Study of Inner Outer Factorization of Hardy Spaces

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Chapter 1

Analytic and Harmonic Functions

1.1 The Cauchy and Poisson Kernels

Proposition 1.1.0.1. *Let $u : \overline{\mathbb{D}} \to \mathbb{C}$ be a harmonic function. Then we have that*

$$u\left(re^{i\theta}\right) = \frac{1}{2\pi} \int_{-\pi}^{\pi} u\left(e^{it}\right) P_r\left(e^{i(\theta-t)}\right) dt \tag{1.1.0.1}$$

1.2 Boundary Values

1.2.1 Weak* convergence of measures

Theorem 1.2.1.1. *Let $\{\varphi_i\}_i$ be an approximate identity on $\mathbb{T}$ and let $\mu \in \mathcal{M}(\mathbb{T})$. Then for all i, $\varphi_i * \mu \in L^1(\mathbb{T})$ with*

$$\|\varphi_i * \mu\|_1 \leq C_\varphi \|\mu\|$$

and

$$\|\mu\| \leq \sup_i \|\varphi_i * \mu\|_1 .$$

Moreover, the measures $d\mu_i = (\varphi_i * \mu)(e^{it})\, dt/2\pi$ *converge to* $d\mu(e^{it})$ *in the weak* topology, i.e.*

$$\lim_i \frac{1}{2\pi} \int_{-\pi}^{\pi} f\left(e^{it}\right) (\varphi_i * \mu)\left(e^{it}\right) dt = \int_{\mathbb{T}} \varphi\left(e^{it}\right) d\mu\left(e^{it}\right)$$

for all $f \in \mathcal{C}(\mathbb{T})$.

1.2.2 Convergence in norm

Theorem 1.2.2.1. *Let* $\{\varphi_i\}_i$ *be an approximate identity on* $\mathbb{T}$ *and let* $f \in L^p(\mathbb{T})$ *with* $p \in [1, \infty)$. *Then for all* i, $\varphi_i * f \in L^p(\mathbb{T})$ *with*

$$\|\varphi_i * f\|_p \leq C_\varphi \|f\|_p$$

and

$$\lim_i \|\varphi_i * f - f\|_p = 0.$$

1.2.3 Weak* convergence of bounded functions

Theorem 1.2.3.1. *Let* $\{\varphi_i\}_i$ *be an approximate identity on* $\mathbb{T}$ *and let* $f \in L^\infty(\mathbb{T})$. *Then for all* i, $\varphi_i * \mu \in \mathcal{C}(\mathbb{T})$ *with*

$$\|\varphi_i * \mu\|_\infty \leq C_\varphi \|\mu\|_\infty$$

and

$$\|f\|_{+\infty} \leq \sup_i \|\varphi_i * f\|_\infty .$$

Moreover, $\varphi_i * f$ *converge to* f *in the weak* topology, i.e.*

$$\lim_i \int_{-\pi}^{\pi} g\left(e^{it}\right) (\varphi_i * f)\left(e^{it}\right) dt = \int_{\mathbb{T}} g\left(e^{it}\right) f\left(e^{it}\right) dt$$

for all $g \in L^1(\mathbb{T})$.

1.2.4 The entire picture!

Definition 1.2.4.1 (Poisson integral of some function or measure)**.** Let $\tilde{f} : \mathbb{D} \to \mathbb{C}$ be a harmonic function. Then $\tilde{f}$ is said to be the *Poisson integral* of the function $f : \mathbb{T} \to \mathbb{C}$ if

$$\tilde{f}(re^{i\theta}) = \frac{1}{2\pi} \int_T f\left(e^{it}\right) P_r\left(e^{i(\theta-t)}\right) dt$$

In such a case, we will denote the function $\tilde{f}$ by $P[f]$. Similarly, f is said to be the *Poisson integral* of a complex measure μ on T if

$$\tilde{f}(re^{i\theta}) = \frac{1}{2\pi} \int_T P_r\left(e^{i(\theta-t)}\right) d\mu\left(e^{it}\right)$$

In such a case, we will denote the function $\tilde{f}$ by $P[\mu]$.

Theorem 1.2.4.2 (Ultimate Convergence)**.** *Let* $f : \mathbb{D} \to \mathbb{C}$ *be a harmonic function. Define for each* $r \in [0,1)$, *the function* $f_r : \mathbb{T} \to \mathbb{C}$ *by*

$$f_r\left(e^{i\theta}\right) = f\left(re^{i\theta}\right)$$

The following statements holds:

1. *If* $1 < p \leq \infty$ *then* $f = P[g]$ *for some* $g \in L^p[g]$ *iff for each* $r > 0$, $\|f_r\|_p < +\infty$.

2. *If p=1 then* $f = P[g]$ *for some* $g \in L^p[g]$ *iff* f_r *converge in the* L^1 *norm.*

3. $f = P[\mu]$ *for some* $\mu \in \mathcal{M}(\mathbb{T})$ *iff for each* $r > 0$, $\|f_r\|_1 < +\infty$

1.3 Fatou's Theorem

Theorem 1.3.0.1. *Let μ be a complex measure on the unit circle $\mathbb{T}$, and let $f : \mathbb{D} \to \mathbb{C}$ be the harmonic function defined by*

$$f(re^{i\theta}) = \frac{1}{2\pi} \int_{\mathbb{T}} P_r \left(e^{i(\theta - t)}\right) d\mu \left(e^{it}\right)$$

Let $e^{i\theta_0}$ be any point where μ is differentiable with respect to the normalised Lebesgue measure. Then

$$\lim_{r \to 1} f\left(re^{i\theta_0}\right) = \left(\frac{d\mu}{d\theta}\right)\left(e^{i\theta_0}\right) = \mu'\left(e^{i\theta_0}\right)$$

In fact, $f(re^{i\theta}) \to \mu'\left(e^{i\theta_0}\right)$ as $re^{i\theta}$ approaches $e^{i\theta_0}$ along any path in the open disc within the region of the form $|\theta - \theta_0| \leq c\,(1 - r)$ for some $c > 0$.

Corollary 1.3.0.2. *Let μ be a complex measure on $\mathbb{T}$. Then $P[\mu]$ has nontangential limits equal everywhere to the Radon Nikodym derivative of μ with respect to the normalised Lebesgue measure.*

Corollary 1.3.0.3. *Let $f : \mathbb{T} \to \mathbb{C}$ be L^1. Then $P[f]$ has nontangential limits at almost everywhere and these limits equal to f almost everywhere.*

Corollary 1.3.0.4. *Let $f : \mathbb{D} \to \mathbb{C}$ be a harmonic function and $1 \leq p < \infty$. Suppose that for all $0 \leq r < 1$, we have that*

$$\|f_r\|_p < +\infty$$

Then for almost every θ the radial limits

$$\tilde{f}(e^{i\theta}) = \lim_{r \to 1} f\left(re^{i\theta}\right)$$

exist and define a function $\tilde{f}$ in $L^p(\mathbb{T})$. The following also holds:

1. *If $p > 1$ then $f = P[\tilde{f}]$.*

2. *If $p = 1$ then $f = P[\mu]$ for some complex measure μ whose absolutely continuous part is $f d\theta$.*

3. *IF f is bounded then the boundary values exist almost everywhere and define a bounded measurable function $\tilde{f}$ on $\mathbb{T}$ such that $f = P[\tilde{f}]$.*

Proof. Suppose that for each $r \in [0,1)$, we have $\|f_r\|_p < +\infty$. We need to prove that for almost every θ, $\lim_{r\to 1} f\left(re^{i\theta}\right)$ exists. Then by Theorem 1.2.4.2, we have that $f = P[g]$ for some $g \in L^p(\mathbb{T})$. Since $L^p(\mathbb{T}) \subset L^1(\mathbb{T})$, we can use the previous corollary. By the previous corollary, we have that $P[g]$ has nontangential limits almost everywhere, we have that

$$\tilde{f}\left(e^{i\theta}\right) = \lim_{r\to 1} f(re^{i\theta}) = \lim_{r\to 1} P[g]\left(re^{i\theta}\right) \tag{1.3.0.1}$$

exists almost everywhere.

Now we proceed to prove part (1). Also by Theorem 1.2.4.2, we have that $f = P[g]$ for some $g \in L^p(\mathbb{T})$. Hence, we have that by Equation 1.3.0.1 that $\tilde{f}(e^{i\theta}) = \lim_{r\to 1} P[g]\left(re^{i\theta}\right)$ holds at almost every θ.

Also, by the previous corollary, $\lim_{r\to 1} P[g]\left(re^{i\theta}\right) = g(e^{i\theta})$ for almost every θ. Hence, we have that $\tilde{f} = g$.

$\ddot\smile$

Corollary 1.3.0.5. *Let $f : \mathbb{D} \to \mathbb{R}_{\geq 0}$ be a harmonic function. Then f has nontangential limits at almost every point of $\mathbb{T}$. (Why demand nonnegative?)*

Let $h(\mathbb{D})$ denote the set of all harmonic functions on $\mathbb{D}$. Let $p \in [1,\infty]$. Define

$$h^p(\mathbb{D}) = \left\{ f \in h(\mathbb{D}) \mid \{f_r\}_{0\leq r<1} \text{ is uniformly bounded in } L^p \text{ norm } \right\}$$

We define a norm on $h^p(\mathbb{D})$ by

$$\|f\|_{h^p(\mathbb{D})} = \sup_{0\leq r<1} \|f_r\|_{L^p(\mathbb{D})} = \begin{cases} \sup_{0\leq r<1} \left(\frac{1}{2\pi}\int_{-\pi}^{\pi} \left|f\left(re^{i\theta}\right)\right|^p d\theta\right)^{\frac{1}{p}} & \text{if } p \in [1,\infty) \\ \sup_{0\leq r<1} \left\|f(re^{i\theta})\right\|_{L^\infty(\mathbb{D})} & \text{if } p = \infty \end{cases}$$

It is easy to see why $\|f\| < +\infty$ for any $f \in h^p(D)$. So we now proceed to show that $h^p(D)$ is a Banach space with this norm. First we show that it is indeed a normed linear space.

Clearly, $h(\mathbb{D})$ is a vector space. To show that $h^p(\mathbb{D})$ is a vector space, it suffices to check that $h^p(\mathbb{D})$ is a subspace.

