STAT 305: Chapter 6

Introduction to formal statistical inference

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Chapter 6.1: Large-sample confidence intervals for a mean

Large Sample Inference

Large Sample Confidence Interval

Formal statistical inference uses probability theory to quantify the reliability of data-based conclusions. We want information on a population.e.g

- true mean fill weight of food jams
- true mean strength of metal bars
- true mean of the number of accidents on a highway in Iowa

We can then use:

1. Point estimates:

   e.g **sample mean** $\overline{X}$ of the strength of metal bars is 4.83.

   > We would then say that $\overline{X}$ is an estimate for true (population) mean μ.

Large Sample Inference

Large Sample Confidence Interval

1. Interval estimates:
   μ is likely to be inside an interval. (e.g
   $\mu \in (2.84, 5.35))$

Then we can say **we are confident that the true mean of the strength of metal bars (μ) is somewhere in the** $(2.84, 5.35)$

But the question is *how confident*?

Large Sample Inference

Large Sample Confidence Interval

Many important engineering applications of statistics fit the following mold. Values for parameters of a data-generating process are unknown. Based on data, the goal is

> 1.identify an inteval of values likely to contain an *unknown parameter*
>
> 2.qualify "how likely" the interval is to cover the correct value of the unknown parameter.

Confidence Interval

Definition and the use

Confidence Interval

Definition: confidence interval for a *parameter* (or function of one or more parameters) is a *data-based interval* of numbers thought likely to contain the parameter (or function of one or more parameters) possessing a stated probability-based confidence or reliability.

> A confidence interval is a realization of a **random interval**, an interval on the real line with a random variable at one or both of the endpoints.

Confidence
Interval

Example:[Instrumental drift]

Let Z be a measure of instrumental drift of a random voltmeter that comes out of a certain factory. Say $Z \sim N(0, 1)$. Define a random interval:

$$(Z - 2, Z + 2)$$

What is the probability that -1 is inside the interval?

$$
\begin{aligned}
P(-1 \text{ is in } (Z - 2, Z + 2)) &= P(Z - 2 < -1 < Z + 2) \\
&= P(Z - 1 < -1 < Z + 3) \\
&= P(-1 < -Z < 3) \\
&= P(-3 < Z < 1) \\
&= \Phi(1) - \Phi(-3) \\
&= 0.84.
\end{aligned}
$$

Large Sample Inference

Confidence Interval

Example:[More practice]

Calculate:

- $P(2 \text{ in } (X - 1, X + 1)), X \sim N(2, 4)$

$$
\begin{aligned}
P(2 \in (X - 1, X + 1)) &= P(X - 1 < 2 < X + 1) \\
&= P(-1 < 2 - X < 1) \\
&= P(-1/2 < X < 1/2) \\
&= \Phi(1/2) - \Phi(-1/2) \\
&= 0.6915 - 0.3085 \\
&= 0.383
\end{aligned}
$$

**Confidence
Interval**

Example:[Abstract random intervals]

Let's say $X_1, X_2, \ldots, X_n$ are iid with $n \geq 25$, mean μ, variance σ^2. We can find a random interval that provides a lower bound for μ with $1 - \alpha$ probability:

We want **A** such that $P(\mu \in (A, +\infty)) = 1 - \alpha$.
We know by CLT:

$$\overline{X} \approx N(\mu, \sigma^2/n)$$

Therefore,

$$\frac{\overline{X} - \mu}{\sigma/\sqrt{n}} \sim N(0, 1).$$

Example:[Abstract random intervals]

Then

$$P(\frac{\overline{X} - \mu}{\sigma/\sqrt{n}} \leq Z_{1-\alpha}) \approx 1 - \alpha$$

$$\Rightarrow P(\overline{X} - Z_{1-\alpha}\frac{\sigma}{\sqrt{n}} \leq \mu) \approx 1 - \alpha$$

$$\Rightarrow P(\mu \in (\overline{X} - Z_{1-\alpha}\frac{\sigma}{\sqrt{n}}, +\infty)) \approx 1 - \alpha$$

