Banach Spaces of Analytic Functions

Ashish Kujur

Last Updated: January 24, 2023

Contents

§1 Analytic and Harmonic Functions

§1.1 Boundary Values

§1.1.1 Weak* convergence of measures

Theorem §1.1.1. *Let* $\{\varphi_i\}_i$ *be an approximate identity on* $\mathbb{T}$ *and let* $\mu \in \mathscr{M}(\mathbb{T})$*. Then for all* i*,* $\varphi_i * \mu \in L^1(\mathbb{T})$ *with*

$$\left\|\varphi_i * \mu\right\|_1 \leq C_\varphi \|\mu\|$$

and

$$\|\mu\| \leq \sup_i \left\|\varphi_i * \mu\right\|_1.$$

Moreover, the measures $d\mu_i = (\varphi_i * \mu)\left(e^{it}\right) dt/2\pi$ *converge to* $d\mu\left(e^{it}\right)$ *in the weak* topology, i.e.*

$$\lim_i \frac{1}{2\pi} \int_{-\pi}^{\pi} f\left(e^{it}\right)(\varphi_i * \mu)\left(e^{it}\right) dt = \int_{\mathbb{T}} \varphi\left(e^{it}\right) d\mu\left(e^{it}\right)$$

for all $f \in \mathscr{C}(\mathbb{T})$*.*

§1.1.2 Convergence in norm

Theorem §1.1.2. *Let $\{\varphi_i\}_i$ be an approximate identity on $\mathbb{T}$ and let $f \in L^p(\mathbb{T})$ with $p \in [1,\infty)$. Then for all i, $\varphi_i * f \in L^p(\mathbb{T})$ with*

$$\left\|\varphi_i * f\right\|_p \le C_\varphi \|f\|_p$$

and

$$\lim_i \left\|\varphi_i * f - f\right\|_p = 0.$$

§1.1.3 Weak* convergence of bounded functions

Theorem §1.1.3. *Let $\{\varphi_i\}_i$ be an approximate identity on $\mathbb{T}$ and let $f \in L^\infty(\mathbb{T})$. Then for all i, $\varphi_i * \mu \in \mathscr{C}(\mathbb{T})$ with*

$$\left\|\varphi_i * \mu\right\|_\infty \le C_\varphi \|\mu\|_\infty$$

and

$$\left\|f\right\|_{+\infty} \le \sup_i \left\|\varphi_i * f\right\|_\infty .$$

Moreover, $\varphi_i * f$ converge to f in the weak topology, i.e.*

$$\lim_i \int_{-\pi}^{\pi} g\left(e^{it}\right)(\varphi_i * f)\left(e^{it}\right) dt = \int_{\mathbb{T}} g\left(e^{it}\right) f\left(e^{it}\right) dt$$

for all $g \in L^1(\mathbb{T})$.

§1.1.4 The entire picture!

Definition §1.1.4 (Poisson integral of some function or measure)**.** Let $\tilde{f} : \mathbb{D} \to \mathbb{C}$ be a harmonic function. Then $\tilde{f}$ is said to be the *Poisson integral* of the function $f : \mathbb{T} \to \mathbb{C}$ if

$$\tilde{f}(re^{i\theta}) = \frac{1}{2\pi} \int_T f\left(e^{it}\right) P_r\left(e^{i(\theta-t)}\right) dt$$

In such a case, we will denote the function $\tilde{f}$ by $P[f]$. Similarly, f is said to be the *Poisson integral* of a complex measure μ on T if

$$\tilde{f}(re^{i\theta}) = \frac{1}{2\pi} \int_T P_r\left(e^{i(\theta-t)}\right) d\mu\left(e^{it}\right)$$

In such a case, we will denote the function $\tilde{f}$ by $P[\mu]$.

