Quiz III

STAT 305, Section 3
FALL 2019

Instructions

- The quiz is scheduled for 80 minutes, from 09:30 to 10:50 AM. At 10:50 AM the exam will end.
- Total points for the exam is ~~60~~ 52. Points for individual questions are given at the beginning of each problem. Show all your calculations clearly to get full credit. Put final answers in the box at the right (except for the diagrams!).
- A forumula sheet is attached to the end of the exam. Feel free to tear it off.
- Normal quantile table is attached to the end of the exam. Feel free to tear it off.
- You may use a calculator during this exam.
- Answer the questions in the space provided. If you run out of room, continue on the back of the page.
- If you have any questions about, or need clarification on the meaning of an item on this exam, please ask your instructor. No other form of external help is permitted attempting to receive help or provide help to others will be considered cheating.
- **Do not cheat on this exam.** Academic integrity demands an honest and fair testing environment. Cheating will not be tolerated and will result in an immediate score of 0 on the exam and an incident report will be submitted to the dean's office.

Name: key

Student ID: ____________________

1. Suppose that X is a continuous random variable with cumulative density function (cdf):

$$F(x) = \begin{cases} 0 & x < 0 \\ 1 - e^{-3x} & x \geq 0 \end{cases}$$

(a) (2 points) What is the probability that X takes a value greater than 2? $\boxed{P = 0.0024}$

$P(X>2) = 1 - P(X \leq 2) = 1 - F(2)$

$= 1 - 1 + e^{-3(2)}$

$= e^{-6} = 0.0024$

(b) (2 points) Derive f(x), the probability density function.

$f(x) = \frac{d}{dx} F(x) = \frac{d}{dx}(1 - e^{-3x}) = 3e^{-3x}, \quad x \geq 0$

(c) (4 points) Find the expected value of X. $\boxed{E(X) = 1/3}$

① If you note, this is the pdf of an Exponential dist. with parameter $\alpha = 1/3$ $\Rightarrow$ $EX = 1/3$

② $EX = \int_0^\infty \underbrace{x}_{u} \underbrace{3e^{-3x}\,dx}_{dv} = -xe^{-3x}\Big|_0^\infty + \int_0^\infty e^{-3x}\,dx$

(integral by parts)

$\left\{ \begin{array}{l} u = x \rightarrow du = dx \\ dv = 3e^{-3x}dx \\ \Rightarrow v = -3e^{-3x} \end{array} \right\} \Rightarrow$

$= 0 + \int_0^\infty e^{-3x}\,dx$

$= -\frac{1}{3}e^{-3x}\Big|_0^\infty$

$= 0 - (-\frac{1}{3}e^{-3(0)}) = 1/3$

2. Suppose that X_1, X_2, X_3, X_4, X_5 and Y_1, Y_2, Y_3, Y_4, Y_5 are all independent random variables where for any i

$$E(X_i) = \mu_X$$
$$Var(X_i) = \sigma_X^2$$
$$E(Y_i) = \mu_Y$$
$$Var(Y_i) = \sigma_Y^2$$

Suppose that we define a random variable U to help compare the values taken by X_is and the values taken by the Y_is by pairing the random variables like this:

$$U = \frac{1}{5}(X_1 - Y_1) + \frac{1}{5}(X_2 - Y_2) + \frac{1}{5}(X_3 - Y_3) + \frac{1}{5}(X_4 - Y_4) + \frac{1}{5}(X_5 - Y_5)$$

(a) (3 points) Find the mean of U (hint: it will include μ_X and μ_Y.) $\boxed{E(U) = \mu_X - \mu_Y}$

$$U = \frac{1}{5}\sum_{i=1}^{5} X_i - \frac{1}{5}\sum_{i=1}^{5} Y_i$$

$$\Rightarrow EU = E\left(\frac{1}{5}(X_1+X_2+X_3+X_4+X_5)\right) - E\left(\frac{1}{5}(Y_1+Y_2+Y_3+Y_4+Y_5)\right)$$

$$= \frac{1}{5}\left[E(X_1) + \cdots + E(X_5)\right] - \frac{1}{5}\left[E(Y_1) + \cdots + E(Y_5)\right]$$

$$= \frac{1}{5}5(\mu_X) - \frac{1}{5}5(\mu_Y)$$

$$= \mu_X - \mu_Y$$

(b) (3 points) Find the standard deviation of U (hint: it will include σ_X^2 and σ_Y^2) $\boxed{SD(U) = \sqrt{\frac{\sigma_X^2 + \sigma_Y^2}{5}}}$

$$var(U) = var\left(\frac{1}{5}\sum_{i=1}^{5} X_i - \frac{1}{5}\sum_{i=1}^{5} Y_i\right)$$

$$= var\left(\frac{1}{5}\sum_{i=1}^{5} X_i\right) + var\left(-\frac{1}{5}\sum_{i=1}^{5} Y_i\right)$$

$$= \left(\frac{1}{5}\right)^2 \sum_{i=1}^{5} var(X_i) + \left(-\frac{1}{5}\right)^2 \sum_{i=1}^{5} var(Y_i)$$

$$= \frac{1}{25}(5\sigma_X^2) + \left(\frac{1}{25}\right)(5\sigma_Y^2)$$

$$= \frac{\sigma_X^2 + \sigma_Y^2}{5} \Rightarrow SD(U) = \sqrt{\frac{\sigma_X^2 + \sigma_Y^2}{5}}$$

