

# STAT 305: Chapter 6

## Introduction to formal statistical inference

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# Chapter 6 1 · Large-sample confidence intervals for a mean

## Large Sample Confidence Interval

Formal statistical inference uses probability theory to quantify the reliability of data-based conclusions. We want information on a population.e.g

- true mean fill weight of food jams
- true mean strength of metal bars
- true mean of the number of accidents on a highway in Iowa

We can then use:

### 1. Point estimates:

e.g **sample mean**  $\bar{X}$  of the strength of metal bars is 4.83.

We would then say that  $\bar{X}$  is an estimate for true (population ) mean  $\mu$ .

# Large Sample Inference

## Large Sample Confidence Interval

1. Interval estimates:

$\mu$  is likely to be inside an interval. (e.g  
 $\mu \in (2.84, 5.35)$ )

Then we can say **we are confident that the true mean of the strength of metal bars ( $\mu$ ) is somewhere in the (2.84, 5.35)**

But the question is *how confident?*

## Large Sample Inference

### Large Sample Confidence Interval

Many important engineering applications of statistics fit the following mold. Values for parameters of a data-generating process are unknown. Based on data, the goal is

1. identify an interval of values likely to contain an *unknown parameter*
2. qualify "how likely" the interval is to cover the correct value of the unknown parameter.

# Confidence Interval

Definition and the use

# Large Sample Inference

## Confidence Interval

### Confidence Interval

**Definition:** confidence interval for a *parameter* (or function of one or more parameters) is a *data-based interval* of numbers thought likely to contain the parameter (or function of one or more parameters) possessing a stated probability-based confidence or reliability.

A confidence interval is a realization of a **random interval**, an interval on the real line with a random variable at one or both of the endpoints.

$$P\left(\bar{x} - Z_{1-\alpha/2} \frac{\sigma}{\sqrt{n}} < \mu < \bar{x} + Z_{1-\alpha/2} \frac{\sigma}{\sqrt{n}}\right) = 1-\alpha$$

$$P(5.5'' < \mu \leq 6.3'') = 0.95$$

# Large Sample Inference

## Confidence Interval

**Example:** [Instrumental drift]

Let  $Z$  be a measure of instrumental drift of a random voltmeter that comes out of a certain factory. Say  $Z \sim N(0, 1)$ . Define a random interval:

$$(Z - 2, Z + 2)$$

What is the probability that  $-1$  is inside the interval?

$$\begin{aligned} P(-1 \text{ is in } (Z - 2, Z + 2)) &= P(Z - 2 < -1 < Z + 2) \\ &= P(Z - 1 < -1 < Z + 3) \\ &= P(-1 < -Z < 3) \\ &= P(-3 < Z < 1) \\ &= \Phi(1) - \Phi(-3) \\ &= 0.84. \end{aligned}$$

with 84% confidence,  $-1$  falls within the interval

$$(Z - 2, Z + 2) \text{ when } Z \sim N(0, 1)$$

# Large Sample Inference

## Confidence Interval

Example:[More practice]

Calculate:

- $P(2 \text{ in } (X - 1, X + 1)), X \sim N(2, 4)$

$$\mu \quad \sigma^2 = 4$$

$$\begin{aligned} P(2 \in (X - 1, X + 1)) &= P(X - 1 < 2 < X + 1) \\ &= P(-1 < 2 - X < 1) \\ &= P(-1/2 < Z < 1/2) \\ &= \Phi(1/2) - \Phi(-1/2) \\ &= 0.6915 - 0.3085 \\ &= 0.383 \end{aligned}$$

$P(-1 - 2 \leq -x < 1 - 2)$

$= P(-3 < -x < -1)$

$= P(3 > x > 1) = P(1 < x < 3) = 0.383$

$= P\left(\frac{1-2}{2} < \frac{x-2}{2} < \frac{3-2}{2}\right) = P\left(-\frac{1}{2} < Z < +\frac{1}{2}\right)$

with 38.1 confidence, 2 falls within the interval  $(X-1, X+1)$  when  $X \sim N(2, 4)$

