# Functional Analysis Assignment 5

Ashish Kujur

## Note

A checkmark ✓ indicates the question has been done.

## Contents

# 1 Question 1

Suppose $M$ and $N$ are two topologically complimentary closed subspace of a Banach space $(X, \|\cdot\|_X)$. Now consider $M \oplus_1 N$, the external direct sum, defined in the following way )

$$M \oplus_1 N = \{(m,n) : m \in M, n \in N\}, \ \|(m,n)\|_1 = \|m\|_X + \|n\|_X.$$

(a) Show that $M \oplus_1 N$ is a Banach space w.r.t the norm $\|\cdot\|_1$ mentioned above.

(b) Show that $X$ is isomorphic to $M \oplus_1 N$.

(c) Show that the quotient space $X/M$ is isomorphic to the Banach space $N$.

---

*Proof of item (a).* We proceed to prove (a). Let $((m_k, n_k))_{k\in\mathbb{N}}$ be a Cauchy sequence in $M \oplus_1 N$. We show that $(m_k)$ is Cauchy in $X$. Consider the following:

$$\|m_k - m_l\|_X \leq \|(m_k, n_k) - (m_l, n_l)\|_1 .$$

Now since $((m_k, n_k))$ is Cauchy, we have that $(m_k)$ is Cauchy in $X$. Since $X$ is a Banach space, we have that $(m_k)$ converges to some $m \in M$ as $M$ is closed. Likewise it can be shown that $(n_k)$ converges to some $n \in N$. We now show that $((m_k, n_k))$ converges to $(m, n)$ in $M \oplus_1 N$. Consider the following:

$$\|(m_k, n_k) - (m, n)\| = \|m_k - m\|_X + \|n_k - n\|_X$$

Since $(m_k)$ converges to $m$ and $(n_k)$ converges to $n$, we are done. ⌣̈

*Proof of item (b).* To show that $X$ is isomorphic to $M \oplus_1 N$, consider the map $T : M \oplus_1 N \to X$ given by

$$T(m,n) = m + n$$

for every $m \in M$ and every $n \in N$. First, we show that $T$ is a normed linear space isomorphism, that is, both $T$ and $T^{-1}$ are bounded linear operators. It is immediate that $T$ is bijective and linear. Since the projection maps $m+n \to m$ and $m+n \to n$ are continuous, there are some constant $\mu$ and $\nu$ such that $\|m\|_X \leq \mu \|m+n\|_X$ and $\|n\|_X \leq \nu \|m+n\|_X$. Now, let $m \in M$ and $n \in N$. Then

$$\begin{aligned} \|T(m,n)\|_X &= \|m+n\|_X \\ &\leq \|m\|_X + \|n\|_X \\ &= \|(m,n)\|_1 \end{aligned}$$

and

$$\begin{aligned} \left\|T^{-1}(m+n)\right\|_1 &= \|(m,n)\|_1 \\ &= \|m\|_1 + \|n\|_1 \\ &\leq (\mu+\nu)\|m+n\|_X \end{aligned}$$

This shows that $X$ is isomorphic to $M \oplus_1 N$. ⌣̈

*Proof of item (c).* Let $P_N : X \to N$ be the projection of $X$ into $N$. Since $P_N$ is onto, by the first isomorphism theorem for vector spaces, we have that $X/M \cong N$. It remains to show that map $[x]_M \mapsto P_N(x)$ and its inverse is continuous (note this is the isomorphism given by the first isomorphism theorem). We show that the map $P_N(x) \mapsto [x]_M$ is continuous. Let $x \in X$. Suppose $x = m + n$. Then we have that $P_N(x) = n$. Then

$$\begin{aligned} \|[x]_M\| &\leq \|x - m\| && \text{(by definition of quotient norm)} \\ &= \|n\| \\ &= \|P_N(x)\|_X \end{aligned}$$

This shows that the aforementioned map is continuous and bijective, by the Banach isomorphism theorem, we are done. $\ddot\smile$

## 2 Question 2

---

Let $H$ be a Hilbert space with an orthonormal basis $\{e_j : j \in \mathbb{N}\}$. Consider the set

$$A = \{e_k + ke_l : k < l,\ k, l \in \mathbb{N}, \}.$$

Show that 0 belongs to the weak closure of $A$. Also show that there is no sequence in $A$ which converge weakly to 0.

---

*Proof.* Recall the fact that in a topological space, we have that $x \in \overline{A}$ if there is a sequence $(x_n)_{n\in\mathbb{N}}$ in $A$ converging to $x \in A$.

