Partial Definite Integral

The partial derivatives of a function $f(x,y)$ are calculated by holding one of the variables fixed and differentiating with respect to the other variable. Let us consider the reverse of this process, partial integration. The symbols $\int_a^b f(x,y)\,dx$ and $\int_c^d f(x,y)\,dy$ denote partial ~~integration~~ definite integrals; the first integral, called the partial definite integral with respect to x, is evaluated by holding y fixed and integrating with respect to x, and the second integral, called the partial definite integral with respect to y, is evaluated by holding x fixed and integrating with respect to y.

A partial definite integral with respect to x is a function of y and hence can be integrated with respect to y; similarly, a partial definite integral with respect to y can be integrated with respect to x. This two-stage integration process is called iterated (or repeated) integration.

We introduce the following notation

$$\int_c^d \int_a^b f(x,y)\,dx\,dy = \int_c^d \left[\int_a^b f(x,y)\,dx \right] dy$$

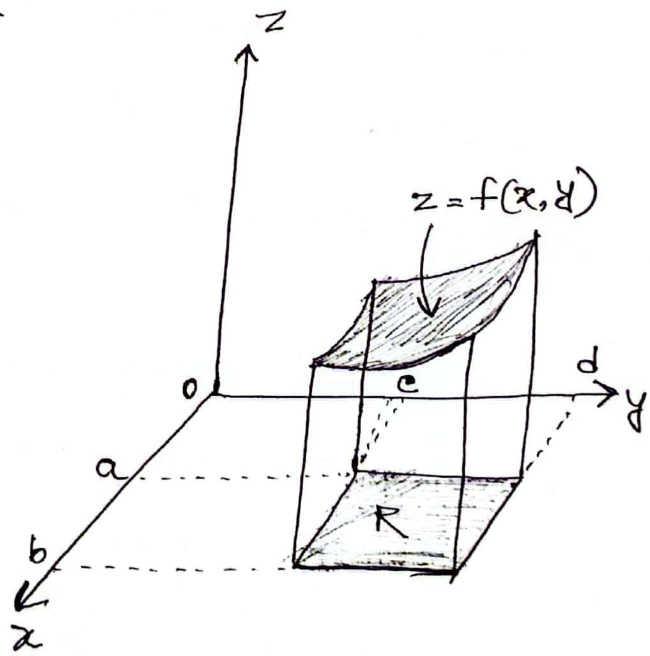
$$\int_a^b \int_c^d f(x,y)\,dy\,dx = \int_a^b \left[\int_c^d f(x,y)\,dy \right] dx$$

These integrals are called iterated integrals.

Double Integral

If $f(x,y) \geq 0$ then the volume V of the solid that lies above the rectangle R and below the surface $z = f(x,y)$ is

$$V = \iint_R f(x,y)\,dA$$



It can be written as

$$V = \iint_R f(x,y)\,dy\,dx$$

$$V = \iint_R f(x,y)\,dx\,dy$$

Example: Evaluate the iterated integrals

a) $\int_0^3\int_1^2 x^2y\,dy\,dx$ b) $\int_1^2\int_0^3 x^2y\,dx\,dy$

Solution: a) $\int_0^3\int_1^2 x^2y\,dy\,dx$

$$= \int_0^3\left[\int_1^2 x^2y\,dy\right]dx$$

$$= \int_0^3\left[\frac{x^2y^2}{2}\right]_1^2 dx = \int_0^3\left[\frac{4x^2}{2} - \frac{x^2}{2}\right]dx$$

$$= \frac{3}{2}\int_0^3 x^2\,dx = \frac{3}{2}\left[\frac{x^3}{3}\right]_0^3 = \frac{27}{2}$$

Ans.

b) $\int_1^2\int_0^3 x^2y\,dx\,dy$

$$= \int_1^2\left[\frac{x^3y}{3}\right]_0^3 dy$$

$$= \int_1^2 9y\,dy = \left[\frac{9y^2}{2}\right]_1^2$$

$$= \frac{36}{2} - \frac{9}{2} = \frac{27}{2}$$

Ans.

