Functional Analysis Assignment 7

Ashish Kujur

Note

Then end of a proof is denoted by $\smile$.

Contents

1 Question 1

Let H be a Hilbert space. Suppose $\{T_n\}_{n\in\mathbb{N}}$ is a sequence operators in $\mathcal{B}(H)$, satisfying $\|T_n - T\| \to 0$ as $n \to \infty$ for some $T \in \mathcal{B}(H)$. If every T_n is self adjoint then show that T is self adjoint. Moreover if every $T_n \geqslant 0$ then show that $T \geqslant 0$.

Proof. Let (T_n) be a sequence of self adjoint operators in $\mathcal{B}(H)$ converging to some $T \in \mathcal{B}(H)$. We wish to show that T is again self-adjoint. Let $f, g \in H$ be arbitrary. We have that

$$\langle T_n f, g\rangle = \langle f, T_n g\rangle$$

for each $n \in \mathbb{N}$ as T_n's are self-adjoint. Taking limits both sides, we have that

$$\lim_n \langle T_n f, g\rangle = \lim_n \langle f, T_n g\rangle .$$

We will be done if we show that $\lim_n \langle T_n f, g\rangle = \langle Tf, g\rangle$. To prove this, consider the following:

$$\begin{aligned} \langle T_n f - Tf, g\rangle &\leq \|T_n f - Tf, g\| \\ &\leq \|T_n - T\| \, \|f\| \, \|g\| \end{aligned}$$

Since $T_n \to T$ in the operator norm, we have that $\lim_n T_n f = Tf$. Likewise, it follows that $\lim_n \langle f, T_n g\rangle = \langle f, Tg\rangle$.

Thus, we have that

$$\langle Tf, g\rangle = \langle f, Tg\rangle .$$

Since f, g were arbitrary, we have that T is self-adjoint.

Now, we proceed to show that a sequence of positive operators (T_n) converge to a positive operator T in the operator norm. Let $f \in H$. Then we claim that $\langle Tf, f\rangle \geq 0$. As in the previous argument, we have that $\lim_n \langle T_n f, f\rangle = \langle Tf, f\rangle$. Since $\langle T_n f, f\rangle \geq 0$ for each $n \in \mathbb{N}$, we have $\langle Tf, f\rangle \geq 0$. $\ddot\smile$

2 Question 2

Let H be a Hilbert space and $T, S \in \mathcal{B}(H)$, satisfying $T \geqslant 0$ and $TS = ST$. Show that $\sqrt{T}S = S\sqrt{T}$.

Proof. Since $T \geq 0$ we have that $\sqrt{T}$ is a limit of sequence of polynomials in T, that is, there exists polynomials p_n such that $p_n(T)$ converges to $\sqrt{T}$ in the operator norm. [1]

Since S and T commute by hypothesis, we have that

$$p_n(T)\, S = S p_n(T)$$

for each $n \in \mathbb{N}$. Taking limit $n \to \infty$ both sides we have that

$$\sqrt{T}S = S\sqrt{T}.$$

$\smile$

---

[1] See Rynne Johnson Theorem 6.58. for a proof

3 Question 3

3.1 Proof of Polar Decomposition as in Bhatia with GORY DETAILS!

Theorem 3.1.1. *Let A be a linear operator in a Hilbert space H. Then there exists a partial isometry W such that*

$$A = W\,|A|$$

whose initial space is $(\ker A)^{\perp}$ and final space is $\overline{\mathrm{im}\ (A)}$. The decomposition is unique in the following sense: if $A = UP$ where U is a partial isometry, $P \geq 0$ and $\ker U = \ker P$ then $P = |A|$ and $U = W$.

Proof. We begin the proof by showing that $\|Ax\| = \||A|\,x\|$ for every $x \in H$. This also shows that $\ker A = \ker |A|$ (prove this!). Indeed, for $x \in H$, we have that

$$\begin{aligned}
\|Ax\|^2 &= \langle Ax, Ax \rangle && \text{(by definition of norm)} \\
&= \langle A^*Ax, x \rangle \\
&= \left\langle |A|^2\,x, x \right\rangle && \text{(by definition of modulus of an operator)} \\
&= \langle |A|\,x, |A|\,x \rangle && \text{(You're mature enough to figure this out!)} \\
&= \||A|\,x\|^2
\end{aligned}$$

Well this shows what we wanted to prove.

