Lecture 09

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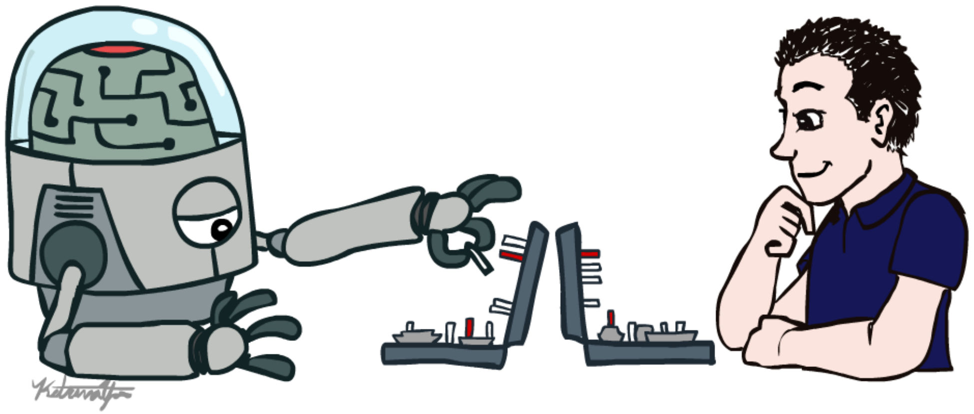
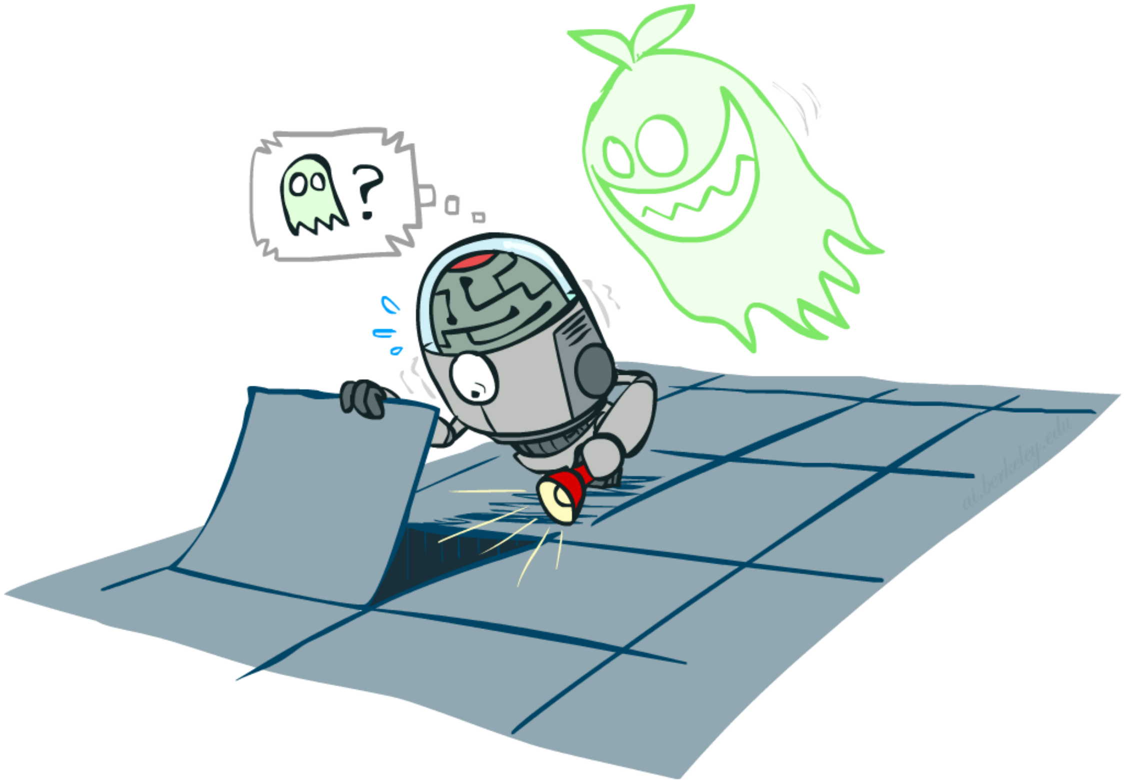
Slides adapted from the official site of “AI: A Modern Approach” book by S. Russell and P. Norvig



