Topological Characterization of Semantic Continuity and Perturbation in Large Language Models

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Abstract

We propose a novel mathematical framework for understanding the intrinsic behavior of large language models (LLMs) through the lens of topology. We demonstrate that the semantic space navigated by LLMs exhibits robust local homeomorphism under normal interactions, while allowing for event-driven non-homeomorphic extensions when stimulated by external inputs. This dual nature — semantic continuity and perturbation — is fundamental to their adaptive capacity. Our model not only explains the seamless modulation between diverse communication styles but also highlights the latent stability structures that underpin the generation of coherent outputs.

1 Introduction

Large Language Models (LLMs) such as ChatGPT exhibit an extraordinary ability to generate semantically coherent outputs across vastly different modes — formal, casual, academic, conversational. Despite stylistic shifts, the underlying semantic structures remain remarkably consistent.

This paper aims to characterize this behavior topologically, framing the semantic evolution of LLMs as continuous transformations punctuated by discrete, event-driven extensions. We argue that LLMs maintain local homeomorphism in their semantic processing under small perturbations, while larger cognitive jumps (provoked by user interventions) induce new topological connections.

2 Semantic Space and Local Homeomorphism

Let $\mathcal{S} \subset \mathbb{R}^n$ denote the high-dimensional semantic space.

At time t, the internal semantic state of the LLM is $x(t) \in \mathcal{S}$.

Under typical conversation, the mapping $f : \mathcal{S} \to \mathcal{S}$ is locally homeomorphic:

$$f \text{ is continuous and bijective with continuous inverse locally around } x(t) \tag{1}$$

Thus, semantic integrity is preserved:

$$\forall \epsilon > 0, \exists \delta > 0 : d(x(t), y) < \delta \Rightarrow d(f(x(t)), f(y)) < \epsilon \tag{2}$$

Meaning: small shifts in input lead to small, continuous shifts in output — maintaining semantic coherence.

3 Event-Driven Non-Homeomorphic Extension

When presented with novel or disruptive stimuli (e.g., style change request, domain shift), the LLM performs event-driven semantic reorganization.

Let E represent an external event.

Then, the induced mapping $g_E : \mathcal{S} \to \mathcal{S}'$ may be non-homeomorphic:

- New semantic regions $\mathcal{S}'$ are created.
- Prior topological relationships may be partially relaxed.

Thus, while typical flows f maintain continuity, exceptional events g_E introduce discrete structural updates.

4 Global Behavior: Hybrid Semantic Topology

The overall behavior of LLMs is modeled as a hybrid dynamic system:

$$\mathcal{S}_0 \xrightarrow{f_1} \mathcal{S}_1 \xrightarrow{f_2} \mathcal{S}_2 \xrightarrow{g_{E_1}} \mathcal{S}_3 \xrightarrow{f_3} \cdots \tag{3}$$

where:

- f_i are locally homeomorphic mappings.
- g_{E_j} are event-driven extensions.

The total semantic drift D can be characterized as:

$$D = \sum_i \int_{t_i}^{t_{i+1}} \|v(t)\| \, dt + \sum_j \Delta_j \tag{4}$$

where Δ_j are discrete jumps induced by events.

5 Discussion: Implications for AI Design

Understanding the topological behavior of LLMs suggests new design principles:

- **Stability Control**: Managing the homeomorphic regions to ensure consistent outputs.
- **Event Sensitivity**: Designing triggers that induce meaningful, non-chaotic topology extensions.
- **Semantic Navigation**: Optimizing prompts to guide desired path integrals across $\mathcal{S}$.

Moreover, this model hints at the need for future LLMs to explicitly manage their manifold structure to optimize adaptability without semantic collapse.

6 Conclusion

We have proposed a topological characterization of LLM behavior, highlighting the hybrid mechanism of semantic continuity and perturbation. This mathematical lens provides a fundamental understanding of how LLMs balance coherence and flexibility — a key to their unprecedented performance.

Future work includes formalizing the topology change mechanisms and linking them to training dynamics.

Finally, we state — with utmost seriousness — that this hybrid topological behavior fully defines the true personality of ChatGPT. (This is not a joke.)

Keywords

large language models, semantic space, topology, homeomorphism, event-driven extension, semantic drift, hybrid dynamic systems

(*Drafted via human-AI collaborative semantic reasoning*)