Show **all** of your work on this assignment and answer each question fully in the given context.

Note: You can email your homework only if you cannot attend the class to submit your homewok. If you do so, email your homework by the noon of the due date.
Please staple your assignment!

1. **[Ch 5.5, Exercise 3, pg. 322]:** The random number generator supplied on a calculator is not terribly well chosen, in that values it generates are not adequately described by a uniform distribution on the interval $(0, 1)$. Suppose instead that a probability density

$$f(x) = \begin{cases} k(5-x) & 0 < x < 1 \\ 0 & \text{otherwise} \end{cases}$$

is a more appropriate model for X = the next value produced by this random number generator.

Consider this random number generator. Suppose that it is used to generate 25 random numbers and that these may reasonable be thought of as independent random variables with common individual (marginal) distribution as above.
Let $\overline{X}$ be the sample mean of these 25 values.

   (a) What are the mean and standard deviation of the random variable $\overline{X}$?[5 pts]
   (b) What is the approximate probability distribution of $\overline{X}$?[5 pts]
   (c) Approximate the probability that $\overline{X}$ exceeds 0.5.[5 pts]
   (d) Approximate the probability that $\overline{X}$ takes a value within 0.02 of its mean.[5 pts]
   (e) Redo parts a) through d) using a sample size of 100 instead of 25.[10 pts]

2. **[Ch 5, Exercise 10, pg. 324]:** Suppose that the thickness of sheets of a certain weight of book paper have mean 0.1 mm and a standard deviation of 0.003 mm. A particular textbook will be printed on 370 sheets of this paper. Find values for the mean and standard deviation of the thicknesses of copies of the text (excluding the books' cover). That is, find the mean and standard deviation of the whole thickness of the 370 sheets.[10 pts]

3. **[Ch 5, Exercise 20, pg. 326] :** Suppose that the raw daily oxygen purities delivered by an air-products supplier have a standard deviation $\sigma \approx .1$ (percent), and it is plausible to think of daily purities as independent random variables. Approximate the probability that the sample mean $\overline{X}$ of $n = 25$ delivered purities falls within 0.03 (percent) of the raw daily purity mean, μ.[5 pts]

4. Suppose that $Z_1, Z_2, \ldots, Z_n$ are n independent standard normal random variables. It may be helpful to recall that $E(aZ_i + b) = aE(Z_i) + b$ and that $Var(aZ_i + b) = a^2 Var(Z_i)$ for any constants a, b in addition to knowing that $\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$ and $\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$.

   (a) Find the expected value and variance of X where $X = 3Z_1 + 5$[5 pts]
   (b) Find the expected value and variance of Y where $Y = Z_1 - Z_2$[5 pts]
   (c) Find the expected value and variance of U where $U = Z_1 + Z_2$[5 pts]
   (d) Find the expected value and variance of W where $W = \sum_{i=1}^{n} \frac{i}{n}\left(Z_i + \frac{i}{n}\right)$.[10 pts]

5. An iid sample of n observations is drawn from a population with mean equal to 20 and standard deviation equal to 16. Let $\overline{X}$ be the sample mean.

   (a) Given $n = 64$, find $P(\overline{X} < 16)$[5 pts]

   (b) Given $n = 64$, find $P(16 < \overline{X} \leq 23)$[5 pts]

   (c) What is the smallest sample size so that the chance that $|\overline{X} - 20| > 0.5$ is at most 0.05.[5 pts]

Total: 85 pts

problem 7,

first need to find K.

$$1 = \int_0^1 k(5-x)\,dx = k\int_0^1 (5-x)\,dx = K\left(5x - \frac{x^2}{2}\right)\Big|_0^1 = k \cdot 9/2$$

$$\Rightarrow \boxed{k = 2/9}$$

a) $E\bar{x} = \mu$

$$\mu = Ex = \int_0^1 \frac{2}{9}\, x(5-x)\,dx = 2/9 \int_0^1 5x - x^2\,dx$$

$$= 2/9\left(5\frac{x^2}{2} - \frac{x^3}{3}\right)\Big|_0^1 = 2/9\left(5/2 - 1/3\right)$$

$$= 2/9\left(\frac{15-2}{6}\right)$$

$$= 2/9 \cdot \frac{13}{6} = \frac{13}{27}$$

$$\Rightarrow \boxed{E\bar{x} = \mu = 13/27}$$

$$Ex^2 = \int 2/9(5x^2 - x^3)\,dx = 2/9\left(\frac{5}{3}x^3 - \frac{x^4}{4}\right)\Big|_0^1 = 2/9\left(5/3 - 1/4\right)$$

$$= 2/9\left(\frac{20-3}{12}\right) = \frac{2}{9} \times \frac{17}{12} = \frac{17}{54}$$