Rowan University

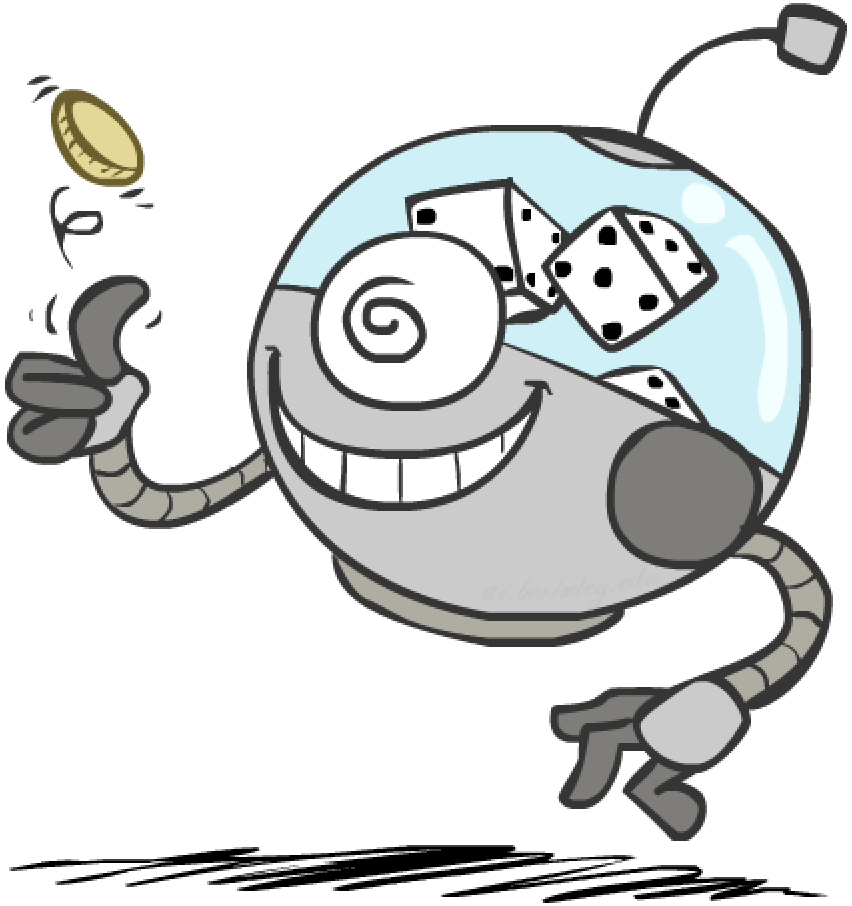
Our Status in AI Course

- **We' re done with Part I (Search and Planning!)**
- **Part II: Probabilistic Reasoning**
  - Diagnosis
  - Spam recognition
  - Tracking objects
  - Robot mapping
  - … lots more!
- **Part III: Machine Learning**



Today

- Probability
  - Random Variables
  - Joint and Marginal Distributions
  - Conditional Distribution
  - Product Rule, Bayes' Rule
- You'll need all this stuff A LOT for this class, so make sure you go over it now!

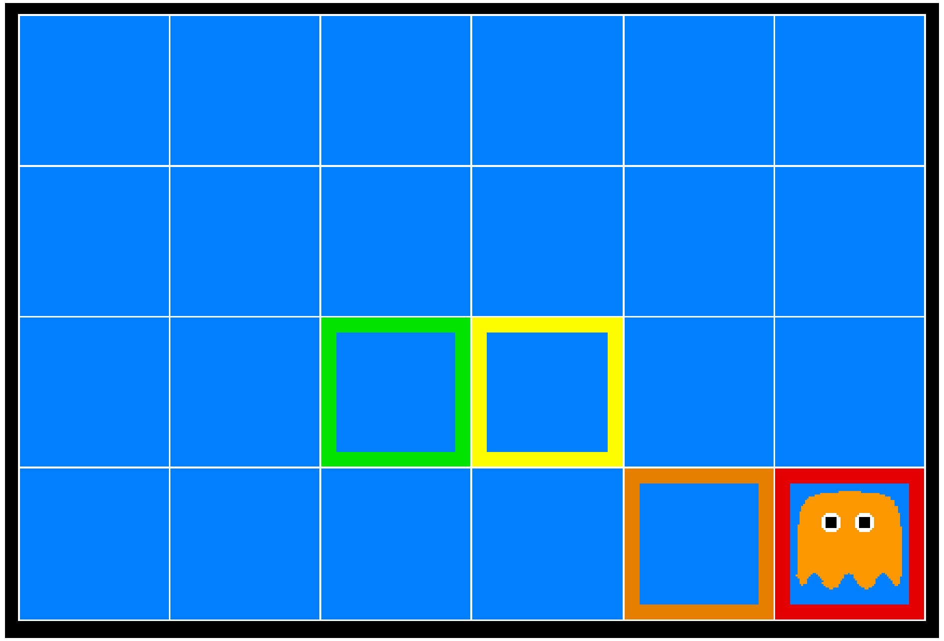


What is probability?

how likely something is to happen

Inference in Ghostbusters

- A ghost is in the grid somewhere
- Sensor readings tell how close a square is to the ghost
  - On the ghost: red
  - 1 or 2 away: orange
  - 3 or 4 away: yellow
  - 5+ away: green



- Sensors are noisy, but we can know the probability of a Colored cell having a distance.

Uncertainty

- General situation:
  - **Observed variables (evidence)**: Agent knows certain things about the state of the world (e.g., sensor readings or symptoms)
  - **Unobserved variables**: Agent needs to reason about other aspects (e.g. where an object is or what disease is present)
  - **Model**: Agent knows something about how the known variables relate to the unknown variables
- Probabilistic reasoning gives us a framework for managing our beliefs and knowledge

