Functional Analysis Assignment 4

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Note

A checkmark ✓ indicates the question has been done.

Contents

1 Question 1

Let $(V, \|\cdot\|_1)$ and $(V, \|\cdot\|_2)$ be two Banach spaces. Suppose there exist a $c > 0$ such that $\|x\|_1 \leqslant c\|x\|_2$ for every $x \in V$. Then show that the two norms $\|\cdot\|_1$ and $\|\cdot\|_2$ are equivalent.

Proof. First, we prove that $I : (V, \|\cdot\|_2) \to (V, \|\cdot\|_1)$ is continuous. To do so , let $x \in V$. Then [1]

$$\begin{aligned} \|Ix\|_1 &= \|x\|_1 \\ &\leq c\,\|x\|_2 . && \text{(by hypothesis)} \end{aligned}$$

Since I is a bounded linear operator, we have that it is invertible by the inverse mapping theorem. Thus $I : (V, \|\cdot\|_1) \to (V, \|\cdot\|_2)$ is bounded. Thus, we have that

$$\begin{aligned} \|x\|_2 &= \|I(Ix)\|_2 \\ &\leq \|I\|\,\|Ix\|_1 && \text{(viewing "first" I as a mapping from } (V, \|\cdot\|_2) \text{ to } (V, \|\cdot\|_1)) \\ &= \|I\|\,\|x\|_1 \end{aligned}$$

Hence, we have that the both norms are equivalent. ///

[1] Yes, I know that there was nothing to show!

2 Question 2

Let X and Y be two Banach spaces and $T : X \to Y$ be a continuous linear transformation. Show that there exist a constant $c > 0$ such that $\|Tx\| \geqslant c\|x\|$ for all $x \in X$ if and only if $\ker T = \{0\}$ and im (T) is closed.

Solution. ($\Longrightarrow$) Suppose that there is a constant $c > 0$ such that $\|Tx\| \geq c\,\|x\|$ for all $x \in X$.

First, let us show that $\ker T = \{0\}$. Let $x \in \ker T$. Then $Tx = 0$. Then we have that $0 = \|Tx\| \geq c\,\|x\|$ and hence $x = 0$.

To show that that the image of T is closed, let $(Tx_n)_{n\in\mathbb{N}}$ be a sequence converging to some $y \in Y$. We need to show that $y = Tx$ for some $x \in X$.

Since (Tx_n) is convergent, it is Cauchy in Y. Therefore, we have that

$$\|x_n - x_m\| \leq \frac{1}{c}\,\|Tx_n - Tx_m\|$$

for all $m, n \in \mathbb{N}$. This shows that $(x_n)_{n\in\mathbb{N}}$ is Cauchy in X. Since X is Banach, we have that $(x_n)_{n\in\mathbb{N}}$ converges to some $x \in X$. By continuity, we have that $(Tx_n)_{n\in\mathbb{N}}$ converges to Tx. By uniqueness of limits, we have that $Tx = y$.

($\Longleftarrow$) If $X = \{0\}$ then the result is trivial. Suppose that $X \neq \{0\}$. Since $T : X \to Y$ is injective, we consider the map $T^{-1} : \text{im } T \to X$. Note that T is bounded linear transformation, thus, T^{-1} is a bounded linear transformation by the inverse mapping theorem. (Quick remark: im T is Banach by virtue of being closed).

Thus, we have that

$$\begin{aligned}\|x\| &= \left\|T^{-1}(Tx)\right\| \\ &\leq \left\|T^{-1}\right\|\|Tx\|\end{aligned}$$

for any $x \in X$. We will be done if we show that $\|T^{-1}\| \neq 0$. Since X is nonzero, im T is nonzero. Select a nonzero vector $y \in$ im (T) such that $\|y\| \leq 1$. Thus, we have that $\|T^{-1}\| \geq \|T^{-1}(y)\|$. Hence $y = Tx$ for some nonzero $x \in X$. Thus, $\|T^{-1}\| \geq \|x\| > 0$. This completes the proof. ///

3 Question 3

Let $(X, \|\cdot\|)$ be a Banach space with a Schauder basis, say $\{v_j : j \in \mathbb{N}\}$. Define a new norm on X in the following manner : For any $x \in X$, there exist unique scalars $\{c_i(x) : i \in \mathbb{N}\}$ such that $x = \sum_{i=1}^{\infty} c_i(x)v_i$. Now consider

$$\|x\|_n := \sup_k \left\{ \| \sum_{i=1}^{k} c_i(x)v_i \| \right\}$$

Show that $\|\cdot\|_n$ is indeed a norm on X and the two norms $\|\cdot\|_n$ and $\|\cdot\|$ are equivalent.

Solution. The first question is whether $\|\cdot\|_n$ is well defined. Let's proceed to show that. Let $x \in X$. Then there exists scalars $\{c_i(x) : i \in \mathbb{N}\}$ such that

$$\begin{aligned} x &= \lim_{k\to\infty} \sum_{i=1}^{k} c_i(x) v_i \\ \rightsquigarrow \|x\| &= \left\| \lim_{k\to\infty} \sum_{i=1}^{k} c_i(x) v_i \right\| \\ &= \lim_{k\to\infty} \left\| \sum_{i=1}^{k} c_i(x) v_i \right\|. \end{aligned}$$

Since $\left\| \sum_{i=1}^{k} c_i(x) v_i \right\|$ is convergent and hence is bounded. Thus the norm $\|\cdot\|_n$ is well defined.

