Notes on Machine Learning (work in progress)

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Contents

1 Overview

I made this document as a way to learn ML for my Maters' thesis . Its not an original work, but a compilation of scribed notes of the ML course by Dr Sanjoy Dasgupta(UCSD), [SD] which I audited. Some parts are drawn directly/inspired from Andrew NG's Stanford CS229 course notes, [ANG] and Kilian weinberger's Cornell CS4780 course notes[KW]. My primary reference was 'The Elements of Statistical Learning' by Trevor Hastie, Robert Tibshirani, Jerome Friedman[HTF]. So there will be considerable overlap with the aforementioned materials. This note might lack mathematical rigor but I have tried to give proofs and supplementary topics wherever necessary. For each algorithm, I have given a url to Github repo containing the implemetation using one of the three datasets viz Fisher's Iris dataset, Wisconsin Breast Cancer Dataset and MNIST handwritten digits dataset. Please write to me if you find any inaccuracies.I hope this proves at least moderately interesting or useful for you.

2 Intro to Neural Networks [1]

The linear models mentioned earlier have serious shortcomings as in inability to learn interaction between any two input variables. To extend linear models to represent nonlinear functions of x, we can apply the linear model not to x itself but to a transformed input $\phi(x)$, where ϕ is a nonlinear transformation. In the linear models, say for example logistic regression the model learns the parameters (weights and bias) but neural network learns the representation ϕ in addition to the parameters.

The strategy of deep learning is to learn ϕ. In this approach, we have a model $y = f(x;\theta,w) = \phi(x;\theta)^T w$. We now have parameters θ that we use to learn ϕ from a broad class of functions, and parameters w that map from $\phi(x)$ to the desired output. This is an example of a deep feedforward network, with ϕ defining a hidden layer.We parametrize the representation as $\phi(x;\theta)$ and use the optimization algorithm to find the θ that corresponds to a good representation. In general we can write

$$h : \phi(x) \rightarrow \phi(x)^T w + b$$

where $\phi(x) = \sigma(U\vec{x})$, σ is a transition function. But we can argue that the transformation $x \rightarrow \phi(x)$ can be attained by a cascade of arbitrary number of transformation say, $x \rightarrow \phi^{''}(x) \rightarrow \phi^{'}(x) \rightarrow \phi(x)$

$$\phi(x) = \sigma(U^{'}\phi^{'}(x))$$
$$\phi^{'}(x) = \sigma(U^{''}\phi^{''}(x))$$
$$\phi^{''}(x) = \sigma(Ux)$$

[1] A compilation of scribed notes of 11-785 Intro to DL, CMU and Andrew NG's Coursera course along with inputs from the book 'Deep Learning' by Goodfellow, Bengio and Courville

3 Perceptron Recap

In a binary classification problem , where dataset(D) is $(x, y) \in \mathbb{R}^d \times \{-1, 1\}$, the learning problem is to find a hyperplane which separates the data into two classes, assuming the data is linearly classifiable. The hyperplane is parametrized by $w \in \mathbb{R}^d$ and $b \in \mathbb{R}$ such that $w.x + b = 0$, so the poblem is equivalent to learning the parameters w and b.On point x, we predict the label as **sign(w.x + b)**.

$$(w.x + b) > 0 \implies y = +1 \therefore y(w.x + b) > 0$$

$$(w.x + b) < 0 \implies y = -1 \therefore y(w.x + b) > 0$$

i.e., If the true label of x is y, then $y(w.x + b) > 0$, whereas for a misclassified pointi$y(w.x + b) \leq 0$.

Perceptron can be loosely thought of as a threshold gate very much like a biological neuron as in when the weighted sum of the inputs (voltage analogue) exceeds the threshold T, the perceptron's output is identically one (neuron fires) else zero always(rest state neuron). This is shown in Fig.1.

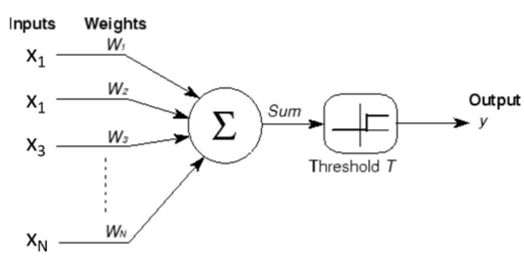


Figure 1

The output of such a perceptron will be a step function, such that

$$y = \begin{cases} 1, & \text{if } \sum_i w_i x_i \geq T \\ 0, & else \end{cases} \tag{1}$$

