Banach Spaces of Analytic Functions

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§1 Analytic and Harmonic Functions

§1.1 The Cauchy and Poisson Kernels

Proposition §1.1.1. *Let $u : \overline{\mathbb{D}} \to \mathbb{C}$ be a harmonic function. Then we have that*

$$u\left(re^{i\theta}\right) = \frac{1}{2\pi}\int_{-\pi}^{\pi} u\left(e^{it}\right) P_r\left(e^{i(\theta-t)}\right) dt \tag{§1.1.1}$$

§1.2 Boundary Values

§1.2.1 Weak* convergence of measures

Theorem §1.2.1. *Let $\{\varphi_i\}_i$ be an approximate identity on $\mathbb{T}$ and let $\mu \in \mathcal{M}(\mathbb{T})$. Then for all i, $\varphi_i * \mu \in L^1(\mathbb{T})$ with*

$$\|\varphi_i * \mu\|_1 \leq C_\varphi \|\mu\|$$

and

$$\|\mu\| \leq \sup_i \|\varphi_i * \mu\|_1 .$$

Moreover, the measures $d\mu_i = (\varphi_i * \mu)(e^{it})\, dt/2\pi$ converge to $d\mu(e^{it})$ in the weak topology, i.e.*

$$\lim_i \frac{1}{2\pi}\int_{-\pi}^{\pi} f\left(e^{it}\right)(\varphi_i * \mu)\left(e^{it}\right) dt = \int_{\mathbb{T}} \varphi\left(e^{it}\right) d\mu\left(e^{it}\right)$$

for all $f \in \mathcal{C}(\mathbb{T})$.

§1.2.2 Convergence in norm

Theorem §1.2.2. *Let $\{\varphi_i\}_i$ be an approximate identity on $\mathbb{T}$ and let $f \in L^p(\mathbb{T})$ with $p \in [1,\infty)$. Then for all i, $\varphi_i * f \in L^p(\mathbb{T})$ with*

$$\|\varphi_i * f\|_p \leq C_\varphi \|f\|_p$$

and

$$\lim_i \|\varphi_i * f - f\|_p = 0.$$

§1.2.3 Weak* convergence of bounded functions

Theorem §1.2.3. *Let $\{\varphi_i\}_i$ be an approximate identity on $\mathbb{T}$ and let $f \in L^\infty(\mathbb{T})$. Then for all i, $\varphi_i * \mu \in \mathcal{C}(\mathbb{T})$ with*

$$\|\varphi_i * \mu\|_\infty \leq C_\varphi \|\mu\|_\infty$$

and

$$\|f\|_{+\infty} \leq \sup_i \|\varphi_i * f\|_\infty .$$

Moreover, $\varphi_i * f$ converge to f in the weak topology, i.e.*

$$\lim_i \int_{-\pi}^{\pi} g\left(e^{it}\right)\left(\varphi_i * f\right)\left(e^{it}\right) dt = \int_{\mathbb{T}} g\left(e^{it}\right) f\left(e^{it}\right) dt$$

for all $g \in L^1(\mathbb{T})$.

§1.2.4 The entire picture!

Definition §1.2.4 (Poisson integral of some function or measure)**.** Let $\tilde{f} : \mathbb{D} \to \mathbb{C}$ be a harmonic function. Then $\tilde{f}$ is said to be the *Poisson integral* of the function $f : \mathbb{T} \to \mathbb{C}$ if

$$\tilde{f}(re^{i\theta}) = \frac{1}{2\pi} \int_T f\left(e^{it}\right) P_r\left(e^{i(\theta - t)}\right) dt$$

In such a case, we will denote the function $\tilde{f}$ by $P[f]$. Similarly, f is said to be the *Poisson integral* of a complex measure μ on T if

$$\tilde{f}(re^{i\theta}) = \frac{1}{2\pi} \int_T P_r\left(e^{i(\theta - t)}\right) d\mu\left(e^{it}\right)$$

In such a case, we will denote the function $\tilde{f}$ by $P[\mu]$.

