

Functional Analysis Assignment 6

ASHISH KUJUR

Note

Then end of a proof is denoted by \smile .

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1 Question 1

Let H be a Hilbert space and $T \in \mathcal{B}(H)$. Show that $T^*T \geq 0$.

Proof. First, we need to show that T^*T is self-adjoint. Observe that

$$\begin{aligned}(T^*T)^* &= T^* (T^*)^* & (ST)^* &= T^* S^* \\ &= T^*T. & (T^*)^* &= T\end{aligned}$$

To show that $T^*T \geq 0$, we need to show that $\langle T^*Tx, x \rangle \geq 0$ for each $x \in H$. To this end, let $x \in H$. Then we have

$$\begin{aligned}\langle T^*Tx, x \rangle &= \langle Tx, (T^*)^* x \rangle && \text{by definition} \\ &= \langle Tx, Tx \rangle \geq 0 && \text{by definition of inner product}\end{aligned}$$

This completes the proof.

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2 Question 2

Let H be a Hilbert space and $T \in \mathcal{B}(H)$. Suppose $A \in \mathcal{B}(H)$ and $A \geq 0$. Then show that $T^*AT \geq 0$.

Proof. Let $A, T \in \mathcal{B}(H)$ and suppose that $A \geq 0$. Then we need to show that $T^*AT \geq 0$. First, we need to show that T^*AT is self adjoint. To this end, consider the following:

$$\begin{aligned}(T^*AT)^* &= T^*A^*(T^*)^* \\ &= T^*AT\end{aligned}$$

Since $A \geq 0$, we have that A is self-adjoint by definition. Now, let us show that $\langle T^*ATx, x \rangle \geq 0$ for each $x \in H$. To this end, let $x \in H$ and consider the following:

$$\langle T^*ATx, x \rangle = \langle ATx, Tx \rangle \geq 0 \text{ since } Tx \in H \text{ and } A \geq 0.$$

This completes the proof.

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3 Question 3

Let H be a Hilbert space and $T, S \in \mathcal{B}(H)$, satisfying $T \geq 0$ and $S \geq 0$. Show that $T + S \geq 0$.

Proof. Routine check. Follows immediately.

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4 Question 4

Let H be a Hilbert space and $P_1, P_2 \in \mathcal{B}(H)$, are two orthogonal projections, that is, $P_j^2 = P_j$ and $P_j^* = P_j$ for each $j = 1, 2$. Show that P_1P_2 is an orthogonal projection if and only if $P_1P_2 = P_2P_1$. In this case $\text{ran}(P_1P_2) = \text{ran}(P_1) \cap \text{ran}(P_2)$.

Proof. Let $P_1, P_2 \in \mathcal{B}(H)$ be two orthogonal projections. We show that P_1P_2 is an orthogonal projection iff $P_1P_2 = P_2P_1$.

(\implies) Suppose that P_1P_2 is an orthogonal projection. We wish to show that $P_1P_2 = P_2P_1$. To do so, consider the following:

$$\begin{aligned} P_1P_2 &= (P_1P_2)^* && P_1P_2 \text{ being orthogonal projection} \\ &= P_2^*P_1^* && \text{property of the adjoint} \\ &= P_2P_1 && P_1, P_2 \text{ being orthogonal projection} \end{aligned}$$

(\impliedby) Suppose that $P_1P_2 = P_2P_1$. We wish to show that P_1P_2 is an orthogonal projection.

P_1P_2 is self adjoint: Consider the following:

$$\begin{aligned} (P_1P_2)^* &= P_2^*P_1^* \\ &= P_2P_1 \\ &= P_1P_2. && \text{by assumption} \end{aligned}$$

P_1P_2 is idempotent: Consider the following:

$$\begin{aligned} (P_1P_2)^2 &= P_1P_2P_1P_2 \\ &= P_1P_1P_2P_2 && \text{since } P_1P_2 = P_2P_1 \\ &= P_1^2P_2^2 \\ &= P_1P_2 && \text{since } P_1, P_2 \text{ are orthogonal projection} \end{aligned}$$

This shows that the both directions of the iff. Let us assume that $P_1P_2 = P_2P_1$. Now, we proceed to show that $\text{im } (P_1P_2) = \text{im } P_1 \cap \text{im } P_2$.

(\subset) : Let $y \in \text{im } P_1P_2$. Then we have that $y = P_1P_2x$ for some $x \in H$. Since $P_2x \in H$, we have that $y = P_1x$. Also, note that $y = P_2P_1x$ since P_1 and P_2 commute. Hence, we have that $y \in \text{im } P_2$. This shows that $\text{im } P_1P_2 \subset \text{im } P_1 \cap \text{im } P_2$.