Now if we set

$$A = \overline{X} - Z_{1-\alpha}\frac{\sigma}{\sqrt{n}}$$

then we have

$$P(\mu \in (A, +\infty)) \approx 1 - \alpha$$

Large Sample Inference

Confidence Interval

Example:[Abstract random intervals]

Calculate:

-
$$P(\mu \in (\overline{X} - z_{1-\alpha/2}\tfrac{\sigma}{\sqrt{n}}, \overline{X} + z_{1-\alpha/2}\tfrac{\sigma}{\sqrt{n}})), X \sim N(\mu, \sigma^2)$$

$$
\begin{aligned}
&P(\mu \in (\overline{X} - z_{1-\alpha/2}\frac{\sigma}{\sqrt{n}}, \overline{X} + z_{1-\alpha/2}\frac{\sigma}{\sqrt{n}})) \\
=&P(\overline{X} - z_{1-\alpha/2}\frac{\sigma}{\sqrt{n}} < \mu < \overline{X} + z_{1-\alpha/2}\frac{\sigma}{\sqrt{n}}) \\
=&P(-z_{1-\alpha/2}\frac{\sigma}{\sqrt{n}} < \mu - \overline{X} < z_{1-\alpha/2}\frac{\sigma}{\sqrt{n}}) \\
=&P(-z_{1-\alpha/2}\frac{\sigma}{\sqrt{n}} < \overline{X} - \mu < z_{1-\alpha/2}\frac{\sigma}{\sqrt{n}}) \\
=&P(-z_{1-\alpha/2} < \frac{\overline{X} - \mu}{\sigma/\sqrt{n}} < z_{1-\alpha/2}) \\
=&P(-z_{1-\alpha/2} < Z < z_{1-\alpha/2}) \\
\approx&1 - \alpha
\end{aligned}
$$

The last result is by **CLT** assuming that $n \geq 25$

A large-n confidence interval

for μ involving σ

A Large-n confidence interval for μ involving σ

A **1-α confidence interval** for an unknown parameter is the realization of a random interval that contains that parameter with probability $1 - \alpha$.

> α is called the **confidence level**

For random variables $X_1, X_2, \ldots, X_n$ iid with $E(X_1) = \mu, Var(X_1) = \sigma^2$, a $1 - \alpha$ confidence interval for μ is

$$\left(\overline{x} - z_{1-\alpha/2} \frac{\sigma}{\sqrt{n}}, \overline{x} + z_{1-\alpha/2} \frac{\sigma}{\sqrt{n}} \right)$$

which is a **realization** from the random interval

$$\left(\overline{X} - z_{1-\alpha/2} \frac{\sigma}{\sqrt{n}}, \overline{X} + z_{1-\alpha/2} \frac{\sigma}{\sqrt{n}} \right).$$

Large Sample Inference

Confidence Interval

CI for μ

- Two-sided $1 - \alpha$ confidence interval for μ

$$\left(\overline{x} - z_{1-\alpha/2}\frac{\sigma}{\sqrt{n}}, \overline{x} + z_{1-\alpha/2}\frac{\sigma}{\sqrt{n}}\right)$$

- One-sided $1 - \alpha$ confidence interval for μ with a upper confidence bound

$$\left(-\infty, \overline{x} + z_{1-\alpha/2}\frac{\sigma}{\sqrt{n}}\right)$$

- One-sided $1 - \alpha$ confidence interval for μ with a lower confidence bound

$$\left(\overline{x} - z_{1-\alpha/2}\frac{\sigma}{\sqrt{n}}, +\infty\right)$$

Example:[Fill weight of jars]

