Intro to Statistial Inference

Confidence Intervals

and

Hypothesis Tests

Statistical Inference

Many applications of statistics in engineering take the following basic mold:

1. Identify an interval of values that is likely to contain some unknown parameter
2. Quantify "how likely" the interval we identified is to cover the true value of the parameter

Example
A machine filling a container with a liquid will, across all possible containers, dispense a mean volume of liquid, μ. We suspect that μ is between 10.001 and 10.002 liters.

Example
Two methods of performing a surgery are being compared. The first has success rate of r_1 and the second has a success rate of r_2. We suspect that $r_1 - r_2$ is between 0.2 and 0.3

Inference

Motivation

Concerns

Statistical Inference

Of course how confident we are about the range we have depends on the strength of our evidence and the size of the range:

With smaller intervals we would naturally be less confident. For instance,

- I am very sure that the average weight of a squirrel is between 0 and 100 pounds.
- I am less confident that the average weight of a squirrel is between 1.245 and 1.246 poinds.

With more evidence (or data) we are more confident. For instance,

- If I have a data set consisting of the weight 10,000 captured squirrels, I would be more confident that an interval around the average weight of the captured squirrels would contain the true average weight of a squirrel.
- If I have a data set consisting of the weight of 3 captured squirrels, I would have little confidence that an interval around

Statistical Inference

More importantly, the amount of data we have to work with influences our confidence in one of the most important rules in statistics:

> **Central Limit Theorem**
> If $X_1, X_2, \ldots, X_n$ are independent and identically distributed (iid) random variables each with mean μ and variance σ^2 and let the random variable $\bar{X} = \frac{1}{n}X_1 + \frac{1}{n}X_2 + \ldots + \frac{1}{n}X_n$. Then
>
> 1. $E(\bar{X}) = \mu$
> 2. $Var(\bar{X}) = \sigma^2/n$
> 3. For large n, $\bar{X}$ is approximately normally distributed (limit goes to normal…)

In otherwords, if n is small we may *not* be approximately normal…

Large Sample Confidence Intervals

Statistical Inference

Large n

In the case of our book, if n is larger than 30, then we have enough data to say that $\bar{x}$ is approximately normal and that it has a mean μ and a variance σ^2/n that match our population mean and variance.

In other words, if $n \geq 30$

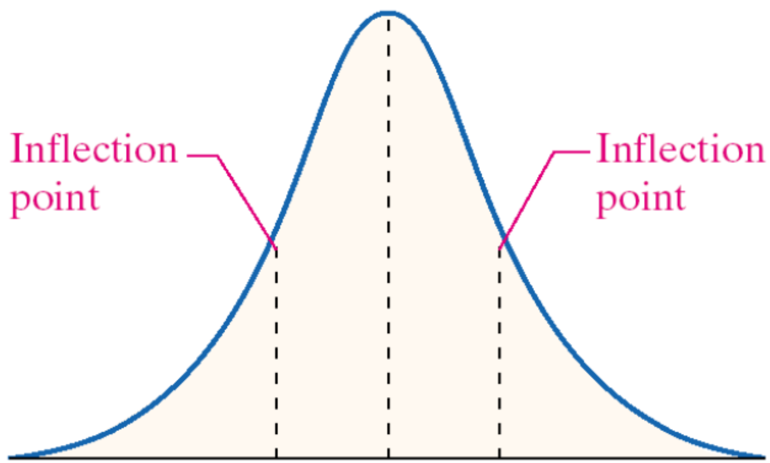
- $E(\bar{X}_n) = \mu$
- $Var(\bar{X}_n) = \frac{1}{n}\sigma^2$
- $\bar{X}_n$ follows a normal distribution

This means that if we know σ we can make probability statements about how close the value of $\bar{X}_n$ will be to μ - even if we don't know μ!