# Large Sample Inference

## Confidence Interval

Find  $A$  s.t.

$$P(\mu \in (A, +\infty)) = 1 - \alpha$$

**Example:** [Abstract random intervals]

Let's say  $X_1, X_2, \dots, X_n$  are iid with  $n \geq 25$ , mean  $\mu$ , variance  $\sigma^2$ . We can find a random interval that provides a lower bound for  $\mu$  with  $1 - \alpha$  probability:

We want  $A$  such that  $P(\mu \in (A, +\infty)) = 1 - \alpha$ .

We know by CLT:

$$\bar{X} \approx N(\mu, \sigma^2/n)$$

Therefore,

→ (Standardization)

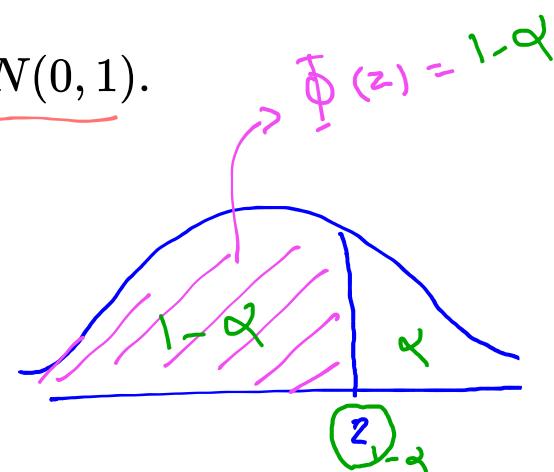
$$\frac{\bar{X} - \mu}{\sigma/\sqrt{n}} \sim N(0, 1).$$

$$\Phi(z) = 1 - \alpha$$

+  
we know

$$P(Z \leq z) \approx 1 - \alpha$$

$$\Phi(z)$$



# Large Sample Inference

## Confidence Interval

Example:[Abstract random intervals]

Then

$Z$

$$P\left(\frac{\bar{X} - \mu}{\sigma/\sqrt{n}} \leq Z_{1-\alpha}\right) \approx 1 - \alpha$$

solve it for  $\mu$   $\Rightarrow P\left(\bar{X} - Z_{1-\alpha} \frac{\sigma}{\sqrt{n}} \leq \mu\right) \approx 1 - \alpha$

(Recall: we're looking for a CI for  $\mu$ )  $\Rightarrow P\left(\mu \in \left(\bar{X} - Z_{1-\alpha} \frac{\sigma}{\sqrt{n}}, +\infty\right)\right) \approx 1 - \alpha$

Now if we set

$$A = \bar{X} - Z_{1-\alpha} \frac{\sigma}{\sqrt{n}}$$

then we have

$$P(\mu \in (A, +\infty)) \approx 1 - \alpha$$

# Large Sample Inference

## Confidence Interval

✓  
 $\mu$  is a parameter. It  
 doesn't have any dist.  
 But  $Z \sim N(0,1)$

Recall: we find  $CI$   
 For Unknown parameters  
 of the population.

**Example:** [Abstract random intervals]

Calculate:

$$P(\mu \in (\bar{X} - z_{1-\alpha/2} \frac{\sigma}{\sqrt{n}}, \bar{X} + z_{1-\alpha/2} \frac{\sigma}{\sqrt{n}})), \bar{X} \sim N(\mu, \sigma^2)$$

$$P(\underline{\mu} \in (\bar{X} - z_{1-\alpha/2} \frac{\sigma}{\sqrt{n}}, \bar{X} + z_{1-\alpha/2} \frac{\sigma}{\sqrt{n}}))$$

$$\text{Solve it for } \mu = P(\bar{X} - z_{1-\alpha/2} \frac{\sigma}{\sqrt{n}} < \mu < \bar{X} + z_{1-\alpha/2} \frac{\sigma}{\sqrt{n}})$$

$$= P(-z_{1-\alpha/2} \frac{\sigma}{\sqrt{n}} < \mu - \bar{X} < z_{1-\alpha/2} \frac{\sigma}{\sqrt{n}})$$

