H^p spaces

A Study of H^p spaces and Inner Outer Factorization of functions in H^p spaces

by Ashish Kujur (MSC21304, Indian Institute of Science Education and Research, Thiruvananthapuram)

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Notations

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- *nth Fourier coefficient* of $f \in \mathcal{L}^1(\mathbb{T})$ and $\mu \in \mathcal{M}(\mathbb{T})$, $n \in \mathbb{Z}$:

$$\hat{f}(n) := \frac{1}{2\pi} \int_{-\pi}^{\pi} f\left(e^{it}\right) e^{-int} dt.$$

$$\hat{\mu}(n) := \int_{\mathbb{T}} e^{-int} d\mu\left(e^{it}\right).$$

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Hardy Spaces

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For $0 \leq r < 1$, define $f_r : \mathbb{T} \to \mathbb{C}$, $f_r\left(e^{i\theta}\right) = f\left(re^{i\theta}\right)$ for each $e^{i\theta} \in \mathbb{T}$.

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Define $H^p(\mathbb{D})$, *Hardy class of analytic functions*, by

$$H^p(\mathbb{D}) = \left\{ f \in H(\mathbb{D}) : \left\{ \|f_r\|_{L^p(\mathbb{T})} \right\}_{0 \leq r < 1} \text{ is bounded} \right\}$$

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Theorem ([Har15])

If $f \in H^p(\mathbb{D})$ then $\|f_{r_1}\|_{L^p(\mathbb{T})} \leq \|f_{r_2}\|_{L^p(\mathbb{T})}$ for $0 < r_1 \leq r_2 < 1$.

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Theorem

For $1 \le p \le \infty$, $H^p(\mathbb{D})$ is a Banach space with the norm

$$\|f\|_{H^p(\mathbb{D})} := \sup_{0<r<1} \|f_r\|_{L^p(\mathbb{T})} = \lim_{r \to 1} \|f_r\|_{L^p(\mathbb{T})}.$$

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Definition (Poisson Kernel)

For each $r \in [0,1)$, we define $P_r : \mathbb{T} \to \mathbb{R}$ by

$$P_r\left(e^{it}\right) = \frac{1-r^2}{1+r^2-2r\cos t}$$

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* The solution to the Dirichlet problem on the $\mathbb{D}$ is:

$$u\left(re^{i\theta}\right) = \begin{cases} \frac{1}{2\pi}\int_{-\pi}^{\pi} P_r\left(e^{i(\theta - t)}\right) f\left(e^{it}\right) dt & re^{i\theta} \in \mathbb{D} \\ f\left(e^{i\theta}\right) & e^{i\theta} \in \mathbb{T} \end{cases}$$

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Let $\mu \in \mathcal{M}(\mathbb{T})$ and $f \in L^1(\mathbb{T})$. Then Poisson integral of μ, denoted by $P[\mu] : \mathbb{D} \to \mathbb{C}$ is given by

$$P[\mu]\left(re^{i\theta}\right) = \int_{\mathbb{T}} P_r\left(e^{i(\theta-t)}\right) d\mu\left(e^{it}\right)$$

and Poisson integral of f, denoted by $P[f] : \mathbb{D} \to \mathbb{C}$ is given by

$$P[f]\left(re^{i\theta}\right) = \frac{1}{2\pi}\int_{-\pi}^{\pi} P_r\left(e^{i(\theta-t)}\right) f\left(e^{i\theta}\right) dt.$$

» Fatou's theorem (1906)

Corollary ([Fat06])

Let $\mu \in \mathcal{M}(\mathbb{T})$. Then

$$\lim_{r \to 1} P[\mu]\left(re^{i\theta}\right)$$

exists for almost all $e^{i\theta} \in \mathbb{T}$ and equals "the" Radon Nikodym derivative of the absolutely continuous part of μ with respect to the Lebesgue measure. As a consequence, we have that if $f \in L^1(\mathbb{T})$ then

$$\lim_{r \to 1} P[f]\left(re^{i\theta}\right) = f\left(e^{i\theta}\right)$$

for almost all $e^{i\theta} \in \mathbb{T}$.

» Interaction of $\mathbb{D}$ and $\mathbb{T}$

Let $u : \mathbb{D} \to \mathbb{C}$ be a harmonic function and $1 \leq p \leq \infty$. Suppose that for all $0 \leq r < 1$, we have that

$$\|u_r\|_p < M < +\infty$$

for some $M > 0$.

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$$\tilde{u}(e^{i\theta}) = \lim_{r \to 1} u\left(re^{i\theta}\right)$$

exist and define a function $\tilde{u}$ in $L^p\left(\mathbb{T}\right)$. The following also holds:

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1. If $p > 1$ then $u = P[\tilde{u}]$.
2. If $p = 1$ then $f = P[\mu]$ for some complex measure μ whose absolutely continuous part is $\frac{1}{2\pi}\tilde{u}dt$.