(\supset) : Let $y \in \text{im } P_1 \cap \text{im } P_2$. Then we have $y = P_1x_1 = P_2x_2$ for some $x_1 \in H$ and some $x_2 \in H$. It follows that

$$P_1y = P_1^2x_1 \rightsquigarrow P_1y = P_1x_1 \rightsquigarrow y = P_1y$$

and

$$P_2y = P_2^2x_2 \rightsquigarrow P_2y = P_2x_2 \rightsquigarrow y = P_2y.$$

The above two equalities show that $y \in \text{im } P_1 \cap \text{im } P_2$.

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5 Question 5

Let H be a Hilbert space and $P_1, P_2 \in \mathcal{B}(H)$, are two orthogonal projections in $\mathcal{B}(H)$. Show that $P_1 + P_2$ is an orthogonal projection if and only if $\text{ran}(P_1)$ is orthogonal to $\text{ran}(P_2)$, that $\langle P_1x, P_2y \rangle = 0$ for every $x, y \in H$. In this case $\text{ran}(P_1 + P_2) = \text{ran}(P_1) + \text{ran}(P_2)$.

Proof. Let H be a Hilbert space and let P_1, P_2 be two orthogonal projections in $\mathcal{B}(H)$. We proceed to show that $P_1 + P_2$ is an orthogonal projection iff $\text{im } P_1$ is orthogonal to $\text{im } P_2$.

(\implies) Suppose that $P_1 + P_2$ is an orthogonal projection. Let $y_1 \in \text{im } P_1$ and $y_2 \in \text{im } P_2$. Then we have that $y_1 = P_1x_1$ and $y_2 = P_2x_2$ for some $x_1 \in H$ and some $x_2 \in H$.

First, we show that $P_1P_2 = -P_2P_1$. Note that

$$\begin{aligned} P_1 + P_2 &= (P_1 + P_2)^2 \\ &= P_1 + P_1P_2 + P_2P_1 + P_2 \\ &\leadsto P_1P_2 = -P_2P_1. \end{aligned}$$

(\impliedby) Suppose that $\text{im } P_1$ is orthogonal to $\text{im } P_2$. We wish to show that $P_1 + P_2$ is an orthogonal projection. The fact that $P_1 + P_2$ is selfadjoint is immediate. It remains to show that $P_1 + P_2$ is idempotent.

But before that, we show that $P_1P_2 = 0$. Let $f \in H$. Observe that

$$\begin{aligned} \|P_1P_2f\|^2 &= \langle P_1P_2f, P_1P_2f \rangle \\ &= \langle P_1^*P_1P_2f, P_2f \rangle \\ &= \langle P_1^2P_2f, P_2f \rangle \\ &= \langle P_1P_2f, P_2f \rangle \\ &= 0. \end{aligned}$$

The first equality is by definition, the second is by definition of adjoint operator and the third is because P_1 is an orthogonal projection and hence adjoint, the the third is because P_1 is selfadjoint and the last is due to our assumption that $\text{im } P_1$ is orthogonal to $\text{im } P_2$. This shows that $P_1P_2 = 0$.

Similarly, it can be shown that $P_2P_1 = 0$. Now, we proceed to $(P_1+P_2)^2 = P_1 + P_2$. Observe that

$$\begin{aligned}(P_1 + P_2)^2 &= P_1^2 + P_1P_2 + P_2P_1 + P_2^2 \\ &= P_1 + P_2.\end{aligned}$$

The first equality is obvious and in the second one, we are using the fact that P_1, P_2 are idempotent and $P_1P_2 = P_2P_1 = 0$.

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6 Question 6

Let H be a Hilbert space and M be a closed subspace of H . Let P_M be the orthogonal projection onto M . Suppose $T \in \mathcal{B}(H)$. Show that M is an invariant subspace for T if and only if $TP_M = P_MTP_M$. Show that M is a reducing subspace for T if and only if $P_MT = TP_M$.

Proof. Let P_M be the orthogonal projection onto the closed subspace M and let $T \in \mathcal{B}(H)$. First, we show that if $x \in M$ then $P_Mx = x$.

We wish to show that M is an invariant subspace of T iff $TP_M = P_MTP_M$.

(\implies) Suppose that M is an invariant subspace of T , that is, $TM \subset M$. We wish to show that


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7 Question 7

Let H be a Hilbert space and $T \in \mathcal{B}(H)$, satisfying $T^k = 0$ for some $k \in \mathbb{N}$. Show that $\sigma(T) = \{0\}$.