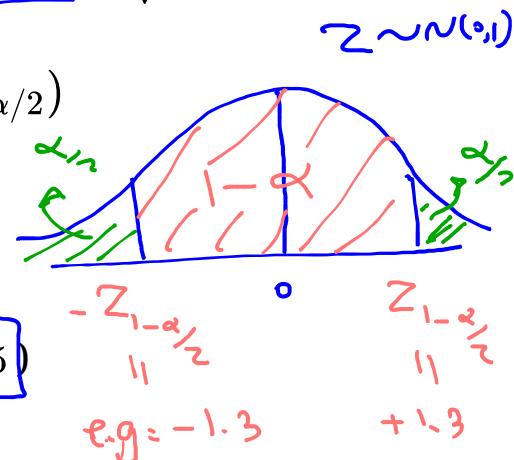
$$= P(-z_{1-\alpha/2} \frac{\sigma}{\sqrt{n}} < \bar{X} - \mu < z_{1-\alpha/2} \frac{\sigma}{\sqrt{n}})$$

$$= P(-z_{1-\alpha/2} < \frac{\bar{X} - \mu}{\sigma/\sqrt{n}} < z_{1-\alpha/2})$$

$$= P(-z_{1-\alpha/2} < Z < z_{1-\alpha/2})$$

$$\approx 1 - \alpha$$

(The last result is by CLT assuming that  $n \geq 25$ )



# Large Sample Inference

## Confidence Interval

**Example:** [Abstract random intervals]

So,  $\mu$  falls within the interval

\*  $(\bar{X} - z_{1-\alpha/2} \frac{\sigma}{\sqrt{n}}, \bar{X} + z_{1-\alpha/2} \frac{\sigma}{\sqrt{n}})$  with the probability of

$1 - \alpha$  for  $X \sim N(\mu, \sigma^2)$

$$\alpha = 0.05$$

i.e. we are  $100(1 - \alpha)\%$  (e.g. 95%) confident

that the true parameter  $\mu$  in the dist.

$X \sim N(\mu, \sigma^2)$  falls within the interval

$$(\bar{x} - z_{1-\alpha/2} \frac{\sigma}{\sqrt{n}}, \bar{x} + z_{1-\alpha/2} \frac{\sigma}{\sqrt{n}})$$

## A large- $n$ confidence interval

for  $\mu$  involving  $\sigma$

# Large Sample Inference

## Confidence Interval

CI for  $\mu$

$$\text{e.g.: } \alpha = 0.02$$

$$\Rightarrow 100(1 - 0.02) \% = 98\%$$

### A Large- $n$ confidence interval for $\mu$ involving $\sigma$

A  $100(1 - \alpha)\%$  confidence interval for an unknown parameter is the realization of a random interval that contains that parameter with probability  $1 - \alpha$ .

|  $\alpha$  is called the **confidence level**

For random variables  $X_1, X_2, \dots, X_n$  iid with  $E(X_1) = \mu$ ,  $Var(X_1) = \sigma^2$ , a  $100(1 - \alpha)\%$  confidence interval for  $\mu$  is

For large  $n$  ( $n \geq 25$ )  
 $\rightarrow (\bar{x} - z_{1-\alpha/2} \frac{\sigma}{\sqrt{n}}, \bar{x} + z_{1-\alpha/2} \frac{\sigma}{\sqrt{n}})$

which is a **realization** from the **random interval**

$$(\bar{X} - z_{1-\alpha/2} \frac{\sigma}{\sqrt{n}}, \bar{X} + z_{1-\alpha/2} \frac{\sigma}{\sqrt{n}}).$$

