Show **all** of your work on this assignment and answer each question fully in the given context.

If you cannot submit your homework in the class, you can drop it at my office door in 3220 Snedecore Hall by Thursday at 03:30 PM.

Please staple your assignment and write your name !

1. **[Ch. 5.1, Exercise 9, pg. 244]** Transmission line interruptions in a telecommunication network occur at an average rate of one per day.
   (a) Use a Poisson distribution as a model for
   $$X = \text{the number of interruptions in the next five-day work week}$$
   Now, precisely specify the probability distribution.[5pts]
   (b) Find $P[X = 0]$[5pts]
   (c) Now consider the random variable
   $$Y = \text{the number of work weeks in the next four in which there are no interruptions}$$
   What is a reasonable probability model for Y?[5pts]
   hint:i.e. precisely specify the probability distribution.
   (d) Find $P[Y = 2]$.[5pts]
2. Suppose a standup comedian plans to give a total of $n = 5$ jokes in an entire 2-hour performance. Call a joke a success if at least one audience member laughs. If no audience member laughs, the joke is a failure.
   Assume that all the jokes are equally funny, with $p = P(\text{success}) = 0.2$. Let X be the random variable associated with the number of successful jokes out of the total 5.
   (a) Preciisely state the distribution of X, giving the values of any parameters necessary.[2pts]
   (b) Calculate the probability that the whole night is a failure. i.e. find the $P(\text{no success})$. [5pts]
   (c) Calculate the probability that the comedian tells at least 4 successful jokes.[5pts]
   (d) Calculate the expected number of successful jokes.[5pts]
   (e) Calculate the standard deviation of the successful jokes.[5pts]
3. The number of computer shutdowns during any month has a Poisson distribution, averaging 0.25 shutdowns per month.
   (a) What is the probability of at least 2 computer shutdowns during the next year?[5pts]
   (b) What is the probability of at most 2 computer shutdowns during the next 6 month? [5pts]
   (c) What is the variance of the number of computer shutdowns during the next year? [2pts]
4. Suppose that X is a random variable with probability density function
   $$f(x) = \begin{cases} c \cdot x^2 & -2 \le x \le 2 \\ 0 & o.w. \end{cases}$$

(a) Find the value of c that makes $f(x)$ a valid probability density function.[5 pts]

(b) Find the CDF of the random variable X.[5 pts]

(c) What is $P(|X| \geq -1)$ [5 pts].

(d) Find the expected value of X.[5 pts]

5. Consider a continuously distributed random variable, W, with a probability density function given by

$$f(w) = \begin{cases} \frac{1}{5(1-e^{-2})}e^{-w/5} & 0 \leq w \leq 10 \\ 0 & \text{otherwise} \end{cases}$$

(a) Show that the function $f(w)$ is a valid probability density function (i.e., show that (i) $f(w)$ is non-negative and (ii) $\int_{-\infty}^{\infty} f(w)dw = 1$). [5 pts]

(b) Find $P(W \leq 2)$ [5 pts]

(c) Find $P(2 \leq W \leq 10)$ [5 pts]

(d) Find $E(W)$ [5 pts]

6. **[Ch. 5.2, Exercise 1, pg. 263]** The random number generator supplied on a calculator is not terribly well chosen, in that values it generates are not adequately described by a uniform distribution on the interval $(0, 1)$. Suppose instead that a probability density

$$f(x) = \begin{cases} k(5-x) & 0 < x < 1 \\ 0 & \text{otherwise} \end{cases}$$

is a more appropriate model for $X =$ the next value produced by this random number generator.

(a) Find the value of k.[5pts]

(b) Evaluate $P[.25 < X < .75]$[5pts]

(c) Compute the cumulative probability distribution function for X, $F(x)$. [5pts]

(d) Calculate E(X) and the standard deviation of X.[6pts]

[Total: 115 pts]

problem 7)

a) $X \sim Poisson(5)$

b) $P(X=0) = F(0) = \dfrac{e^{-5} 5^{0}}{0!} = 0.0067$

c) in the next 4 weeks, $n=4$ & Y is the number of success (No interruptions).