Let $f, g \in h^p(\mathbb{D})$ and let $\alpha \in \mathbb{C}$. Then for any $r \in [0,1)$, we have that

$$\begin{aligned} \|(f+\alpha g)_r\|_p &= \|f_r + \alpha g_r\| \\ &= \|f_r\|_p + \alpha \|g_r\|_p \end{aligned}$$

Take note of the use of Holder's inequality. After this is done, since $\{f_r\}_{r\in[0,1)}$ and $\{g_r\}_{r\in[0,1)}$ is uniformly bounded, we have that $\{f+\alpha g\}_{r\in[0,1)}$ is uniformly bounded in L^p norm.

Now, we need to show that it is a normed linear space but this follows almost immediately.

To show that it is a Banach space, we show that

Theorem 1.3.0.6. *Let $p \in [1,\infty]$. If $u \in L^p(\mathbb{T})$ then $f = P * u \in h^p(\mathbb{D})$ and $\|f\|_p = \|u\|_p$. If $\mu \in \mathcal{M}(\mathbb{T})$ then $f = P * \mu \in h^1(\mathbb{D})$ and $\|f\|_1 = \|\mu\|$.*

Proof. We consider the case $p \in [1,\infty)$. The other cases can be dealt similarly. Consider the map

$$u \overset{T}{\mapsto} U$$

where $U\left(re^{i\theta}\right) = \frac{1}{2\pi}\int_{-\pi}^{\pi} P_r\left(e^{i(\theta-t)}\right) u\left(e^{it}\right) dt$. By Theorem 1.2.2.1, we have that $\|U\| = \|u\|_p < +\infty$. Hence $U \in h^p(\mathbb{D})$.

Linearity is obvious. We need to check injectivity and surjectivity.

To check injectivity, let $u \in L^p(\mathbb{T})$ and suppose that $T(u) = P[u] = 0$. Now $\lim_{r\to 1} P[u]\left(re^{i\theta}\right) = u$ for almost θ by Corollary 1.3.0.3 and hence $u = 0$ almost everywhere.

Surjectivity is clear from Theorem 1.2.4.2. $\ddot\smile$

1.4 Hardy Spaces – H^p spaces

Let us denote the set of all analytic functions on $\mathbb{D}$ by $H(\mathbb{D})$. Hence, $H(\mathbb{D}) \subset h(\mathbb{D})$. For $p \in (0, \infty]$, we consider the *Hardy classes* of analytic functions on the unit disc

$$H^p(\mathbb{D}) = \left\{ F \in H(\mathbb{D}) \mid \|F\|_p < \infty \right\}$$

Clearly,

$$H^p(\mathbb{D}) \subset h^p(\mathbb{D})$$

We will see that $H^p(\mathbb{D})$, $1 \leq p \leq +\infty$, is also a Banach spaces isomorphic to a closed subspace of $L^p(\mathbb{T})$ denotes by $H^p(\mathbb{T})$.

To prove that $H^p(\mathbb{D})$ is a closed subspace of $h^p(\mathbb{D})$, we are going to identify $H^p(\mathbb{D})$ with the closed subspace

$$\left\{ u \in L^p(\mathbb{T}) : \int_{-\pi}^{\pi} u\left(e^{it}\right) e^{ikt} = 0 \textit{ for all } k \in \mathbb{N} \right\}$$

Let $\{u_n\}$ be a sequence of functions in the above subspace; suppose that $\{u_n\}$ converge to $u \in L^p(\mathbb{T})$. Now, let $k \in \mathbb{N}$ be arbitrary. Since $\{u_n\}$ converge to u in p-norm, we have that $\{u_n\}$ converge to u in 1-norm. Hence we have the following:

$$\left| \int_{-\pi}^{\pi} u_n\left(e^{it}\right) e^{ikt} dt - \int_{-\pi}^{\pi} u\left(e^{it}\right) e^{ikt} dt \right| \leq \int_{-\pi}^{\pi} \left| u_n\left(e^{it}\right) - u\left(e^{it}\right) \right| dt$$

From the above inequality, it is evident that u is in the subspace mentioned above!

Series Representation of Harmonic Functions

Theorem 1.4.0.1. *Let U be a harmonic on the disc $D_R = \{|z| < R\}$. Then, for each $n \in \mathbb{Z}$, the quantity*

$$a_n = \frac{\rho^{-|n|}}{2\pi} \int_{-\pi}^{\pi} U\left(\rho e^{it}\right) e^{-int} dt \qquad (0 < \rho < R) \tag{1.4.0.1}$$

is independent of ρ *and we have*

$$U\left(re^{i\theta}\right) = \sum_{n=-\infty}^{\infty} a_n r^{|n|} e^{in\theta} \qquad \left(re^{i\theta} \in \mathbb{D}\right) \tag{1.4.0.2}$$

The function

$$V\left(re^{i\theta}\right) = \sum_{n=-\infty}^{\infty} -isgn\,(n) a_n r^{|n|} e^{in\theta} \qquad \left(re^{i\theta} \in \mathbb{D}\right) \tag{1.4.0.3}$$

is the unique harmonic conjugate of U *such that* $V(0) = 0$*. The series in 1.4.0.2 and 1.4.0.3 are absolutely and uniformly convergent on compact subsets of* D_R

Chapter 2

The space H^1

2.1 Brief Recap!

Theorem 2.1.0.1. *Let $u : \overline{\mathbb{D}} \to \mathbb{C}$ be a harmonic function. Then we have that*

$$u\left(re^{i\theta}\right) = \frac{1}{2\pi}\int_{-\pi}^{\pi} u\left(e^{it}\right) P_r\left(e^{i(\theta - t)}\right)$$

2.2 The Helson-Lowdenslager Approach

Let $\mathcal{C}\left(\overline{\mathbb{D}}\right)$ be the set of all continuous functions on $\overline{\mathbb{D}}$ and let $H\left(\mathbb{D}\right)$ be the set of all holomorphic functions on the open disc $\mathbb{D}$. We define $\mathcal{A} = \mathcal{C}\left(\overline{\mathbb{D}}\right) \cap H\left(\mathbb{D}\right)$.

We show that $\mathcal{A}$ is an uniformly closed algebra of $\mathcal{C}\left(\overline{\mathbb{D}}\right)$. Let $\{f_n\}$ be a sequence in $\mathcal{A}$ converging uniformly to $f \in \mathcal{C}\left(\overline{\mathbb{D}}\right)$.

We recall Morera's Theorem for analytic functions at this point:

Theorem 2.2.0.1 (Morera)**.** *A continuous, complex valued function $f : D \to \mathbb{C}$ that satisfies $\oint_\gamma f\left(z\right) dz = 0$ for any closed piecewise C^1 path γ in D must be holomorphic on D.*

We use this theorem to prove what we want to prove. Now, let C be any closed curve in $\mathbb{D}$. Then for any $n \in \mathbb{N}$,

$$\oint_C f_n(z)\, dz = 0$$

So,

$$\oint_C f(z) dz = \oint_C \lim_{n\to\infty} f_n(z)\, dz = \lim_{n\to\infty} \oint_C f_n(z)\, dz = 0$$

Since C was arbitrary, f must be holomorphic. This shows that $\mathcal{A}$ is uniformly closed. The fact that it is an algebra is easy to check ✓.

Now, note that since $\mathbb{D}$ is a compact metric space, we have that $\mathcal{C}(\mathbb{D})$ is a complete metric space with supremum metric. Since the supremum metric can also be induced by a norm, namely the supremum norm, we have that $\mathcal{C}(\mathbb{D})$ is a Banach space with the supremum norm.

Thus, this is what we have proved so far:

Theorem 2.2.0.2. *The disc algebra $\mathcal{A} = \mathcal{C}\left(\overline{\mathbb{D}}\right) \cap H(\mathbb{D})$ is a Banach space under the* sup *norm*

$$\|f\|_\infty = \sup_{|z|\leq 1} |f(z)|$$

We make a couple of observations at this point:

1. Each $f \in \mathcal{A}$ is the Poisson integral of its boundary values:

$$f\left(re^{i\theta}\right) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f\left(e^{it}\right) P_r\left(e^{i(\theta-t)}\right) dt$$

2. It follows from the Maximum Modulus Theorem that

$$\|f\|_\infty = \sup\left|f\left(e^{it}\right)\right|$$

Theorem 2.2.0.3 (Correspondence of $\mathcal{A}$ with a closed subspace of $\mathcal{C}(\mathbb{T})$)**.** *Consider the subspace*

$$\tilde{\mathcal{A}} = \left\{ f \in \mathcal{C}(\mathbb{T}) \,:\, \int_{-\pi}^{\pi} f\left(e^{it}\right) e^{int} = 0 \text{ for } n = 1, 2, \ldots \right\}$$

of $\mathcal{C}(\mathbb{T})$*. Then there is an isometric isomorphism of* $\mathcal{A}$ *with* $\tilde{\mathcal{A}}$*.*

Proof. First, we show that $\tilde{\mathcal{A}}$ is a closed subspace of $\mathcal{C}(\mathbb{T})$. Let $\{f_n\}$ be a sequence of functions in $\tilde{\mathcal{A}}$ converging to $f \in \mathcal{C}(\mathbb{T})$. Consider the following:

$$\begin{aligned}
\left| \int_{-\pi}^{\pi} f\left(e^{it}\right) e^{ikt} dt \right| &= \left| \int_{-\pi}^{\pi} f\left(e^{it}\right) e^{ikt} dt - \int_{-\pi}^{\pi} f_n\left(e^{it}\right) e^{ikt} dt \right| \\
&= \int_{-\pi}^{\pi} \left| f\left(e^{it}\right) - f_n\left(e^{it}\right) \right| dt \\
&\leq 2\pi \left\| f_n - f \right\|_{\infty} \to 0 \text{ as } n \to \infty
\end{aligned}$$

This shows that $\tilde{\mathcal{A}}$ is closed under $\mathcal{C}(\mathbb{T})$ with supremum norm.

Now consider the linear map $T : \mathcal{A} \to \tilde{\mathcal{A}}$ given by

$$f \overset{T}{\longmapsto} f\,|_{\mathbb{T}}$$

For the sake of convenience, we will write $f\,|_{\mathbb{T}}$ as $f_{\mathbb{T}}$. We first need to show this map is well defined! That is, we need to show that

$$\int_{-\pi}^{\pi} f_{\mathbb{T}}\left(e^{it}\right) e^{ikt} dt = 0$$

for all $k \in \mathbb{N}$ but this immediately follows from Cauchy's theorem.