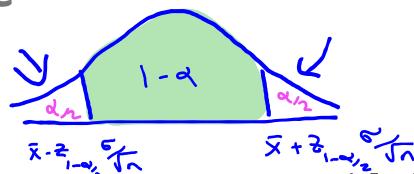
# Large Sample Inference

In General (for large  $n$ )

## Confidence Interval

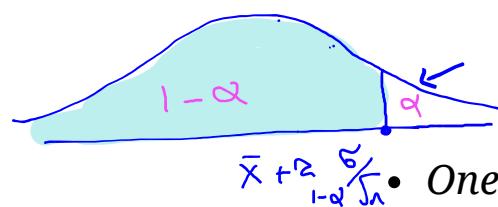
### CI for $\mu$

- Two-sided  $100(1 - \alpha)\%$  confidence interval for  $\mu$



$$(\bar{x} - z_{1-\alpha/2} \frac{\sigma}{\sqrt{n}}, \bar{x} + z_{1-\alpha/2} \frac{\sigma}{\sqrt{n}})$$
✓

- One-sided  $100(1 - \alpha)\%$  confidence interval for  $\mu$  with a upper confidence bound

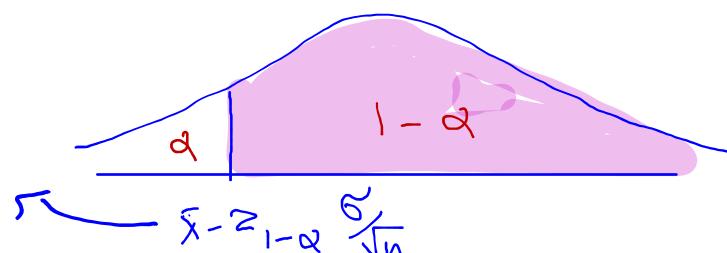


$$(-\infty, \bar{x} + z_{1-\alpha} \frac{\sigma}{\sqrt{n}})$$

- One-sided  $100(1 - \alpha)\%$  confidence interval for  $\mu$  with a lower confidence bound

$$(\bar{x} - z_{1-\alpha} \frac{\sigma}{\sqrt{n}}, +\infty)$$

lower bound  
for  $\mu$



# Large Sample Inference

## Confidence Interval

CI for  $\mu$

$$100(1-\alpha)\% = 90\%$$

$$\Rightarrow 1-\alpha = 0.9$$

$$\alpha = 0.1$$

$$\left\{ \begin{array}{l} z_{0.95} = 1.64 \\ z_{0.9} = 1.96 \\ (\text{check!}) \end{array} \right.$$

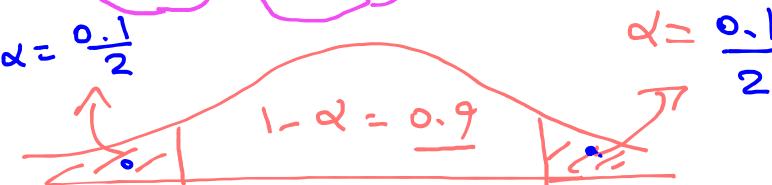
### Example:[Fill weight of jars]

Suppose a manufacturer fills jars of food using a stable filling process with a known standard deviation of  $\sigma = 1.6\text{g}$ . We take a sample of  $n = 47$  jars and measure the sample mean weight  $\bar{x} = 138.2\text{g}$ . A two-sided 90% confidence interval,  $\alpha = 0.1$ , for the true mean weight  $\mu$  is:

$$\begin{aligned} & \left( \bar{x} - z_{1-\alpha/2} \frac{\sigma}{\sqrt{n}}, \bar{x} + z_{1-\alpha/2} \frac{\sigma}{\sqrt{n}} \right) \\ &= \left( \bar{x} - z_{1-0.1/2} \frac{\sigma}{\sqrt{n}}, \bar{x} + z_{1-0.1/2} \frac{\sigma}{\sqrt{n}} \right) \\ &= \left( 138.2 - z_{.95} \frac{1.6}{\sqrt{47}}, 138.2 + z_{.95} \frac{1.6}{\sqrt{47}} \right) \\ &= \left( 138.2 - 1.64(.23), 138.2 + 1.64(.23) \right) \\ &= (137.82, 138.58) \end{aligned}$$

