Show **all** of your work on this assignment and answer each question fully in the given context.

Please staple your assignment!

1. Suppose that X is a random variable with probability density function

$$f(x) = \begin{cases} c \cdot x^2 & -2 \leq x \leq 2 \\ 0 & o.w. \end{cases}$$

   (a) Find the value of c that makes $f(x)$ a valid probability density function.[5 pts]
   (b) Find the CDF of the random variable X.[5 pts]
   (c) What is $P(|X| \geq -1)$ [5 pts].
   (d) Find the expected value of X.[5 pts]

2. Consider a continuously distributed random variable, W, with a probability density function given by

$$f(w) = \begin{cases} \frac{1}{5(1-e^{-2})} e^{-w/5} & 0 \leq w \leq 10 \\ 0 & \text{otherwise} \end{cases}$$

   (a) Show that the function $f(w)$ is a valid probability density function (i.e., show that (i) $f(w)$ is non-negative and (ii) $\int_{-\infty}^{\infty} f(w)dw = 1$). [5 pts]
   (b) Find $P(W \leq 2)$ [5 pts]
   (c) Find $P(2 \leq W \leq 10)$ [5 pts]
   (d) Find $E(X)$ [5 pts]

3. The mileage to first failure for a model of military personnel carrier can be modeled as exponential with mean $1,000$ miles.

   (a) Find the probability that a vehicle of this type gives less than 500 miles of service before first failure.[5 pts]
   (b) Find the probability that a vehicle of this type gives less than 2000 miles of service before first failure. [5 pts]

4. (Ch. 5.2, Exercise 2, pg. 263) Suppose that Z is a standard normal random variable. Evaluate the following probabilities involving Z:

   (a) $P[Z < -.62]$ [3 pts]
   (b) $P[Z > 1.06]$ [3 pts]
   (c) $P[-.37 < Z < .51]$ [3 pts]
   (d) $P[|Z| \leq .47]$ [3 pts]
   (e) $P[|Z| > .93]$ [3 pts]
   (f) $P[-3.0 < Z < 3.0]$ [3 pts]

   Now find numbers # such that the following statements involving Z are true:

   (a) $P[Z \leq \#] = .90$ [3 pts]
   (b) $P[|Z| \leq \#] = .90$ [3 pts]
   (c) $P[|Z| > \#] = .03$ [3 pts]

Total: 77 pts

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(a) $\int_{-\infty}^{\infty} f(x)\,dx = 1 \longrightarrow \int_{-2}^{2} cx^2\,dx = 1$

$$\Rightarrow c\int_{-2}^{2} x^2\,dx = 1$$

$$\Rightarrow c\cdot\left(\frac{x^3}{3}\Big|_{-2}^{2}\right) = 1$$

$$\Rightarrow c\left(\frac{8}{3} - \frac{(-8)}{3}\right) = 1$$

$$\Rightarrow c\left(\frac{16}{3}\right) = 1 \Longrightarrow \boxed{c = \frac{3}{16}}$$

(b) Note that the pdf is now

$$f_X(x) = \begin{cases} \frac{3}{16}x^2 & -2 \le x \le 2 \\ 0 & \text{o.w} \end{cases}$$

Finding CDF :

$$F_X(t) = P(X \le t) = \int_{-2}^{t} \frac{3}{16}x^2\,dx$$

$$= \frac{3}{16}\left[\frac{x^3}{3}\Big|_{-2}^{t}\right]$$

$$= \frac{1}{16}\left[x^3\Big|_{-2}^{t}\right]$$

$$= \frac{1}{16}\left(t^3 - (-8)\right) = \boxed{\frac{1}{16}\left(t^3 + 8\right)}$$

(c)

$P(|X| \geq -1) = 1$

(Note that no matter what pdf X has, the statement $|X| \geq -1$ always holds true!

(d)

