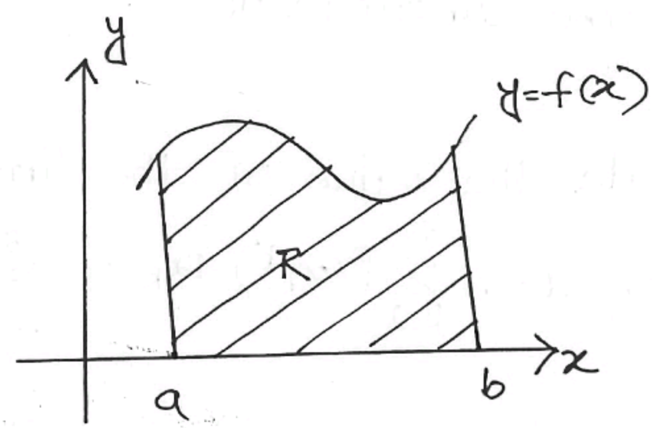
Definition of Area :-

Formulas for the areas of polygons, such as squares, rectangles, triangles, and trapezoids are well known in many early civilizations. However, the problem of finding formulas for regions with curved boundaries caused difficulties for early mathematicians.

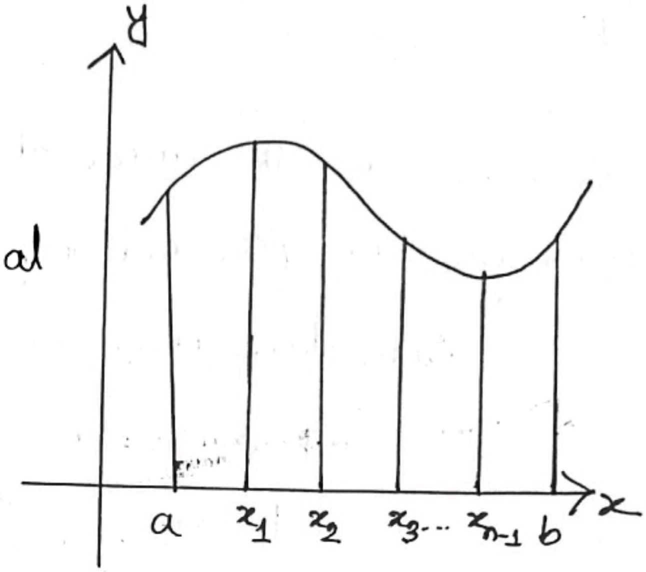
Suppose that the function f is continuous and nonnegative on the interval $[a, b]$ and let R denote the region bounded below by the x-axis, bounded on the sides by the vertical lines $x=a$ and $x=b$, bounded above by the curve $y=f(x)$.



We can motivate a definition for area of R by using Rectangle method (Riemann sum).

Riemann Sum :-

Let $y=f(x)$ be a function defined on the closed interval $[a,b]$.



① Divide the interval $[a,b]$ into n-subintervals $[x_{k-1}, x_k]$ of width $\Delta x_k = x_k - x_{k-1}$ where $a = x_0 < x_1 < x_2 < \cdots < x_n = b$ ... ①

② Choose a number x_k^* in each subinterval $[x_{k-1}, x_k]$. The n-numbers $x_1^*, x_2^*, \ldots, x_n^*$ are called sample points in the subintervals.

③ We calculate the value of the function $f(x)$ at $x = x_k^*$. Then we sum i.e. $\sum_{k=1}^{n} f(x_k^*) \Delta x_k$ ... ②

④ Sum of the kind given in ② corresponding to various portions of $[a,b]$ are known as **Riemann sum**.

If we repeat the process using more and more divisions, and we define the area of R to be the 'limit' of the areas of the approximating regions R_n as n increases without bound.