» Answering the Question $p > 1$

For $p > 1$, consider the map

$$\begin{aligned} H^p(\mathbb{T}) &\to H^p(\mathbb{D}) \\ u &\mapsto P[u] \end{aligned}$$

This is an isometric isomorphism.

» Answering the Question $p = 1$

For $p = 1$, consider the map

$$\begin{aligned} \mathcal{M}_a(\mathbb{T}) &\to H^1(\mathbb{D}) \\ \mu &\mapsto P[\mu] \end{aligned}$$

This turns out to be a isometric isomorphism.

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Theorem (F and M Riesz (1916))

Let $\mu \in \mathcal{M}_a(\mathbb{T})$ then μ is absolutely continuous.

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Theorem (F and M Riesz (1916))

Let $\mu \in \mathcal{M}_a(\mathbb{T})$ then μ is absolutely continuous.

$$\begin{aligned} H^1(\mathbb{T}) &\to H^1(\mathbb{D}) \\ f &\mapsto P[f] \end{aligned}$$

is again an isometric isomorphism.

H^1 is nice!

» Szegő's theorem

Theorem (Szegő)

Let $f \in H^1(\mathbb{T})$, $f \not\equiv 0$. *Then the function* $\log|f|$ *is integrable and*

$$\frac{1}{2\pi}\int_{\mathbb{T}} \log\left|f\left(e^{it}\right)\right| dt \geq \log\left|f\left(e^{i0}\right)\right|$$

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Corollary

Let $f \in H^1(\mathbb{T})$. If $f \not\equiv 0$ then f cannot vanish on a (measurable) subset of $\mathbb{T}$ with positive Lebesgue measure.

The Factorization

» Inner Function

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Definition (Inner Function)

Let $f : \mathbb{D} \to \mathbb{C}$. Then f is said to be an *inner function* if $f \in H^\infty(\mathbb{D})$ and the corresponding boundary function $\tilde{f} \in H^\infty(\mathbb{T})$ has unit modulus almost everywhere on $\mathbb{T}$. In other words, the function $\tilde{f}$ defined almost everywhere on $\mathbb{T}$ by

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Finite Blaschke Product

Let $z_1, z_2, \ldots, z_n \in \mathbb{D}$ and $\alpha \in \mathbb{R}$. Then the finite Blaschke product is the function given by

$$B(z) = e^{i\alpha} \prod_{k=1}^{n} \frac{z - z_j}{1 - \bar{z}_j z}.$$

Outer Function

Definition (Outer Function)

An *outer function* is a holomorphic function $f\colon \mathbb{D} \to \mathbb{C}$ of the form

$$f\left(re^{it}\right) = \alpha \exp\left[\frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{e^{it} + re^{i\theta}}{e^{it} - re^{i\theta}} k\left(e^{it}\right)\, dt\right]$$

where $\alpha \in \mathbb{C}$ with $|\alpha| = 1$ and k is a real valued integrable function on $\mathbb{T}$.

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where $\alpha \in \mathbb{C}$ with $|\alpha| = 1$ and k is a real valued integrable function on $\mathbb{T}$.

Proposition

Let $f : \mathbb{D} \to \mathbb{C}$ be an outer function with above form. Then

$$f \in H^1(\mathbb{D}) \Leftrightarrow e^k \in L^1(\mathbb{T})$$

» Inner Outer Factorization

Theorem ([Beu49])

Let $f \in H^1(\mathbb{D})$ and $f \not\equiv 0$. Then f has a factorization $\theta \cdot u$ where θ is inner and u is outer. This factorization is unique up to a constant of modulus 1.

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Idea of Proof: Define

$$u(z) = \exp\left(\frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{e^{it}+z}{e^{it}-z} \log\left|\tilde{f}\left(e^{it}\right)\right| dt\right)$$

$$\theta = \frac{f}{u}.$$

» Disintegrating *Inner* part

Theorem (nonzero H^1 functions satisfy the Blaschke condition)

Let $f \in H^1(\mathbb{D})$ and $f \not\equiv 0$. Then the zeroes of f are countable in number and satisfy the ***Blaschke condition****, that is, if $z_1, z_2, \ldots$ are the zeroes of f, then*

$$\sum_{k=1}^{\infty} (1 - |z_n|) < \infty \Longleftrightarrow \prod_{n=1}^{\infty} |z_n| < \infty$$

Infinite Blaschke Products

Theorem ([Bla15])

Let $\{z_n\}_{n\in\mathbb{N}} \subset \mathbb{D} \setminus \{0\}$ be a sequence. The infinite product

$$B(z) = \prod_{n=1}^{\infty} \frac{\bar{z}_n}{z_n} \frac{z_n - z}{1 - \bar{z}_n z}$$

converges uniformly on compact subsets of $\mathbb{D}$ iff the product $\prod_{n=1}^{\infty} |z_n|$ converges iff

$$\sum_{n=1}^{\infty} (1 - |z_n|) < \infty.$$

When either of these is satisfied, B defines an inner function whose zeroes are $\{z_n : n \in \mathbb{N}\}$.