Theorem §1.2.5 (Ultimate Convergence)**.** *Let $f : \mathbb{D} \to \mathbb{C}$ be a harmonic function. Define for each $r \in [0, 1)$, the function $f_r : \mathbb{T} \to \mathbb{C}$ by*

$$f_r\left(e^{i\theta}\right) = f\left(re^{i\theta}\right)$$

The following statements holds:

1. *If $1 < p \leq \infty$ then $f = P[g]$ for some $g \in L^p[g]$ iff for each $r > 0$, $\|f_r\|_p < +\infty$.*
2. *If p=1 then $f = P[g]$ for some $g \in L^p[g]$ iff f_r converge in the L^1 norm.*
3. *$f = P[\mu]$ for some $\mu \in \mathcal{M}(\mathbb{T})$ iff for each $r > 0$, $\|f_r\|_1 < +\infty$*

§1.3 Fatou's Theorem

Theorem §1.3.1. *Let μ be a complex measure on the unit circle $\mathbb{T}$, and let $f : \mathbb{D} \to \mathbb{C}$ be the harmonic function defined by*

$$f(re^{i\theta}) = \frac{1}{2\pi} \int_{\mathbb{T}} P_r\left(e^{i(\theta - t)}\right) d\mu\left(e^{it}\right)$$

Let $e^{i\theta_0}$ be any point where μ is differentiable with respect to the normalised Lebesgue measure. Then

$$\lim_{r \to 1} f\left(re^{i\theta_0}\right) = \left(\frac{d\mu}{d\theta}\right)\left(e^{i\theta_0}\right) = \mu'\left(e^{i\theta_0}\right)$$

In fact, $f(re^{i\theta}) \to \mu'\left(e^{i\theta_0}\right)$ as $re^{i\theta}$ approaches $e^{i\theta_0}$ along any path in the open disc within the region of the form $|\theta - \theta_0| \leq c\,(1 - r)$ for some $c > 0$.

Corollary §1.3.2. *Let μ be a complex measure on $\mathbb{T}$. Then $P[\mu]$ has nontangential limits equal everywhere to the Radon Nikodym derivative of μ with respect to the normalised Lebesgue measure.*

Corollary §1.3.3. *Let $f : \mathbb{T} \to \mathbb{C}$ be L^1. Then $P[f]$ has nontangential limits at almost everywhere and these limits equal to f almost everywhere.*

Corollary §1.3.4. *Let $f : \mathbb{D} \to \mathbb{C}$ be a harmonic function and $1 \leq p < \infty$. Suppose that for all $0 \leq r < 1$, we have that*

$$\|f_r\|_p < +\infty$$

Then for almost every θ the radial limits

$$\tilde{f}(e^{i\theta}) = \lim_{r\to 1} f\left(re^{i\theta}\right)$$

exist and define a function $\tilde{f}$ in $L^p(\mathbb{T})$. The following also holds:

1. *If $p > 1$ then $f = P[\tilde{f}]$.*
2. *If $p = 1$ then $f = P[\mu]$ for some complex measure μ whose absolutely continuous part is $f d\theta$.*
3. *IF f is bounded then the boundary values exist almost everywhere and define a bounded measurable function $\tilde{f}$ on $\mathbb{T}$ such that $f = P[\tilde{f}]$.*

Proof. Suppose that for each $r \in [0, 1)$, we have $\|f_r\|_p < +\infty$. We need to prove that for almost every θ, $\lim_{r\to 1} f\left(re^{i\theta}\right)$ exists. Then by Theorem §1.2.5, we have that $f = P[g]$ for some $g \in L^p(\mathbb{T})$. Since $L^p(\mathbb{T}) \subset L^1(\mathbb{T})$, we can use the previous corollary. By the previous corollary, we have that $P[g]$ has nontangential limits almost everywhere, we have that

$$\tilde{f}\left(e^{i\theta}\right) = \lim_{r\to 1} f(re^{i\theta}) = \lim_{r\to 1} P[g]\left(re^{i\theta}\right) \tag{§1.3.1}$$

exists almost everywhere.

Now we proceed to prove part (1). Also by Theorem §1.2.5, we have that $f = P[g]$ for some $g \in L^p(\mathbb{T})$. Hence, we have that by Equation §1.3.1 that $\tilde{f}(e^{i\theta}) = \lim_{r\to 1} P[g]\left(re^{i\theta}\right)$ holds at almost every θ.

Also, by the previous corollary, $\lim_{r\to 1} P[g]\left(re^{i\theta}\right) = g(e^{i\theta})$ for almost every θ. Hence, we have that $\tilde{f} = g$.

