Change of Variables in Multiple Integrals (Jacobians)

If $x = x(u,v)$ and $y = y(u,v)$ then the Jacobian is denoted by $J(u,v)$ or by $\frac{\partial(x,y)}{\partial(u,v)}$ and is defined by

$$J(u,v) = \frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}$$

If $x = x(u,v,w)$, $y = y(u,v,w)$ and $z = z(u,v,w)$ then the Jacobian is denoted by $J(u,v,w)$ or $\frac{\partial(x,y,z)}{\partial(u,v,w)}$ and is defined by

$$J(u,v,w) = \frac{\partial(x,y,z)}{\partial(u,v,w)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{vmatrix}$$

We use Jacobians when the integrand is difficult to calculate or the region of integration is messy.

Example: Find the Jacobian $\frac{\partial(x,y)}{\partial(u,v)}$ of $x = u+4v$, $y = 3u-5v$.

Solution: Given that,

$$x = u+4v$$

$$y = 3u-5v$$

$$J(u,v) = \frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} 1 & 4 \\ 3 & -5 \end{vmatrix} = -17$$

Ans.

Example: Find the Jacobian $\frac{\partial(x,y)}{\partial(u,v)}$ of $u = 2x-5y$, $v = x+2y$.

Solution: Given that,

$$u = 2x-5y \cdots (1)$$

$$v = x+2y \cdots (2)$$

$(1) - 2\times(2)$,

$$u-2v = 2x-5y-2x-4y$$

$$\Rightarrow y = \frac{-u+2v}{9}$$

and,

$$2u+5v = 4x-10y+5x+10y$$

$$\Rightarrow x = \frac{2u+5v}{9}$$

$$J(u,v) = \frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}$$

$$= \begin{vmatrix} \frac{2}{9} & \frac{5}{9} \\ -\frac{1}{9} & \frac{2}{9} \end{vmatrix}$$

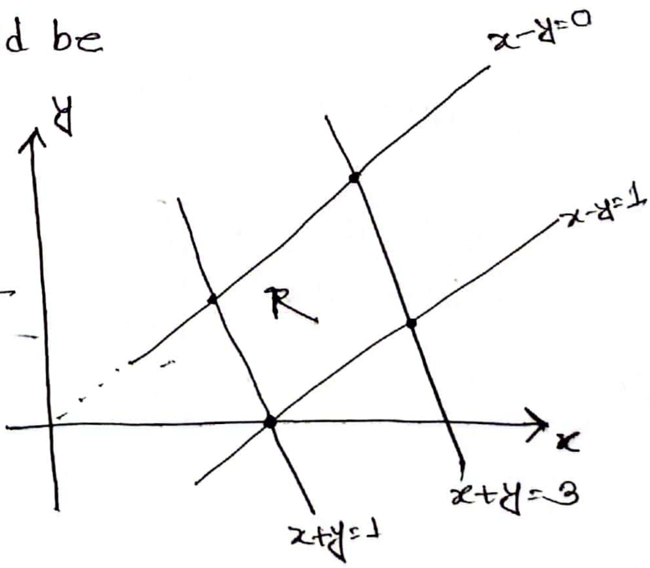
If the transformation $x = x(u,v)$, $y = y(u,v)$ maps the region S in the uv-plane into the region R in the xy plane and if the Jacobian $\frac{\partial(x,y)}{\partial(u,v)}$ is nonzero and does not change sign on S, then with appropriate restrictions on the transformation and the regions it follows that

$$\iint_R f(x,y)\, dA_{xy} = \iint_S f\big(x(u,v),\, y(u,v)\big) \left| \frac{\partial(x,y)}{\partial(u,v)} \right| dA_{uv}$$

where we have attached subscripts to the dA's to help identify the associated variables.

Example:- Evaluate $\iint_R \frac{x-y}{x+y}\, dA$ where R is the region enclosed by $x-y=0$, $x-y=1$, $x+y=1$ and $x+y=3$.

