

STAT 305: Chapter 5

Part III

Amin Shirazi

Course page:

ashirazist.github.io/stat305_s2020.github.io

Continuous Random Variables

Terminology, Use, and Common Distributions

What is a Continuous Random Variable?

Background

What?

Background on Continuous Random Variable

Along with discrete random variables, we have continuous random variables. While discrete random variables take one specific values from a *discrete* (aka countable) set of possible real-number values, continuous random variables take values over intervals of real numbers.

def: Continuous random variable

A continuous random variable is a random variable which takes values on a continuous interval of real numbers.

The reason we treat them differently has mainly to do with the differences in how the math behaves: now that we are dealing with interval ranges, we change summations to integrals.

Background

Examples of continuous random variable:

What?

Z is the amount of torque required to loosen the next bolt (not rounded)

T is the time you will wait for the next bus

C is the outside temperature at 11:49 pm tomorrow

L is the length of the next manufactured metal bar

V is the yield of the next run of process

Terminology and Usage

Background

Probability Density Function

Terminology

pdf

Since we are now taking values over an interval, we can not "add up" probabilities with our probability function anymore. Instead, we need a new function to describe probability:

def: probability density function

A probability density function (pdf) defines the way the probability of a continuous random variable is distributed across the interval of values it can take. Since it represents probability, the probability function must always be non-negative. Regions of higher density have higher probability.

Background Probability Density Function

Terminology Validity of a *pdf*

pdf

Any function that satisfies the following can be a probability density function:

1.

2. for all in

and such that for all ,

Background

Probability Density Function

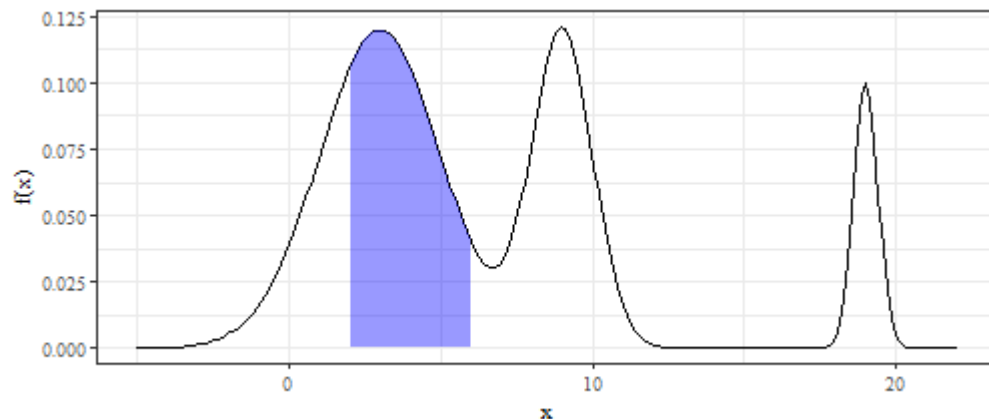
Terms and Use

pdf

With continuous random variables, we use pdfs to get probabilities as follows:

For a continuous random variable with probability density function ,

for any real values such that



Background

Example

Terms and Use

Consider a de-magnetized compass needle mounted at its center so that it can spin freely. It is spun clockwise and when it comes to rest the angle, θ , from the vertical, is measured. Let

pdf

What values can θ take?

What form makes sense for $P(\theta)$?

Background

Example

Terms and Use

pdf

If this form is adopted, that what must the pdf be?

Using this pdf, calculate the following probabilities:

- —

Background Example

Terms and Use

pdf

• –

• –

Background

Cumulative Density Function (CDF)

Terms and Use

pdf

cdf

We also have the cumulative density function for continuous random variables:

def: Cumulative density function (cdf) For a continuous random variable, X , with pdf $f(x)$ the cumulative density function $F(x)$ is defined as the probability that X takes a value less than or equal to x which is to say

TRUE FACT: the Fundamental Theorem of Calculus applies here:

Background

Cumulative Density Function (CDF)

Terms and Use

Properties of CDF for continuous random variables

As with discrete random variables, has the following properties:

pdf

- F is monotonically increasing (i.e it is never decreasing)

cdf

- $\lim_{x \rightarrow -\infty} F(x) = 0$ and $\lim_{x \rightarrow \infty} F(x) = 1$
 - This means that $0 \leq F(x) \leq 1$ for **any** CDF
- F is *continuous*. (instead of just right continuous in discrete form)

