# Beurling Factorisation of Hardy Spaces

## Final Presentation

**by** Ashish Kujur
**on** 16th May, 2023

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* *nth Fourier coefficient* of $f \in \mathcal{L}^1(\mathbb{T})$ and $\mu \in \mathcal{M}(\mathbb{T})$, $n \in \mathbb{Z}$:

$$\hat{f}(n) := \frac{1}{2\pi} \int_{-\pi}^{\pi} f\left(e^{it}\right) e^{-int} dt.$$

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Define $H^p(\mathbb{D})$, *Hardy class of analytic functions*, by

$$H^p(\mathbb{D}) = \left\{ f \in H(\mathbb{D}) : \left\{ \|f_r\|_p \right\}_{0 \leq r < 1} \text{ is bounded} \right\}$$

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**Theorem (G.H. Hardy, 1915)**

*If $f \in H^p(\mathbb{D})$ then $\|f_{r_1}\|_p \leq \|f_{r_2}\|_p$ for $0 < r_1 \leq r_2 < 1$.*

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**Theorem**

*For $1 \leq p < \infty$, $H^p(\mathbb{D})$ is a Banach space with the norm*

$$\|f\|_p := \sup_{0 < r < 1} \|f_r\|_p = \lim_{r \to 1} \|f_r\|_p .$$

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**Question**

*$H^p(\mathbb{D})$ and $H^p(\mathbb{T})$ are both Banach spaces. Are they related?*

YES!

## » Poisson Kernel & Integral (Recover!)

**Definition (Poisson Kernel)**

For each $r \in [0,1)$, we define $P_r : \mathbb{T} \to \mathbb{R}$ by

$$P_r\left(e^{it}\right) = \frac{1 - r^2}{1 + r^2 - 2r\cos t}$$

## » Poisson Kernel & Integral (Recover!)

**Definition (Poisson Integral)**

Let $\mu \in \mathcal{M}(\mathbb{T})$ and $f \in L^1(\mathbb{T})$. Then Poisson integral of $\mu$, denoted by $P[\mu] : \mathbb{D} \to \mathbb{C}$ is given by

$$P[\mu]\left(re^{i\theta}\right) = \int_{\mathbb{T}} P_r\left(e^{i(\theta-t)}\right) d\mu\left(e^{it}\right)$$

and Poisson integral of $f$, denoted by $P[f] : \mathbb{D} \to \mathbb{C}$ is given by

$$P[f]\left(re^{i\theta}\right) = \frac{1}{2\pi}\int_{-\pi}^{\pi} P_r\left(e^{i(\theta-t)}\right) f\left(e^{i\theta}\right) dt.$$

# » Representation Theorems (Recovery)

Let $u \in h(\mathbb{D})$. Then $u$ is a Poisson integral of

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