Functional Analysis Assignment 3

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Note

A checkmark ✓ indicates the question has been done.

Contents

1 Question 1

Let V and W be two NLS and $T : V \to W$ be a linear map. Show that T is continuous if and only if T maps every Cauchy sequence of V to a Cauchy sequence of W.

Proof. Let V, W be two NLS and let $T : V \to W$ be a linear map.

($\Longrightarrow$) Suppose that T is continuous. Let $\{x_n\}$ be a Cauchy sequence in X. We want to show that $\{Tx_n\}$ is Cauchy sequence in Y. To do so, let $\varepsilon > 0$ be given. By the continuity of T, there is some $k > 0$ such that

$$\|Tx\| \leq k \|x\| \text{ for every } x \in X. \tag{1.0.1}$$

Since $\{x_n\}$ is Cauchy, there is some $N \in \mathbb{N}$ such that

$$\|x_n - x_m\| < \frac{\varepsilon}{k} \text{ for every } n, m \geq N \tag{1.0.2}$$

Thus, for every $n, m \geq N$, we have that

$$\begin{aligned} \|Tx_n - Tx_m\| &\leq k \|x_n - x_m\| && \text{from 1.0.1} \\ &< \varepsilon && \text{from 1.0.2} \end{aligned}$$

This shows that $\{Tx_n\}$ is Cauchy in Y.

($\Longleftarrow$) We prove it by contrapostitively. Suppose that T is not continuous. Then for every $k > 0$,

$$\|Tx\| > k \|x\| \text{ for some } x \in X.$$

Thus, for each $n \in \mathbb{N}$, we can find some $x_n \in X$ such that $\|Tx_n\| > n^2 \|x_n\|$. Consider the sequence $\{y_n\}$ in V defined by

$$y_n = \frac{x_n}{n \|x_n\|} \text{ for each } n \in \mathbb{N}$$

We now show that $\{y_n\}$ is Cauchy. Let $\varepsilon > 0$ be given. Select $N \in \mathbb{N}$ such that $\frac{2}{N} < \varepsilon$. For $k \in \mathbb{N}$ and $n \geq N$, we have that

$$\begin{aligned} \|y_{n+k} - y_m\| &= \left\| \frac{x_{n+k}}{(n+k)\|x_{n+k}\|} - \frac{x_n}{n\|x_n\|} \right\| \\ &\leq \frac{1}{n+k} + \frac{1}{n} \\ &= \frac{2}{n} \leq \frac{2}{N} < \varepsilon \end{aligned}$$

This shows that $\{y_n\}$ is Cauchy but on the other hand, we have that

$$\|Ty_n\| = \left\| T\left(\frac{x_n}{n\|x_n\|}\right) \right\| > n$$

This shows that $\{Ty_n\}$ is unbounded, a property which Cauchy sequences cannot have. □

2 Question 2

Let X be a real NLS and $T : X \to \mathbb{R}$ be a non continuous linear functional. Then show that $T(U) = \mathbb{R}$ for any non empty open subset $U \subseteq X$.

Proof. We first show that $T(B_X(0,1)) = \mathbb{R}$ and we will show that this is all we need. First, suppose that T is not continuous. Therefore, for every $k > 0$,

$$|Tx| > k \text{ for some } x \in \overline{B_X(0,1)}. \tag{2.0.1}$$

It is clear that $T(B_X(0,1)) \subset \mathbb{R}$. To show the reverse inclusion, let $\alpha \in \mathbb{R}$ then by 2.0.1, we have that there is some $x \in X$ with $\|x\| \leq 1$ and $|Tx| > |\alpha| + 1$. Now, now define the vector $y = \frac{\alpha}{Tx}x$. Observe that

$$Ty = \alpha\frac{Tx}{Tx} = \alpha$$

and

$$\begin{aligned} \|y\| &= \left|\frac{\alpha}{Tx}\right| \|x\| \\ &< \frac{\alpha}{|\alpha| + 1} \|x\| \\ &\leq \|x\| = 1 \end{aligned}$$

Hence, we have that $\alpha \in T(B(0,1))$. It remains to show that it suffices to work on the unit ball.