The prob. of success (No interruptions) is given by part (b): $P(X=0) = 0.0067$.

So, $Y \sim bin(n=4, p=0.0067)$

d) $P(Y=2) = \dfrac{4!}{2!(4-2)!}(0.0067)^2(1-0.0067)^{4-2}$

$= 0.0027$

problem 2)

a) $X \sim binomial(n=5,\ p=0.2)$

b) $P(\text{no success}) = P(X=0)$

$$= \frac{5!}{0!(5-0)!}(0.2)^0(1-0.2)^{5-0}$$

$$= 0.3277$$

c) $P(\text{at least 4 successful jokes}) = P(X \geq 4)$

$$= P(X=4) + P(X=5)$$

$$= f(4) + f(5)$$

$$= \frac{5!}{4!(5-4)!}(0.2)^4(0.8)^{5-4} + \frac{5!}{5!(5-5)!}(0.2)^5(0.8)^{5-5}$$

$$= 0.00672$$

d) $E(X) = n.p = 5(0.2) = 1$

e) $SD(X) = \sqrt{Var(X)} = \sqrt{np(1-p)}$

$$= \sqrt{5(0.2)(0.8)}$$

$$= \sqrt{0.8} = 0.8944$$

Problem 3)

a) let X be # of shut downs in a year.

First need to find λ for a year

0.25 shutdowns	1 month	$\Rightarrow \lambda = 12(0.25) = 3$
λ	12	

$X \sim \text{Poissoin}(\lambda = 3)$

$P(\text{at least 2 computer shutdowns}) = P(X \geq 2)$

$= 1 - P(X \leq 1) = 1 - f(0) - f(1)$

$= 1 - \left(\frac{e^{-3} 3^0}{0!}\right) - \left(\frac{e^{-3} 3^1}{1!}\right)$

$= 0.8009$

b) 6 month $\longrightarrow$ $\lambda = 6(0.25) = 1.5$

$P(X \leq 2) = P(X=0) + P(X=1) + P(X=2)$

$= f(0) + f(1) + f(2)$

$= \frac{e^{-1.5} 1.5^0}{0!} + \frac{e^{-1.5} 1.5^1}{1!} + \frac{e^{-1.5} 1.5^2}{2!}$

$= 0.8088$

c) $X \sim \text{Poissoin}(\lambda = 3)$ (for a year)

$\boxed{Var(X) = \lambda = 3}$

Problem 4)

(a) $\int_{-\infty}^{\infty} f(x)\,dx = 1 \longrightarrow \int_{-2}^{2} cx^2\,dx = 1$

$$\Rightarrow c\int_{-2}^{2} x^2\,dx = 1$$

$$\Rightarrow c.\left(\frac{x^3}{3}\Big|_{-2}^{2}\right) = 1$$

$$\Rightarrow c\left(\frac{8}{3} - \frac{(-8)}{3}\right) = 1$$

$$\Rightarrow c\left(\frac{16}{3}\right) = 1 \Longrightarrow \boxed{c = \frac{3}{16}}$$

(b) Note that the pdf is now

$$f_X(x) = \begin{cases} \frac{3}{16}x^2 & -2 \le x \le 2 \\ 0 & \text{o.w} \end{cases}$$

Finding CDF :

$$F_X(t) = P(X \le t) = \int_{-2}^{t} \frac{3}{16}x^2\,dx$$

$$= \frac{3}{16}\left[\frac{x^3}{3}\Big|_{-2}^{t}\right]$$

$$= \frac{1}{16}\left[x^3\Big|_{-2}^{t}\right]$$

$$= \frac{1}{16}\left(t^3 - (-8)\right) = \boxed{\frac{1}{16}\left(t^3 + 8\right)}$$

(c)

$$P(|X| \geq -1) = 1$$

(Note that no matter what pdf X has, the statement $|X| \geq -1$ always holds true!