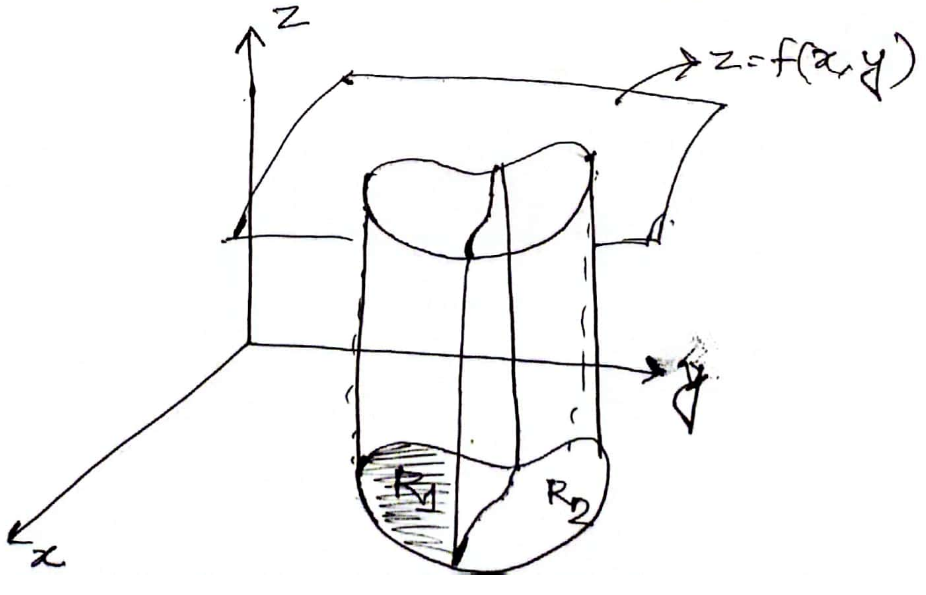
Properties of Double Integrals

① $$\iint_R cf(x,y)\,dA = c\iint_R f(x,y)\,dA$$

② $$\iint_R [f(x,y) \pm g(x,y)]\,dA = \iint_R f(x,y)\,dA \pm \iint_R g(x,y)\,dA$$

③ It is evident intuitively that if $f(x,y)$ is nonnegative on a region R, then subdividing R into two regions R_1 and R_2 has the effect of subdividing the solid between R and $z=f(x,y)$ into two solids, the sum of whose volumes is the volume of the entire solid. This suggests that the following result, which holds ~~it~~ even if f has negative values:–

$$\iint_R f(x,y)\,dA = \iint_{R_1} f(x,y)\,dA + \iint_{R_2} f(x,y)\,dA$$



Triple Integral

If $f(x,y,z) \geq 0$ then the triple integral of the function is defined as

$$B = \iiint_G f(x,y,z)\, dV$$

It can be written as

$$B = \iiint_G f(x,y,z)\, dx\, dy\, dz$$

$$\text{or,} \quad B = \iiint_G f(x,y,z)\, dy\, dx\, dz$$

$$\text{or,} \quad B = \iiint_G f(x,y,z)\, dz\, dy\, dx$$

You can also combine the last three terms in different ways as you want.

Example: Evaluate $\iiint 12xy^2z^3\,dV$ where the variables are defined as $-1 \leq x \leq 2$, $0 \leq y \leq 3$ and $0 \leq z \leq 2$.

Solution:

$$\int_0^2\int_0^3\int_{-1}^2 12xy^2z^3\,dx\,dy\,dz$$

$$= \int_0^2\int_0^3 \left[\frac{12x^2}{2}\right]_{-1}^2 y^2z^3\,dy\,dz$$

$$= \int_0^2\int_0^3 \left[\frac{48}{2} - \frac{12}{2}\right] y^2z^3\,dy\,dz$$

$$= \int_0^2\int_0^3 18y^2z^3\,dy\,dz$$

$$= \int_0^2 \left[\frac{18y^3}{3}\right]_0^3 z^3\,dz$$

$$= \int_0^2 162z^3\,dz = 162\left[\frac{z^4}{4}\right]_0^2$$

$$= 648$$

Ans.