Let U be any nonempty open set in X. Then there is some point $x_0 \in U$ and some $r > 0$ such that $B(x_0, r) \subset U$. Observe that

$$\begin{aligned} T(B(x_0, r)) &= T(x_0 + rB(0,1)) \\ &= T(x_0) + rB(0,1) \end{aligned}$$

Since by the previous argument, we have $B(0,1) = \mathbb{R}$. Hence, we have that $\mathbb{R} \subset U$ and thus, we are done. $\square$

3 Question 3

Let $T : \mathscr{C}^1[0,1] \to \mathscr{C}[0,1]$ be the linear map defined by $T(f) = f'$, $f \in \mathscr{C}^1[0,1]$, where $\mathscr{C}[0,1]$ equipped with the usual sup norm $\|\cdot\|_\infty$. Show that T is not continuous if $\mathscr{C}^1[0,1]$ is equipped with the usual sup norm $\|\cdot\|_\infty$. But T is a continuous linear transformation and $\|T\| = 1$, if $\mathscr{C}^1[0,1]$ endowed with the following norm

$$\|f\| = \max\{\|f\|_\infty, \|f'\|_\infty\}. \tag{3.0.1}$$

Solution. Let $T : \mathscr{C}^1[0,1] \to \mathscr{C}[0,1]$ be given by $Tf = f'$. Let $\mathscr{C}^1[0,1]$ be given the sup norm first and $\mathscr{C}[0,1]$ be given the same sup norm. Consider the sequence of functions $f_n : [0,1] \to \mathbb{R}$ given by

$$f_n(x) = \sin(nx)$$

for every $x \in [0,1]$. Then we have that

$$f_n'(x) = n\cos(nx)$$

for each $x \in [0,1]$. Hence, we have that $\|f_n\|_\infty = 1$ and

$$\begin{aligned} \|Tf\| = \|f_n'\|_\infty &= \|n\cos(nx)\|_\infty \\ &= n \end{aligned}$$

for each $n \in \mathbb{N}$. Hence, we have that T is not a bounded linear operator. On the other hand, let's suppose that $\mathscr{C}^1[0,1]$ is given the norm specified in Equation 3.0.1. We now that that T is continuous with the specified norm. Let $f \in \mathscr{C}^1[0,1]$ with $\|f\| \leq 1$ then we have that

$$\|Tf\|_\infty = \|f'\|_\infty \leq \|f\| \leq 1.$$

Hence, this shows that T is continuous. $\square$

4 Question 4

Let X and Y be two NLS and T be a continuous linear map from X into Y. Show that following holds:

$$\underbrace{\sup\{\|Tx\|_Y : \|x\|_X \leqslant 1\}}_{:=\alpha} = \underbrace{\sup\{\|Tx\|_Y : \|x\|_X < 1\}}_{:=\beta} \tag{4.0.1}$$

$$= \underbrace{\sup\{\|Tx\|_Y : \|x\|_X = 1\}}_{:=\chi} \tag{4.0.2}$$

$$= \underbrace{\sup\{\frac{\|Tx\|_Y}{\|x\|_X} : x \in X, x \neq 0\}}_{:=\delta}. \tag{4.0.3}$$

Solution. We first prove that $\alpha = \beta$. Observe that

$$\{\|Tx\|_Y : \|x\|_X < 1\} \subset \{\|Tx\|_Y : \|x\|_X \leq 1\}$$
$$\rightsquigarrow \sup\{\|Tx\|_Y : \|x\|_X < 1\} \leq \sup\{\|Tx\|_Y : \|x\|_X \leq 1\}$$
$$\rightsquigarrow \beta \leq \alpha$$