Mean and Variance of Continuous Random Variables

Background

Expected Value and Variance

Terms and Use

Expected Value

As with discrete random variables, continuous random variables have expected values and variances:

pdf

cdf

$E(X)$, $V(X)$

def: Expected Value of Continuous Random Variable

For a continuous random variable, X , with pdf $f(x)$ the expected value (also known as the mean) is defined as

We often use the symbol μ for the mean of a random variable, since writing $E(X)$ can get confusing when lots of other parenthesis are around. We also sometimes write μ_X .

Background

Expected Value and Variance

Terms and Use

pdf

cdf

$E(X)$, $V(X)$

Variance

def: Variance of Continuous Random Variable

For a continuous random variable, X , with pdf $f(x)$ and expected value μ , the variance is defined as

which is identical to saying

We will sometimes use the symbol σ^2 to refer to the variance and you may see the notation σ or σ_x as well.

Background

Expected Value and Variance

Terms and Use

Standard Deviation (SD)

We can also use the variance to get the standard deviation of the random variable:

pdf

cdf

$E(X)$, $V(X)$

def: Standard Deviation of Continuous Random Variable

For a continuous random variable, X , with pdf $f(x)$ and expected value μ , the standard deviation is defined as:

$$\sigma = \sqrt{V(X)}$$

Background

Expected Value and Variance: Example

Terms and Use

pdf

Library books

Let T denote the amount of time for which a book on - hour hold reserve at a college library is checked out by a randomly selected student and suppose its density function is

Calculate $E(T)$ and $Var(T)$.

An important point about Expected Value and Variance of Random Variables

Background

Expected Value and Variance:

Terms and Use

pdf

For a linear function,
constants,

, where and are



e.g Let
and variance of $4X - 3$?

. What is the expected value

Common Distributions

Uniform Distribution

Background

Common continuous Distributions

Terms and Use

Uniform Distribution

Common Dists

For cases where we only know/believe/assume that a value will be between two numbers but know/believe/assume *nothing* else.

Uniform

Origin: We know a the random variable will take a value inside a certain range, but we don't have any belief that one part of that range is more likely than another part of that range.

Definition: Uniform random variable

The random variable X is a uniform random variable on the interval $[a, b]$ if it's density is constant on $[a, b]$ and the probability it takes a value outside $[a, b]$ is 0. We say that X follows a uniform distribution or $X \sim \text{Uniform}(a, b)$.

Background

Uniform Distribution

Terms and Use

Common Dists

Uniform

Definition: Uniform pdf

If X is a uniform random variable on $[a, b]$, then the probability density function of X is given by

$$f(x) = \frac{1}{b-a} \quad \text{for } a \leq x \leq b$$

With this, we can find the for any value of a and b , if the mean and variance are:

—

—

Background

Uniform Distribution

Terms and Use

Common Dists

Uniform

Definition: Uniform cdf

If X is a uniform random variable on $[a, b]$ then the cumulative density function of X is given by

,

Background

Uniform Distribution

Terms and Use

A few useful notes:

- The most commonly used uniform random variable is $U(0, 1)$.

Common Dists

- Again, this is useful if we want to use a random variable that takes values within an interval, but we don't think it is likely to be in any certain region.

Uniform

- The values a and b used to determine the range in which X is not 0 are parameters of the distribution.

Common Continuous Distributions

Exponential Distribution

Background

Exponential Distribution

Terms and
Use

Definition: Exponential random variable

An random variable measures the waiting time until a specific event that has an equal chance of happening at any point in time. (it can be considered the continuous version of geometric distribution)

Common
Dists

Examples:

- Time between your arrival at the bus station and the moment that bus arrives
- Time until the next person walks inside the park's library
- The time (in hours) until a light bulb burns out.