That is, we define the area A as,

$$A = \lim_{n\to\infty} \sum_{k=1}^{n} f(x_k^*)\,\Delta x_k \quad \cdots ③$$

The values of $x_1^*, x_2^*, \ldots, x_n^*$ in ③ can be chosen arbitrarily, so it is conceivable that different choices of these values might produce different values of A. Fortunately, this does not happen; it is provided in advanced courses that if f is continuous, then the same value of A results no matter how the x_k^* are chosen. Practically, they are chosen in some systematic fashion, some common choices are being

Points	Choice of sample points
Left endpoint	$x_k^* = a + (k-1)\Delta x$
Right endpoint	$x_k^* = a + k\Delta x$
Midpoint	$x_k^* = a + \left(k - \frac{1}{2}\right)\Delta x$

Example :- Calculate the Riemann sum of the function $f(x) = 3x+1$ over the interval $[2,6]$ with four subintervals using

i) Left endpoints

ii) Right endpoints

iii) Mid points.

Solution:- The width of the subinterval is,

$$\Delta x = \frac{b-a}{n} = \frac{6-2}{4} = 1$$

i) The sample (left endpoints) points and the corresponding functional values are.

$x_1^* = 2$; $f(x_1^*) = 3\times2+1=7$.

$x_2^* = 3$; $f(x_2^*) = 3\times3+1=10$

$x_3^* = 4$; $f(x_3^*) = 3\times4+1 = 13$

$x_4^* = 5$; $f(x_4^*) = 3\times5+1 = 16.$

Area, $A = \sum_{k=1}^{4} f(x_k^*)\Delta x_k$

$= f(x_1^*)\Delta x_1 + f(x_2^*)\Delta x_2 + f(x_3^*)\Delta x_3 + f(x_4^*)\Delta x_4$

$= 7\times1 + 10\times1 + 13\times1 + 16\times1$.

$= 46$.

ii) The sample (right endpoints) points and the corresponding functional values are,

$x_1^* = 3$; $f(x_1^*) = 3\times3+1 = 10$

$x_2^* = 4$; $f(x_2^*) = 3\times4+1 = 13$

$x_3^* = 5$; $f(x_3^*) = 3\times5+1 = 16$

$x_4^* = 6$; $f(x_4^*) = 3\times6+1 = 19$

Area, $A = \sum_{k=1}^{4} f(x_k^*) \Delta x_k$.

$= f(x_1^*) \Delta x_1 + f(x_2^*) \Delta x_2 + f(x_3^*) \Delta x_3 + f(x_4^*) \Delta x_4$

$= 10 \times 1 + 13 \times 1 + 16 \times 1 + 19 \times 1$

$= 58.$

iii) The sample (midpoints) points and the corresponding functional values are,

$x_1^* = 2.5 \qquad f(x_1^*) = \frac{17}{2}$

$x_2^* = 3.5 \qquad f(x_2^*) = \frac{23}{2}$

$x_3^* = 4.5 \qquad f(x_3^*) = \frac{29}{2}$

$x_4^* = 5.5 \qquad f(x_4^*) = \frac{35}{2}$

Area, $A = \sum_{k=1}^{4} f(x_k^*) \Delta x_k$

$= f(x_1^*) \Delta x_1 + f(x_2^*) \Delta x_2 + f(x_3^*) \Delta x_3 + f(x_4^*) \Delta x_4$

$= \frac{17}{2} \times 1 + \frac{23}{2} \times 1 + \frac{29}{2} \times 1 + \frac{35}{2} \times 1$

$= 52.$

Ans.

Example: Find the ~~area~~ area/Riemann sum under the curve $f(x) = x-1$ over the interval $[0, 1]$ using

i) left endpoint

ii) right endpoint

iii) midpoint

(When the number of subintervals is not mentioned).

Solution: Since the number of subintervals are not mentioned we take n-subintervals. The length of each subinterval is,

$$\Delta x = \frac{b-a}{n} = \frac{1-0}{n} = \frac{1}{n}$$

i) For n-subintervals, the left endpoints can be chosen in the following manner

$$x_k^* = a + (k-1)\Delta x = 0 + \frac{(k-1)}{n} = \frac{k-1}{n}.$$

Then, $\sum_{k=1}^{n} f(x_k^*)\Delta x_k$.

