Study of Closed Ideals of Disc Algebra

Indian Institute of Science Education and Research – Thiruvananthapuram

IISER
THIRUVANANTHAPURAM

Ashish Kujur

Last Updated: February 10, 2023

Contents

1 Analytic and Harmonic Functions **3**
1.1 The Cauchy and Poisson Kernels . . . 3
1.2 Boundary Values . . . 3
1.2.1 Weak* convergence of measures . . . 3
1.2.2 Convergence in norm . . . 4
1.2.3 Weak* convergence of bounded functions . . . 4
1.2.4 The entire picture! . . . 4
1.3 Fatou's Theorem . . . 5
1.4 Hardy Spaces – H^p spaces . . . 8

2 The space H^1 **10**
2.1 Brief Recap! . . . 10
2.2 The Helson-Lowdenslager Approach . . . 10
2.3 Szegö's Theorem . . . 23

Chapter 1

Analytic and Harmonic Functions

1.1 The Cauchy and Poisson Kernels

Proposition 1.1.0.1. *Let $u : \overline{\mathbb{D}} \to \mathbb{C}$ be a harmonic function. Then we have that*

$$u\left(re^{i\theta}\right) = \frac{1}{2\pi}\int_{-\pi}^{\pi} u\left(e^{it}\right) P_r\left(e^{i(\theta-t)}\right) dt \tag{1.1.0.1}$$

1.2 Boundary Values

1.2.1 Weak* convergence of measures

Theorem 1.2.1.1. *Let $\{\varphi_i\}_i$ be an approximate identity on $\mathbb{T}$ and let $\mu \in \mathcal{M}(\mathbb{T})$. Then for all i, $\varphi_i * \mu \in L^1(\mathbb{T})$ with*

$$\|\varphi_i * \mu\|_1 \leq C_\varphi \|\mu\|$$

and

$$\|\mu\| \leq \sup_i \|\varphi_i * \mu\|_1 .$$

Moreover, the measures $d\mu_i = (\varphi_i * \mu)(e^{it})\, dt/2\pi$ converge to $d\mu(e^{it})$ in the weak topology, i.e.*

$$\lim_i \frac{1}{2\pi}\int_{-\pi}^{\pi} f\left(e^{it}\right)(\varphi_i * \mu)\left(e^{it}\right) dt = \int_{\mathbb{T}} \varphi\left(e^{it}\right) d\mu\left(e^{it}\right)$$

for all $f \in \mathcal{C}(\mathbb{T})$.

1.2.2 Convergence in norm

Theorem 1.2.2.1. *Let $\{\varphi_i\}_i$ be an approximate identity on $\mathbb{T}$ and let $f \in L^p(\mathbb{T})$ with $p \in [1, \infty)$. Then for all i, $\varphi_i * f \in L^p(\mathbb{T})$ with*

$$\|\varphi_i * f\|_p \leq C_\varphi \|f\|_p$$

and

$$\lim_i \|\varphi_i * f - f\|_p = 0.$$

1.2.3 Weak* convergence of bounded functions

Theorem 1.2.3.1. *Let $\{\varphi_i\}_i$ be an approximate identity on $\mathbb{T}$ and let $f \in L^\infty(\mathbb{T})$. Then for all i, $\varphi_i * \mu \in \mathcal{C}(\mathbb{T})$ with*

$$\|\varphi_i * \mu\|_\infty \leq C_\varphi \|\mu\|_\infty$$

and

$$\|f\|_{+\infty} \leq \sup_i \|\varphi_i * f\|_\infty .$$

Moreover, $\varphi_i * f$ converge to f in the weak topology, i.e.*

$$\lim_i \int_{-\pi}^{\pi} g\left(e^{it}\right)(\varphi_i * f)\left(e^{it}\right) dt = \int_{\mathbb{T}} g\left(e^{it}\right) f\left(e^{it}\right) dt$$

for all $g \in L^1(\mathbb{T})$.

1.2.4 The entire picture!

Definition 1.2.4.1 (Poisson integral of some function or measure)**.** Let $\tilde{f} : \mathbb{D} \to \mathbb{C}$ be a harmonic function. Then $\tilde{f}$ is said to be the *Poisson integral* of the function $f : \mathbb{T} \to \mathbb{C}$ if

$$\tilde{f}(re^{i\theta}) = \frac{1}{2\pi} \int_T f\left(e^{it}\right) P_r\left(e^{i(\theta - t)}\right) dt$$

In such a case, we will denote the function $\tilde{f}$ by $P[f]$. Similarly, f is said to be the *Poisson integral* of a complex measure μ on T if

$$\tilde{f}(re^{i\theta}) = \frac{1}{2\pi} \int_T P_r\left(e^{i(\theta - t)}\right) d\mu\left(e^{it}\right)$$

In such a case, we will denote the function $\tilde{f}$ by $P[\mu]$.

Theorem 1.2.4.2 (Ultimate Convergence)**.** *Let $f : \mathbb{D} \to \mathbb{C}$ be a harmonic function. Define for each $r \in [0, 1)$, the function $f_r : \mathbb{T} \to \mathbb{C}$ by*

$$f_r\left(e^{i\theta}\right) = f\left(re^{i\theta}\right)$$

The following statements holds:

1. *If $1 < p \leq \infty$ then $f = P[g]$ for some $g \in L^p[g]$ iff for each $r > 0$, $\|f_r\|_p < +\infty$.*

2. *If p=1 then $f = P[g]$ for some $g \in L^p[g]$ iff f_r converge in the L^1 norm.*

3. *$f = P[\mu]$ for some $\mu \in \mathcal{M}(\mathbb{T})$ iff for each $r > 0$, $\|f_r\|_1 < +\infty$*

1.3 Fatou's Theorem

Theorem 1.3.0.1. *Let μ be a complex measure on the unit circle $\mathbb{T}$, and let $f : \mathbb{D} \to \mathbb{C}$ be the harmonic function defined by*

$$f(re^{i\theta}) = \frac{1}{2\pi} \int_{\mathbb{T}} P_r\left(e^{i(\theta - t)}\right) d\mu\left(e^{it}\right)$$

Let $e^{i\theta_0}$ be any point where μ is differentiable with respect to the normalised Lebesgue measure. Then

$$\lim_{r \to 1} f\left(re^{i\theta_0}\right) = \left(\frac{d\mu}{d\theta}\right)\left(e^{i\theta_0}\right) = \mu'\left(e^{i\theta_0}\right)$$

In fact, $f(re^{i\theta}) \to \mu'\left(e^{i\theta_0}\right)$ as $re^{i\theta}$ approaches $e^{i\theta_0}$ along any path in the open disc within the region of the form $|\theta - \theta_0| \leq c\,(1 - r)$ for some $c > 0$.