Uniform

Exponential

Background

Exponential Distribution

Terms and Use

Common Dists

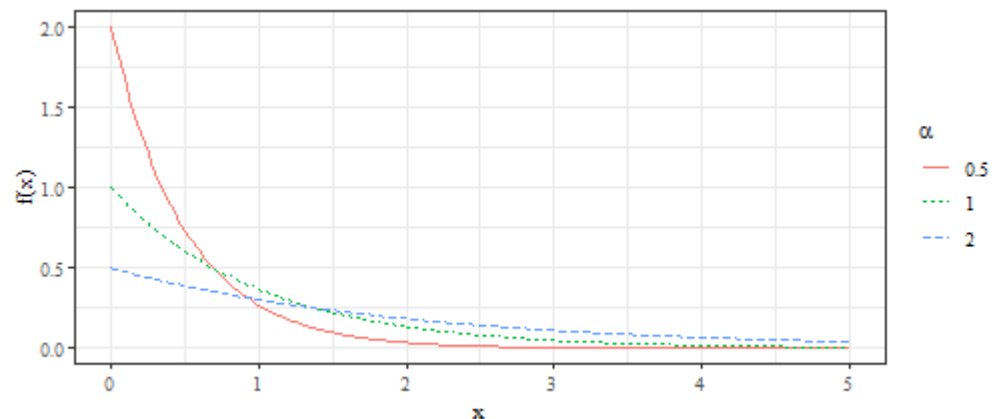
Uniform

Exponential

Definition: Exponential pdf

If X is an exponential random variable with rate α — then the probability density function of X is given by

$$f(x) = \alpha e^{-\alpha x}$$



Background

Exponential Distribution

Terms and Use

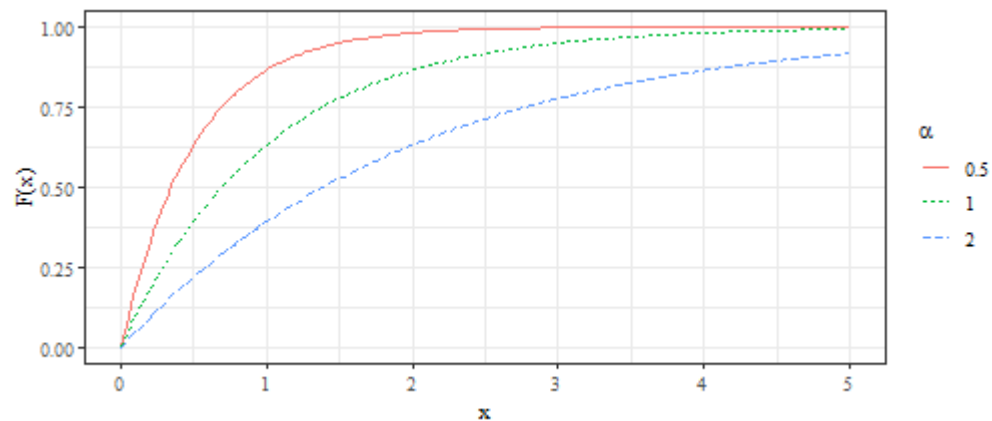
Common Dists

Uniform

Exponential

Definition: Exponential CDF

If X is a exponential random variable with rate α then the cumulative density function of X is given by



Mean and Variance of Exponential Distribution

Background Exponential Distribution

Terms and Use

Common Dists

Uniform

Exponential

Definition: Exponential pdf

If X is an exponential random variable with rate λ — then the probability density function of X is given by

$$f(x) = \lambda e^{-\lambda x}$$

` From this, we can derive:

Background

Exponential Distribution

Terms and Use

Example: Library arrivals, cont'd

Recall the example the arrival rate of students at Parks library between 12:00 and 12:10pm early in the week to be about students per minute. That translates to a minute average waiting time between student arrivals.

Common Dists

Consider observing the entrance to Parks library at exactly noon next Tuesday and define the random variable

Uniform

Exponential

Using , what is the probability of waiting more than seconds ($1/6$ min) for the first arrival?

Background

Exponential Distribution

Terms and
Use

Example: Library arrivals, cont'd

Common
Dists

What is the probability of waiting less than seconds?

Uniform

Exponential

Common Continuous Distributions

Normal Distribution

Background

The Normal distribution

Terms and Use

We have already seen the normal distribution as a "bell shaped" distribution, but we can formalize this.

The **normal** or **Gaussian** distribution is a continuous probability distribution with probability density function (pdf)

Common Dists



Uniform

for .

We then show that by

Exponential

Normal

Background

The Normal distribution

Terms and Use

A normal random variable is (often) a finite average of many repeated, independent, identical trials.