0.11	0.11	0.11
0.11	0.11	0.11
0.11	0.11	0.11

$$\frac{1}{9} = 0.11$$

0.17	0.10	0.10
0.09	0.17	0.10
<0.01	0.09	0.17

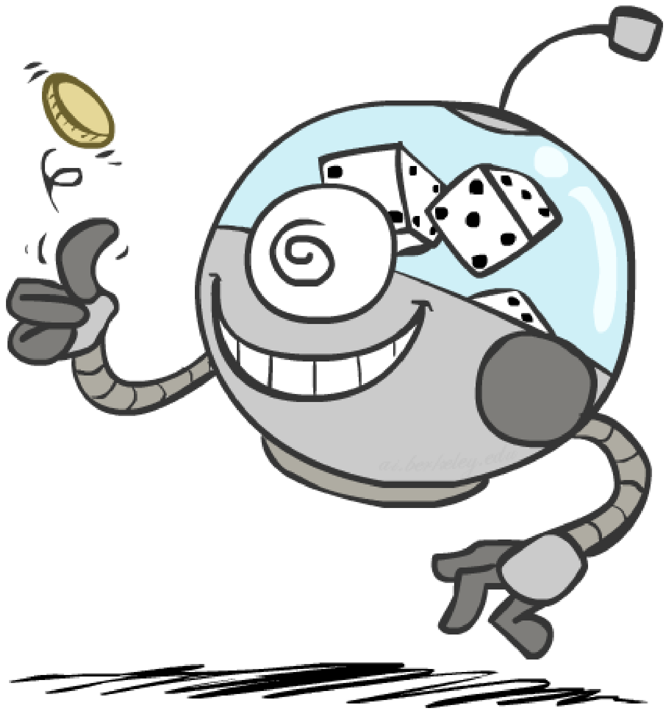
<0.01	<0.01	0.03
<0.01	0.05	0.05
<0.01	0.05	0.81

Random Variables

- A random variable is some aspect of the world about which we (may) have uncertainty
  - R = Is it raining?
  - T = Is it hot or cold?
  - D = How long will it take to drive to work?
  - L = Where is the ghost?

- We denote random variables with capital letters

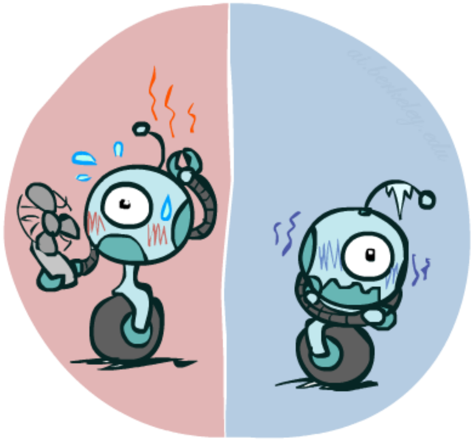
- Random variables have domains
  - R in {true, false}   (often write as {+r, -r})
  - T in {hot, cold}
  - D in [0, ∞)
  - L in possible locations, maybe {(0,0), (0,1), …}



Probability Distributions

- Associate a probability with each value

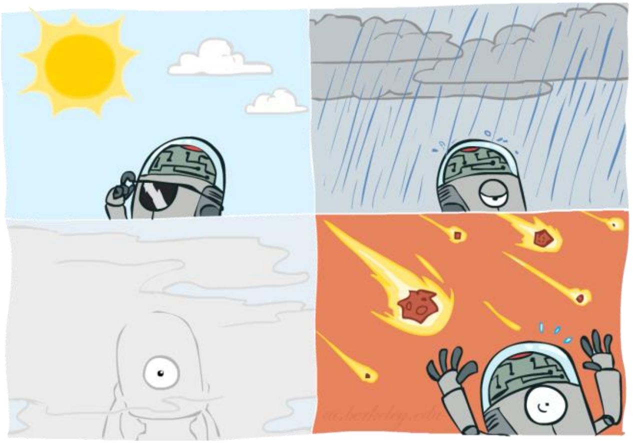
  - Temperature:



$P(T)$

T	P
hot	0.5
cold	0.5

  - Weather:



$P(W)$

W	P
sun	0.6
rain	0.1
fog	0.3
meteor	0.0

Probability Distributions

- Unobserved random variables have distributions

$P(T)$

T	P
hot	0.5
cold	0.5

$P(W)$

W	P
sun	0.6
rain	0.1
fog	0.3
meteor	0.0

Shorthand notation:

$$P(hot) = P(T = hot),$$
$$P(cold) = P(T = cold),$$
$$P(rain) = P(W = rain),$$
$$\ldots$$

OK *if* all domain entries are unique

- A distribution is a TABLE of probabilities of values

- A probability (lower case value) is a single number $P(W = rain) = 0.1$

- Must have: $\forall x \;\; P(X = x) \geq 0$ and $\sum_x P(X = x) = 1$

Joint Distributions

- A *joint distribution* over a set of random variables: $X_1, X_2, \ldots X_n$ specifies a real number for each assignment (or *outcome*):

$$P(X_1 = x_1, X_2 = x_2, \ldots X_n = x_n)$$

$$P(x_1, x_2, \ldots x_n)$$

  - Must obey:

$$P(x_1, x_2, \ldots x_n) \geq 0$$

$$\sum_{(x_1, x_2, \ldots x_n)} P(x_1, x_2, \ldots x_n) = 1$$

$$P(T, W)$$

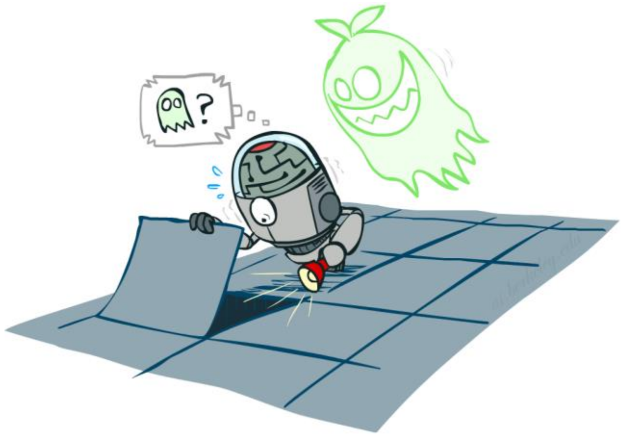
T	W	P
hot	sun	0.4
hot	rain	0.1
cold	sun	0.2
cold	rain	0.3

