Functional Analysis Assignment 5

Ashish Kujur

Note

A checkmark ✓ indicates the question has been done.

Contents

1 Question 1

Suppose M and N are two topologically complimentary closed subspace of a Banach space $(X, \|\cdot\|_X)$. Now consider $M \oplus_1 N$, the external direct sum, defined in the following way)

$$M \oplus_1 N = \{(m, n) : m \in M, n \in N\},\ \|(m, n)\|_1 = \|m\|_X + \|n\|_X.$$

(a) Show that $M \oplus_1 N$ is a Banach space w.r.t the norm $\|\cdot\|_1$ mentioned above.

(b) Show that X is isomorphic to $M \oplus_1 N$.

(c) Show that the quotient space X/M is isomorphic to the Banach space N.

Proof of item (a). We proceed to prove (a). Let $((m_k, n_k))_{k\in\mathbb{N}}$ be a Cauchy sequence in $M \oplus_1 N$. We show that (m_k) is Cauchy in X. Consider the following:

$$\|m_k - m_l\|_X \leq \|(m_k, n_k) - (m_l, n_l)\|_1 .$$

Now since $((m_k, n_k))$ is Cauchy, we have that (m_k) is Cauchy in X. Since X is a Banach space, we have that (m_k) converges to some $m \in M$ as M is closed. Likewise it can be shown that (n_k) converges to some $n \in N$. We now show that $((m_k, n_k))$ converges to (m, n) in $M \oplus_1 N$. Consider the following:

$$\|(m_k, n_k) - (m, n)\| = \|m_k - m\|_X + \|n_k - n\|_X$$

Since (m_k) converges to m and (n_k) converges to n, we are done. ☻

Proof of item (b). To show that X is isomorphic to $M \oplus_1 N$, consider the map $T : M \oplus_1 N \to X$ given by

$$T(m, n) = m + n$$

for every $m \in M$ and every $n \in N$. First, we show that T is a normed linear space isomorphism, that is, both T and T^{-1} are bounded linear operators. It is immediate that T is bijective and linear. Since the projection maps $m+n \to m$ and $m+n \to n$ are continuous, there are some constant μ and ν such that $\|m\|_X \leq \mu \|m + n\|_X$ and $\|n\|_X \leq \nu \|m + n\|_X$. Now, let $m \in M$ and $n \in N$. Then

$$\begin{aligned}\|T(m, n)\|_X &= \|m + n\|_X \\ &\leq \|m\|_X + \|n\|_X \\ &= \|(m, n)\|_1\end{aligned}$$

and

$$\begin{aligned}\left\|T^{-1}(m + n)\right\|_1 &= \|(m, n)\|_1 \\ &= \|m\|_1 + \|n\|_1 \\ &\leq (\mu + \nu) \|m + n\|_X\end{aligned}$$

This shows that X is isomorphic to $M \oplus_1 N$. ☻

Proof of item (c). Let $P_N : X \to N$ be the projection of X into N. Since P_N is onto, by the first isomorphism theorem for vector spaces, we have that $X/M \cong N$. It remains to show that map $[x]_M \mapsto P_N(x)$ and its inverse is continuous (note this is the isomorphism given by the first isomorphism theorem). We show that the map $P_N(x) \mapsto [x]_M$ is continuous. Let $x \in X$. Suppose $x = m + n$. Then we have that $P_N(x) = n$. Then

$$\begin{aligned} \|[x]_M\| &\leq \|x - m\| && \text{(by definition of quotient norm)} \\ &= \|n\| \\ &= \|P_N(x)\|_X \end{aligned}$$

This shows that the aforementioned map is continuous and bijective, by the Banach isomorphism theorem, we are done. ☻

2 Question 2

Let H be a Hilbert space with an orthonormal basis $\{e_j : j \in \mathbb{N}\}$. Consider the set

$$A = \{e_k + ke_l : k < l,\, k, l \in \mathbb{N}, \}.$$

Show that 0 belongs to the weak closure of A. Also show that there is no sequence in A which converge weakly to 0.

Proof. Recall the fact that in a topological space, we have that $x \in \overline{A}$ if there is a sequence $(x_n)_{n\in\mathbb{N}}$ in A converging to $x \in A$.

For each $k \in \mathbb{N}$, we have that the sequence $(e_k + ke_l)_{l\geq k}$ converges to e_k. Thus, we have that $e_k \in \overline{A}^w$. Also, $(e_k) \in \overline{A}^w$ converges to 0. Therefore, we have that $0 \in \overline{A}^w$.

Let $(\tilde{e}_n)$ be a sequence in A converging to 0. Then $\tilde{e}_n = e_{k_n} + k_n e_{l_n}$ for some $k_n, l_n \in \mathbb{N}$ with $k_n < l_n$. Then $(\tilde{e}_n)$ must be norm bounded. Let $M > 0$ such that $\|\tilde{e}_n\| \leq M$ for each $n \in \mathbb{N}$.

