Functional Analysis Assignment 3

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Note

A checkmark ✓ indicates the question has been done.

Contents

1 Question 1

Let V and W be two NLS and $T : V \to W$ be a linear map. Show that T is continuous if and only if T maps every Cauchy sequence of V to a Cauchy sequence of W.

Proof. Let V, W be two NLS and let $T : V \to W$ be a linear map.

($\Longrightarrow$) Suppose that T is continuous. Let $\{x_n\}$ be a Cauchy sequence in X. We want to show that $\{Tx_n\}$ is Cauchy sequence in Y. To do so, let $\varepsilon > 0$ be given. By the continuity of T, there is some $k > 0$ such that

$$\|Tx\| \leq k \|x\| \text{ for every } x \in X. \tag{1.0.1}$$

Since $\{x_n\}$ is Cauchy, there is some $N \in \mathbb{N}$ such that

$$\|x_n - x_m\| < \frac{\varepsilon}{k} \text{ for every } n, m \geq N \tag{1.0.2}$$

Thus, for every $n, m \geq N$, we have that

$$\begin{aligned} \|Tx_n - Tx_m\| &\leq k \|x_n - x_m\| && \text{from 1.0.1} \\ &< \varepsilon && \text{from 1.0.2} \end{aligned}$$

This shows that $\{Tx_n\}$ is Cauchy in Y.

($\Longleftarrow$) We prove it by contrapostitively. Suppose that T is not continuous. Then for every $k > 0$,

$$\|Tx\| > k \|x\| \text{ for some } x \in X.$$

Thus, for each $n \in \mathbb{N}$, we can find some $x_n \in X$ such that $\|Tx_n\| > n^2 \|x_n\|$. Consider the sequence $\{y_n\}$ in V defined by

$$y_n = \frac{x_n}{n \|x_n\|} \text{ for each } n \in \mathbb{N}$$

We now show that $\{y_n\}$ is Cauchy. Let $\varepsilon > 0$ be given. Select $N \in \mathbb{N}$ such that $\frac{2}{N} < \varepsilon$. For $k \in \mathbb{N}$ and $n \geq N$, we have that

$$\begin{aligned} \|y_{n+k} - y_m\| &= \left\| \frac{x_{n+k}}{(n+k)\|x_{n+k}\|} - \frac{x_n}{n\|x_n\|} \right\| \\ &\leq \frac{1}{n+k} + \frac{1}{n} \\ &= \frac{2}{n} \leq \frac{2}{N} < \varepsilon \end{aligned}$$

This shows that $\{y_n\}$ is Cauchy but on the other hand, we have that

$$\|Ty_n\| = \left\| T\left(\frac{x_n}{n\|x_n\|}\right) \right\| > n$$

This shows that $\{Ty_n\}$ is unbounded, a property which Cauchy sequences cannot have. $\square$

2 Question 2

Let X be a real NLS and $T : X \to \mathbb{R}$ be a non continuous linear functional. Then show that $T(U) = \mathbb{R}$ for any non empty open subset $U \subseteq X$.

Proof. We first show that $T(B_X(0,1)) = \mathbb{R}$ and we will show that this is all we need. First, suppose that T is not continuous. Therefore, for every $k > 0$,

$$|Tx| > k \text{ for some } x \in \overline{B_X(0,1)}. \tag{2.0.1}$$

It is clear that $T(B_X(0,1)) \subset \mathbb{R}$. To show the reverse inclusion, let $\alpha \in \mathbb{R}$ then by 2.0.1, we have that there is some $x \in X$ with $\|x\| \leq 1$ and $|Tx| > |\alpha| + 1$. Now, now define the vector $y = \frac{\alpha}{Tx}x$. Observe that

$$Ty = \alpha \frac{Tx}{Tx} = \alpha$$

and

$$\begin{aligned} \|y\| &= \left| \frac{\alpha}{Tx} \right| \|x\| \\ &< \frac{\alpha}{|\alpha| + 1} \|x\| \\ &\leq \|x\| = 1 \end{aligned}$$

Hence, we have that $\alpha \in T(B(0,1))$. It remains to show that it suffices to work on the unit ball.

Let U be any nonempty open set in X. Then there is some point $x_0 \in U$ and some $r > 0$ such that $B(x_0, r) \subset U$. Observe that

$$\begin{aligned} T(B(x_0, r)) &= T(x_0 + rB(0,1)) \\ &= T(x_0) + rB(0,1) \end{aligned}$$

Since by the previous argument, we have $B(0,1) = \mathbb{R}$. Hence, we have that $\mathbb{R} \subset U$ and thus, we are done. □