Properties of Triple Integral

① $$\iiint_G c f(x,y,z)\,dV = c\iiint_G f(x,y,z)\,dV.$$

② $$\iiint_G \left[f(x,y,z) \pm g(x,y,z)\right] dV = \iiint_G f(x,y,z)\,dV \pm \iiint_G g(x,y,z)\,dV$$

③ $$\iiint_G f(x,y,z)\,dV = \iiint_{G_1} f(x,y,z)\,dV + \iiint_{G_2} f(x,y,z)\,dV$$

Practice Problem

chapter 14.1 → 1-16, 29-32

Chapter 14.5 → 1-8, 9-12

Homework :-

1. Find the volume V for the following problems:

   a) $z = 8x + 6y$ over the rectangle $R = [0,1] \times [0,2]$

   b) $z = e^{x+y}$ over the rectangle $R = [2,3] \times [1,2]$

   c) $\int_0^{2\pi} \int_0^{\pi} \sin(x+y)\, dx\, dy$

2. Integrate $\iiint_G \cos(z/y)\, dV$, where G is the solid defined by the inequalities $\frac{\pi}{6} \leq y \leq \frac{\pi}{2}$, $y \leq x \leq \frac{\pi}{2}$, $0 \leq z \leq xy$.

3. Find the volume of the wedge in the first octant that is cut from the solid cylinder $y^2 + z^2 \leq 1$ by the planes $y = x$ and $x = 0$

4. ~~f~~. Evaluate the following integrals:

   a) $\int_0^1 \int_0^{z^2} \int_0^3 y \cos(z^5)\, dx\, dy\, dz$

   b) $\int_2^3 \int_{-1}^4 \int_1^0 (4x^2y - z^3)\, dz\, dy\, dx$

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Double Integrals over General Regions

Type-I Region:- A type-I region is bounded on the left and right by vertical lines $x=a$ and $x=b$ and is bounded below and above by continuous curves $y=g_1(x)$ and $y=g_2(x)$, where $g_1(x) \leq g_2(x)$ for $a \leq x \leq b$.

In other words a plane region R is said to be of type I if it lies between the graphs of two continuous functions of x, that is,

$$R = \{(x,y) \mid a \leq x \leq b,\ g_1(x) \leq y \leq g_2(x)\}$$

If R is a type I region on which $f(x,y)$ is continuous then

$$\iint_R f(x,y)\,dA = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x,y)\,dy\,dx$$