Note that injectivity is clear from Theorem 2.1.0.1. To show surjectivity, let $f \in \tilde{A}$. We need to show that there is a function $u \in \mathcal{A}$ such that $u_{\mathbb{T}} = f$. Consider the function

$$u\left(re^{i\theta}\right) = \begin{cases} (P * f)(re^{i\theta}) & \text{if } 0 \leq r < 1 \\ f\left(e^{i\theta}\right) & \text{if } r = 1 \end{cases}$$

This is the Dirichlet problem on the unit disc! So, u is continuous on $\overline{\mathbb{D}}$. It remains to show that u is analytic on $\mathbb{D}$. But note that for $r \in [0,1)$,

$$\begin{aligned} u\left(re^{i\theta}\right) &= \sum_{n=-\infty}^{\infty} r^{|n|} \hat{f}\left(n\right) e^{int} \\ &= \sum_{n=0}^{\infty} r^{|n|} \hat{f}\left(n\right) e^{int} \end{aligned}$$

This completes the proof of the theorem! $\ddot\smile$

In view of the previous theorem, we will simply write $\tilde{\mathcal{A}}$ as $\mathcal{A}$.

Definition 2.2.0.4. An analytic trignometric polynomial p on the circle $\mathbb{T}$ is of the form

$$p\left(e^{it}\right) = \sum_{k=0}^{n} a_k e^{ik\theta}$$

Proposition 2.2.0.5. *The set of the trignometric polynomials is a dense subset of $\mathcal{A}$.*

Proof. It is clear that any trignometric polynomial on the circle is a member of the disc algebra. Now, if $f : \mathbb{T} \to \mathbb{C}$ is in $\mathcal{A}$, then its negative Fourier coefficients are zero! Since, the Cesaro sum of f

$$s_n\left(x\right) = \sum_{k=-n}^{k=n} \hat{f}(n)e^{ikx} = \sum_{k=0}^{n} \hat{f}(n)e^{ikx}$$

converge to f uniformly and is a sequence of trignometric polynomial, we are done! $\ddot\smile$

The following result is used in the proof of the next theorem, so, we prove it here:

Theorem 2.2.0.6. *The real parts of functions in $\mathcal{A}$ are uniformly dense in $\mathcal{C}\left(\mathbb{T}, \mathbb{R}\right)$. In other words, if μ is finite signed Borel measure on $\mathbb{T}$ such that $\int f \, d\mu = 0$ for every $f \in \mathcal{A}$ then μ is the zero measure.*

Proof. We first show that any trignometric polynomial of the form

$$p\left(e^{it}\right) = \sum_{k=-n}^{n} c_k e^{ikt} \tag{2.2.0.1}$$

where $c_{-k} = \overline{c_k}$ for each $k \in \{1, \ldots, n\}$ is a real part of a function $f \in \mathcal{A}$. Note that $p\left(e^{it}\right)$ in Equation 2.2.0.1 is the real part of the function:

$$f\left(e^{it}\right) = c_0 + 2c_1 e^{it} + 2c_2 e^{2it} + \ldots + 2c_n e^{int}$$

Now, we claim that every function $f \in C\left(\mathbb{T}, \mathbb{R}\right)$ is a uniform limit of a trignometric polynomial of the form 2.2.0.1. We will be done if we show that the negative Fourier coefficients of real valued function is the conjugate of the its positive counterpart, that is, for each $n \in \mathbb{Z}_{\geq 0}$, we have that $\hat{f}\left(-n\right) = \overline{\hat{f}\left(n\right)}$. To show this, take any $n \in \mathbb{Z}_{\geq 0}$ and then observe that

$$\begin{aligned}
\hat{f}\left(-n\right) &= \int_{\mathbb{T}} f\left(e^{it}\right) e^{int} \frac{dt}{2\pi} \\
&= \overline{\int_{\mathbb{T}} f\left(e^{it}\right) e^{-int} \frac{dt}{2\pi}} \\
&= \overline{\hat{f}\left(n\right)}
\end{aligned}$$

This shows that the Cesaro means of a real valued function is a trignometric polynomial of the form 2.2.0.1 and since the Cesaro means converges to f uniformly. Thus, the closure of the real parts of $\mathcal{A}$ is indeed $\mathcal{C}\left(\mathbb{T}, \mathbb{R}\right)$.

Now, let μ be a finite signed measure on $\mathbb{T}$ such that $\int f \, d\mu = 0$ for every $f \in \mathcal{A}$. We show that μ is the zero measure. Notice that if $f \in \mathcal{A}$ then

$$0 = \int_{\mathbb{T}} f \, d\mu = \int_{\mathbb{T}} \Re(f) \, d\mu + i \int_{\mathbb{T}} \Im(f) \, d\mu$$

Hence, it follows that

$$\int_{\mathbb{T}} \Re(f) \, d\mu = 0$$

for every $f \in \mathcal{A}$. Now, if $g \in \mathcal{C}(\mathbb{T}, \mathbb{R})$ then by the first part of this theorem, there is a sequence $\{f_n\} \in \mathcal{A}$ such that $\Re(f_n)$ converges to g uniformly. By the Dominated Convergence Theorem (which holds, thanks to Jordan Decomposition Theorem), we have that $\int_{\mathbb{T}} g\, d\mu = 0$.

Now to prove that every $\mu = 0$, it suffices to show that $\hat{\mu}(n) = 0$ for every $n \in \mathbb{Z}$[1]. Now, notice that for any $n \in \mathbb{Z}$, we have

$$\begin{aligned}\hat{\mu}(n) &= \int_{\mathbb{T}} e^{-int} d\mu\left(e^{it}\right) \\ &= \int_{\mathbb{T}} \left(\cos(nt) - i\sin(nt)\right) d\mu\left(e^{it}\right) \\ &= \int_{\mathbb{T}} \cos(nt)\, d\mu\left(e^{it}\right) - i\int_{\mathbb{T}} \sin(nt)\, d\mu\left(e^{it}\right) \\ &= 0\end{aligned}$$

This completes the proof! ⌣̈

Corollary 2.2.0.7. *Let μ be a finite signed measure on $\mathbb{T}$ such that $\int f d\mu = 0$ for every $f \in \mathcal{A}$ which vanishes at the origin then μ is a constant multiple of Lebesgue measure.*

Proof. We first prove the following claim: If $f \in \mathcal{A}$ then $\int_{\mathbb{T}} f d\mu = \frac{f(0)}{2\pi}$. Since the negative Fourier cofficients are zero and f is continuous, we have that the Cesaro means converge uniformly to f, that is,

$$\sum_{k=0}^{\infty} \hat{f}(n)\, e^{int} \to f \text{ uniformly}$$

[1]See Page 41, Corollary 2.3 of Mashreghi.

Thus,

$$\begin{aligned}\int_T f\left(e^{it}\right)\frac{dt}{2\pi} &= \frac{1}{2\pi}\int_{\mathbb{T}}\left(\sum_{k=0}^{\infty}\hat{f}\left(n\right)e^{int}\right)dt\\ &= \frac{1}{2\pi}\sum_{k=0}^{\infty}\int_{\mathbb{T}}\left(\hat{f}\left(n\right)e^{int}dt\right)\\ &= \frac{\hat{f}(0)}{2\pi}\end{aligned}$$

Now, we proceed to the proof. We define a measure $d\nu = d\mu - \frac{1}{2\pi}\mu\left(\mathbb{T}\right)dt$. Now, we have that

$$\begin{aligned}\int_{\mathbb{T}} f\left(e^{it}\right)d\nu\left(e^{it}\right) &= \int_{\mathbb{T}}\left[f - f\left(0\right)\right]\left(e^{it}\right)d\mu\left(e^{it}\right) + f\left(0\right)\int_{\mathbb{T}}d\nu\left(e^{it}\right)\\ &= 0\end{aligned}$$

Hence, we have that $d\mu = \frac{1}{2\pi}\mu\left(\mathbb{T}\right)dt$. $\ddot\smile$

We will be working entirely on the $\mathbb{T}$. So, $\mathcal{A}$ and H^2 will be the spaces on the unit circle rather on the open unit disc.

Now, consider $\mathcal{A}$ as a subset of $L^2\left(\mathbb{T},\mathscr{B}\left(\mathbb{T}\right),\mu\right)$ where μ is any finite positive measure. Let $\mathcal{A}_0 = \left\{f\in\mathcal{A} : \int_{\mathbb{T}} f\left(e^{it}\right)\frac{dt}{2\pi} = \frac{\hat{f}(0)}{2\pi} = 0\right\}$. It is easily seen that $\mathcal{A}_0$ is a subspace of $L^2\left(d\mu\right)$. Therefore, we have the closed subspace spanned by $\mathcal{A}$ is $[\![\mathcal{A}_0]\!] = \overline{\operatorname{span}(\mathcal{A}_0)} = \overline{\mathcal{A}_0}$. By a theorem of Hilbert spaces, we have that there is some vector $F \in [\![\mathcal{A}_0]\!]$ such that

$$\inf_{f\in[\![\mathcal{A}_0]\!]}\int\left|1 - f^2\right|d\mu = \int\left|1 - F^2\right|d\mu$$

But since $d\left(1,[\![\mathcal{A}_0]\!]\right) = d\left(1,\overline{\mathcal{A}_0}\right) \overset{2}{=} d\left(1,\mathcal{A}_0\right)$, we have

$$\inf_{f\in\mathcal{A}_0}\int\left|1 - f^2\right|d\mu = \int\left|1 - F^2\right|d\mu$$

Note that this F is the orthogonal projection of 1 into the closed subspace spanned by $\mathcal{A}_0$.