Suppose a manufacturer fills jars of food using a stable filling process with a known standard deviation of $\sigma = 1.6$g. We take a sample of $n = 47$ jars and measure the sample mean weight $\overline{x} = 138.2$g. A two-sided 90% confidence interval, $\alpha = 0.1$, for the true mean weight μ is:

$$\left(\overline{x} - z_{1-\alpha/2}\frac{\sigma}{\sqrt{n}}, \overline{x} + z_{1-\alpha/2}\frac{\sigma}{\sqrt{n}}\right)$$

$$= \left(\overline{x} - z_{1-0.1/2}\frac{\sigma}{\sqrt{n}}, \overline{x} + z_{1-0.1/2}\frac{\sigma}{\sqrt{n}}\right)$$

$$= \left(138.2 - z_{.95}\frac{1.6}{\sqrt{47}}, 138.2 + z_{.95}\frac{1.6}{\sqrt{47}}\right)$$

$$= \left(038.2 - 1.64(.23), 138.2 + 1.64(.23)\right)$$

$$= \left(137.82, 138.58\right)$$

or we can write it as 138.2 ± 0.38g

Large Sample Inference

Confidence Interval

CI for μ

Example: [Fill weight of jars]

Interpretation:

> We are 90% **confident** that the **true mean** is between 137.82g and 138.58g
>
> or we can say
>
> If we took 100 more samples of 47 jams each, roughly *90 of those samples would have a confidence interval containing the true mean fill weight*

Example:[Fill weight of jars]

What if we just want to be sure that the true mean fill weight is high enough?

We could use a one-sided 90% CI with a lower bound:

$$\left(\overline{x} - z_{1-\alpha}\frac{\sigma}{\sqrt{n}} \; , \; +\infty\right)$$

$$= \left(138.2 - z_{.9}\frac{1.6}{\sqrt{47}} \; , \; +\infty\right)$$

$$= \left(137.91 \; , \; +\infty\right)$$

Then we would say:

> We are 90% confident that the true mean fill weight is above 137.91

Example:[Hard disk failures]

F. Willett, in the article "The Case of the Derailed Disk Drives?" (**Mechanical Engineering**, 1988), discusses a study done to isolate the cause of link code A failure in a model of Winchester hard disk drive.

For each disk, the investigator measured the breakaway torque (in. oz.) required to loosen the drive's interrupter flag on the stepper motor shaft.

Breakaway torques for 26 disk drives were recorded, with a sample mean of 11.5 in. oz. Suppose you know the true standard deviation of the breakaway torques is 5.1 in. oz.

CI for μ

Example:[Hard disk failures]

Calculate and interpret:

- A two-sided 90% confidence interval for the true mean breakaway torque of the relevant type of Winchester drive.

$$(\overline{x} - z_{1-\alpha/2}\frac{\sigma}{\sqrt{n}}, \overline{x} + z_{1-\alpha/2}\frac{\sigma}{\sqrt{n}})$$

$$= (\overline{x} - z_{1-0.1/2}\frac{\sigma}{\sqrt{n}}, \overline{x} + z_{1-0.1/2}\frac{\sigma}{\sqrt{n}})$$

$$= (11.5 - z_{.95}\frac{5.1}{\sqrt{26}} \ , \ 11.5 + z_{.95}\frac{5.1}{\sqrt{26}})$$

$$= (11.5 - 1.64(1.0002) \ , \ 11.5 + 1.64(1.0002))$$

$$= (9.86 \ , \ 13.14)$$

Interpretation: we are 90% confident that the true mean breaking torque lies between 9.86 and 13.14 in.oz.

Example:[Width of a CI]

If you want to estimate the breakaway torque with a 2-sided, 95% confidence interval with ± 2.0 in. oz. of precision, what sample size would you need?