For each $k \in \mathbb{N}$, we have that the sequence $(e_k + ke_l)_{l\geq k}$ converges to $e_k$. Thus, we have that $e_k \in \overline{A}^w$. Also, $(e_k) \in \overline{A}^w$ converges to 0. Therefore, we have that $0 \in \overline{A}^w$.

Let $(\tilde{e}_n)$ be a sequence in $A$ converging to 0. Then $\tilde{e}_n = e_{k_n} + k_n e_{l_n}$ for some $k_n, l_n \in \mathbb{N}$ with $k_n < l_n$. Then $(\tilde{e}_n)$ must be norm bounded. Let $M > 0$ such that $\|\tilde{e}_n\| \leq M$ for each $n \in \mathbb{N}$.

We claim that $\{k_n : n \in \mathbb{N}\}$ is finite. This is easy to see:

$$\begin{aligned} M &\geq \|k_n e_{l_n} + e_{k_n}\| \\ &\geq |k_n \|e_{l_n}\| - \|e_{k_n}\|| \\ &\geq k_n - 1 \end{aligned}$$

for each $n \in \mathbb{N}$.

Since the aforementioned set is finite, we may let $\{k_n : n \in \mathbb{N}\} = \{k_{n_1}, \ldots, k_{n_l}\}$ for some $n_1, n_2, \ldots, n_l \in \mathbb{N}$. It is a consequence of Riesz Representation theorem that in a Hilbert Space, $x_n \to x$ weakly iff $\langle x_n, y\rangle \to \langle x, y\rangle$ for each $y \in H$. We use this to achieve a contradiction. Now, let $y = e_{k_{n_1}} + e_{k_{n_2}} + \ldots + e_{k_{n_l}}$. It can be seen that $\langle \tilde{e}_n, y\rangle \geq 1$ for each $n \in \mathbb{N}$ and cannot converge to 0 as $n \to \infty$. $\ddot\smile$

## 3 Question 3

---

Let $H$ be a Hilbert space. Suppose $\{x_n\}_{n\in\mathbb{N}}$ is a sequence of vector in $H$ which converges weakly to a vector $x$ in $H$ and $\|x_n\| \to \|x\|$ as $n \to \infty$. Then show that $\|x_n - x\| \to 0$ as $n \to \infty$.

---

*Proof.* Let $(x_n)$ be a sequence in $H$ converging to $x \in H$ and furthermore suppose that $\|x_n\| \to \|x\|$ as $n \to \infty$. We wish to show that $x_n \to x$ strongly.

Since $x_n \to x$ weakly, we have that $\langle x_n, y\rangle \to \langle x, y\rangle$ for each $y \in H$ by the definition and Riesz Representation theorem. Thus, we have that $\langle x_n, x\rangle \to \|x\|^2$ in particular.

Now,

$$\|x_n - x\|^2 = \|x_n\|^2 - 2\Re \langle x_n, x\rangle + \|x\|^2 .$$

Taking limits both sides, we have that

$$\lim_{n\to\infty} \|x_n - x\|^2 = 0$$

because $\|x_n\|^2 \to \|x\|^2$ and $\langle x_n, x\rangle \to \|x\|^2$.

This completes the proof as square root function is continuous. $\ddot\smile$

# 4 Question 4

---

Let $\{e_n : n \in \mathbb{N}\}$ be the standard Schauder basis for the Banach space $\ell^p(\mathbb{N})$ where $1 \leqslant p < \infty$. Show that $e_n \to 0$ in the weak topology of $\ell^p(\mathbb{N})$ for every $p > 1$. But for $p = 1$, the sequence $e_n$ does not converges to 0 in the weak topology of $\ell^1(\mathbb{N})$.

---

*Proof.* First, we deal with the case when $1 < p < +\infty$. It can be shown that

$$(\ell^p (N))^* = \{L_y : y \in \ell^q (\mathbb{N})\}$$

where $L_y(x) = \sum_{i=1}^{\infty} x_i y_i, x \in \ell^q (\mathbb{N})$. Note that for each $y \in \ell^q (\mathbb{N})$ with $1 \leq q < \infty$, we have that $y_i \to 0$ as $i \to \infty$. This is because $\sum_{i=1}^{\infty} |y_i|^q < \infty$ for $y \in \ell^q (\mathbb{N})$.

Now, let $y \in \ell^q (\mathbb{N})$. We have that

$$L_y (e_n) = y_n \to 0 \text{ as } n \to \infty.$$

This shows that $(e_n)$ converges to 0 in the weak topology.