$$= \sum_{k=1}^{n}\left(\frac{k-1}{n} - 1\right)\frac{1}{n}.$$

$$= \sum_{k=1}^{n} \frac{k-1}{n^2} - \sum_{k=1}^{n} \frac{1}{n}.$$

$$= \frac{1}{n^2}\sum_{k=1}^{n}(k-1) - \sum_{k=1}^{n}\frac{1}{n}.$$

$$= \frac{1}{n^2} \sum_{k=1}^{n} k - \frac{1}{n^2} \sum_{k=1}^{n} 1 - \sum_{k=1}^{n} \frac{1}{n}$$

Area, $A = \lim_{n \to \infty} \sum_{k=1}^{n} f(x_k^*) \Delta x_k$.

$$= \lim_{n\to\infty} \left(\frac{1}{n^2} \sum_{k=1}^{n} k - \frac{1}{n^2} \sum_{k=1}^{n} 1 - \sum_{k=1}^{n} \frac{1}{n} \right)$$

$$= \lim_{n\to\infty} \left(\frac{1}{n^2} \frac{n(n+1)}{2} - \frac{1}{n^2} \cdot n - \frac{1}{n} \cdot n \right)$$

$$= \lim_{n\to\infty} \left(\frac{n^2+n}{2n^2} - \frac{1}{n} - 1 \right)$$

$$= \lim_{n\to\infty} \left(\frac{1}{2} + \frac{1}{2n} - \frac{1}{n} - 1 \right)$$

$$= \frac{1}{2} - 1 = -\frac{1}{2}$$ Ans.

ii) For n-subintervals, the right endpoints can be chosen in the following manner

$$x_k^* = a + k\Delta x = 0 + \frac{k}{n} = \frac{k}{n}$$

Then, $\sum_{k=1}^{n} f(x_k^*) \Delta x_k = \sum_{k=1}^{n} \left(\frac{k}{n} - 1 \right) \frac{1}{n}$

$$= \sum_{k=1}^{n} \left(\frac{k}{n^2} - \frac{1}{n} \right)$$

$$= \sum_{k=1}^{n} \frac{k}{n^2} - \sum_{k=1}^{n} \frac{1}{n}$$

$$= \frac{1}{n^2} \sum_{k=1}^{n} k - \frac{1}{n} \sum_{k=1}^{n} 1$$

$$= \frac{1}{n^2}\sum_{k=1}^{n} k - \frac{1}{n}\sum_{k=1}^{n} 1 .$$

Therefore, Area $= \lim_{n\to\infty} \sum_{k=1}^{n} f(x_k^*) \Delta x_k$

$$= \lim_{n\to\infty} \left(\frac{1}{n^2}\sum_{k=1}^{n} k - \frac{1}{n}\sum_{k=1}^{n} 1 \right)$$

$$= \lim_{n\to\infty} \left(\frac{1}{n^2} \frac{n(n+1)}{2} - \frac{1}{n} \cdot n \right)$$

$$= \lim_{n\to\infty} \left(\frac{1}{n^2} \frac{n^2+n}{2} - 1 \right)$$

$$= \frac{1}{2} - 1 = -\frac{1}{2} .$$

iii) For n-subintervals, the midpoints can be chosen in the following manner

$$x_k^* = a + \left(k - \tfrac{1}{2}\right)\Delta x$$

$$= a + \left(k - \tfrac{1}{2}\right)\frac{1}{n} = 0 + \left(k - \tfrac{1}{2}\right)\frac{1}{n} = \frac{1}{n}\left(k - \tfrac{1}{2}\right)$$

Then, $\sum_{k=1}^{n} f(x_k^*) \Delta x_k = \sum_{k=1}^{n} \left\{ \frac{1}{n}\left(k - \frac{1}{2}\right) - 1 \right\} \frac{1}{n}$

$$= \sum_{k=1}^{n} \left(\frac{k}{n^2} - \frac{1}{2n^2} - \frac{1}{n} \right)$$

$$= \sum_{k=1}^{n} \frac{k}{n^2} - \sum_{k=1}^{n} \frac{1}{2n^2} - \sum_{k=1}^{n} \frac{1}{n}$$

