Recap :

- Binomial distribution : $X \sim \text{binomial}(n, p)$

n: # of experiments

p: probability of success.

$X :=$ The number of successes out of "n" Bernoulli trials. $\boxed{x = 0, 1, 2, \ldots, n}$

- each trial is independent of the other trials

- The probability of success, p, is the same over all n trials.

- No closed form CDF. (e.g $P(X \le 4) = P(x=0$ or $x=1$ or $x=2$ or $x=3$ or $x=4)$)

- $EX = n \cdot p$

- $\text{var}X = np(1-p)$

- $SD(X) = \sqrt{\text{var}X} = \sqrt{np(1-p)}$

- Geometric distribution : $X \sim \text{Geom}(p)$

$X :=$ The number of trials until observing the first success. $\boxed{x = 1, 2, 3, \ldots}$

- Each trial is independent of others.

- The prob. of success, p, is the same for all trials.

(e.g $P(X \leq 20) = 1 - (1-p)^{20}$) ← - Closed form CDF : $F_X(x) = 1 - (1-p)^x$

- $EX = 1/p$

- $\text{Var}\, X = \dfrac{1-p}{p^2}$

- $SD(X) = \sqrt{\text{Var}\, X} = \sqrt{\dfrac{1-p}{p^2}}$

Mean

and

Variance

of Geometric Distrbution

The Geometric Distribution

$X \sim Geom.(p)$

Expected value:

$$E(X) = \frac{1}{p}$$

Variance:

$$Var(X) = \frac{1-p}{p^2}$$

Common Distributions

Background

Bernoulli

Binomial

Geometric

Example

NiCad batteries: An experimental program was successful in reducing the percentage of manufactured NiCad cells with internal shorts to around 1%. Let T be the test number at which the first short is discovered. Then, $T \sim \text{Geom}(p)$.

Calculate

$T \sim \text{Geom}(0.01)$, $f_X(x) = p(1-p)^{x-1}$, $x = 1, 2, 3, \ldots$

- $P(\text{1st or 2nd cell tested has the 1st short})$

$$P(T=1 \text{ or } T=2) = P(T=1) + P(T=2)$$

$$= f(1) + f(2) = p(1-p)^{1-1} + p(1-p)^{2-1} = (0.01)(0.99)^0 + (0.01)(0.99)^1 = 0.02$$

- $P(\text{at least 50 cells tested without finding a short})$

$$P(T > 50) = 1 - P(T \leq 50)$$

$$= 1 - F_X(50) = 1 - \left[1 - (1-0.01)^{50}\right]$$

$$= (1-0.01)^{50} = (0.99)^{50} = 0.61$$

Common Distributions

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Example

NiCad batteries: $T \sim \text{Geom}(p=0.01)$

Calculate the expected test number at which the first short is discovered and the variance in test numbers at which the first short is discovered.

$$E(T) = \frac{1}{p} = \frac{1}{0.01} = 100$$

(on average, you need to test 100 batteries until you observe the first short)

$$\text{Var}\,T = \frac{1-p}{p^2} = \frac{1-0.01}{(0.01)^2} = \frac{0.99}{(0.01)^2} = 9900$$

$$SD(T) = \sqrt{\text{Var}\,T} = \sqrt{9900}$$

Example

A shipment of 200 widgets arrives from a new widget distributor. The distributor has claimed that the widgets there is only a 10% defective rate on the widgets. Let X be the random variable asociated with the number of trials untill finding the first defective widgets.

- What is the probability distribution associated with this random variable X? Precisely specify the parameter(s).

  $X \sim \text{Geom}(p = 0.1)$

- How many widgets would you expect to test before finding the first defective widget?

$$EX = \frac{1}{p} = \frac{1}{0.1} = 10$$

So, we expect to test 10 widgets (on average) until finding the first defective one.

Common Distributions

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Bernoulli

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Example

You find your first defective widget while testing the thrid widget.

- What is the probability that a the first defective widget would be found **on** the third test if there are only 10% defective widgets from in the shipment?

$$P(x = 3) = p(1-p)^{x-1}$$

$$= 0.1(1-0.1)^{3-1}$$

$$= 0.1(0.9)^2 = \boxed{0.081}$$

Common Distributions

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Geometric

Example

- What is the probability that a the first defective widget would be found **by** the third test if there are only 10% defective widgets from in the shipment?

$$P(\cancel{x} \leq 3) = F_X(3) = 1 - (1 - p)^3$$

$$= 1 - (1 - .1)^3$$

$$= 1 - (0.9)^3 = \boxed{0.271}$$

in Geom: $F_X(x) = 1 - (1-p)^x$

in Geom: pmf: $f_X(x) := P_X(x) = p(1-p)^{x-1}$; $x = 1, 2, 3, \ldots$

The Poisson Distribution

Common Distributions

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The Poisson Distribution

Origin: A rare occurance is watched for over a specified interval of time or space.