Common Dists

Mean width of the next 50 hexamine pallets

Mean height of 30 students

Total yield of the next 10 runs of a chemical process

Uniform

Exponential

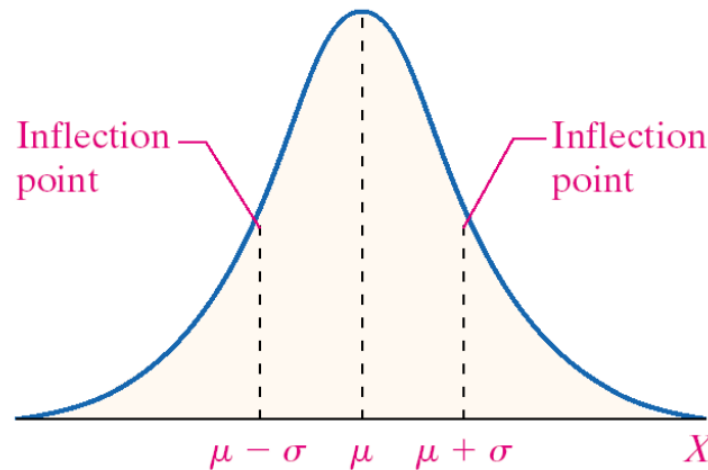
Normal

Background

Normal Distribution's Center and Shape

Terms and Use

Regardless of the values of μ and σ , the normal pdf has the following shape:



Common Dists

Uniform

Exponential

Normal

In other words, the distribution is centered around μ and has an inflection point at $\mu \pm \sigma$.

In this way, the value of μ determines the center of our distribution and the value of σ determines the spread.

