General
Info

Reminder:
RVs

Discrete?

Terms &
Notation

Summaries

Almost all of the devices for describing relative frequency (empirical) distributions in Ch. 3 have versions that can describe (theoretical) probability distributions.

1. Measures of location == Mean
   population : μ
   sample : $\bar{x}$
2. Measures of spread == variance
   pop : σ^2
   sample : s^2
3. Histogram == probability histograms based on theoretical probabilities

Mean

and

Variance

of Discrete Random Variables

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Mean of a Discrete Random Variable

For a discrete random varable, X, which can take values $x_1, x_2, \ldots$ we define **the mean of X** (also known as **the expected value of X**) as:

value

$$E(X) = \sum_{i=1}^{n} x_i \cdot f(x_i)$$

probability

We often use the symbol μ instead of $E(X)$.

Also, just to be confusing, you will often see $\boxed{EX}$ instead of $\boxed{E(X)}$. Use context clues.

Example:

Suppose that the we roll a die and let T be the number of dots facing up. Find the expected value of T.

x	1	2	3	4	5	6
$P(X=x)$	1/6	1/6	1/6	1/6	1/6	1/6

$$E(X) = \sum_{x=1}^{6} x\,P(X=x)$$

$$= 1\,P(X=1) + 2\,P(X=2) + 3\,P(X=3) + 4\,P(X=4) + 5\,P(X=5)$$

$$+ 6\,P(X=6)$$

$$= 1 \cdot \tfrac{1}{6} + 2 \cdot \tfrac{1}{6} + 3 \cdot \tfrac{1}{6} + 4 \cdot \tfrac{1}{6} + 5 \cdot \tfrac{1}{6} + 6 \cdot \tfrac{1}{6}$$

$$= \tfrac{1}{6}(1+2+3+4+5+6) = \frac{21}{6} = 3.5$$

Interpretation:

If rolling a fair die once, on average, we see 3.5.

Expected value of "anything" with a

probability function $f_X(x)$ < continuous / discrete

$$E(\text{anything}) = \sum \int (\text{anything}) \times (\text{Prob.})$$

$$X \sim f_X(x) = P(X=x) \; : \; \text{discrete}$$

$$E(X) = \sum_{x \in S_X} x \cdot P(X=x)$$

support of X

$$E(X^2) = \sum_{x \in S_X} x^2 P(X=x)$$

X	0	1	2	3
$P(X=x)$	0.2	0.1	0.5	0.2

$$E(X) = 0 \cdot P(X=0) + 1 \cdot P(X=1) + 2P(X=2) + 3 \cdot P(X=3)$$

$$= 0 + 1 \cdot 0.1 + 2(0.5) + 3(0.2) = 1.7$$

$$E(\sqrt{X}) = \sqrt{0}\, P(X=0) + \sqrt{1}\, P(X=1) + \sqrt{2}\, P(X=2) + \sqrt{3}\, P(X=3)$$

$=$

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Recall: population variance:

$$\sigma^2 = \frac{1}{n}\sum_{i=1}^{n}(X_i - \underline{\mu})^2$$

Variance of a Discrete Random Variable

For a discrete random varable, X, which can take values $x_1, x_2, \ldots$ and has mean μ we define **the variance of X** as:

$$Var(X) = \sum_{i=1}^{n}(x_i - \overset{E(X)}{\mu})^2 \cdot f(x_i)$$

There are other usefule ways to write this, most importantly:

$$Var(X) = \sum_{i=1}^{n} x_i^2 \cdot f(x_i) - \mu^2$$

which is the same as

$$\mathrm{Var} X = \sum_x (x - \mathrm{E}X)^2 f(x) = \boxed{\mathrm{E}(X^2) - (\mathrm{E}X)^2}.$$

Variance of a Discrete Random Variable

Example:

Suppose that the we roll a die and let T be the number of dots facing up. What is the variance of T?

t	1	2	3	4	5	6
$P(T=t)$	$1/6$	$1/6$	$1/6$	$1/6$	$1/6$	$1/6$

$$E(T) = 21/6 = 3.5$$

$$Var(T) = E(X^2) - [E(X)]^2$$

$$E(X^2) = \sum_{x=1}^{6} x^2 P(X=x)$$

$$= 1^2 \overbrace{P(X=1)}^{1/6} + 2^2 \overbrace{P(X=2)}^{1/6} + 3^2 \overbrace{P(X=3)}^{1/6} + 4^2 \overbrace{P(X=4)}^{1/6}$$

$$+ 5^2 \overbrace{P(X=5)}^{1/6} + 6^2 \overbrace{P(X=6)}^{1/6}$$

$$= \frac{1}{6}\left(1^2 + 2^2 + 3^2 + 4^2 + 5^2 + 6^2\right)$$

$$= 15.17$$

$$\operatorname{var}(\underline{X}) = E(\underline{X}^2) - [E(\underline{X})]^2 = 15.17 - (3.5)^2$$

$$= 2.92$$

The standard deviation of X is

$$S = \sqrt{\operatorname{var}(X)} = \sqrt{2.92} = 1.7088$$

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Variance of a Discrete Random Variable