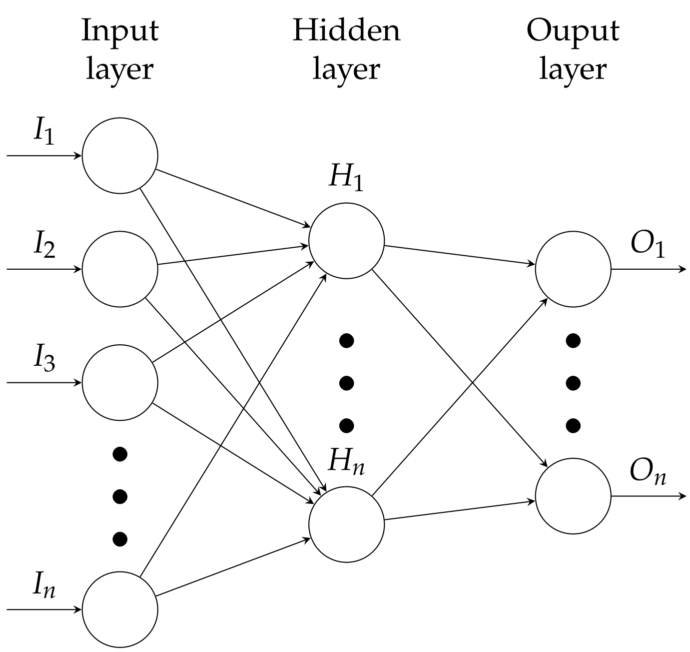
If we define $z = \sum_i w_i x_i - T$, then it is equivalent to,

$$y = \begin{cases} 1, & \text{if } z \geq 0 \\ 0, & else \end{cases} \tag{2}$$

This threshold gate can realize AND and OR GATE, but not XOR gate.

3.1 Multi-layer Perceptron

MLP is a network of perceptrons.Generically a MLP can be said to have three different types of layers, the input layer, output layer and the hidden layer.The input layer doesnt contain perceptrons. The nodes in the input layer represent each components of the input data vector.The output layer gives the output. Flanked by these two are the hidden layer. The inpput and output of a MLP can be real valued or Boolean.



Why MLPs? Why are these better than a single perceptron? What kind of computation they can undertake? What are there limitations?

Naively speaking, the answer to these questions lies in the fact that MLPs can compose Boolean functions and real valued functions and they are universal function approximators.

3.2 Perceptron as Boolean gate

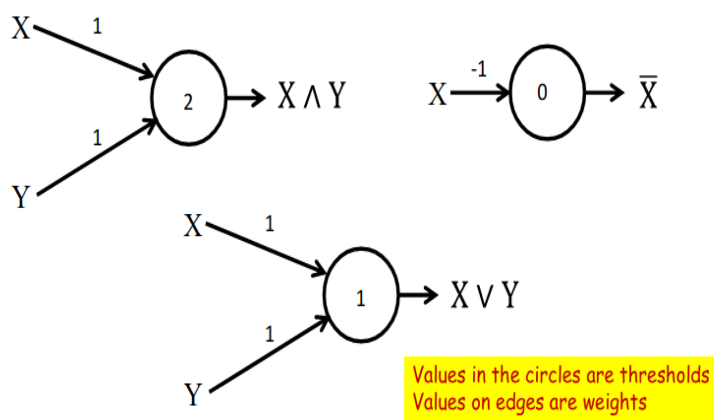


Figure 2

Figure 2 shows perceptron can compute AND, OR and NOT gates.Consider a perceptron for XOR gate with threshold t. XOR gate is a digital logic gate that gives a true (1 or HIGH) output when the number of true inputs is odd. Let the weights of input be w_1 and w_2 and inputs be X and Y. The input pairs $(1,0)$, $(0,1)$, $(1,1)$, $(0,0)$ along with above threshold and weights have to hold following inequalities for the perceptron to act as a XOR gate.

$$1w_1 + 0w_2 \geq t \implies w_1 \geq t$$
$$0w_1 + 1w_2 \geq t \implies w_2 \geq t$$
$$0w_1 + 0w_2 < t \implies 0 < t$$
$$1w_1 + 1w_2 < t \implies w_1 + w_2 < t$$

But these equations are in contradiction to each other. So a perceptron cannot compute an XOR gate.

A single perceptron(here, threshold gate) can be a Universal AND , OR and a majority gate.

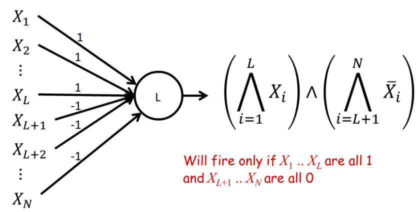


Figure 3: Universal AND gate - AND any arbitrary number of inputs, of which any subset is negated

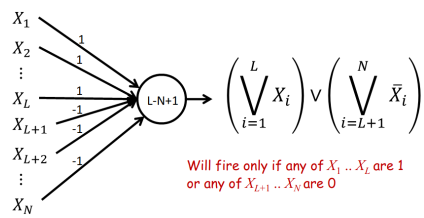


Figure 4: Universal OR gate - OR any arbitrary number of inputs, of which any subset is negated

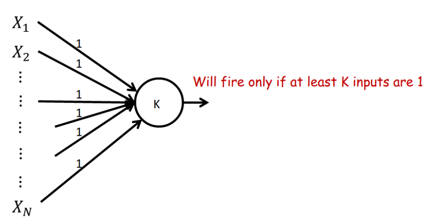


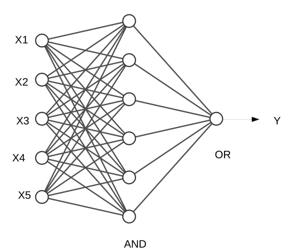
Figure 5: Generalized Majority gate - For implementing the same with standard boolean circuit would require exponentially (in input) sized ci-cuit, but can be done with single threshold gate

For any arbitrary Boolean function, how many layers of MLP is needed to compute it ? This can be explained using Disjunctive Normal Forms(DNF) which is a canonical normal form of a logical formula consisting of disjunction of conjunctions i.e OR of ANDs(sum of products)

Consider this Boolean function given as a truth table.This boolean function can be expressed as a DNF formula.This boolean function can be computed using a perceptron with a threshold activation.