Now, we define a map $\bar{W} : \mathrm{im}\ (|A|) \to \mathrm{im}\ (A)$ by $\bar{W}\,|A|\,x = Ax$ for each $x \in A$. Is this well-defined? YES! Suppose that $|A|\,x = |A|\,y$. Then we have that $\||A|\,(x-y)\| = 0$. Thus, $\|A\,(x-y)\| = 0$ by what we had proved earlier. Hence, $Ax = Ay$. Thus, $\bar{W}$ is well defined. In fact, this is an isometry by what we had proved earlier.

Now, we claim that $\overline{\mathrm{im}\ (|A|)} = (\ker A)^{\perp}$. To prove this, recall that for subspaces M of Hilbert spaces, we have that $\left(M^{\perp}\right)^{\perp} = \overline{M}$. Now, consider the following:

$$\begin{aligned}
\overline{\text{im }(|A|)} &= \left((\text{im }(|A|))^{\perp}\right)^{\perp} \\
&= (\ker(|A|^*))^{\perp} \\
&= (\ker|A|)^{\perp} \\
&= (\ker A)^{\perp} && \text{(see the second line of this proof!)}
\end{aligned}$$

Since im (A) is dense in $(\ker A)^{\perp}$, $\bar{W}$ extends linearly to an operator $\tilde{W}$: $(\ker A)^{\perp} \to \text{im } A$ such that $\tilde{W}\ |_{\text{im }(|A|)} = \bar{W}$. Read the TILDE and BAR carefully lest you may get lost.

Now, we can extend $\tilde{W}$ to all of H by defining

$$Wx = \tilde{W} P_{(\ker A)^{\perp}} x$$

for each $x \in H$.

Let us now show that W is a partial isometry. Now, if $x \in (\ker A)^{\perp}$ then $P_{(\ker A)^{\perp}} x = x$ and thus $\|Wx\| = \left\|\tilde{W}x\right\|$. We will be done if we show that $\left\|\tilde{W}x\right\| = \|x\|$.

$\ddot\smile$

3.2 The Question

Let H be a Hilbert space and $T \in \mathcal{B}(H)$. Suppose $T = U|T|$ be the polar decomposition of T. Show that

(a) $U^*T = |T|$.

(b) $U|T| = |T|U$ if and only if $T(T^*T) = (T^*T)T$.

Proof. Let $T = U\,|T|$ where U is a partial isometry whose initial space is $(\ker T)^{\perp}$ and final space is $\overline{(\text{im } T)}$.

1. Observe that $U^*T = U^*U\,|T|$. But then $U^*U = P_{(\ker T)^\perp}$ because if U is a partial isometry then it is an orthogonal projection onto its initial space. Also, note that

$$\begin{aligned} \text{im }\,(|T|) &\subset \overline{\text{im }\,(|T|)} \\ &= (\ker |T|^*)^\perp \\ &= (\ker |T|)^\perp \\ &= (\ker T)^\perp \end{aligned}$$

   This shows that $U^*U\,|T| = |T|$.

2.

$\ddot\smile$

4 Question 4

Let H be a Hilbert space and $T \in \mathcal{B}(H)$. Suppose that $T \geqslant 0$. Show that $T^k \geqslant 0$ and $\ker T = \ker T^k$ for every $k \in \mathbb{N}$.

Proof. Suppose that $T \geq 0$. We show by induction that $T^k \geq 0$ for each $k \in \mathbb{N}$. It easily follows by induction that $\left(T^k\right)^* = (T^*)^k$ for each $k \in \mathbb{N}$. Let us show that $T^2 \geq 0$. Let $f \in H$. Then we have that

$$\begin{aligned} \left\langle T^2 f, f \right\rangle &= \langle Tf, Tf \rangle \\ &\geq 0 \end{aligned}$$

Now, assume that we have shown that $T^k \geq 0$ for all $k \leq n$ where $n \geq 3$. We now show that $T^n \geq 0$. Let $f \in H$. Then

$$\begin{aligned} \langle T^n f, f \rangle &= \left\langle T^{n-1} f, Tf \right\rangle \\ &= \left\langle T^{n-2}(Tf), Tf \right\rangle \\ &\geq 0 \end{aligned}$$

This shows that $T^k \geq 0$ for all $k \in \mathbb{N}$.

