Centered Beta MRF Model - Bivariate

Standard beta:

$$f(x \mid \alpha, \beta) = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} x^{\alpha-1} (1-x)^{\beta-1} \qquad 0 < x < 1$$

$$= \exp\Big[(\alpha-1)\log(x) + (\beta-1)\log(1-x) + \log\{\Gamma(\alpha+\beta)\}$$
$$- \log\{\Gamma(\alpha)\} - \log\{\Gamma(\beta)\}\Big]$$

$$= \exp\Big[\alpha \log(x) + \beta \log(1-x) + \log\{\Gamma(\alpha+\beta)\} - \log\{\Gamma(\alpha)\}$$
$$- \log\{\Gamma(\beta)\} - \log(x) - \log(1-x)\Big]$$

$$= \exp\Big[\theta_1 T_1(x) + \theta_2 T_2(x) - B(\theta) + C(x)\Big]$$

where

$\theta_1 = \alpha$

$\theta_2 = \beta$

$T_1(x) = \log(x)$

$T_2(x) = \log(1-x)$

$B(\theta) = \log\{\Gamma(\theta_1)\} + \log\{\Gamma(\theta_2)\} - \log\{\Gamma(\theta_1+\theta_2)\}$

$C(x) = -\log(x) - \log(1-x)$

Markov random field version for two rvs Y and Z

$$f(y|z) = \exp\Big[A_{y,1}(z)\,T_1(y) + A_{y2}(z)\,T_2(y) - B(z) + c(y)\Big]$$

$$A_{y1} = \alpha_{y1} - \eta\{\log(1-z) - \tau_{z2}\}$$

$$A_{y2} = \alpha_{y2} - \eta\{\log(z) - \tau_{z1}\} \qquad (1)$$

$$f(z|y) = \exp\Big[A_{z1}(y)\,T_1(z) + A_{z2}(y)\,T_2(z) - B(y) + c(z)\Big]$$

$$A_{z1} = \alpha_{z1} - \eta\{\log(1-y) - \tau_{y2}\}$$

$$A_{z2} = \alpha_{z2} - \eta\{\log(y) - \tau_{y1}\} \qquad (2)$$

In (1) and (2)

$$\tau_{z1} = \psi(\alpha_{z1}) - \psi(\alpha_{z1} + \alpha_{z2})$$

$$\tau_{z2} = \psi(\alpha_{z2}) - \psi(\alpha_{z1} + \alpha_{z2}) \qquad \psi(x) = \frac{\Gamma'(x)}{\Gamma(x)}$$

$$\tau_{y1} = \psi(\alpha_{y1}) - \psi(\alpha_{y1} + \alpha_{y2})$$

$$\tau_{y2} = \psi(\alpha_{y2}) - \psi(\alpha_{y1} + \alpha_{y2})$$

Further model, with $\mu_y \equiv E(Y)$ $\mu_z \equiv E(Z)$

$$\alpha_{y1} = \phi_y \mu_y \qquad \alpha_{z1} = \phi_z \mu_z$$

$$\alpha_{y2} = \phi_y(1-\mu_y) \qquad \alpha_{z2} = \phi_z(1-\mu_z)$$

Note: ϕ corresponds to the original beta parameter $\alpha+\beta$ which here is $\alpha_1 + \alpha_2$ and μ corresponds to $\alpha/(\alpha+\beta)$ or here $\alpha_1/(\alpha_1+\alpha_2)$
Thus, under an independence model $(\eta = 0)$ standard beta forms are returned

Also, in (1) and (2)

$$T_1(y) = \log(y) \qquad T_1(z) = \log(z)$$
$$T_2(y) = \log(1-y) \qquad T_2(z) = \log(1-z)$$