We claim that $\{k_n : n \in \mathbb{N}\}$ is finite. This is easy to see:

$$\begin{aligned} M &\geq \|k_n e_{l_n} + e_{k_n}\| \\ &\geq |k_n \|e_{l_n}\| - \|e_{k_n}\|| \\ &\geq k_n - 1 \end{aligned}$$

for each $n \in \mathbb{N}$.

Since the aforementioned set is finite, we may let $\{k_n : n \in \mathbb{N}\} = \{k_{n_1}, \ldots, k_{n_l}\}$ for some $n_1, n_2, \ldots, n_l \in \mathbb{N}$. It is a consequence of Riesz Representation theorem that in a Hilbert Space, $x_n \to x$ weakly iff $\langle x_n, y\rangle \to \langle x, y\rangle$ for each $y \in H$. We use this to achieve a contradiction. Now, let $y = e_{k_{n_1}} + e_{k_{n_2}} + \ldots + e_{k_{n_l}}$. It can be seen that $\langle \tilde{e}_n, y\rangle \geq 1$ for each $n \in \mathbb{N}$ and cannot converge to 0 as $n \to \infty$. ☻

3 Question 3

Let H be a Hilbert space. Suppose $\{x_n\}_{n\in\mathbb{N}}$ is a sequence of vector in H which converges weakly to a vector x in H and $\|x_n\| \to \|x\|$ as $n \to \infty$. Then show that $\|x_n - x\| \to 0$ as $n \to \infty$.

Proof. Let (x_n) be a sequence in H converging to $x \in H$ and furthermore suppose that $\|x_n\| \to \|x\|$ as $n \to \infty$. We wish to show that $x_n \to x$ strongly.

Since $x_n \to x$ weakly, we have that $\langle x_n, y\rangle \to \langle x, y\rangle$ for each $y \in H$ by the definition and Riesz Representation theorem. Thus, we have that $\langle x_n, x\rangle \to \|x\|^2$ in particular.

Now,

$$\|x_n - x\|^2 = \|x_n\|^2 - 2\Re\langle x_n, x\rangle + \|x\|^2 .$$

Taking limits both sides, we have that

$$\lim_{n\to\infty} \|x_n - x\|^2 = 0$$

because $\|x_n\|^2 \to \|x\|^2$ and $\langle x_n, x\rangle \to \|x\|^2$.

This completes the proof as square root function is continuous. ☻

4 Question 4

Let $\{e_n : n \in \mathbb{N}\}$ be the standard Schauder basis for the Banach space $\ell^p(\mathbb{N})$ where $1 \leqslant p < \infty$. Show that $e_n \to 0$ in the weak topology of $\ell^p(\mathbb{N})$ for every $p > 1$. But for $p = 1$, the sequence e_n does not converges to 0 in the weak topology of $\ell^1(\mathbb{N})$.

Proof. First, we deal with the case when $1 < p < +\infty$. It can be shown that

$$(\ell^p (N))^* = \{L_y : y \in \ell^q (\mathbb{N})\}$$

where $L_y(x) = \sum_{i=1}^{\infty} x_i y_i, x \in \ell^q (\mathbb{N})$. Note that for each $y \in \ell^q (\mathbb{N})$ with $1 \leq q < \infty$, we have that $y_i \to 0$ as $i \to \infty$. This is because $\sum_{i=1}^{\infty} |y_i|^q < \infty$ for $y \in \ell^q (\mathbb{N})$.

Now, let $y \in \ell^q (\mathbb{N})$. We have that

$$L_y (e_n) = y_n \to 0 \text{ as } n \to \infty.$$

This shows that (e_n) converges to 0 in the weak topology.

Now, consider the case where $p = 1$. Then we have

$$\left(\ell^1 (\mathbb{N})\right)^* = \{L_y : y \in \ell^\infty (\mathbb{N})\}$$

where L_y is as specified in the previous case. Let $y = (1, 1, 1, \ldots)$. Then we have that

$$L_y (e_n) = 1$$

for each $n \in \mathbb{N}$. Hence, we have that (e_n) does not converge to 0 in the weak topology. ☻

5 Question 5

Let M be a norm closed subspace of a normed linear space X. Show that M is also closed in the weak topology of X.

Proof. Let M be a strongly closed subspace of X. We wish to show that it is weakly closed. To do so, we will show that $X \setminus M$ is weakly open.

Let $x \in X \setminus M$. We will be done if we show that there is a weakly open set U such that $x \in U \subset X \setminus M$.

Recall a result about metric spaces: in a metric space, distance between a closed set and a compact set which are disjoint is strictly positive. Since $\{x\}$ is compact and M is closed, we have that the distance d between the point x and M is strictly positive.

We claim that there is linear functional $f \in X^*$ such that $f(M) = 0$ and $f(x) = d$.

For the timebeing, let us assume this claim. Let $f \in X^*$ be such a functional. Then we have that $U := f^{-1}((d/2, \infty))$ is weakly open (because weak topology is the smallest topology which makes every linear functional continuous), $a \in U$ and $U \subset X \setminus M$. This shows that $X \setminus M$ is weakly open and hence M is weakly closed.