Now, let $\varepsilon > 0$ be given. Then there exists $x_0 \in X$ satisfying $\|x_0\|_X \leq 1$ such that

$$\alpha - \varepsilon < \|Tx_0\|_Y.$$

For each $n \in \mathbb{N}$, we have that

$$\left(1 - \frac{1}{n}\right)(\alpha - \varepsilon) < \left\|T\left(\left(1 - \frac{1}{n}\right)x_0\right)\right\| \leq \beta. \tag{4.0.4}$$

Note that last inequality is true because

$$\left\|\left(1 - \frac{1}{n}\right)x_0\right\| = \left(1 - \frac{1}{n}\right)\|x_0\| < 1.$$

Let $n \to \infty$ in 4.0.4, we have that

$$(\alpha - \varepsilon) \leq \beta.$$

Since $\varepsilon > 0$ is arbitrary, we have that $\alpha \leq \beta$ and this completes the proof of the first equality.

We now proceed to show the equality $\alpha = \chi$. By subset argument, it is easy to see that $\chi \leq \alpha$. To show the reverse inequality, let $\varepsilon > 0$ be given. Then there exists $x \in X$ with $\|x\|_X \leq 1$ such that

$$\alpha - \varepsilon < \|Tx_0\|_Y$$

If $\|x_0\| = 0$ then we would have that $\alpha - \varepsilon < 0 \leq \beta$ and since $\varepsilon > 0$ is arbitrary, we would be done. So, assume that $\|x_0\| > 0$. Then we would have that

$$\frac{\alpha - \varepsilon}{\|x_0\|} < \left\|T\left(\frac{x_0}{\|x_0\|}\right)\right\| \leq \chi \rightsquigarrow \alpha - \varepsilon \leq \|x_0\|\chi \rightsquigarrow \alpha - \varepsilon \leq \chi.$$

Since $\varepsilon > 0$ is arbitrary, we would be done.

We finally show that $\chi = \delta$. Observe that the sets

$$\sup\{\|Tx\|_Y : \|x\|_X = 1\} = \sup\left\{\frac{\|Tx\|_Y}{\|x\|_X} : x \in X, x \neq 0\right\}.$$

and hence we are done. $\square$

5 Question 5

Let T be a finite rank (say of rank k) continuous linear operator from a Hilbert space H into itself. Show that there exist a linearly independent set $\{x_1, \ldots, x_k\}$ and $\{y_1, \ldots, y_k\}$ in H such that

$$T = (x_1 \otimes y_1) + \cdots + (x_k \otimes y_k).$$

Solution. Let $k \in \operatorname{ran} T$. Then $k = Tf$ for some f, i.e. $k = \lambda g$, where $\lambda = \langle f, h \rangle$. So every element in $\operatorname{ran} T$ is a scalar multiple of g. Thus, $\operatorname{ran} T$ has a basis consisting of $\{g\}$, i.e. it has dimension 1.

Now assume that T is finite rank. Let $g_1', \ldots, g_n'$ be an orthonormal basis of $\operatorname{ran} T$. Then, for every $f \in H$, $Tf = \sum_j \lambda_j(f)\, g_j'$, with the coefficients $\lambda_1(f), \ldots, \lambda_n(f)$ uniquely determined for for each f. So, for each j, the map $f \mapsto \lambda_j(f)$ is a linear functional on H. Note that

$$|\lambda_j(g)| \le \left(\sum_{k=1}^{n} |\lambda_k(f)|^2 \right)^{1/2} = \|Tf\| \le \|T\|\, \|f\|,$$

so every λ_j is a bounded functional. By the Riesz Representation Theorem, there exist vectors $e_1', \ldots, e_n'$ such that $\lambda_j(f) = \langle f, e_j' \rangle$. So