$$EX = \int_x x f(x)\, dx = \int_{-2}^{2} x \cdot \tfrac{3}{16} x^2\, dx$$

$$= \int_{-2}^{2} \tfrac{3}{16} x^3\, dx$$

$$= \tfrac{3}{16} \cdot \frac{x^4}{4} \Big|_{-2}^{2}$$

$$= \tfrac{3}{16} \left(\tfrac{2^4}{4} - \frac{(-2)^4}{4} \right)$$

$$\boxed{= 0}$$

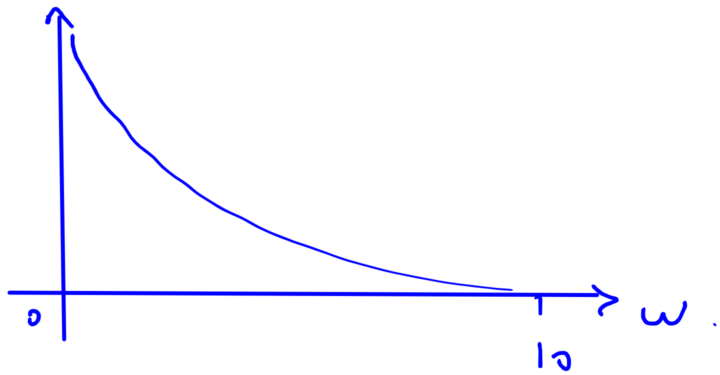
2,

(a) i: need to show $f(\omega) \geq 0 \ \forall \ \omega \in [0, 10]$

$$f(\omega) = \frac{1}{5(1-e^{-2})} e^{-\omega/5}$$

$\approx 0.231 \geq 0$

On the other hand, $e^{-w/5}$ for $0 \leq w \leq 10$ has a shape like



which is always a positive quantity.

(i.e $e^{-w/5} \geq 0$ for $w \in [0\ \ 10]$)

Therefore, $f(w) = \dfrac{1}{5(1-e^{-2})} e^{-w/5}, \quad 0 \leq w \leq 10$

>0 >0

>0

ii) need to show $\int_{-\infty}^{\infty} f(w)\, dw = 1$

$$\Rightarrow \int_0^{10} \frac{1}{5(1-e^{-2})} e^{-w/5}\, dw = \frac{1}{5(1-e^{-2})} \int_0^{10} e^{-w/5}\, dw$$

$$= \frac{1}{5(1-e^{-2})} \left(-5e^{-w/5} \Big|_0^{10} \right)$$

$$= \frac{1}{5(1-e^{-2})} \left(5\left(1 - e^{-10/5}\right) \right) = 1$$

(b) $$P(W \le 2) = \int_0^2 \frac{1}{5(1-e^{-2})} e^{-w/5}\, dw$$

$$= \frac{1}{5(1-e^{-2})} \int_0^2 e^{-w/5}\, dw$$

$$= \frac{1}{5(1-e^{-2})} \left(-5 e^{-w/5} \Big|_0^2\right)$$

$$= \frac{1}{5(1-e^{-2})} \left(5(1-e^{-2/5})\right)$$

$$= \frac{1-e^{-0.4}}{1-e^{-2}} = 0.381$$

(c) $$P(2 \le W \le 10) = \int_2^{10} \frac{1}{5(1-e^{-2})} e^{-w/5}\, dw$$

$$= \frac{1}{5(1-e^{-2})} \int_2^{10} e^{-w/5}\, dw$$

$$= \frac{1}{\cancel{5}(1-e^{-2})} \left(-\cancel{5}\, e^{-w/5} \Big|_2^{10}\right)$$

$$= \frac{1}{1-e^{-2}} \left(e^{-2/5} - e^{-10/5}\right)$$

$$= \frac{e^{-0.4} - e^{-2}}{1-e^{-2}} = 0.618$$

d) $E(w) = \int_0^{10} \frac{w}{5(1-e^{-2})} e^{-w/5}\, dw$

$$= \frac{1}{5(1-e^{-2})} \int_0^{10} w\, e^{-w/5}\, dw$$

Integration by part :

$\int u\,dv = uv - \int v\,du$

In This problem :

$u = w \longrightarrow du = dw$

$dv = e^{-w/5} \longrightarrow v = \int e^{-w/5}\, dw = -5\, e^{-w/5}$

$$\Rightarrow E.W = \frac{1}{5(1-e^{-2})} \left[w.(-5\, e^{-w/5}) \Big|_0^{10} - \int_0^{10} (-5 e^{-w/5})\, dw \right]$$

$$= \frac{1}{5(1-e^{-2})} \left[10(-5\, e^{-10/5}) + 0 + \int_0^{10} 5\, e^{-w/5}\, dw \right]$$

$$= \frac{1}{5(1-e^{-2})} \left[-10(5)\, e^{-2} + 5 \underbrace{\left[-5\, e^{-w/5} \Big|_0^{10} \right]}_{-5(e^{-10/2} - 1)} \right]$$

$$= \frac{1}{5(1-e^{-2})} \left[-50\, e^{-2} + 5(5)(1 - e^{-2}) \right]$$

$$= \frac{1}{5(1-e^{-2})} \left(-50\, e^{-2} + 25 - 25\, e^{-2} \right)$$

$$= \frac{1}{5(1-e^{-2})}\left(25 - 75e^{-2}\right)$$

$$= \frac{+25}{5(1-e^{-2})}\left(1 - 3e^{-2}\right)$$

$$= 5\ \frac{1-3e^{-2}}{1-e^{-2}} = 5(0.6869)$$

$$= 3.4345$$

3, X: The random variable associated with the mileage until the first failure.