Corollary 1.3.0.2. *Let μ be a complex measure on $\mathbb{T}$. Then $P[\mu]$ has nontangential limits equal everywhere to the Radon Nikodym derivative of μ with respect to the normalised Lebesgue measure.*

Corollary 1.3.0.3. *Let $f : \mathbb{T} \to \mathbb{C}$ be L^1. Then $P[f]$ has nontangential limits at almost everywhere and these limits equal to f almost everywhere.*

Corollary 1.3.0.4. *Let $f : \mathbb{D} \to \mathbb{C}$ be a harmonic function and $1 \le p < \infty$. Suppose that for all $0 \le r < 1$, we have that*

$$\|f_r\|_p < +\infty$$

Then for almost every θ the radial limits

$$\tilde{f}(e^{i\theta}) = \lim_{r\to 1} f\left(re^{i\theta}\right)$$

exist and define a function $\tilde{f}$ in $L^p(\mathbb{T})$. The following also holds:

1. *If $p > 1$ then $f = P[\tilde{f}]$.*
2. *If $p = 1$ then $f = P[\mu]$ for some complex measure μ whose absolutely continuous part is $f d\theta$.*
3. *IF f is bounded then the boundary values exist almost everywhere and define a bounded measurable function $\tilde{f}$ on $\mathbb{T}$ such that $f = P[\tilde{f}]$.*

Proof. Suppose that for each $r \in [0, 1)$, we have $\|f_r\|_p < +\infty$. We need to prove that for almost every θ, $\lim_{r\to 1} f\left(re^{i\theta}\right)$ exists. Then by Theorem 1.2.4.2, we have that $f = P[g]$ for some $g \in L^p(\mathbb{T})$. Since $L^p(\mathbb{T}) \subset L^1(\mathbb{T})$, we can use the previous corollary. By the previous corollary, we have that $P[g]$ has nontangential limits almost everywhere, we have that

$$\tilde{f}\left(e^{i\theta}\right) = \lim_{r\to 1} f(re^{i\theta}) = \lim_{r\to 1} P[g]\left(re^{i\theta}\right) \tag{1.3.0.1}$$

exists almost everywhere.

Now we proceed to prove part (1). Also by Theorem 1.2.4.2, we have that $f = P[g]$ for $g \in L^p(\mathbb{T})$. Hence, we have that by Equation 1.3.0.1 that $\tilde{f}(e^{i\theta}) = \lim_{r\to 1} P[g]\left(re^{i\theta}\right)$ holds at almost every θ.

Also, by the previous corollary, $\lim_{r\to 1} P[g]\left(re^{i\theta}\right) = g(e^{i\theta})$ for almost every θ. Hence, we have that $\tilde{f} = g$.

□

Corollary 1.3.0.5. *Let $f : \mathbb{D} \to \mathbb{R}_{\ge 0}$ be a harmonic function. Then f has nontangential limits at almost every point of $\mathbb{T}$. (Why demand nonnegative?)*

Let $h(\mathbb{D})$ denote the set of all harmonic functions on $\mathbb{D}$. Let $p \in [1, \infty]$. Define

$$h^p(\mathbb{D}) = \left\{ f \in h(\mathbb{D}) \mid \{f_r\}_{0 \leq r < 1} \text{ is uniformly bounded in } L^p \text{ norm } \right\}$$

We define a norm on $h^p(\mathbb{D})$ by

$$\|f\|_{h^p(\mathbb{D})} = \sup_{0 \leq r < 1} \|f_r\|_{L^p(\mathbb{D})} = \begin{cases} \sup_{0 \leq r < 1} \left(\frac{1}{2\pi} \int_{-\pi}^{\pi} \left| f\left(re^{i\theta}\right)\right|^p d\theta \right)^{\frac{1}{p}} & \text{if } p \in [1, \infty) \\ \sup_{0 \leq r < 1} \left\| f(re^{i\theta}) \right\|_{L^\infty(\mathbb{D})} & \text{if } p = \infty \end{cases}$$

It is easy to see why $\|f\| < +\infty$ for any $f \in h^p(D)$. So we now proceed to show that $h^p(D)$ is a Banach space with this norm. First we show that it is indeed a normed linear space.

Clearly, $h(\mathbb{D})$ is a vector space. To show that $h^p(\mathbb{D})$ is a vector space, it suffices to check that $h^p(\mathbb{D})$ is a subspace.

Let $f, g \in h^p(\mathbb{D})$ and let $\alpha \in \mathbb{C}$. Then for any $r \in [0, 1)$, we have that

$$\begin{aligned} \|(f + \alpha g)_r\|_p &= \|f_r + \alpha g_r\| \\ &= \|f_r\|_p + \alpha \|g_r\|_p \end{aligned}$$

Take note of the use of Holder's inequality. After this is done, since $\{f_r\}_{r \in [0,1)}$ and $\{g_r\}_{r \in [0,1)}$ is uniformly bounded, we have that $\{f + \alpha g\}_{r \in [0,1)}$ is uniformly bounded in L^p norm.

Now, we need to show that it is a normed linear space but this follows almost immediately.