Let us proceed to show that $\ker T^k = \ker T^{k+1}$ for all $k \in \mathbb{N}$. Let us do it for the case $k = 1$. It is always true that $\ker T \subset \ker T^2$, so, let $f \in H$ such that $T^2 f = 0$. To show $f = 0$, consider the following

$$\begin{aligned} \|Tf\| &= \langle Tf, Tf \rangle \\ &= \left\langle T^2 f, f \right\rangle \\ &= 0 \end{aligned}$$

This shows that $Tf = 0$. Hence $\ker T^2 \subset \ker T$.

Now, suppose that the claim is true for $k = n$. Then we need to show that $\ker T^{n+1} = \ker T^{n+2}$. It is trivial that $\ker T^{n+1} \subset \ker T^{n+2}$. Now, let $f \in H$ and suppose that $T^{n+2} f = 0$. We wish to show that $T^{k+2} f = 0$. To that end, consider the following:

$$\begin{aligned} \left\|T^{k+1} f\right\|^2 &= \left\langle T^{k+1} f, T^{k+1} f \right\rangle \\ &= \left\langle T^{k+2} f, T^k f \right\rangle \\ &= 0 \end{aligned}$$

Hence, we are done. $\ddot\smile$

5 Question 5

Let H be a Hilbert space and $T, S \in \mathcal{B}(H)$, satisfying $T \geqslant 0$, $S \geqslant 0$ and $TS = ST$. Show that $TS \geqslant 0$.

Proof. Let $S, T \geq 0$ and suppose that $ST = TS$. Then we have that

$$\begin{aligned}(ST)^* &= T^* S^* \\ &= TS \\ &= ST.\end{aligned}$$

Consider the product ST. We claim that $ST = \sqrt{S}T\sqrt{S}$. This follows from Question 2. Since $ST = TS$, we have that $\sqrt{S}T = T\sqrt{S}$. Multiplying $\sqrt{S}$ both sides, we have that $ST = \sqrt{S}T\sqrt{S}$. Now, let $f \in H$. Then we have that

$$\begin{aligned}\langle STf, f\rangle &= \left\langle \sqrt{S}T\sqrt{S}f, f\right\rangle \\ &= \left\langle T\sqrt{S}f, \sqrt{S}f\right\rangle \\ &\geq 0 && (T \geq 0)\end{aligned}$$

$\smile$

6 Question 6

Let $T \in \mathcal{B}(H)$ and $T^*T = TT^*$. Suppose that $T^k = 0$ for some $k \in \mathbb{N}$. Then show that $T = 0$.

Proof. Let T be a normal operator. Then we have that $\sigma(T) \subset \mathbb{R}$. If $T^k = 0$ for some $k \in \mathbb{N}$ then we have that $\sigma\left(T^k\right) = (\sigma(T))^k = \{0\}$. Let $\lambda \in \sigma(T)$. Then we have that $\lambda^k = 0$ implies $\lambda = 0$. This shows that $\sigma(T) = \{0\}$ because spectrum is nonempty.

Since T is normal, we have that $\|T\| = r(\rho(T)) = 0$. Thus, we have that $T = 0$. ☺

7 Question 7

Let $R > 0$ be the radius of convergence of the power series

$$f(z) = \sum_{k=0}^{\infty} a_k z^k.$$

Suppose $T \in \mathcal{B}(H)$ and $R > \|T\|$. Then show that the sequence $S_n = a_0 + a_1 T + a_2 T^2 + \cdots + a_n T^n$ is a Cauchy sequence in $\mathcal{B}(H)$. This gives us that the series $\sum\limits_{k=0}^{\infty} a_k T^k$ converges in $\mathcal{B}(H)$. We denote the limit operator by $f(T)$.

Proof. Since $\|T\| < R$, we have that

$$\sum_{k=0}^{\infty} |a_k| \, \|T\|^k < \infty \tag{7.1}$$

because the power series converges absolutely and uniformly on disks of radius smaller than the radius of convergence.

To show that S_n is Cauchy in $\mathcal{B}(H)$, consider the following for $n \geq m$:

$$\begin{aligned} \|S_n - S_m\| &= \left\| a_{m+1} T^{m+1} + + \ldots a_n T_n \right\| \\ &\leq |a_{m+1}| \, \|T\|^{m+1} + \ldots + |a_n| \, \|T\|^n \end{aligned}$$

From 7.1, we have

$$|a_{m+1}| \, \|T\|^{m+1} + \ldots + |a_n| \, \|T\|^n \to 0 \text{ as } n, m \to \infty.$$

Thus, (S_n) is Cauchy, Hence, the series $\sum_{k=0}^{\infty} a_k T^k$ converges in $\mathcal{B}(H)$. ⌣̈