Hypothesis Testing

Null

Alternative

P-value

CI method

Matched Pairs

Two-sample

Small samples

Also assuming

1
- H_0 is true, 2
- The sample 1 points are iid $N(\mu_1, \sigma_1^2)$, the 3 sample 2 points are iid $N(\mu_2, \sigma_2^2)$,
- 4 and the sample 1 points are independent of the sample 2 points and $\boxed{\sigma_1^2 \approx \sigma_2^2.}$ 5

Then

5 assumptions to use →

$$K = \frac{\overline{X}_1 - \overline{X}_2 - (\#)}{S_p\sqrt{(\frac{1}{n_1} + \frac{1}{n_2})}} \sim \boxed{t_{(n_1+n_2-2)}}$$

Small samples

$1-\alpha$ confidence intervals (2-sided, 1-sided upper, and 1-sided lower, respectively) for $\mu_1-\mu_2$ under these assumptions are of the form:

(let $\boxed{\nu = n_1+n_2-2}$)

- *Two-sided* $100(1-\alpha)\%$ confidence interval for $\mu_1-\mu_2$

$$(\bar{x}_1-\bar{x}_2) \pm \boxed{t_{(\nu,1-\alpha/2)}} * \boxed{S_p\sqrt{(\frac{1}{n_1}+\frac{1}{n_2})}}$$

- *One-sided* $100(1-\alpha)\%$ confidence interval for $\mu_1-\mu_2$ with a upper confidence bound

$$(-\infty\ ,\ (\bar{x}_1-\bar{x}_2)+t_{(\nu,1-\alpha)} * S_p\sqrt{(\frac{1}{n_1}+\frac{1}{n_2})}$$

n_1+n_2-2

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- *One-sided* $100(1-\alpha)\%$ confidence interval for μ with a lower confidence bound

$$\left((\overline{x}_1 - \overline{x}_2) - t_{(\nu,1-\alpha)} * S_p\sqrt{\left(\frac{1}{n_1}+\frac{1}{n_2}\right)},\ +\infty\right)$$

$\nu = n_1 + n_2 - 2$

In general: CI:

estimate $\pm$ population quantiles $\times$ SD(estimate)

$\mu = 0$

$\mu \neq 0$

large sample : $\bar{x} \pm Z_{1-\alpha/2}\, S/\sqrt{n}$

* HW 9 posted (optional)
  Due Thursday Dec. 12.

* Final exam:
  - Wednesday, Dec 18
    9:45 – 11:45 (in-class)
  - Comprehensive with focus on
    materials after quiz 3.

Small samples

Example:[Springs]

The data of W. Armstrong on spring lifetimes (appearing in the book by Cox and Oakes) not only concern spring longevity at a 950 N/ mm^2 stress level but also longevity at a 900 N/ mm^2 stress level.

Let sample 1 be the 900 N/ mm^2 stress group and sample 2 be the 950 N/ mm^2 stress group.

900 N/mm2 Stress	950 N/mm2 Stress
216, 162, 153, 216, 225, 216, 306, 225, 243, 189	225, 171, 198, 189, 189, 135, 162, 135, 117, 162

$n_1 = 10$ $n_2 = 10$

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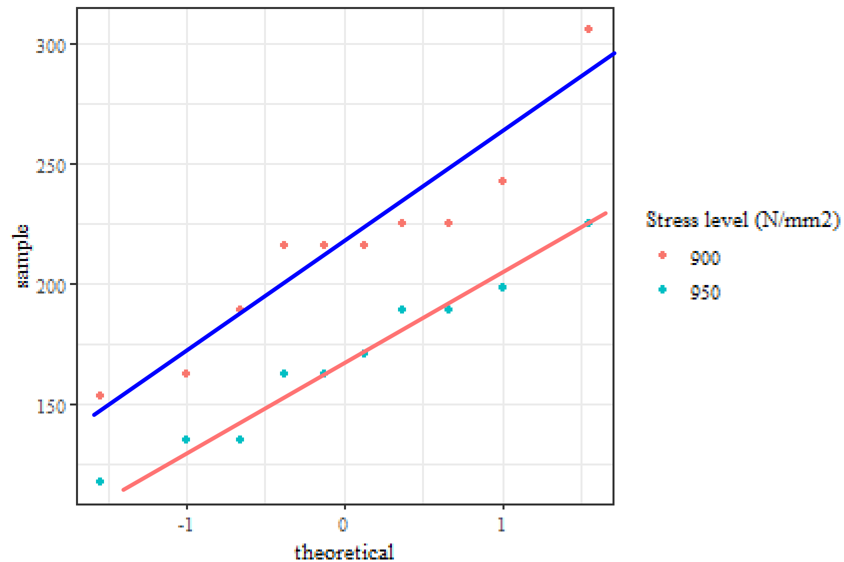
$\boxed{\text{Normal}}$ plots of Spring lifetime under two different stress level.