$$Tf = \sum_{j=1}^{n} \langle f, e_j' \rangle\, g_j', \qquad f \in H.$$

Now, using Gram-Schmidt, there exist $e_1, \ldots, e_n$, orthonormal, such that

$$e_k' = \sum_{j=1}^{k} \lambda_{kj} e_j$$

for coefficients $\{\lambda_{kj}\}_{k=1,\ldots,n; j=1,\ldots,k}$ (note that these are not the equalities from Gram-Schmidt, but rather the *inverse* form, where we express the old vectors in terms of the new orthonormal ones). Then

$$Tf = \sum_{k=1}^{n} \langle f, e_k' \rangle\, g_k' = \sum_{k=1}^{n} \langle f, \sum_{j=1}^{k} \lambda_{kj} e_j \rangle\, g_k' = \sum_{k=1}^{n} \sum_{j=1}^{k} \lambda_{kj} \langle f, e_j \rangle\, g_k' = \sum_{j=1}^{n} \langle f, e_j \rangle \left(\sum_{k=j}^{n} \lambda_{kj} g_k' \right).$$

Letting $g_j = \sum_{k=j}^{n} \lambda_{kj} g_k'$, $j = 1, \ldots, n$, we get the desired expression. □

This solution was borrowed from here!

6 Question 6

For each $y = (y_j)_{j\in\mathbb{N}}$ in $\ell^\infty(\mathbb{N})$, consider the map $T_y : \ell^1(\mathbb{N}) \to \mathbb{C}$ defined by

$$T_y(x) = \sum_{j=1}^{\infty} x_j y_j, \quad x = (x_j)_{j\in\mathbb{N}} \in \ell^1(\mathbb{N}).$$

Show that the map $y \to T_y$ is an isometry from $\ell^\infty(\mathbb{N})$ onto $\left(\ell^1(\mathbb{N})\right)^*$. Thus $\left(\ell^1(\mathbb{N})\right)^*$ is isometrically isomorphic to $\ell^\infty(\mathbb{N})$.

Solution. Fix $y = (y_j)_{j\in\mathbb{N}} \in \ell^\infty(\mathbb{N})$. Consider the map

$$T_y(x) = \sum_{j=1}^{\infty} x_j y_j$$

for each $x = (x_j)_{j\in\mathbb{N}} \in \ell^1(\mathbb{N})$.

It is easy to see that this map is well defined, continuous linear functional by the Holder's inequality. Hence, we have that $T_y \in (\ell^1(\mathbb{N}))^*$.

Now, we show that the map $F : \ell^\infty(\mathbb{N}) \to (\ell^1(\mathbb{N}))^*$ given by

$$y \xmapsto{F} T_y$$

It is easy to see that the map is linear and all we need to show is that this map is an isometry and an isomorphism as well. First, fix a $y \in \ell^\infty(\mathbb{N})$ and observe that for any $x \in \ell^1(\mathbb{N})$ with $\|x\|_1 = 1$, we have that

$$\begin{aligned} |T_y(x)| &= \left|\sum_{j=1}^{\infty} x_j y_j\right| \\ &\leq \|x\|_1 \|y\|_\infty && \text{Holder's inequality} \\ &= \|y\|_\infty \end{aligned}$$

Thus, taking supremum, we have from Question 4 that

$$\|T_y\|_{(\ell^1(\mathbb{N}))^*} \leq \|y\|_\infty$$

To show the reverse inequality, observe that for each $i \in \mathbb{N}$, we have that $\|e_1\|_1 = 1$ and hence, we have that

$$|T_y(e_i)| = |y_i| \leq \|T_y\|_{(\ell^1(\mathbb{N}))^\infty}$$

for each $i \in \mathbb{N}$. Taking supremums over $i \in \mathbb{N}$, we have that

$$\|y\|_\infty \leq \|T_y\|_{(\ell^1(\mathbb{N}))^\infty}$$

This shows that $y \mapsto T_y$ is an isometry. It remains to show that F is an isomorphism. It suffices to show that F is onto.