Now, consider the case where $p = 1$. Then we have

$$\left(\ell^1 (\mathbb{N})\right)^* = \{L_y : y \in \ell^\infty (\mathbb{N})\}$$

where $L_y$ is as specified in the previous case. Let $y = (1, 1, 1, \ldots)$. Then we have that

$$L_y (e_n) = 1$$

for each $n \in \mathbb{N}$. Hence, we have that $(e_n)$ does not converge to 0 in the weak topology. $\ddot\smile$

# 5 Question 5

Let $M$ be a norm closed subspace of a normed linear space $X$. Show that $M$ is also closed in the weak topology of $X$.

*Proof.* Let $M$ be a strongly closed subspace of $X$. We wish to show that it is weakly closed. To do so, we will show that $X \setminus M$ is weakly open.

Let $x \in X \setminus M$. We will be done if we show that there is a weakly open set $U$ such that $x \in U \subset X \setminus M$.

Recall a result about metric spaces: in a metric space, distance between a closed set and a compact set which are disjoint is strictly positive. Since $\{x\}$ is compact and $M$ is closed, we have that the distance $d$ between the point $x$ and $M$ is strictly positive.

We claim that there is linear functional $f \in X^*$ such that $f(M) = 0$ and $f(x) = d$.

For the timebeing, let us assume this claim. Let $f \in X^*$ be such a functional. Then we have that $U := f^{-1}((d/2, \infty))$ is weakly open (because weak topology is the smallest topology which makes every linear functional continuous), $a \in U$ and $U \subset X \setminus M$. This shows that $X \setminus M$ is weakly open and hence $M$ is weakly closed.

We now proceed to prove that claim. Consider the subspace:

$$N := \{\lambda x + m : \lambda \in \mathbb{F}, m \in M\}$$

of $X$. We now define a continuous linear functional $f_N$ on $N$ and extend it to $X$ via Hahn Banach. So, consider the linear functional $f_N : N \to \mathbb{F}$ given by $f_N(\lambda x + m) = \lambda d$. It is easy to see that this functional is well defined and linear. We now show that $\|f_N\|_{N^*} \leq 1$. Let $\lambda \in \mathbb{F}$ and $m \in M$. We have that

$$\|\lambda x + m\| = |\lambda| \left\| x - \left(-\frac{m}{\lambda}\right) \right\| \geq |\lambda| \, d = \|f_N(\lambda x + m)\|$$

This shows that $f_N$ is continuous. Hence, by Hahn Banach, we are done. $\ddot\smile$

## 6 Question 6

---

Let $H$ be a Hilbert space. Show that closed unit ball in $H$ is compact in the weak topology.

---

*Proof.* First, we show that any Hilbert space is isometrically isomorphic to its dual. Let $H$ be a Hilbert space. Let $H^*$ be its dual. We establish that there is a isometry between $H$ and $H^*$. For each $y \in H$, define $L_y : H \to \mathbb{C}$ by $L_y(x) = \langle x, y \rangle$.

Now, consider the map $\varphi : H \to H^*$ given by $\varphi(y) = L_y$ for each $y \in H$. We claim that this map is an isometric isomorphism. It is easy to see this map is linear. To see that this map is one one, let $y \in H$ such that $L_y = 0$. Then we have that $\langle y, y \rangle = 0$. Thus, $y = 0$. This shows that $y = 0$. Onto and isometry follows from Riesz Representation theorem.

Now, we prove that if $X$ and $Y$ are isometric normed linear spaces then there is a homeomorphism between the weak topology on $X$ and the weak topology on $Y$. Let $\varphi : X \to Y$ be isometry between $X$ and $Y$. (Complete the proof . . . )

Therefore, we have the closed unit ball in $H$ is compact in the weak topology. $\ddot\smile$

# 7 Question 7

Suppose $X$ is a finite dimensional normed linear space. Then show that the weak topology on $X$ and the norm topology on $X$ coincides.

*Proof.* It is clear that the norm topology contains the norm topology. To show the reverse inclusion, we show that every open ball contains a basis element of the weak topology.

Consider the open ball $B(0,1)$. Suppose that $X$ is of dimension $n$. Consider the linear functionals $f_i(x) = x_i$ for each $i = 1, 2, \ldots, n$. Then it is easy to see that $\cap_{i=1}^{n} f_i^{-1}(B(0,1/2)) \subset B(0,1)$. This completes the proof. $\ddot\smile$

# 8 Question 8

---

Let $V$ be a vector space over $\mathbb{F}$, where ($\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$). Suppose $g, f_1, f_2, \ldots, f_k$ are non zero linear functional on $V$ satisfying

$$\bigcap_{j=1}^{k} \ker f_j \subseteq \ker g.$$

Then show that $g$ belong to $\operatorname{span}\{f_1, f_2, \ldots, f_k\}$.