Example:[Springs]



Let's do a hypothesis test to see if the sample 1 springs lasted significantly longer than the sample 2 springs. (For $\alpha = 0.05$)

Note: however the sample sizes are small, the data look pretty Normal.

1, $\begin{cases} H_0: \mu_1 - \mu_2 = 0 \\ H_a: \mu_1 - \mu_2 > 0 \end{cases}$

2, $\alpha = 0.05$

3, If we assume that (1) H_0 is true & (2) sample 1 is iid $N(\mu_1, \sigma_1^2)$ (3) independent from sample 2 (4) iid $N(\mu_2, \sigma_2^2)$ and (5) $\sigma_1^2 \approx \sigma_2^2$, then the test statistic is

$$\rightarrow K = \frac{(\bar{x}_1 - \bar{x}_2) - 0}{S_p \sqrt{\frac{1}{n_1} + \frac{1}{n_2}}} \;\dot\sim\; t_{(n_1+n_2-2)}$$

4, calculate

$\bar{x}_1 = 215.1$, $S_1 = 42.9$, $S_1^2 = 1840.41$

$\bar{x}_2 = 168.3$, $S_2 = 33.1$, $S_2^2 = 1095.61$

$$\Rightarrow S_p = \sqrt{\frac{(n_1-1)S_1^2 + (n_2-1)S_2^2}{n_1+n_2-2}} = \sqrt{\frac{(10-1)1840.41 + (10-1)1095.61}{(10+10-2)}}$$

$$\Rightarrow S_p = 38.3$$

Then

$$K = \frac{(\bar{x}_1 - \bar{x}_2) - 0}{S_p\sqrt{\frac{1}{n_1}+\frac{1}{n_2}}} = \frac{(215.1 - 163.3)}{38.3\sqrt{\frac{1}{10}+\frac{1}{10}}} = 2.7$$

Recall:

We need to calculate p-value now.

* Corresponding to α: $t_{(n_1+n_2-2,\ 1-\alpha)} = t_{(18,\ 0.95)}$

by table $= \underline{1.73}$

(By the methods we learned about p-value and α, we just need to decide if p-value $\gtrless \alpha$)