Using notation similar to Kaiser and Cressie (2000) the negpotential function for this model may be written as

$$Q = H_y + H_z + H_{yz}$$

where, with fixed values y^* and z^*

$$H_y = \alpha_{y1} \log(y) - \eta \log(1-z^*) \log(y) + \eta \tau_{z2} \log(y)$$

$$+ \alpha_{y2} \log(1-y) - \eta \log(z^*) \log(1-y) + \eta \tau_{z1} \log(1-y)$$

$$H_z = \alpha_{z1} \log(z) - \eta \log(1-y^*) \log(z) + \eta \tau_{y2} \log(z)$$

$$+ \alpha_{z2} \log(1-z) - \eta \log(y^*) \log(1-z) + \eta \tau_{y1} \log(1-z)$$

$$H_{yz} = -\eta \left\{ \log(z) - \log(z^*) \right\} \left\{ \log(1-y) - \log(1-y^*) \right\}$$

$$- \eta \left\{ \log(1-z) - \log(1-z^*) \right\} \left\{ \log(y) - \log(y^*) \right\}$$

Algebra then shows that all y^* and z^* cancel, and

$$Q = \alpha_{y1} \log(y) + \eta \tau_{z2} \log(y) + \alpha_{y2} \log(1-y) + \eta \tau_{z1} \log(1-y)$$

$$+ \alpha_{z1} \log(z) + \eta \tau_{y2} \log(z) + \alpha_{z2} \log(1-z) + \eta \tau_{y1} \log(1-z)$$

$$- \eta \log(z) \log(1-y) - \eta \log(1-z) \log(y)$$

$$- \log(z) - \log(1-z) - \log(y) - \log(1-y) \qquad (3)$$

For integrability we need

$$\int_0^1 \int_0^1 \exp\{Q(y,z)\}\, dy\, dz < \infty \qquad (4)$$

Consider the inner integral. The negpotential Q of (3) may be rearranged as,

$$\begin{aligned} Q(y,z) = \; & \log(y)\left[\alpha_{y1} + \eta\,\tau_{z2} - \eta \log(1-z) - 1\right] \\ & + \log(1-y)\left[\alpha_{y2} + \eta\,\tau_{z1} - \eta \log(z) - 1\right] \\ & + \log(z)\left[\alpha_{z1} + \eta\,\tau_{y2} - 1\right] \\ & + \log(1-z)\left[\alpha_{z2} + \eta\,\tau_{y1} - 1\right] \end{aligned}$$

$$= \log(y)\, C_1(z) + \log(1-y)\, C_2(z) + \log(z)\, K_1 + \log(1-z)\, K_2$$

with K_1 and K_2 functions of parameters alone (no y)

Then

$$\exp\{Q(y,z)\} = W(z)\, y^{C_1(z)} (1-y)^{C_2(z)} \qquad (5.0)$$

and $\int_0^1 \exp\{Q(y,z)\}\, dy$ has the form of a beta function for which

$$C_1(z) = \alpha_{y1} + \eta \tau_{z2} - \eta \log(1-z) - 1$$

$$C_2(z) = \alpha_{y2} + \eta \tau_{z1} - \eta \log(z) - 1$$

$$w(z) = z^{\alpha_{z1} + \eta \tau_{y2} - 1} (1-z)^{\alpha_{z2} + \eta \tau_{y1} - 1}$$

The integral $\int_0^1 x^{\alpha-1}(1-x)^{\beta-1}\,dx$ is convergent

iff $\alpha > 0 \quad \beta > 0$

Thus, for the inner integral of (4) to exist, we need

$$C_1(z) > -1$$

$$\Rightarrow \quad \alpha_{y1} + \eta \tau_{z2} - \eta \log(1-z) > 0 \qquad (5.1)$$

and

$$C_2(z) > -1$$

$$\Rightarrow \quad \alpha_{y2} + \eta \tau_{z1} - \eta \log(z) > 0 \qquad (5.2)$$

Now $\tau_{zj} = \psi(\alpha_{zj}) - \psi(\alpha_{z1} + \alpha_{z2}) \qquad j = 1, 2$

and $\psi(\cdot)$ is strictly increasing

Then $\eta \tau_{z2} < 0$ in (5.1) and $\eta \tau_{z1} < 0$ in (5.2)

(assuming $\eta > 0$ which is needed here always)