0.38

or we can write it as  $138.2 \pm 0.38\text{g}$



## Large Sample Inference

## Confidence Interval

## CI for $\mu$

**Example:** [Fill weight of jars]

\* Interpretation:

(μ)

We are 90% confident that the **true mean** is between 137.82g and 138.58g

or we can say

If we took 100 more samples of 47 jams each,  
roughly 90 of those samples would have a confidence interval containing the true mean fill weight

# Large Sample Inference

## Confidence Interval

### CI for $\mu$

**Example:** [Fill weight of jars]

What if we just want to be sure that the true mean fill weight is high enough?

We could use a one-sided 90% CI with a lower bound:

$$\begin{aligned} & \underbrace{\alpha = 0.1}_{\text{one-sided}} \quad \left( \bar{x} - z_{1-\alpha} \frac{\sigma}{\sqrt{n}}, +\infty \right) \\ &= \left( 138.2 - z_{.9} \frac{1.6}{\sqrt{47}}, +\infty \right) \\ &= (137.91, +\infty) \end{aligned}$$

Then we would say:

We are 90% confident that the true mean fill weight is above  $\bar{x}$

# Large Sample Inference

## Confidence Interval

### CI for $\mu$

#### Example:[Hard disk failures]

F. Willett, in the article "The Case of the Derailed Disk Drives?" (**Mechanical Engineering**, 1988), discusses a study done to isolate the cause of link code A failure in a model of Winchester hard disk drive.

For each disk, the investigator measured the breakaway torque (in. oz.) required to loosen the drive's interrupter flag on the stepper motor shaft.

Breakaway torques for 26 disk drives were recorded, with a sample mean of 11.5 in. oz. Suppose you know the true standard deviation of the breakaway torques is 5.1 in. oz.

# Large Sample Inference

## Confidence Interval

CI for  $\mu$

Example:[Hard disk failures]

Calculate and interpret:

- A two-sided 90% confidence interval for the true mean breakaway torque of the relevant type of Winchester drive.

$$\begin{aligned} 1_{\alpha}(1-\alpha) &= 0.9 \\ \Rightarrow 1-\alpha &= 0.9 \\ \boxed{\alpha = 0.1} \end{aligned}$$
$$\begin{aligned} (\bar{x} - z_{1-\alpha/2} \frac{\sigma}{\sqrt{n}}, \bar{x} + z_{1-\alpha/2} \frac{\sigma}{\sqrt{n}}) \\ = (\bar{x} - z_{1-0.1/2} \frac{\sigma}{\sqrt{n}}, \bar{x} + z_{1-0.1/2} \frac{\sigma}{\sqrt{n}}) \\ = (11.5 - z_{0.95} \frac{5.1}{\sqrt{26}}, 11.5 + z_{0.95} \frac{5.1}{\sqrt{26}}) \\ = (11.5 - 1.64(1.0002), 11.5 + 1.64(1.0002)) \\ = (9.86, 13.14) \end{aligned}$$

- \* **Interpretation:** we are 90% confident that the true mean breaking torque lies between 9.86 and 13.14 in.oz.

scale

# Large Sample Inference

## Confidence Interval

### CI for $\mu$

#### Example: [Width of a CI]

If you want to estimate the breakaway torque with a 2-sided, 95% confidence interval with  $\pm 2.0$  in. oz. of precision, what sample size would you need?