$X \sim Exp(1000)$

$$f_X(x) = \begin{cases} \frac{1}{1000}\, e^{-\frac{x}{1000}} & , x \geq 0 \\ 0 & o.w \end{cases}$$

(a) $P(X \leq 500) = ?$ (we know in Exponential distribution, $F_X(x) = 1 - e^{-x/\alpha}$)

So, $P(X \leq 500) = F(500) = 1 - e^{-\frac{500}{1000}} = 1 - e^{-0.5}$

$= 0.3934$

(b) $P(X \le 2000) = F(2000) = 1 - e^{-\frac{2000}{1000}}$

$= 1 - e^{-2}$

$= 0.8646$

4, $Z \sim N(0,1)$

by the table

(a) $P(Z < -0.63) = \Phi(-0.63) = 0.2643$

(b) $P(Z > 1.06) = ?$

method ① : $P(Z > 1.06) = 1 - P(Z \le 1.06)$

$= 1 - \Phi(1.06)$

by the table $= 1 - 0.8554$

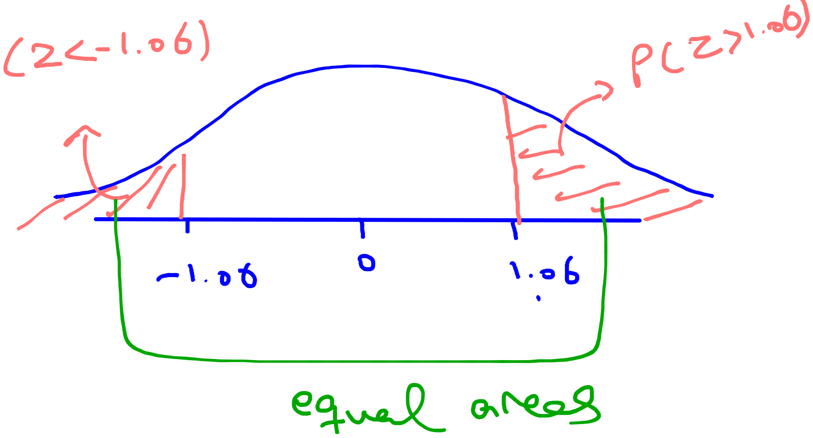
$= 0.1445$

method ② : Normal is a symmetric distribution.