□

Corollary §1.3.5. *Let $f : \mathbb{D} \to \mathbb{R}_{\geq 0}$ be a harmonic function. Then f has nontangential limits at almost every point of $\mathbb{T}$. (Why demand nonnegative?)*

Let $h(\mathbb{D})$ denote the set of all harmonic functions on $\mathbb{D}$. Let $p \in [1, \infty]$. Define

$$h^p(\mathbb{D}) = \left\{ f \in h(\mathbb{D}) \mid \{f_r\}_{0\leq r<1} \text{ is uniformly bounded in } L^p \text{ norm } \right\}$$

We define a norm on $h^p(\mathbb{D})$ by

$$\|f\|_{h^p(\mathbb{D})} = \sup_{0\le r<1} \|f_r\|_{L^p(\mathbb{D})} = \begin{cases} \sup_{0\le r<1} \left(\frac{1}{2\pi}\int_{-\pi}^{\pi} \left|f\left(re^{i\theta}\right)\right|^p d\theta\right)^{\frac{1}{p}} & \text{if } p\in[1,\infty) \\ \sup_{0\le r<1} \left\|f(re^{i\theta})\right\|_{L^\infty(\mathbb{D})} & \text{if } p=\infty \end{cases}$$

It is easy to see why $\|f\| < +\infty$ for any $f \in h^p(D)$. So we now proceed to show that $h^p(D)$ is a Banach space with this norm. First we show that it is indeed a normed linear space.

Clearly, $h(\mathbb{D})$ is a vector space. To show that $h^p(\mathbb{D})$ is a vector space, it suffices to check that $h^p(\mathbb{D})$ is a subspace.

Let $f, g \in h^p(\mathbb{D})$ and let $\alpha \in \mathbb{C}$. Then for any $r \in [0,1)$, we have that

$$\begin{aligned} \|(f+\alpha g)_r\|_p &= \|f_r + \alpha g_r\| \\ &= \|f_r\|_p + \alpha \|g_r\|_p \end{aligned}$$

Take note of the use of Holder's inequality. After this is done, since $\{f_r\}_{r\in[0,1)}$ and $\{g_r\}_{r\in[0,1)}$ is uniformly bounded, we have that $\{f+\alpha g\}_{r\in[0,1)}$ is uniformly bounded in L^p norm.

Now, we need to show that it is a normed linear space but this follows almost immediately.

To show that it is a Banach space, we show that

Theorem §1.3.6. *Let $p \in [1,\infty]$. If $u \in L^p(\mathbb{T})$ then $f = P * u \in h^p(\mathbb{D})$ and $\|f\|_p = \|u\|_p$. If $\mu \in \mathcal{M}(\mathbb{T})$ then $f = P * \mu \in h^1(\mathbb{D})$ and $\|f\|_1 = \|\mu\|$.*

Proof. We consider the case $p \in [1,\infty)$. The other cases can be dealt similarly. Consider the map

$$u \overset{T}{\mapsto} U$$

where $U\left(re^{i\theta}\right) = \frac{1}{2\pi}\int_{-\pi}^{\pi} P_r\left(e^{i(\theta-t)}\right) u\left(e^{it}\right) dt$. By Theorem §1.2.2, we have that $\|U\| = \|u\|_p < +\infty$. Hence $U \in h^p(\mathbb{D})$.

Linearity is obvious. We need to check injectivity and surjectivity.

To check injectivity, let $u \in L^p(\mathbb{T})$ and suppose that $T(u) = P[u] = 0$. Now $\lim_{r\to1} P[u]\left(re^{i\theta}\right) = u$ for almost θ by Corollary §1.3.3 and hence $u = 0$ almost everywhere.

Surjectivity is clear from Theorem §1.2.5. □

§1.4 Hardy Spaces – H^p spaces

Let us denote the set of all analytic functions on $\mathbb{D}$ by $H(\mathbb{D})$. Hence, $H(\mathbb{D}) \subset h(\mathbb{D})$. For $p \in (0,\infty]$, we consider the *Hardy classes* of analytic functions on the unit disc

$$H^p(\mathbb{D}) = \left\{F \in H(\mathbb{D}) \mid \|F\|_p < \infty\right\}$$

Clearly,

$$H^p(\mathbb{D}) \subset h^p(\mathbb{D})$$

We will see that $H^p(\mathbb{D})$, $1 \le p \le +\infty$, is also a Banach spaces isomorphic to a closed subspace of $L^p(\mathbb{T})$ denotes by $H^p(\mathbb{T})$.

