Example: Example 19 in text book

Suppose that S and R have joint probability density function:

$$f_{SR}(s,r) = \frac{1}{16.5}e^{\left(-\frac{s}{16.5}\right)}\frac{1}{\sqrt{2\pi(.25)}}e^{\left(-(r-s)^2/2(.25)\right)}$$

if $s > 0$ and is 0 otherwise.

1. Find $f_{S|R}(s|r)$. What is the distribution of S if $R = r$?

In order to do this, we can start with the formula for conditional probability density functions:

$$f_{S|R}(s|r) = \frac{f_{SR}(s,r)}{f_R(r)}$$

Since we are being given r then we can treat it like a constant - however, we still need $f_R(r)$ in this problem. Again, we have

$$f_R(r) = \int_{-\infty}^{\infty} f_{SR}(s,r)ds$$

$$= \int_{-\infty}^{\infty} \frac{1}{16.5}e^{\left(-\frac{s}{16.5}\right)}\frac{1}{\sqrt{2\pi(.25)}}e^{\left(-(r-s)^2/2(.25)\right)}ds$$

$$= \frac{1}{16.5}\frac{1}{\sqrt{2\pi(.25)}}\int_{-\infty}^{\infty} e^{\left(-\frac{s}{16.5}\right)}e^{\left(-(r^2-2rs+s^2)/(.5)\right)}ds$$

$$= \frac{1}{16.5}\frac{1}{\sqrt{2\pi(.25)}}\int_{-\infty}^{\infty} e^{\left(-\frac{1}{16.5}s\right)}e^{\left(-2r^2+4rs-2s^2)\right)}ds$$

$$= \frac{1}{16.5}\frac{1}{\sqrt{2\pi(.25)}}\int_{-\infty}^{\infty} e^{\left(-\frac{1}{16.5}s-2r^2+4rs-2s^2)\right)}ds$$

$$= e^{\left(-2r^2\right)}\frac{1}{16.5}\frac{1}{\sqrt{2\pi(.25)}}\int_{-\infty}^{\infty} e^{\left((4r-\frac{1}{16.5})s-2s^2)\right)}ds$$

From the handout about pdf integration, we can see that if we set $\sigma^2 = 0.25$ and $\mu = 0.25\left(4r - \frac{1}{16.5}\right) = \left(r - \frac{1}{64}\right) = \left(\frac{64r-1}{64}\right)$ then the integral on the left becomes

$$\int_{-\infty}^{\infty} e^{\left((4r-\frac{1}{16.5})s-2s^2)\right)}ds = \int_{-\infty}^{\infty} e^{\left(\frac{\mu}{\sigma^2}s-\frac{1}{2\sigma^2}s^2)\right)}ds$$

$$= \sqrt{2\pi\sigma^2}\exp\left(\frac{\mu^2}{2\sigma^2}\right)$$

$$= \sqrt{\pi(0.5)}\exp\left(\frac{(64r-1)^2}{(64^2)(0.5)}\right)$$

which allows us to return to finding $f_R(r)$:

$$f_R(r) = e^{\left(-2r^2\right)}\frac{1}{16.5}\frac{1}{\sqrt{2\pi(.25)}}\left[\int_{-\infty}^{\infty} e^{\left((4r-\frac{1}{16.5})s-2s^2)\right)}ds\right]$$

$$= e^{\left(-2r^2\right)}\frac{1}{16.5}\frac{1}{\sqrt{2\pi(.25)}}\left[\sqrt{\pi(0.5)}\exp\left(\frac{(64r-1)^2}{(64^2)(0.5)}\right)\right]$$

Since we aren't interested in this function in particular, we may not want to spend valuable time simplifying it. Instead, we can go back to our real problem:

$$f_{S|R}(s|r) = \frac{f_{SR}(s,r)}{f_R(r)} = \frac{\frac{1}{16.5}e^{\left(-\frac{s}{16.5}\right)}\frac{1}{\sqrt{2\pi(.25)}}e^{\left(-(r-s)^2/2(.25)\right)}}{e^{(-2r^2)}\frac{1}{16.5}\frac{1}{\sqrt{2\pi(.25)}}\left[\int_{-\infty}^{\infty} e^{\left((4r-\frac{1}{16.5})s-2s^2)\right)}ds\right]}$$

$$= \frac{\frac{1}{16.5}e^{\left(-\frac{s}{16.5}\right)}\frac{1}{\sqrt{2\pi(.25)}}e^{\left(-(r-s)^2/2(.25)\right)}}{e^{(-2r^2)}\frac{1}{16.5}\frac{1}{\sqrt{2\pi(.25)}}\left[\sqrt{\pi(0.5)}\exp\left(\frac{(64r-1)^2}{(64^2)(0.5)}\right)\right]}$$

and by canceling out as much as possible:

$$f_{S|R}(s|r) = \frac{\frac{1}{16.5}e^{\left(-\frac{s}{16.5}\right)}\frac{1}{\sqrt{2\pi(.25)}}e^{\left(-(r-s)^2/2(.25)\right)}}{e^{(-2r^2)}\frac{1}{16.5}\frac{1}{\sqrt{2\pi(.25)}}\left[\sqrt{\pi(0.5)}\exp\left(\frac{(64r-1)^2}{(64^2)(0.5)}\right)\right]}$$

$$= \frac{e^{\left(-\frac{s}{16.5}\right)}e^{\left(-(r-s)^2/2(.25)\right)}}{e^{(-2r^2)}\left[\sqrt{\pi(0.5)}\exp\left(\frac{(64r-1)^2}{(64^2)(0.5)}\right)\right]}$$

$$= \frac{1}{\sqrt{\pi(0.5)}}e^{\left(-\frac{s}{16.5}\right)}e^{\left(-(r-s)^2/2(.25)\right)}e^{\left(2r^2\right)}\exp\left(-\frac{(64r-1)^2}{(64^2)(0.5)}\right)$$

$$= \frac{1}{\sqrt{\pi(0.5)}}\exp\left(-\frac{4}{64}s-2(r-s)^2+2r^2-\frac{(64r-1)^2}{(64^2)(0.5)}\right)$$

$$= \frac{1}{\sqrt{\pi(0.5)}}\exp\left(-\frac{4}{64}s+4rs-2s^2-\frac{(64r-1)^2}{(64^2)(0.5)}\right)$$

$$= \frac{1}{\sqrt{\pi(0.5)}}\exp\left(-2s^2+\left(4r-\frac{4}{64}\right)s-\frac{(64r-1)^2}{(64^2)(0.5)}\right)$$

$$= \frac{1}{\sqrt{2\pi(0.25)}}\exp\left(-\frac{\left(s-\frac{64r-1}{64}\right)^2}{2(0.25)}\right)$$

If you are clever, you may recognize this as a normal distribution with variance 0.25 and mean $\frac{64r-1}{64}$ (i.e., $S|R \sim N((64r-1)/64, 0.25)$).

2. Find the expected value of S given that $R = 2$.

Since $S|R = r$ follows a normal distribution with mean $\frac{64r-1}{64}$ and then for $R = 2$ we get $\frac{64(2)-1}{64} = 127/64$

3. Find the expected value of S given that $R = 3$.

Since $S|R = r$ follows a normal distribution with mean $\frac{64r-1}{64}$ and then for $R = 3$ we get $\frac{64(3)-1}{64} = 191/64$