Applications of 1st Order ODE :

Example :

It is often desirable to describe the behavior of some real life system or phenomenon, whether physical, sociological, or even economic in mathematical terms. The mathematical description of a system of phenomenon is called a mathematical model and is constructed with certain goals in mind.

Example: A culture initially has P_0 number of bacteria. At $t = 1h$ the number of bacteria is measured to be $\frac{3}{2}P_0$. If the rate of growth is proportional to the number of bacteria $P(t)$ present at time t, determine the time necessary for the number of bacteria to triple.

Solution: From the question we can define the relation as

$$\frac{dP}{dt} \propto P$$

$$\Rightarrow \frac{dP}{dt} = kP \quad ; \quad k = \text{constant of proportionality}$$

Let $P(t)$ be the population at any time t

with conditions $P(0) = P_0$

$$P(1) = \frac{3}{2} P_0$$

Now,

$$\frac{dP}{dt} = kP$$

$$\Rightarrow \int \frac{dP}{P} = \int k\, dt$$

$$\Rightarrow \ln P = kt + c$$

$$\Rightarrow P = e^{kt} \cdot e^{c} \Rightarrow P(t) = c_1 e^{kt}$$

At $t = 0$, $P = P_0$

$$\Rightarrow c_1 e^{k \times 0} = P_0$$

$$\Rightarrow c_1 = P_0$$

At $t = 1$, $P = \frac{3}{2} P_0$.

$$\Rightarrow c_1 e^{k} = \frac{3}{2} P_0$$

$$\Rightarrow P_0 e^{k} = \frac{3}{2} P_0$$

$$\Rightarrow e^{k} = \frac{3}{2}$$

$$\Rightarrow k = \ln(3/2)$$

$$\Rightarrow k = 0.4055$$

Therefore, $P(t) = P_0 e^{0.4055t}$.

Since the population has become $3P_0$, then

$$3P_0 = P_0 e^{0.4055t}.$$

$$\Rightarrow t = 2.71\text{ h}.$$ Ans.

Example :- A breeder reactor converts relatively stable uranium-238 into the isotope plutonium -239. After 15 years it is determined that 0.043% of the initial amount A_0 of plutonium has disintegrated. Find the half life of the isotope if the rate of disintegration is proportional to the amount remaining.

Solution :- Let $A(t)$ denote the amount of plutonium remaining at time t. Therefore, the relation shows

$$\frac{dA}{dt} \propto A$$

$$\Rightarrow \frac{dA}{dt} = kA$$

with conditions $A(0) = A_0$

$$A(15) = A_0 - 0.043\% A_0$$

$$= \cancel{99.57\% A_0}\ 99.957\% A_0$$

If $\frac{dA}{dt} = kA$

$$\Rightarrow \int \frac{dA}{A} = \int k\,dt$$

$$\Rightarrow A = c_1 e^{kt}$$

When $t=0$, $A(0) = c_1 e^{k \times 0}$.

$$\Rightarrow A_0 = c_1$$

When $t=15$, $A(15) = 0.99957 A_0$

$$\Rightarrow A_0 e^{15k} = 0.99957 A_0.$$

$$\Rightarrow k = -2.86 \times 10^{-5}$$

Hence, $A = A_0 e^{-2.86 \times 10^{-5} t}$

For half life, $A(t) = \frac{1}{2} A_0$

$$\Rightarrow A_0 e^{-2.86 \times 10^{-5} t} = \frac{1}{2} A_0$$

$$\Rightarrow t \approx 24,180 \text{ years}.$$

Ans.

Homework:- A fossilized bone is found to contain one-thousandth of the C-14 level found in living matter. The half life is 5600 years. Estimate the age of the fossil.

Newton's Law of Cooling / Warming

The rate at which the temperature of a body changes is proportional to the difference between the temperature of the body and the temperature of the surrounding of the body and the temperature of the surrounding medium. If $T(t)$ represents the temperature of the body at time t, T_m the temperature of the surrounding medium, $\frac{dT}{dt}$ the rate at which the temperature of the body changes, then Newton's law of cooling/warming translates into the mathematical statement

$$\frac{dT}{dt} \propto T - T_m$$

$$\text{or, } \frac{dT}{dt} = k(T - T_m)$$

where k is a constant of proportionality. In either case, cooling or warming, if T_m is a constant, it stands to reason that $k<0$.

Example: When a cake is removed from an oven, its temperature is measured at 300°F. 3 minutes later, its temperature is 200°F. Find the temperature function $T(t)$ when the room temperature is 70°F.

Solution :- Let $T(t)$ be the temperature function at any time t.

Given that,

Surrounding temperature, $T_m = 70°F$

At $t=0$, $T = 300°F$

At $t=3$, $T = 200°F$

From the question

$$\frac{dT}{dt} \propto (T-70)$$

$$\Rightarrow \frac{dT}{dt} = k(T-70)$$

$$\Rightarrow \int \frac{dT}{T-70} = \int k\,dt$$

$$\Rightarrow \ln|T-70| = kt + c_1$$

$$\Rightarrow T = 70 + ce^{kt}.$$

At $t=0$, $T = 300$

$$\Rightarrow 70 + c\,e^0 = 300$$

$$\Rightarrow c = 230.$$

At $t=3$, $T = 200$

$$\Rightarrow 70 + ce^{k} = 206$$

$$\Rightarrow 230e^{k} = 130$$

$$\Rightarrow k = -0.19018$$

Therefore, $T(t) = 70 + 230e^{-0.19018t}$

Ans.

Practice Problem

Chapter 318 $\rightarrow$ 1-5, 6, 7, 8, 13-15, 17.