METHOD OF SEPARABLE VARIABLES

A first-order differential equation of the form

$$\frac{dy}{dx} = g(x)\,h(y)$$

is said to separable or to have separable variables.

For example, the equation $\frac{dy}{dx} = y^2 x e^{3x+4y}$ is separable but in the equation $\frac{dy}{dx} + y + \sin x = 0$ is nonseparable because there is no way in expressing $-y-\sin x$ as a product of a function of x times a function of y.

Method of Solution

For a separable equation

$$\frac{dy}{dx} = g(x)\,h(y)$$

$$\Rightarrow \frac{1}{h(y)}\frac{dy}{dx} = g(x) \qquad \left[\frac{1}{h(y)} = P(y)\right]$$

$$\Rightarrow P(y)\frac{dy}{dx} = g(x) \quad \cdots (1)$$

If $y = \varphi(x)$ represents a solution of (1), we must have $P(\varphi(x))\,\varphi'(x) = g(x)$ and therefore,

$$\int P(\varphi(x))\varphi'(x)\,dx = \int g(x)\,dx \quad \cdots (2)$$

But $dy = \varphi'(x)dx$, and so (2) is the same as

$$\int P(y)\,dy = \int g(x)\,dx$$

or, $H(y) = G(x) + c \quad \cdots (3)$ $\left[\int P(y)\,dy = H(y),\ \int g(x)dx = G(x)\right]$

There is no need to use two constants in the integration of a separable equation, because if we write $H(y) + c_1 = G(x) + c_2$, then the difference $c_2 - c_1$ can be replaced by a single constant c, as in (3).

Example:— Solve $(1+x)\,dy - y\,dx = 0$.

Solution:— Given that,

$$(1+x)\,dy - y\,dx = 0.$$

$$\Rightarrow (1+x)\,dy = y\,dx$$

$$\Rightarrow \int \frac{dy}{y} = \int \frac{dx}{1+x} \quad \left[\text{Dividing by } (1+x)y\right]$$

$$\Rightarrow \ln|y| = \ln|1+x| + c_1.$$

$$\Rightarrow y = e^{\ln|1+x| + c_1} = e^{\ln|1+x|} \cdot e^{c_1}$$

$$\Rightarrow y = |1+x| \cdot e^{c_1}$$

$$\Rightarrow y = e^{c_1}|1+x| \qquad |1+x| = \begin{cases} 1+x, & x \geqslant -1 \\ -(1+x) & x < -1 \end{cases}$$

$$\Rightarrow y = \pm e^{c_1}(1+x)$$

$$\Rightarrow y = \pm c(1+x).$$

Ans.

Example:- Solve the initial-value problem $\frac{dy}{dx} = -\frac{x}{y}$, $y(4) = -3$.

Solution:- Rewriting the equation as

$$y\,dy = -x\,dx$$

$$\Rightarrow \int y\,dy = -\int x\,dx$$

$$\Rightarrow \frac{y^2}{2} = -\frac{x^2}{2} + c_1$$

As, $y(4) = -3$.

then, $\frac{9}{2} = -\frac{16}{2} + c_1$

$$\Rightarrow c_1 = \frac{25}{2}.$$

$$\therefore \frac{y^2}{2} = -\frac{x^2}{2} + \frac{25}{2} \Rightarrow x^2 + y^2 = 25$$

Ans.

Practice Problem

Chapter 2.2 → 1-30 .

Homework :-

① Solve the following differential equations by the method of separable variables:-

a) $y' = e^{y-x}$, ~~$y(0,0)$~~ $y(0) = 0$

b) $\frac{dy}{dx} = y^3 x e^{x^2}$, $y(0) = 1$

c) $\frac{dx}{dt} = \ln(t)\sqrt{1-x^2}$, $x(1) = 0$

d) $(1+x)y' = (x+2)(y-1)$

e) $y' = e^y 5^x$, $y(0) = \ln(\ln(5))$

f) $y' = -2x\tan(y)$, $y(0) = \frac{\pi}{2}$.

METHOD OF INTEGRATING FACTOR

Linear Equation

A first order differential equation of the form

$$a_1(x)\frac{dy}{dx} + a_0(x)y = g(x)$$

is said to be linear equation in the dependent variable y.

Now, $a_1(x)\frac{dy}{dx} + a_0(x)y = g(x)$

$$\Rightarrow \frac{dy}{dx} + \frac{a_0(x)}{a_1(x)}y = \frac{g(x)}{a_1(x)}$$

$$\Rightarrow \frac{dy}{dx} + P(x)y = Q(x) \quad \ldots (1)$$

$$\left[\begin{array}{l} \frac{a_0(x)}{a_1(x)} = P(x) \\ \frac{g(x)}{a_1(x)} = Q(x) \end{array}\right]$$

Therefore, (1) is the standard form of a linear equation.