Example

Say we have a random variable Q with pmf:

q	f(q)	$q f(q)$	$q^2 f(q)$
1	0.34	0.34	0.34
2	0.10	0.2	$2^2(0.1)$
3	0.22	0.66	$3^2(0.22)$
7	0.34	2.28	$7^2(0.34)$

Find the variance and standard deviation

$$E(Q) = ? \longrightarrow \sum_{q=\{1,2,3,7\}} q f_Q(q) = 0.34 + 0.2 + .66 + 2.28 = 3.58$$

$$E(Q^2) = ? \longrightarrow \sum_{q} q^2 f(q) = 19.38$$

$$\mathrm{Var}(Q) = E(Q^2) - [E(Q)]^2$$

$$= 19.38 - (3.58)^2$$

$$= 6.56.$$

Variance is ALWAYS ≥ 0

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Summary

Discrete Random Variables

- Discrete RVs are RVs that will take one of a countable set of values.
- When working with a discrete random variable, it is common to need or use the RV's
  - probability distribution: the values the RV can take and their probabilities
  - probability function: a function where $f(x) = P(X = x)$
  - cumulative probability function: a function where $F(x) = P(X \leq x)$.
  - mean: a value for X defined by $EX = \sum_x x \cdot f(x)$
  - variance: a value for X defined by $VarX = \sum_x (x - \mu)^2 \cdot f(x)$

Your Turn:

Chapter 5 Handout 1

Common Distributions

Working with Off The Shelf Random Variables

General Info

Common Distributions

Background

Common Distributions

Why Are Some Distributions Worth Naming?

Even though you may create a random variable in a unique scenario, the way that it's probability distribution behaves (mathematically) may have a lot in common with other random variables in other scenarios. For instance,

> I roll a die until I see a 6 appear and then stop. I call X the number of times I have to roll the die in total.
>
> I flip a coin until I see heads appear and then stop. I call Y the number of times I have to flip the coin in total.
>
> I apply for home loans until I get accepted and then I stop. I call Z the number of times I have to apply for a loan in total.

Why Are Some Distributions Worth Naming? (cont)

In each ot the above cases, we count the number of times we have to do some action until we see some specific result. The only thing that really changes from the random variables perspective is the likelyhood that we see the specific result each time we try.

Mathematically, that's not a lot of difference. And if we can really understand the probability behavior of one of these scenarios then we can move our understanding to the different scenario pretty easily.

By recognizing the commonality between these scenarios, we have been able to identify many random variables that behave very similarly. We describe the similarity in the way the random variables behave by saying that they have a common/shared distribution.

We study the most useful ones by themselves.

The Bernoulli Distribution

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Common Distributions

Background

Bernoulli

The Bernoulli Distribution

Origin: A random experiment is performed that results in one of two possible outcomes: success or failure. The probability of a successful outcome is p.

Definition: X takes the value 1 if the outcome is a success. X takes the value 0 if the outcome is a failure.

probability function:

$$f(x) = \begin{cases} p & x = 1, \\ 1-p & x = 0, \\ 0 & o.w. \end{cases}$$

success

failure

otherwise.

which can also be written as

$$f(x) = \begin{cases} p^x(1-p)^{1-x} & x = 0, 1 \\ 0 & o.w. \end{cases}$$

Example:

- In fliping a coin:

Define success as observing a "Head" in fliping a coin.

let X be a r.v associated with observing a "Head".

Then X is a binary trial (r.v) with probability of success $\boxed{p = 1/2}$

and probability of failure $\boxed{1 - p = 1 - 1/2 = \frac{1}{2}}$.

- what if we flip the coin twice! what is the prob. that we see Heads in both trials? (each trial is independent)

P(Heads in both trials) = P(X=1 in both trials) [success]

= P(X=1 on the first trial $\boxed{\text{and}}$ X=1 on the second trial)

$= P(X=1 \text{ on the first trial}) \cdot P(X=1 \text{ on the second trial})$

$= P(X=1) \cdot P(X=1)$

Bernoulli trial $= \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4}$