To prove that $H^p(\mathbb{D})$ is a closed subspace of $h^p(\mathbb{D})$, we are going to identify $H^p(\mathbb{D})$ with the closed subspace

$$\left\{u \in L^p(\mathbb{T}) : \int_{-\pi}^{\pi} u\left(e^{it}\right) e^{ikt} = 0 \textit{ for all } k \in \mathbb{N}\right\}$$

Let $\{u_n\}$ be a sequence of functions in the above subspace; suppose that $\{u_n\}$ converge to $u \in L^p(\mathbb{T})$. Now, let $k \in \mathbb{N}$ be arbitrary. Since $\{u_n\}$ converge to u in p-norm, we have that $\{u_n\}$ converge to u in 1-norm. Hence we have the following:

$$\left|\int_{-\pi}^{\pi} u_n\left(e^{it}\right) e^{ikt} dt - \int_{-\pi}^{\pi} u\left(e^{it}\right) e^{ikt} dt\right| \leq \int_{-\pi}^{\pi} \left|u_n\left(e^{it}\right) - u\left(e^{it}\right)\right| dt$$

From the above inequality, it is evident that u is in the subspace mentioned above!

§1.4.1 Series Representation of Harmonic Functions

Theorem §1.4.1. *Let U be a harmonic on the disc $D_R = \{|z| < R\}$. Then, for each $n \in \mathbb{Z}$, the quantity*

$$a_n = \frac{\rho^{-|n|}}{2\pi} \int_{-\pi}^{\pi} U\left(\rho e^{it}\right) e^{-int} dt \qquad (0 < \rho < R) \tag{§1.4.1}$$

is independent of ρ and we have

$$U\left(re^{i\theta}\right) = \sum_{n=-\infty}^{\infty} a_n r^{|n|} e^{in\theta} \qquad \left(re^{i\theta} \in \mathbb{D}\right) \tag{§1.4.2}$$

The function

$$V\left(re^{i\theta}\right) = \sum_{n=-\infty}^{\infty} -isgn\,(n) a_n r^{|n|} e^{in\theta} \qquad \left(re^{i\theta} \in \mathbb{D}\right) \tag{§1.4.3}$$

is the unique harmonic conjugate of U such that $V(0) = 0$. The series in §1.4.2 and §1.4.3 are absolutely and uniformly convergent on compact subsets of D_R

§2 The space H^1

§2.1 Brief Recap!

Theorem §2.1.1. *Let $u : \overline{\mathbb{D}} \to \mathbb{C}$ be a harmonic function. Then we have that*

$$u\left(re^{i\theta}\right) = \frac{1}{2\pi} \int_{-\pi}^{\pi} u\left(e^{it}\right) P_r\left(e^{i(\theta - t)}\right)$$

§2.2 The Helson-Lowdenslager Approach

Let $\mathcal{C}\left(\overline{\mathbb{D}}\right)$ be the set of all continuous functions on $\overline{\mathbb{D}}$ and let $H\left(\mathbb{D}\right)$ be the set of all holomorphic functions on the open disc $\mathbb{D}$. We define $\mathcal{A} = \mathcal{C}\left(\overline{\mathbb{D}}\right) \cap H\left(\mathbb{D}\right)$.

We show that $\mathcal{A}$ is an uniformly closed algebra of $\mathcal{C}\left(\overline{\mathbb{D}}\right)$. Let $\{f_n\}$ be a sequence in $\mathcal{A}$ converging uniformly to $f \in \mathcal{C}\left(\overline{\mathbb{D}}\right)$.

We recall Morera's Theorem for analytic functions at this point:

Theorem §2.2.1 (Morera)**.** *A continuous, complex valued function $f : D \to \mathbb{C}$ that satisfies $\oint_\gamma f\left(z\right) dz = 0$ for any closed piecewise C^1 path γ in D must be holomorphic on D.*

We use this theorem to prove what we want to prove. Now, let C be any closed curve in $\mathbb{D}$. Then for any $n \in \mathbb{N}$,

$$\oint_C f_n\left(z\right) dz = 0$$

So,

$$\oint_C f(z) dz = \oint_C \lim_{n\to\infty} f_n\left(z\right) dz = \lim_{n\to\infty} \oint_C f_n\left(z\right) dz = 0$$

Since C was arbitrary, f must be holomorphic. This shows that $\mathcal{A}$ is uniformly closed. The fact that it is an algebra is easy to check ✓.

Now, note that since $\mathbb{D}$ is a compact metric space, we have that $\mathcal{C}\left(\mathbb{D}\right)$ is a complete metric space with supremum metric. Since the supremum metric can also be induced by a norm, namely the supremum norm, we have that $\mathcal{C}\left(\mathbb{D}\right)$ is a Banach space with the supremum norm.