We now proceed to prove that claim. Consider the subspace:

$$N := \{\lambda x + m : \lambda \in \mathbb{F}, m \in M\}$$

of X. We now define a continuous linear functional f_N on N and extend it to X via Hahn Banach. So, consider the linear functional $f_N : N \to \mathbb{F}$ given by $f_N(\lambda x + m) = \lambda d$. It is easy to see that this functional is well defined and linear. We now show that $\|f_N\|_{N^*} \leq 1$. Let $\lambda \in \mathbb{F}$ and $m \in M$. We have that

$$\|\lambda x + m\| = |\lambda| \left\| x - \left(-\frac{m}{\lambda}\right) \right\| \geq |\lambda| \, d = \|f_N(\lambda x + m)\|$$

This shows that f_N is continuous. Hence, by Hahn Banach, we are done. ☻

6 Question 6

Let H be a Hilbert space. Show that closed unit ball in H is compact in the weak topology.

Proof. First, we show that any Hilbert space is isometrically isomorphic to its dual. Let H be a Hilbert space. Let H^* be its dual. We establish that there is a isometry between H and H^*. For each $y \in H$, define $L_y : H \to \mathbb{C}$ by $L_y(x) = \langle x, y \rangle$.

Now, consider the map $\varphi : H \to H^*$ given by $\varphi(y) = L_y$ for each $y \in H$. We claim that this map is an isometric isomorphism. It is easy to see this map is linear. To see that this map is one one, let $y \in H$ such that $L_y = 0$. Then we have that $\langle y, y \rangle = 0$. Thus, $y = 0$. This shows that $y = 0$. Onto and isometry follows from Riesz Representation theorem.

Now, we prove that if X and Y are isometric normed linear spaces then there is a homeomorphism between the weak topology on X and the weak topology on Y. Let $\varphi : X \to Y$ be isometry between X and Y. (Complete the proof ...)

Therefore, we have the closed unit ball in H is compact in the weak topology. ☻

7 Question 7

Suppose X is a finite dimensional normed linear space. Then show that the weak topology on X and the norm topology on X coincides.

Proof. It is clear that the norm topology contains the norm topology. To show the reverse inclusion, we show that every open ball contains a basis element of the weak topology.

Consider the open ball $B(0,1)$. Suppose that X is of dimension n. Consider the linear functionals $f_i(x) = x_i$ for each $i = 1, 2, \ldots, n$. Then it is easy to see that $\cap_{i=1}^{n} f_i^{-1}(B(0,1/2)) \subset B(0,1)$. This completes the proof.

8 Question 8

Let V be a vector space over $\mathbb{F}$, where ($\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$). Suppose $g, f_1, f_2, \ldots, f_k$ are non zero linear functional on V satisfying

$$\bigcap_{j=1}^{k} \ker f_j \subseteq \ker g.$$

Then show that g belong to $\operatorname{span}\{f_1, f_2, \ldots, f_k\}$.

Proof. We proceed by induction. First suppose that $\ker f \subset \ker g$. We show that $g = \lambda f$ for some $\lambda \in \mathbb{F}$.

Observe that if $\ker g = V$ then $g = 0 = 0f$ and we are done. Suppose not. Then we can select $v_0 \in V$ such that $g(v_0) = 1$. Then $f(v_0) \neq 0$ for otherwise $v_0 \in \ker f$ which would imply that $g(v_0) = 0$ as $\ker f \subset \ker g$. Define $\lambda_1 = \frac{1}{f(v_0)}$.

We have that $V = \ker f \oplus \operatorname{span}\{v_0\}$. We have that $v = v_f + \lambda v_0$ for some $\lambda \in \mathbb{F}$.

Therefore, we have that

$$\begin{aligned} g(v) &= g(v_f + \lambda v_0) \\ &= 1 \end{aligned}$$

Also, note that

$$\begin{aligned} \lambda_1 f(v) &= \lambda_1 \lambda f(v_0) \\ &= \lambda \end{aligned}$$

This shows the theorem is true for the case $n = 1$. ☻

9 Question 9

Let X be an infinite dimensional normed linear space and $S = \{x \in X : \|x\| = 1\}$ be the unit sphere in X. Show that if $y \in X$ with $\|y\| \leqslant 1$, then every weak neighbourhood of y must intersect S. Finally show that weak closure of S is equal to the closed unit Ball $B = \{x \in X : \|x\| \leqslant 1\}$.

Proof. We proceed to show that if $y \in X$ with $\|y\| \leq 1$ and U is a weak neighbourhood of y then $U \cap S \neq \emptyset$. ☻

10 Question 10

Let $T : X^* \to \mathbb{F}$ be a linear functional such that T is continuous w.r.t the weak star topology (X, τ_w). Show that $T = J_x$ for some $x \in X$.

11 Question 11

Suppose X be an infinite dimensional normed linear space. Then show that the weak topology (X, τ_w) is never first countable and hence (X, τ_w) is not metrizable.