Let $T \in \left(\ell^1(\mathbb{N})\right)^*$. We need to find a $y \in \ell^\infty(\mathbb{N})$ such that $T = T_y$.

For each $i \in \mathbb{N}$, we define

$$y_i = T(e_i).$$

We now claim that $T = T_y$. It is easy to see that

$$T(e_i) = T_y(e_i)$$

Note that $\operatorname{span}\{e_i : i \in \mathbb{N}\} = c_{00}$ and since $\overline{c_{00}} = \ell^1(\mathbb{N})$, we have that $T = T_y$ as they agree on a dense subset.

This completes the proof of the claim. $\square$

7 Question 7

For each $y = (y_j)_{j\in\mathbb{N}}$ in $\ell^1(\mathbb{N})$, consider the map $T_y : \ell^\infty(\mathbb{N}) \to \mathbb{C}$ defined by

$$T_y(x) = \sum_{j=1}^{\infty} x_j y_j, \quad x \in \ell^\infty(\mathbb{N}).$$

Show that the map $y \to T_y$ is an isometry from $\ell^1(\mathbb{N})$ into $\big(\ell^\infty(\mathbb{N})\big)^*$, but not surjective.

Solution. Fix $y = (y_j)_{j\in\mathbb{N}} \in \ell^1(\mathbb{N})$. Consider the map $T_y : \ell^\infty(\mathbb{N}) \to \mathbb{C}$ given by

$$T_y x = \sum_{j=1}^{\infty} x_j y_j$$

for all $x = (x_j)_{j\in\mathbb{N}} \in \ell^1$. It is easy to show that this map is welldefined and that $\|T_y\|_{(\ell^\infty)^*} \leq \|y\|_{\ell^1}$ by Holder's inequality.

Also, we need to show that $\|y\|_{\ell^1} \leq \|T_y\|_{(\ell^\infty)^*}$. To show the reverse inequality, let us denote $y = (y_n)_{n\in\mathbb{N}}$. Now, define for each $n \in \mathbb{N}$,

$$\psi_n := \sum_{k=1}^{n} e^{-i \arg y_k} e_k.$$

It is easy to see that $\psi_n \in c_0$ for each $n \in \mathbb{N}$ and $\|\psi_n\|_{c_0} = 1$. Now observe that for each $n \in \mathbb{N}$, we have that

$$\begin{aligned} f_y(\psi_n) &= f_y\left(\sum_{k=1}^{n} e^{-i\arg y_k} e_k\right) \\ &= \sum_{k=1}^{n} e^{-i\arg y_k} f_y(e_k) \\ &= \sum_{k=1}^{n} e^{-i\arg y_k} y_k \\ &= \sum_{k=1}^{n} |y_k|. \end{aligned}$$

Thus, we have that $\|f_y\|_{(\ell\infty)^*} \geq \sum_{k=1}^{n} |y_k|$. Letting $n \to \infty$, we have that $\|f_y\|_{(\ell^\infty)^*} \geq \sum_{k=1}^{\infty} |y_k| = \|y\|_{\ell^1}$.

If it happened that $y \to T_y$ was an isometric isomorphism then $(\ell^\infty(\mathbb{N}))^*$ would be separable since $\ell^1(\mathbb{N})$ is separable. Then $(\ell^\infty)(\mathbb{N})$ being separable would imply $\ell^\infty(\mathbb{N})$ would be separable. But $\ell^\infty(\mathbb{N})$ is not separable.[1]

[1] X^* separable $\rightsquigarrow$ X separable.

Here's how one can show that ℓ^∞ is not separable. It is easy to see that the set of binary sequences in ℓ^∞ is uncountable. If ℓ^∞ was separable then there would be a sequence $\{x_n : n \in \mathbb{N}\}$ such that $\overline{\{x_n : n \in \mathbb{N}\}} = \ell^\infty$. Now, if y is a binary sequence, then there would be some $k(y) \in \mathbb{N}$ such that $\left\|y - x_{k(y)}\right\|_\infty < \frac{1}{2}$.