Interval precision = interval half width

Therefore, for a two-sided 95% CI we have

$$(\bar{x} - z_{1-\alpha/2} \frac{\sigma}{\sqrt{n}}, \bar{x} + z_{1-\alpha/2} \frac{\sigma}{\sqrt{n}})$$

which means that the precision is  $z_{1-\alpha/2} \frac{\sigma}{\sqrt{n}}$

$$\text{We want } z_{1-\alpha/2} \frac{\sigma}{\sqrt{n}} \leq 2$$

The length of this interval:  $\bar{x} + z_{1-\alpha/2} \frac{\sigma}{\sqrt{n}} - \bar{x} - z_{1-\alpha/2} \frac{\sigma}{\sqrt{n}}$

$$= 2 z_{1-\alpha/2} \frac{\sigma}{\sqrt{n}}$$

$\approx$

# Large Sample Inference

## Confidence Interval

### CI for $\mu$

Example:[Width of a CI]

So,

$$1 - \alpha = 0.95$$

$$\underline{\alpha = 0.05}$$

$$\sigma = 5.1$$

$$z_{1-\alpha/2} \frac{5.1}{\sqrt{n}} \leq 2$$

$$z_{.975} \frac{5.1}{\sqrt{n}} \leq 2$$

$$1.96 \frac{5.1}{\sqrt{n}} \leq 2$$

$$\frac{9.996}{\sqrt{n}} \leq 2$$

$$\Rightarrow n \geq 24.98$$

$$\Rightarrow n \geq 25$$

We would need a sample of at least 25 disks to have at least a precision of 2 in.oz

A large- $n$  confidence interval

for  $\mu$  when  $\sigma$  is Unknown

# Large Sample Inference

## Confidence Interval

### CI for $\mu$

### CI for $\mu$

### unknown $\sigma^2$

A generally applicable large-n confidence interval for  $\mu$

Although the equations for a  $(1 - \alpha)\%$  confidence interval is mathematically correct, it is severely limited in its usefulness because it requires us to know  $\sigma$  (the population variance). It is unusual to have to estimate  $\mu$  and know  $\sigma$  in real life.

If  $n \geq 25$  and  $\sigma$  is unknown,  $Z = \frac{\bar{X} - \mu}{\frac{s}{\sqrt{n}}}$ , where

$$s = \sqrt{\frac{1}{n-1} \sum_{i=1}^n (x_i - \bar{x})^2}.$$

is still approximately standard normally distributed.

So, you can replace  $\sigma$  in the confidence interval formula with the sample standard deviation,  $s$ .

# Large Sample Inference

## Confidence Interval

### CI for $\mu$

### CI for $\mu$ unknown $\sigma^2$

A generally applicable large-n confidence interval for  $\mu$

- Two-sided  $100(1 - \alpha)\%$  confidence interval for  $\mu$

$$(\bar{x} - z_{1-\alpha/2} \frac{s}{\sqrt{n}}, \bar{x} + z_{1-\alpha/2} \frac{s}{\sqrt{n}})$$

- One-sided  $100(1 - \alpha)\%$  confidence interval for  $\mu$  with a upper confidence bound

$$(-\infty, \bar{x} + z_{1-\alpha} \frac{s}{\sqrt{n}})$$

- One-sided  $100(1 - \alpha)\%$  confidence interval for  $\mu$  with a lower confidence bound

$$*\quad (\bar{x} - z_{1-\alpha} \frac{s}{\sqrt{n}}, +\infty)$$

# Large Sample Inference

## Confidence Interval

CI for  $\mu$

CI for  $\mu$

unknown  $\sigma^2$

### Example:

Suppose you are a manufacturer of construction equipment. You make 0.0125 inch wire rope and need to determine how much weight it can hold before breaking so that you can label it clearly. Here are breaking strengths, in kg, for 41 sample wires:

```
[1] 100.37  96.31  72.57  88.02 105.89  
[6] 107.80  75.84  92.73  67.47  94.87  
[11] 122.04 115.12  95.24 119.75 114.83  
[16] 101.79  80.90  96.10 118.51 109.66  
[21]  88.07  56.29  86.50  57.62  74.70  
[26]  92.53  86.25  82.56  97.96  94.92  
[31]  62.00  93.00  98.44 119.37 103.70  
[36]  72.40  71.29 107.24  64.82  93.51  
[41]  86.97
```