Theorem §1.1.5 (Ultimate Convergence)**.** *Let $f : \mathbb{D} \to \mathbb{C}$ be a harmonic function. Define for each $r \in [0,1)$, the function $f_r : \mathbb{T} \to \mathbb{C}$ by*

$$f_r\left(e^{i\theta}\right) = f\left(re^{i\theta}\right)$$

The following statements holds:

1. *If $1 < p \le \infty$ then $f = P[g]$ for some $g \in L^p[g]$ iff for each $r > 0$, $\left\|f_r\right\|_p < +\infty$.*
2. *If p=1 then $f = P[g]$ for some $g \in L^p[g]$ iff f_r converge in the L^1 norm.*
3. *$f = P[\mu]$ for some $\mu \in \mathscr{M}(\mathbb{T})$ iff for each $r > 0$, $\left\|f_r\right\|_1 < +\infty$*

§1.2 Fatou's Theorem

Theorem §1.2.1. *Let μ be a complex measure on the unit circle $\mathbb{T}$, and let $f : \mathbb{D} \to \mathbb{C}$ be the harmonic function defined by*

$$f(re^{i\theta}) = \frac{1}{2\pi}\int_{\mathbb{T}} P_r\left(e^{i(\theta-t)}\right) d\mu\left(e^{it}\right)$$

Let $e^{i\theta_0}$ be any point where μ is differentiable with respect to the normalised Lebesgue measure. Then

$$\lim_{r\to 1} f\left(re^{i\theta_0}\right) = \left(\frac{d\mu}{d\theta}\right)\left(e^{i\theta_0}\right) = \mu'\left(e^{i\theta_0}\right)$$

In fact, $f(re^{i\theta}) \to \mu'\left(e^{i\theta_0}\right)$ as $re^{i\theta}$ approaches $e^{i\theta_0}$ along any path in the open disc within the region of the form $|\theta - \theta_0| \le c(1-r)$ for some $c > 0$.

Corollary §1.2.2. *Let μ be a complex measure on $\mathbb{T}$. Then $P[\mu]$ has nontangential limits equal everywhere to the Radon Nikodym derivative of μ with respect to the normalised Lebesgue measure.*

Corollary §1.2.3. *Let $f : \mathbb{T} \to \mathbb{C}$ be L^1. Then $P[f]$ has nontangential limits at almost everywhere and these limits equal to f almost everywhere.*

Corollary §1.2.4. *Let $f : \mathbb{D} \to \mathbb{C}$ be a harmonic function and $1 \le p < \infty$. Suppose that for all $0 \le r < 1$, we have that*

$$\|f_r\|_p < +\infty$$

Then for almost every θ the radial limits

$$\tilde{f}(e^{i\theta}) = \lim_{r\to 1} f\left(re^{i\theta}\right)$$

exist and define a function $\tilde{f}$ in $L^p(\mathbb{T})$. The following also holds:

1. *If $p > 1$ then $f = P[\tilde{f}]$.*
2. *If $p = 1$ then $f = P[\mu]$ for some complex measure μ whose absolutely continuous part is $f\,d\theta$.*
3. *IF f is bounded then the boundary values exist almost everywhere and define a bounded measurable function $\tilde{f}$ on $\mathbb{T}$ such that $f = P[\tilde{f}]$.*

Proof. Suppose that for each $r \in [0,1)$, we have $\|f_r\|_p < +\infty$. We need to prove that for almost every θ, $\lim_{r\to 1} f\left(re^{i\theta}\right)$ exists. Then by Theorem §1.1.5, we have that $f = P[g]$ for some $g \in L^p(\mathbb{T})$. Since $L^p(\mathbb{T}) \subset L^1(\mathbb{T})$, we can use the previous corollary. By the previous corollary, we have that $P[g]$ has nontangential limits almost everywhere, we have that

$$\tilde{f}\left(e^{i\theta}\right) = \lim_{r\to 1} f(re^{i\theta}) = \lim_{r\to 1} P[g]\left(re^{i\theta}\right) \tag{§1.2.1}$$

exists almost everywhere.

Now we proceed to prove part (1). Also by Theorem §1.1.5, we have that $f = P[g]$ for some $g \in L^p(\mathbb{T})$. Hence, we have that by Equation §1.2.1 that $\tilde{f}(e^{i\theta}) = \lim_{r\to 1} P[g]\left(re^{i\theta}\right)$ holds at almost every θ.

Also, by the previous corollary, $\lim_{r\to 1} P[g]\left(re^{i\theta}\right) = g(e^{i\theta})$ for almost every θ. Hence, we have that $\tilde{f} = g$.