Theorem 2.2.0.8. *Let μ be a finite positive Borel measure on $\mathbb{T}$ and suppose that the constant function 1 is not in $[\![\mathcal{A}_0]\!]$. Then let $f = P_{[\![\mathcal{A}_0]\!]}(1)$. Then the following holds:*

1. *The measure $d\nu = |1 - F^2|\, d\mu$ is a nonzero constant multiple of the Lebesgue measure. In particular, Lebesgue measure is absolutely continuous with respect to μ.*
2. *The function $(1 - F)^{-1} \in H^2$.*
3. *If $h = \left(\frac{d\mu}{d\theta}\right)$ then $(1 - F)\, h \in L^2 = L^2\left(\frac{d\theta}{2\pi}\right)$.*

Proof. 1. Let $S = [\![\mathcal{A}_0]\!]$. We begin to prove part one of the theorem. Let $F = P_S(1)$. Then we have by the uniqueness of the decomposition that

$$1 = \underbrace{F}_{P_S(1)} + \underbrace{1 - F}_{P_{S^\perp}(1)}$$

Thus, we have that $(1 - F)$ is orthogonal to every element in S and hence, in particular, any element in $\mathcal{A}_0$ (because $\mathcal{A}_0 \subset S \rightsquigarrow S^\perp \subset \mathcal{A}_0^\perp$). We claim that $1 - F$ is orthogonal to $(1 - F)\, f$ for every $f \in \mathcal{A}_0$. But before, we do this, we need to show that $(1 - F)f \in L^2(d\mu)$. Observe that

$$\int_{\mathbb{T}} \left|(1 - F)f^2\right| d\mu \leq \|f\|_\infty^2 \|1 - F\|_2^2 < \infty$$

To prove this, note that we showed that $S = \overline{\mathcal{A}_0}$ in the paragraph before the statement of this theoremand since $F \in S$, there is a sequence $\{f_n\} \in \mathcal{A}_0$ converging to F. Hence, we have that $\{f\,(1 - f_n)\}$ is a sequence in $\mathcal{A}_0$[3] converges to $f\,(1 - F)$ in the L^2-norm. Hence, we have that

[3] $\mathcal{A}_0$ is an algebra!

$$
\begin{aligned}
\langle f(1-F),(1-F)\rangle &= \left\langle \lim_{n\to\infty} f\left(1-f_n\right),(1-F)\right\rangle \\
&= \lim_{n\to\infty} \left\langle f\left(1-f_n\right),(1-F)\right\rangle && \text{continuity of the inner product} \\
&= 0 && 1-F \text{ is orthogonal to } \mathcal{A}_0
\end{aligned}
$$

Now, let $d\nu = |1-F|^2\, d\mu$. We have shown that for any $f \in \mathcal{A}_0$,

$$\int_{\mathbb{T}} f d\nu = \int_{\mathbb{T}} f\,|1-F|^2\, d\mu = \langle f(1-F),(1-F)\rangle = 0$$

Hence, by Corollary 2.2.0.7, we have that $d\mu = k\,d\lambda$ for some $k \geq 0$.

Now, we claim that this $k \neq 0$. If $k=0$ then we would have that

$$\int_{\mathbb{T}} d\nu = 0 \rightsquigarrow \int_{\mathbb{T}} |1-F|^2\, d\mu = 0$$

Hence, we have that $F = 1$ μ-almost everywhere[4]. But then we have that $1 \in S$ which contradicts our assumption. Hence $k \neq 0$.

2. Observe that part 1 of the theorem tells us that

$$|1-F|^2\, d\mu = k\,d\lambda \text{ where } k \neq 0$$

Then we have that by Lebesgue Decomposition Theorem

$$d\mu = hd\lambda + d\mu_s$$

for some positive $\mathcal{L}^1$-function h and some singular measure μ_s. Hence, we have

[4]See Corollary 2.3.12 of Cohn's Measure Theory.

that

$$|1-F|^2\,d\mu = |1-F|^2\,(hd\lambda + d\mu_s) \tag{2.2.0.2}$$
$$= \underbrace{|1-F|^2\,hd\lambda}_{(1)} + \underbrace{|1-F|^2\,d\mu_s}_{(2)} \tag{2.2.0.3}$$

By the uniqueness of the Lebesgue Decomposition Theorem (one needs to verify that the measures obtained in (1) and (2) are absolutely continuous and singular), we have that

$$|1-F|^2\,h = k \ \lambda\text{-almost everywhere.}$$
$$\rightsquigarrow \frac{1}{|1-F|^2} = \frac{h}{k} \ \lambda\text{-almost everywhere.}$$

This tells us that $(1-F)^{-1} \in L^2(d\lambda)$ where λ is the Lebesgue measure on $\mathbb{T}$. Also, note that $F = 1$ $d\mu_s$-almost everwhere. Hence, we have that

$$|1-F|^2\,d\mu = |1-F|^2\,hd\lambda = kd\lambda \rightsquigarrow d\mu = |1-F|^{-2}\,kd\lambda \tag{2.2.0.4}$$

Now, we proceed to show that the negative Fourier coefficients of $(1-F)^{-1}$ is 0 then we would have shown that $(1-F)^{-1} \in H^2$. Consider the following for $f \in \mathcal{A}_0$:

$$\begin{aligned} k\int_{\mathbb{T}} (1-F)^{-1}\,fd\lambda &= k\int_{\mathbb{T}} \left(1-\overline{F}\right) f\,|1-F|^{-2}\,d\lambda \\ &= k\int_{\mathbb{T}} \left(1-\overline{F}\right) fd\mu \\ &= 0 \end{aligned}$$

The last line follows from the fact that $1-F \perp f$ for any $f \in \mathcal{A}_0$. Hence this holds for any $e^{in\theta}$ and hence we are done.

3. Note that the derivative of μ with respect to normalized Lebesgue meaures is

h as demonstrated in Equation 2.2.0.3. Now, Equation 2.2.0.4 shows that

$$|1-F|\,h = k\,|1-F|^{-1} \;\; \lambda\text{-almost everywhere.}$$

Since $|1-F|^{-1} \in L^2(\lambda)$, so is $(1-F)h$.

$\ddot{\smile}$

Corollary 2.2.0.9. *Let μ be a finite positive measure on $\mathbb{T}$ with absolutely continuous part μ_a, then*

$$\inf_{f\in\mathcal{A}_0}\int_{\mathbb{T}} |1-f|^2\,d\mu = \inf_{f\in\mathcal{A}_0}\int_{\mathbb{T}} |1-f|^2\,d\mu_a$$

where μ_a is the absolutely continuous part of μ. In particular, for any singular measure μ, the function 1 is in the $L^2(d\mu)$ closure of $\mathcal{A}_0$.

Proof. Let F be the orthogonal projection of 1 into the closed subspace spanned by $\mathcal{A}_0$. Then we have that

$$\inf_{f\in\mathcal{A}_0}\int_{\mathbb{T}} |1-f|^2\,d\mu = \|1-F\|^2 = \int_{\mathbb{T}} |1-F|^2\,d\mu$$

From Equation 2.2.0.4, we see that

$$|1-F|^2\,d\mu = |1-F|^2\,hd\lambda = |1-F|^2\,d\mu_a$$

where h is the Radon Nikodym derivative of μ. Integrating both sides, we have that

$$\inf_{f\in\mathcal{A}_0}\int_{\mathbb{T}} |1-f|^2\,d\mu = \inf_{f\in\mathcal{A}_0}\int_{\mathbb{T}} |1-f|^2\,d\mu_a$$

which is what we wanted to prove. Now if μ is a singular measure then we have that the absolutely continuous part of μ is the zero measure and hence we have that

$$\inf_{f\in\mathcal{A}_0}\int_{\mathbb{T}} |1-f|^2\,d\mu = \|1-F\|^2 = 0$$

From the fact that $\|1 - F\| = 0$, we conclude that $1 = F$ almost everywhere and hence 1 is in the closed subspace spanned by $\mathcal{A}_0$ which is in fact $\overline{\mathcal{A}_0}$. This completes the proof of the Corollary. $\ddot{\smile}$

Corollary 2.2.0.10. *Let μ be a finite complex Borel measure on $\mathbb{T}$ which is orthogonal to $\mathcal{A}_0$, that is, $\int_{\mathbb{T}} f d\mu = 0$ for all $f \in \mathcal{A}_0$. Then the absolutely continuous part and singular parts of μ are separately orthogonal to $\mathcal{A}_0$. That is, if μ_a and μ_s are the absolutely continuous and the singular parts of the measure μ then we have that*

$$\int_{\mathbb{T}} f d\mu_a = 0 \text{ and } \int_{\mathbb{T}} f d\mu_s = 0$$

for every $f \in \mathcal{A}_0$.

Proof. Let ρ be any finite positive measure satisfying

1. $\mu \ll \rho$ and $\frac{d\mu}{d\rho}$ is bounded, and

2. $\frac{d\rho}{d\theta} \geq \frac{1}{2\pi}$, that is, the Radon Nikodym derivative with respect to the normalised Lebesgue measure is bounded below by $1/2\pi$.

Let $d\mu = \frac{1}{2\pi} h d\theta + d\mu_s$ be the Lebesgue decomposition of the measure μ. We define a new measure on $\mathbb{T}$ by

$$d\rho = (1 + |h|) \frac{d\theta}{2\pi} + d|\mu_s|$$

where $|\mu_s|$ is the variation measure of the complex measure μ_s, the singular part of μ.

We show that ρ is an finite positive measure satisfying properties as in items 1 and 2 above. First, we show that ρ is a finite positive measure. But before that note that the variation measure of any complex measure is a finite measure.[5]. Now,

[5]See Cohn's Measure Theory – Proposition 4.1.7

consider the following:

$$\begin{aligned}
\rho(\mathbb{T}) &= \int_{\mathbb{T}} d\rho \\
&= \int_{\mathbb{T}} (1+|h|)\frac{d\theta}{2\pi} + \int_{\mathbb{T}} d|\mu_s| && \text{by definition} \\
&= 1 + |\mu_a|(\mathbb{T}) + |\mu_s|(\mathbb{T}) && \mu_a \text{ is the absolutely continuous part of } \mu \\
&< \infty
\end{aligned}$$

Now we proceed to show that item 1 holds for ρ. First, we prove that $\mu \ll \rho$. Let $A \in \mathscr{B}(\mathbb{T})$ with $\rho(A) = 0$. Then we have that

$$\int_A (1+|h|)\frac{d\theta}{2\pi} = 0 \text{ and}$$
$$\int_A d|\mu_s| = 0$$

From this, we have that

$$\begin{aligned}
|\mu_a|(A) &= \int_A |h| \frac{d\theta}{2\pi} \\
&\leq \int_A (1+|h|)\frac{d\theta}{2\pi} \\
&= 0
\end{aligned}$$

Since $|\mu(A)| \leq |\mu_a|(A)$, we have that $\mu_a(A) = 0$. Also, it follows by the definition of the variation measure that $\mu_s(A) = 0$. Hence, we have that μ is absolutely continuous with respect to ρ. Now, we need to show that $\frac{d\mu}{d\rho}$ is bounded. Needs work! We proceed to show that item 2 holds. Observe that

$$\frac{d\rho}{d\theta} = (1+|h|)\frac{1}{2\pi} \geq \frac{1}{2\pi}$$

and hence we are done showing that ρ is such a measure.