X

The sample mean breaking strength is 91.85 kg and the sample standard deviation is 17.6 kg.

s

# Large Sample Inference

## Confidence Interval

CI for  $\mu$

CI for  $\mu$

unknown  $\sigma^2$

**Example:** Using the appropriate 95% confidence interval, try to determine whether the breaking strengths meet the requirement of at least 85 kg.

$$(1 - \alpha = .95, \bar{x} = 91.85, s = 17.6, n = 41)$$

The CI is then

$$\begin{aligned} & (\bar{x} - z_{1-\alpha} \frac{s}{\sqrt{n}}, +\infty) \\ &= (91.85 - z_{.95} \frac{17.6}{\sqrt{41}}, +\infty) \\ &= (91.85 - 1.64 \frac{17.6}{\sqrt{41}}, +\infty) \\ &= (87.3422, +\infty) \end{aligned}$$

With 95% confidence, we have shown that the true mean breaking strength is above 87.3422 kg.

Hence, we meet the 85kg requirement with 95% confidence

# Small-sample Confidence Interval for a Mean

## Large Sample Inference

### Confidence Interval

#### CI for $\mu$

#### CI for $\mu$ unknown $\sigma^2$

## Small-sample confidence intervals for a mean

- The most important practical limitation on the use of the methods of the previous sections is the requirement that  **$n$  must be large ( $\geq 25$ )**
- That restriction comes from the fact that without it, there is no way (in general) to calculate  $\frac{\bar{X} - \mu}{S/n}$  that is approximately  $N(0, 1)$ . (i.e we cannot use CLT when sample size is small)
- So, if one mechanically uses the large-  $n$  interval formula  $\bar{x} \pm z \frac{s}{\sqrt{n}}$  with a small sample, there is no way of assessing what actual level of confidence should be declared.

## Large Sample Inference

## Confidence Interval

### CI for $\mu$

### CI for $\mu$ unknown $\sigma^2$

## Small-sample confidence intervals for a mean

- If it is sensible to model the observations as iid normal random variables, then we can arrive at inference methods for small-~~n~~<sup>n</sup> sample means.

In this case (small sample size),  $\bar{x} \pm z \frac{s}{\sqrt{n}}$  is not standard Normal anymore, BUT it is a different normed distribution!

# The Student t Distribution

## Large Sample Inference

## Confidence Interval

## CI for $\mu$

## CI for $\mu$

## unknown $\sigma^2$

## t Distribution

## t Student distribution

**Definition:** The (Student)  $t$  distribution with degrees of freedom parameter  $\nu$  is a continuous probability distribution with probability density

$$f(t) = \frac{\Gamma\left(\frac{\nu+1}{2}\right)}{\Gamma\left(\frac{\nu}{2}\right)\sqrt{\pi\nu}} \left(1 + \frac{t^2}{\nu}\right)^{-(\nu+1)/2} \quad \text{for all } t \in R.$$

The  $t$  distribution

- is bell-shaped and symmetric about 0
- has fatter tails than the normal, but approaches the shape of the normal as  $\nu \rightarrow \infty$ .

