Functional Analysis Assignment 5

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Note

Then end of a proof is denoted by ⌣̈.

Contents

1 Question 1

Suppose M and N are two topologically complimentary closed subspace of a Banach space $(X, \|\cdot\|_X)$. Now consider $M \oplus_1 N$, the external direct sum, defined in the following way)

$$M \oplus_1 N = \{(m, n) : m \in M, n \in N\}, \ \|(m, n)\|_1 = \|m\|_X + \|n\|_X.$$

(a) Show that $M \oplus_1 N$ is a Banach space w.r.t the norm $\|\cdot\|_1$ mentioned above.

(b) Show that X is isomorphic to $M \oplus_1 N$.

(c) Show that the quotient space X/M is isomorphic to the Banach space N.

Proof of item (a). We proceed to prove (a). Let $((m_k, n_k))_{k\in\mathbb{N}}$ be a Cauchy sequence in $M \oplus_1 N$. We show that (m_k) is Cauchy in X. Consider the following:

$$\|m_k - m_l\|_X \leq \|(m_k, n_k) - (m_l, n_l)\|_1 .$$

Now since $((m_k, n_k))$ is Cauchy, we have that (m_k) is Cauchy in X. Since X is a Banach space, we have that (m_k) converges to some $m \in M$ as M is closed. Likewise it can be shown that (n_k) converges to some $n \in N$. We now show that $((m_k, n_k))$ converges to (m, n) in $M \oplus_1 N$. Consider the following:

$$\|(m_k, n_k) - (m, n)\| = \|m_k - m\|_X + \|n_k - n\|_X$$

Since (m_k) converges to m and (n_k) converges to n, we are done. $\ddot\smile$

Proof of item (b). To show that X is isomorphic to $M \oplus_1 N$, consider the map $T : M \oplus_1 N \to X$ given by

$$T(m, n) = m + n$$

for every $m \in M$ and every $n \in N$. First, we show that T is a normed linear space isomorphism, that is, both T and T^{-1} are bounded linear operators. It is immediate that T is bijective and linear. Since the projection maps $m+n \to m$ and $m+n \to n$ are continuous, there are some constant μ and ν such that $\|m\|_X \leq \mu \|m + n\|_X$ and $\|n\|_X \leq \nu \|m + n\|_X$. Now, let $m \in M$ and $n \in N$. Then

$$\begin{aligned} \|T\,(m, n)\|_X &= \|m + n\|_X \\ &\leq \|m\|_X + \|n\|_X \\ &= \|(m, n)\|_1 \end{aligned}$$

and

$$\begin{aligned} \left\|T^{-1}(m + n)\right\|_1 &= \|(m, n)\|_1 \\ &= \|m\|_1 + \|n\|_1 \\ &\leq (\mu + \nu)\, \|m + n\|_X \end{aligned}$$

This shows that X is isomorphic to $M \oplus_1 N$. $\ddot\smile$

Proof of item (c). Let $P_N : X \to N$ be the projection of X into N. Since P_N is onto, by the first isomorphism theorem for vector spaces, we have that $X/M \cong N$. It remains to show that map $[x]_M \mapsto P_N(x)$ and its inverse is continuous (note this is the isomorphism given by the first isomorphism theorem). We show that the map $P_N(x) \mapsto [x]_M$ is continuous. Let $x \in X$. Suppose $x = m + n$. Then we have that $P_N(x) = n$. Then

$$\begin{aligned}
\|[x]_M\| &\leq \|x - m\| && \text{(by definition of quotient norm)} \\
&= \|n\| \\
&= \|P_N(x)\|_X
\end{aligned}$$

This shows that the aforementioned map is continuous and bijective, by the Banach isomorphism theorem, we are done. $\ddot\smile$

2 Question 2

Let H be a Hilbert space with an orthonormal basis $\{e_j : j \in \mathbb{N}\}$. Consider the set

$$A = \{e_k + ke_l : k < l,\ k, l \in \mathbb{N}, \}.$$

Show that 0 belongs to the weak closure of A. Also show that there is no sequence in A which converge weakly to 0.

Proof. Recall the fact that in a topological space, we have that $x \in \overline{A}$ if there is a sequence $(x_n)_{n\in\mathbb{N}}$ in A converging to $x \in A$.