It's often important to keep track of the total number of occurrences of some relatively rare phenomenon.

Definition

Consider a variable

> X : the count of occurences of a phenomenon across a specified interval of time or space

or

> X: the number of times the rare occurance is observed

Common Distributions

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The Poisson Distribution

Parameter

Poisson(λ)

probability function:

The **Poisson**~~(λ)~~ (λ) distribution is a discrete probability distribution with pmf

$$f(x) = \begin{cases} \boxed{\dfrac{e^{-\lambda}\lambda^x}{x!}} & x = 0, 1, \ldots \\ 0 & o.w. \end{cases}$$

For $\boxed{\lambda > 0}$

The Poisson Distribution

These occurrences must:

- be independent
- be sequential in time (no two occurances at once)
- occur at the same constant rate λ

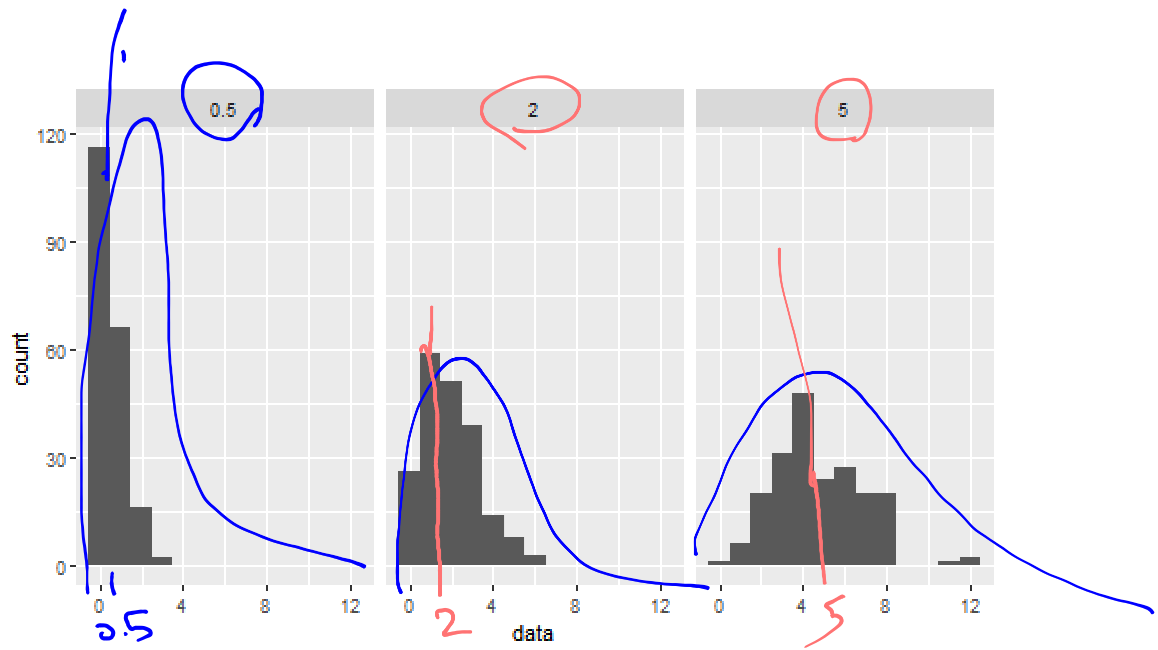
λ the $\boxed{\textit{rate parameter}}$, is the expected number of occurances in **the specified interval of time or space** (i.e $\mathrm{E(X)} = \lambda$)