$$\text{Area, } A = \lim_{n \to \infty} \sum_{k=1}^{n} f(x_k^*) \Delta x_k$$

$$= \lim_{n \to \infty} \left(\sum_{k=1}^{n} \frac{k}{n^2} - \sum_{k=1}^{n} \frac{1}{2n^2} - \sum_{k=1}^{n} \frac{1}{n} \right)$$

$$= \lim_{n \to \infty} \left(\frac{1}{n^2} \sum_{k=1}^{n} k - \frac{1}{2n^2} \sum_{k=1}^{n} 1 - \frac{1}{n} \sum_{k=1}^{n} 1 \right)$$

$$= \lim_{n \to \infty} \left(\frac{1}{n^2} \frac{n(n+1)}{2} - \frac{1}{2n^2} \cdot n - \frac{1}{n} \cdot n \right)$$

$$= \frac{1}{2} - 1 = -\frac{1}{2} \quad \text{Ans.}$$

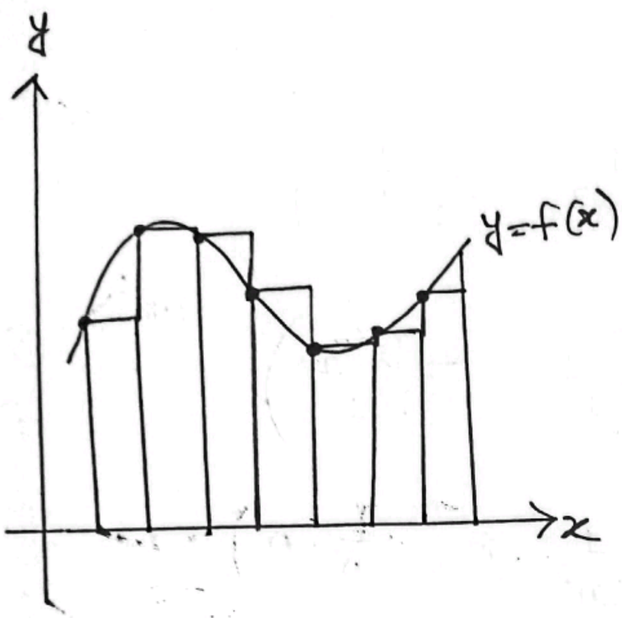
Some known Summation Formula:-

$$\sum_{k=1}^{n} 1 = 1 + 1 + \cdots + 1 = n \longrightarrow \frac{1}{n} \lim_{n \to \infty} \sum_{k=1}^{n} 1 = 1$$

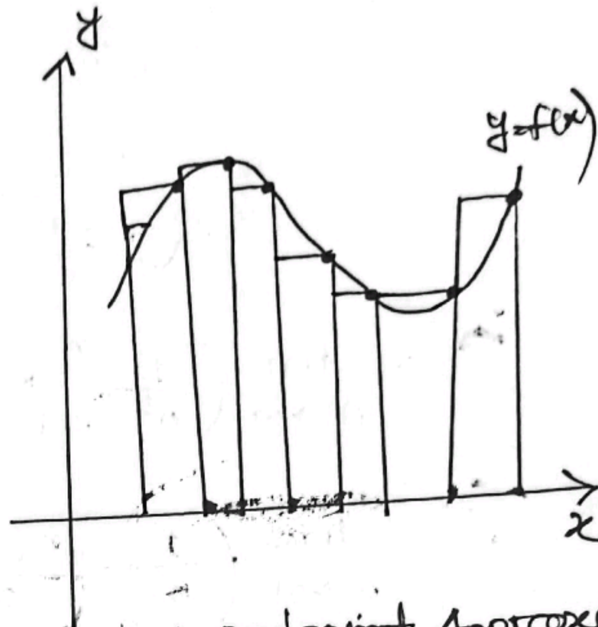
$$\sum_{k=1}^{n} k = 1 + 2 + \cdots + n = \frac{n(n+1)}{2}. \longrightarrow \frac{1}{n^2} \lim_{n \to \infty} \sum_{k=1}^{n} k = \frac{1}{2}$$

$$\sum_{k=1}^{n} k^2 = 1^2 + 2^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}. \longrightarrow \frac{1}{n^3} \lim_{n \to \infty} \sum_{k=1}^{n} k^2 = \frac{1}{3}$$