For each $k \in \mathbb{N}$, we have that the sequence $(e_k + ke_l)_{l \geq k}$ converges to e_k. Thus, we have that $e_k \in \overline{A}^w$. Also, $(e_k) \in \overline{A}^w$ converges to 0. Therefore, we have that $0 \in \overline{A}^w$.

Let $(\tilde{e}_n)$ be a sequence in A converging to 0. Then $\tilde{e}_n = e_{k_n} + k_n e_{l_n}$ for some $k_n, l_n \in \mathbb{N}$ with $k_n < l_n$. Then $(\tilde{e}_n)$ must be norm bounded. Let $M > 0$ such that $\|\tilde{e}_n\| \leq M$ for each $n \in \mathbb{N}$.

We claim that $\{k_n : n \in \mathbb{N}\}$ is finite. This is easy to see:

$$\begin{aligned} M &\geq \|k_n e_{l_n} + e_{k_n}\| \\ &\geq |k_n \|e_{l_n}\| - \|e_{k_n}\|| \\ &\geq k_n - 1 \end{aligned}$$

for each $n \in \mathbb{N}$.

Since the aforementioned set is finite, we may let $\{k_n : n \in \mathbb{N}\} = \{k_{n_1}, \ldots, k_{n_l}\}$ for some $n_1, n_2, \ldots, n_l \in \mathbb{N}$. It is a consequence of Riesz Representation theorem that in a Hilbert Space, $x_n \to x$ weakly iff $\langle x_n, y\rangle \to \langle x, y\rangle$ for each $y \in H$. We use this to achieve a contradiction. Now, let $y = e_{k_{n_1}} + e_{k_{n_2}} + \ldots + e_{k_{n_l}}$. It can be seen that $\langle \tilde{e}_n, y\rangle \geq 1$ for each $n \in \mathbb{N}$ and cannot converge to 0 as $n \to \infty$. ⌣̈

3 Question 3

Let H be a Hilbert space. Suppose $\{x_n\}_{n\in\mathbb{N}}$ is a sequence of vector in H which converges weakly to a vector x in H and $\|x_n\| \to \|x\|$ as $n \to \infty$. Then show that $\|x_n - x\| \to 0$ as $n \to \infty$.

Proof. Let (x_n) be a sequence in H converging to $x \in H$ and furthermore suppose that $\|x_n\| \to \|x\|$ as $n \to \infty$. We wish to show that $x_n \to x$ strongly.

Since $x_n \to x$ weakly, we have that $\langle x_n, y\rangle \to \langle x, y\rangle$ for each $y \in H$ by the definition and Riesz Representation theorem. Thus, we have that $\langle x_n, x\rangle \to \|x\|^2$ in particular.

Now,

$$\|x_n - x\|^2 = \|x_n\|^2 - 2\Re\langle x_n, x\rangle + \|x\|^2 .$$

Taking limits both sides, we have that

$$\lim_{n\to\infty} \|x_n - x\|^2 = 0$$

because $\|x_n\|^2 \to \|x\|^2$ and $\langle x_n, x\rangle \to \|x\|^2$.

This completes the proof as square root function is continuous. $\ddot\smile$

4 Question 4

Let $\{e_n : n \in \mathbb{N}\}$ be the standard Schauder basis for the Banach space $\ell^p(\mathbb{N})$ where $1 \leqslant p < \infty$. Show that $e_n \to 0$ in the weak topology of $\ell^p(\mathbb{N})$ for every $p > 1$. But for $p = 1$, the sequence e_n does not converges to 0 in the weak topology of $\ell^1(\mathbb{N})$.

Proof. First, we deal with the case when $1 < p < +\infty$. It can be shown that

$$(\ell^p(\mathbb{N}))^* = \{L_y : y \in \ell^q(\mathbb{N})\}$$

where $L_y(x) = \sum_{i=1}^{\infty} x_i y_i, x \in \ell^p(\mathbb{N})$. Note that for each $y \in \ell^q(\mathbb{N})$ with $1 \leq q < \infty$, we have that $y_i \to 0$ as $i \to \infty$. This is because $\sum_{i=1}^{\infty} |y_i|^q < \infty$ for $y \in \ell^q(\mathbb{N})$.

Now, let $y \in \ell^q(\mathbb{N})$. We have that

$$L_y(e_n) = y_n \to 0 \text{ as } n \to \infty.$$

This shows that (e_n) converges to 0 in the weak topology.