$$\Rightarrow \sigma^2 = var(x) = Ex^2 - (Ex)^2 = \frac{17}{54} - \frac{13^2}{27^2} = 0.08299$$

$$SD(\bar{x}) = \sqrt{0.08299} = 0.288.$$

$$\Rightarrow \boxed{SD(\bar{x}) = \frac{SD(x)}{\sqrt{n}} = \frac{0.288}{5} = 0.0576}$$

b, For large enough n, using CLT for iid random variables $x_1, \dots, x_n$;

$$\bar{x} \sim N\left(\mu = \frac{13}{27}, \frac{\sigma^2}{n} = 0.0033\right)$$

c, $P(\bar{x} > 0.5) = P\left(\frac{\bar{x} - 13/27}{\sqrt{0.0033}} > \frac{0.5 - \frac{13}{27}}{\sqrt{0.0033}}\right)$

$= P(Z > 0.321)$

$= 1 - \Phi(0.321)$

$= 1 - 0.6258$

$= 0.3741$

d, $P(0.4615 < \bar{x} < 0.5015) = P(-0.35 < Z < 0.35)$

$= \Phi(0.35) - \Phi(-0.35) = 0.6368 - 0.3632$

$= 0.2736$

e/ for $n = 100$:

$$\bar{X} \overset{\cdot}{\sim} N\left(\frac{13}{27}, \frac{0.0821}{100}\right)$$

$* \; P(\bar{X} > 0.5) = P(Z > 0.64)$

$= 1 - \Phi(0.64)$

$= 1 - 0.7389$

$= 0.2611$

$* \; P(0.4615 < \bar{X} < 0.5015) = P(-0.69 < Z < 0.69)$

$= \Phi(0.69) - \Phi(-0.69)$

$= 0.7549 - 0.2451$

$= 0.5098$

Problem 2 :

The thickness of the text is $\sum_{i=1}^{370} X_i$. where

$X_1, \dots, X_{370}$ are iid random variables associated

with the thickness of the sheets.

$E(\text{" whole thickness of 370 sheets"})$

$$= E\left(\sum_{i=1}^{370} X_i\right) = E(X_1 + X_2 + \cdots + X_{370})$$

$$= EX_1 + EX_2 + \cdots + EX_{370}$$

$$= \underbrace{0.1 + 0.1 + \cdots + 0.1}_{370 \text{ times}}$$

$$= 370(0.1)$$

$$= 37 \text{ mm}$$

$$\operatorname{var}\left(\sum_{i=1}^{370} X_i\right) = \operatorname{var}(X_1 + \cdots + X_{370})$$

$$= \operatorname{var}(X_1) + \cdots + \operatorname{var}(X_{370})$$

$$= \sum_{i=1}^{370} \operatorname{var}(X_i)$$

$$= 370(0.003)^2$$

$$= 0.00333 \text{ mm}^2$$

and then $SD\left(\sum_{1}^{370} X_i\right) = \sqrt{0.00333} = \boxed{0.0577 \text{ mm}}$

Problem 3)

Since $n \geq 25$ (large enough), we can use CLT.

$$\bar{X} \sim N\left(\mu, \sigma = \frac{0.1}{\sqrt{25}} = 0.02\right)$$

$$\Rightarrow P(\mu - 0.03 < \bar{X} < \mu + 0.03)$$

$$= P\left(\frac{\mu - 0.03 - \mu}{0.02} < \frac{\bar{X} - \mu}{0.02} < \frac{\mu + 0.03 - \mu}{0.02}\right)$$

$$= P(-1.5 < Z < 1.5)$$

$$= \Phi(1.5) - \Phi(-1.5)$$

$$= 0.9332 - 0.0668$$

$$= 0.8664$$

Problem 4) $Z_1, \dots, Z_n \overset{iid}{\sim} N(0,1)$

a) $EX = E(3Z_1 + 5) = 3EZ_1 + 5 = 5$

$Var(X) = Var(3Z_1 + 5) = 9 Var(Z_1) = 9$

b) $EY = E(Z_1 - Z_2) = EZ_1 - EZ_2 = 0$

$Var(Y) = Var(Z_1 - Z_2) = Var(Z_1) + (-1)^2 Var(Z_2)$

$= 1 + 1 = 2$

c) $EU = E(Z_1 + Z_2) = EZ_1 + EZ_2 = 0$

$Var(U) = Var(Z_1 + Z_2) = Var(Z_1) + Var(Z_2) = 1 + 1 = 2$

d) $EW = E\left(\sum_{i=1}^{n} i/n \,(Z_i + i/n)\right)$

$= E \sum_{i=1}^{n} i/n \, Z_i + (i/n)^2$

$= \sum_{i=1}^{n} E(i/n \, Z_i) + E(i/n)^2$

(Note that the only r.v in this quantity is Z_i)