Background

Terms and
Use

Common
Dists

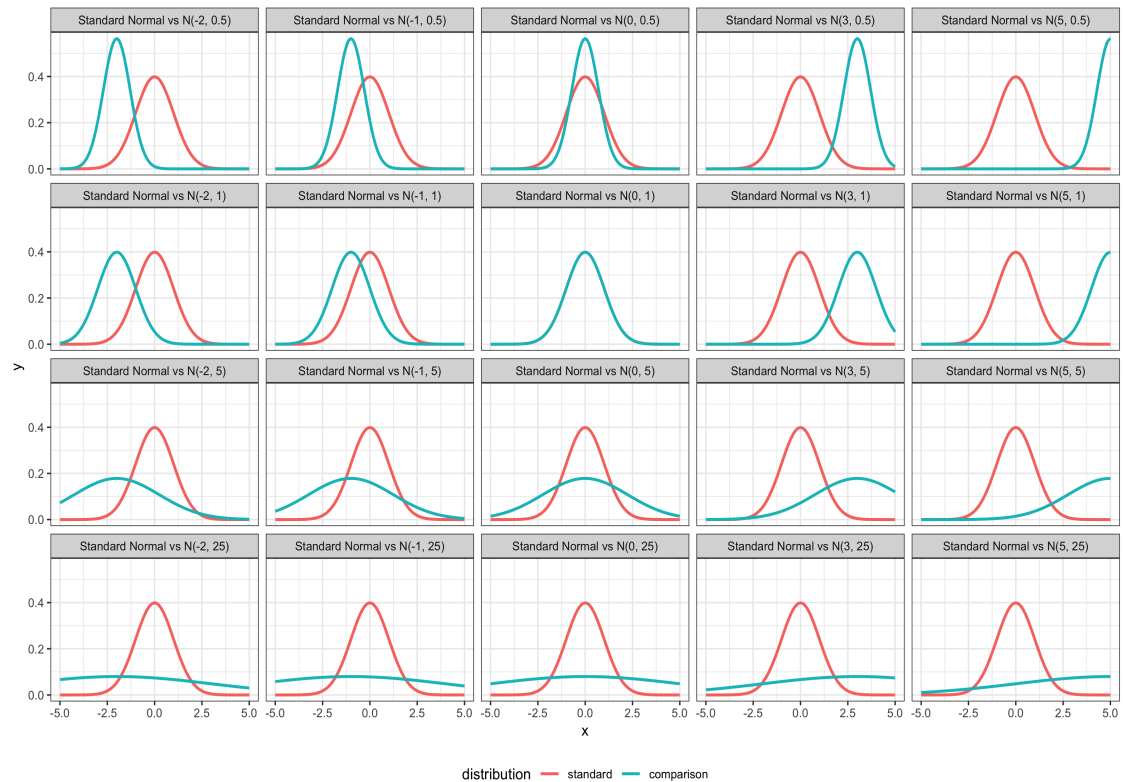
Uniform

Exponential

Normal

Normal Distribution's Center and Shape

Here we can see what differences in μ and σ do to the shape of the distribution



Mean and Variance of Normal Distribution

Background The Normal distribution

Terms and It is not obvious, but

Use

•

Common
Dists

•

Uniform

Exponential

•

Normal

One poine before we go on

Standardization

Background

Definition

Terms and Use

Standardization is the process of transforming a random variable, X , into the signed number of standard deviations by which it is above its mean value.

Common Dists

has mean

Uniform

Exponential

has variance (and standard deviation)

Normal

Background

Terms and Use

The Calculus I methods of evaluating integrals via anti-differentiation will fail when it comes to normal densities. They do not have anti-derivatives that are expressible in terms of elementary functions.

Common Dists

This means we cannot find probabilities of a Normally distributed random variable by hand.

So, what is the solution?

Use computers or tables of values.

Uniform

Exponential

Normal

Background

Terms and Use

The use of tables for evaluating normal probabilities depends on the following relationship. If

,

Common Dists

Uniform

Exponential

Normal

where

.

Background

Standard Normal Distribution

Terms and Use

The parameters are important in determining the probability, but because the pdf of a normal random variable is difficult to work with we often use the distribution with $\mu = 0$ and $\sigma = 1$ as a reference point.

Common Dists

Uniform

Exponential

Normal

Std. Normal

Definition: Standard Normal Distribution

The standard normal distribution is a normal distribution with $\mu = 0$ and $\sigma = 1$. It has pdf

$$f(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}$$

We say that a random variable is a "standard normal random variable" if it follows a standard normal distribution or that $Z \sim N(0, 1)$.

Background

Standard Normal Distribution (cont)

Terms and Use

It's worth pointing out the reason why the standard normal distribution is important. There is no "closed form" for the cdf of a normal distribution.

In other words, since we can't finish this step:

Common Dists

$$\frac{\phi(z)}{\sigma}$$

Uniform

we have to estimate the value each time. However, we have already done this for *standard* normal random variables already in **Table B.3**

Exponential

So if $\lambda = 1$ then $P(X \leq x) = 1 - e^{-x}$.

Normal

The good news is that we can connect any normal probabilities to the values we have for the standard normal probabilities.

Std. Normal

Background

Terms and
Use

Common
Dists

Uniform

Exponential

Normal

Std. Normal

Standard Normal Distribution (cont)

These facts drive the connection between different normal random variables:

Key Facts: Converting Normal Distributions

If $Z \sim N(\mu, \sigma^2)$ and $Y \sim N(\mu, \sigma^2)$ then

If $Z \sim N(\mu, \sigma^2)$ and $Y \sim N(\mu, \sigma^2)$ then

We use this connection as a way to avoid working with the normal pdf directly.

Background

Standard Normal Distribution (cont)

Terms and Use

A rule of thumb in dealing with questions about finding probabilities of Normally distributed probabilities of

:

Common Dists

(1) Translate that question to standard Normal distribution. i.e.

(2) Look it up in a table

Uniform

Exponential

Normal

Standard Normal

Background

CDF of Standard Normal Distribution

Terms and Use

The standard Normal distribution $Z \sim N(0,1)$ plays an important role in finding probabilities associated with a Normal random variable. The **CDF** of a standard Normal distribution is

Common Dists

—
=

Uniform

Therefore, we can find probabilities for all normal distributions by tabulating probabilities for only the standard normal distribution. We will use a table of the **standard normal cumulative probability function**.