Proof. By the Spectral Mapping Theorem, we have that

$$\begin{aligned}(\sigma(T))^k &= \sigma(T^k) \\ &= \sigma(0) \\ &= \{0\}\end{aligned}$$

This shows that $\sigma(T) \subset \{0\}$. Since the spectrum is always nonempty, we have that $\sigma(T) = \{0\}$. 

8 Question 8

Let H be a Hilbert space and M be a closed subspace of H . Let P_M be the orthogonal projection onto M . Compute the spectrum $\sigma(P_M)$.

Proof. Let P_M be the orthogonal projection onto M . Define a polynomial $p(z) \in \mathbb{C}[z]$ given by $p(z) = z^2 - z$. It is easy to see that $p(P_M) = 0$.

It follows that by the Spectral Mapping Theorem that

$$p(\sigma(P_M)) = \sigma(p(P_M)) = \sigma(0) = \{0\}.$$

Hence, if $\alpha \in \sigma(P_M)$, we have that $\alpha^2 - \alpha = 0$. Thus, $\alpha \in \{0, 1\}$. This shows that $\sigma(P_M) \subset \{0, 1\}$.

Since the spectrum is nonempty, we must have that exactly one of the following holds: $\sigma(P_M) = \{0\}$ or $\sigma(P_M) = \{1\}$ or $\sigma(P_M) = \{0, 1\}$. Since P_M is a projection, we have that $H = \ker P_M \oplus \operatorname{im} P_M$. This shows that this corresponds to $P_M = I$, $P_M = 0$ and "something in between"! \smile

9 Question 9

Let H be a Hilbert space and $T \in \mathcal{B}(H)$. Suppose X is an invertible operator in $\mathcal{B}(H)$. Then show that $\sigma(X^{-1}TX) = \sigma(T)$. (In other words similar operators have same spectrum).

Proof. Let X be an invertible operator in $\mathcal{B}(H)$. Consider the following equivalence:

$$\begin{aligned}\alpha \notin \sigma(T) &\iff T - \alpha I \text{ is invertible} \\ &\iff X^{-1}T - \alpha X^{-1} \text{ is invertible} \\ &\iff X^{-1}TX - \alpha I \text{ is invertible} \\ &\iff \alpha \notin \sigma(X^{-1}TX).\end{aligned}$$

This shows that $\sigma(X^{-1}TX) = \sigma(T)$.

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10 Question 10

Let H be a Hilbert space and $T \in \mathcal{B}(H)$. Suppose X is a unitary operator in $\mathcal{B}(H)$. Then show that

- (i) $\|X^{-1}TX\| = \|T\|$.
 - (ii) T is normal if and only if $X^{-1}TX$ is normal.
 - (iii) T is self adjoint if and only if $X^{-1}TX$ is self adjoint.
 - (iv) $T \geq 0$ if and only if $X^{-1}TX \geq 0$.
 - (v) T is an orthogonal projection if and only if $X^{-1}TX$ is orthogonal projection.
-

Proof. Let H be a Hilbert space and $T \in \mathcal{B}(H)$. Let X be a unitary operator in $\mathcal{B}(H)$.

- (i) We intend to show that $\|X^{-1}TX\| = \|T\|$. Since X is a unitary operator, we have that X is an isometric isomorphism. To this end, let $f \in H$ and consider the following:

$$\begin{aligned}\|X^{-1}TXf\|^2 &= \|X^*TXf\|^2 \\ &= \langle X^*TXf, X^*TXf \rangle \\ &= \langle TXf, XX^*TXf \rangle \\ &= \langle TXf, TXf \rangle \\ &= \|TXf\|^2.\end{aligned}$$

Thus, we have that

$$\|X^{-1}TXf\| = \|TXf\| \tag{10.1}$$

for each $f \in H$. Now, we claim that

$$\{\|TXf\| : \|f\| = 1\} = \{\|Tf\| : \|f\| = 1\}. \tag{10.2}$$

To prove this:

- (\subset) : Let $y = \|TXf\|$ for some $f \in H$ with $\|f\| = 1$. Since X is a unitary operator, we have that $\|Xf\| = \|f\| = 1$, hence, y is in the set of the right side of the equality of the aforementioned claim.
- (\supset) : Let $y = \|Tg\|$ for some $g \in H$ with $\|g\| = 1$. Since X is unitary, it is invertible and hence $Xf = g$ for some $f \in H$. Since X is unitary, we have that $\|f\| = \|Xf\| = \|g\| = 1$. Thus, y is in the left side of the set in the aforementioned equality.

