Chapter 5.4: Continuing Joint Distributions and Independence

Working with Multiple Random Variables

Functions of Random Variables

Linear Combinations

Linear Combinations

Defined

Linear Combinations

Suppose that $X_1, X_2, \ldots, X_n$ are all random variables.

Then $U = g(X_1, X_2, \ldots, X_n)$ is also a random variable.

However, the probability function (or density function) for U can be very, *very*, **very** difficult to find. However, there are still some things we can say generally about a certain kind of function: a linear combination

> **Linear Combination of Random Variables**
> For constants $a_0, a_1, a_2, \ldots, a_n$ and independent random variables $X_1, X_2, \ldots, X_n$, let $U = a_0 + a_1X_1 + a_2X_2 + \ldots + a_nX_n$. Then U is a *linear combination* of the random variables.

Linear Combinations

Defined

Example

Linear Combination Example I

Suppose that X and Y are two random variables.

Let $U = 5 - 2X + \sin(\pi/8)Y$

Then U is a linear combination of X and Y. Here's why:

- $a_0 = 5$
- $a_1 = -2$
- $a_2 = \sin(\pi/8)$

Linear Combinations

Defined

Example

Expectation

Expected Value of Linear Combination

In most cases, if $U = g(X_1, X_2, \ldots, X_n)$ then to get the expected value we have to find the joint probability function of $X_1, X_2, \ldots, X_n$ is also a random variable ($f(x_1, x_2, \ldots, x_n)$). This is (as we said before) really, really hard! Then we have to do a sum/integration over all the random variables in the distribution!

However, there is a special result for linear combinations, as long as the random variables are independent:

> **Expected Value of Lin. Comb. of RVs**
> For constants $a_0, a_1, a_2, \ldots, a_n$ and independent random variables $X_1, X_2, \ldots, X_n$, let
>
> $$U = a_0 + a_1X_1 + a_2X_2 + \ldots + a_nX_n$$
>
> Then
>
> $$E(U) = a_0 + a_1E(X_1) + a_2E(X_2) + \ldots a_nE(X_n)$$

Variance of Linear Combination

The process for getting the variance of a function of random variables is usually even more difficult than the process of getting the expected value.

However, there is a special result for linear combinations, as long as the random variables are independent:

> **Variance of Lin. Comb. of RVs**
> For constants $a_0, a_1, a_2, \ldots, a_n$ and independent random variables $X_1, X_2, \ldots, X_n$, let
>
> $$U = a_0 + a_1X_1 + a_2X_2 + \ldots + a_nX_n$$
>
> Then
>
> $$Var(U) = a_1^2 Var(X_1) + a_2^2 Var(X_2) + \ldots a_n^2 Var(X_n)$$

Notice that the a_0 drops away. Since it's just a constant and doesn't connect to one of the random variables, it never varies - so the variance of that part is 0.

Linear Combination Example II

Suppose that $X \sim N(5, 4)$ and $Yi \sim N(-3, 9)$ are independent random variables.

Let $U = 5 - 2X + \sin(\pi/8)Y$. Then

$$
\begin{aligned}
E(U) &= 5 - 2E(X) + \sin(\pi/8)E(Y) \\
&= 5 - 2(5) + \sin(\pi/8)(-3) \\
&= -5 - 3\sin(\pi/8)(-3)
\end{aligned}
$$

$$
\begin{aligned}
Var(U) &= (-2)^2 Var(X) + (\sin(\pi/8))^2 Var(Y) \\
&= 4(4) + \sin^2(\pi/8)(9)
\end{aligned}
$$

Modeling A System With Random Variables

Suppose that River U has two tributaries, River X and River Y with the following additional pieces of information:

- River X has a baseline volume that flows into River U (say, V_X)
- River Y has a baseline volume that flows into River U (say, V_Y)
- The volume flowing from each river could exceed the baseline
- The volume flowing from River X and River Y are not related
- If the volume flowing into River U from the two tributaries exceeds M, River U will overflow

Model this system and determine a formula for the probability that River U will overflow.

Central Limit Theorem

The most important result in statistics

> **Central Limit Theorem**
> If $X_1, X_2, \ldots, X_n$ are independent and identically distributed (iid) random variables each with mean μ and variance σ^2 and let the random variable $\bar{X} = \frac{1}{n}X_1 + \frac{1}{n}X_2 + \ldots + \frac{1}{n}X_n$. Then
>
> 1. $E(\bar{X}) = \mu$
> 2. $Var(\bar{X}) = \sigma^2/n$
> 3. For large n, $\bar{X}$ is approximately normally distributed (limit goes to normal…)

Example 25 (page 317) in the book provides a wonderful illustration of this