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§2 Functions and Integrals

§2.1 Measurable Functions

§2.1.1 Question 1

Observe that all this question wants us to prove is that

$$\chi_{\limsup A_n} = \limsup \chi_{A_n}$$

This is easy and simply follows from the definition.

§2.1.2 Question 2

Let Y be a subset of $\mathbb{R}$ which is not Borel measurable. Then χ_Y cannot be Borel measurable (See Example 2.1.2 (b) in the book). Observe that Y cannot be countable for otherwise we could write Y as union of its singleton members of Y and hence Y would be Borel set. Thus, Y is uncountable. Now, consider the set of functions $J := \left\{\chi_{\{y\}} : y \in Y\right\}$. It is easy to see that $\chi_Y = \sup\left\{\chi_{\{y\}} : y \in Y\right\}$. Clearly, J is set of Borel measurable functions whose supremum is not Borel measurable.

§2.1.3 Question 3

Let $f : \mathbb{R} \to \mathbb{R}$ be a function which is differentiable everywhere. Consider the sequence of function $\{f_n\}_{n\in\mathbb{N}}$ which is given by $f_n(x) = \frac{f\left(x+\frac{1}{n}\right)-f(x)}{1/n}$ for every $x \in \mathbb{R}$. Clearly, $f_n \to f$ pointwise on $\mathbb{R}$ and each f_n is measurable. Since limit of Borel measurable functions is measurable, we have that f is Borel measurable.

§2.1.4 Question 4

Let $A := \left\{x \in X : \lim_n f_n(x) \text{ exists and finite}\right\}$. Let $B := \left\{x \in X : \limsup_n f_n(x) = \liminf_n f_n(x)\right\}$. Then B is measurable by virtue of Proposition 2.1.5 and 2.1.3. Note that B is the set of all points x in X where $\lim_n f_n(x)$ exists. Now $|f|$ is a measurable function. So the set $C := \bigcap_n \left\{x \in X : |f|(x) \leq n\right\}$ is measurable. Notice that set $A = B \cap C$ and this completes the proof.

§2.1.5 Question 5

Let $(X, \mathscr{A})$ be a measurable space and let $f, g : X \to \mathbb{R}$ be simple, measurable functions. We first show the following statement is true: A measurable function $f : X \to \mathbb{R}$ is simple iff there exists $\alpha_1, \ldots, \alpha_n \in \mathbb{R}$ and $A_1, \ldots, A_n \in \mathscr{A}$ such that $f = \sum_{i=1}^n a_i \chi_{A_i}$.

Let $f : X \to \mathbb{R}$ be measurable. We start the proof of the above statement. ($\Longrightarrow$) Suppose that f is simple. Then, by definition, f takes only finitely many values, say, $\alpha_1, \ldots, \alpha_n$. Since f is measurable, the set $A_i := \left\{x \in X : f(x) = \alpha_i\right\}$ is measurable. Then it is easy to check that $f = \sum_{i=1}^n a_i \chi_{A_i}$. ($\Longleftarrow$) Suppose that there exists $\alpha_1, \ldots, \alpha_n \in \mathbb{R}$ and $A_1, \ldots, A_n \in \mathscr{A}$ such that $f = \sum_{i=1}^n a_i \chi_{A_i}$. Suppose that $\alpha_1 < \alpha_2 < \ldots < \alpha_n$. It is fairly easy to check by cases that f is measurable (Just see what happens when $\alpha_i < t$ for each i).

1. Let $f, g : X \to \mathbb{R}$ be simple measurable functions. Then there exists $\alpha_1, \dots, \alpha_n \in \mathbb{R}$ and $A_1, \dots, A_n \in \mathscr{A}$ such that $f = \sum_{i=1}^{n} a_i \chi_{A_i}$ and there exists $\beta_1, \dots, \beta_m$ and measurable sets $B_1, B_2, \dots, B_m$ such that $g = \sum_{j=1}^{m} b_j \chi_{B_j}$. It is then easy to see that $f+g = \sum_i \sum_j \left(a_i + b_j\right) \chi_{A_i \cap B_j}$ and $fg = \sum_i \sum_j a_i b_j \chi_{A_i \cap B_j}$. By equivalent statement that we proved above, we are done.

2. Let $f, g : X \to \mathbb{R}$ be measurable functions. Then by Proposition 2.1.8, there exist a nondecreasing sequences of real valued measurable simple functions $\{f_n\}$ and $\{g_n\}$ converging pointwise to f and g respectively. By part (1) of this problem $\{f_n + g_n\}$ is a nondecresasing sequence of real valued measurable simple functions converging pointwise to $f + g$. By Proposition 2.1.5, we have that limits of measurable functions are measurable and we are done.

§2.1.6 Question 6

Let $f : X \to \mathbb{R}$ be measurable function. We first show that $x \mapsto t - f(x)$ is a measurable function for every $t \in \mathbb{R}$. Let $t \in \mathbb{R}$ be fixed, let $s \in \mathbb{R}$ be some arbitrary real number. Then $\left\{x \in X : t - f(x) < s\right\} = \left\{x \in X : f(x) > t - s\right\}$. Since the set on the right hand side of the previous inequality is measurable as f is measurable, we are done.