We claim that this map $y \mapsto k(y)$ is injective. Suppose not. Then for y_1, y_2 with $y_1 \neq y_2$ we have $x_{y_1} = x_{y_2}$. Then we have that

$$\begin{aligned}
1 = \|y_1 - y_2\|_\infty &= \|y_1 - x_{y_1} + x_{y_2} - y_2\| && \text{adding zero} \\
&\leq \|y_1 - x_{y_1}\|_\infty + \|y_2 - x_{y_2}\| && \text{triangle inequality} \\
&< \frac{1}{2} + \frac{1}{2} \\
&= 1 && \text{Boom! } 1 < 1!
\end{aligned}$$

Do I need to say more? □

8 Question 8

Let c denotes the set of all convergent sequence and c_0 denotes the set of all convergent sequences whose limit is 0.

(a) Show that c and c_0 is a closed subspace of $\ell^\infty(\mathbb{N})$.

(b) Show that c_0 admits a Schauder basis, namely, $\{e_j : j \in \mathbb{N}\}$.

(c) Let e be the sequence $(1, 1, 1, \ldots)$. Show that $\{e, e_1, e_2, e_3, \ldots\}$ forms a Schauder basis for c.

(d) Show that c_0^* is isometrically isomorphic to $\ell^1(\mathbb{N})$.

(e) Show that c^* is isometrically isomorphic to $\ell^1(\mathbb{N})$ as well.

(f)* Show that the space c_0 and c are not isometrically isomorphic. (Hint: A point p of a closed convex set S in a normed linear space X is called an extreme point of S if p can not be written as convex combination of two distinct points in S. An isometry must take an extreme point to an extreme point. Note that closed unit ball of c_0 has no extreme point but closed unit ball of c has extreme points.)

Proof. Well, well:

(a) Let $(x_n)_{n\in\mathbb{N}}$ be a sequence in c_0 which converges to some $y \in \ell^\infty(\mathbb{N})$. We need to show that $y \in c_0$.

For each $n \in \mathbb{N}$, let us denote

$$x_n = (x_{nk})_{k\in\mathbb{N}}\,.$$

Since $(x_n)_{n\in\mathbb{N}}$ is a sequence in c_0, we have that for each $n \in \mathbb{N}$, the sequence $(x_{nk})_{k\in\mathbb{N}}$ converges to 0.

Now, we proceed to show that the sequence $(y_k)_{k\in\mathbb{N}}$ converges to $0 \in \mathbb{C}$. First, let $\varepsilon > 0$ be given. Select an $N \in \mathbb{N}$ such that

$$\|y - x_N\|_\infty < \frac{\varepsilon}{2}.$$

This can be done because $(x_n)_{n\in\mathbb{N}}$ converges to y in the $\ell^\infty(\mathbb{N})$ norm. Since $(x_{Nk})_{k\in\mathbb{N}}$ converges to $0 \in \mathbb{C}$, we can find a $M \in \mathbb{N}$ such that

$$|x_{Nk}| < \frac{\varepsilon}{2} \text{ for every } k \geq N.$$

Consider the following for $k \geq N$:

$$\begin{aligned} |y_k| &\leq |y_k - x_{Nk}| + |x_{Nk}| \\ &\leq \|y - x_N\|_\infty + |x_{Nk}| \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon. \end{aligned}$$

This shows that $y \in c_0$. Hence, c_0 is closed.

Now, we proceed to show that c is closed. Let $(x_n)_{n\in\mathbb{N}}$ be a sequence in c converging to some $y \in \ell^\infty(\mathbb{N})$. We want to show that $y \in c$. Since for each $n \in \mathbb{N}$, $x_n \in c$, we can let $\xi_n = \lim_{k\to\infty} x_{nk}$.