3. Consider the following joint distribution for two random variables X and Y:

Y \X	0	1	2	3
0	0.1	0.2	0.1	0
1	0.4	0	0	0.2

(a) (4 points) Find the marginal pmfs of X and Y

x	0	1	2	3
f(x)	0.5	0.2	0.1	0.2

y	0	1
f(y)	0.4	0.6

(b) (2 points) Find the conditional distribution of $f_{X|Y}(x|y=1)$

x	0	1	2	3
$f_{X\|Y}(x\|y=1)$	2/3	0	0	1/3

$f_{x|y}(0|1) = \frac{f(0,1)}{f_y(1)} = \frac{0.4}{0.6} = 2/3$

$f_{x|y}(2|1) = \frac{f(2,1)}{f_y(1)} = 0$

$f_{x|y}(1|1) = \frac{f(1,1)}{f_y(1)} = 0$

$f_{x|y}(3|1) = \frac{f(3,1)}{f_y(1)} = \frac{0.2}{0.6} = 1/3$

(c) (3 points) Find the conditional expected value of $x|y=1$. That is $E_{(X|Y)}(x|y=1)$

$E_{(X|Y)}(x|y=1) =$ 1

by part b: $E_{x|y}(x|y=1) = 0(2/3) + 1(0) + 2(0) + 3(1/3) = 1$

(d) (4 points) Find the conditional variance of $x|y=1$. That is $Var_{(X|Y)}(x|y=1)$

$Var_{(X|Y)}(x|y=1) =$ 2

$E_{x|y}(x^2|y=1) = 0^2(2/3) + 1^2(0) + 2^2(0) + 3^2(1/3) = 3$

$Var_{x|y}(x|y=1) = E_{x|y}(x^2|y=1) - (E_{x|y}(x|y=1))^2$

$= 3 - 1 = 2$

(e) (2 points) Are X and Y independent? Why or why not?

No.

For example: $f_{X,Y}(0,0) = 0.1$

$f_X(0) = 0.5$

$f_Y(0) = 0.4$

$\Rightarrow 0.1 = f_{X,Y}(0,0) \neq \underbrace{f_X(0) f_Y(0)}_{0.5\times 0.4 = 0.2}$

4. Let X be a normal random variable with a mean of -2 and a varaince of 16 (i.e., $X \sim N(-2, 16)$) Find the following probabilities (note: the attached standard normal probability table may be helpful):

(a) (2 points) $P(-8 \leq X < 1)$ $\boxed{P = 0.7065}$

$= P\left(\frac{-8-(-2)}{4} \leq \frac{X-(-2)}{4} < \frac{1-(-2)}{4}\right)$

$= P(-3/2 \leq Z < 3/4) = \Phi(0.75) - \Phi(-1.5)$

$= 0.7733 - 0.0668 = 0.7065$

(b) (3 points) $P(|X + 2| \geq 4)$ $\boxed{P = 0.3173}$

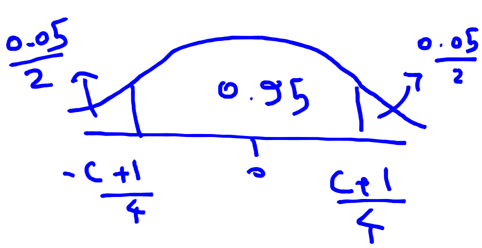
$= P(X+2 \geq 4) + P(X+2 \leq -4) = P\left(\frac{X+2}{4} \geq \frac{4}{4}\right) + P\left(\frac{X+2}{4} \leq \frac{-4}{4}\right)$

$= P(Z \geq 1) + P(Z \leq -1) = 1 - \Phi(1) + \Phi(-1)$

$= 0.3173$

(c) (4 points) Find the value of c such that $P(|X + 1| \leq c) = 0.95$ $\boxed{c = 8.84}$

$0.95 = P(-c \leq X+1 \leq c) = P(-c-1 \leq X \leq c-1)$



$= P\left(\frac{-c-1+2}{4} \leq \frac{X+2}{4} \leq \frac{c-1+2}{4}\right)$

$= P\left(Z \leq \frac{c+1}{4}\right) - P\left(Z \leq \frac{-c+1}{4}\right)$

$= 1 - 2\Phi\left(\frac{-c+1}{4}\right)$

$\Rightarrow 2\Phi\left(\frac{-c+1}{4}\right) = 1 - 0.95 = 0.05$

$\Rightarrow \Phi\left(\frac{-c+1}{4}\right) = \frac{0.05}{2} = 0.025 \Rightarrow \frac{-c+1}{4} = -1.96 \Rightarrow c = 8.84$

5. Seventy independent messages are sent from an electronic transmission center. Messages are processed sequentially, one after another. Transmission time of each message is Exponential with parameter $\alpha =$ 5min.

(a) (2 points) what are the expected value and variance of the sample mean of all 70 messages?

$E(\overline{X}) =$ 5

$\mathrm{Var}(\overline{X}) =$ 0.357

$X_1, \ldots, X_{70} \sim Exp(\alpha = 5)$

$\begin{cases} Ex = 5 \\ var(x) = 5^2 = 25 \end{cases}$

by CLT for $n = 70$ (>25) $\Rightarrow \bar{x} \overset{\cdot}{\sim} N\left(5, \frac{25}{70} = \frac{5}{14} = 0.357\right)$

(b) (3 points) Find the probability that the average of all 70 messages are transmitted in less than 12 minutes.

P = 1

$$P(\bar{x} < 12) = P\left(\frac{\bar{x} - 5}{\sqrt{0.357}} < \frac{12 - 5}{\sqrt{0.357}}\right)$$

$$= P\left(Z \le \frac{7}{0.5974}\right)$$

$$= P(Z \le 11.71) = 1$$