Triple Integrals in Spherical Coordinates

In spherical coordinates

$$r = \rho \sin\phi, \quad z = \rho \cos\phi$$

The relationship between the rectangular and spherical coordinates is,

$$x = \rho \sin\phi \cos\theta$$

$$y = \rho \sin\phi \sin\theta$$

$$z = \rho \cos\phi$$

Here ϕ is the angle between the z-axis and the projection.

So, $x^2+y^2+z^2$

$$= (\rho \sin\phi \cos\theta)^2 + (\rho \sin\phi \sin\theta)^2 + (\rho \cos\phi)^2$$

$$= \rho^2$$

The Jacobian is defined as

$$J(\rho, \phi, \theta) = \frac{\partial(x, y, z)}{\partial(\rho, \phi, \theta)} = \begin{vmatrix} \frac{\partial x}{\partial \rho} & \frac{\partial x}{\partial \phi} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial \rho} & \frac{\partial y}{\partial \phi} & \frac{\partial y}{\partial \theta} \\ \frac{\partial z}{\partial \rho} & \frac{\partial z}{\partial \phi} & \frac{\partial z}{\partial \theta} \end{vmatrix}$$

$$= \begin{vmatrix} \sin\varphi\cos\theta & \rho\cos\varphi\cos\theta & -\rho\sin\varphi\sin\theta \\ \sin\varphi\sin\theta & \rho\cos\varphi\sin\theta & \rho\sin\varphi\cos\theta \\ \cos\varphi & -\rho\sin\varphi & 0 \end{vmatrix}$$

$$= \rho^2 \sin\varphi$$

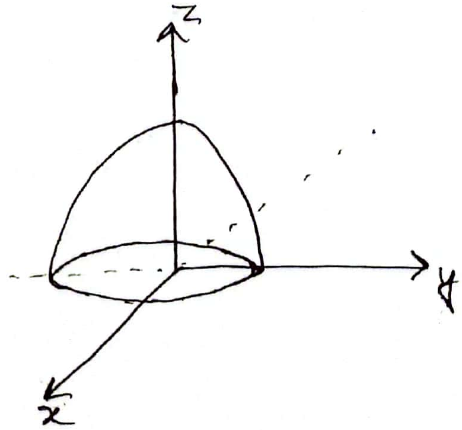
The triple integrals can be converted from rectangular coordinates to spherical coordinates by making the substitution $x = \rho\sin\varphi\cos\theta$, $y = \rho\sin\varphi\sin\theta$, $z = \rho\cos\varphi$.
The two integrals are related by the equation

$$\iiint_G f(x,y,z)\,dV = \underset{\substack{\text{appropriate}\\ \text{limits}}}{\iiint} f(\rho\sin\varphi\cos\theta, \rho\sin\varphi\sin\theta, \rho\cos\varphi)\,\rho^2\sin\varphi\,d\rho\,d\varphi\,d\theta$$

Example:- Evaluate $\displaystyle\int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}}\int_{0}^{\sqrt{4-x^2-y^2}} z^2\sqrt{x^2+y^2+z^2}\,dz\,dy\,dx$.

Solution:- Given that,

$$\int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}}\int_{0}^{\sqrt{4-x^2-y^2}} z^2\sqrt{x^2+y^2+z^2}\,dz\,dy\,dx$$



$$= \int_{0}^{2\pi}\int_{0}^{\pi/2}\int_{0}^{2} \rho^2\cos^2\varphi\sqrt{\rho^2}\,\rho^2\sin\varphi\,d\rho\,d\varphi\,d\theta$$

$$= \int_0^{2\pi}\int_0^{\pi/2}\int_0^2 \rho^5 \cos^2\varphi \sin\varphi \, d\rho \, d\varphi \, d\theta$$

Let,

$u = \cos\varphi$

$\Rightarrow du = -\sin\varphi \, d\varphi$

When $\varphi = 0$ $u = 1$

When $\varphi = \pi/2$ $u = 0$.

$$= \int_0^{2\pi}\int_0^{\pi/2} \left[\frac{\rho^6}{6}\right]_0^2 \cos^2\varphi \sin\varphi \, d\rho \, d\varphi \, d\theta$$

$$= \frac{64}{6}\int_0^{2\pi}\int_0^{\pi/2} \cos^2\varphi \sin\varphi \, d\varphi \, d\theta$$

$$= \frac{64}{6}\int_0^{2\pi}\int_0^{1} u^2 \, du \, d\theta$$

$$= \frac{64}{6}\int_0^{2\pi} \left[\frac{u^3}{3}\right]_0^1 d\theta$$

$$= \frac{64}{18}\left[\theta\right]_0^{2\pi} = \frac{64}{18} \times 2\pi = \frac{64\pi}{9}$$

Ans.