---

*Proof.* We proceed by induction. First suppose that $\ker f \subset \ker g$. We show that $g = \lambda f$ for some $\lambda \in \mathbb{F}$.

Observe that if $\ker g = V$ then $g = 0 = 0f$ and we are done. Suppose not. Then we can select $v_0 \in V$ such that $g(v_0) = 1$. Then $f(v_0) \neq 0$ for otherwise $v_0 \in \ker f$ which would imply that $g(v_0) = 0$ as $\ker f \subset \ker g$. Define $\lambda_1 = \frac{1}{f(v_0)}$.

We have that $V = \ker f \oplus \operatorname{span}\{v_0\}$. We have that $v = v_f + \lambda v_0$ for some $\lambda \in \mathbb{F}$.

Therefore, we have that

$$\begin{aligned} g(v) &= g(v_f + \lambda v_0) \\ &= 1 \end{aligned}$$

Also, note that

$$\begin{aligned} \lambda_1 f(v) &= \lambda_1 \lambda f(v_0) \\ &= \lambda \end{aligned}$$

This shows the theorem is true for the case $n = 1$.

# 9 Question 9

---

Let $X$ be an infinite dimensional normed linear space and $S = \{x \in X : \|x\| = 1\}$ be the unit sphere in $X$. Show that if $y \in X$ with $\|y\| \leqslant 1$, then every weak neighbourhood of $y$ must intersect $S$. Finally show that weak closure of $S$ is equal to the closed unit Ball $B = \{x \in X : \|x\| \leqslant 1\}$.

---

*Proof.* We proceed to show that if $y \in X$ with $\|y\| \leq 1$ and $U$ is a weak neighbourhood of $y$ then $U \cap S \neq \emptyset$.

Since $U$ is a nonempty set, we claim that there exists $x_0 \neq 0$ such that $y + \operatorname{span}\{x_0\} \subset U$. Since $U$ is a weakly open set, there exists $f_1, f_2, \ldots, f_n \in X^*$ and $\varepsilon > 0$ such that

$$\bigcap_{i=1}^{n} \{x \in X : |f_i(y - x)| < \varepsilon\} \subset U.$$

We show that there is an $x_0 \in X \setminus \{0\}$ such that $f_1(x_0) = f_2(x_0) = \ldots = f_n(x_0) = 0$. If there are none, we consider the map

$$\begin{aligned} X &\to \mathbb{C}^n \\ x &\mapsto (f_1(x), f_2(x), \ldots, f_n(x)). \end{aligned}$$

and this map would be injective which is a contradiction as $C^n$ is finite dimensional while OTOH, $X$ is infinite dimensional.

Thus, we can let $x_0 \in X \setminus \{0\}$ such that $f_1(x_0) = f_2(x_0) = \ldots = f_n(x_0) = 0$. Now, for any $t \in \mathbb{C}$ and $i \in \{1, 2, \ldots, n\}$, we have that

$$f_i(y + tx_0) = f_i(y).$$

Hence, we have that

$$y + tx_0 \in \cap_{i=1}^{n} f_i^{-1}(\{f_i(y)\}) \subset \bigcap_{k=1}^{n} f_i^{-1}(B(f_i(y), \varepsilon)).$$

This shows that $y + tx_0 \in U$ for each $t \in \mathbb{C}$. This completes the proof of our claim.

Now, observe that the function $f\colon t \mapsto \|y + tx_0\|$ is continuous on $[0, +\infty)$, $f(0) \leq 1$, and $f(t) = |t|\|x_0 + \frac{1}{t}x\| \to +\infty$, $t \to +\infty$ if $\|x_0\| \neq 0$. By the intermediate value theorem, there is a $t_0 \in [0, +\infty)$ such that $f(t_0) = \|y + t_0 x_0\| = 1$.

Thus, we have that $S \cap U \neq \emptyset$. ⌣̈

# 10 Question 10

Let $T : X^* \to \mathbb{F}$ be a linear functional such that $T$ is continuous w.r.t the weak star topology $(X, \tau_w)$. Show that $T = J_x$ for some $x \in X$.

# 11 Question 11

---

Suppose $X$ be an infinite dimensional normed linear space. Then show that the weak topology $(X, \tau_w)$ is never first countable and hence $(X, \tau_w)$ is not metrizable.

---