To show that it is a Banach space, we show that

Theorem 1.3.0.6. *Let $p \in [1, \infty]$. If $u \in L^p(\mathbb{T})$ then $f = P * u \in h^p(\mathbb{D})$ and $\|f\|_p = \|u\|_p$. If $\mu \in \mathcal{M}(\mathbb{T})$ then $f = P * \mu \in h^1(\mathbb{D})$ and $\|f\|_1 = \|\mu\|$.*

Proof. We consider the case $p \in [1, \infty)$. The other cases can be dealt similarly. Consider the map

$$u \overset{T}{\mapsto} U$$

where $U\left(re^{i\theta}\right) = \frac{1}{2\pi} \int_{-\pi}^{\pi} P_r\left(e^{i(\theta - t)}\right) u\left(e^{it}\right) dt$. By Theorem 1.2.2.1, we have that $\|U\| = \|u\|_p < +\infty$. Hence $U \in h^p(\mathbb{D})$.

Linearity is obvious. We need to check injectivity and surjectivity.

To check injectivity, let $u \in L^p(\mathbb{T})$ and suppose that $T(u) = P[u] = 0$. Now $\lim_{r \to 1} P[u]\left(re^{i\theta}\right) = u$ for almost θ by Corollary 1.3.0.3 and hence $u = 0$ almost everywhere.

Surjectivity is clear from Theorem 1.2.4.2. $\square$

1.4 Hardy Spaces – H^p spaces

Let us denote the set of all analytic functions on $\mathbb{D}$ by $H(\mathbb{D})$. Hence, $H(\mathbb{D}) \subset h(\mathbb{D})$. For $p \in (0, \infty]$, we consider the *Hardy classes* of analytic functions on the unit disc

$$H^p(\mathbb{D}) = \left\{ F \in H(\mathbb{D}) \mid \|F\|_p < \infty \right\}$$

Clearly,

$$H^p(\mathbb{D}) \subset h^p(\mathbb{D})$$

We will see that $H^p(\mathbb{D})$, $1 \leq p \leq +\infty$, is also a Banach spaces isomorphic to a closed subspace of $L^p(\mathbb{T})$ denotes by $H^p(\mathbb{T})$.

To prove that $H^p(\mathbb{D})$ is a closed subspace of $h^p(\mathbb{D})$, we are going to identify $H^p(\mathbb{D})$ with the closed subspace

$$\left\{ u \in L^p(\mathbb{T}) : \int_{-\pi}^{\pi} u\left(e^{it}\right) e^{ikt} = 0 \textit{ for all } k \in \mathbb{N} \right\}$$

Let $\{u_n\}$ be a sequence of functions in the above subspace; suppose that $\{u_n\}$ converge to $u \in L^p(\mathbb{T})$. Now, let $k \in \mathbb{N}$ be arbitrary. Since $\{u_n\}$ converge to u in p-norm, we have that $\{u_n\}$ converge to u in 1-norm. Hence we have the following:

$$\left| \int_{-\pi}^{\pi} u_n\left(e^{it}\right) e^{ikt} dt - \int_{-\pi}^{\pi} u\left(e^{it}\right) e^{ikt} dt \right| \leq \int_{-\pi}^{\pi} \left| u_n\left(e^{it}\right) - u\left(e^{it}\right) \right| dt$$

From the above inequality, it is evident that u is in the subspace mentioned above!

Series Representation of Harmonic Functions

Theorem 1.4.0.1. *Let U be a harmonic on the disc $D_R = \{|z| < R\}$. Then, for each $n \in \mathbb{Z}$, the quantity*

$$a_n = \frac{\rho^{-|n|}}{2\pi} \int_{-\pi}^{\pi} U\left(\rho e^{it}\right) e^{-int} dt \qquad (0 < \rho < R) \tag{1.4.0.1}$$

is independent of ρ and we have

$$U\left(re^{i\theta}\right) = \sum_{n=-\infty}^{\infty} a_n r^{|n|} e^{in\theta} \qquad \left(re^{i\theta} \in \mathbb{D}\right) \tag{1.4.0.2}$$

The function

$$V\left(re^{i\theta}\right)=\sum_{n=-\infty}^{\infty}-isgn\ (n)a_nr^{|n|}e^{in\theta}\qquad\left(re^{i\theta}\in\mathbb{D}\right)\tag{1.4.0.3}$$

is the unique harmonic conjugate of U such that $V(0)=0$. The series in 1.4.0.2 and 1.4.0.3 are absolutely and uniformly convergent on compact subsets of D_R

Chapter 2

The space H^1

2.1 Brief Recap!

Theorem 2.1.0.1. *Let $u : \overline{\mathbb{D}} \to \mathbb{C}$ be a harmonic function. Then we have that*

$$u\left(re^{i\theta}\right) = \frac{1}{2\pi}\int_{-\pi}^{\pi} u\left(e^{it}\right) P_r\left(e^{i(\theta-t)}\right)$$

2.2 The Helson-Lowdenslager Approach

Let $\mathcal{C}\left(\overline{\mathbb{D}}\right)$ be the set of all continuous functions on $\overline{\mathbb{D}}$ and let $H(\mathbb{D})$ be the set of all holomorphic functions on the open disc $\mathbb{D}$. We define $\mathcal{A} = \mathcal{C}\left(\overline{\mathbb{D}}\right) \cap H(\mathbb{D})$.

We show that $\mathcal{A}$ is an uniformly closed algebra of $\mathcal{C}\left(\overline{\mathbb{D}}\right)$. Let $\{f_n\}$ be a sequence in $\mathcal{A}$ converging uniformly to $f \in \mathcal{C}\left(\overline{\mathbb{D}}\right)$.

We recall Morera's Theorem for analytic functions at this point:

Theorem 2.2.0.1 (Morera)**.** *A continuous, complex valued function $f : D \to \mathbb{C}$ that satisfies $\oint_\gamma f(z)\,dz = 0$ for any closed piecewise C^1 path γ in D must be holomorphic on D.*

We use this theorem to prove what we want to prove. Now, let C be any closed curve in $\mathbb{D}$. Then for any $n \in \mathbb{N}$,

$$\oint_C f_n(z)\,dz = 0$$

So,

$$\oint_C f(z)dz = \oint_C \lim_{n\to\infty} f_n\left(z\right)dz = \lim_{n\to\infty}\oint_C f_n\left(z\right)dz = 0$$

Since C was arbitrary, f must be holomorphic. This shows that $\mathcal{A}$ is uniformly closed. The fact that it is an algebra is easy to check ✓.

