Beta Function

The Beta function $\beta(m,n)$ is defined by the definite integral

$$\beta(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}\,dx, \quad m,n>0$$

Gamma Function

The Gamma function is defined as

$$\Gamma n = \int_0^\infty x^{n-1}e^{-x}\,dx$$

$$\Gamma n+1 = n! \quad \left[\text{When } n \text{ is an integer}\right]$$

$$\Gamma n+1 = n\Gamma n \quad \left[\text{When } n \text{ is fraction}\right]$$

$$\Gamma \tfrac{1}{2} = \sqrt{\pi}$$

Relation Between Beta and Gamma Function

$$\beta(m,n) = \frac{\Gamma m \cdot \Gamma n}{\Gamma m+n}$$

$$\int_0^{\pi/2} \sin^p\theta \cos^q\theta \, d\theta = \frac{1}{2}\beta\left(\frac{p+1}{2}, \frac{q+1}{2}\right) = \frac{1}{2}\,\frac{\Gamma\frac{p+1}{2}\,\Gamma\frac{q+1}{2}}{\Gamma\frac{p+q+2}{2}}$$

Example: Evaluate $\int_0^{\pi/4} \sin^2 4\theta \cos^3 2\theta \, d\theta$

Solution: $\int_0^{\pi/4} \sin^2 4\theta \cos^3 2\theta \, d\theta$

Let

$z = 2\theta$

$\Rightarrow dz = 2d\theta$

$\Rightarrow \frac{dz}{2} = d\theta$

When $\theta = 0, \quad z = 0$

When $\theta = \frac{\pi}{4}, z = \pi/2$

$$= \int_0^{\pi/2} \sin^2 2z \cos^3 z \, \frac{dz}{2}$$

$$= \frac{1}{2} \int_0^{\pi/2} (2 \sin z \cos z)^2 \cos^3 z \, dz$$

$$= 2 \int_0^{\pi/2} \sin^2 z \cos^5 z \, dz$$

$$= 2 \cdot \frac{1}{2} \, \beta\left(\frac{2+1}{2}, \frac{5+1}{2} \right)$$

$$= \beta\left(\frac{3}{2}, 3 \right) = \frac{\Gamma\frac{3}{2} \, \Gamma 3}{\Gamma \frac{9}{2}} = \frac{\Gamma\left(\frac{1}{2}+1\right) \Gamma(2+1)}{\Gamma\left(1+\frac{7}{2}\right)}$$

$$= \frac{\Gamma\frac{1}{2} \, 2!}{\frac{7}{2} \, \Gamma\frac{7}{2}}$$

$$= \frac{2\sqrt{\pi}}{\frac{7}{2} \cdot \frac{5}{2} \, \frac{3}{2} \, \frac{1}{2} \sqrt{\pi}}$$

Ans

Example:- Evaluate $\int_0^1 \frac{x^3}{\sqrt{1-x^3}}\,dx$

Solution:- $\int_0^1 \frac{x^3}{\sqrt{1-x^3}}\,dx$

Let

$x^3 = z$

$\Rightarrow x = z^{1/3}$

$\Rightarrow dx = \frac{1}{3} z^{-2/3} dz$

When $x=0$, $z=0$

When $x=1$, $z=1$

$$= \int_0^1 \frac{z}{\sqrt{1-z}} \cdot \frac{1}{3} z^{-2/3} dz$$

$$= \frac{1}{3}\int_0^1 z^{1/3}(1-z)^{-1/2} dz$$

$$= \frac{1}{3}\int_0^1 z^{4/3-1}(1-z)^{1/2-1} dz$$

$$= \frac{1}{3}\beta\left(\frac{4}{3},\frac{1}{2}\right) = \frac{1}{3}\frac{\Gamma\frac{4}{3}\,\Gamma\frac{1}{2}}{\Gamma\left(\frac{4}{3}+\frac{1}{2}\right)}$$

$$= \frac{1}{3}\frac{\Gamma 4/3\,\sqrt{\pi}}{\Gamma\frac{11}{6}}$$

Ans.