Now, let $f \in \mathcal{A}_0$. Then we have that

$$\begin{aligned}
\int_{\mathbb{T}} |1-f|^2 \, d\rho &= \int_{\mathbb{T}} |1-f|^2 \frac{d\rho}{d\theta} d\theta + \int_{\mathbb{T}} |1-f|^2 \, d\mu_s \\
&\geq \int_{\mathbb{T}} |1-f|^2 \frac{d\rho}{d\theta} d\theta \\
&\geq \int_{\mathbb{T}} |1-f|^2 \frac{d\theta}{2\pi} \\
&= \|1-f\|^2 \\
&= \|1\|^2 + \|f\|^2 && \langle f, 1 \rangle = 0 \text{ as } f \in \mathcal{A}_0 \\
&\geq 1
\end{aligned}$$

Now, let F be the orthogonal projection of 1 into the closed subspace spanned by $\mathcal{A}_0$ of $L^2(d\rho)$. Thus, by definition, we have that

$$\int_{\mathbb{T}} |1-F|^2 = \inf_{f \in \mathcal{A}_0} \int_{\mathbb{T}} |1-f|^2 \, d\rho \geq 1$$

Hence, we have that 1 is not in the closed subspace spanned by $\mathcal{A}_0$ of $L^2(d\rho)$. Hence, we have by Theorem 2.2.0.8 that $(1-F)^{-1} \in H^2$ and that

$$(1-F)(1+|h|) \in L^2 = L^2\left(\frac{d\theta}{2\pi}\right)$$

From this, we can conclude that $h \in L^2$ by the following fashion:

$$|h| = \underbrace{(1-F)(1+|h|)}_{\in L^2} \underbrace{\left(|1-F|^{-1}\right)}_{\in H^2 \rightsquigarrow \in L^2} - 1$$

Hence, we have that $(1-F)h \in L^2$.

Suppose $g \in \mathcal{A}_0$. We claim that

$$\int_{\mathbb{T}} (1-F) \, g d\mu = 0$$

$\ddot\smile$

Since $F \in \overline{\mathcal{A}}_0$, there is a sequence $f_n \in \mathcal{A}_0$ such that $f_n \to F$ in $L^2(d\rho)$. Hence,

$$(1 - f_n)g \to (1 - F)\, g$$

in $L^2\,(d\rho)$.

Since $\frac{d\mu}{d\rho}$ is bounded, we have that

$$\int_{\mathbb{T}} (1 - F)\, g d\mu = \int_{\mathbb{T}} (1 - F)\, g \frac{d\mu}{d\rho} d\rho = \lim_{n\to\infty} \int_{\mathbb{T}} (1 - f_n)\, \frac{d\mu}{d\rho} d\rho = 0.$$

Now, $1 - F = 0$ with respect to $d\,|\mu_s|$ which implies that $1 - F = 0$ with respect to $d\mu_s$.

Thus, we have that

$$(1 - F)\, d\mu = \frac{1}{2\pi}\,(1 - F)\, h d\theta$$

Hence, we have for $g \in \mathcal{A}_0$

$$\begin{aligned}\int_{\mathbb{T}} (1 - F)\, g h d\theta &= \int_{\mathbb{T}} (1 - F) g d\mu \\ &= 0\end{aligned}$$

Since $\overline{A} = H^2$. There is a sequence of functions g_n in $\mathcal{A}$ converging to $(1 - F)^{-1} \in L^2\left(\frac{d\theta}{2\pi}\right)$ in L^2-norm. Hence, we have for any $f \in \mathcal{A}_0$,

$$\int_{\mathbb{T}} g_n f\,(1 - F)\, h d\theta = 0$$

Also, note that $(1 - F)\, h \in L^2$. From this, we can conclude that

$$\int_{\mathbb{T}} f h d\theta = 0$$

Theorem 2.2.0.11 (Frigyez and Marcel Riesz, 1916)**.** *Let μ be a complex Borel* *measure on the unit circle* $\mathbb{T}$ *such that*

$$\int_{\mathbb{T}} e^{in\theta} d\mu\left(e^{i\theta}\right) = 0$$

for all $n \in \mathbb{N}$*. Then* μ *is absolutely continuous with respect to the Lebesgue measure.*

Proof. Suppose that μ is orthogonal to $\mathcal{A}_0$. Then from the previous corollary, we have that μ_a is orthogonal to $\mathcal{A}_0$ and μ_s is orthogonal to $\mathcal{A}_0$. Since $|\mu_s|$ is singular (definition?), we can find a sequence of functions $\{f_n\}$ converging to 1 in $L^2\left(d\left|\mu_s\right|\right)$. Since μ_s is orthogonal to $\mathcal{A}_0$, we have that

$$\int_{\mathbb{T}} d\mu_s = \lim_{n\to\infty} \int_{\mathbb{T}} f_n d\mu_s = 0$$

Thus, μ_s is orthogonal to 1. The singular measure $e^{-i\theta} d\mu_s$ is now orthogonal to $\mathcal{A}_0$; hence it is orthogonal to 1, that is,

$$\int_{\mathbb{T}} e^{-i\theta} d\mu_s = 0$$

Similarly, $e^{-2i\theta} d\mu_s$ is orthogonal to $\mathcal{A}_0$, and consequently, orthogonal to 1. Repeating thei process, we conclude that

$$\int_{\mathbb{T}} e^{in\theta} = 0$$

for $n \in \mathbb{Z}$. Hence μ_s must be the zero measure. Therefore our original measure μ is absolutely continuous. $\ddot\smile$

2.3 Szegö's Theorem

Theorem 2.3.0.1 (Szegö). *Let $h \in L^1(\mathbb{T})$ and suppose that $h \geq 0$ on $\mathbb{T}$ (that makes h real valued!). Then the following holds:*

$$\exp\left[\frac{1}{2\pi}\int_{-\pi}^{\pi}\log h\left(e^{it}\right)dt\right] = \inf_{f\in\mathcal{A}_0}\frac{1}{2\pi}\int_{-\pi}^{\pi} he^{\Re(f)}dt$$

Proof. We first claim that $\log h$ is integrable iff $\int \log hd\theta = -\infty$. To see this, first we decompose

$$\int \log hd\theta = \underbrace{\int (\log h)^+ d\theta}_{(i)} - \underbrace{\int (\log h)^- d\theta}_{(ii)}$$

First observe that $\log h \leq h$ (this follows by definition of log) and hence, we have that $(\log h)^+ \leq h^+ = h$. Thus, we have that integral (i) is finite. Therefore, $\log h$ is not integrable iff $\int \log hd\theta = -\infty$.

Regardless of integrability of $\log h$, we have the following relation:

$$\exp\left[\frac{1}{2\pi}\int_{\mathbb{T}}\log h\left(e^{it}\right)dt\right] \leq \frac{1}{2\pi}\int_{\mathbb{T}} h\left(e^{it}\right)dt \tag{2.3.0.1}$$

One can prove the famililar arithmetic and geometric means inequality from the above inequality. Let $n \in \mathbb{N}$ be arbitrary and $a_1, \ldots, a_n$ be positive real numbers. Let $A_1, A_2, \ldots, A_n$ be the disjoint subsets of $\mathbb{T}$ which cover it. Therefore, we have that

$$\mu(A_i) = \frac{2\pi}{n} \text{ for all } i = 1, 2, \ldots, n.$$

Observe the following:

$$\begin{aligned}\log\left(a_1\chi_{A_1}+\ldots+a_n\chi_{A_n}\right) &= \log\left(a_1\chi_{A_1}+\ldots+a_n\chi_{A_n}\right)\\ &= \log\left(a_1\right)\chi_{A_1}+\ldots+\log\left(a_n\right)\chi_{A_n}\end{aligned}$$

Integrating both sides, we get

$$\begin{aligned}\frac{1}{2\pi}\int_{\mathbb{T}}\log\left(a_1\chi_{A_1}+\ldots+a_n\chi_{A_n}\right)dt &= \frac{1}{2\pi}\int_{\mathbb{T}}\log\left(a_1\right)\chi_{A_1}+\ldots+\log\left(a_n\right)\chi_{A_n}dt\\ &= \frac{1}{2\pi}\left(\log\left(a_1\right)\mu\left(A_1\right)+\ldots+\log\left(a_n\right)\mu\left(A_n\right)\right)\\ &= \log\left(\sqrt[n]{a_1\ldots a_n}\right)\end{aligned}$$

It can be seen now that it follows immediately.