Now, consider the case where $p = 1$. Then we have

$$\left(\ell^1(\mathbb{N})\right)^* = \{L_y : y \in \ell^\infty(\mathbb{N})\}$$

where L_y is as specified in the previous case. Let $y = (1, 1, 1, \ldots)$. Then we have that

$$L_y(e_n) = 1$$

for each $n \in \mathbb{N}$. Hence, we have that (e_n) does not converge to 0 in the weak topology. $\ddot\smile$

5 Question 5

Let M be a norm closed subspace of a normed linear space X. Show that M is also closed in the weak topology of X.

Proof. Let M be a strongly closed subspace of X. We wish to show that it is weakly closed. To do so, we will show that $X \setminus M$ is weakly open.

Let $x \in X \setminus M$. We will be done if we show that there is a weakly open set U such that $x \in U \subset X \setminus M$.

Recall a result about metric spaces: in a metric space, distance between a closed set and a compact set which are disjoint is strictly positive. Since $\{x\}$ is compact and M is closed, we have that the distance d between the point x and M is strictly positive.

We claim that there is linear functional $f \in X^*$ such that $f(M) = 0$ and $f(x) = d$.

For the timebeing, let us assume this claim. Let $f \in X^*$ be such a functional. Then we have that $U := f^{-1}((d/2, \infty))$ is weakly open (because weak topology is the smallest topology which makes every linear functional continuous), $a \in U$ and $U \subset X \setminus M$. This shows that $X \setminus M$ is weakly open and hence M is weakly closed.

We now proceed to prove that claim. Consider the subspace:

$$N := \{\lambda x + m : \lambda \in \mathbb{F}, m \in M\}$$

of X. We now define a continuous linear functional f_N on N and extend it to X via Hahn Banach. So, consider the linear functional $f_N : N \to \mathbb{F}$ given by $f_N(\lambda x + m) = \lambda d$. It is easy to see that this functional is well defined and linear. We now show that $\|f_N\|_{N^*} \leq 1$. Let $\lambda \in \mathbb{F}$ and $m \in M$. We have that

$$\|\lambda x + m\| = |\lambda| \left\| x - \left(-\frac{m}{\lambda}\right) \right\| \geq |\lambda|\, d = \|f_N(\lambda x + m)\|$$

This shows that f_N is continuous. Hence, by Hahn Banach, we are done. $\ddot\smile$

6 Question 6

Let H be a Hilbert space. Show that closed unit ball in H is compact in the weak topology.

Proof (sketch). First, we show that any REAL Hilbert space is isometrically isomorphic to its dual. Let H be a Hilbert space. Let H^* be its dual. We establish that there is a isometry between H and H^*. For each $y \in H$, define $L_y : H \to \mathbb{C}$ by $L_y(x) = \langle x, y \rangle$.

Now, consider the map $\varphi : H \to H^*$ given by $\varphi(y) = L_y$ for each $y \in H$. We claim that this map is an isometric isomorphism. It is easy to see this map is linear. To see that this map is one one, let $y \in H$ such that $L_y = 0$. Then we have that $\langle y, y \rangle = 0$. Thus, $y = 0$. This shows that $y = 0$. Onto and isometry follows from Riesz Representation theorem.

Now, we prove that if X and Y are isometric normed linear spaces then there is a homeomorphism between the weak topology on X and the weak topology on Y. To, this end, let $\varphi : X \to Y$ be isometry between two normed linear spaces X and Y. Let $f \in X^*$ and $B(a, r) \subset \mathbb{C}$. Then $f^{-1}(B(a, r))$ is an subbasis element of the weak topology on X. Then $f \circ T^{-1} \in Y^*$ as composition of bounded linear maps is bounded. Also, we have that

$$T\left(f^{-1}(B(a, r))\right) = \left(f \circ T^{-1}\right)(B(a, r))$$

which shows that T sends an subbasis element of weak topology of X to an subbasis element of weak topology of Y. A symmetric argument shows that T^{-1} sends a subbasis element of weak topology of Y to a weak topology of X. Thus, T induces a homeomorphism between the weak topology of X and the weak topology of Y.

Therefore, we have the closed unit ball in H is compact in the weak topology.