(heavy tails)

$$\pi(x) = (x)$$

# Large Sample Inference

## Confidence Interval

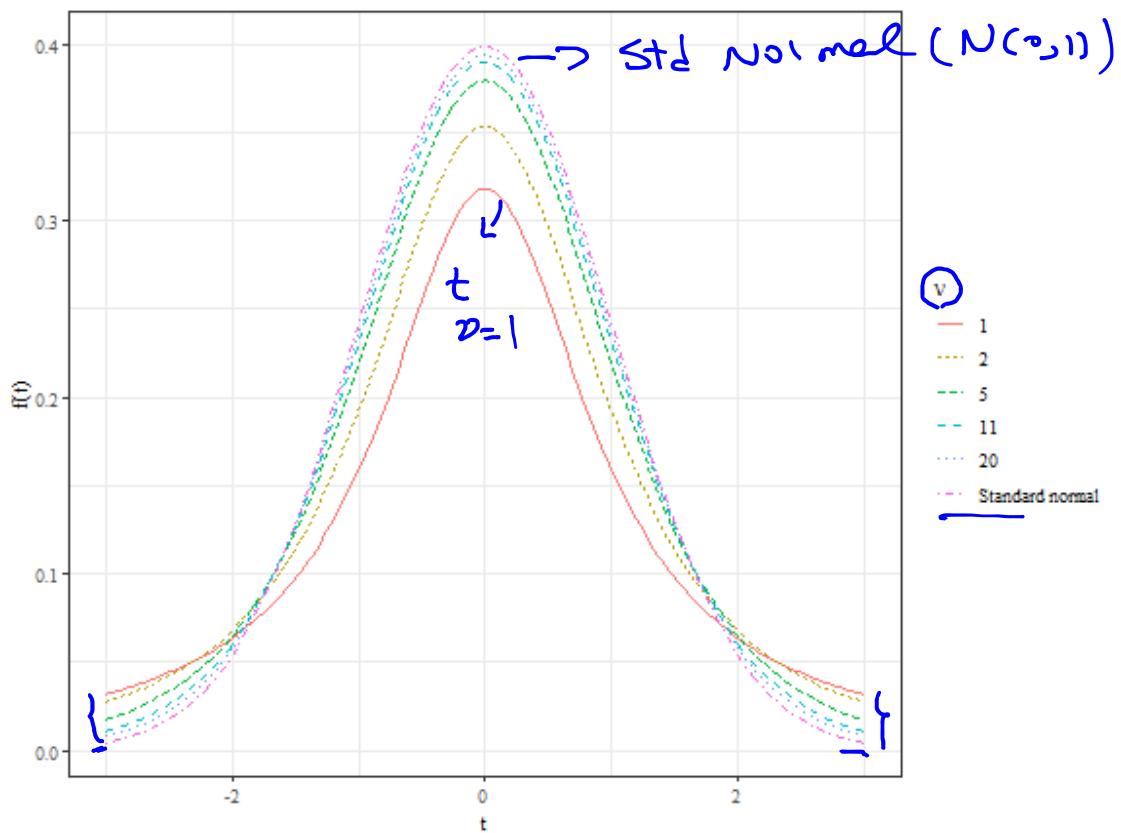
### CI for $\mu$

### CI for $\mu$ unknown $\sigma^2$

## t Distribution

# t Student distribution

We use the  $t$  table (Table B.4 in Vardeman and Jobe) to calculate quantiles.



# Large Sample Inference

## Confidence Interval

CI for  $\mu$

CI for  $\mu$

unknown  $\sigma^2$

## t Distribution

## t Student distribution

**Example:** Say  $T \sim t_5$ . Find  $c$  such that  $P(T \leq c) = 0.9$ .

**Table B.4**  
t Distribution Quantiles

$v$	$Q(.9)$	$Q(.95)$	$Q(.975)$	$Q(.99)$	$Q(.995)$	$Q(.999)$	$Q(.9995)$
1	3.078	6.314	12.706	31.821	63.657	318.317	636.607
2	1.886	2.920	4.303	6.965	9.925	22.327	31.598
3	1.638	2.353	3.182	4.541	5.841	10.215	12.924
4	1.533	2.132	2.776	3.747	4.604	7.173	8.610
5	1.476	2.015	2.571	3.365	4.032	5.893	6.869

So,  $P(T \leq c) = 0.9$  holds true if  $c = 1.476$  (by the table).