Thus, this is what we have proved so far:

Theorem §2.2.2. *The disc algebra $\mathcal{A} = \mathcal{C}\left(\overline{\mathbb{D}}\right) \cap H\left(\mathbb{D}\right)$ is a Banach space under the* sup *norm*

$$\|f\|_\infty = \sup_{|z|\le 1} \left|f\left(z\right)\right|$$

We make a couple of observations at this point:

1. Each $f \in \mathcal{A}$ is the Poisson integral of its boundary values:

$$f\left(re^{i\theta}\right) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f\left(e^{it}\right) P_r\left(e^{i(\theta-t)}\right) dt$$

2. It follows from the Maximum Modulus Theorem that

$$\|f\|_\infty = \sup\left|f\left(e^{it}\right)\right|$$

Theorem §2.2.3 (Correspondence of $\mathcal{A}$ with a closed subspace of $\mathcal{C}(\mathbb{T})$)**.** *Consider the subspace*

$$\tilde{\mathcal{A}} = \left\{ f \in \mathcal{C}(\mathbb{T}) \,:\, \int_{-\pi}^{\pi} f\left(e^{it}\right) e^{in\theta} \textit{ for } n = 1, 2, \ldots \right\}$$

of $\mathcal{C}(\mathbb{T})$. Then there is an isomorphism of $\mathcal{A}$ with $\tilde{\mathcal{A}}$.

Proof. First, we show that $\tilde{\mathcal{A}}$ is a closed subspace of $\mathcal{C}(\mathbb{T})$. Let $\{f_n\}$ be a sequence of functions in $\tilde{\mathcal{A}}$ converging to $f \in \mathcal{C}(\mathbb{T})$. Consider the following:

$$\begin{aligned}\left|\int_{-\pi}^{\pi} f\left(e^{it}\right) e^{ikt} dt\right| &= \left|\int_{-\pi}^{\pi} f\left(e^{it}\right) e^{ikt} dt - \int_{-\pi}^{\pi} f_n\left(e^{it}\right) e^{ikt} dt\right| \\ &= \int_{-\pi}^{\pi} \left|f\left(e^{it}\right) - f_n\left(e^{it}\right)\right| dt \\ &\leq 2\pi \|f_n - f\|_\infty \to 0 \text{ as } n \to \infty\end{aligned}$$

This shows that $\tilde{\mathcal{A}}$ is closed under $\mathcal{C}(\mathbb{T})$ with supremum norm.

Now consider the linear map $T : \mathcal{A} \to \tilde{\mathcal{A}}$ given by

$$f \overset{T}{\longmapsto} f\,|_{\mathbb{T}}$$

For the sake of convenience, we will write $f\,|_{\mathbb{T}}$ as $f_{\mathbb{T}}$. We first need to show this map is well defined! That is, we need to show that

$$\int_{-\pi}^{\pi} f_{\mathbb{T}}\left(e^{it}\right) e^{ikt} dt = 0$$

for all $k \in \mathbb{N}$ but this immediately follows from Cauchy's theorem.

Note that injectivity is clear from Theorem §2.1.1. To show surjectivity, let $f \in \tilde{A}$. We need to show that there is a function $u \in \mathcal{A}$ such that $u_{\mathbb{T}} = f$. Consider the function

$$u\left(re^{i\theta}\right) = \begin{cases} (P * f)(re^{i\theta}) & \text{if } 0 \leq r < 1 \\ f\left(e^{i\theta}\right) & \text{if } r = 1 \end{cases}$$

This is the Dirichlet problem on the unit disc! So, u is continuous on $\overline{\mathbb{D}}$. It remains to show that u is analytic on $\mathbb{D}$. But note that for $r \in [0,1)$,

$$\begin{aligned} u\left(re^{i\theta}\right) &= \sum_{n=-\infty}^{\infty} r^{|n|} \hat{f}\left(n\right) e^{int} \\ &= \sum_{n=0}^{\infty} r^{|n|} \hat{f}\left(n\right) e^{int} \end{aligned}$$

This completes the proof of the theorem! $\square$

In view of the previous theorem, we will simply write $\tilde{\mathcal{A}}$ as $\mathcal{A}$.

Theorem §2.2.4 (F and M. Riesz)**.**

§2.3 Szegö's Theorem