Exponential

Normal

Std. Normal

Background

Standard Normal Distribution (cont)

Terms and
Use

Example: Normal to Standard Normal

If _____ then:

Common
Dists

Uniform

Exponential

where the value 0.9332 is found from **Table B.3**

Normal

Std. Normal

Background

Standard Normal Distribution (cont)

Terms and
Use

Example: Standard normal probabilities

Common
Dists

Uniform

Exponential

Normal

Std. Normal

Background

Terms and
Use

Common
Dists

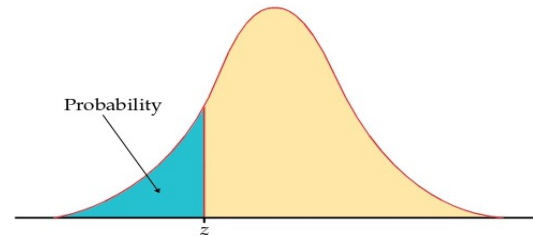
Uniform

Exponential

Normal

Std. Normal

Table entry for z is
the area under the
standard normal curve
to the left of z .



| TABLE A Standard normal probabilities | | | | | | | | | | |
|--|-------|-------|-------|-------|-------|-------|-------|-------|-------|-------|
| z | .00 | .01 | .02 | .03 | .04 | .05 | .06 | .07 | .08 | .09 |
| -3.4 | .0003 | .0003 | .0003 | .0003 | .0003 | .0003 | .0003 | .0003 | .0003 | .0002 |
| -3.3 | .0005 | .0005 | .0005 | .0004 | .0004 | .0004 | .0004 | .0004 | .0004 | .0003 |
| -3.2 | .0007 | .0007 | .0006 | .0006 | .0006 | .0006 | .0006 | .0005 | .0005 | .0005 |
| -3.1 | .0010 | .0009 | .0009 | .0009 | .0008 | .0008 | .0008 | .0008 | .0007 | .0007 |
| -3.0 | .0013 | .0013 | .0013 | .0012 | .0012 | .0011 | .0011 | .0011 | .0010 | .0010 |
| -2.9 | .0019 | .0018 | .0018 | .0017 | .0016 | .0016 | .0015 | .0015 | .0014 | .0014 |
| -2.8 | .0026 | .0025 | .0024 | .0023 | .0023 | .0022 | .0021 | .0021 | .0020 | .0019 |
| -2.7 | .0035 | .0034 | .0033 | .0032 | .0031 | .0030 | .0029 | .0028 | .0027 | .0026 |
| -2.6 | .0047 | .0045 | .0044 | .0043 | .0041 | .0040 | .0039 | .0038 | .0037 | .0036 |
| -2.5 | .0062 | .0060 | .0059 | .0057 | .0055 | .0054 | .0052 | .0051 | .0049 | .0048 |
| -2.4 | .0082 | .0080 | .0078 | .0075 | .0073 | .0071 | .0069 | .0068 | .0066 | .0064 |
| -2.3 | .0107 | .0104 | .0102 | .0099 | .0096 | .0094 | .0091 | .0089 | .0087 | .0084 |
| -2.2 | .0139 | .0136 | .0132 | .0129 | .0125 | .0122 | .0119 | .0116 | .0113 | .0110 |
| -2.1 | .0179 | .0174 | .0170 | .0166 | .0162 | .0158 | .0154 | .0150 | .0146 | .0143 |
| -2.0 | .0228 | .0222 | .0217 | .0212 | .0207 | .0202 | .0197 | .0192 | .0188 | .0183 |
| -1.9 | .0287 | .0281 | .0274 | .0268 | .0262 | .0256 | .0250 | .0244 | .0239 | .0233 |
| -1.8 | .0359 | .0351 | .0344 | .0336 | .0329 | .0322 | .0314 | .0307 | .0301 | .0294 |
| -1.7 | .0446 | .0436 | .0427 | .0418 | .0409 | .0401 | .0392 | .0384 | .0375 | .0367 |
| -1.6 | .0548 | .0537 | .0526 | .0516 | .0505 | .0495 | .0485 | .0475 | .0465 | .0455 |
| -1.5 | .0668 | .0655 | .0643 | .0630 | .0618 | .0606 | .0594 | .0582 | .0571 | .0559 |
| -1.4 | .0808 | .0793 | .0778 | .0764 | .0749 | .0735 | .0721 | .0708 | .0694 | .0681 |
| -1.3 | .0968 | .0951 | .0934 | .0918 | .0901 | .0885 | .0869 | .0853 | .0838 | .0823 |
| -1.2 | .1151 | .1131 | .1112 | .1093 | .1075 | .1056 | .1038 | .1020 | .1003 | .0985 |
| -1.1 | .1357 | .1335 | .1314 | .1292 | .1271 | .1251 | .1230 | .1210 | .1190 | .1170 |
| -1.0 | .1587 | .1562 | .1539 | .1515 | .1492 | .1469 | .1446 | .1423 | .1401 | .1379 |
| -0.9 | .1841 | .1814 | .1788 | .1762 | .1736 | .1711 | .1685 | .1660 | .1635 | .1611 |
| -0.8 | .2119 | .2090 | .2061 | .2033 | .2005 | .1977 | .1949 | .1922 | .1894 | .1867 |
| -0.7 | .2420 | .2389 | .2358 | .2327 | .2296 | .2266 | .2236 | .2206 | .2177 | .2148 |
| -0.6 | .2743 | .2709 | .2676 | .2643 | .2611 | .2578 | .2546 | .2514 | .2483 | .2451 |
| -0.5 | .3085 | .3050 | .3015 | .2981 | .2946 | .2912 | .2877 | .2843 | .2810 | .2776 |
| -0.4 | .3446 | .3409 | .3372 | .3336 | .3300 | .3264 | .3228 | .3192 | .3156 | .3121 |
| -0.3 | .3821 | .3783 | .3745 | .3707 | .3669 | .3632 | .3594 | .3557 | .3520 | .3483 |
| -0.2 | .4207 | .4168 | .4129 | .4090 | .4052 | .4013 | .3974 | .3936 | .3897 | .3859 |
| -0.1 | .4602 | .4562 | .4522 | .4483 | .4443 | .4404 | .4364 | .4325 | .4286 | .4247 |
| 0.0 | .5000 | .4960 | .4920 | .4880 | .4840 | .4801 | .4761 | .4721 | .4681 | .4641 |