▎ Interval precision = intervald half width

Therefore, for a two-sided 95% CI we have

$$\left(\overline{x} - z_{1-\alpha/2} \frac{\sigma}{\sqrt{n}}, \overline{x} + z_{1-\alpha/2} \frac{\sigma}{\sqrt{n}} \right)$$

which means that the precision is $z_{1-\alpha/2} \frac{\sigma}{\sqrt{n}}$

We want $z_{1-\alpha/2} \frac{\sigma}{\sqrt{n}} \leq 2$

Example:[Width of a CI]

So,

$$z_{1-.05/2}\frac{5.1}{\sqrt{n}} \leq 2$$

$$z_{.975}\frac{5.1}{\sqrt{n}} \leq 2$$

$$1.96\frac{5.1}{\sqrt{n}} \leq 2$$

$$\frac{9.996}{\sqrt{n}} \leq 2$$

$$\Rightarrow n \geq 24.98$$

$$\Rightarrow n \geq 25$$

We would need a sample of at least 25 disks to have at least a precision of 2 in.oz

A large-n confidence interval

for μ when σ is Unknown

Large Sample
Inference

Confidence
Interval

CI for μ

CI for μ
unknown σ^2

A generally applicable large-n confidence interval for μ

Although the equations for a $1 - \alpha$ confidence interval is mathematically correct, it is severely limited in its usefulness because it requres us to know σ (the population variance). It is unusual to have to estimate μ and know σ in real life.

If $n \geq 25$ and σ is *unknown*, $Z = \frac{\overline{X} - \mu}{s/\sqrt{n}}$, where

$$s = \sqrt{\frac{1}{n-1} \sum_{i=1}^{n} (x_i - \overline{x})^2}.$$

is still **approximately standard normally distributed**.

So, you can replace σ in the confidence interval formula with the sample standard deviation, s.

Large Sample
Inference

Confidence
Interval

CI for μ

CI for μ
unknown σ^2

A generally applicable large-n confidence interval for μ

- Two-sided $1 - \alpha$ confidence interval for μ

$$\left(\overline{x} - z_{1-\alpha/2}\frac{s}{\sqrt{n}}, \overline{x} + z_{1-\alpha/2}\frac{s}{\sqrt{n}}\right)$$

- One-sided $1 - \alpha$ confidence interval for μ with a upper confidence bound

$$\left(-\infty\,,\, \overline{x} + z_{1-\alpha/2}\frac{s}{\sqrt{n}}\right)$$

- One-sided $1 - \alpha$ confidence interval for μ with a lower confidence bound

$$\left(\overline{x} - z_{1-\alpha/2}\frac{s}{\sqrt{n}}\,,\, +\infty\right)$$

Large Sample
Inference

Confidence
Interval

CI for μ

CI for μ
unknown σ^2

Example:

Suppose you are a manufacturer of construction equipment. You make 0.0125 inch wire rope and need to determine how much weight it can hold before breaking so that you can label it clearly. Here are breaking strengths, in kg, for 41 sample wires:

```
 [1] 100.37  96.31  72.57  88.02 105.89
 [6] 107.80  75.84  92.73  67.47  94.87
[11] 122.04 115.12  95.24 119.75 114.83
[16] 101.79  80.90  96.10 118.51 109.66
[21]  88.07  56.29  86.50  57.62  74.70
[26]  92.53  86.25  82.56  97.96  94.92
[31]  62.00  93.00  98.44 119.37 103.70
[36]  72.40  71.29 107.24  64.82  93.51
[41]  86.97
```

The sample mean breaking strength is 91.85 kg and the sample standard deviation is 17.6 kg.

Large Sample
Inference

Confidence
Interval

CI for μ

CI for μ
unknown σ^2

Example: Using the appropriate 95% confidence interval, try to determine whether the breaking strengths meet the requirement of at least 85 kg.

$$(1 - \alpha = .95 \quad, \quad \overline{x} = 91.85 \quad, \quad s = 17.6 \quad, \quad n = 41)$$

The CI is then

$$(\overline{x} - z_{1-\alpha/2}\frac{s}{\sqrt{n}} \, , \ +\infty)$$

$$= (91.85 - z_{.95}\frac{17.6}{\sqrt{41}} \, , \ +\infty)$$

$$= (91.85 - 1.64\frac{17.6}{\sqrt{41}} \, , \ +\infty)$$

$$= (87.3422 \, , \ +\infty)$$

> With 95% confidence, we have shown that the true mean breaking strength is above 87.3422 kg.
> Hence, we meet the 85kg requirement with 95% confidence