Integration by Parts :-

$$\int uv\,dx = u\int v\,dx - \int \left\{ \frac{d}{dx}(u) \int v\,dx \right\} dx$$

Usually we choose the u-function according to the following rule :-

Tri. Inverse / Inverse ← I

Trigonometric ← T

L I A T E → Exponential

L → Logarithmic

A → Algebraic

When there is a combination of trigonometric and exponential functions there is no hard and fast rule for choosing u.

Example :- Calculate $\int 3xe^{-x}dx$

Solution :- $\int 3xe^{-x}dx$

$u = 3x$

$v = e^{-x}$

$$= 3x\int e^{-x}dx - \int \left\{ \frac{d}{dx}(3x) \int e^{-x}dx \right\} dx$$

$$= -3xe^{-x}\cancel{dx} + \int 3e^{-x}dx$$

$$= -3xe^{-x} \cancel{+} - 3e^{-x} + c$$

Ans.

Example:- Calculate $\int x \tan^2 x dx$.

Solution:- Let,

$$u = x$$

$$v = \tan^2 x$$

Then, $\int x \tan^2 x dx$

$$= x \int \tan^2 x dx - \int \left\{ \frac{d}{dx}(x) \int \tan^2 x dx \right\} dx$$

$$= x \int (\sec^2 x - 1) dx - \int \left\{ 1 \cdot \int (\sec^2 x - 1) dx \right\} dx$$

$$= x(\tan x - x) - \int (\tan x - x) dx$$

$$= x \tan x - x^2 - \int \left(\frac{\sin x}{\cos x} - x \right) dx$$

$$= x \tan x - x^2 + \ln|\cos x| + \frac{x^2}{2} + c$$

$$= x \tan x + \ln|\cos x| - \frac{x^2}{2} + c$$

Ans.

Example:- Calculate $\int e^x \sin 2x dx$.

Solution:- Let,

$$I = \int e^x \sin 2x dx$$

$$\Rightarrow I = e^x \int \sin 2x dx - \int \left\{ \frac{d}{dx}(e^x) \int \sin 2x dx \right\} dx$$

$$\Rightarrow I = -\frac{1}{2} e^x \cos 2x + \int \frac{e^x \cos 2x}{2} dx$$

$$\Rightarrow I = -\frac{1}{2}e^x\cos 2x + \frac{1}{2}\left[e^x\int\cos 2x\,dx - \int\left\{\frac{d}{dx}(e^x)\int\cos 2x\,dx\right\}dx\right]$$

$$\Rightarrow I = -\frac{1}{2}e^x\cos 2x + \frac{1}{4}e^x\sin 2x - \frac{1}{4}\int e^x\sin 2x\,dx$$

$$\Rightarrow I = -\frac{1}{2}e^x\cos 2x + \frac{1}{4}e^x\sin 2x - \frac{1}{4}I + c$$

$$\Rightarrow \frac{5}{4}I = -\frac{1}{2}e^x\cos 2x + \frac{1}{4}e^x\sin 2x + c$$

$$\Rightarrow I = -\frac{2}{5}e^x\cos 2x + \frac{1}{5}e^x\sin 2x + \frac{4}{5}c$$

$$\Rightarrow \int e^x\sin 2x\,dx = \frac{-2}{5}e^x\cos 2x + \frac{1}{5}e^x\sin 2x + \underbrace{\frac{4}{5}c}_{c_1}$$ Ans.

Practice Problem :-

chapter $\cancel{7}$.2 $\rightarrow$ 1-38

Homework :- (Integration by Parts)

1. $\int 2x^{17} e^{1+x^9} dx$

2. $\int (5+x^4) \sin(x/2)\, dx$

3. $\int \sqrt{x^3} \ln\left(\sqrt[3]{x}\right) dx$

4. $\int_0^{\pi/8} e^{-x} \sin(4x)\, dx$

5. $\int -3 \sin^{-1}(10x)\, dx$

The Chain Rule: If g is differentiable at x and f is differentiable at $g(x)$, then the composition $f \circ g$ is differentiable at x. Moreover, if $y = f(g(x))$ and $u = g(x)$ then $y = f(u)$ and