Now, if $h\in L^{\mathbb{T}}$ and is nonnegative, g is any real valued continuous function on $\mathbb{T}$ then we have that he^g is an nonnegative $L^1(\mathbb{T})$ because product of an L^1 function with a bounded funtion on a finite measure space retains it back and therefore, we can apply Equation 2.3.0.1. Futhermore, if we assume that $\int_{\mathbb{T}} g dt = 0$ then we have that

$$\begin{aligned}\exp\left[\frac{1}{2\pi}\int_{\mathbb{T}}\log\left(h\left(e^{it}\right)e^{g\left(e^{it}\right)}\right)dt\right] &\le \frac{1}{2\pi}\int_{\mathbb{T}}h\left(e^{it}\right)e^{g\left(e^{it}\right)}dt\\ \rightsquigarrow \exp\left[\frac{1}{2\pi}\int_{\mathbb{T}}\log\left(h\left(e^{it}\right)\right)dt+\cancelto{0}{\int_{\mathbb{T}}g\left(e^{it}\right)dt}\right] &\le \frac{1}{2\pi}\int_{\mathbb{T}}h\left(e^{it}\right)e^{g\left(e^{it}\right)}dt\\ \rightsquigarrow \exp\left[\frac{1}{2\pi}\int_{\mathbb{T}}\log\left(h\left(e^{it}\right)\right)dt\right] &\le \frac{1}{2\pi}\int_{\mathbb{T}}h\left(e^{it}\right)e^{g\left(e^{it}\right)}dt\end{aligned}$$

Now, let $g=\Re f$ for some $f\in\mathcal{A}_0$. Then we have that g is continuous and real valued and hence the previous inequality holds, therefore, we have that

$$\exp\left[\frac{1}{2\pi}\int_{\mathbb{T}}\log\left(h\left(e^{it}\right)\right)dt\right]\le\frac{1}{2\pi}\int_{\mathbb{T}}h\left(e^{it}\right)e^{\Re\left(f\left(e^{it}\right)\right)}dt.$$

Since $f \in \mathcal{A}_0$ is arbitrary, we have that

$$\exp\left[\frac{1}{2\pi}\int_{\mathbb{T}} \log\left(h\left(e^{it}\right)\right) dt\right] \leq \inf_{f\in\mathcal{A}_0} \frac{1}{2\pi}\int_{\mathbb{T}} h\left(e^{it}\right) e^{\Re\left(f\left(e^{it}\right)\right)} dt.$$

Now, we claim the set of functions $\Re f$ where $f \in \mathcal{A}$ is dense subset of $L^1_{\mathbb{R}}(\mathbb{T})$. This is easy to see. Let $f \in L^1_{\mathbb{R}}(\mathbb{T})$. Then denseness of continuous functions in $L^1_{\mathbb{R}}(\mathbb{T})$, we have that there is some function $g \in \mathcal{C}_{\mathbb{R}}(\mathbb{T})$ such that

$$\|f - g\|_1 < \frac{\varepsilon}{2}$$

Also, by Theorem 2.2.0.6, we have that there is some function $h \in \mathcal{A}$ such that

$$\|g - \Re h\|_\infty < \frac{\varepsilon}{2}$$

Therefore, we have that

$$\begin{aligned} \|f - \Re h\|_1 &\leq \|f - g\|_1 + \|g - \Re h\| && \text{Minkowski inequality} \\ &\leq \|f - g\|_1 + \|f - g\|_\infty && \text{finite measure space} \\ &\leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon. \end{aligned}$$

This shows the denseness.

Now, we show that the $\left\{f \in L^1_{\mathbb{R}}(\mathbb{T}) : \int_{\mathbb{T}} f d\theta = 0\right\}$ is dense in set of all functions $f \in L^1_{\mathbb{R}}(\mathbb{T})$ such that $\int f d\theta = 0$. This is immediate because closure distributes over intersections and $\left\{f \in L^1_{\mathbb{R}}(\mathbb{T}) : \int_{\mathbb{T}} f d\theta = 0\right\}$ is a closed subspace of $L^1_{\mathbb{R}}(\mathbb{R})$.

Now, we claim that $\underbrace{\inf\{\int he^g d\theta : g \in L^1, \int g = 0\}}_{:=\alpha} = \underbrace{\inf\left\{\int he^g d\theta : g \in L^\infty, \int g = 0\right\}}_{:=\beta}$.

Since $L^\infty \subset L^1$, we have that $\alpha \leq \beta$ is clear. To show the reverse inequality, let $g \in L^1$ with $\int g d\theta = 0$. By the Simple Function Approximation Theorem, there is a sequence $g_n \in L^\infty$ such that $0 \leq |g_1| \leq |g_2| \leq \ldots \leq |g|$ and g_n converging to g pointwise.

$\ddot\smile$

Theorem 2.3.0.2 (Szegő, Kolmogoroff-Krein)**.** *Let μ be a finite Borel measure on the unit circle $\mathbb{T}$ and h be the derivative of μ with respect to the normalised Lebesgue measure. Then*

$$\inf_{f\in\mathcal{A}_0} \inf |1-f|^2 \, d\mu = \exp\left[\frac{1}{2\pi}\int_{\pi}^{\pi} \log h\left(e^{i\theta}\right)\, d\theta\right]$$

2.4 Completing the Discussion of H^1

Corollary 2.4.0.1. *If $f \in H^1(\mathbb{T})$, f cannot vanish on a set of positive Lebesgue measure unless $f \equiv 0$ on $\mathbb{T}$*

Proof. Assume $f \not\equiv 0$. We wish to show that f cannot vanish on a set of positive Lebesgue measure on $\mathbb{T}$. Since $\log|f|$ is integrable on $\mathbb{T}$ (by Szego's theorem), we have that

$$\lambda\left(\left\{e^{i\theta} \in \mathbb{T} \,:\, \left|\log\left|f\left(e^{i\theta}\right)\right|\right| = +\infty\right\}\right) = 0$$

since integrability implies finite almost everywhere. Therefore we have that

$$\lambda\left(\left\{e^{i\theta} \in \mathbb{T} \,:\, f\left(e^{i\theta}\right) = 0\right\}\right) = \lambda\left(\left\{e^{i\theta} \in \mathbb{T} \,:\, \log\left|f\left(e^{i\theta}\right)\right| = -\infty\right\}\right) = 0$$

Thus, if S is any measurable subset of the $\mathbb{T}$ where f vanishes then we must be have that $\lambda(S) = 0$. This shows that f cannot vanish on a set of positive Lebesgue measure. $\ddot\smile$

Chapter 3

Factorization for H^p Functions

3.1 Inner and Outer Functions

Let $f \in H^1(\mathbb{D})$ be nonzero, that is, not identically zero. Then f has nontangential limits almost everywhere, due to Fatou's theorem. Define

$$\tilde{f}(e^{i\theta}) = \lim_{z \to e^{i\theta}} f(z)$$

for almost every $e^{i\theta} \in \mathbb{T}$ and

$$f(re^{i\theta}) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \tilde{f}\left(e^{it}\right) P_r\left(e^{i(\theta - t)}\right) dt.$$

As a consequence of Szegö's theorem, we have that $\log\left(\left|f\left(e^{it}\right)\right|\right)$ is Lebesgue integrable. Let

$$F(z) = \exp\left[\frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{e^{i\theta} + z}{e^{i\theta} - z} \log|f(e^{i\theta})|\right] d\theta$$

Then F is an analytic function in the unit disc. To prove this, let $g(e^{i\theta}) = \log\left|f\left(e^{i\theta}\right)\right|$. Then we have that Let (h_n) be sequence in $\mathbb{C}$ of nonzero terms converging to 0 and $z \in \mathbb{D}$ Then $|e^{i\theta} - z - h_n| \to |e^{i\theta} - z| > 0$. Thus there exists $M > 0$ and $N \in \mathbb{N}$ such that $|e^{i\theta} - z - h_n| > M$ for all $n \geq N$. Let $g(z) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{e^{i\theta}+z}{e^{i\theta}-z} u(e^{i\theta})$Consider the following for $n \geq N$:

$$\left|\frac{g(z+h_n)-g(z)}{h_n}\right| \leq \frac{1}{2\pi}\int_{-\pi}^{\pi}\left|\frac{2h_n e^{i\theta}}{(e^{i\theta}-z-h_n)(e^{i\theta}-z)}\right||u(e^{i\theta})|d\theta$$
$$\leq \frac{2h_n}{M\inf_{e^{i\theta}\in\mathbb{T}}|e^{i\theta}-z|}\|u\|_1$$

This shows that g is analytic and hence e^g is analytic.

Definition 3.1.0.1 (Inner & Outer Function)**.** Let $g \in H(\mathbb{D})$. Then g is said to be an **inner function** if

- $|g(z)| \leq 1$ for every $z \in \mathbb{D}$ and
- $|g\left(e^{i\theta}\right) = 1$ for every $e^{i\theta} \in \mathbb{T}$.

A function $F : \mathbb{D} \to \mathbb{C}$ is said to be an **outer function** if

$$F(z) = \lambda \exp\left[\frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{e^{i\theta}+z}{e^{i\theta}-z}k\left(e^{i\theta}\right)\, d\theta\right]$$

for some integrable function $k : \mathbb{T} \to \mathbb{R}$ and some $\lambda \in \mathbb{C}$ with $|\lambda| = 1$.

Remark 3.1.0.2. Few remarks follow for Definition 3.1.0.1:

- An outer function $F \in H^{z1}(\mathbb{D})$ iff e^k is integrable.

  Proof. It immediately follows by definition that

  $$|F| = \exp\left[P\left[k\right]\right] \text{ on } \mathbb{D}$$

  Hence $\exp(k\left(e^{i\theta}\right)) = \hat{F}\left(e^{i\theta}\right)$ almost eveywhere by Fatou's lemma. By the isometric isomorphism, we have that

  $$\|\exp\left(k\right)\|_{H^1(T)} = \|F\|_{H^1(\mathbb{D})}$$

  Thus, $\exp\left(k\right)$ is integrable.

To prove the converse, assume that e^k is integrable. Then we have that

$$\begin{aligned} |F(re^{i\theta})| &= \exp\left(\frac{1}{2\pi}\int_{-\pi}^{\pi} P_r\left(e^{i(\theta-t)}k\left(e^{i\theta}\right)d\theta\right)\right) \\ &\leq \frac{1}{2\pi}\int_{-\pi}^{\pi}\exp\left(k\left(e^{i\theta}\right)\right)d\theta && \text{(Jensen Inequality)} \\ &= \|\exp(k)\|_1 \end{aligned}$$

This completes the proof of the remark. ⌣̈

- If F is an outer function in H^1 then we have zthat

$$k\left(e^{i\theta}\right) = \log\left|F\left(e^{i\theta}\right)\right| \text{ almost everyhere}$$

Proof. This is evident from Fatou's lemma. ⌣̈

3.1.1 Characterising Outer Functions

Theorem 3.1.1.1. *Let F be a nonzero function in H^1. TFAE:*

(i) F *is an outer function.*

(ii) If f is any function in H^1 such that $|f| = |F|$ almost everywhere on $\mathbb{T}$ then $|F(z)| \geq |f(z)|$ on $\mathbb{D}$.

(iii) $\log|F(0)| = \frac{1}{2\pi}\int_{-\pi}^{\pi}\log\left|F\left(e^{i\theta}\right)\right|d\theta.$

Proving this (i) implies (ii) will take a while. To prove this direction, we prove a result due to Jensen (1915).