Now, let $f, g : X \to \mathbb{R}$ be measurable functions. To show that $f + g$ is measurable, let $t \in \mathbb{R}$. Then the set $\left\{x \in X : f(x) + g(x) < t\right\} = \left\{x \in X : g(x) < t - f(x)\right\}$. Since g is measurable and the function $x \mapsto t - f(x)$ is measurable by the previous paragraph, Proposition 2.1.6 implies that $f + g$ is measurable.

§2.1.7 Question 7

§2.1.8 Question 8

§2.1.9 Question 9

§2.1.10 Question 10

Let $\mathscr{V}_0$ be the collection of all Borel functions $f : \mathbb{R} \to \mathbb{R}$. Proposition 2.1.7. shows that $\mathscr{V}_0$ is a vector space. Since every continuous function is Borel measurable, we have the set of allcontinuous functions from $\mathbb{R}$ to $\mathbb{R}$ is contained in $\mathscr{V}_0$. The third property is satisfied due to Proposition 2.1.5.

We follow the hint given in the book. Let $\mathscr{V}$ be a collection of functions satisfying the three criteria given. Let $S(\mathscr{V}) = \left\{A \subseteq \mathbb{R} : \chi_A \in \mathscr{V}\right\}$. Clearly

§2.2 Properties That Hold Almost Everywhere

§2.2.1 Question 1

Consider the function $\chi_{\mathbb{R}\setminus\mathbb{Q}}$ and the constant function 1. Both are Borel measurable. and they both agree on a dense set of $\mathbb{R}$, namely, $\mathbb{R}\setminus\mathbb{Q}$. But the set $\left\{x \in \mathbb{R} : \chi_{\mathbb{R}\setminus\mathbb{Q}} \neq 1\right\} = \mathbb{Q}$ which is a countable set and hence λ-negligible.

§2.2.2 Question 2

Let $\{x_n\}$ be a sequence of real numbers and let μ be the measure defined by $\mu = \sum_n \delta_{x_n}$. Exercise 1.2.6 shows that μ is indeed a measure on $(\mathbb{R}, \mathscr{B}(\mathbb{R}))$. Let $f, g : \mathbb{R} \to \mathbb{R}$ be two functions.

We want to show that f, g agree μ-almost everywhere iff $f(x_n) = g(x_n)$ holds for every n.

($\Longrightarrow$) Suppose that f, g agree almost everywhere. Then $A := \left\{x \in \mathbb{R} : f(x) \neq g(x)\right\}$ is μ negligible. Since $A \in \mathscr{B}(\mathbb{R})$, we have that $\mu(A) = 0$. So if there was some $k \in \mathbb{N}$ such that $f(x_k) \neq g(x_k)$ then $x_k \in A$ and then $\mu(\{x_k\}) = 1$. But since $\{x_k\} \subseteq A$, we have that $1 < 0$ which is absurd. Hence, we are done!

($\Longleftarrow$) Suppose that $f(x_n) = g(x_n)$ for every $n \in \mathbb{N}$. We need to prove that A is μ-negligible. Since $A \cap \{x_k : k \in \mathbb{N}\} = \emptyset$, we have that $\mu(A) = \sum_n \delta_{x_n}(A) = 0$.

§2.2.3 Question 3

We show the following is true: If f is a continuous real-valued function on $\mathbb{R}$ and $f = 0$ λ-almost everywhere then f is the zero function.

By assumption, $\mu\left(\left\{x \in \mathbb{R} : f(x) \neq 0\right\}\right) = 0$.

Let $x \in \mathbb{R}$. We claim that for every $\varepsilon > 0$ there is some $y \in \mathbb{R}$ such that $f(y) = 0$ and $|y - x| < \varepsilon$.

Suppose not. Then there is some $\varepsilon > 0$ such that for every $y \in \mathbb{R}$ with $|y - x| < \varepsilon$ we have $f(y) \neq 0$.

But note that previous statement is a contradiction as $(x - \varepsilon, x + \varepsilon) \subset \left\{x \in \mathbb{R} : f(x) \neq 0\right\}$.

§2.2.4 Question 4

§2.2.5 Question 5

Let $(X, \mathscr{A}, \mu)$ be a measure space. Let $f, f_1, f_2, \ldots$ be a sequence of $\mathscr{A}$-measurable $[-\infty, \infty]$-valued functions on X. Suppose that $f_n \to f$ converges μ-almost everywhere.

Let $N = \left\{x \in X : \lim_n f_n(x) \neq f(x)\right\}$. Define $g_n : X \to [-\infty, \infty]$ by $g_n = f_n \circ \chi_{N^c} + f \circ \chi_N$ for every $x \in X$ and every $n \in \mathbb{N}$.

We claim that $g_n \to f$ everywhere. This is easy to see. If $x \in N$ then $g_n(x) = f(x)$ for every $n \in \mathbb{N}$ and if $x \in N^c$ then $g_n(x) = f_n(x)$ and $\lim_n g_n(x) = \lim_n f_n(x)$. Since $x \in N^c$, $\lim_n g_n(x) = f(x)$. It is easy to see that $N^c \subset \left\{x \in X : g_n(x) = f_n(x)\right\}$ so $\left\{x \in X : g_n(x) \neq f_n(x)\right\} \subseteq N$. Since N is μ-negligible, we are done.

§2.2.6 Question 6

See Wikipedia! Thomae's function!