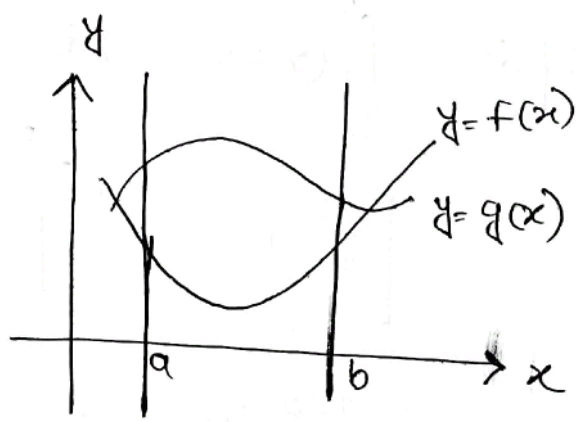
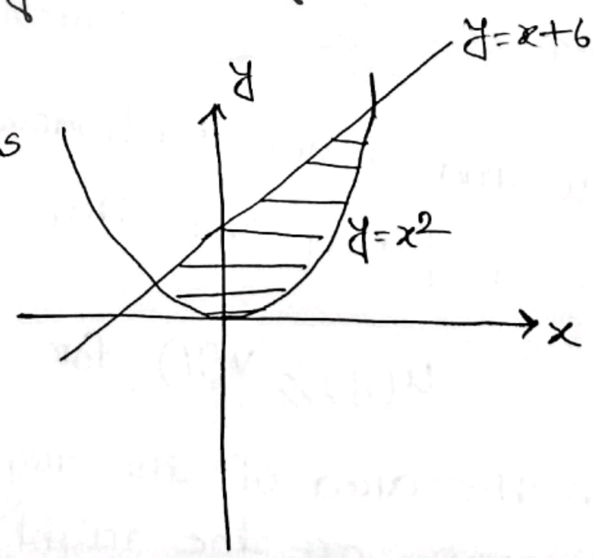
Area Between the Curves $y = f(x)$ and $y = g(x)$

If f and g are continuous functions on the interval $[a, b]$ and if $f(x) \geqslant g(x)$ for all $x \in [a, b]$ then the area of the region bounded by above by $y = f(x)$, below by $y = g(x)$, on the left by the line $x = a$, and on the right by the line $x = b$ is

$$A = \int_a^b [f(x) - g(x)]\, dx$$



Example: Find the area of the region that is enclosed by the curves $y = x^2$ and $y = x + 6$.



Solution: The given curves are,

$$y = x + 6 = f(x)$$

$$y = x^2 = g(x)$$

To find out the intersection points we set

$$x^2 = x+6$$

$$\Rightarrow x^2 - x - 6 = 0$$

$$\Rightarrow (x-3)(x+2) = 0$$

$$\Rightarrow x = -2, 3.$$

The area between the curves $y = f(x)$ and $y = g(x)$ over the interval $[-2, 3]$ is,

$$A = \int_{-2}^{3} [f(x) - g(x)]\, dx = \int_{-2}^{3} (x+6-x^2)\, dx$$

$$= \left[\frac{x^2}{2} + 6x - \frac{x^3}{3}\right]_{-2}^{3}$$

$$= \frac{125}{6} \text{ units}^2.$$

Ans.

Area Between the Curves $x = u(y)$ and $x = v(y)$

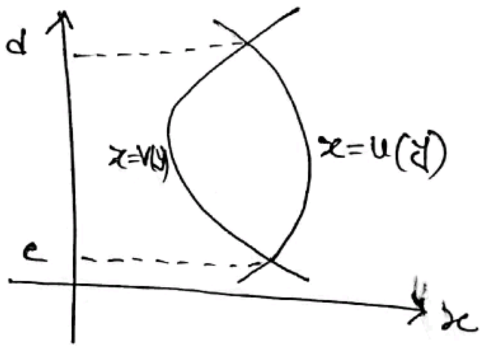
If u and v are continuous functions of y on an interval $[c, d]$ such that

$$u(y) \geq v(y) \text{ for } c \leq y \leq d$$

then the area of the region bounded on the left by $x = v(y)$, on the right $x = u(y)$, below by $y = c$

and above by $y = d$ is.