In the case of the complex Hilbert space, we have that map $\varphi : H \to H^*$ as defined previously is an antilinear map, which again is a homeomorphism. (Note in the previous paragraph, we never used the fact that T is a isometry, so, the same argument works!) $\smile$

7 Question 7

Suppose X is a finite dimensional normed linear space. Then show that the weak topology on X and the norm topology on X coincides.

Proof (sketch). It is clear that the norm topology contains the norm topology. To show the reverse inclusion, we show that every open ball contains a basis element of the weak topology.

Consider the open ball $B(0,1)$. Suppose that X is of dimension n. Consider the linear functionals $f_i(x) = x_i$ for each $i = 1, 2, \ldots, n$. Then it is easy to see that $\cap_{i=1}^{n} f_i^{-1}(B(0,1/2)) \subset B(0,1)$. This completes the proof. Since weak topology is translation invariant, we are done.

(To show that the weak topology is translation invariant, show that the translation map maps the subbasis element to a subbasis.)

8 Question 8

Let V be a vector space over $\mathbb{F}$, where ($\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$). Suppose $g, f_1, f_2, \ldots, f_k$ are non zero linear functional on V satisfying

$$\bigcap_{j=1}^{k} \ker f_j \subseteq \ker g.$$

Then show that g belong to $\operatorname{span}\{f_1, f_2, \ldots, f_k\}$.

Proof. We proceed by induction. First suppose that $\ker f \subset \ker g$. We show that $g = \lambda f$ for some $\lambda \in \mathbb{F}$.

Observe that if $\ker g = V$ then $g = 0 = 0f$ and we are done. Suppose not. Then we can select $v_0 \in V$ such that $g(v_0) = 1$. Then $f(v_0) \neq 0$ for otherwise $v_0 \in \ker f$ which would imply that $g(v_0) = 0$ as $\ker f \subset \ker g$. Define $\lambda_1 = \frac{1}{f(v_0)}$.

We have that $V = \ker f \oplus \operatorname{span}\{v_0\}$. We have that $v = v_f + \lambda v_0$ for some $\lambda \in \mathbb{F}$.

Therefore, we have that

$$\begin{aligned} g(v) &= g(v_f + \lambda v_0) \\ &= 1 \end{aligned}$$

Also, note that

$$\begin{aligned} \lambda_1 f(v) &= \lambda_1 \lambda f(v_0) \\ &= \lambda \end{aligned}$$

This shows the theorem is true for the case $n = 1$. $\ddot\smile$

9 Question 9

Let X be an infinite dimensional normed linear space and $S = \{x \in X : \|x\| = 1\}$ be the unit sphere in X. Show that if $y \in X$ with $\|y\| \leqslant 1$, then every weak neighbourhood of y must intersect S. Finally show that weak closure of S is equal to the closed unit Ball $B = \{x \in X : \|x\| \leqslant 1\}$.

Proof. We proceed to show that if $y \in X$ with $\|y\| \leq 1$ and U is a weak neighbourhood of y then $U \cap S \neq \emptyset$.

Since U is a nonempty set, we claim that there exists $x_0 \neq 0$ such that $y + \operatorname{span}\{x_0\} \subset U$. Since U is a weakly open set, there exists $f_1, f_2, \ldots, f_n \in X^*$ and $\varepsilon > 0$ such that

$$\bigcap_{i=1}^{n} \{x \in X : |f_i(y - x)| < \varepsilon\} \subset U.$$

We show that there is an $x_0 \in X \setminus \{0\}$ such that $f_1(x_0) = f_2(x_0) = \ldots = f_n(x_0) = 0$. If there are none, we consider the map

$$\begin{aligned} X &\to \mathbb{C}^n \\ x &\mapsto (f_1(x), f_2(x), \ldots, f_n(x)). \end{aligned}$$

and this map would be injective which is a contradiction as C^n is finite dimensional while OTOH, X is infinite dimensional.

Thus, we can let $x_0 \in X \setminus \{0\}$ such that $f_1(x_0) = f_2(x_0) = \ldots = f_n(x_0) = 0$. Now, for any $t \in \mathbb{C}$ and $i \in \{1, 2, \ldots, n\}$, we have that

$$f_i(y + tx_0) = f_i(y).$$

Hence, we have that

$$y + tx_0 \in \cap_{i=1}^{n} f_i^{-1}(\{f_i(y)\}) \subset \bigcap_{k=1}^{n} f_i^{-1}(B(f_i(y), \varepsilon)).$$

This shows that $y + tx_0 \in U$ for each $t \in \mathbb{C}$. This completes the proof of our claim.

