Exact Equation

If $z = f(x, y)$ is a function of two variables with continuous first partial derivatives in a region R of the xy-plane, then its differential is

$$dz = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy \quad \ldots (1)$$

In the special case when $f(x,y) = c$, where c is a constant then (1) implies,

$$\frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy = 0.$$

$$\Rightarrow M(x,y)dx + N(x,y)dy = 0$$

Not every first-order differential equation written in differential form $M(x,y)dx + N(x,y)dy = 0$ corresponds to a differential of $f(x,y) = c$.

Criterion for an Exact Differential

Let $M(x,y)$ and $N(x,y)$ be continuous and have continuous first partial derivatives in a rectangular region R defined by $a < x < b$, $c < y < d$. Then a necessary and sufficient condition that $M(x,y)dx + N(x,y)dy$ be an exact differential is $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$. $\ldots (2)$

Method of Solution

Given an equation in the differential form $M(x,y)dx + N(x,y)dy = 0$, determine whether the equality in ② holds. If it does, then there exists a function f for which

$$\frac{\partial f}{\partial x} = M(x,y)$$

We can find f by integrating $M(x,y)$ with respect to x while holding y constant

$$f(x,y) = \int M(x,y)\,dx + g(y) \quad \cdots ③$$

where the arbitrary function $g(y)$ is the constant of integration. Now differentiate ③ with respect to y and assume that $\frac{\partial f}{\partial y} = N(x,y)$

$$\frac{\partial f}{\partial y} = \frac{\partial}{\partial y}\int\left[M(x,y)dx + g'(y)\right] = N(x,y)$$

This gives, $g'(y) = N(x,y) - \frac{\partial}{\partial y}\int M(x,y)\,dx \quad \cdots ④$

Finally, integrate ④ with respect to y and substitute the result in ③. The implicit solution of the equation is $f(x,y) = c$.

Example:- Solve $2xy\,dx + (x^2-1)\,dy = 0$.

Solution:- Given that,

$$2xy\,dx + (x^2-1)\,dy = 0$$

Comparing the equation with $M(x,y)dx + N(x,y)dy = 0$ we get,

$$M(x,y) = 2xy$$

$$N(x,y) = x^2-1$$

Now, $\frac{\partial M}{\partial y} = 2x = \frac{\partial N}{\partial x}$

Therefore, the equation is exact.

By setting,

$$\frac{\partial f}{\partial x} = M(x,y)$$

$$\Rightarrow \frac{\partial f}{\partial x} = 2xy$$

$$\Rightarrow f(x,y) = \int 2xy\,dx$$

$$\Rightarrow f(x,y) = x^2y + g(y)$$

We know that,

$$\frac{\partial f}{\partial y} = N(x,y)$$

$$\Rightarrow \frac{\partial f}{\partial y} = x^2-1$$

$\Rightarrow x^2 + g'(y) = x^2 - 1$

$\Rightarrow g'(y) = -1$

$\Rightarrow g(y) = -y + c.$

Thereforce, $f(x, y) = x^2y - y + c$ Ans.

A Nonexact DE made Exact

If the nonlinear DE is not exact, then it can be made exact.

* If $\frac{M_y - N_x}{N}$ is a function of x alone then an integrating factor can be calculated as

$$IF = e^{\int \frac{M_y - N_x}{N} dx}$$

* If $\frac{N_x - M_y}{M}$ is a function of y alone then an integrating factor can be calculated as

$$IF = e^{\int \frac{N_x - M_y}{M} dy} = e^{\int -\frac{(M_y - N_x)}{M} dy}$$

Then multiply the differential equation by the integrating factor. Thereforce, you have to solve the [illegible] equation in the previous manner.

Example :- Solve $xy\,dx + (2x^2+3y^2-20)\,dy = 0$

Solution :- Given that,

$$xy\,dx + (2x^2+3y^2-20)\,dy = 0 \quad \cdots (1)$$

Here, $M(x,y) = xy$

$N(x,y) = 2x^2 + 3y^2 - 20$

We have to check whether the equation is exact or not.

$$\frac{\partial M}{\partial y} = x \quad ; \quad \frac{\partial N}{\partial x} = 4x$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$ the equation is not exact. Now we have to transform the nonexact DE to an exact DE.

$$\frac{M_y - N_x}{N} = \frac{x-4x}{2x^2+3y^2-20} = \frac{-3x}{2x^2+3y^2-20}$$

$$\frac{N_x - M_y}{M} = \frac{4x-x}{xy} = \frac{3x}{xy} = \frac{3}{y}$$

The first quotient depends on x and y but the second quotient depends only on y. So we need to use the second quotient term to evaluate the integrating factor.

$$IF = e^{\int \frac{3}{y} dy} = e^{3 \ln y} = e^{\ln y^3} = y^3.$$

Multiplying ① by the integrating factor.

$$xy^4 dx + (2x^2y^3 + 3y^5 - 20y^3) dy = 0 \quad \ldots (2)$$

This equation is now exact. Now the equation ② can be solved by the method of exact equation.

Practice Problem

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Homework:

Determine ~~the~~ the exactness of the given equations and then solve by proper method.

a) $\left(3y^3e^{3xy}-1\right)dx+\left(2ye^{3xy}+3xy^2e^{3xy}\right)dy=0,\ y(0)=1$

b) $\left(\frac{2ty}{t^2+1}-2t\right)dt-\left(2-\ln(t^2+1)\right)dy=0,\ y(5)=0$.

c) $\frac{x\,dx}{(x^2+y^2)^{3/2}}+\frac{y\,dy}{(x^2+y^2)^{3/2}}=0$

d) $\left(\frac{1}{t}+\frac{1}{t^2}-\frac{y}{t^2+y^2}\right)dt+\left(ye^y+\frac{t}{t^2+y^2}\right)dy=0$

In the following problems find the value of k so that the given differential equation is exact. Then solve the problem with the corresponding value of k.

a) $(y^3+kxy^4-2x)dx+(3xy^2+20x^2y^3)dy=0$

b) $(6xy^3+\cos y)dx+(2kx^2y^2-x\sin y)dy=0$