» Infinite Blaschke Products

Definition

An *(infinite) Blaschke product* is a holomorphic function B of the form

$$B(z) = z^p \prod_{n=1}^{\infty} \left[\frac{\bar{z}_n}{|z_n|} \cdot \frac{z_n - z}{1 - \bar{z}_n z} \right]^{p_n}$$

where

1. $p, p_1, p_2, \ldots \in \mathbb{N}$;
2. $\{z_n : n \in \mathbb{N}\} \subset \mathbb{D} \setminus \{0\}$;
3. the product $\prod_{n=1}^{\infty} |z_n|^{p_n}$ is convergent.

Corollary (Factoring all zeroes of $H^1(\mathbb{D})$)

The Blaschke product formed out of the zeroes of a nonzero $H^1(\mathbb{D})$ function is an inner function.

» So far...

Let $f \in H^1(\mathbb{D})$, $f \not\equiv 0$. Let

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be its inner outer factorisation where θ is inner and u is outer. The outer part u has no zeroes of F, so, θ has all the zeroes. Let B be the Blaschke product formed out of zeroes of u (which is same as that of f). Is

$$\theta / B$$

another inner function?

» Riesz Decomposition Theorem

Theorem ([Rie23])

Let $f \in H^p(\mathbb{D})\,, 1 \leq p \leq \infty$, $f \not\equiv 0$ and let B be the Blaschke product formed with the zeroes of f in $\mathbb{D}$. Let

$$g = f/B$$

Then $g \in H^p(\mathbb{D})$, g is zerofree in $\mathbb{D}$ and

$$\|g\|_p = \|f\|_p\,.$$

» Singular Inner Function

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Any nonzero H^1 function f can be written as $B \cdot S \cdot u$ where B is a Blaschke product, S is a zerofree inner function and u is the outer part of f. This representation is *unique* upto multiplication by unimodular constant.

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Definition (Singular Inner function)

A inner function S which is zerofree and $S(0) > 0$ is called singular function.

Theorem ([Her11])

Let g be a singular inner function. Then there is a unique singular positive measure μ such that

$$g(z) = \exp\left[-\int_{\mathbb{T}} \frac{e^{it}+z}{e^{it}-z} d\mu\left(e^{it}\right)\right]$$

» The Factorization Theorem

Let $f \not\equiv 0$ be an H^1 function in the unit disc. Then f is uniquely expressible in the form of $f = B \cdot S \cdot u$ where B is a Blaschke product, S is a singular inner function and u is an outer function (in H^1).

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Let p be the order of zero of f at the origin and let $p_1, p_2, \ldots$ be the multiplicities of the remaining zeroes $\alpha_1, \alpha_2, \ldots$ of f.

Then we have that

$$B(z) = z^p \prod_{n=1}^{\infty} \left[\frac{\overline{\alpha_n}}{|\alpha_n|} \frac{\alpha_n - z}{1 - \overline{\alpha_n} z} \right]^{p_n}$$

$$u(z) = \exp\left[\frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{e^{i\theta} + z}{e^{i\theta} - z} \left(\log \left| \tilde{u}\left(e^{i\theta}\right) \right| + ia \right) d\theta \right]$$

$$S(z) = \frac{f(z)}{B(z)\, u(z)} = \exp\left[- \int \frac{e^{i\theta} + z}{e^{i\theta} - z} d\mu(\theta) \right]$$

for some positive singular measure μ and where $a = \arg(f/B)(0)$.

» References

[Ash14] Robert B Ash. *Complex variables*. Academic Press, 2014.

[Beu49] Arne Beurling. "On two problems concerning linear transformations in Hilbert space". In: (1949).

[Bla15] Wilhelm Blaschke. "Eine erweiterung des satzes von vitali über folgen analytischer funktionen". In: *Leipzig Ber* 67 (1915), pp. 194–200.

[Coh13] Donald L Cohn. *Measure theory*. Vol. 1. Springer, 2013.

[Fat06] Pierre Fatou. "Séries trigonométriques et séries de Taylor". In: *Acta mathematica* 30.1 (1906), pp. 335–400.

[Har15] Godfrey H Hardy. "The mean value of the modulus of an analytic function". In: *Proceedings of the London Mathemathical Society* 2.1 (1915), pp. 269–277.

» References (cont.)

[Her11] Gustav Herglotz. "Uber potenzreihen mit positivem, reelen teil im einheitskreis". In: *Ber. Verhandl. Sachs Akad. Wiss. Leipzig, Math.-Phys. Kl.* 63 (1911), pp. 501–511.

[Hof07] Kenneth Hoffman. *Banach spaces of analytic functions*. Courier Corporation, 2007.

[Mas09] Javad Mashreghi. *Representation theorems in Hardy spaces*. Vol. 74. Cambridge University Press, 2009.

[Rie23] Freidrich Riesz. "Über die Randwerte einer analytischen Funktion". In: *Mathematische Zeitschrift* 18.1 (1923), pp. 87–95.

Thank You!