Proof of (ii) implies (i). Suppose that (ii) holds. Since F is nonzero function and is in H^1, we have that $\log\left|F\left(e^{i\theta}\right)\right|$ is integrable and hence, we have the following function

$$G(z) = \exp\left[\frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{e^{i\theta}+z}{e^{i\theta}-z}\log\left|F\left(e^{i\theta}\right)\right|\,d\theta\right] \tag{3.1.1.1}$$

is an outer function. Since G is an outer function, we have by Remark 3.1.0.2 that $\left|F(e^{i\theta})\right| = \left|G\left(e^{i\theta}\right)\right|$ for almost every θ. Consequently, $\|G\|_{L^1(\mathbb{D})} = \|F\|_{H^1(\mathbb{D})} < +\infty$. By (ii), we have that $|F(z)| \geq |G(z)|$ for each $z \in \mathbb{D}$. Since G is nonzero, we have that $|F(z)| \leq |G(z)|$ on $\mathbb{D}$. By interchanging the roles of F and G, we have that $|F(z)| = |G(z)|$ on $\mathbb{D}$. Thus, F/G is analytic and everywhere of absolute value 1. Define the function $\lambda : \mathbb{D} \to \mathbb{C}$ by $\lambda = F/G$. Then we have by the open mapping theorem, that $\lambda(\mathbb{D})$ must be an open set. However, we have that λ maps an open set to a subset of $\mathbb{T}$. Hence, λ must be a constant. ☺

We proceed to show that $(i) \Leftrightarrow (iii)$.

Proof of $(i) \Leftrightarrow (iii)$. Note that $(\Rightarrow)$ is immediate. We proceed to show that $(iii) \Rightarrow (i)$. Define G as in 3.1.1.1. Then we have that F/G is bounded by 1 in the disc. This is a direct consequence of definition of G and Fubini's theorem. Also, F/G has modulus 1 at zero. Thus, by Maximum modulus theorem, we have that $F/G = \lambda$, a constant of modulus 1 in $\mathbb{D}$. ☺

We finally prove (i) implies (ii). We state the following theorem without proof.

Theorem 3.1.1.2 (Poisson-Jensen (1915))**.** *Suppose that $f \in \mathcal{A}(\mathbb{D})$ and f has no zeroes on $\mathbb{T}$. Let $a_1, a_2, \ldots, a_n$ be the distinct zeros of f in $\mathbb{D}$ with multiplicities $k_1, \ldots, k_n$ respectively. Then for $re^{i\theta} \in \mathbb{D} \setminus \{a_1, a_2, \ldots, a_n\}$, we have*

$$\ln\left|f\left(re^{i\theta}\right)\right| = \sum_{j=1}^{n} k_j \ln\left|\frac{z - a_j}{1 - \overline{a_j}z}\right| + \frac{1}{2\pi}\int_{-\pi}^{\pi} P_r\left(e^{i(\theta - t)}\right)\ln\left|f\left(e^{it}\right)\right| dt$$

3.2 Infinite Products, Blaschke Products & Singular Functions

Theorem 3.2.0.1. *Let f be a bounded analytic function in the unit disc and further suppose that $f(0) \neq 0$. If (α_n) is the sequence of zeros in the open disc repeated as* *often as the multiplicity of the zero of f then the product is convergent, that is,*

$$\sum_{n=1}^{\infty}(1-|\alpha_n|)<\infty$$

Proof. We may assume wlog that $|f| \leq 1$ for convenience. If f has finitely many zeroes there is no question of convergence. Now, suppose that there are infinitely many zeroes. We claim that there can be only countably many of them.

Let $r < 1$. We claim that there are only finitely many zeroes of f in $\overline{B(0,r)}$. Suppose not. Then by Bolzano Weierstrass theorem, we have that $Z(f)$ have a limit point in $\overline{B(0,r)}$. By the identity theorem, we have that f must be identically zero in $\overline{B(0,r)}$.But then this would contradict the fact that $f(0) \neq 0$. Thus, every closed disk contains at most finitely many zeroes of f. Let (r_n) be a strictly increasing sequence converging to 1. Then zeroes of f in $\mathbb{D}$ is the union of zeroes of f in the disk $\overline{B(0,r_n)}$. This shows that the zeroes of the f is countable.

Let (α_n) be the sequence of zeroes counting multiplicities. We define a partial product $B_n(z)$ in the following fashion:

$$B_n(z)=\prod_{k=1}^{n}\frac{z-\alpha_k}{1-\overline{\alpha_k}z_k}$$

Observe that each B_n is a a rational function, analytic on $\mathbb{D}$ and whose absolute value is 1 on the boundary of the disc.

We now claim that f/B_n is a bounded analytic function on the unit disc. This can be seen as follows. First observe that f/B_n is bounded above by 1 on the boundary of the disc:

$$\left|\frac{f\left(e^{i\theta}\right)}{B_n\left(e^{i\theta}\right)}\right|=\left|f\left(e^{i\theta}\right)\right|\leq 1$$

for almost all θ. Hence, by the maximum modulus theorem, we have that $|f(z)| \leq$

$|B_n(z)|$ on $\mathbb{D}$. Thus, we have that

$$0 < |f(0)| \leq |B_n(0)| = \prod_{k=1}^{n} |\alpha_k|$$

Since $|\alpha_k| < 1$ for each $k \in \mathbb{N}$, we have that the product is decreasing and is bounded by a nonnegative number and hence the product converges. $\smile$

We now prove a necessary and sufficient condition for the convergence of product of zeroes of an analytic function.

Theorem 3.2.0.2. *Let (α_n) be a sequence of nonzero complex numbers in the open unit disc. A necessary and sufficient condition that the infinite product*

$$\prod_{n=1}^{\infty} \frac{\overline{\alpha_n}}{|\alpha_n|} \frac{\alpha_n - z}{1 - \overline{\alpha_n} z}$$

should converge uniformly on compact subsets of the disc is that the product $\prod |\alpha_n|$ should converge, that is, that

$$\sum_{n=1}^{\infty} (1 - |\alpha_n|) < \infty$$

When either of the conditions are satisfied, the product defines an inner function whose zeros are exactly $\alpha_1, \alpha_2, \ldots$.

Proof. $\smile$

Definition 3.2.0.3 (Blaschke product)**.** A Blaschke product is an analytic function of the form

$$B(z) = z^p \prod_{n=1}^{\infty} \left[\frac{\overline{\alpha_n}}{|\alpha_n|} \frac{\alpha_n - z}{1 - \overline{\alpha_n} z} \right]^{p^n}$$

where

(i) $p, p_1, p_2, \ldots$ are nonnegative integers;

(ii) the α_n are distinct nonzero integers in the open unit disc;

(iii) the product $\prod |\alpha_n|^{p_n}$ is convergent.

Theorem 3.2.0.4. *Let $f \in H^\infty$ and suppose that $f \not\equiv 0$, that is, not identically zero. Then f is uniquely expressible in the form $f = Bg$ where B is a Blaschke product and g is a bounded analytic function without any zeros.*

Proof. Suppose that $f \in H^\infty$ and suppose that f is not identically zero. We wish to show that f can factored as a Blaschke product as in the previous definition and a bounded analytic function which is zerofree.

Since f is not identically zero, we write $f(z) = z^p h(z)$ where h is analytic and $h(0) \neq 0$ by Taylor's theorem on $\mathbb{D}$. Let $\alpha_1, \alpha_2, \ldots, \alpha_n$ be the roots of h with multiplicities $p_1, p_2, \ldots, p_n$ respectively. We then consider the Blaschke product B formed by the zeroes of h. Hence,

$$B(z) = z^p \prod_{n=1}^{\infty} \left[\frac{\overline{\alpha_n}}{|\alpha_n|} \frac{\alpha_n - z}{1 - \alpha_n z} \right]^{p^n}$$

Let us suppose that $\|f\|_{H^\infty} \leq M$. Then by the isometry isomorphism, we have that $\|f\|_{H^\infty(T)} \leq M$. Hence, we have that

$$\left| \frac{f\left(e^{i\theta}\right)}{B\left(e^{i\theta}\right)} \right| = \left| f\left(e^{i\theta}\right) \right| \leq M$$

almost all $e^{i\theta} \in \mathbb{T}$. Thus by the maximum modulus theorem, we have that $|f(z)| \leq |B_n(z)|$ for $z \in \mathbb{D}$. Thus $g = f/B$ is bounded and analytic on $\mathbb{D}$. Hence, we have that $f = Bg$ is unique because a Blaschke product is uniquely determined by its zeros. ☺

Theorem 3.2.0.5. *Let g be an inner function without zeroes, and suppose that $g(0)$ is positive. Then there is a unique singular positive measure μ on $\mathbb{T}$ such that*

$$g(z) = exp \left[- \int_T \frac{e^{i\theta} + z}{e^{i\theta} - z} d\mu\left(e^{i\theta}\right) \right]$$

Proof. Let g be analytic function in the disc which is zerofree. Then g has an analytic logarithm. Therefore, we may write

$$g = e^{-h}$$

for some analytic function h in the disc. Since g is bounded by 1, it must be that h must be nonnegative on the disc. Let $h = u + iv$ for some real valued functions u, v. Then $u \geq 0$.

Now, the nonnegative harmonic function u is uniquely expressible in the form

$$u\left(re^{i\theta}\right) = \int P_r\left(\theta - t\right) d\mu\left(t\right)$$

where μ is a positive measure on the circle. SIince $g\left(0\right) > 0$ we have that $v\left(0\right) = 0$. Thus, we have that

$$h\left(z\right) = \int \frac{e^{i\theta} + z}{e^{i\theta} - z} d\theta.$$

Now, $|g| = 1$ almost everywhere on the circle. Since $|g| = e^{-u}$, this means the non-tangential limits of u must vanish almost everywhere on $\mathbb{T}$. But these non-tangential limits are equal to $\frac{1}{2\pi}\frac{d\mu}{d\theta}$. So μ is singular and this completes the proof. ☺

Theorem 3.2.0.6. *Let $f \neq 0$ be an H^1 function in the unit disc. Then f is uniquely expressible in the form of $f = BSF$ where B is a Blaschke product, S is a singular function and F is an outer function (in H^1).*

Proof. Since $f \neq 0$ and is in H^1, we have that $f = gF$ for some inner function g and outer function F. We know that this factorization is unique up to constant multiple of modulus 1. If B is the Blaschke product formed from the zeroes of g (that is, the zeroes of f) then $g = BS$, where $S$is an inner function without zeroes. By multiplying g by a constant of modulus 1, we can arrange that so that $S\left(0\right) > 0$, that is a singular function. WE can absorb that into the outer function F and we are done. ☺

3.2.1 Final Description of the Factorization

Let $f \in H^1(\mathbb{D})$, $f \not\equiv 0$. By the previous theorem, we have that f can be factorized to BSF as in the previous theorem. Let p be the order of zero of f at the origin and let $p_1, p_2, \ldots$ be the multiplicities of the remaining zeroes $\alpha_1, \alpha_2, \ldots$ of f.