Now, note that since $\mathbb{D}$ is a compact metric space, we have that $\mathcal{C}\left(\mathbb{D}\right)$ is a complete metric space with supremum metric. Since the supremum metric can also be induced by a norm, namely the supremum norm, we have that $\mathcal{C}\left(\mathbb{D}\right)$ is a Banach space with the supremum norm.

Thus, this is what we have proved so far:

Theorem 2.2.0.2. *The disc algebra $\mathcal{A} = \mathcal{C}\left(\overline{\mathbb{D}}\right) \cap H\left(\mathbb{D}\right)$ is a Banach space under the* sup *norm*

$$\|f\|_\infty = \sup_{|z|\leq 1}\left|f\left(z\right)\right|$$

We make a couple of observations at this point:

1. Each $f \in \mathcal{A}$ is the Poisson integral of its boundary values:

$$f\left(re^{i\theta}\right) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f\left(e^{it}\right)P_r\left(e^{i(\theta-t)}\right)dt$$

2. It follows from the Maximum Modulus Theorem that

$$\|f\|_\infty = \sup\left|f\left(e^{it}\right)\right|$$

Theorem 2.2.0.3 (Correspondence of $\mathcal{A}$ with a closed subspace of $\mathcal{C}\left(\mathbb{T}\right)$)**.** *Consider the subspace*

$$\tilde{\mathcal{A}} = \left\{f \in \mathcal{C}\left(\mathbb{T}\right)\,:\, \int_{-\pi}^{\pi} f\left(e^{it}\right)e^{int} = 0 \text{ for } n = 1, 2, \ldots\right\}$$

of $\mathcal{C}\left(\mathbb{T}\right)$. Then there is an isometric isomorphism of $\mathcal{A}$ with $\tilde{\mathcal{A}}$.

Proof. First, we show that $\tilde{\mathcal{A}}$ is a closed subspace of $\mathcal{C}(\mathbb{T})$. Let $\{f_n\}$ be a sequence of functions in $\tilde{\mathcal{A}}$ converging to $f \in \mathcal{C}(\mathbb{T})$. Consider the following:

$$\begin{aligned}\left|\int_{-\pi}^{\pi} f\left(e^{it}\right) e^{ikt} dt\right| &= \left|\int_{-\pi}^{\pi} f\left(e^{it}\right) e^{ikt} dt - \int_{-\pi}^{\pi} f_n\left(e^{it}\right) e^{ikt} dt\right| \\ &= \int_{-\pi}^{\pi} \left|f\left(e^{it}\right) - f_n\left(e^{it}\right)\right| dt \\ &\leq 2\pi \left\|f_n - f\right\|_{\infty} \to 0 \text{ as } n \to \infty\end{aligned}$$

This shows that $\tilde{\mathcal{A}}$ is closed under $\mathcal{C}(\mathbb{T})$ with supremum norm.

Now consider the linear map $T : \mathcal{A} \to \tilde{\mathcal{A}}$ given by

$$f \xmapsto{T} f\mid_{\mathbb{T}}$$

For the sake of convenience, we will write $f\mid_{\mathbb{T}}$ as $f_{\mathbb{T}}$. We first need to show this map is well defined! That is, we need to show that

$$\int_{-\pi}^{\pi} f_{\mathbb{T}}\left(e^{it}\right) e^{ikt} dt = 0$$

for all $k \in \mathbb{N}$ but this immediately follows from Cauchy's theorem.

Note that injectivity is clear from Theorem 2.1.0.1. To show surjectivity, let $f \in \tilde{A}$. We need to show that there is a function $u \in \mathcal{A}$ such that $u_{\mathbb{T}} = f$. Consider the function

$$u\left(re^{i\theta}\right) = \begin{cases} (P * f)(re^{i\theta}) & \text{if } 0 \leq r < 1 \\ f\left(e^{i\theta}\right) & \text{if } r = 1 \end{cases}$$

This is the Dirichlet problem on the unit disc! So, u is continuous on $\overline{\mathbb{D}}$. It remains to show that u is analytic on $\mathbb{D}$. But note that for $r \in [0,1)$,

$$\begin{aligned}u\left(re^{i\theta}\right) &= \sum_{n=-\infty}^{\infty} r^{|n|} \hat{f}(n)\, e^{int} \\ &= \sum_{n=0}^{\infty} r^{|n|} \hat{f}(n)\, e^{int}\end{aligned}$$

This completes the proof of the theorem! $\square$

In view of the previous theorem, we will simply write $\tilde{\mathcal{A}}$ as $\mathcal{A}$.

Definition 2.2.0.4. An analytic trignometric polynomial p on the circle $\mathbb{T}$ is of the form

$$p\left(e^{it}\right) = \sum_{k=0}^{n} a_k e^{ik\theta}$$

Proposition 2.2.0.5. *The set of the trignometric polynomials is a dense subset of $\mathcal{A}$.*

Proof. It is clear that any trignometric polynomial on the circle is a member of the disc algebra. Now, if $f : \mathbb{T} \to \mathbb{C}$ is in $\mathcal{A}$, then its negative Fourier coefficients are zero! Since, the Cesaro sum of f

$$s_n\left(x\right) = \sum_{k=-n}^{k=n} \hat{f}(n)e^{ikx} = \sum_{k=0}^{n} \hat{f}(n)e^{ikx}$$

converge to f uniformly and is a sequence of trignometric polynomial, we are done! $\square$

The following result is used in the proof of the next theorem, so, we prove it here:

Theorem 2.2.0.6. *The real parts of functions in $\mathcal{A}$ are uniformly dense in $\mathcal{C}\left(\mathbb{T}, \mathbb{R}\right)$. In other words, if μ is finite signed Borel measure on $\mathbb{T}$ such that $\int f\, d\mu = 0$ for every $f \in \mathcal{A}$ then μ is the zero measure.*