Probabilistic Models

- A probabilistic model is a joint distribution over a set of random variables
- Probabilistic models:
  - (Random) variables with domains
  - Assignments are called *outcomes*
  - Joint distributions: say whether assignments (outcomes) are likely
  - *Normalized:* sum to 1.0
  - Ideally: only certain variables directly interact
- Constraint satisfaction problems:
  - Variables with domains
  - Constraints: state whether assignments are possible
  - Ideally: only certain variables directly interact

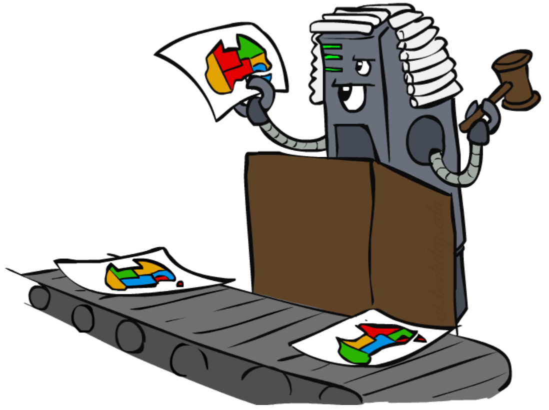
Distribution over T,W

T	W	P
hot	sun	0.4
hot	rain	0.1
cold	sun	0.2
cold	rain	0.3



Constraint over T,W

T	W	P
hot	sun	T
hot	rain	F
cold	sun	F
cold	rain	T



Events

- An *event* is a set E of outcomes

$$P(E) = \sum_{(x_1 \ldots x_n) \in E} P(x_1 \ldots x_n)$$

- From a joint distribution, we can calculate the probability of any event
  - Probability that it's hot AND sunny?
  - Probability that it's hot?
  - Probability that it's hot OR sunny?

- Typically, the events we care about are *partial assignments*, like P(T=hot)

$P(T, W)$

T	W	P
hot	sun	0.4
hot	rain	0.1
cold	sun	0.2
cold	rain	0.3

Quiz: Events

- P(+x, +y) ?

- P(+x) ?

- P(-y OR +x) ?

$$P(X, Y)$$

X	Y	P
+x	+y	0.2
+x	-y	0.3
-x	+y	0.4
-x	-y	0.1

Quiz: Events

- P(+x, +y) ?

  .2

- P(+x) ?

  .2+.3=.5

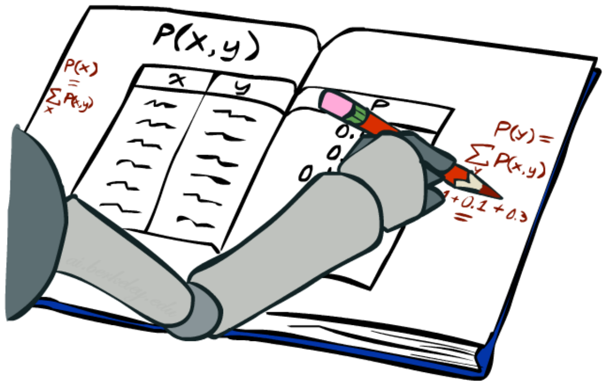
- P(-y OR +x) ?

  .1+.3+.2=.6

$$P(X, Y)$$

X	Y	P
+x	+y	0.2
+x	-y	0.3
-x	+y	0.4
-x	-y	0.1

Marginal Distributions



- Marginal distributions are sub-tables which eliminate variables
- Marginalization (summing out): Combine collapsed rows by adding

