Chapter 5.4: Continuing Joint Distributions and Independence

Working with Multiple Random Variables

Conditional Distributions

and

Independence

Conditional Distributions

Recall Lecture 12 and 13 and the idea of conditional probability. We have the same concept in distributions: if we know information about the random variable Y, then we may be changing how likely we are to see certain values for X.

> **Conditional Distributions for Discrete RVs**
>
> For discrete random variables X and Y, the conditional probability function of X given that $Y = y$ is
>
> $$f_{X|Y}(x) = \frac{f_{XY}(x, y)}{f_Y(y)}$$

This is how we find the probability that $X = x$ if we know that $Y = y$.

Conditional Distributions

The same rule applies for the continuous random variables:

Conditional Distributions for continuous RVs

For continuous random variables X and Y, the conditional probability density function of X given that $Y = y$ is

$$f_{X|Y}(x|y) = \frac{f_{XY}(x, y)}{f_Y(y)}$$

This is how we express probability density for X if we know that $Y = y$.

Example: Example 19 in text book

Suppose that S and R have joint probability density function:

$$f_{SR}(s,r) = \frac{1}{16.5} e^{\left(-\frac{s}{16.5}\right)} \frac{1}{\sqrt{2\pi(.25)}} e^{\left(-(r-s)^2/2(.25)\right)}$$

if $s > 0$ and is 0 otherwise.

1. Find the $f_{S|R}(s|r)$. What is the distribution of S if $R = r$?
2. Find the expected value of S given $R = 2$.
3. Find the expected value of S given $R = 3$.

Example

Suppose that X and Y have joint probability function:

$$f_{XY}(x,y) = \begin{cases} c\exp\left(-\frac{2x+3y}{6}\right) & x \geq 0, y \geq 0 \\ 0 & o.w. \end{cases}$$

where c is a constant.

1. Find the $f_{X|Y}(x|y)$. What is the distribution of X if $Y = y$?
2. Find the expected value of X given $Y = 2$.
3. Find the expected value of X given $Y = 3$.
4. What is the difference between the way these two examples?

Independence

The big difference between the distributions in the last two examples is that while changing the value taken by R changes the likelihood of values S can take, changing Y has no impact on the likelihood of the values X can take. We call X and Y **independent** random variables.

> **Independence**
>
> For discrete or continuous random variables X and Y, we say X and Y are independent if and only ifif the conditional probability density function of X given that $Y = y$ is
>
> $$f_{XY}(x, y) = f_X(x) \cdot f_Y(y)$$

This concept is hugely important in statistics.

Independence (cont)

Example

Suppose that Z_1, Z_2, Z_3, Z_4 are independent random variables and that each follows a standard normal distribution.

Find the joint pdf $f(z_1, z_2, z_3, z_4)$.

Wrap Up Example

Example: Section 5.4 Exercise 5

Chapter 5.5: Functions of Random Variables

Results and Theorems

Functions of Random Variables

A random variable can be thought of as a function whose input is an outcome and whose output is a real number. When we take a function of the value the random variable takes, the resulting value is still depends on the outcome of a random experiment - in other words: functions of random variables are random variables.

This means that a function of a random variable will have probabilities attached to the value it takes, based on the value taken by the random variable. It also means functions of random variables will have:

- probability functions (if discrete) or probability density functions (if continuous)
- cumulative probability functions (if discrete) or cumulative density functions (if continuous)
- expected values and variances
- …

In other words, everything that normal random variables have

Functions of Random Variables

Suppose that X is a random variable with the following probability values: $P(X = -1) = 0.2$, $P(X = 0) = 0.6$, $P(X = 1) = 0.2$. Find the probabilities associated with $Y = X^2$.

Functions of RVs

Inverting Functions

For continuous variables, it is possible to use inverses to get the distribution for a function of a random variable:

Suppose that X is a random variable and $Y = g(X)$ is a function of the random variable and that g is invertable. Then

$$\begin{aligned}
F_Y(y) &= P(Y \leq y) \\
&= P(g(X) \leq y) \\
&= P(X \leq g^{-1}(y)) \\
&= F_X(g^{-1}(y))
\end{aligned}$$

Functions of RVs

Meaning

Single RVs

Taking Inverses

Example

Suppose that $X \sim exp(3)$. Let $Y = X^3$. Find the probability density function of Y.

A Note of Caution

When doing the inversion, be careful that you don't lose probability. For instance, consider this example

> Suppose that X is uniform on the interval $(-1, 1)$. For the random variable $Y = X^2$, find the $P(Y \leq 0.5)$.

Functions of RVs

Meaning

Single RVs

Taking Inverses

Multiple RVs

Functions of Multiple Random Variables

Suppose that $X_1, X_2, \ldots, X_n$ are all random variables. Then $U = g(X_1, X_2, \ldots, X_n)$ is also a random variable. However, the probability function (or density function) for U can be very, _very_, *very* difficult to find. However, there are still some things we can say generally about a certain kind of function: a linear combination

> **Linear Combination of Random Variables**
> For constants $a_0, a_1, a_2, \ldots, a_n$ and independent random variables $X_1, X_2, \ldots, X_n$, let $U = a_0 + a_1X_1 + a_2X_2 + \ldots + a_nX_n$. Then U is a **linear combination** of the random variables.

Functions of RVs

Meaning

Single RVs

Taking Inverses

Multiple RVs

Functions of Multiple Random Variables

We can say the following about linear combinations of random variables:

Mean Linear Combination of Random Variables
For constants $a_0, a_1, a_2, \ldots, a_n$ and independent random variables $X_1, X_2, \ldots, X_n$, let $U = a_1X_1 + a_2X_2 + \ldots + a_nX_n$.
Then

$$E(U) = a_0 + a_1E(X_1) + a_2E(X_2) + \ldots + a_nE(X_n)$$

Functions of Multiple Random Variables

Variance of Linear Combination of Random Variables
For constants $a_0, a_1, a_2, \ldots, a_n$ and independent random variables $X_1, X_2, \ldots, X_n$, let
$U = a_1X_1 + a_2X_2 + \ldots + a_nX_n$.
Then and

$$Var(U) = a_0 + a_1^2 Var(X_1) + a_2^2 Var(X_2) + \ldots + a_n^2 Var(X_n)$$

Functions of RVs

Meaning

Single RVs

Taking Inverses

Multiple RVs

Central Limit Theorem

The most important result in statistics

> **Central Limit Theorem**
> If $X_1, X_2, \ldots, X_n$ are independent and identically distributed (iid) random variables each with mean μ and variance σ^2 and let the random variable $\bar{X} = \frac{1}{n}X_1 + \frac{1}{n}X_2 + \ldots + \frac{1}{n}X_n$. Then
>
> 1. $E(\bar{X}) = \mu$
> 2. $Var(\bar{X}) = \frac{\sigma^2}{n}$
> 3. For large n, $\bar{X}$ is approximately normally distributed (limit goes to normal…)

Example 25 (page 317) in the book provides a wonderful illustration of this