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx} = \frac{dy}{dg}\frac{dg}{dx} = f'(g(x))\, g'(x)$$

which we can write in integral form as

$$\int f'(g(x))\, g'(x)\,dx = F(g(x)) + c \quad \ldots (1)$$

Integration by Substitution:
For our purposes it will be useful to let $u = g(x)$ and to write $\frac{du}{dx} = g'(x)$ in the differential form $du = g'(x)\,dx$. With this notation (1) can be expressed as

$$\int f(u)\,du = F(u) + c \quad \ldots (2)$$

The process of evaluating an integral of form (1) by converting it into form (2) with the substitution

$$u = g(x) \text{ and } du = g'(x)dx$$

is called the method of u-substitution.

It is important to realize that in the method of u-substitution you have control over the choice of u, but once you make that choice you have no control over the resulting expression for du. However, in general, the method of u-substitution will fail if the chosen u and the computed du can not be used to produce an integrand in which no expressions involving x remain, or if you cannot evaluate the resulting integral.

Guidelines for u-substitution :-

Step 1 :- Look for some composition $f(g(x))$ within the integrand for which the substitution $u = g(x)$, $du = g'(x)dx$ produces an integral that is expressed entirely in terms of u and its differential du. This may or may not be possible.

Step 2 :- If you are successful in step 1, then try to evaluate the resulting integral in terms of u. Again, this may or may not be possible.

Step 3:- If you are successful in step 2, then replace u by $g(x)$ to express your final answer in terms of x.

Example :- Evaluate $\int_0^2 x(x^2+1)^3 dx$

Solution :- Here, $\int_0^2 x(x^2+1)^3 dx$

$= \int_1^5 u^3 \frac{du}{2}$

Let,

$u = x^2+1$

$\Rightarrow du = 2x dx$

$\Rightarrow \frac{du}{2} = x dx$

When $x=0$, $u=1$

When $x=2$, $u=5$

$= \frac{1}{2}\left[\frac{u^4}{4}\right]_1^5$

$= \frac{1}{8}\left[5^4 - 1^4\right]$

$= 78.$

Again we can solve the problem in the following way :

$\int_0^2 x(x^2+1)^3 dx$

Now, $\int x(x^2+1)^3 dx$

Let $u = x^2+1$

$\Rightarrow du = 2x dx$

$\Rightarrow \frac{du}{2} = x dx$

$= \int u^3 \frac{du}{2} = \frac{1}{8} u^4 = \frac{1}{8}(x^2+1)^4$

$$\therefore \int_0^2 x(x^2+1)^3 dx = \frac{1}{8}\left[(x^2+1)^4\right]_0^2 = \frac{1}{8} \times 5^4 = 78$$

Ans.

Example: Evaluate $\int_0^{\pi/8} \sin^5 2x \cos 2x \, dx$.

Solution: $\int_0^{\pi/8} \sin^5 2x \cos 2x \, dx$

Let,

$u = \sin 2x$

$\Rightarrow \frac{du}{dx} = 2\cos 2x$

$\Rightarrow \frac{du}{2} = \cos 2x \, dx$

When $x=0$, $u=0$

When $x = \frac{\pi}{8}$ $u = \frac{1}{\sqrt{2}}$

$$= \int_0^{1/\sqrt{2}} u^5 \frac{du}{2}$$

$$= \frac{1}{2}\left[\frac{u^6}{6}\right]_0^{1/\sqrt{2}}$$

$$= \frac{1}{12}\left[\left(\frac{1}{\sqrt{2}}\right)^6 - 0\right]$$

$$= \frac{1}{96}$$

Ans.