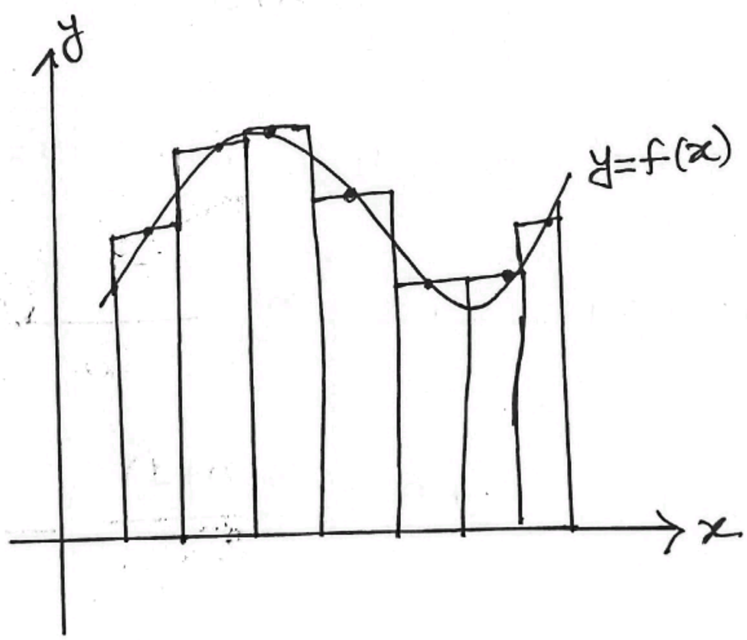
$$\sum_{k=1}^{n} k^3 = 1^3 + 2^3 + \cdots + n^3 = \left\{ \frac{n(n+1)}{2} \right\}^2 \longrightarrow \frac{1}{n^4} \lim_{n \to \infty} \sum_{k=1}^{n} k^3 = \frac{1}{4}$$



Left endpoint Approximation



Right endpoint Approximation



Midpoint Approximation

Let's again recall the expression,

$$S = \lim_{n \to \infty} \sum_{k=1}^{n} f(x_k^*) \Delta x_k$$

The fundamental idea here is that as the lengths of the intervals of subdivision all tend to zero so that the number of intervals tend to infinity, the sum converges to a common limiting value which is precisely the area under the curve. There is no hard and fast rules for the choice of the height of the rectangles. We can write the expression in the integral form $\int_a^b f(x)dx$.

$$\int_{a}^{b} \boxed{\boxed{f(x)}\,dx}$$

$b \to$ Upper limit of integration

$\boxed{f(x)\,dx} \to$ Integral

$f(x) \to$ Integrand

$a \to$ Lower limit of integration

The expression $\int_a^b f(x)\,dx$ generally calculates the area under the curve $y = f(x)$ over the interval $[a, b]$. Here we integrate the function $f(x)$ with respect to x. In a similar way, $A(y) = \int_c^d f(y)\,dy$ generally expresses the integration of the function $f(y)$ with respect to y over the interval $[c, d]$.

The process of finding a function from its derivatives is called antidifferentiation or integration.

Homework :-

1. Estimate the area of the region between the following functions and the x-axis using i) the left endpoints, ii) the right endpoints and iii) the midpoints of the subintervals.

a) $f(x) = 15 + 4x - x^3$ on $[1, 3]$

b) $g(x) = -3x^2+2x-1$ on $[-4, 0]$

c) $h(x) = \sin^2\left(\frac{x}{2}\right)$ on $[0, 3]$

2. Divide the specified interval into $n=4$ subintervals of equal length and then compute the Riemann sum using the i) left endpoints ii) right endpoints and iii) midpoints of the following functions.

a) $f(x) = 2x - x^2$; $[-1, 3]$

b) $f(x) = \cos x$; $[0, \pi]$

c) $f(x) = 1 - x^3$; $[-3, -1]$

PRACTICE PROBLEM

Chapter 5.4 → 27-30, 35-40, 41-44, 45-48