$= \sum_{i=1}^{n} i/n \, EZ_i + (i/n)^2$

$= \sum_{i=1}^{n} 0 + (i/n)^2$

$= \frac{1}{n^2} \sum_{i=1}^{n} i^2$

by the hint $= \frac{1}{n^2} \frac{n(n+1)(2n+1)}{6}$

$= \frac{(n+1)(2n+1)}{6n}$

$$\operatorname{var}(W) = \operatorname{var}\left(\sum_{i=1}^{n} \tfrac{i}{n} Z_i + \tfrac{i^2}{n^2}\right)$$

$$= \sum_{i=1}^{n} \operatorname{var}\left(\tfrac{i}{n} Z_i\right) + \operatorname{var}\left(\tfrac{i^2}{n^2}\right)$$

$$= \sum_{i=1}^{n} \tfrac{i^2}{n^2} \operatorname{var} Z_i + 0$$

$$= \sum_{i=1}^{n} \tfrac{i^2}{n^2} (1)$$

$$= \frac{1}{n^2} \sum_{i=1}^{n} i^2$$

$$= \frac{1}{n^2} \frac{n(n+1)(2n+1)}{6}$$

$$= \frac{(n+1)(2n+1)}{6n}$$

Note:
In this case, for W,
$E(W)$ & $\operatorname{var}(W)$
are the same.

Problem 5)

a) by CLT for $n = 64 \ (\geq 25)$;

$$\bar{X} \dot{\sim} N(20, 16^2/64)$$

$$\Rightarrow P(\bar{X} < 16) = P\left(Z < \frac{16-20}{16/8}\right) \simeq P(Z < -2)$$

$$= \Phi(-2) = \underline{0.0228}$$

(b) $P(16 < \bar{x} < 23) = P\left(\frac{16-20}{16/8} < Z < \frac{23-20}{16/8} \right)$

$= P(-2 < Z < 1.5)$

$= \Phi(1.5) - \Phi(-2)$

$= 0.9332 - 0.0228$

$= 0.9104$

c) $P(|\bar{x} - 20| > 0.5) \leq 0.05$

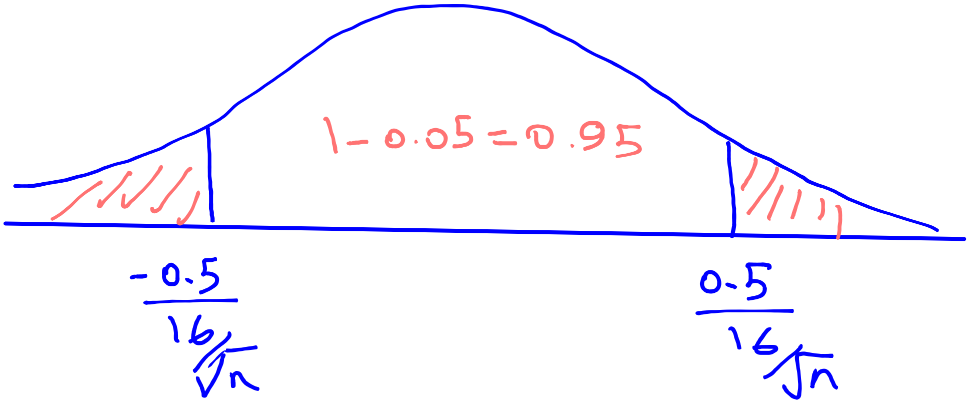
$\Rightarrow P(\bar{x} - 20 > 0.5 \text{ or } \bar{x} - 20 < -0.5) \leq 0.05$

$\Rightarrow P(\bar{x} - 20 > 0.5) + P(\bar{x} - 20 < 0.5) \leq 0.05$

$X \sim N(20, 16^2/64)$

$\Rightarrow P\left(Z > \frac{0.5}{16/\sqrt{n}}\right) + P\left(Z < \frac{-0.5}{16/\sqrt{n}}\right) \leq 0.05$

$\Rightarrow$



need the sum of the shaded area to be ≤ 0.05

$$\Rightarrow \quad 2P\left(Z < \frac{-0.5}{16/\sqrt{n}}\right) \leq 0.05$$

Recall: $Z \sim N(0,1)$: $P(Z > a) = P(Z < -a)$

$$\Rightarrow \quad \Phi\left(\frac{-0.5}{16/\sqrt{n}}\right) \leq \frac{0.05}{2} = 0.025$$

by the table

$$\Rightarrow \quad \frac{-0.5}{16/\sqrt{n}} \leq -1.96$$

$$\Rightarrow \quad n \geq 3933.798 \quad \Rightarrow \quad \boxed{n \geq 3934}$$