We now show that $(\xi_n)_{n\in\mathbb{N}}$ is Cauchy in $\mathbb{C}$ (hence convergent). Let $\varepsilon > 0$ be given. Select $N \in \mathbb{N}$ such that

$$\|x_n - x_m\|_\infty < \frac{\varepsilon}{3} \text{ for each } n, m \geq N.$$

This can be done because $(x_n)_{n\in\mathbb{N}}$ is convergent, hence, Cauchy in $\ell^\infty(\mathbb{N})$.

Now, let $n, m \geq N$. Select $K \in \mathbb{N}$ large enough so that

$$|\xi_n - x_{nK}| < \frac{\varepsilon}{3} \text{ and } |\xi_m - x_{mK}| < \frac{\varepsilon}{3}.$$

This can be done because $\xi_n = \lim_{k\to\infty} x_{nk}$ for each $n \in \mathbb{N}$.

Therefore, we have

$$\begin{aligned}
|\xi_n - \xi_m| &\leq |\xi_n - x_{nK}| + |\xi_{mK} - x_{mK}| + |x_{mK} - x_{nK}| \\
&\leq |\xi_n - x_{nK}| + |\xi_{mK} - x_{mK}| + \|x_n - x_m\|_\infty \\
&< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon.
\end{aligned}$$

This shows that $(\xi_n)_{n\in\mathbb{N}}$ is Cauchy. Hence, $(\xi_n)_{n\in\mathbb{N}}$ converges to some $\xi \in \mathbb{C}$.

We now show that $(y_k)_{k\in\mathbb{N}}$ converges to ξ. Let $\varepsilon > 0$ be given. Select $N \in \mathbb{N}$ large enough so that

$$\|y - x_N\|_\infty < \frac{\varepsilon}{3} \text{ and } |\xi_N - \xi| < \frac{\varepsilon}{3}.$$

Now, select $K \in \mathbb{N}$ such that

$$|x_{Nk} - \xi_N| < \frac{\varepsilon}{3} \text{ for every } k \geq K.$$

For $k \geq K$, we have

$$\begin{aligned}
|y_k - \xi| &= |y_k - x_{Nk} + x_{Nk} - \xi_N + \xi_N - \xi| \\
&\leq |y_k - x_{Nk}| + |x_{Nk} - \xi_N| + |\xi_N - \xi| \\
&< \|y - x_n\|_\infty + |x_{Nk} - \xi_N| + |\xi_N - \xi| \\
&< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon.
\end{aligned}$$

This shows that c is closed.

(b) Let $x \in c_0$. We show that there exists unique scalars $(\alpha_n)_{n\in\mathbb{N}} \subset \mathbb{F}$ such that

$$x = \sum_{i=1}^{\infty} \alpha_i e_i.$$

Let $x = (x_n)_{n\in\mathbb{N}}$. We claim that

$$x = \lim_{n\to\infty} \sum_{i=1}^{n} x_i e_i$$

where the convergence is the ℓ^∞-convergence.

Let $\varepsilon > 0$ be given. Select $N \in \mathbb{N}$ such that

$$|x_i| < \frac{\varepsilon}{2} \text{ for all } i \geq N \rightsquigarrow \sup_{i\geq N} |x_i| \leq \frac{\varepsilon}{2} < \varepsilon.$$

Thus for $n \geq N$, we have

$$\begin{aligned} \|x - x_1 e_1 - x_2 e_2 - \ldots - x_n e_n\|_\infty &\leq \sup_{i\geq n+1} |x_i| \\ &\leq \sup_{i\geq N} |x_i| < \varepsilon. \end{aligned}$$

This proves our claim.