It is easy to see that $\|\cdot\|_n$ is a norm on X. Let $x \in X$ be arbitrary. Since $\{v_j : j \in \mathbb{N}\}$ is a Schauder basis, there exists unique scalars $\{c_i(x) : i \in \mathbb{N}\}$ such that

$$x = \sum_{i=1}^{\infty} c_i(x) v_i$$

Observe that for each $k \in \mathbb{N}$, we have that

$$\left\| \sum_{i=1}^{k} c_i(x) x_i \right\| \leq \|x\|_n .$$

Letting $k \to \infty$, we have that

$$\|x\| \leq \|x\|_n .$$

In view of Question 1 of this Assignment, we are done. ///

4 Question 4

Let $(X, \|\cdot\|)$ be a Banach space with a Schauder basis, say $\{v_j : j \in \mathbb{N}\}$. Thus for any $x \in X$, there exist unique scalars $\{c_i(x) : i \in \mathbb{N}\}$ such that $x = \sum_i c_i(x) v_i$. Now consider the family of linear functional $P_i : X \to \mathbb{F}$ defined by $P_i(x) = c_i(x)$ for every $x \in X$. Show that P_i is a continuous linear functional on X for each $i \in \mathbb{N}$.

Proof. By the uniqueness part of the Schauder basis, it is easy to see that each $P_i : X \to \mathbb{F}$ is indeed an linear functional. We are left to show that it is continuous.

We use the previous problem to complete this problem. Since $\|\cdot\|$ and $\|\cdot\|_n$ is equivalent, there exists constants $\alpha_1, \alpha_2 > 0$ such that

$$\alpha_1 \|x\|_n \leq \|x\| \leq \alpha_2 \|x\|_n$$

for each $x \in X$.

We show that the map $\varphi_k : (V, \|\cdot\|) \to (V, \|\cdot\|)$ given by $\varphi_k(x) = \sum_{i=1}^{k} c_i(x) v_i$ is continuous. Let $x \in X$ and consider the following:

$$\begin{aligned} \|\varphi_k(x)\| &= \left\| \sum_{i=1}^{k} c_i(x) v_i \right\| \\ &\leq \|x\|_n \\ &\leq \frac{1}{\alpha_1} \|x\|. \end{aligned}$$

This shows that φ_k is continuous for each $k \in \mathbb{N}$. Define $\varphi_0(x) = 0$ for each $x \in X$. Notice that now, $P_k = \frac{1}{\|v_k\|} \|\varphi_k - \varphi_{k-1}\|$ for each $k \in \mathbb{N}$. By continuity of norm and φ_k's, we are done. ///

5 Question 5

Let $T : \ell^2(\mathbb{N}) \to \ell^2(\mathbb{N})$ be the linear map given by $T\Big((x_j)_{j\in\mathbb{N}}\Big) = \Big((\frac{x_j}{j})_{j\in\mathbb{N}}\Big)$.

1. Show that T is continuous and injective.
2. Consider the map $T^{-1} : range(T) \to \ell^2(\mathbb{N})$ given by $T^{-1}(Tf) = f$ for $f \in \ell^2(\mathbb{N})$. Show that T^{-1} is not continuous.
3. Conclude that $range(T)$ is not closed in $\ell^2(\mathbb{N})$.

Solution. 1. This is clear from Holder's inequality.

2. Let $k > 0$. Select $N \in \mathbb{N}$ such that $k < N$. Consider the sequence

$$N = \left\| T^{-1}\left(\frac{e_N}{N}\right) \right\| > k \left\| e_N \right\|$$

   This shows that T is discontinuous.

3. If im T was closed then the Banach isomorphism theorem would tell us that T^{-1} : im $T \to \ell^2\left(\mathbb{N}\right)$ is continuous which would contradict item 2.

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6 Question 6

Suppose φ is a Borel measurable function on $[0,1]$ such that $\varphi f \in L^2[0,1]$ for every $f \in L^2[0,1]$. Consider the map $M_\varphi : L^2[0,1] \to L^2[0,1]$ defined by $M_\varphi(f) = \varphi f$ for every $f \in L^2[0,1]$. Prove that M_φ is continuous linear transformation and $\varphi \in L^\infty[0,1]$.

Proof. Let φ be a Borel measurable function. Since $\varphi f \in L^2[0,1]$ for each $f \in L^2[0,1]$, we have by taking $f = 1$ that $\varphi \in L^2[0,1]$. Consider the sequence of functions given by $\varphi_n = \varphi \chi_{\{|\varphi| \leq n\}}$. Now note that $|\varphi_n - \varphi| \leq |\varphi|$ on $[0,1]$. Also, note that $|\varphi_n| \leq n$ on $[0,1]$ so $M_{\varphi_n} : L^2[0,1] \to L^2[0,1]$ defined by $M_{\varphi_n}(f) = \varphi_n f$ for every $f \in L^2[0,1]$ is continuous.

We now show that $M_{\varphi_n}(f) \to M_\varphi(f)$ in 2-norm. Now, $|\varphi_n - \varphi|^2 |f|^2 \leq |\varphi f|^2$ on $[0,1]$. By Dominated Convergence Theorem, we have that

$$\lim_{n\to\infty} \int_0^1 |\varphi_n - \varphi|^2 |f|^2 \, dt = 0$$

as $|\varphi_n - \varphi| \to 0$ pointwise.

By the Corollary 2 in this webpage, we conclude that M_φ is continuous. ///

7 Question 7

(*) Let X and Y be two normed linear spaces and $dim(Y) < \infty$. Suppose $T : X \to Y$ be onto linear transformation. Show that T is an open map.