$$A = \int_c^d \left[u(y) - v(y) \right] dy$$

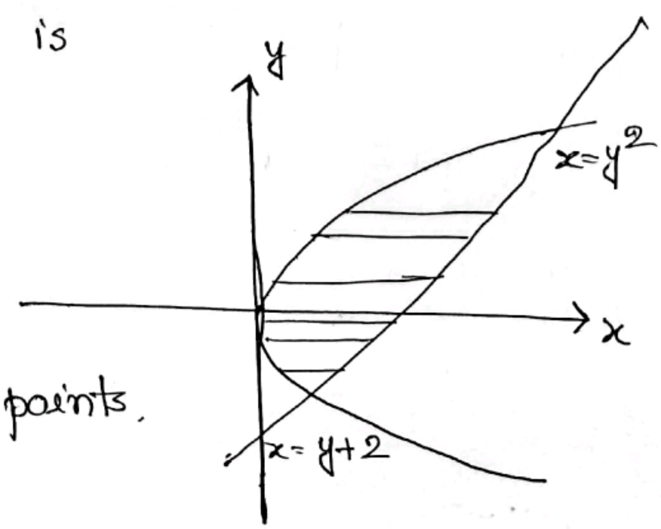


Example: Find the area of the region enclosed by $x = y^2$ and $y = x - 2$, integrating with respect to y.

Solution: The given curve is

$$x = y + 2 = u(y)$$

$$x = y^2 = v(y).$$



To find the intersection points.

$$y + 2 = y^2$$

$$\Rightarrow y^2 - y - 2 = 0$$

$$\Rightarrow (y+1)(y-2) = 0$$

Therefore, the enclosed area is,

$$A = \int_{-1}^{2} [u(y) - v(y)]\, dy = \int_{-1}^{2} (y + 2 - y^2)\, dy$$

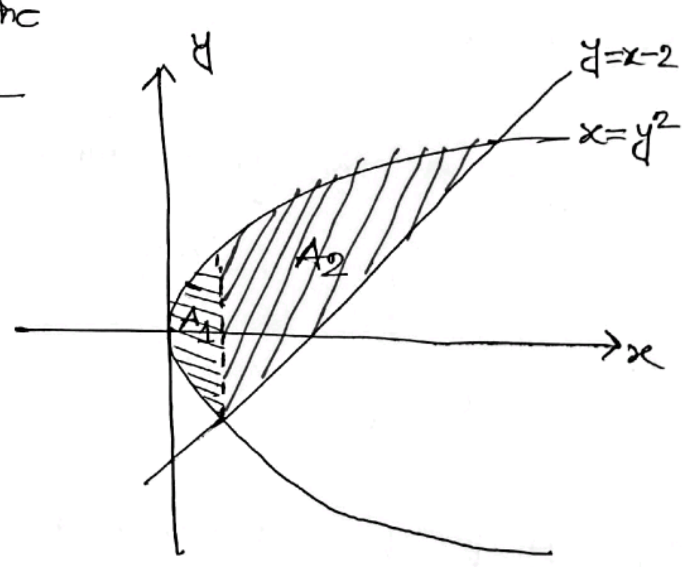
$$= \left[\frac{y^2}{2} + 2y - \frac{y^3}{3} \right]_{-1}^{2}$$

$$= \frac{9}{2} \text{ units}^2$$

Ans.

Example:- Solve the previous example integrating with respect to x.

Solution:- To find out the intersection points we set



$$y^2 = y + 2$$

$$\Rightarrow y^2 - y - 2 = 0$$

$$\Rightarrow (y-2)(y+1) = 0$$

$$\Rightarrow y = -1, 2.$$

When $y = -1$, $x = 1$

When $y = 2$, $x = 4$

The upper boundary can be written as $y = \sqrt{x}$ and the lower boundary consists of two parts: $y = -\sqrt{x}$ for $0 \leq x \leq 1$ and $y = x - 2$ for $1 \leq x \leq 4$. Because of this changer in the lower boundary, it is necessary to divide the region into two parts and find the area of the each part separately.

So setting $f(x) = \sqrt{x}$, $g(x) = -\sqrt{x}$, $a = 0$ and $b = 1$ we obtain

$$A_1 = \int_0^1 \left[\sqrt{x} - (-\sqrt{x})\right] dx = 2\int_0^1 \sqrt{x}\, dx = \frac{4}{3}.$$

With $f(x) = \sqrt{x}$ and $g(x) = x - 2$, $a = 1$, $b = 4$ we obtain

$$A_2 = \int_1^4 \left[\sqrt{x} - (x-2)\right] dx = \int_1^4 (\sqrt{x} - x + 2) dx$$

$$= \left[\frac{x^{3/2}}{3/2} - \frac{x^2}{2} + 2x\right]_1^4$$

$$= \frac{19}{6}$$

Thus the entire area of the region is

$$A = A_1 + A_2 = \frac{4}{3} + \frac{19}{6} = \frac{9}{2}$$ Ans.