Thus, we have

$$\begin{aligned}
\|X^*TX\| &= \sup_{\|f\|=1} \|X^*TXf\| && \text{(by definition)} \\
&= \sup_{\|f\|=1} \|TXf\| && \text{(see 10.1)} \\
&= \sup_{\|f\|=1} \|Tf\| && \text{(see 10.2)} \\
&= \|T\|. && \text{(by definition)}
\end{aligned}$$

This completes the proof.

- (ii) (\implies) Assume that T is normal. Consider the following:

$$\begin{aligned}
(X^*TX)(X^*TX)^* &= X^*TX X^*T^*X \\
&= X^*TT^*X
\end{aligned}$$

and on the other hand, we have

$$\begin{aligned}
(X^*TX)^*(X^*TX) &= X^*T X X^*T X \\
&= X^*T T^* X
\end{aligned}$$

Note that the two computations above show that X^*TX is normal by assuming T is normal.

- (\impliedby) The computation in the other direction, normality of X^*TX and the invertibility of X imply that T is normal.

This completes the proof.

- (iii) Skipping because EASY!.

- (iv) Skipping because EASY!.

(v) Skipping because EASY!.



11 Question 11

Let H be a Hilbert space and $T \in \mathcal{B}(H)$. Show that the following statements are equivalent :

- (i) T is an isometry, that is, $T^*T = I$.
 - (ii) $\|Tx\| = \|x\|$ for every $x \in H$.
 - (iii) $\langle Tx, Ty \rangle = \langle x, y \rangle$ for every $x, y \in H$.
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Proof. (i) \implies (ii): Suppose that $T^*T = I$. Let $x \in H$. Then

$$\begin{aligned}\|Tx\|^2 &= \langle Tx, Tx \rangle \\ &= \langle T^*Tx, x \rangle \\ &= \langle x, x \rangle \\ &= \|x\|^2.\end{aligned}$$

(ii) \implies (iii) Suppose that $\|Tx\| = \|x\|$ for each $x \in H$. See Axler's Theorem 7.42.

(iii) \implies (i) Suppose that $\langle Tx, Ty \rangle = \langle x, y \rangle$ for each $x, y \in H$. Let $x, y \in H$. Then we have

$$\begin{aligned}\langle (T^*T - I)x, y \rangle &= \langle T^*Tx - x, y \rangle \\ &= \langle T^*Tx, y \rangle - \langle x, y \rangle \\ &= \langle Tx, Ty \rangle - \langle x, y \rangle \\ &= 0\end{aligned}$$

This shows that $T^*T = I$.

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12 Question 15

For $\varphi \in L^\infty[0, 1]$, consider the operator $M_\varphi : L^2[0, 1] \rightarrow L^2[0, 1]$, defined by $M_\varphi f = \varphi f$, $f \in L^2[0, 1]$. The essential range of φ (w.r.t the Lebesgue measure) denoted as $\text{ess ran}(\varphi)$ is defined as follows:

A point $p \in \mathbb{C}$ is said to be not in the $\text{ess ran}(\varphi)$ if there exist a $\delta > 0$ such that the Lebesgue measure $m(\varphi^{-1}(B(p, \delta))) = 0$.

Prove that $\sigma(M_\varphi) = \text{ess ran}(\varphi)$.

Proof. We first show that α is an eigenvalue of M_φ iff $m(\{t \in [0, 1] : \varphi(t) = \alpha\}) > 0$.

(\implies) Let α be an eigenvalue of M_φ . Then there exists $f \neq 0$ such that $M_\varphi f = \alpha f$, that is, $\varphi f = \alpha f$.

Assume for the sake of contradiction that $m(\{\varphi = \alpha\}) = 0$.

Observe that $\{f \neq 0\} \subset \{\varphi = \alpha\} \cup \{\varphi f \neq \alpha f\}$. This is easy to verify: let $t \in [0, 1]$ be such that $f(t) \neq 0$. If $\varphi(t) = \alpha$, we are done because then $t \in \{\varphi = \alpha\}$. So suppose not, that is, $\varphi(t) \neq \alpha$. Then $\varphi(t)f(t) \neq \alpha f(t)$ because we are multiplying by a nonzero number.

Now, $m(f \neq 0) \leq m(\{\varphi = \alpha\}) + m(\{\varphi f \neq \alpha f\}) = 0$. Hence, $m(f \neq 0) = 0$ as $\varphi(x)f(x) = \alpha f(x)$ for almost all $x \in [0, 1]$.