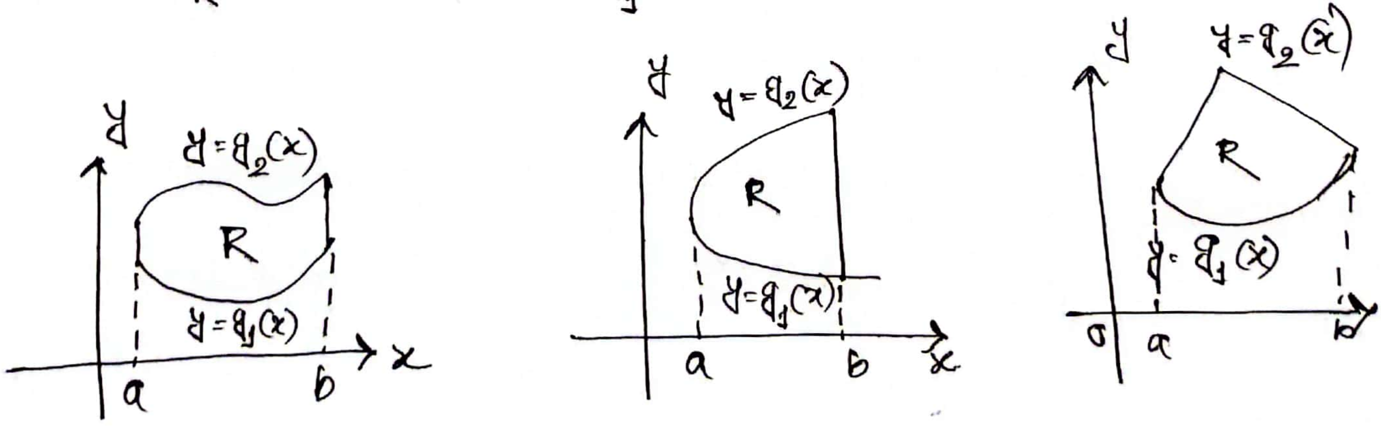
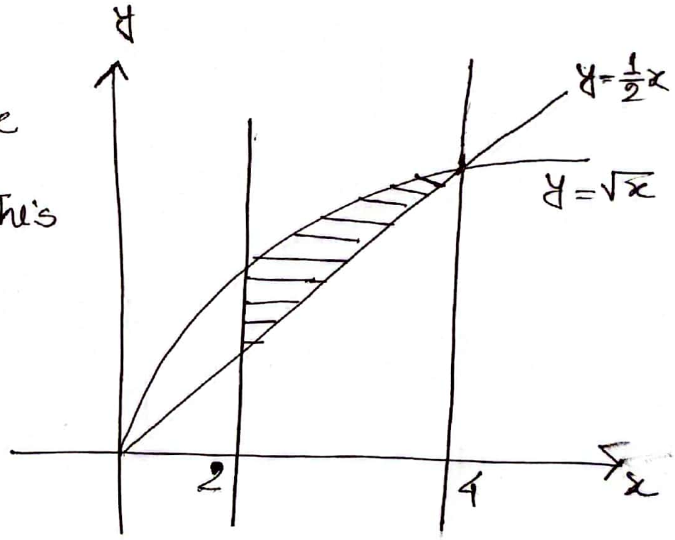


Figure:- Some type I regions.

For a type I region, the functions g_1 and g_2 must be continuous by they do not need to be defined by a single formula. For instance, in the third region of the figure, g_2 is a continuous piecewise defined function.

Example: Evaluate $\iint_R xy\,dA$ over the region R enclosed between $y = \frac{1}{2}x$, $y = \sqrt{x}$, $x=2$ and $x=4$.

Solution: We view R as a type I region. The region R and a vertical line corresponding to a fixed x are shown in the figure. This line meets the region R at the lower boundary $y = \frac{1}{2}x$ and the upper boundary $y = \sqrt{x}$. These are the y-limits of integration. Moving this line first left and then right yields the x-limits of integration, $x=2$ and $x=4$. Thus,



$$\iint_R xy\,dA = \int_2^4 \int_{x/2}^{\sqrt{x}} xy\,dy\,dx = \int_2^4 \left[\frac{xy^2}{2}\right]_{x/2}^{\sqrt{x}} dx = \int_2^4 \left(\frac{x^2}{2} - \frac{x^3}{8}\right) dx$$

$$= \left[\frac{x^3}{6} - \frac{x^4}{32}\right]_2^4 = \frac{11}{6}$$

Type II Region

A type II region is bounded below and above by the horizontal lines $y = c$ and $y = d$ and is bounded on the left and right by continuous curves $x = h_1(y)$ and $x = h_2(y)$ satisfying $h_1(y) \leqslant h_2(y)$ for $c \leqslant y \leqslant d$.