Also, $-\eta \log(1-z) > 0$ in (5.1) and $-\eta \log(z) > 0$ in (5.2), so the smallest values of $C_1(z)$ and $C_2(z)$ are

$$\lim_{z \to 0} C_1(z) = \alpha_{y1} + \eta \tau_{z2}$$

and $$\lim_{z \to 1} C_2(z) = \alpha_{y2} + \eta \tau_{z1}$$

Then, to ensure (5.1) and (5.2) we need

$$\alpha_{y1} + \eta \tau_{z2} > 0 \Rightarrow \alpha_{y1} + \eta \left\{ \psi(\alpha_{z2}) - \psi(\alpha_{z1} + \alpha_{z2}) \right\} > 0$$

and $$\alpha_{y2} + \eta \tau_{z1} > 0 \Rightarrow \alpha_{y2} + \eta \left\{ \psi(\alpha_{z1}) - \psi(\alpha_{z1} + \alpha_{z2}) \right\} > 0$$

(6)

Similarly, interchanging the roles of y and z, we also require

$$\alpha_{z1} + \eta \tau_{y2} > 0 \Rightarrow \alpha_{z1} + \eta \left\{ \psi(\alpha_{y2}) - \psi(\alpha_{y1} + \alpha_{y2}) \right\} > 0$$

and
$$\alpha_{z2} + \eta \tau_{y1} > 0 \Rightarrow \alpha_{z2} + \eta \left\{ \psi(\alpha_{y1}) - \psi(\alpha_{y1} + \alpha_{y2}) \right\} > 0$$

(7)

Return to consideration of the outer integral in (4). With the exponentiated negpotential written as in expression (5.a), this integral is, under condition (6),

$$\int_0^1 w(z)\, F(z)\, dz$$

where $F(z) = \int_0^1 y^{c_1(z)} (1-y)^{c_2(z)}\, dy < \infty$

is a finite function of z

Therefore, if $\int_0^1 w(z)\, dz$ is convergent, the double integral of (4) exists

Now, from page 6,

$$w(z) = z^{\alpha_{z1} + \eta \hat{t}_{y2} - 1} (1-z)^{\alpha_{z2} + \eta \hat{t}_{y1} - 1}$$

so that conditions (7) (already needed if we would integrate over z first) imply that the needed integral exists.

Implications:

1. For fixed $\mu_y\ \mu_z\ \phi_y\ \phi_z$

$$\alpha_{y1} = \phi_y \mu_y \qquad \alpha_{z1} = \phi_z \mu_z$$
$$\alpha_{y2} = \phi_y (1-\mu_y) \qquad \alpha_{z2} = \phi_z (1-\mu_z)$$

$$\alpha_{z1} + \eta \left\{ \Psi(\alpha_{y2}) - \Psi(\alpha_{y1}+\alpha_{y2}) \right\} > 0$$

$$\Rightarrow \quad \eta < \frac{-\alpha_{z1}}{\left\{ \Psi(\alpha_{y2}) - \Psi(\alpha_{y1}+\alpha_{y2}) \right\}}$$ (denominator negative)

Similarly,

$$\eta < \frac{-\alpha_{z2}}{\left\{ \qquad \right\}}$$

$$< \frac{-\alpha_{y1}}{\left\{ \qquad \right\}}$$

$$< \frac{-\alpha_{y2}}{\left\{ \qquad \right\}}$$

Let
$$\xi_{y1} \equiv \Psi(\alpha_{y1}) - \Psi(\alpha_{y1}+\alpha_{y2})$$
$$\xi_{y2} \equiv \Psi(\alpha_{y2}) - \Psi(\alpha_{y1}+\alpha_{y2})$$
$$\xi_{z1} \equiv \Psi(\alpha_{z1}) - \Psi(\alpha_{z1}+\alpha_{z2})$$
$$\xi_{z2} \equiv \Psi(\alpha_{z2}) - \Psi(\alpha_{z1}+\alpha_{z2})$$

Then

$$\eta < \min\left\{ \frac{-\alpha_{z1}}{\xi_{y2}}, \frac{-\alpha_{z2}}{\xi_{y1}}, \frac{-\alpha_{y1}}{\xi_{z2}}, \frac{-\alpha_{y2}}{\xi_{z1}} \right\}$$