(\impliedby) Let $\alpha \in \mathbb{C}$ such that $m(\{t \in [0, 1] : \varphi(t) = \alpha\}) > 0$. We wish to show that α is an eigenvalue of M_φ . Note that $\chi_{\{\varphi=\alpha\}} \neq 0$ and also we have that $\varphi \chi_{\{\varphi=\alpha\}} = \alpha \chi_{\{\varphi=\alpha\}}$. This shows that α is an eigenvalue of M_φ as $\chi_{\{\varphi=\alpha\}} \in L^2[0, 1]$.

This describes the point spectrum of M_φ .

Now, we show that $\sigma(M_\varphi) = \text{essran } \varphi$. This is equivalent to showing that $\alpha \notin \sigma(M_\varphi)$ iff $m(\{|h - \alpha| < \varepsilon\}) = 0$ for some $\varepsilon > 0$. Before we prove this, we prove a preliminary lemma:

Lemma 1. Let (X, \mathcal{A}, μ) be a finite measure space. Let $h \in \mathcal{L}^\infty(d\mu)$, $\varepsilon > 0$ and suppose that $|h| \geq \varepsilon$ for μ -almost everywhere. Then we have that there is a $\eta \in \mathcal{L}^\infty(\mu)$ such that $h(x)\eta(x) = \eta(x)h(x) = 1$ for μ -almost all $x \in X$.

Proof of Lemma 1. Let h be as in the hypothesis of the lemma. We can define a function. Define a function $\eta : X \rightarrow \mathbb{C}$ given by

$$\eta(x) = \begin{cases} \frac{1}{h(x)} & h(x) \neq 0 \\ 0 & x = 0 \end{cases}$$

Note that $\{x \in X : h(x) = 0\}$ is a set of measure zero because it is contained in $\{x \in X : |h(x)| < \varepsilon\}$ which is a set of measure zero by assumption. Thus, $\eta(x) = \frac{1}{h(x)}$ for μ -almost all $x \in X$. Hence, we have that $h(x)\eta(x) = \eta(x)h(x) = 1$ for μ almost all $x \in X$. Since $|\eta(x)| \leq \frac{1}{\varepsilon}$ for μ -almost all $x \in X$. We have that $\|\eta\|_\infty \leq \frac{1}{\varepsilon}$. This completes the proof of the lemma. \smile

(\implies) Let $\alpha \in \mathbb{C}$ and suppose that there exist $\varepsilon > 0$ such that $m(\{|\varphi - \alpha| < \varepsilon\}) = 0$. Since $\varphi \in \mathcal{L}^\infty[0, 1]$, we have that $\varphi - \alpha \in \mathcal{L}^\infty[0, 1]$. By hypothesis, we have that $|\varphi - \alpha| \geq \varepsilon$ for almost all $x \in [0, 1]$. Hence by the previous lemma, there exists a function (which we call) $(\varphi - \alpha)^{-1} \in \mathcal{L}^\infty[0, 1]$ such that $(\varphi - \alpha)(x) \cdot (\varphi - \alpha)^{-1}(x) = (\varphi - \alpha)^{-1}(x) \cdot (\varphi - \alpha)(x) = 1$ for almost all $x \in [0, 1]$. It is easy to see now that $M_{\varphi - \alpha} M_{(\varphi - \alpha)^{-1}} = I$ and $M_{(\varphi - \alpha)^{-1}} M_{\varphi - \alpha} = I$ and hence $\alpha \notin \sigma(M_\varphi)$.

(\impliedby) Suppose that $\alpha \notin \sigma(M_\varphi)$. Thus, we have that $M_{\varphi - \alpha}$ is invertible. Hence, by Question 12, we have that there exists $\varepsilon > 0$ such that

$$\|M_{\varphi - \alpha}x\| \geq \varepsilon \|x\|$$

for each $x \in X$. As a consequence we have that

$$\|M_{\varphi - \alpha}\|_{(L^2[0,1])^*} \geq \varepsilon.$$

Also, it can be easily shown that

$$\|M_{\varphi - \alpha}\| \leq \|\varphi - \alpha\|_\infty.$$

From the above two lines, we have that

$$\|\varphi - \alpha\|_\infty \geq \varepsilon.$$

Thus by definition of esssup , we have that

$$|\varphi(x) - \alpha| \geq \varepsilon$$

for almost all $x \in X$. Hence, we have that

$$m(\{|\varphi - \alpha| < \varepsilon\}) = 0.$$

This completes the proof.

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