right)$$

From the condition $y(0) = -1$.

$$\Rightarrow c_1 + c_2 \times 0 = -1.$$

$$\Rightarrow c_1 = -1$$

and $y'(0) = 2$.

$$\Rightarrow -\frac{1}{2}c_1 + 2c_2 = 2.$$

$$\Rightarrow 2c_2 = \frac{3}{2} \Rightarrow c_2 = \frac{3}{4}.$$

Therefore,

$$y = e^{-x/2}\left(-\cos 2x + \frac{3}{4}\sin 2x\right)$$

Ans.

Example :- Solve $y^{(4)}+y'''+y''=0$.

Solution :- Given that,

$$y^{(4)}+y'''+y''=0 \quad \cdots (1)$$

Let the trial solution of equation (1) be of the form

$$y=e^{mx} \quad \cdots (2)$$

Substituting (2) in equation (1) we get,

$$m^4e^{mx}+m^3e^{mx}+m^2e^{mx}=0.$$

$$\Rightarrow m^4+m^3+m^2=0$$

$$\Rightarrow m^2(m^2+m+1)=0.$$

Either,

$$m^2=0$$

$$\Rightarrow m=0,0$$

Or,

$$m^2+m+1=0$$

$$\Rightarrow m=\frac{-1\pm\sqrt{1-4}}{2}$$

$$\Rightarrow m=\frac{-1\pm\sqrt{3}i}{2}$$

The general solution is,

$$y=c_1e^0+c_2xe^0+e^{-1/2x}\left(c_3\cos\frac{\sqrt{3}x}{2}+c_4\sin\frac{\sqrt{3}x}{2}\right)$$

$$=c_1+c_2x+e^{-x/2}\left(c_3\cos\frac{\sqrt{3}x}{2}+c_4\sin\frac{\sqrt{3}x}{2}\right)$$

Ans.

Example:- Solve $16y^{(4)}+24y''+9y=0$

Solution:- Given that,

$$16y^{(4)}+24y''+9y=0 \quad \cdots (1)$$

Let the trial solution of Equation (1) be of the form $y=e^{mx} \cdots (2)$

Substituting (2) in equation (1) we get,

$$16m^4e^{mx}+24m^2e^{mx}+9e^{mx}=0$$

$$\Rightarrow 16m^4+24m^2+9=0.$$

$$\Rightarrow 16(m^2)^2+24m^2+9=0.$$

$$\Rightarrow m^2=\frac{-24\pm\sqrt{(24)^2-4.16.9}}{2.16}$$

$$\Rightarrow m^2=-\frac{24}{32}$$

$$\Rightarrow m^2=-\frac{3}{4}$$

$$\Rightarrow m=\pm\frac{\sqrt{3}i}{2}, \pm\frac{\sqrt{3}i}{2}.$$

The general solution is,

$$y=c_1\cos\frac{\sqrt{3}x}{2}+c_2\sin\frac{\sqrt{3}x}{2}+c_3x\cos\frac{\sqrt{3}x}{2}+c_4x\sin\frac{\sqrt{3}x}{2}$$

Ans.

Practice Problem

Chapter 4.3 → 1 - 40

Homework:-

① Solve the fallowing homogeneous differential equations:-

a) $9y'' + 4y = 0, \quad y\left(\frac{\pi}{4}\right) = 2, \quad y'\left(\frac{\pi}{4}\right) = -2$

b) $y'' + 3y = 0, \quad y\left(\frac{\pi}{3}\right) = 2, \quad y'\left(\frac{\pi}{3}\right) = -1$

c) $y'' - 6y' - 7y = 0, \quad y(2) = -\frac{1}{3}, \quad y'(2) = -5$

d) $y''' + 12y'' + 36y' = 0, \quad y(0) = 0, \quad y'(0) = 1, \quad y''(0) = -7$

e) $2\dfrac{d^5x}{ds^5} - 7\dfrac{d^4x}{ds^4} + 12\dfrac{d^3x}{ds^3} + 8\dfrac{d^2x}{ds^2} = 0$.

Method of Undetermined Coefficients

We use the method of undetermined coefficients to solve the nonhomogeneous ODE. Then the general solution has two parts — complementary solution and particular solution i.e.

$$y = y_c + y_p$$

$y_p \longrightarrow$ Particular solution

$y_c \downarrow$ complementary solution

The function $g(x)$ can be constant, polynomial function, exponential function, sine or cosine function or finite sums and products of these functions.

Example:— Solve $y'' + 4y' - 2y = 2x^2 - 3x + 6$.