The Poisson Distribution

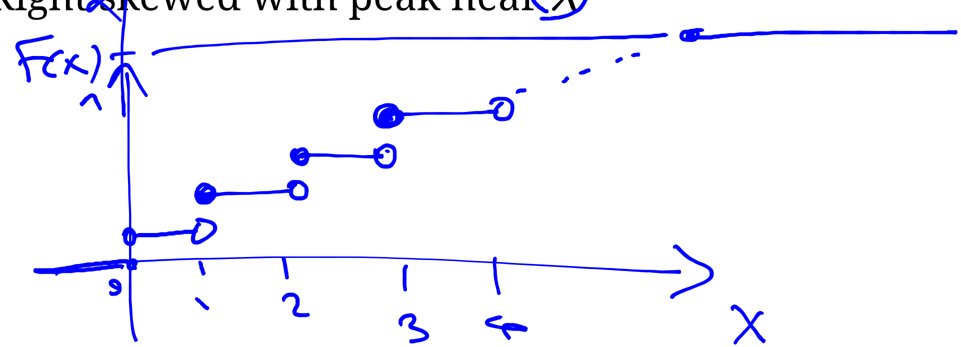
Examples that could follow a Poisson(λ) distribution :

> Y is the number of shark attacks off the coast of CA next **year**, $\lambda = 100$ attacks per year
>
> Z is the number of shark attacks off the coast of CA next **month**, $\lambda = 100/12$ attacks per month
>
> N is the number of α-particles emitted from a small bar of polonium, registered by a counter in a minute, $\lambda = 459.21$ particles per **minute**
>
> J is the number of particles per hour, $\lambda = 459.21 * 60 = 27,552.6$ particles per **hour**.

The Poisson Distribution



> Right skewed with peak near λ



The Poisson Distribution

For X a Poisson$\cancel{(\lambda)}$ (λ) random variable,

$$\mu = \mathrm{E}X = \sum_{x=0}^{\infty} x \frac{e^{-\lambda}\lambda^{x}}{x!} = \lambda$$

$$\sigma^2 = \mathrm{Var}X = \sum_{x=0}^{\infty} (x-\lambda)^2 \frac{e^{-\lambda}\lambda^{x}}{x!} = \lambda$$

Common Distributions

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Example

Arrivals at the library

Some students' data indicate that between 12:00 and (one min.) 12:10pm on Monday through Wednesday, an average of around 125 students entered Parks Library at ISU. Consider modeling

> M : the number of students entering the ISU library between 12:00 and 12:01pm next Tuesday

Model $M \sim \text{Poisson}(\lambda)$. What would a reasonable choice of λ be?

$$\lambda = \frac{125}{10}$$

Common Distributions

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Example

Arrivals at the library

Under this model, the probability that between $\boxed{\text{10 and 15}}$ students arrive at the library between 12:00 and 12:01 PM is:

$$M \sim \text{Poisson}\left(\lambda = \frac{125}{10}\right)$$

$$p_M(m) = \frac{e^{-\lambda}\lambda^m}{m!}, \quad m = 0, 1, 2, \ldots$$

$$P(10 \le M \le 15) = p(10) + p(11) + \cdots + p(15)$$

$$= \frac{e^{-12.5}(12.5)^{10}}{10!} + \frac{e^{-12.5}(12.5)^{11}}{11!} + \cdots \frac{e^{-12.5}(12.5)^{15}}{15}$$

$= 0.6$

Common Distributions

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Shark attacks

No Closed Form for CDF in Poisson!

Let X be the number of unprovoked shark attacks that will occur off the coast of Florida $\boxed{\text{next year}}$. Model

$$X \sim \text{Poisson}(\lambda).$$

From the shark data at http://www.flmnh.ufl.edu/fish/sharks/statistics/FLactivity.html, 246 unprovoked shark attacks occurred from 2000 to 2009.

What would a reasonable choice of λ be?

246 attacks per 10 years.

The rate (λ) for next year (only one year)

$$\lambda = \frac{246}{10} = 24.6$$

Shark attacks

Under this model, calculate the following:

- P(no attacks next year)

$$P(X=0) = P(0) = \frac{e^{-24.6}(24.6)^0}{0!} = e^{-24.6} \approx 2.07 \times 10^{-11}$$

- P(at least 5 attacks)

$$P(X \geq 5) = 1 - P(X < 5) = 1 - P(X \leq 4)$$

$$= 1 - [P(X=0 \text{ or } X=1 \text{ or } X=2 \text{ or } X=3 \text{ or } X=4]$$

$$= 1 - [P(X=0) + P(X=1) + \cdots + P(X=4)]$$

- P(more than 10 attacks)

$$P(X > 10) = 1 - P(X \leq 10)$$

$$1 - \left[\frac{e^{-24.6}(24.6)^0}{0!} + \frac{e^{-24.6}(24.6)^1}{1!} + \cdots + \frac{e^{-24.6}(24.6)^4}{4!} \right]$$

$$= 1 - e^{-24.6}(3626 6812)$$

$$= 0.999249$$