Now we find the integrating factor which is defined as $IF = e^{\int P(x)dx}$

Multiplying (1) by IF we get,

$$e^{\int P(x)dx}\frac{dy}{dx} + P(x)e^{\int P(x)dx}y = Q(x)e^{\int P(x)dx}$$

$$\Rightarrow \frac{d}{dx}\left[e^{\int P(x)dx} y\right] = Q(x)e^{\int P(x)dx}$$

$$\Rightarrow \int d\left[e^{\int P(x)dx} y\right] = \int Q(x)e^{\int P(x)dx}\,dx$$

$$\Rightarrow e^{\int P(x)dx} y = \int Q(x)e^{\int P(x)dx}\,dx + c$$

$$\Rightarrow y = e^{\int P(x)dx}\int Q(x)e^{\int P(x)dx}\,dx + ce^{-\int P(x)dx}$$

Steps

① Put the lineare equation in the standard form.

② calculate the integrating factor $e^{\int P(x)dx}$. No constants need to be used in evaluating the indefinite integral $\int P(x)dx$.

③ Multiply the both sides of the standard equation by the integrating factor. The left hand side can be automatically written as

$$\frac{d}{dx}\left[e^{\int P(x)dx} y\right] = e^{\int P(x)dx} Q(x)$$

④ Integrate both sides and solve for y.

Example :- Solve $x\frac{dy}{dx} - 4y = x^6e^x,\ x>0.$

Solution :- Given that,

$$x\frac{dy}{dx} - 4y = x^6e^x$$

$$\Rightarrow \frac{dy}{dx} - \frac{4}{x}y = x^5e^x \quad \cdots (1)$$

Comparing (1) with the standard equation $\frac{dy}{dx} + P(x)y = Q(x)$

we get,

$$P(x) = -\frac{4}{x},\quad Q(x) = x^5e^x$$

Integrating factor, $IF = e^{\int -\frac{4}{x}dx} = e^{-4\ln|x|}$

$$= e^{\ln|x|^{-4}}$$

$$= x^{-4}e^{\ln} = x^{-4}$$

Multiplying (1) by x^{-4} we get

$$x^{-4}\frac{dy}{dx} - x^{-4}\frac{4}{x}y = x^{-4}x^5e^x$$

$$\Rightarrow x^{-4}\frac{dy}{dx} - 4x^{-5}y = xe^x.$$

$$\Rightarrow \frac{d}{dx}\left(x^{-4}y\right) = xe^x$$

Integrating both sides, $x^{-4}y = xe^x - e^x + c$

$$\Rightarrow y = x^5e^x - x^4e^x + cx^4$$

Ans

Example :– Solve $\frac{dy}{dx} + y = x$, $y(0) = 4$.

Solution :– Given that,

$$\frac{dy}{dx} + y = x \quad \ldots \quad (1)$$

Comparing equation (1) with the standard equation

$\frac{dy}{dx} + P(x)y = Q(x)$ we get,

$$P(x) = 1, \quad Q(x) = x$$

Integrating factor, $IF = e^{\int P(x)dx} = e^{\int 1 dx} = e^{x}$

Multiplying (1) by integrating factore we get,

$$e^{x}\frac{dy}{dx} + e^{x}y = xe^{x}$$

$$\Rightarrow \frac{d}{dx}(ye^{x}) = xe^{x}$$

Integrating both sides, $ye^{x} = xe^{x} - e^{x} + c$

$$\Rightarrow y = x - 1 + ce^{-x}$$

From the initial condition, $4 = 0 - 1 + ce^{0}$

$$\Rightarrow 5 = c .$$

Therefore, $y(x) = x - 1 + 5e^{-x}$

Ans .

Practice Problem

Chapter 2.3 → 1-24

CamScanner

Homework:

1. Solve the following initial value problems:

a) $\frac{dv}{dt} = 9.8 - 0.196v, \quad v(0) = 48$

b) $\cos(x)\, y' + \sin(x)\, y = 2\cos^3(x)\sin(x) - 1, \quad 0 < x < \frac{\pi}{2}$

$y\left(\frac{\pi}{4}\right) = 3\sqrt{2}$

c) $t y' + 2y = t^2 - t + 1, \quad y(1) = \frac{1}{2}$

d) $t y' - 2y = t^5 \sin(2t) - t^3 + 4t^4, \quad y(\pi) = \frac{3}{2}\pi^4$

e) $2y' - y = 4\sin(3t), \quad y(0) = y_0$