The Normal Distribution

Large n

Regardless of the values of μ and σ^2, the normal pdf has the following shape:



Since for $n \geq 30$, $\bar{X}_n$ is approximately normal with mean μ and standard deviation $\sigma/\sqrt{n}$ then we can say:

$$\bar{X}_n \sim N\left(\mu, \frac{1}{n}\sigma^2\right) \rightarrow \frac{\bar{X}_n - \mu}{\sigma/\sqrt{n}} \sim N(0, 1)$$

Creating an Interval

Large n

We can use this relationship between normal and standard normal to start making probability statements about intervals around μ:

$$\begin{aligned}
0.95 &= P\left(-1.96 \le Z \le 1.96\right) \\
&= P\left(-1.96 \le \frac{\bar{X}_n - \mu}{\sigma/\sqrt{n}} \le 1.96\right) \\
&= P\left(-1.96\frac{\sigma}{\sqrt{n}} \le \bar{X}_n - \mu \le 1.96\frac{\sigma}{\sqrt{n}}\right) \\
&= P\left(-\bar{X}_n - 1.96\frac{\sigma}{\sqrt{n}} \le -\mu \le -\bar{X}_n + 1.96\frac{\sigma}{\sqrt{n}}\right) \\
&= P\left(\bar{X}_n - 1.96\frac{\sigma}{\sqrt{n}} \le \mu \le \bar{X}_n + 1.96\frac{\sigma}{\sqrt{n}}\right)
\end{aligned}$$

which means the probability that μ is in the interval $\bar{X} \pm 1.96\frac{\sigma}{\sqrt{n}}$ is 0.95 - i.e., we are 95% confident the interval contains μ.

Creating an Interval

Large n

What this shows is that once we know the value of $\bar{X}_n = \bar{x}_n$ then the interval $(\bar{x}_n - 2\sigma/n, \bar{x}_n + 2\sigma/n)$ has a 95% chance to contain the true value μ. So even though we will not *know* μ, we can be 95% confident that it is in some interval.

> **Two-Sided Confidence Interval**
> If $z_{1-\alpha/2}$ is the value s.t. $P(Z \leq z_{1-\alpha/2}) = 1 - \alpha/2$ then the for a sample mean $\bar{x}_n$ with $n \geq 30$ and known σ, then the interval
>
> $$\bar{x}_n \pm z_{1-\alpha/2}\frac{\sigma}{\sqrt{n}}$$
>
> defines a $100(1 - \alpha)\%$ confidence interval

Creating an Interval

Large n

Notice that the confidence interval width changes based on our need for certainty. By looking at some common values of $z_{1-\alpha/2}$, we can see the trade off:

Desired Confidence	$1-\alpha/2$	$z_{1-\alpha/2}$	2-Sided Interval Width
80%	0.10	1.28	$2.56\frac{\sigma}{\sqrt{n}}$
90%	0.05	1.645	$3.29\frac{\sigma}{\sqrt{n}}$
95%	0.025	1.96	$3.92\frac{\sigma}{\sqrt{n}}$
98%	0.001	2.33	$4.66\frac{\sigma}{\sqrt{n}}$
99%	0.0005	2.58	$5.16\frac{\sigma}{\sqrt{n}}$

Notice that the width depends on three things:

1. The variability of what we are estimating: more variable = larger σ = larger interval
2. The size of our sample: larger n = smaller $1/\sqrt{n}$ = smaller interval
3. How confident we choose to be: more confident = larger $z_{1-\alpha/2}$ = larger interval

Large n

Example

Suppose that we randomly sample 50 ligthbulbs from production and find that the average lifetime of the sample is 420.25 hours. Suppose that we know the variance of the bulbs lifetime is 25.36 hours.

- Provide a 80% confidence interval for the true mean lifetime of all bulbs from production.
- Provide a 95% confidence interval for the true mean lifetime of all bulbs from production.
- Provide a 92% confidence interval for the true mean lifetime of all bulbs from production.