□

Corollary §1.2.5. *Let $f : \mathbb{D} \to \mathbb{R}_{\geq 0}$ be a harmonic function. Then f has nontangential limits at almost every point of $\mathbb{T}$. (Why demand nonnegative?)*

Let $h(\mathbb{D})$ denote the set of all harmonic functions on $\mathbb{D}$. Let $p \in [1, \infty]$. Define

$$h^p(\mathbb{D}) = \left\{ f \in h(\mathbb{D}) \mid \left\{ f_r \right\}_{0\leq r<1} \text{ is uniformly bounded in } L^p \text{ norm} \right\}$$

We define a norm on $h^p(\mathbb{D})$ by

$$\|f\| = \sup_{0\leq r<1} \left\| f_r \right\|_p = \begin{cases} \sup_{0\leq r<1} \left(\frac{1}{2\pi} \int_{-\pi}^{\pi} \left| f\left(re^{i\theta}\right)\right|^p d\theta \right)^{\frac{1}{p}} & \text{if } p \in [1,\infty) \\ \sup_{0\leq r<1} \left\| f(re^{i\theta}) \right\|_\infty \end{cases}$$

It is easy to see why $\|f\| < +\infty$ for any $f \in h^p(D)$. So we now proceed to show that $h^p(D)$ is a Banach space with this norm. First we show that it is indeed a normed linear space.

Clearly, $h(\mathbb{D})$ is a vector space. To show that $h^p(\mathbb{D})$ is a vector space, it suffices to check that $h^p(\mathbb{D})$ is a subspace.

Let $f, g \in h^p(\mathbb{D})$ and let $\alpha \in \mathbb{C}$. Then for any $r \in [0, 1)$, we have that

$$\begin{aligned} \left\| (f + \alpha g)_r \right\|_p &= \left\| f_r + \alpha g_r \right\| \\ &= \left\| f_r \right\|_p + \alpha \left\| g_r \right\|_p \end{aligned}$$

Take note of the use of Holder's inequality. After this is done, since $\left\{ f_r \right\}_{r\in[0,1)}$ and $\left\{ g_r \right\}_{r\in[0,1)}$ is uniformly bounded, we have that $\left\{ f + \alpha g \right\}_{r\in[0,1)}$ is uniformly bounded in L^p norm.

Now, we need to show that it is a normed linear space but this follows almost immediately.

To show that it is a Banach space, we show that

Theorem §1.2.6. *Let $p \in [1, \infty]$. If $u \in L^p(\mathbb{T})$ then $f = P * u \in h^p(\mathbb{D})$ and $\left\| f \right\|_p = \|u\|_p$. If $\mu \in \mathscr{M}(\mathbb{T})$ then $f = P * \mu \in h^1(\mathbb{D})$ and $\left\| f \right\|_1 = \|\mu\|$.*

Proof. We consider the case $p \in [1, \infty)$. The other cases can be dealt similarly. Consider the map

$$u \overset{T}{\mapsto} U$$

where $U\left(re^{i\theta}\right) = \frac{1}{2\pi} \int_{-\pi}^{\pi} P_r\left(e^{i(\theta-t)}\right) u\left(e^{it}\right) dt$. By Theorem §1.1.2, we have that $\|U\| = \|u\|_p < +\infty$. Hence $U \in h^p(\mathbb{D})$.

Linearity is obvious. We need to check injectivity and surjectivity.

To check injectivity, let $u \in L^p(\mathbb{T})$ and suppose that $T(u) = P[u] = 0$. Now $\lim_{r\to 1} P[u]\left(re^{i\theta}\right) = u$ for almost θ by Corollary §1.2.3 and hence $u = 0$ almost everywhere.

Surjectivity is clear from Theorem §1.1.5. □

§1.3 H^p spaces

Let us denote the set of all analytic functions on $\mathbb{D}$ by $H(\mathbb{D})$. Hence, $H(\mathbb{D}) \subset h(\mathbb{D})$. For $p \in (0, \infty]$, we consider the *Hardy classes* of analytic functions on the unit disc

$$H^p(\mathbb{D}) = \left\{F \in H(\mathbb{D}) \mid \|F\|_p < \infty\right\}$$

Clearly,

$$H^p(\mathbb{D}) \subset h^p(\mathbb{D})$$

We will see that $H^p(\mathbb{D})$, $1 \leq p \leq +\infty$, is also a Banach spaces isomorphic to a closed subspace of $L^p(\mathbb{T})$ denotes by $H^p(\mathbb{T})$.