Proof. We first show that any trignometric polynomial of the form

$$p\left(e^{it}\right) = \sum_{k=-n}^{n} c_k e^{ikt} \tag{2.2.0.1}$$

where $c_{-k} = \overline{c_k}$ for each $k \in \{1, \ldots, n\}$ is a real part of a function $f \in \mathcal{A}$. Note that $p\left(e^{it}\right)$ in Equation 2.2.0.1 is the real part of the function:

$$f\left(e^{it}\right) = c_0 + 2c_1e^{it} + 2c_2e^{2it} + \ldots + 2c_ne^{int}$$

Now, we claim that every function $f \in C\left(\mathbb{T}, \mathbb{R}\right)$ is a uniform limit of a trignometric polynomial of the form 2.2.0.1. We will be done if we show that the negative Fourier coefficients of real valued function is the conjugate of the its positive counterpart, that is, for each $n \in \mathbb{Z}_{\geq 0}$, we have that $\hat{f}\left(-n\right) = \overline{\hat{f}\left(n\right)}$. To show this, take any $n \in \mathbb{Z}_{\geq 0}$ and then observe that

$$\begin{aligned}\hat{f}(-n) &= \int_{\mathbb{T}} f\left(e^{it}\right) e^{int} \frac{dt}{2\pi} \\ &= \overline{\int_{\mathbb{T}} f\left(e^{it}\right) e^{-int} \frac{dt}{2\pi}} \\ &= \overline{\hat{f}(n)}\end{aligned}$$

This shows that the Cesaro means of a real valued function is a trignometric polynomial of the form 2.2.0.1 and since the Cesaro means converges to f uniformly. Thus, the closure of the real parts of $\mathcal{A}$ is indeed $\mathcal{C}(\mathbb{T}, \mathbb{R})$.

Now, let μ be a finite signed measure on $\mathbb{T}$ such that $\int f \, d\mu = 0$ for every $f \in \mathcal{A}$. We show that μ is the zero measure. Notice that if $f \in \mathcal{A}$ then

$$0 = \int_{\mathbb{T}} f \, d\mu = \int_{\mathbb{T}} \Re(f) \, d\mu + i \int_{\mathbb{T}} \Im(f) \, d\mu$$

Hence, it follows that

$$\int_{\mathbb{T}} \Re(f) \, d\mu = 0$$

for every $f \in \mathcal{A}$. Now, if $g \in \mathcal{C}(\mathbb{T}, \mathbb{R})$ then by the first part of this theorem, there is a sequence $\{f_n\} \in \mathcal{A}$ such that $\Re(f_n)$ converges to g uniformly. By the Dominated Convergence Theorem (which holds, thanks to Jordan Decomposition Theorem), we have that $\int_{\mathbb{T}} g \, d\mu = 0$.

Now to prove that every $\mu = 0$, it suffices to show that $\hat{\mu}(n) = 0$ for every $n \in \mathbb{Z}$[1]. Now, notice that for any $n \in \mathbb{Z}$, we have

$$\begin{aligned}\hat{\mu}(n) &= \int_{\mathbb{T}} e^{-int} d\mu\left(e^{it}\right) \\ &= \int_{\mathbb{T}} \left(\cos(nt) - i \sin(nt)\right) d\mu\left(e^{it}\right) \\ &= \int_{\mathbb{T}} \cos(nt) \, d\mu\left(e^{it}\right) - i \int_{\mathbb{T}} \sin(nt) \, d\mu\left(e^{it}\right) \\ &= 0\end{aligned}$$

This completes the proof! □

[1] See Page 41, Corollary 2.3 of Mashreghi.

Corollary 2.2.0.7. *Let μ be a finite signed measure on $\mathbb{T}$ such that $\int f d\mu = 0$ for every $f \in \mathcal{A}$ which vanishes at the origin then μ is a constant multiple of Lebesgue measure.*

Proof. We first prove the following claim: If $f \in \mathcal{A}$ then $\int_{\mathbb{T}} f d\mu = \frac{f(0)}{2\pi}$. Since the negative Fourier cofficients are zero and f is continuous, we have that the Cesaro means converge uniformly to f, that is,

$$\sum_{k=0}^{\infty} \hat{f}\left(n\right) e^{int} \rightarrow f \text{ uniformly}$$

Thus,

$$\begin{aligned}
\int_{T} f\left(e^{it}\right) \frac{dt}{2\pi} &= \frac{1}{2\pi} \int_{\mathbb{T}} \left(\sum_{k=0}^{\infty} \hat{f}\left(n\right) e^{int} \right) dt \\
&= \frac{1}{2\pi} \sum_{k=0}^{\infty} \int_{\mathbb{T}} \left(\hat{f}\left(n\right) e^{int} dt \right) \\
&= \frac{\hat{f}(0)}{2\pi}
\end{aligned}$$

Now, we proceed to the proof. We define a measure $d\nu = d\mu - \frac{1}{2\pi}\mu\left(\mathbb{T}\right) dt$. Now, we have that

$$\begin{aligned}
\int_{\mathbb{T}} f\left(e^{it}\right) d\nu\left(e^{it}\right) &= \int_{\mathbb{T}} \left[f - f\left(0\right)\right]\left(e^{it}\right) d\mu\left(e^{it}\right) + f\left(0\right) \int_{\mathbb{T}} d\nu\left(e^{it}\right) \\
&= 0
\end{aligned}$$

Hence, we have that $d\mu = \frac{1}{2\pi}\mu\left(\mathbb{T}\right) dt$. $\square$

We will be working entirely on the $\mathbb{T}$. So, $\mathcal{A}$ and H^2 will be the spaces on the unit circle rather on the open unit disc.