(d)

$$E X = \int_x x f(x)\, dx = \int_{-2}^{2} x \cdot \frac{3}{16} x^2\, dx$$

$$= \int_{-2}^{2} \frac{3}{16} x^3\, dx$$

$$= \frac{3}{16} \cdot \frac{x^4}{4} \Big|_{-2}^{2}$$

$$= \frac{3}{16}\left(\frac{2^4}{4} - \frac{(-2)^4}{4} \right)$$

$$\boxed{= 0}$$

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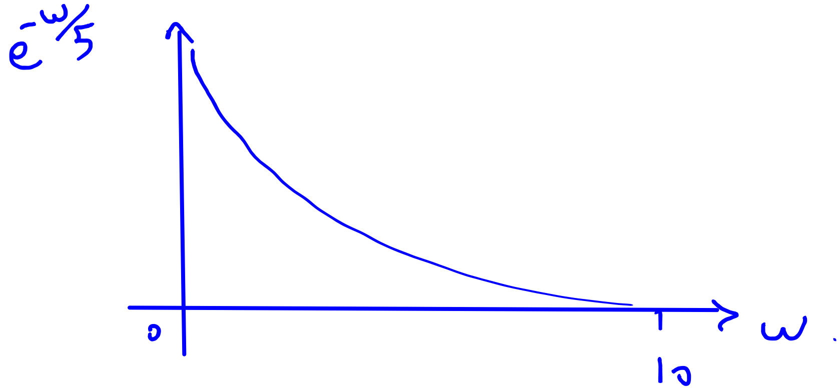
problem 5)

(a) i: need to show $f(w) \geq 0 \;\forall\; w \in [0, 10]$

$$f(w) = \frac{1}{5(1-e^{-2})} e^{-w/5}$$

$\approx 0.231 \geq 0$

On the other hand, $e^{-w/5}$ for $0 \le w \le 10$ has a shape like



which is always a positive quantity.

(i.e $e^{-w/5} \geqslant 0$ for $w \in [0 \ \ 10]$)

Therefore, $f(w) = \underbrace{\underbrace{\frac{1}{5(1-e^{-2})}}_{>0} \underbrace{e^{-w/5}}_{>0}}_{>0}$, $0 \le w \le 10$

ii) need to show $\int_{-\infty}^{\infty} f(w)\, dw = 1$

$$\Rightarrow \int_0^{10} \frac{1}{5(1-e^{-2})} e^{-w/5}\, dw = \frac{1}{5(1-e^{-2})} \int_0^{10} e^{-w/5}\, dw$$

$$= \frac{1}{5(1-e^{-2})} \left(-5e^{-w/5} \Big|_0^{10} \right)$$

$$= \frac{1}{5(1-e^{-2})} \left(5\left(1 - e^{-10/5}\right) \right) = 1$$

(b) $$P(W \le 2) = \int_0^2 \frac{1}{5(1-e^{-2})} e^{-w/5}\, dw$$

$$= \frac{1}{5(1-e^{-2})} \int_0^2 e^{-w/5}\, dw$$

$$= \frac{1}{5(1-e^{-2})} \left(-5 e^{-w/5} \Big|_0^2\right)$$

$$= \frac{1}{5(1-e^{-2})} \left(5\left(1-e^{-2/5}\right)\right)$$

$$= \frac{1-e^{-0.4}}{1-e^{-2}} = 0.381$$

(c) $$P(2 \le W \le 10) = \int_2^{10} \frac{1}{5(1-e^{-2})} e^{-w/5}\, dw$$

$$= \frac{1}{5(1-e^{-2})} \int_2^{10} e^{-w/5}\, dw$$

$$= \frac{1}{\cancel{5}(1-e^{-2})} \left(-\cancel{5}\, e^{-w/5} \Big|_2^{10}\right)$$

$$= \frac{1}{1-e^{-2}} \left(e^{-2/5} - e^{-10/5}\right)$$

$$= \frac{e^{-0.4} - e^{-2}}{1-e^{-2}} = 0.618$$

d) $E(w) = \int_0^{10} \frac{w}{5(1-e^{-2})} e^{-w/5}\, dw$

$$= \frac{1}{5(1-e^{-2})} \int_0^{10} w\, e^{-w/5}\, dw$$

Integration by part :