Now, observe that the function $f\colon t \mapsto \|y + tx_0\|$ is continuous on $[0, +\infty)$, $f(0) \leq 1$, and $f(t) = |t|\|x_0 + \frac{1}{t}x\| \to +\infty$, $t \to +\infty$ if $\|x_0\| \neq 0$. By the intermediate value theorem, there is a $t_0 \in [0, +\infty)$ such that $f(t_0) = \|y + t_0 x_0\| = 1$.

Thus, we have that $S \cap U \neq \emptyset$.

10 Question 10

Let $T : X^* \to \mathbb{F}$ be a linear functional such that T is continuous w.r.t the weak star topology (X, τ_w). Show that $T = J_x$ for some $x \in X$.

Proof (sketch). Let $T : X^* \to \mathbb{F}$ be a linear functional that is continuous wrt weak star topology. Then $\Upsilon = \{\varphi \in X^* : |T(\varphi)| < 1\}$ is weak starly open.

Note that $0 \in \Upsilon$. Thus, there must be some $x_1, x_2, \ldots, x_k \in X$ and $\varepsilon > 0$ such that

$$\bigcap_{i=1}^{k} \{f \in X^* : |J_{x_i}(\varphi)| < \varepsilon\} = \bigcap_{i=1}^{k} \{\varphi \in X^* : |\varphi(x_i)| < \varepsilon\} \subset \Upsilon.$$

Now, we claim that $\bigcap_{i=1}^{n} \ker J_{x_i} \subset \ker T$. Let $\varphi \in J_{x_i}$ for each $i = 1, 2, \ldots, k$. Let $n \in \mathbb{N}$ be arbitrary. Then we have that $J_{x_i}(n\varphi) = 0$, that is, $n\varphi(x_i) = 0$ for each $i = 1, 2, \ldots, k$. By the previous inclusion, we have that $n\varphi \in \Upsilon$.

Thus, we have that

$$|T(n\varphi)| < 1 \text{ for each } n \in \mathbb{N}.$$

which further implies that $T(\varphi) = 0$.

Hence, we have that $\varphi \in \ker T$.

Question 8 in this Assignment then allows us to finish the problem. $\smile$

11 Question 11

Suppose X be an infinite dimensional normed linear space. Then show that the weak topology (X, τ_w) is never first countable and hence (X, τ_w) is not metrizable.

Proof. Let X be an infinite dimensional normed linear space. Then we have that X^* is a Banach space. By BCT, let us keep in mind that X^* cannot have a countable Hamel basis.

Let us assume for the sake of contradiction that $\mathcal{B}_0 = \{B_n : n \in \mathbb{N}\}$ is a countable local basis at the origin 0. For each $n \in \mathbb{N}$, we can find $f_{n,1}, f_{n,2}, \ldots, f_{n,k_n} \in X^*$ and $\varepsilon_n > 0$ such that

$$\underbrace{\cap_{i=1}^{k_n} f_{n,i}^{-1}\left(B\left(0, \varepsilon_n\right)\right)}_{:=\mathscr{U}_n} \subset B_n.$$

It is easy to see that $\{\mathscr{U}_n : n \in \mathbb{N}\}$ forms a countable local basis at 0. So, we may assume that $\mathcal{B}_0 = \{\mathscr{U}_n : n \in \mathbb{N}\}$.

Consider the following countable set of linear functionals:

$$\mathcal{F} := \{f_{i,j} : i \in \mathbb{N}, 1 \leq j \leq k_i\}.$$

Since X^* cannot have a countable Hamel basis, there must be some linear functional $f \in X^*$ such that $f \notin \operatorname{span} \mathcal{F}$.

Consider the weakly open set $f^{-1}(B(0,1))$. Since $\mathcal{B}_0$ is a countable local basis at 0, we must have that there is some $n \in \mathbb{N}$ such that

$$\mathscr{U}_n \subset f^{-1}\left(B\left(0, 1\right)\right).$$

The same proof in Question 10 shows that $\cap_{i=1}^{k_n} \ker f_{n,i} \subset \ker f$. Question 8 as a consequence shows that $f \in \operatorname{span} \mathcal{F}$ which is a contradiction. Hence, we are done. ⌣̈