Then we have that

$$\begin{aligned} B(z) &= z^p \prod_{n=1}^{\infty} \left[\frac{\overline{\alpha_n}}{|\alpha_n|} \frac{\alpha_n - z}{1 - \overline{\alpha_n} z} \right]^{p_n} \\ F(z) &= \exp \left[\frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{e^{i\theta} + z}{e^{i\theta} - z} \left(\log \left| f\left(e^{i\theta}\right) \right| + ia \right) d\theta \right] \\ S(z) &= \frac{f(z)}{B(z) F(z)} = \exp \left[- \int \frac{e^{i\theta} + z}{e^{i\theta} - z} d\mu(\theta) \right] \end{aligned}$$

for some positive singular measure μ and where $a = \arg(f/B)(0)$.

We can deduce a generalised Jensen formula from this factorisation. If $f(0) \neq 0$ then

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} \log \left| f\left(e^{i\theta}\right) \right| d\theta = \log |f(0)| + \sum_n p_n \log |\alpha_n|^{-1} + \int d\mu.$$

Definition 3.2.1.1. If a function f can be factorised into BSF where the notations are as in the previous theorem, we will call F the **outer part** of f and $B \cdot S$ the **inner** part of f.

Theorem 3.2.1.2. *The Blaschke product whose zeroes are*

$$\alpha_1, \alpha_2, \ldots, \ 0 < |\alpha_n| < 1$$

converges at all points z in the complex plane except those in the compact set K consisting of

(i) the points $z = 1/\overline{\alpha_n}$;

(ii) the points z on the unit circle which are accumulation points of the sequence (α_n).

The convergence is uniform on any closed set in the plane which is disjoint from K, and the product $B(z)$ is thus analytic off K.

Proof. First, we need to show that the set K which is defined by

$$K = \left\{ \frac{1}{\bar{\alpha_n}} : n \in \mathbb{N} \right\} \cup \{ z \in \mathbb{T} \, : \, z \text{ is an accumulation point of the sequence } (\alpha_n) \}$$

is indeed a compact set.

By the Blaschke condition, we have that $\sum_{n\in\mathbb{N}} (1 - |\alpha_n|)$ and hence $\lim_{n\to\infty} |\alpha_n| = 1$. This tells us that the sequence K is bounded. To show that K is compact, it suffices to show that the set K is closed. It is easy to see that the sequence $\left(\frac{1}{\bar{\alpha_n}}\right)$ must accumulate on the boundary because (α_n) can only accumulate on the boundary. Thus, K is closed. This shows that the set K is compact.

Let F be any closed set disjoint from K. Then let $M = d(F, K) > 0$. Then we have that for every $z \in F$ and every $n \in \mathbb{N}$,

$$\left| \frac{1}{\bar{\alpha_n} - z} \right| \geq N \Longleftrightarrow |1 - \bar{\alpha_n} z| \geq M.$$

Now, for $z \in K$, we have that

$$\begin{aligned} 1 - f_n(z) &= \frac{1 - |\alpha_n|}{|\alpha_n|} \left[\frac{1 + |\alpha_n|}{1 - \bar{\alpha}_n z} - 1 \right] \\ \rightsquigarrow |1 - f_n(z)| &\leq \frac{1 - |\alpha_n|}{|\alpha_n|} \left[\frac{2}{M} + 1 \right] \end{aligned}$$

Since the sequence (α_n) satisfies the Blaschke condition, we have that $\sum_n (1 - |\alpha_n|)$ converges and as s a consequence we have that $\sum |1 - f_n(z)|$ is uniformly summable and hence the product $\prod_n f_n(z)$ is uniformly and absolutely convergent on the closed sets disjoint from K. This shows that the Blaschke product $B(z)$ is analytic off K. [1]

Before we go on the next theorem, we make the following definition:

[1] See Theorem 6.1.7 in Ash & Novinger's Complex Variables.

Definition 3.2.1.3. Let μ be a finite signed or a complex measure. The support of the measure μ is defined in the following manner:

$$\operatorname{supp}(\mu) = X \setminus \bigcup \{G \subset \mathbb{T} \, : \, G \text{ open}, \mu(G) = 0\}.$$

Theorem 3.2.1.4. *Let $\mu \in \mathcal{M}(\mathbb{T})$ be a positive singular measure. Consider the singular function S that is determined by μ. Note that $S : \mathbb{C} \to \mathbb{C}$ is given by:*

$$S(z) = \exp\left[-\int_{-\pi}^{\pi} \frac{e^{i\theta} + z}{e^{i\theta} - z} d\mu(\theta)\right]$$

for each $z \in \mathbb{C}$. Then we have that S is analytic in $\mathbb{C} \setminus \operatorname{supp}(\mu)$. Also, the function S (or even $|S|$) is not continuously extendable from the interior of the disc to any point in $\operatorname{supp}(\mu)$.

Proof. Let μ be a finite positive measure on $\mathbb{T}$.

3.3 Two theorems due to Hardy

Before we prove the theorems due to Hardy and an another one which due to Haryd and Littlewood, we prove a lemma which is nontrivial, in the sense, that every H^1 function can be factored as a product of two H^2 functions. On the other hand, it can be easily seen that product of two H^2 functions is always a H^1 function.

Theorem 3.3.0.1 (Hardy's inequality). *Let $F \in H^1(\mathbb{D})$. Then there exists $G, K \in H^2(\mathbb{D})$ such that*

$$F = GK$$

and

$$\|F\|_1 = \|G\|_2^2 = \|K\|_2^2.$$

Proof. By Theorem 3.2.0.4, F can be factored into $F = B\Phi$ where B is a Blaschke product and $\Phi \in H^1(\mathbb{D})$ is free of zeroes on $\mathbb{D}$ with $\|\Phi\|_1 = \|F\|_1$. Since Φ is zerofree on $\mathbb{D}$, it must have an analytic square root. [2] Let

$$G = B\Phi^{1/2} \text{ and } K = \Phi^{1/2}.$$

Clearly then $GK = F$ and $\|K\|_2^2 = \|\Phi\|_1 = \|F\|_1$. Moreover, again by Theorem 3.2.0.4, we have that

$$\|G\|_2^2 = \left\|B\Phi^{1/2}\right\|_2^2 = \left\|\Phi^{1/2}\right\|_2^2 = \|\Phi\|_1 = \|F\|_1 .$$

Now, we state Hardy's inequality:

Theorem 3.3.0.2 (Hardy)**.** *Let f be a function in H^1 with power series*

$$\sum_{n=0}^{\infty} a_n z^n.$$

Then we have that

$$\sum_{n=1}^{\infty} \frac{1}{n} |a_n| \leq \pi \|f\|_1 .$$

Proof. First, we consider the case when the "Fourier" coefficients of f are $a_n \geq 0$ for $n \geq 0$. Then we have that

$$\Im f\left(re^{i\theta}\right) = \sum_{n=1}^{\infty} a_n r^n \sin(n\theta) .$$

A simple computation shows that

$$\frac{1}{2\pi}\int_0^{2\pi} (\pi - \theta)\sin(n\theta)\, d\theta = \frac{1}{n}.$$

[2] For a proof of this fact, see Complex Variables by Ash and Novinger.

Using the above, we have that

$$\sum_{n=1}^{\infty} \frac{1}{n} a_n r^n = \frac{1}{2\pi} \int_0^{2\pi} (\pi - \theta) \Im f\left(re^{i\theta}\right) d\theta \leq \frac{1}{2} \int_0^{2\pi} \left| f\left(re^{i\theta}\right)\right| d\theta = \pi \left\| f \right\|_1$$

Letting $r \to 1$, we have the theorem where we assumed that $a_n \geq 0$.

Using the above theorem 3.3.0.1, we have that the function f can be factored into two H^2 functions. Hence, we have that

$$f = gh.$$

Now, we can write

$$g(z) = \sum_{n=0}^{\infty} b_n z^n$$
$$h(z) = \sum_{n=0}^{\infty} c_n z^n$$

Then by the Riesz Fischer theorem, we have that the functions

$$G(z) = \sum |b_n| z^n$$
$$H(z) = \sum |c_n| z^n$$

are also in H^2; in fact,

$$\|G\|_2 = \|g\|_2$$
$$\|H\|_2 = \|h\|_2 .$$

Let $F = GH$. Certainly $F \in H^1$ and we have that

$$F(z) = \sum_{n=0}^{\infty} \tilde{a_n} z^n$$

where $\tilde{a_n} \geq 0$. It is also apparent that $|a_n| \leq \tilde{a_n}$. It follows by the first part of the proof that

$$\sum_{n=1}^{\infty} \frac{1}{n} |a_n| \leq \sum_{n=1}^{\infty} \frac{1}{n} \tilde{a_n} \leq \pi \|F\|_1 .$$

But

$$\|F\|_1 \leq \|G\|_2 \|H\|_2 = \|g\|_2 \|h\|_2 = \|f\|_1$$

and this completes the proof. $\smile$

Theorem 3.3.0.3. *Let f be a function on the unit circle which is both of bounded variation and in H^1. Then*

(i) f is an absolutely continuous function;

(ii) the Fourier series for f is absolutely convergent.

Proof. Since f is of bounded variation, the Fourier coefficients of f are

$$a_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} f\left(e^{i\theta}\right) e^{-in\theta} d\theta = \frac{i}{n} \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{-in\theta} df\left(\theta\right) \text{ for } \neq 0$$

This shows that *df* is analytic. By F and M Riesz, *df* is absolutely continuous, i.e, $df = gd\theta$, where $g \in H^1$. Thus $a_n = \frac{i}{n} b_n$ where b_n is the nth Fourier coeffiecient of g for $n = 1, 2, 3, \ldots$. By the last theorem, we have that

$$\sum_{i=1}^{\infty} |a_n| = \sum_{n=1}^{\infty} \frac{1}{n} |b_n| < \infty$$

$\smile$