$P(T, W)$

T	W	P
hot	sun	0.4
hot	rain	0.1
cold	sun	0.2
cold	rain	0.3

$$P(t) = \sum_s P(t, s)$$

$P(T)$

T	P
hot	0.5
cold	0.5

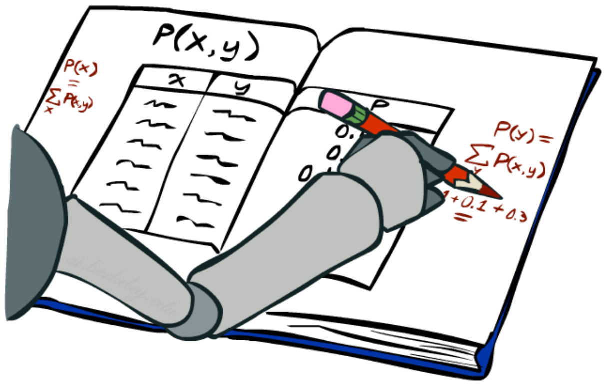
$$P(s) = \sum_t P(t, s)$$

$P(W)$

W	P
sun	0.6
rain	0.4

$$P(X_1 = x_1) = \sum_{x_2} P(X_1 = x_1, X_2 = x_2)$$

Quiz: Marginal Distributions



$P(X,Y)$

X	Y	P
+x	+y	0.2
+x	-y	0.3
-x	+y	0.4
-x	-y	0.1

$$P(x) = \sum_y P(x,y)$$

$$P(y) = \sum_x P(x,y)$$

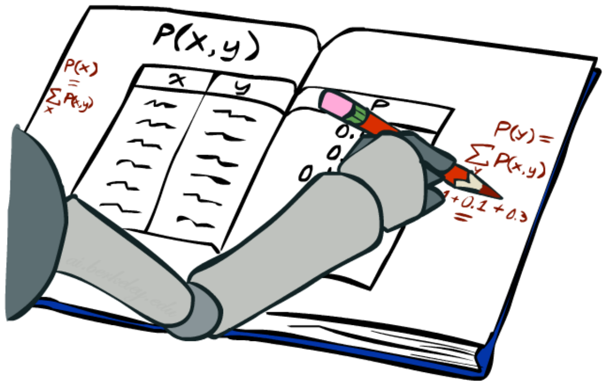
$P(X)$

X	P
+x	
-x	

$P(Y)$

Y	P
+y	
-y	

Quiz: Marginal Distributions



$P(X, Y)$

X	Y	P
+x	+y	0.2
+x	-y	0.3
-x	+y	0.4
-x	-y	0.1

$$P(x) = \sum_y P(x, y)$$

$$P(y) = \sum_x P(x, y)$$

$P(X)$

X	P
+x	.5
-x	.5

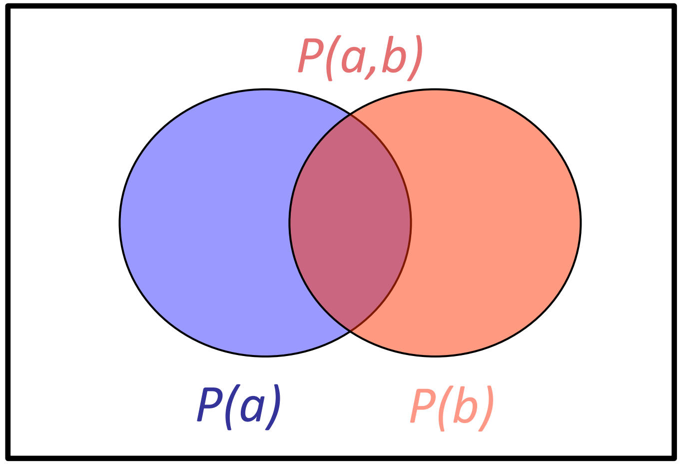
$P(Y)$

Y	P
+y	.6
-y	.4

Conditional Probabilities

- A simple relation between joint and conditional probabilities
  - In fact, this is taken as the *definition* of a conditional probability

$$P(a|b) = \frac{P(a,b)}{P(b)}$$



$$P(T,W)$$

T	W	P
hot	sun	0.4
hot	rain	0.1
cold	sun	0.2
cold	rain	0.3

$$P(W=s|T=c) = \frac{P(W=s,T=c)}{P(T=c)} = \frac{0.2}{0.5} = 0.4$$

$$= P(W=s,T=c) + P(W=r,T=c)$$
$$= 0.2 + 0.3 \quad = 0.5$$

Quiz: Conditional Probabilities

$P(X, Y)$

X	Y	P
+x	+y	0.2
+x	-y	0.3
-x	+y	0.4
-x	-y	0.1

- P(+x | +y) ?
- P(-x | +y) ?
- P(-y | +x) ?

Quiz: Conditional Probabilities

$P(X, Y)$

X	Y	P
+x	+y	0.2
+x	-y	0.3
-x	+y	0.4
-x	-y	0.1

- P(+x | +y) ?

  .2/.6=1/3

- P(-x | +y) ?

  .4/.6=2/3

- P(-y | +x) ?

  .3/.5=.6

Probabilistic Inference

- Probabilistic inference: compute a desired probability from other known probabilities (e.g. conditional from joint)

- We generally compute conditional probabilities
  - P(on time | no reported accidents) = 0.90
  - These represent the agent's *beliefs* given the evider

- Probabilities change with new evidence:
  - P(on time | no accidents, 5 a.m.) = 0.95
  - P(on time | no accidents, 5 a.m., raining) = 0.80
  - Observing new evidence causes *beliefs to be updated*

Inference by Enumeration

- P(W)?

S	T	W	P
summer	hot	sun	0.30
summer	hot	rain	0.05
summer	cold	sun	0.10
summer	cold	rain	0.05
winter	hot	sun	0.10
winter	hot	rain	0.05
winter	cold	sun	0.15
winter	cold	rain	0.20

Inference by Enumeration

- P(W)?

P(sun)=.3+.1+.1+.15=.65

S	T	W	P
summer	hot	sun	0.30
summer	hot	rain	0.05
summer	cold	sun	0.10
summer	cold	rain	0.05
winter	hot	sun	0.10
winter	hot	rain	0.05
winter	cold	sun	0.15
winter	cold	rain	0.20

Inference by Enumeration

- P(W)?

P(sun)=.3+.1+.1+.15=.65
P(rain)=1-.65=.35

S	T	W	P
summer	hot	sun	0.30
summer	hot	rain	0.05
summer	cold	sun	0.10
summer	cold	rain	0.05
winter	hot	sun	0.10
winter	hot	rain	0.05
winter	cold	sun	0.15
winter	cold	rain	0.20

Inference by Enumeration

- P(W | winter, hot)?

S	T	W	P
summer	hot	sun	0.30
summer	hot	rain	0.05
summer	cold	sun	0.10
summer	cold	rain	0.05
winter	hot	sun	0.10
winter	hot	rain	0.05
winter	cold	sun	0.15
winter	cold	rain	0.20

Inference by Enumeration

- P(W | winter, hot)?