Now, consider $\mathcal{A}$ as a subset of $L^2\left(\mathbb{T}, \mathscr{B}\left(\mathbb{T}\right), \mu\right)$ where μ is any finite positive measure. Let $\mathcal{A}_0 = \left\{ f \in \mathcal{A} : \int_{\mathbb{T}} f\left(e^{it}\right) \frac{dt}{2\pi} = \frac{\hat{f}(0)}{2\pi} = 0 \right\}$. It is easily seen that $\mathcal{A}_0$ is a subspace of $L^2\left(d\mu\right)$. Therefore, we have the closed subspace spanned by $\mathcal{A}$ is $[\![\mathcal{A}_0]\!] = \overline{\text{span}(\mathcal{A}_0)} = \overline{\mathcal{A}_0}$. By a theorem of Hilbert spaces, we have that there is some vector $F \in [\![\mathcal{A}_0]\!]$ such that

$$\inf_{f \in [\![\mathcal{A}_0]\!]} \int \left|1 - f^2\right| d\mu = \int \left|1 - F^2\right| d\mu$$

But since $d\left(1, \llbracket \mathcal{A}_0 \rrbracket\right) = d\left(1, \overline{\mathcal{A}_0}\right) \overset{2}{=} d\left(1, \mathcal{A}_0\right)$, we have

$$\inf_{f \in \mathcal{A}_0} \int \left|1 - f^2\right| d\mu = \int \left|1 - F^2\right| d\mu$$

Note that this F is the orthogonal projection of 1 into the closed subspace spanned by $\mathcal{A}_0$.

Theorem 2.2.0.8. *Let μ be a finite positive Borel measure on $\mathbb{T}$ and suppose that the constant function 1 is not in $\llbracket \mathcal{A}_0 \rrbracket$. Then let $f = P_{\llbracket \mathcal{A}_0 \rrbracket}(1)$. Then the following holds:*

1. *The measure $d\nu = \left|1 - F^2\right| d\mu$ is a nonzero constant multiple of the Lebesgue measure. In particular, Lebesgue measure is absolutely continuous with respect to μ.*
2. *The function $(1 - F)^{-1} \in H^2$.*
3. *If $h = \left(\frac{d\mu}{d\theta}\right)$ then $(1 - F)\, h \in L^2 = L^2\left(\frac{d\theta}{2\pi}\right)$.*

Proof. 1. Let $S = \llbracket \mathcal{A}_0 \rrbracket$. We begin to prove part one of the theorem. Let $F = P_S(1)$. Then we have by the uniqueness of the decomposition that

$$1 = \underbrace{F}_{P_S(1)} + \underbrace{1 - F}_{P_{S^\perp}(1)}$$

Thus, we have that $(1 - F)$ is orthogonal to every element in S and hence, in particular, any element in $\mathcal{A}_0$ (because $\mathcal{A}_0 \subset S \rightsquigarrow S^\perp \subset \mathcal{A}_0^\perp$). We claim that $1 - F$ is orthogonal to $(1 - F)\, f$ for every $f \in \mathcal{A}_0$. But before, we do this, we need to show that $(1 - F) f \in L^2(d\mu)$. Observe that

$$\int_{\mathbb{T}} \left|(1 - F) f^2\right| d\mu \leq \|f\|_\infty^2 \, \|1 - F\|_2^2 < \infty$$

To prove this, note that we showed that $S = \overline{\mathcal{A}_0}$ in the paragraph before the statement of this theoremand since $F \in S$, there is a sequence $\{f_n\} \in \mathcal{A}_0$ converging to F. Hence, we have that $\{f\,(1 - f_n)\}$ is a sequence in $\mathcal{A}_0$[3] converges to $f\,(1 - F)$ in the L^2-norm. Hence, we have that

[3] $\mathcal{A}_0$ is an algebra!

$$\begin{aligned}
\langle f(1-F),(1-F)\rangle &= \left\langle \lim_{n\to\infty} f\left(1-f_n\right),(1-F)\right\rangle \\
&= \lim_{n\to\infty} \left\langle f\left(1-f_n\right),(1-F)\right\rangle && \text{continuity of the inner product} \\
&= 0 && 1-F \text{ is orthogonal to } \mathcal{A}_0
\end{aligned}$$

Now, let $d\nu = |1-F|^2\, d\mu$. We have shown that for any $f \in \mathcal{A}_0$,

$$\int_{\mathbb{T}} f d\nu = \int_{\mathbb{T}} f\, |1-F|^2\, d\mu = \langle f(1-F),(1-F)\rangle = 0$$

Hence, by Corollary 2.2.0.7, we have that $d\mu = k\, d\lambda$ for some $k \geq 0$.

Now, we claim that this $k \neq 0$. If $k = 0$ then we would have that

$$\int_{\mathbb{T}} d\nu = 0 \rightsquigarrow \int_{\mathbb{T}} |1-F|^2\, d\mu = 0$$

Hence, we have that $F = 1$ μ-almost everywhere[4]. But then we have that $1 \in S$ which contradicts our assumption. Hence $k \neq 0$.

2. Observe that part 1 of the theorem tells us that

$$|1-F|^2\, d\mu = k\, d\lambda \text{ where } k \neq 0$$

Then we have that by Lebesgue Decomposition Theorem

$$d\mu = h d\lambda + d\mu_s$$

for some positive $\mathcal{L}^1$-function h and some singular measure μ_s. Hence, we have that

$$\begin{aligned}
|1-F|^2\, d\mu &= |1-F|^2\, (h d\lambda + d\mu_s) && (2.2.0.2)\\
&= \underbrace{|1-F|^2\, h d\lambda}_{(1)} + \underbrace{|1-F|^2\, d\mu_s}_{(2)} && (2.2.0.3)
\end{aligned}$$

By the uniqueness of the Lebesgue Decomposition Theorem (one needs to verify that the measures obtained in (1) and (2) are absolutely continuous and singular), we have that

$$\begin{aligned}
&|1-F|^2\, h = k \ \lambda\text{-almost everywhere.}\\
\rightsquigarrow\ &\frac{1}{|1-F|^2} = \frac{h}{k}\ \lambda\text{-almost everywhere.}
\end{aligned}$$

[4]See Corollary 2.3.12 of Cohn's Measure Theory.