Practice Problem

Chapter 6.1 → 7-18

Homework :-

1. Determine the area to the left of $g(y) = 3 - y^2$ and to the right of $x = -1$.

2. Determine the area of the region bounded by the given set of curves.

a) $y = \frac{8}{x}$, $y = 2x$, $x = 4$

b) $x = y^2 - y - 6$, $x = 2y + 4$

c) $y = 4x + 3$, $y = 6 - x - 2x^2$, $x = -4$, $x = 2$

d) $x = y^2 + 1$, $x = 5$, $y = -3$, $y = 3$.

e) Below $f(x) = 10 - 2x^2$ and above the line $y = 3$.

Arc Length

If $y = f(x)$ is a smooth curve on the interval $[a, b]$, then the arc length L of this curve $[a, b]$ is defined as

$$L = \int_a^b \sqrt{1 + [f'(x)]^2}\, dx \quad \dots (1)$$

$$= \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$

Moreover, for a curve expressed in the form $x = g(y)$ where g is continuous on $[c, d]$, the arc length L from $y = c$ to $y = d$ can be expressed as

$$L = \int_c^d \sqrt{1 + [g'(y)]^2}\, dy \quad \dots (2)$$

$$= \int_c^d \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dy$$

Example: Find the arc length of the curve $y = x^{3/2}$ from $(1, 1)$ to $(2, 2\sqrt{2})$

a) using formula (1)

b) using formula (2).

Solution :- Given that,

$$y = x^{3/2}$$

$$\Rightarrow \frac{dy}{dx} = \frac{3}{2} x^{1/2}$$

Since the curve extends from $x = 1$ to $x = 2$, it follows that

a) $L = \int_1^2 \sqrt{1 + \left(\frac{3}{2} x^{1/2}\right)^2}\, dx$

Let,

$u = 4 + 9x$

$$= \int_1^2 \sqrt{1 + \frac{9}{4} x}\, dx$$

$\Rightarrow du = 9\, dx$

$\Rightarrow \frac{du}{9} = dx$

$$= \frac{1}{2} \int_1^2 \sqrt{4 + 9x}\, dx$$

When $x = 1$, $u = 13$

When $x = 2$, $u = 22$

$$= \frac{1}{2} \int_{13}^{22} u^{1/2} \frac{du}{9} = \frac{2}{3} \times \frac{1}{18} \left[u^{3/2} \right]_{13}^{22} = \frac{1}{27} \left(22^{3/2} - 13^{3/2} \right)$$

Ans.

b) We must rewrite the equation as

$$y = x^{3/2}$$

$$\Rightarrow x = y^{2/3}$$

$$\Rightarrow \frac{dx}{dy} = \frac{2}{3} y^{-1/3}.$$

Since the curve extends from $y=1$ to $y=2\sqrt{2}$, it follows that

$$L = \int_{1}^{2\sqrt{2}} \sqrt{1+\frac{4}{9}y^{-2/3}}\, dy$$

$$= \frac{1}{3}\int_{1}^{2\sqrt{2}} y^{-1/3}\sqrt{9y^{2/3}+4}\; dy$$

$$= \frac{1}{18}\int_{13}^{22} \sqrt{u}\, du$$

$$= \frac{1}{18}\times\frac{2}{3}\left[u^{3/2}\right] = \frac{1}{27}\left(22^{3/2}-13^{3/2}\right).$$

Ans.

Let,

$u = 9y^{2/3}+4$

$\Rightarrow du = 6y^{-1/3}dy$

$\Rightarrow \frac{du}{6} = y^{-1/3}dy$

when $y=1$, $u=13$

when $y=2\sqrt{2}$, $u=22$

Example: Find the arc length of the parametric curve, $x=\cos 2t$, $y=\sin 2t$ when $0\leqslant t\leqslant \pi/2$.

Solution: The curve is defined as

$$x = \cos 2t$$

$$y = \sin 2t$$

Arc length, $L = \displaystyle\int_{0}^{\pi/2}\sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2}\, dt$

$$= \int_0^{\pi/2} \sqrt{(-2\sin 2t)^2 + (2\cos 2t)^2}\, dt$$

$$= \int_0^{\pi/2} 2\sqrt{\sin^2 2t + \cos^2 2t}\, dt$$

$$= 2\int_0^{\pi/2} \sqrt{1}\, dt = 2\left[t\right]_0^{\pi/2} = \pi$$

Ans.