S	T	W	P
summer	hot	sun	0.30
summer	hot	rain	0.05
summer	cold	sun	0.10
summer	cold	rain	0.05
winter	hot	sun	0.10
winter	hot	rain	0.05
winter	cold	sun	0.15
winter	cold	rain	0.20

Inference by Enumeration

- P(W | winter, hot)?

P(sun|winter,hot)~.1
P(rain|winter,hot)~.05

S	T	W	P
summer	hot	sun	0.30
summer	hot	rain	0.05
summer	cold	sun	0.10
summer	cold	rain	0.05
winter	hot	sun	0.10
winter	hot	rain	0.05
winter	cold	sun	0.15
winter	cold	rain	0.20

Inference by Enumeration

- P(W | winter)?

S	T	W	P
summer	hot	sun	0.30
summer	hot	rain	0.05
summer	cold	sun	0.10
summer	cold	rain	0.05
winter	hot	sun	0.10
winter	hot	rain	0.05
winter	cold	sun	0.15
winter	cold	rain	0.20

Inference by Enumeration

- P(W | winter)?

P(sun|winter)~.1+.15=.25

S	T	W	P
summer	hot	sun	0.30
summer	hot	rain	0.05
summer	cold	sun	0.10
summer	cold	rain	0.05
winter	hot	sun	0.10
winter	hot	rain	0.05
winter	cold	sun	0.15
winter	cold	rain	0.20

Inference by Enumeration

- $P(W \mid \text{winter})$?

P(rain|winter)~.05+.2=.25

S	T	W	P
summer	hot	sun	0.30
summer	hot	rain	0.05
summer	cold	sun	0.10
summer	cold	rain	0.05
winter	hot	sun	0.10
winter	hot	rain	0.05
winter	cold	sun	0.15
winter	cold	rain	0.20

Inference by Enumeration

- $P(W \mid winter)$?

P(sun|winter)~.25
P(rain|winter)~.25
P(W |winter)=.5

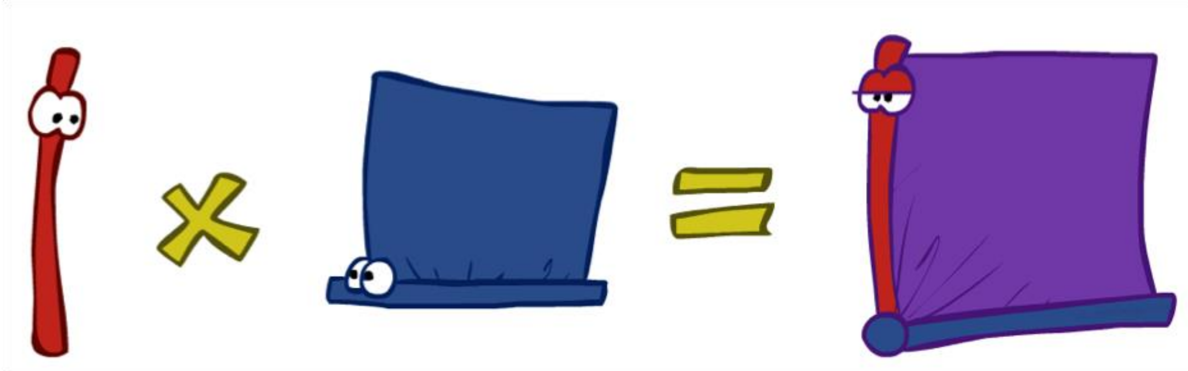
S	T	W	P
summer	hot	sun	0.30
summer	hot	rain	0.05
summer	cold	sun	0.10
summer	cold	rain	0.05
winter	hot	sun	0.10
winter	hot	rain	0.05
winter	cold	sun	0.15
winter	cold	rain	0.20

The Product Rule

- Sometimes have conditional distributions but want the joint

$$P(y)P(x|y) = P(x, y)$$

$$P(x|y) = \frac{P(x,y)}{P(y)}$$



The Product Rule

$$P(y)P(x|y) = P(x,y)$$

- Example:

$P(W)$

R	P
sun	0.8
rain	0.2

$P(D|W)$

D	W	P
wet	sun	0.1
dry	sun	0.9
wet	rain	0.7
dry	rain	0.3

$P(D,W)$

D	W	P
wet	sun	
dry	sun	
wet	rain	
dry	rain	

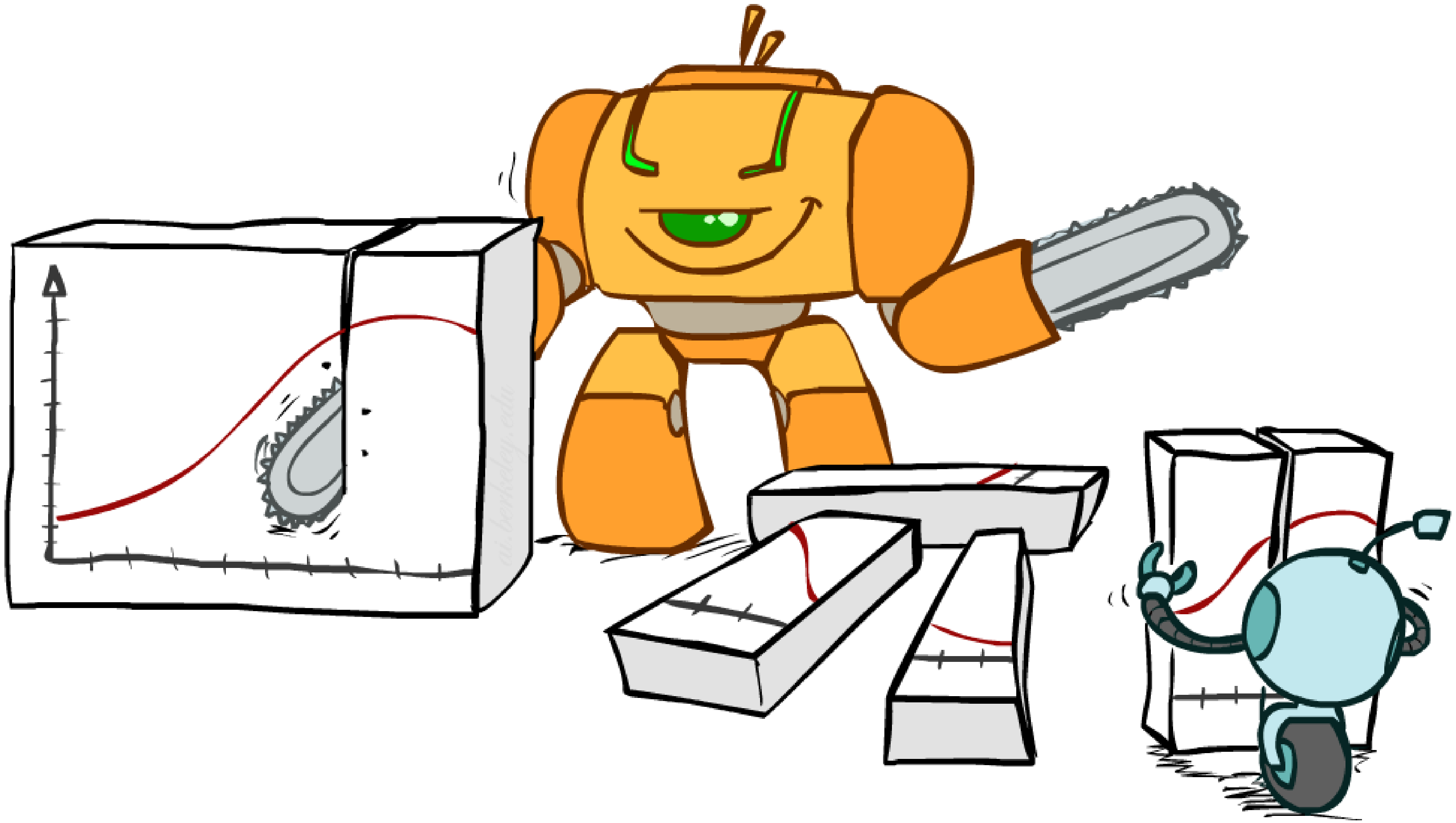
The Chain Rule

- More generally, can always write any joint distribution as an incremental product of conditional distributions

$$P(x_1, x_2, x_3) = P(x_1)P(x_2|x_1)P(x_3|x_1, x_2)$$

$$P(x_1, x_2, \ldots x_n) = \prod_i P(x_i|x_1 \ldots x_{i-1})$$

Bayes Rule



Bayes' Rule

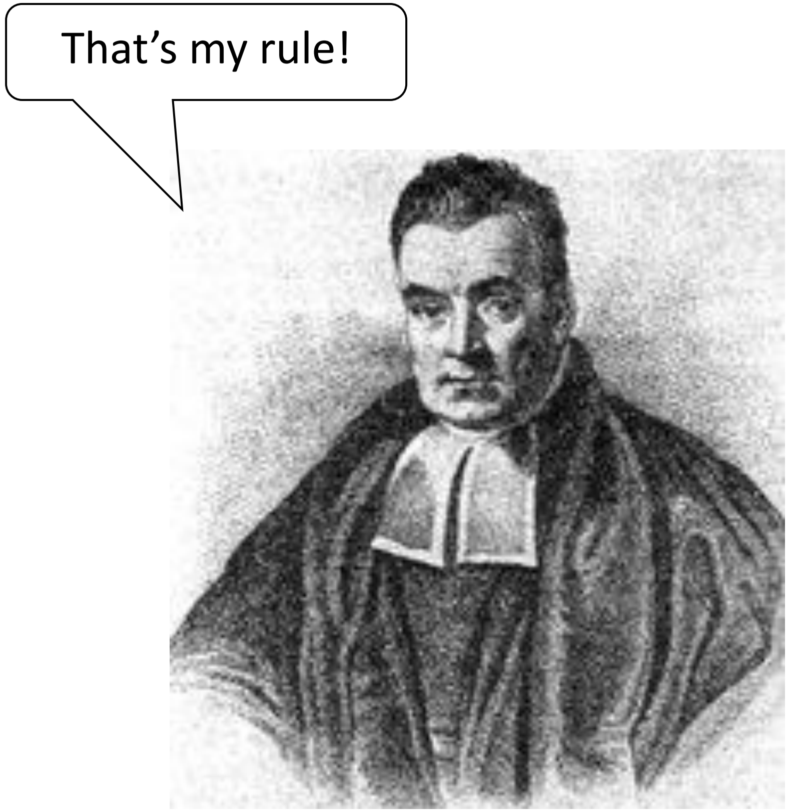
- Two ways to factor a joint distribution over two variables:

$$P(x,y) = P(x|y)P(y) = P(y|x)P(x)$$

- Dividing, we get:

$$P(x|y) = \frac{P(y|x)}{P(y)}P(x)$$

- Why is this at all helpful?
  - Let's us build one conditional from its reverse
  - Often one conditional is tricky but the other one is simple
  - Foundation of many systems we'll see later