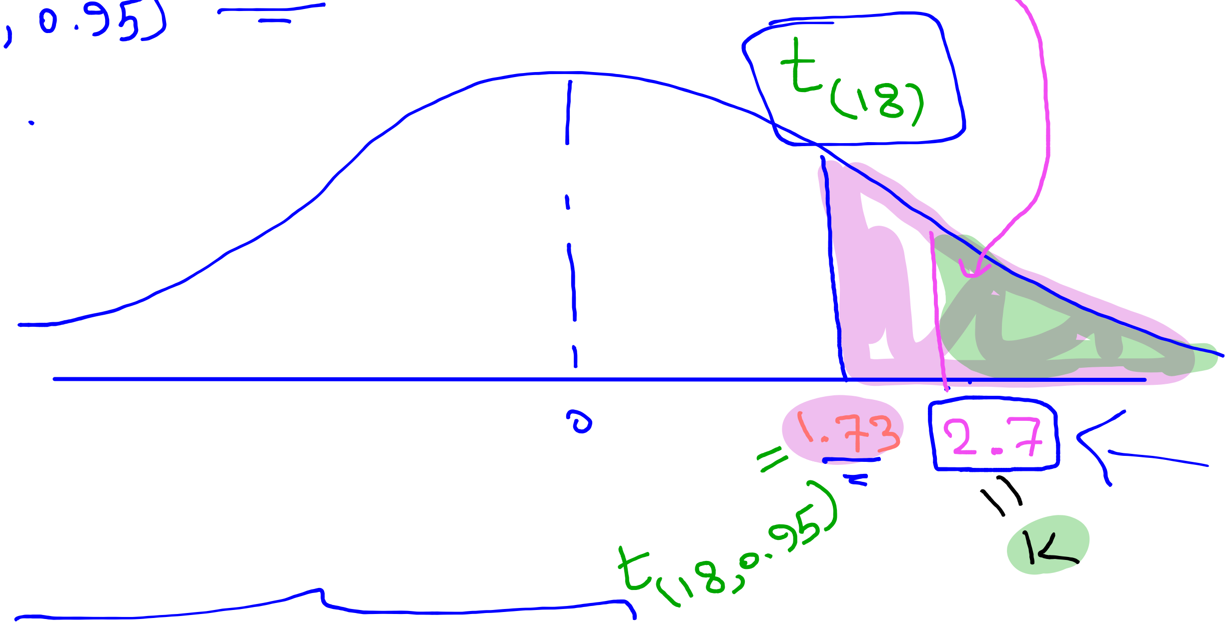
→ * Corresponding to p-value $\equiv k = 2.7$

Now:

one sided test

$$p\text{-value} = P(T > K) = P(T > 2.7)$$

$$t_{(18,\,0.95)} = 1.73$$



5/ Since $K > 1.73 \; (= t_{(18,0.95)})$

$\Rightarrow$ P-value $< \alpha$ $\Rightarrow$ Reject H_0

(area under the curve for $P(T>K)$ is smaller than α)

6, There is enough evidence to conclude that springs on average last longer subjected to $\underline{900}$ N/mm^2 of stress than $\underline{\underline{950}}$ N/mm^2 of stress.

Small samples

Example:[Stopping distance]

Suppose μ_1 and μ_2 are true mean stopping distances (in meters) at 50 mph for cars of a certain type equipped with two different types of breaking systems.

Suppose $n_1 = n_2 = 6, \overline{x}_1 = 115.7, \overline{x}_2 = 129.3, s_1 = 5.08$, and $s_2 = 5.38$.

Use significance level $\boxed{\alpha = 0.01}$ to test
$H_0 : \mu_1 - \mu_2 = -10$ vs. $H_A : \mu_1 - \mu_2 < -10.$

Construct a 2-sided 99% confidence interval for the true difference in stopping distances.

① $\begin{cases} H_0: \mu_1 - \mu_2 = -10 \\ H_a: \mu_1 - \mu_2 < -10 \end{cases}$

② $\alpha = 0.01$

③ Under the assumptions that ① H_0 is true and

② sample 1 is iid $N(\mu_1, \sigma_1^2)$ ③ independent of

④ sample 2 iid $N(\mu_2, \sigma_2^2)$ and ⑤ $\sigma_1^2 \approx \sigma_2^2$ ✓

we use test statistic

$$K = \frac{\bar{x}_1 - \bar{x}_2 - (-10)}{S_p \sqrt{1/n_1 + 1/n_2}}$$

where $\overset{*}{K} \sim t_{(n_1+n_2-2)}$

(4) $S_p = \sqrt{\dfrac{(n_1-1)S_1^2 + (n_2-1)S_2^2}{n_1+n_2-2}} = \sqrt{\dfrac{(6-1)5.08^2 + (6-1)5.38^2}{6+6-2}}$