Background

Terms and
Use

Common
Dists

Uniform

Exponential

Normal

Std. Normal

Table entry for z is the area under the standard normal curve to the left of z .

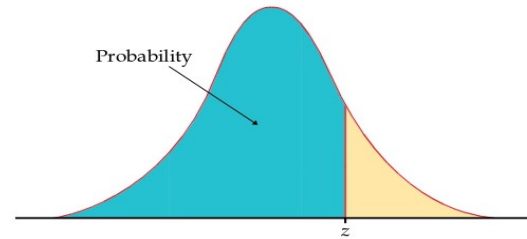


TABLE A

Standard normal probabilities (continued)

| z | .00 | .01 | .02 | .03 | .04 | .05 | .06 | .07 | .08 | .09 |
|-----|-------|-------|-------|-------|-------|-------|-------|-------|-------|-------|
| 0.0 | .5000 | .5040 | .5080 | .5120 | .5160 | .5199 | .5239 | .5279 | .5319 | .5359 |
| 0.1 | .5398 | .5438 | .5478 | .5517 | .5557 | .5596 | .5636 | .5675 | .5714 | .5753 |
| 0.2 | .5793 | .5832 | .5871 | .5910 | .5948 | .5987 | .6026 | .6064 | .6103 | .6141 |
| 0.3 | .6179 | .6217 | .6255 | .6293 | .6331 | .6368 | .6406 | .6443 | .6480 | .6517 |
| 0.4 | .6554 | .6591 | .6628 | .6664 | .6700 | .6736 | .6772 | .6808 | .6844 | .6879 |
| 0.5 | .6915 | .6950 | .6985 | .7019 | .7054 | .7088 | .7123 | .7157 | .7190 | .7224 |
| 0.6 | .7257 | .7291 | .7324 | .7357 | .7389 | .7422 | .7454 | .7486 | .7517 | .7549 |
| 0.7 | .7580 | .7611 | .7642 | .7673 | .7704 | .7734 | .7764 | .7794 | .7823 | .7852 |
| 0.8 | .7881 | .7910 | .7939 | .7967 | .7995 | .8023 | .8051 | .8078 | .8106 | .8133 |
| 0.9 | .8159 | .8186 | .8212 | .8238 | .8264 | .8289 | .8315 | .8340 | .8365 | .8389 |
| 1.0 | .8413 | .8438 | .8461 | .8485 | .8508 | .8531 | .8554 | .8577 | .8599 | .8621 |
| 1.1 | .8643 | .8665 | .8686 | .8708 | .8729 | .8749 | .8770 | .8790 | .8810 | .8830 |
| 1.2 | .8849 | .8869 | .8888 | .8907 | .8925 | .8944 | .8962 | .8980 | .8997 | .9015 |
| 1.3 | .9032 | .9049 | .9066 | .9082 | .9099 | .9115 | .9131 | .9147 | .9162 | .9177 |
| 1.4 | .9192 | .9207 | .9222 | .9236 | .9251 | .9265 | .9279 | .9292 | .9306 | .9319 |
| 1.5 | .9332 | .9345 | .9357 | .9370 | .9382 | .9394 | .9406 | .9418 | .9429 | .9441 |
| 1.6 | .9452 | .9463 | .9474 | .9484 | .9495 | .9505 | .9515 | .9525 | .9535 | .9545 |
| 1.7 | .9554 | .9564 | .9573 | .9582 | .9591 | .9599 | .9608 | .9616 | .9625 | .9633 |
| 1.8 | .9641 | .9649 | .9656 | .9664 | .9671 | .9678 | .9686 | .9693 | .9699 | .9706 |
| 1.9 | .9713 | .9719 | .9726 | .9732 | .9738 | .9744 | .9750 | .9756 | .9761 | .9767 |
| 2.0 | .9772 | .9778 | .9783 | .9788 | .9793 | .9798 | .9803 | .9808 | .9812 | .9817 |