Practice Problem

Chapter 6.4 → 3-8

Chapter 10.1 → 65-70

Homework :-

1. Determine the length of $y = 7(6+x)^{3/2}$, $189 \leq y \leq 875$.

2. Determine the length of $x = 4(3+y)^2$, $1 \leq y \leq 4$.

3. Determine the length of $x = 2+(y-1)^2$, $2 \leq y \leq 5$.

4. Find the arc length of the curve $x = (10-2y)^{3/2}$ over the interval $12^{3/2} \leq x \leq 6^{3/2}$ with respect to y-axis.

5. Find the arc length of the curve $y = (8x+3)^{3/2}$ over the interval $11^{3/2} \leq y \leq 27^{3/2}$ with respect to x-axis.

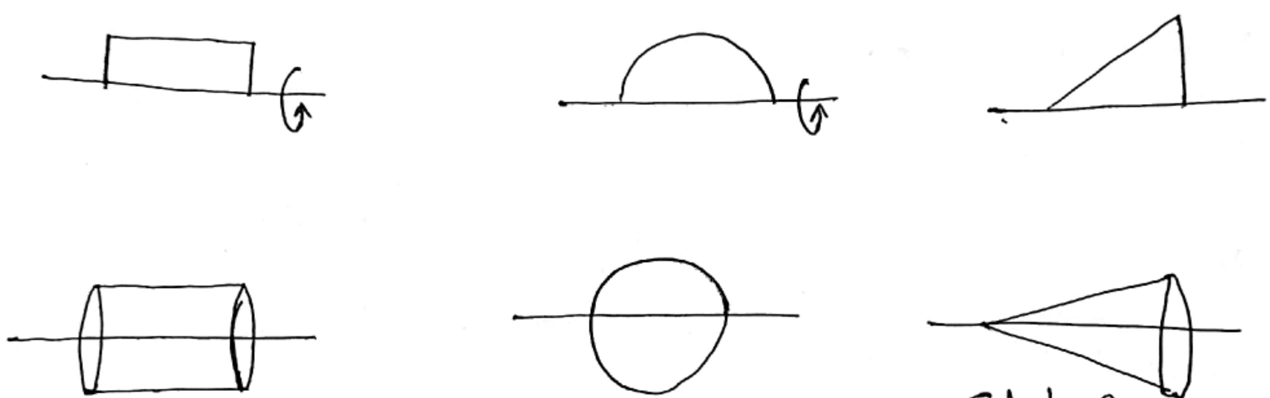
6. Find the arc length of the following curves:

a) $x = 8t^{3/2}$, $y = 3+(8-t)^{3/2}$, $0 \leq t \leq 4$

b) $x = 3t+1$, $y = 4-t^2$, $-2 \leq t \leq 0$.

Solids of Revolution :- A solid of revolution is a solid that is generated by revolving a plane region about a line that lies in the same plane as the region; the line is called the axis of revolution.

Some familiar Solids of Revolution



Right circular cylinder

Solid Sphere

Solid Cone.

Surface of Revolution

A surface of revolution is a surface that is generated by revolving a plane curve about an axis that lies in the same plane as the curve.

If f is a smooth, nonnegative function $[a,b]$, then the surface area S of the surface of revolution that is generated by revolving the portion of the curve

$y=f(x)$ between $x=a$ and $x=b$ about the x-axis is defined as

$$S=\int_a^b 2\pi f(x)\sqrt{1+[f'(x)]^2}\,dx$$

$$=\int_a^b 2\pi f(x)\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx$$

Moreover, if g is nonnegative and $x=g(y)$ is a smooth curve on the interval $[c,d]$, then the area of the surface that is generated by revolving the portion of a curve $x=g(y)$ between $y=c$ and $y=d$ about the y-axis can be expressed as

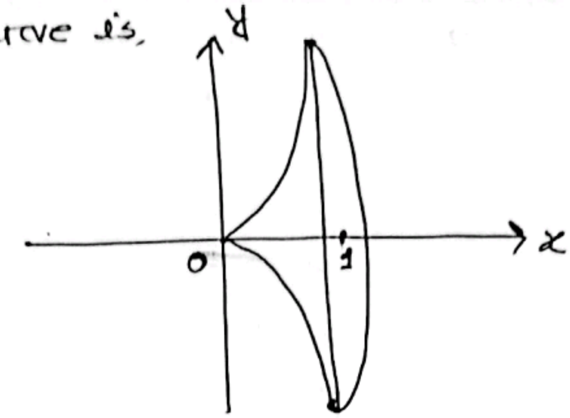
$$S=\int_c^d 2\pi g(y)\sqrt{1+[g'(y)]^2}\,dy=\int_c^d 2\pi x\sqrt{1+[g'(y)]^2}\,dy$$

$$=\int_c^d 2\pi x\sqrt{1+\left[\frac{dx}{dy}\right]^2}\,dy$$

Example:- Find the area of the surface that is generated by revolving the portion of the curve $y=x^3$ between $x=0$ and $x=1$ about the x-axis.