so $P(Z > 1.06) = P(Z < -1.06)$

$= \Phi(-1.06)$

by the table $= 0.1445$



(c) $P(-0.37 < Z < 0.51) = P(Z < 0.51) - P(Z < -0.37)$

$= \Phi(0.51) - \Phi(-0.37)$

by the Normal table $= 0.6949 - 0.3556$

$= 0.3392$

(d) $P(|Z| \leq 0.47) = P(-0.47 < Z < 0.47)$

$= P(0.47) - P(Z < -0.47)$

$= \Phi(0.47) - \Phi(-0.47)$

$= 0.6808 - 0.3191$

$= 0.3616$

(e) $P(|Z| > 0.93) = P(Z > 0.93 \text{ or } Z < -0.93)$

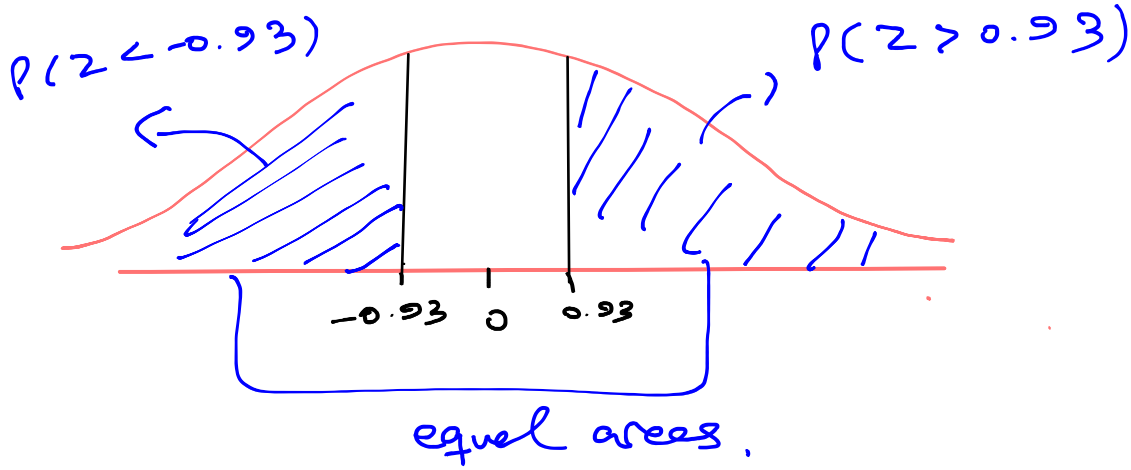
$= P(Z > 0.93) + P(Z < -0.93)$

method 1 : $= 1 - P(Z \leq 0.93) + P(Z < -0.93)$

$= 1 - \Phi(0.93) + \Phi(-0.93)$

$= 1 - 0.8238 + 0.1761 = \boxed{0.3523}$

method 2 :



$P(Z>0.93) = P(Z<-0.93)$

$\Rightarrow P(Z>0.93) + P(Z<-0.93) = 2P(Z<0.93)$

$= 2(.1761)$

$= \boxed{0.3523}$

(f) $P(-3<Z<3) = P(Z<3) - P(Z<-3)$

$= \Phi(3) - \Phi(-3)$

$= 0.9986 - 0.0013$

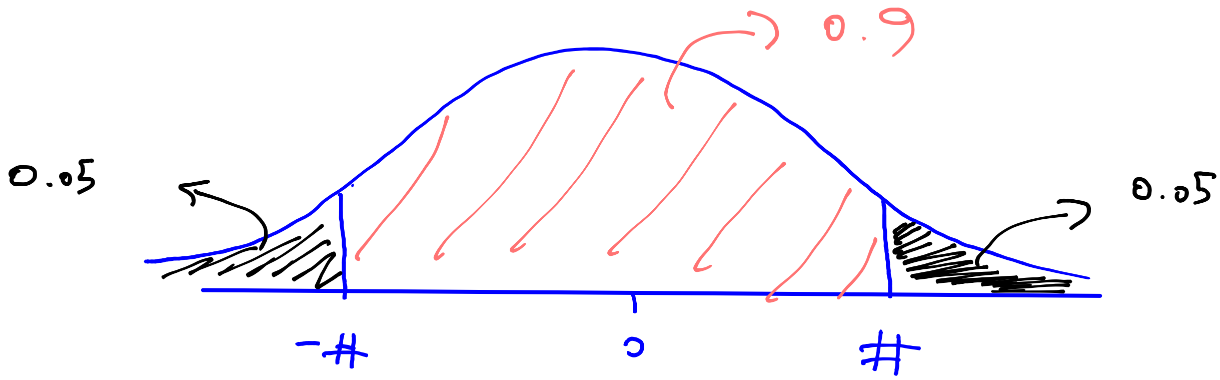
$= 0.9973$

(a) $P(Z \leq \#) = 0.9$ $\longrightarrow$ by table, we should look for a value of Z which has probability 0.9

If we look it up in the standard normal table, $\boxed{\# \simeq 1.28}$

(b) $P(|Z| \leq \#) = 0.9$

$\to P(-\# < Z \leq \#) = 0.9$



because Normal is symmetric, $P(Z \leq -\#) = P(Z > \#) = 0.05$

So, by looking it up in the table,

$$-\# = -1.645$$

$$\boxed{\Rightarrow \# = 1.645}$$

The other method is:

$$P(-\# \leq Z \leq \#) = 0.9$$

by symmetry of Normal $\Rightarrow 1 - 2P(Z \leq -\#) = 0.9$

$\Rightarrow 2P(Z \leq -\#) = 1 - 0.9$

$\Rightarrow P(Z \leq -\#) = \frac{0.1}{2} = 0.05$

$\Rightarrow \Phi(-\#) = 0.05$

by the Normal table $\Rightarrow -\# = -1.645 \Rightarrow \# = 1.645$

(c) $P(|Z| > \#) = 0.03$

$\hookrightarrow P(Z > \# \text{ or } Z \leq -\#) = 0.03$

$\Rightarrow \underbrace{P(Z > \#)}_{P(Z \leq -\#)} + P(Z \leq -\#) = 0.03$

$\Rightarrow 2P(Z \leq -\#) = 0.03$

$\Rightarrow P(Z \leq -\#) = \frac{0.03}{2} = 0.015$

$\Rightarrow \Phi(-\#) = 0.015$

by the Normal table : $-\# = -2.17 \Rightarrow \# = 2.17$