$$\int u\,dv = uv - \int v\,du$$

In This problem :

$$u = w \longrightarrow du = dw$$

$$dv = e^{-w/5} \longrightarrow v = \int e^{-w/5}\, dw = -5\, e^{-w/5}$$

$$\Rightarrow E\,W = \frac{1}{5(1-e^{-2})}\left[w\cdot(-5\, e^{-w/5}) \Big|_0^{10} - \int_0^{10} (-5\, e^{-w/5})\, dw \right]$$

$$= \frac{1}{5(1-e^{-2})}\left[10(-5\, e^{-10/5}) + 0 + \int_0^{10} 5\, e^{-w/5}\, dw \right]$$

$$= \frac{1}{5(1-e^{-2})}\left[-10(5)\, e^{-2} + 5\,\underbrace{\left[-5\, e^{-w/5} \Big|_0^{10} \right]}_{-5(e^{-10/2} - 1)} \right]$$

$$= \frac{1}{5(1-e^{-2})}\left[-50\, e^{-2} + 5(5)(1-e^{-2}) \right]$$

$$= \frac{1}{5(1-e^{-2})}\left(-50\, e^{-2} + 25 - 25\, e^{-2} \right)$$

$$= \frac{1}{5(1-e^{-2})}\left(25 - 75e^{-2}\right)$$

$$= \frac{+25}{5(1-e^{-2})}\left(1 - 3e^{-2}\right)$$

$$= 5\ \frac{1-3e^{-2}}{1-e^{-2}} = 5(0.6869)$$

$$= 3.4345$$

Problem 6)

a) $1 = \int_0^1 k(5-x)\,dx$

$= k\int_0^1 5-x\,dx = k\left[5x\Big|_0^1 - \frac{x^2}{2}\Big|_0^1\right]$

$= k\left[5 - \frac{1}{2}\right]$

$= k \cdot \frac{9}{2}$

$\Rightarrow k = \frac{2}{9}$

So, $f(x) = \begin{cases} \frac{2}{9}(5-x) & 0 < x < 1 \\ 0 & \text{o.w} \end{cases}$

b) $P(0.25 < x < 0.75) = \int_{0.25}^{0.75} \frac{2}{9}(5-x)\,dx$

$= \frac{2}{9}\left(5x - \frac{x^2}{2}\right)\Big|_{0.25}^{0.75}$

$= 0.5$

c) $F(t) = ?$ $\longrightarrow$ $P(X \leq t) = \int_0^t \frac{2}{9}(5-x)\,dx$

$$= \begin{cases} 0 & x < 0 \\ \frac{2}{9}\left(5x - \frac{1}{2}x^2\right) & 0 \leq x < 1 \\ 1 & x \geq 1 \end{cases}$$

d) $E(X) = \int_0^1 x \cdot f(x)\,dx = \int_0^1 x \cdot \frac{2}{9}(5-x)\,dx$

$= \frac{2}{9}\int_0^1 5x - x^2\,dx$

$= \frac{2}{9}\left(\frac{5x^2}{2} - \frac{x^3}{3}\right)\Big|_0^1$

$= \frac{13}{27}$

$E(x^2) = \int_0^1 x^2 f(x)\,dx = \int_0^1 \frac{2}{9}x^2(5-x)\,dx$

$= \frac{2}{9}\int_0^1 5x^2 - x^3\,dx$

$= \frac{2}{9}\left(\frac{5x^3}{3} - \frac{x^4}{4}\right)\Big|_0^1$

$= \frac{2}{9}\left(\frac{5}{3} - \frac{1}{4}\right) = \frac{2}{9} \times \frac{17}{12} = \frac{17}{9 \times 6}$

$$Var(X) = EX^2 - [E(X)]^2$$

$$= \frac{17}{9 \times 6} - \left(\frac{13}{27}\right)^2$$

$$= 0.3148 - 0.2318$$

$$= 0.083$$