Solution:— Given that,

$$y'' + 4y' - 2y = 2x^2 - 3x + 6.$$

We first solve the associated homogeneous equation

$$y'' + 4y' - 2y = 0 \quad \cdots \text{(1)}.$$

Let the trial solution of equation (1) be of the form

$$y = e^{mx} \quad \cdots \text{(2)}$$

Substituting ② in equation ① we get,

$$m^2e^{mx} + 4me^{mx} - 2e^{mx} = 0 .$$

$$\Rightarrow m^2 + 4m - 2 = 0 .$$

$$\Rightarrow m = \frac{-4 \pm \sqrt{16+8}}{2} = -2 \pm \sqrt{6} .$$

The complementary solution is,

$$y_c = c_1 e^{(-2+\sqrt{6})x} + c_2 e^{(-2-\sqrt{6})x}$$

Because the function $g(x)$ is a quadratic polynomial, let us assume a particular solution that is also in the form of quadratic polynomial

$$y_p = Ax^2 + Bx + c$$

We seek to determine the specific coefficients A, B, C. Substituting y_p and its derivatives into ① we get,

$$2A + 8Ax + 4B - 2Ax^2 - 2Bx - 2C = 2x^2 - 3x + 6 .$$

Equating the coefficients of like powers,

$-2A = 2$ $\Rightarrow A = -1$

$8A - 2B = -3$ $\Rightarrow B = -5/2$

$2A + 4B - 2C = 6$ $\Rightarrow C = -9 .$

Thus the particular solution is,

$$y_p = -x^2 - \frac{5x}{2} - 9$$

Therefore, the general solution is,

$$y = y_c + y_p$$

$$= c_1 e^{(-2+\sqrt{6})x} + c_2 e^{(-2-\sqrt{6})x} - x^2 - \frac{5}{2}x - 9$$

Ans.

Example :- Solve $y'' + y = 4x + 10\sin x$, $y(\pi) = 0$, $y'(\pi) = 2$

Solution :- Given that,

$$y'' + y = 4x + 10\sin x$$

The associated homogeneous equation is,

$$y'' + y = 0 \quad \cdots (1)$$

Let the trial solution of equation (1) be of the form

$$y = e^{mx} \quad \cdots (2)$$

Substituting (2) in equation (1) we get,

$$m^2 e^{mx} + e^{mx} = 0$$

$$\Rightarrow m^2 + 1 = 0$$

$$\Rightarrow m = \pm i$$

The complementary solution is

$$y_c = e^{0.x}\left(c_1 \cos x + c_2 \sin x\right) = c_1 \cos x + c_2 \sin x$$

Let the form of the particular solution y_p be

$$y_p = Ax + B + D\cos x + E\sin x \quad \cdots (3)$$

Substituting the particular solution y_p and its derivati in equation (1) we get,

$$-D\cos x - E\sin x + Ax + B + D\cos x + E\sin x = 4x + 10\sin x$$

$$\Rightarrow Ax + B = 4x + 10\sin x$$

It is not possible to determine the values of the coefficients from the last equation. It happens because of the repitition of cosx and sinx in the particular solution.

To avoid the repitition of terms we set the particular solution as

$$y_p = Ax + B + Dx\cos x + Ex\sin x$$

Then substitution of y_p and its derivatives in equatio (1) yields,

$$Ax + B - 2D\sin x + 2E\cos x = 4x + 10\sin x$$

Solving,

$A = 4, \quad B = 0, \quad D = -5, \quad E = 0$

Therefore, the general solution is,

$y = y_c + y_p$

$= \cancel{4x}\, c_1 \cos x + c_2 \sin x + 4x - 5x \cos x$

From the conditions.

$y(\pi) = 0.$

$\Rightarrow -c_1 + 4\pi + 5\pi = 0$

$\Rightarrow c_1 = 9\pi$

and $\quad y'(\pi) = 2$

$\Rightarrow -c_1 \sin\pi + c_2 \cos\pi + 4 - 5\cos\pi + 5\pi \sin\pi = 2$

$\Rightarrow c_2 = 7.$

Therefore, the solution to the initial value problem is,

$y(x) = 9\pi \cos x + 7 \sin x + 4x - 5x \cos x$

Ans.

Practice Problem

chapter 4.4 → 1-36.

Homework :-

Solve the following nonhomogeneous differential equatio

a) $y'' - 8y' + 20y = 100x^2 - 26xe^x$

b) $y'' - 2y' + 2y = e^{2x}(\cos x - 3\sin x)$

c) $y'' + 4y' + 4 = (3+x)e^{-2x}$, $y(0) = 2$, $y'(0) = 5$

d) $y''' + 8y = 2x - 5 + 8e^{-2x}$, $y(0) = -5$, $y'(0) = 3$,
$y''(0) = -4$

e) $y'' + 4y = g(x)$, $y(0) = 1$, $y'(0) = 2$, where

$$g(x) = \begin{cases} \sin x, & 0 \leq x \leq \pi/2 \\ 0, & x > \pi/2 \end{cases}$$

f) $y'' + 3y = 6x$, $y(0) + y'(0) = 0$, $y(1) = 0$.