This tells us that $(1-F)^{-1} \in L^2(d\lambda)$ where λ is the Lebesgue measure on $\mathbb{T}$. Also, note that $F = 1$ $d\mu_s$-almost everwhere. Hence, we have that

$$|1-F|^2 d\mu = |1-F|^2 h d\lambda = k d\lambda \rightsquigarrow d\mu = |1-F|^{-2} k d\lambda \tag{2.2.0.4}$$

Now, we proceed to show that the negative Fourier coefficients of $(1-F)^{-1}$ is 0 then we would have shown that $(1-F)^{-1} \in H^2$. Consider the following for $f \in \mathcal{A}_0$:

$$\begin{aligned} k\int_{\mathbb{T}} (1-F)^{-1} f d\lambda &= k\int_{\mathbb{T}} \left(1-\overline{F}\right) f |1-F|^{-2} d\lambda \\ &= k\int_{\mathbb{T}} \left(1-\overline{F}\right) f d\mu \\ &= 0 \end{aligned}$$

The last line follows from the fact that $1 - F \perp f$ for any $f \in \mathcal{A}_0$. Hence this holds for any $e^{in\theta}$ and hence we are done.

3. Note that the derivative of μ with respect to normalized Lebesgue meaures is h as demonstrated in Equation 2.2.0.3. Now, Equation 2.2.0.4 shows that

$$|1-F| h = k |1-F|^{-1} \ \lambda\text{-almost everywhere.}$$

Since $|1-F|^{-1} \in L^2(\lambda)$, so is $(1-F)h$.

□

Corollary 2.2.0.9. *Let μ be a finite positive measure on $\mathbb{T}$ with absolutely continuous part μ_a, then*

$$\inf_{f\in\mathcal{A}_0} \int_{\mathbb{T}} |1-f|^2 d\mu = \inf_{f\in\mathcal{A}_0} \int_{\mathbb{T}} |1-f|^2 d\mu_a$$

where μ_a is the absolutely continuous part of μ. In particular, for any singular measure μ, the function 1 is in the $L^2(d\mu)$ closure of $\mathcal{A}_0$.

Proof. Let F be the orthogonal projection of 1 into the closed subspace spanned by $\mathcal{A}_0$. Then we have that

$$\inf_{f\in\mathcal{A}_0} \int_{\mathbb{T}} |1-f|^2 d\mu = \|1-F\|^2 = \int_{\mathbb{T}} |1-F|^2 d\mu$$

From Equation 2.2.0.4, we see that

$$|1-F|^2 \, d\mu = |1-F|^2 \, h d\lambda = |1-F|^2 \, d\mu_a$$

where h is the Radon Nikodym derivative of μ. Integrating both sides, we have that

$$\inf_{f \in \mathcal{A}_0} \int_{\mathbb{T}} |1-f|^2 \, d\mu = \inf_{f \in \mathcal{A}_0} \int_{\mathbb{T}} |1-f|^2 \, d\mu_a$$

which is what we wanted to prove. Now if μ is a singular measure then we have that the absolutely continuous part of μ is the zero measure and hence we have that

$$\inf_{f \in \mathcal{A}_0} \int_{\mathbb{T}} |1-f|^2 \, d\mu = \|1-F\|^2 = 0$$

From the fact that $\|1-F\| = 0$, we conclude that $1 = F$ almost everywhere and hence 1 is in the closed subspace spanned by $\mathcal{A}_0$ which is in fact $\overline{\mathcal{A}_0}$. This completes the proof of the Corollary. □

Corollary 2.2.0.10. *Let μ be a finite complex Borel measure on $\mathbb{T}$ which is orthogonal to $\mathcal{A}_0$, that is, $\int_{\mathbb{T}} f d\mu = 0$ for all $f \in \mathcal{A}_0$. Then the absolutely continuous part and singular parts of μ are separately orthogonal to $\mathcal{A}_0$. That is, if μ_a and μ_s are the absolutely continuous and the singular parts of the measure μ then we have that*

$$\int_{\mathbb{T}} f d\mu_a = 0 \text{ and } \int_{\mathbb{T}} f d\mu_s = 0$$

for every $f \in \mathcal{A}_0$.

Proof. Let ρ be any finite positive measure satisfying

1. $\mu \ll \rho$ and $\frac{d\mu}{d\rho}$ is bounded, and

2. $\frac{d\rho}{d\theta} \geq \frac{1}{2\pi}$, that is, the Radon Nikodym derivative with respect to the normalised Lebesgue measure is bounded below by $1/2\pi$.

Let $d\mu = \frac{1}{2\pi} h d\theta + d\mu_s$ be the Lebesgue decomposition of the measure μ. We define a new measure on $\mathbb{T}$ by

$$d\rho = (1 + |h|) \frac{d\theta}{2\pi} + d\,|\mu_s|$$

where $|\mu_s|$ is the variation measure of the complex measure μ_s, the singular part of μ.

We show that ρ is an finite positive measure satisfying properties as in items 1 and 2 above. First, we show that ρ is a finite positive measure. But before that note that the variation measure of any complex measure is a finite measure.[5]. Now, consider the following:

$$\begin{aligned}
\rho(\mathbb{T}) &= \int_{\mathbb{T}} d\rho \\
&= \int_{\mathbb{T}} (1+|h|)\frac{d\theta}{2\pi} + \int_{\mathbb{T}} d|\mu_s| && \text{by definition} \\
&= 1 + |\mu_a|(\mathbb{T}) + |\mu_s|(\mathbb{T}) && \mu_a \text{ is the absolutely continuous part of } \mu \\
&< \infty
\end{aligned}$$

Now we proceed to show that item 1 holds for ρ. First, we prove that $\mu \ll \rho$. Let $A \in \mathscr{B}(\mathbb{T})$ with $\rho(A) = 0$. Then we have that

$$\int_A (1+|h|)\frac{d\theta}{2\pi} = 0 \text{ and}$$
$$\int_A d|\mu_s| = 0$$

From this, we have that

$$\begin{aligned}
|\mu_a|(A) &= \int_A |h|\frac{d\theta}{2\pi} \\
&\le \int_A (1+|h|)\frac{d\theta}{2\pi} \\
&= 0
\end{aligned}$$

Since $|\mu(A)| \le |\mu_a|(A)$, we have that $\mu_a(A) = 0$. Also, it follows by the definition of the variation measure that $\mu_s(A) = 0$. Hence, we have that μ is absolutely continuous with respect to ρ. Now, we need to show that $\frac{d\mu}{d\rho}$ is bounded. Needs work! We proceed to show that item 2 holds. Observe that

$$\frac{d\rho}{d\theta} = (1+|h|)\frac{1}{2\pi} \ge \frac{1}{2\pi}$$

and hence we are done showing that ρ is such a measure.

