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§2 Functions and Integrals

§2.1 Measurable Functions

§2.1.1 Question 1

Observe that all this question wants us to prove is that

$$\chi_{\limsup A_n} = \limsup \chi_{A_n}$$

This is easy and simply follows from the definition.

§2.1.2 Question 2

Let Y be a subset of $\mathbb{R}$ which is not Borel measurable. Then χ_Y cannot be Borel measurable (See Example 2.1.2 (b) in the book). Observe that Y cannot be countable for otherwise we could write Y as union of its singleton members of Y and hence Y would be Borel set. Thus, Y is uncountable. Now, consider the set of functions $J := \left\{\chi_{\{y\}} : y \in Y\right\}$. It is easy to see that $\chi_Y = \sup\left\{\chi_{\{y\}} : y \in Y\right\}$. Clearly, J is set of Borel measurable functions whose supremum is not Borel measurable.

§2.1.3 Question 3

Let $f : \mathbb{R} \to \mathbb{R}$ be a function which is differentiable everywhere. Consider the sequence of function $\{f_n\}_{n\in\mathbb{N}}$ which is given by $f_n(x) = \frac{f\left(x+\frac{1}{n}\right)-f(x)}{1/n}$ for every $x \in \mathbb{R}$. Clearly, $f_n \to f$ pointwise on $\mathbb{R}$ and each f_n is measurable. Since limit of Borel measurable functions is measurable, we have that f is Borel measurable.

§2.1.4 Question 4

Let $A := \left\{x \in X : \lim_n f_n(x) \text{ exists and finite}\right\}$. Let $B := \left\{x \in X : \limsup_n f_n(x) = \liminf_n f_n(x)\right\}$. Then B is measurable by virtue of Proposition 2.1.5 and 2.1.3. Note that B is the set of all points x in X where $\lim_n f_n(x)$ exists. Now $|f|$ is a measurable function. So the set $C := \bigcap_n \left\{x \in X : |f|(x) \leq n\right\}$ is measurable. Notice that set $A = B \cap C$ and this completes the proof.

§2.1.5 Question 5

Let $(X, \mathscr{A})$ be a measurable space and let $f, g : X \to \mathbb{R}$ be simple, measurable functions. We first show the following statement is true: A measurable function $f : X \to \mathbb{R}$ is simple iff there exists $\alpha_1, \ldots, \alpha_n \in \mathbb{R}$ and $A_1, \ldots, A_n \in \mathscr{A}$ such that $f = \sum_{i=1}^n a_i \chi_{A_i}$.

Let $f : X \to \mathbb{R}$ be measurable. We start the proof of the above statement. ($\Longrightarrow$) Suppose that f is simple. Then, by definition, f takes only finitely many values, say, $\alpha_1, \ldots, \alpha_n$. Since f is measurable, the set $A_i := \left\{x \in X : f(x) = \alpha_i\right\}$ is measurable. Then it is easy to check that $f = \sum_{i=1}^n a_i \chi_{A_i}$. ($\Longleftarrow$) Suppose that there exists $\alpha_1, \ldots, \alpha_n \in \mathbb{R}$ and $A_1, \ldots, A_n \in \mathscr{A}$ such that $f = \sum_{i=1}^n a_i \chi_{A_i}$. Suppose that $\alpha_1 < \alpha_2 < \ldots < \alpha_n$. It is fairly easy to check by cases that f is measurable (Just see what happens when $\alpha_i < t$ for each i).

1. Let $f, g : X \to \mathbb{R}$ be simple measurable functions. Then there exists $\alpha_1, \ldots, \alpha_n \in \mathbb{R}$ and $A_1, \ldots, A_n \in \mathscr{A}$ such that $f = \sum_{i=1}^{n} a_i \chi_{A_i}$ and there exists $\beta_1, \ldots, \beta_m$ and measurable sets $B_1, B_2, \ldots, B_m$ such that $g = \sum_{j=1}^{m} b_j \chi_{B_j}$. It is then easy to see that $f+g = \sum_i \sum_j \left(a_i + b_j\right) \chi_{A_i \cap B_j}$ and $fg = \sum_i \sum_j a_i b_j \chi_{A_i \cap B_j}$. By equivalent statement that we proved above, we are done.

2. Let $f, g : X \to \mathbb{R}$ be measurable functions. Then by Proposition 2.1.8, there exist a nondecreasing sequences of real valued measurable simple functions $\{f_n\}$ and $\{g_n\}$ converging pointwise to f and g respectively. By part (1) of this problem $\{f_n + g_n\}$ is a nondecresasing sequence of real valued measurable simple functions converging pointwise to $f + g$. By Proposition 2.1.5, we have that limits of measurable functions are measurable and we are done.

§2.1.6 Question 6

Let $f : X \to \mathbb{R}$ be measurable function. We first show that $x \mapsto t - f(x)$ is a measurable function for every $t \in \mathbb{R}$. Let $t \in \mathbb{R}$ be fixed, let $s \in \mathbb{R}$ be some arbitrary real number. Then $\left\{x \in X : t - f(x) < s\right\} = \left\{x \in X : f(x) > t - s\right\}$. Since the set on the right hand side of the previous inequality is measurable as f is measurable, we are done.

Now, let $f, g : X \to \mathbb{R}$ be measurable functions. To show that $f + g$ is measurable, let $t \in \mathbb{R}$. Then the set $\left\{x \in X : f(x) + g(x) < t\right\} = \left\{x \in X : g(x) < t - f(x)\right\}$. Since g is measurable and the function $x \mapsto t - f(x)$ is measurable by the previous paragraph, Proposition 2.1.6 implies that $f + g$ is measurable.