Example: Evaluate $\int_0^{\ln 2} e^x(1+e^x)^{1/2} dx$

Solution: Here, $\int_0^{\ln 2} e^x(1+e^x)^{1/2} dx$

Let,

$u = 1+e^x$

$\Rightarrow du = e^x dx$

When $x=0$, $u=2$

When $x=\ln 2$, $u=3$

Now, $\int_0^{\ln 2} e^x(1+e^x)^{1/2}dx = \int_2^3 u^{1/2}du = \left[\frac{u^{3/2}}{\frac{3}{2}}\right]_2^3$

$$= \frac{2}{3}\left(3^{3/2} - 2^{3/2}\right)$$

Ans.

Example: Evaluate $\int 7\cos\left(\frac{\pi z}{2}\right)\left(4+\sin\frac{\pi z}{2}\right)^5 dz$.

Solution: Here,

$$\int 7\cos\left(\frac{\pi z}{2}\right)\left(4+\sin\frac{\pi z}{2}\right)^5 dz$$

Let,

$u = 4+\sin\frac{\pi z}{2}$

$\Rightarrow \frac{du}{dz} = \frac{\pi}{2}\cos\frac{\pi z}{2}$

$\Rightarrow \frac{2}{\pi}du = \cos\frac{\pi z}{2}dz$

$$= 7\int \cos\frac{\pi z}{2}\left(4+\sin\frac{\pi z}{2}\right)^5 dz$$

$$= 7\int \frac{2}{\pi}u^5 du$$

$$= \frac{14}{\pi}\,\frac{u^6}{6} + c$$

$$= \frac{14}{6\pi}\left(\cos\frac{\pi z}{2}\right)^6 + c$$

Ans.

Example :- Evaluate $\int x^8 \sqrt{2-x^3}\, dx$.

Solution :- Here, $\int x^8 \sqrt{2-x^3}\, dx$

Let,

$u = 2 - x^3$

$\Rightarrow \frac{du}{dx} = -3x^2$

$\Rightarrow \frac{du}{-3} = x^2 dx$

$x^3 = 2 - u$

$= \int x^6 . x^2 \sqrt{2-x^3}\, dx$

$= \int (x^3)^2 x^2 \sqrt{2-x^3}\, dx$

$= \int (2-u)^2 \sqrt{u} \frac{du}{-3}$

$= \frac{1}{-3} \int (4 - 4u + u^2) \sqrt{u}\, du$

$= \frac{-1}{3} \int (4\sqrt{u} - 4u^{3/2} + u^{5/2})\, du$

$= -\frac{1}{3} \left[\frac{4u^{3/2}}{3/2} - 4 \frac{u^{5/2}}{5/2} + \frac{u^{7/2}}{7/2} \right] + c$

$= -\frac{1}{3} \left[\frac{8}{3} (2-x^3)^{3/2} - \frac{8}{5} (2-x^3)^{5/2} + \frac{2}{7} (2-x^3)^{7/2} \right] + c$

Ans.

Practice Problem :-

Chapter 5.9 → 5-22, 31-50

Homework :-

1. $\int 12v(7+6v^2)^9 dv$ and $\int x^2\sqrt{x-1}\, dx$

2. $\int_0^{\pi/3} \left\{ \frac{4\sin(3t)}{2+\cos(3t)} + \frac{7\sin(3t)}{(2+\cos(3t))^2} \right\} dt$

3. $\int \left\{ \sqrt{1+2y} + (4-y)(y^2-8y+5)^4 \right\} dy$

4. $\int_0^1 \left\{ e^{2z}\sin(e^{2z}-1) + \sin(z)\, e^{2-\cos z} \right\} dz$

5. $\int \left\{ 8w^2 + \frac{\sin w + \cos w}{\sin w - \cos w} \right\} dw$

6. $\int \frac{\sin(1+\ln(2x)) - \sqrt{1+\ln(2x)}}{x}\, dx$

7. $\int \frac{4}{\cancel{20}\, 25+9w^2}\, dw$

8. $\int 17\left\{ (xe^x+e^x)\sin(xe^x) - 14\sin x \right\} dx$

9. $\int \frac{3+7y}{y^2+3}\, dy$

10. $\int_1^9 \left(\sqrt{x^5} + \frac{\sin(\sqrt{x})}{\sqrt{x}} \right) dx$.