X_1	X_2	X_3	X_4	X_5	Y
0	0	1	1	0	1
0	1	0	1	1	1
0	1	1	0	0	1
1	0	0	0	1	1
1	0	1	1	1	1
1	1	0	0	1	1

$$\begin{aligned} Y = \overline{X_1}\ \overline{X_2}X_3X_4X_5 + \overline{X_1}X_2\overline{X_3}X_4X_5 \\ + \overline{X_1}X_2X_3\overline{X_4}\ \overline{X_5} + X_1\overline{X_2}\ \overline{X_3}\ \overline{X_4}X_5 \\ + X_1\overline{X_2}X_3X_4X_5 + X_1X_2\overline{X_3}\ \overline{X_4}X_5 \end{aligned} \tag{3}$$



To be continued...

4 Learning

Deep feedforward networks, also often called feedforward neural networks, or multilayer perceptrons (MLPs), are the quintessential deep learning models. The goal of a feedforward network is to approximate some function **g** . For example, for a classifier, $y = g(x)$ maps an input x to a category y. A feedforward network defines a mapping $y = f(x; W)$ and learns the value of the parameters W that result in the best function approximation $f \approx g$.

So the neural network(NN) is a function approximator with parameters W, which must be set to appropriate values to get desired behaviour from the NN.For a given Neural Network (NN), the parameters are the weights and biases of the individual neuron. The learning process is finding the vales of the parameters such that the network computes the desired function.

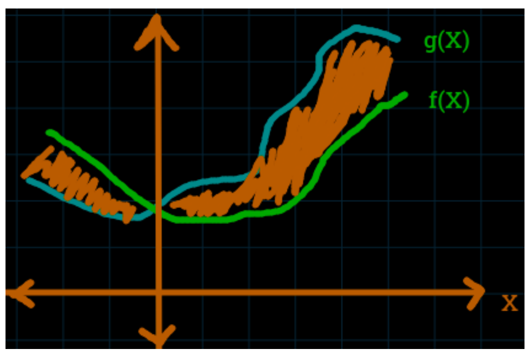


Figure 6: Here, g(x) is the function we want the NN to compute and f(x) is the funnction that we are computing

Error is the difference between $g(X)$ and $f(X)$ and the error waries with parameters. So the question of learnjng has reducredm to finding the parameters such that the error is less.When $f(x; W)$ has the capacity to exacly represent $g(x)$, we need to find $\hat{W}$, such that

$$\hat{W} = argmin_W \int_x div(f(X; W), g(X))dX$$

Where the $div(f(X; W), g(X)) = 0$ when $f(X; W) = g(X)$. For finding the integral, $g(X)$ has to be fully specified inn the domain, but thats never the case.So we sample from the domain.

- Get input-output pairs for a number of samples of X_i.
- Samples (x_i, d_i) where $d_i = g(x_i) + noise$
- Optimum sampling when the samples of X will be drawn using the distribution $P(X)$.
- eg: set of their images and their labels, speech recording and transcribes

Although $g(X)$ exists, we dont know the value of $g(X)$ throughout the domain of g, We only have value for the sampled points. What we do is, we learn the parameters of the NN such that it can fit the input-output pair we sampled i.e is the NN can correctly compute the ys for the xs we sampled and we hope the learned function $f(X; W)$ is same as the the original function $g(X)$ in the entire domain.

We impose a requirement - regions of the input space which has higher probability must have lower error than the regions of lower probabiltity.So we weigh the value of X with their probability distribution.

$$\hat{W} = argmin_W \int_x div(f(X; W), g(X))P(X)dX$$
$$\hat{W} = argmin_W \mathbb{E}\ [div(f(X; W), g(X))]$$

The expected error(risk) is defined as the averaged error over the entire input space i.e,

$$\mathbb{E}\ [div(f(X; W), g(X))] = \int_x div(f(X; W), g(X))P(X)dX$$

But its impossible to calculate the risk as $g(X)$ is unknown and what we have is just samples in the input space and their corresponding output. The emperical estimate of the expected error is the average error over all the samples, so,

$$\mathbb{E}\ [div(f(X; W), g(X))] = \frac{1}{N} \sum_{i=1}^{N} div(f(X_i; W), d_i = g(X_i))$$

This emperical average is known commonly as Loss,

$$L(W) = \frac{1}{N} \sum_{i=1}^{N} div(f(X_i; W), d_i = g(X_i))$$

So the problem now is to estimate the parameters to minimize the emperical estimate of expected error,

$$\hat{W} = argmin_W Loss(W)$$

, So the problem now is an Optimization problem.