| 2.1 | .9821 | .9826 | .9830 | .9834 | .9838 | .9842 | .9846 | .9850 | .9854 | .9857 |
| 2.2 | .9861 | .9864 | .9868 | .9871 | .9875 | .9878 | .9881 | .9884 | .9887 | .9890 |
| 2.3 | .9893 | .9896 | .9898 | .9901 | .9904 | .9906 | .9909 | .9911 | .9913 | .9916 |
| 2.4 | .9918 | .9920 | .9922 | .9925 | .9927 | .9929 | .9931 | .9932 | .9934 | .9936 |
| 2.5 | .9938 | .9940 | .9941 | .9943 | .9945 | .9946 | .9948 | .9949 | .9951 | .9952 |
| 2.6 | .9953 | .9955 | .9956 | .9957 | .9959 | .9960 | .9961 | .9962 | .9963 | .9964 |
| 2.7 | .9965 | .9966 | .9967 | .9968 | .9969 | .9970 | .9971 | .9972 | .9973 | .9974 |
| 2.8 | .9974 | .9975 | .9976 | .9977 | .9977 | .9978 | .9979 | .9979 | .9980 | .9981 |
| 2.9 | .9981 | .9982 | .9982 | .9983 | .9984 | .9984 | .9985 | .9985 | .9986 | .9986 |
| 3.0 | .9987 | .9987 | .9987 | .9988 | .9988 | .9989 | .9989 | .9989 | .9990 | .9990 |
| 3.1 | .9990 | .9991 | .9991 | .9991 | .9992 | .9992 | .9992 | .9992 | .9993 | .9993 |
| 3.2 | .9993 | .9993 | .9994 | .9994 | .9994 | .9994 | .9994 | .9995 | .9995 | .9995 |
| 3.3 | .9995 | .9995 | .9995 | .9996 | .9996 | .9996 | .9996 | .9996 | .9996 | .9997 |
| 3.4 | .9997 | .9997 | .9997 | .9997 | .9997 | .9997 | .9997 | .9997 | .9997 | .9998 |

Background

Some useful tips about standard Normal distribution

Terms and Use

By symmetry of the standard Normal distribution around zero

Common Dists

Uniform

We can also do it reverse, find x such that

Exponential

Normal

Std. Normal

Background

Example: Baby food

Terms and
Use

J. Fisher, in his article Computer Assisted Net Weight Control (**Quality Progress**, June 1983), discusses the filling of food containers with strained plums and tapioca by weight. The mean of the values portrayed is about _____ g, the standard deviation is about _____ g, and data look bell-shaped. Let

Common
Dists

Let _____ . Find the probability that the next jar contains less food by mass than it's supposed to (declared weight = _____ g).

Uniform

Exponential

Normal

Std. Normal

Background

More example

Terms and
Use

Using the standard normal table, calculate the following:

Common
Dists

Uniform

Exponential

Normal

Std. Normal

Background

More example

Terms and
Use

Find x such that

where

Common
Dists

Uniform

Exponential

Normal

Std. Normal