Inference with Bayes' Rule

- Example: Diagnostic probability from causal probability:

$$P(\text{cause}|\text{effect}) = \frac{P(\text{effect}|\text{cause})P(\text{cause})}{P(\text{effect})}$$

- Example:
  - M: meningitis, S: stiff neck

$$\left.\begin{aligned} P(+m) &= 0.0001 \\ P(+s|+m) &= 0.8 \\ P(+s|-m) &= 0.01 \end{aligned}\right\} \text{Example givens}$$

$$P(+m|+s) = \frac{P(+s|+m)P(+m)}{P(+s)} = \frac{P(+s|+m)P(+m)}{P(+s|+m)P(+m) + P(+s|-m)P(-m)} = \frac{0.8 \times 0.0001}{0.8 \times 0.0001 + 0.01 \times 0.999}$$

  - Note: posterior probability of meningitis still very small
  - Note: you should still get stiff necks checked out! Why?

Quiz: Bayes' Rule

- Given:

$P(W)$

R	P
sun	0.8
rain	0.2

$P(D|W)$

D	W	P
wet	sun	0.1
dry	sun	0.9
wet	rain	0.7
dry	rain	0.3

- If the day is dry, do we have chances of rain?
- What is P(W | dry) ?

Quiz: Bayes' Rule

- Given:

$P(W)$

R	P
sun	0.8
rain	0.2

$P(D|W)$

D	W	P
wet	sun	0.1
dry	sun	0.9
wet	rain	0.7
dry	rain	0.3

- What is P(W | dry) ?

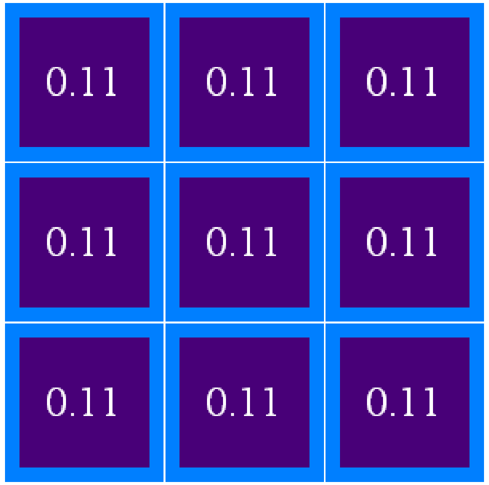
P(sun|dry) ~ P(dry|sun)P(sun) = .9*.8 = .72
P(rain|dry) ~ P(dry|rain)P(rain) = .3*.2 = .06

P(sun|dry) > P(rain|dry)   No rain

Look, we don't need P(dry), since it is same on both cases.

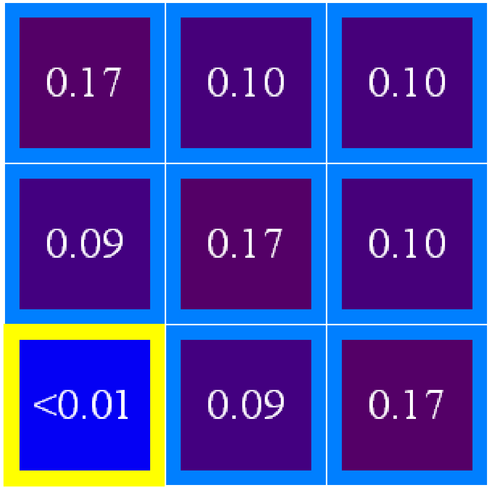
Ghostbusters, Revisited

- Let's say we have two distributions:
  - Prior distribution over ghost location: P(G)
    - Let's say this is uniform
  - Sensor reading model: P(R | G)
    - Given: we know what our sensors do
    - R = reading color measured at (1,1)
    - E.g. P(R = yellow | G=(1,1)) = 0.1

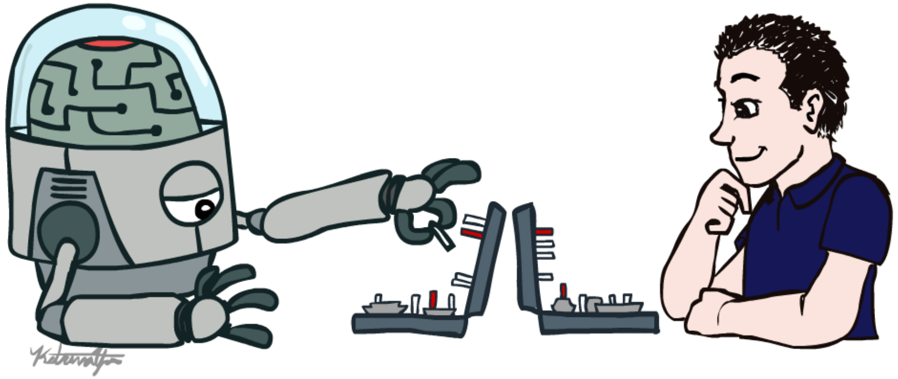


- We can calculate the posterior distribution P(G|r) over ghost locations given a reading using Bayes' rule:

$$P(g|r) \propto P(r|g)P(g)$$



[Demo: Ghostbuster – with probability (L12D2)]



Thanks!