Practice Problem

Chapter 14.6 → 1-4, 9-16, 17-20

Homework :-

① Evaluate the following triple integrals using cylindrical or spherical coordinates :-

a) $\displaystyle\int_0^a \int_0^{\sqrt{a^2-x^2}} \int_0^{a^2-x^2-y^2} x^2 \, dz\, dy\, dx \quad (a>0)$

b) $\displaystyle\int_{-3}^{3} \int_{-\sqrt{9-y^2}}^{\sqrt{9-y^2}} \int_{-\sqrt{9-x^2-y^2}}^{\sqrt{9-x^2-y^2}} \sqrt{x^2+y^2+z^2} \, dz\, dx\, dy$

c) $\displaystyle\iiint_E 4xy \, dV$ where E is the region bounded by $z = 2x^2+2y^2-7$ and $z=1$.

d) $\displaystyle\int_0^{\sqrt{5}} \int_{-\sqrt{5-x^2}}^{0} \int_{x^2+y^2-11}^{9-3x^2-3y^2} (2x-3y) \, dz\, dy\, dx$

e) $\displaystyle\iiint_E (10xz+3) \, dV$ where E is the region portion of $x^2+y^2+z^2=16$ with $z \geq 0$.

f) $\displaystyle\int_{-1}^{0} \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \int_{\sqrt{6x^2+6y^2}}^{\sqrt{7-x^2-y^2}} 18y \, dz\, dy\, dx$

Differential Equation

An equation containing the derivatives of one or more dependent variables, with respect to one or more independent variables is said to be a differential equation (DE).

For example, $\frac{dy}{dx} = 0.2xy$

$$\frac{dx}{dt} + \frac{dy}{dt} = 2x+y$$

$$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}.$$

Ordinary Differential Equation :– If a differential equation contains only ordinary derivatives of one or more unknown functions with respect to a single independent variable, it is said to be ordinary differential equation (ODE).

For example, $\frac{dy}{dx} + 5y = e^x$

$$\frac{d^2y}{dx^2} + \frac{dy}{dx} - 6y = 0$$

$$\frac{dx}{dt} + \frac{dy}{dt} = 2x+y$$

Partial Differential Equation

An equation involving partial derivatives of one or more dependent variables of two or more independent variable is called a partial differential equation (PDE).

For example, $$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

$$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

Different Types of Notation

Leibniz notation $\rightarrow \frac{dy}{dx}, \frac{d^2y}{dx^2}, \frac{d^3y}{dx^3}, \frac{d^ny}{dx^n}$

Prime notation $\rightarrow y', y'', y''', y^{(n)}$

The prime notation is used to denote only the first three derivatives; the fourth derivative is written $y^{(4)}$ instead of y''''.

Dot notation $\rightarrow \dot{y}, \ddot{y}, \dddot{y}$

Subscript notation $\rightarrow u_{xx}, u_{tt}, u_t$

The Leibniz notation has an advantage over the prime notation in that it clearly displays both the dependent and independent variables.

~~Order of~~

Classification by Order :– The order of a differential equation (either ODE or PDE) is the order of the highest derivative in the equation.

For example,

$$\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^3 = 0 \rightarrow \text{second order}$$

$$\frac{d^4y}{dx^4} + \sin y = 0 \rightarrow \text{Fourth order}$$

$$x^3 \frac{d^3y}{dx^3} + x\frac{dy}{dx} - 5y = e^x \rightarrow \text{Third order}$$

Classification by Degree :– The degree of a differential equation (either ODE or PDE) is the power of the highest derivative in the equation.

For example,

$$\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^3 + 4y = e^x \rightarrow \text{Degree one}$$

$$\frac{d^3y}{dx^3} = \sqrt{1+\left(\frac{dy}{dx}\right)^2} \rightarrow \text{Degree one}$$

$$\left(\frac{dy}{dx}\right)^2 + 6y = 0 \rightarrow \text{Degree two.}$$

Practice Problem

chapter 1·1 → 1 − 14

Classification by Linearcity

An nth order ordinary differential equation is given as

$$a_n(x)\frac{d^n y}{dx^n} + a_{n-1}(x)\frac{d^{n-1} y}{dx^{n-1}} + a_{n-2}(x)\frac{d^{n-2} y}{dx^{n-2}} + \cdots$$

$$+ a_1(x)\frac{dy}{dx} + a_0(x)y = g(x) \quad \cdots (1)$$

The equation (1) is said to be linear if

i) the dependent variable y and all its derivatives $y', y'', \ldots, y^{(n)}$ are of the first degree, i.e. the power of each term involving y is 1.

ii) the coefficients $a_0, a_1, a_2, \ldots, a_n$ of $y, y', \ldots, y^{(n)}$ depend at most the independent variable x.

A nonlinear ordinary differential equation is simply one that is not linear. Non linear functions of the dependent variables or its derivatives such as $\sin y$ or $e^{y'}$, can not appear in a linear equation.

For example, $(y-x)\,dx + 4xy\,dy = 0 \longrightarrow$ Linear

$$x^3\frac{d^3y}{dx^3} + x\frac{dy}{dx} - 5y = e^x \longrightarrow \text{Linear}$$

$$(1-y)y' + 2y = e^x \longrightarrow \text{Nonlinear}$$

$$\frac{d^2y}{\ldots} \quad \sin y = 0 \longrightarrow \text{Non linear}$$

Homogeneous Equations

A linear n-th order differential equation of the form

$$a_n(x)\frac{d^n y}{dx^n} + a_{n-1}(x)\frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_1(x)\frac{dy}{dx} \cancel{+ a(x) - y \neq 0}$$
$$+ a_0(x) y = g(x)$$

is said to be homogeneous with $g(x)$ identically zero, whereas an equation with $g(x)$ not identically zero, is said to be nonhomogeneous.

For example, $2y'' + 3y' - 5y = 0 \rightarrow$ Homogeneous ODE

$x^3 y''' + 6y' + 10y = e^x \rightarrow$ Nonhomogeneous ODE