Banach Spaces of Analytic Functions

Ashish Kujur

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Contents

§1 Analytic and Harmonic Functions

§1.1 Boundary Values

Definition §1.1.1 (Poisson integral of some function or measure)**.** Let $\tilde{f}:\mathbb{D}\to\mathbb{C}$ be a harmonic function. Then $\tilde{f}$ is said to be the *Poisson integral* of the function $f:\mathbb{T}\to\mathbb{C}$ if

$$\tilde{f}(re^{i\theta}) = \frac{1}{2\pi}\int_T f\left(e^{it}\right) P_r\left(e^{i(\theta-t)}\right) dt$$

Similarly, f is said to be the *Poisson integral* of a complex measure μ on T if

$$\tilde{f}(re^{i\theta}) = \frac{1}{2\pi}\int_T P_r\left(e^{i(\theta-t)}\right) d\mu\left(e^{it}\right)$$

§1.2 Fatou's Theorem

Theorem §1.2.1. *Let μ be a complex measure on the unit circle $\mathbb{T}$, and let $f:\mathbb{D}\to\mathbb{C}$ be the harmonic function defined by*

$$f(re^{i\theta}) = \frac{1}{2\pi}\int_{\mathbb{T}} P_r\left(e^{i(\theta-t)}\right) d\mu\left(e^{it}\right)$$

Let $e^{i\theta_0}$ be any point where μ is differentiable with respect to the normalised Lebesgue measure. Then

$$\lim_{r\to1} f\left(re^{i\theta_0}\right) = \left(\frac{d\mu}{d\theta}\right)\left(e^{i\theta_0}\right) = \mu'\left(e^{i\theta_0}\right)$$

In fact, $f(re^{i\theta}) \to \mu'\left(e^{i\theta_0}\right)$ as $re^{i\theta}$ approaches $e^{i\theta_0}$ along any path in the open disc within the region of the form $|\theta-\theta_0| \le c(1-r)$ for some $c>0$.