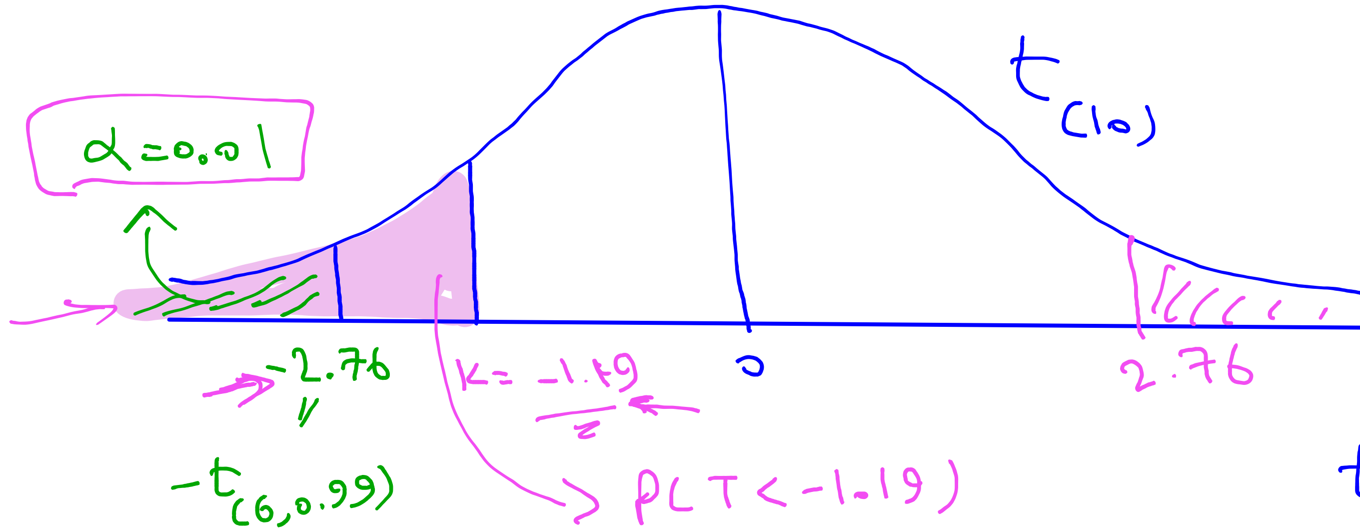
$\rightarrow S_p = 5.23$

$\rightarrow K = \dfrac{(\bar{x}_1 - \bar{x}_2) - (-10)}{S_p\sqrt{\frac{1}{n_1}+\frac{1}{n_2}}} = \dfrac{(115.7 - 129.3)+10}{5.23\sqrt{\frac{1}{6}+\frac{1}{6}}} = -1.19$

$T \sim t_{(n_1+n_2-2)}$

$\begin{cases} H_0: \mu_1 - \mu_2 = -10 \\ H_a: \mu_1 - \mu_2 < -10 \end{cases} \rightarrow$ p-value $= P(T < K) = P(T < -1.19)$

by the table: $t_{(6+6-2,\ 1-\alpha)} = t_{(10,\ 0.99)} = \boxed{2.76}$



$t_{(6,\ 0.99)}$

$$\text{Since } \underbrace{P(T < K) = P(T < -1.19)}_{p\text{-value}} > \underbrace{P(T < -2.76)}_{\alpha}$$

we fail to reject H_0.

⑤ Since p-value > α, we fail to reject H_0

⑥ There is Not enough evidence to conclude that stopping distances for breaking system 1 are on average less than those of breaking system 2 by over 10 m.

Construct $\boxed{\text{2-Sided}}$ 99% CI for $\mu_1 - \mu_2$ (the true difference of mean stopping distance):

using formula:

$$\rightarrow \left((\bar{x}_1 - \bar{x}_2) - t_{(n_1+n_2-2,\, 1-\alpha/2)}\, s_p \sqrt{\frac{1}{n_1} + \frac{1}{n_2}},\ (\bar{x}_1 - \bar{x}_2) + t_{(n_1+n_2-2,\, 1-\alpha/2)}\, s_p \sqrt{\frac{1}{n_1} + \frac{1}{n_2}} \right)$$

10 $\quad 1 - \frac{0.01}{2} = 0.995$

$$= \left((115.7 - 129.3) - 3.17 \cdot (5.23)\sqrt{\frac{1}{6} + \frac{1}{6}},\ (115.7 - 129.3) + 3.17 (5.23)\sqrt{\frac{1}{6} + \frac{1}{6}} \right)$$

$$= \boxed{(-23.17, -4.03)}$$

we are 99% confident that the true mean stopping distance of system 1 is anywhere between 23.17m to 4.03m less than that of system 2.