If R is a type II region on which $f(x, y)$ is continuous, then

$$\iint_R f(x, y)\, dA = \int_c^d \int_{h_1(y)}^{h_2(y)} f(x, y)\, dx\, dy .$$

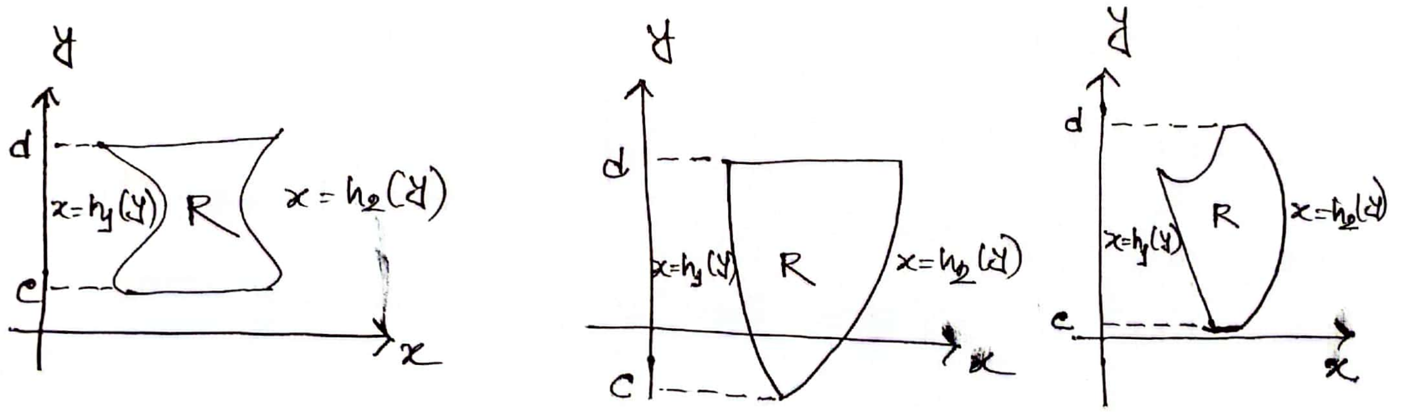


Figure :- Some Type II Regions

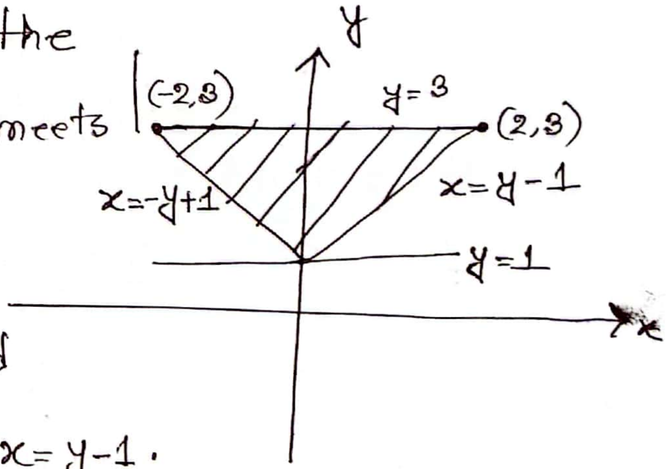
Practice Problem

Chapter 14·2 → 1-8, 15-26

Example :- Evaluate $\iint_R (2x-y^2)\,dA$ over the triangular region R enclosed between the lines $y=-x+1$, $y=x+1$ and $y=3$.

Homework :- Solve the problem considering type I region.

Solution :- We view R as a type II region. The region R and a horizontal line corresponding to a fixed y are shown in the given figure. This line meets the region R at its left hand boundary $x=1-y$ and its right hand boundary $x=y-1$.



These are the x-limits of integration. Moving the line first down and then up yields the y-limits. $y=1$ and $y=3$. Thus.

$$\iint_R (2x-y^2)\,dA = \int_1^3 \int_{-y+1}^{y-1} (2x-y^2)\,dx\,dy$$

$$= \int_1^3 \left[x^2-y^2x\right]_{-y+1}^{y-1} dy$$

$$= \int_1^3 (2y^2-2y^3)\,dy$$

$$= \left[\frac{2y^3}{3}-\frac{2y^4}{4}\right]_1^3 = \frac{-68}{3}, \quad \text{Ans}.$$

Homework :-

① Evaluate the following integrals viewing R as type-I or type-II region :-

a) $\iint_R x(1+y^2)^{-1/2}\,dA$; R is the region in the first quadrant enclosed by $y=x^2$, $y=4$, $x=0$.

b) $\iint_R (x-1)\,dA$; R is the region in the first ~~quadrant~~ quadrant enclosed between $y=x$ and $y=x^3$.

c) $\iint_R \sin(y^3)\,dA$, R is the region bounded by $y=\sqrt{x}$, $y=2$, $x=0$.

② Evaluate $\iint_D (2yx^2+9y^3)\,dA$ where D is the region bounded by $x=-2y^2$ and $x=y^3$