[5] See Cohn's Measure Theory – Proposition 4.1.7

Now, let $f \in \mathcal{A}_0$. Then we have that

$$\begin{aligned}
\int_{\mathbb{T}} |1-f|^2 \, d\rho &= \int_{\mathbb{T}} |1-f|^2 \frac{d\rho}{d\theta} d\theta + \int_{\mathbb{T}} |1-f|^2 \, d\mu_s \\
&\geq \int_{\mathbb{T}} |1-f|^2 \frac{d\rho}{d\theta} d\theta \\
&\geq \int_{\mathbb{T}} |1-f|^2 \frac{d\theta}{2\pi} \\
&= \|1-f\|^2 \\
&= \|1\|^2 + \|f\|^2 && \langle f, 1 \rangle = 0 \text{ as } f \in \mathcal{A}_0 \\
&\geq 1
\end{aligned}$$

Now, let F be the orthogonal projection of 1 into the closed subspace spanned by $\mathcal{A}_0$ of $L^2(d\rho)$. Thus, by definition, we have that

$$\int_{\mathbb{T}} |1-F|^2 = \inf_{f \in \mathcal{A}_0} \int_{\mathbb{T}} |1-f|^2 \, d\rho \geq 1$$

Hence, we have that 1 is not in the closed subspace spanned by $\mathcal{A}_0$ of $L^2(d\rho)$. Hence, we have by Theorem 2.2.0.8 that $(1-F)^{-1} \in H^2$ and that

$$(1-F)(1+|h|) \in L^2 = L^2\left(\frac{d\theta}{2\pi}\right)$$

From this, we can conclude that $h \in L^2$ by the following fashion:

$$|h| = \underbrace{(1-F)(1+|h|)}_{\in L^2} \underbrace{\left(|1-F|^{-1}\right)}_{\in H^2 \rightsquigarrow \in L^2} -1$$

Hence, we have that $(1-F)h \in L^2$.

Suppose $g \in \mathcal{A}_0$. We claim that

$$\int_{\mathbb{T}} (1-F) \, g d\mu = 0$$

□

Since $F \in \overline{\mathcal{A}}_0$, there is a sequence $f_n \in \mathcal{A}_0$ such that $f_n \to F$ in $L^2(d\rho)$. Hence,

$$(1-f_n)g \to (1-F)\, g$$

in $L^2(d\rho)$.

Since $\frac{d\mu}{d\rho}$ is bounded, we have that

$$\int_{\mathbb{T}} (1-F)\, g d\mu = \int_{\mathbb{T}} (1-F)\, g \frac{d\mu}{d\rho} d\rho = \lim_{n\to\infty} \int_{\mathbb{T}} (1-f_n) \frac{d\mu}{d\rho} d\rho = 0.$$

Now, $1-F=0$ with respect to $d|\mu_s|$ which implies that $1-F=0$ with respect to $d\mu_s$.

Thus, we have that

$$(1-F)\, d\mu = \frac{1}{2\pi}(1-F)\, h d\theta$$

Hence, we have for $g \in \mathcal{A}_0$

$$\begin{aligned}\int_{\mathbb{T}} (1-F)\, g h d\theta &= \int_{\mathbb{T}} (1-F) g d\mu \\ &= 0\end{aligned}$$

Since $\overline{A} = H^2$. There is a sequence of functions g_n in $\mathcal{A}$ converging to $(1-F)^{-1} \in L^2\left(\frac{d\theta}{2\pi}\right)$ in L^2-norm. Hence, we have for any $f \in \mathcal{A}_0$,

$$\int_{\mathbb{T}} g_n f\, (1-F)\, h d\theta = 0$$

Also, note that $(1-F)\, h \in L^2$. From this, we can conclude that

$$\int_{\mathbb{T}} f h d\theta = 0$$

Theorem 2.2.0.11 (Frigyez and Marcel Riesz, 1916)**.** *Let μ be a complex Borel measure on the unit circle $\mathbb{T}$ such that*

$$\int_{\mathbb{T}} e^{in\theta} d\mu\left(e^{i\theta}\right) = 0$$

for all $n \in \mathbb{N}$. Then μ is absolutely continuous with respect to the Lebesgue measure.

Proof. Suppose that μ is orthogonal to $\mathcal{A}_0$. Then from the previous corollary, we have that μ_a is orthogonal to $\mathcal{A}_0$ and μ_s is orthogonal to $\mathcal{A}_0$. Since $|\mu_s|$ is singular (definition?), we can find a sequence of functions $\{f_n\}$ converging to 1 in $L^2(d|\mu_s|)$. Since μ_s is orthogonal to $\mathcal{A}_0$, we have that

$$\int_{\mathbb{T}} d\mu_s = \lim_{n\to\infty}\int_{\mathbb{T}} f_n d\mu_s = 0$$

Thus, μ_s is orthogonal to 1. The singular measure $e^{-i\theta}d\mu_s$ is now orthogonal to $\mathcal{A}_0$; hence it is orthogonal to 1, that is,

$$\int_{\mathbb{T}} e^{-i\theta}d\mu_s = 0$$

Similarly, $e^{-2i\theta}d\mu_s$ is orthogonal to $\mathcal{A}_0$, and consequently, orthogonal to 1. Repeating thei process, we conclude that

$$\int_{\mathbb{T}} e^{in\theta} = 0$$

for $n \in \mathbb{Z}$. Hence μ_s must be the zero measure. Therefore our original measure μ is absolutely continuous. $\square$

2.3 Szegö's Theorem