Solution:- The given curve is,

$$y = x^3 = f(x)$$



Area of the surface,

$$S = \int_0^1 2\pi f(x)\sqrt{1+[f'(x)]^2}\,dx$$

$$= \int_0^1 2\pi x^3 \sqrt{1+9x^4}\,dx$$

$$= \int_1^{10} 2\pi\sqrt{u}\,\frac{du}{36}$$

$$= \frac{2\pi}{36}\,\frac{2}{3}\left[u^{3/2}\right]$$

$$= \frac{\pi}{27}\left[10^{3/2}-1\right]$$ Ans.

Let,

$u = 1+9x^4$

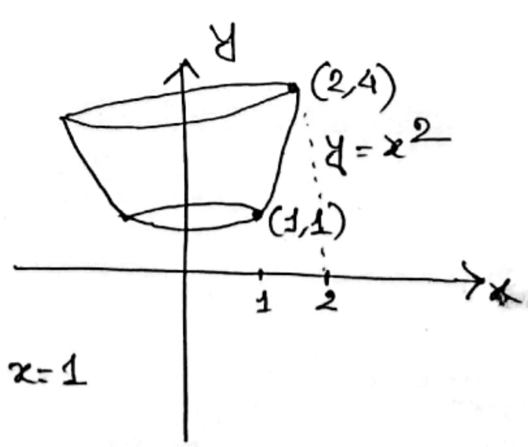
$\Rightarrow \frac{du}{dx} = 36x^3$

$\Rightarrow \frac{du}{36} = x^3 dx$

When $x=0$, $u=1$

When $x=1$, $u=10$.

Example:- Find the area of the surface that is generated by revolving the portion of the curve $y=x^2$ between $x=1$ and $x=2$ about y-axis.



Solution:- We rewrite

$$y = x^2$$

$$\Rightarrow x = \pm\sqrt{y}$$

Since we are considering about $x=1$

and $x=2$, the ~~$y=\sqrt{\ }$~~ $x=\sqrt{y}$

When $x=1$, $y=1$

When $x=2$, $y=4$

The Surface Area, $S=\int_1^4 2\pi g(y)\sqrt{1+[g'(y)]^2}\,dy$

$$=\int_1^4 2\pi\sqrt{y}\sqrt{1+\left(\frac{1}{2\sqrt{y}}\right)^2}\,dy$$

$$=\int_1^4 2\pi\sqrt{y}\sqrt{1+\frac{1}{4y}}\,dy$$

$$=\int_1^4 2\pi\sqrt{y}\,\frac{\sqrt{4y+1}}{2\sqrt{y}}\,dy$$

$$=\int_1^4 \pi\sqrt{4y+1}\,dy$$

Let,

$u=4y+1$

$\Rightarrow du=4dy$

$$=\frac{\pi}{4}\int_5^{17}\sqrt{u}\,du$$

When $y=1$, $u=5$

When $y=4$, $u=17$

$$=\frac{\pi}{4}\cdot\frac{2}{3}\left[u^{3/2}\right]_5^{17}$$

$$=\frac{\pi}{6}\left(17^{3/2}-5^{3/2}\right)$$

Ans.

Practice Problem:-

Chapter 6.5 → 1-8

Homework :-

1. Determine the surface area of the object obtained by revolving the curve $x = \frac{y^4}{4} + \frac{1}{8y^2}$ over the interval $1 \leq y \leq 2$ with respect to y-axis.

2. Determine the surface area of the object obtained by revolving the curve $y = 4 + 3x^2$ over the interval $1 \leq x \leq 2$ with respect to y-axis.

3. Determine the surface area of the object obtained by revolving the curve $y = \sqrt{4 - x^2}$ over the interval $-1 \leq x \leq 1$ with respect to x-axis.

4. Find the surface area of the object obtained by rotating $y = \frac{1}{4}\sqrt{6x + 2}$, $\frac{\sqrt{2}}{2} \leq y \leq \frac{\sqrt{5}}{2}$ about the x-axis.

5. Find the surface area of the object obtained by rotating $x = e^{2y}$, $-1 \leq y \leq 0$ about the y-axis.