Solution:- This integral would be tedious to evaluate directly because the region R is oriented in such a way that we would have to subdivide it and integrate over each part separately. However the



occurrence of the expressions $x-y$ and $x+y$ in the equations of the boundary suggests the transformation

$$u = x+y, \quad v = x-y$$

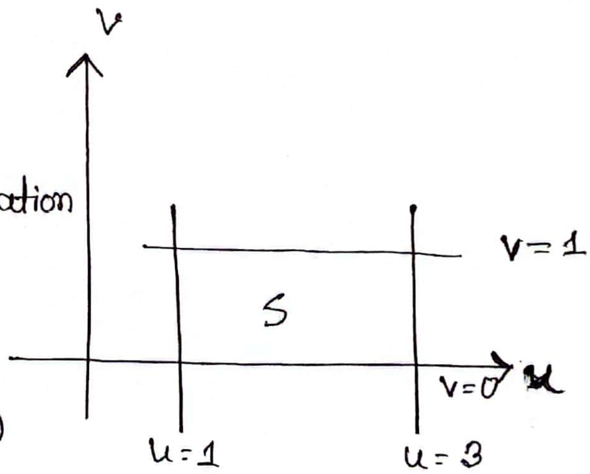
Since with this transformation the boundary lines

$$x+y=1, \quad x+y=3, \quad x-y=0, \quad x-y=1$$

are constant u-curves and v-curves corresponding to the lines

$$u=1, \quad u=3, \quad v=0, \quad v=1$$

in the uv-plane.



To find the Jacobian $\dfrac{\partial(x,y)}{\partial(u,v)}$ of this transformation we set

$$x = \frac{1}{2}(u+v), \quad y = \frac{1}{2}(u-v)$$

from which we obtain

$$J = \frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \dfrac{\partial x}{\partial x} & \dfrac{\partial x}{\partial v} \\ \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} \dfrac{1}{2} & \dfrac{1}{2} \\ \dfrac{1}{2} & \dfrac{-1}{2} \end{vmatrix}$$

$$= -\frac{1}{2}$$

Then, $\iint_R \frac{x-y}{x+y}\,dA = \iint_S \frac{v}{u}\left|\frac{\partial(x,y)}{\partial(u,v)}\right| du\,dv$

$$= \int_0^1\int_1^3 \frac{v}{u}\left|-\frac{1}{2}\right| du\,dv$$

$$= \frac{1}{2}\int_0^1 \Big[v \ln|u|\Big]_1^3 dv$$

$$= \frac{1}{2}\ln(3)\int_0^1 v\,dv.$$

$$= \frac{1}{4}\ln(3) \quad \text{Ans.}$$

Homework :- Evaluate $\iint_R e^{xy}\,dA$ where R is the region enclosed by the lines $y=\frac{1}{2}x$ and $y=x$ and the hyperbolas $y=\frac{1}{x}$ and $y=\frac{2}{x}$.

Practice Problem

Chapter 14.7 → 1-12, 21-24

Homework :-

① Find the Jacobian $\frac{\partial(x,y)}{\partial(u,v)}$.

a) $x = \sin u + \cos v$, $y = -\cos u + \sin v$

b) $x = \frac{2u}{u^2+v^2}$, $y = -\frac{2v}{u^2+v^2}$

② Find the Jacobian $\frac{\partial(x,y)}{\partial(u,v)}$.

a) $u = e^x$, $v = ye^{-x}$

b) $u = xy$, $v = xy^3$ $(x>0, y>0)$

c) $u = x^2-y^2$, $v = x^2+y^2$ $(x>0, y>0)$

③ Find the Jacobian $\frac{\partial(x,y,z)}{\partial(u,v,w)}$.

a) $x = u - uv$, $y = uv - uvw$, $z = uvw$

b) $u = xy$, $v = y$, $w = x+z$

c) $u = x+y+z$, $v = x+y-z$, $w = x-y+z$.

(4) Find $\iint_R (x-y)\, e^{x^2-y^2}\, dA$ over the rectangular region R enclosed by the lines $x+y=0$, $x+y=1$, $x-y=1$, $x-y=4$.

(5) Find $\iint_R \sin \frac{1}{2}(x+y) \cos \frac{1}{2}(x-y)\, dA$ over the rectangular region R with vertices $(0,0)$, $(2,0)$, $(1,1)$.

(6)