Now, we prove uniqueness. Suppose that $x \in c_0$ has two different representations:

$$x = \sum_{i=1}^{\infty} \alpha_i e_i \text{ and } x = \sum_{i=1}^{\infty} \beta_i e_i$$

We show that $\alpha_i = \beta_i$ for each $i \in \mathbb{N}$. Let $i \in \mathbb{N}$ be arbitrary. Then for any $n \geq i$, we have that

$$\begin{aligned} |\alpha_i - \beta_i| &\leq \left\| \sum_{k=1}^{n} (\alpha_k - \beta_k) e_k \right\|_\infty && \text{by definition of the } \infty \text{ norm} \\ &\leq \left\| \sum_{k=1}^{\infty} (\alpha_k - \beta_k) e_k \right\|_\infty && \text{let } n \to \infty \\ &= \|0\|_\infty = 0. \end{aligned}$$

Since $i \in \mathbb{N}$ was arbitrary, we are done.

(c) Let $e = (1, 1, 1, \ldots)$. We wish to show that $\{e, e_1, e_2, \ldots\}$ is a Schauder basis for c. To do so, let $x \in c$. Suppose that $x = (x_n)_{n\in\mathbb{N}}$ converges to ξ. Define a new sequence $x_0 = x - \xi e$. It is easy to see that $x_0 \in c_0$. Thus, we have from part (b) that for some unique $(\alpha_i)_{i\in\mathbb{N}} \subset \mathbb{F}$,

$$x_0 = \sum_{i=1}^{\infty} \alpha_i e_i \rightsquigarrow x = \xi e + \sum_{i=1}^{\infty} \alpha_i e_i.$$

It remains to prove uniqueness. Let $x = (x_n)_{n\in\mathbb{N}} \in c$. Suppose that

$$x = \alpha e + \sum_{i=1}^{\infty} \alpha_i e_i \text{ and } x = \beta e + \sum_{i=1}^{\infty} \beta_i e_i.$$

It is easy to see that

$$\lambda e = \sum_{i=1}^{\infty} \lambda e_i \text{ for any } \lambda \in \mathbb{F}.$$

Therefore, we have that

$$x = \alpha e + \sum_{i=1}^{\infty} \alpha_i e_i = \sum_{i=1}^{\infty} \alpha e_i + \sum_{i=1}^{\infty} \alpha_i e_i = \sum_{i=1}^{\infty} (\alpha + \alpha_i) e_i$$

Likewise, we have that

$$x = \sum_{i=1}^{\infty} (\beta + \beta_i) e_i.$$

The same argument at the end of item (b) shows that $\alpha + \alpha_i = \beta + \beta_i$ for each $i \in \mathbb{N}$. Taking limit $i \to \infty$, we have that $\alpha = \beta$. [2] Hence, we are done.

(d) We need to show that there is an isometric isomorphism between c_0^* and $\ell^1(\mathbb{N})$. For $y \in \ell^1(\mathbb{N})$, consider the linear map $f_y : c_0 \to \mathbb{F}$ given by $f_y(x) = \sum_{i=1}^{\infty} x_i y_i$ for each $y \in c_0$.

We will show that the linear map $T : \ell^1(\mathbb{N}) \to c_0^*$ given by

$$y \xmapsto{T} f_y$$

is an isometric isomorphism.

First, let $y \in \ell^1(\mathbb{N})$. For $x \in c_0$ with $\|x\|_{c_0} \leq 1$, we have by Holder's inequality that (one does not even need Holder :))

$$|f_y(x)| \leq \|x\|_{c_0} \|y\|_{\ell^1} \leq \|y\|_{\ell^1}$$

To show the reverse inequality, let us denote $y = (y_n)_{n \in \mathbb{N}}$. Now, define for each $n \in \mathbb{N}$,

$$\psi_n := \sum_{k=1}^{n} e^{-i \arg y_k} e_k.$$

It is easy to see that $\psi_n \in c_0$ for each $n \in \mathbb{N}$ and $\|\psi_n\|_{c_0} = 1$. Now observe that for each $n \in \mathbb{N}$, we have that

$$\begin{aligned} f_y(\psi_n) &= f_y\left(\sum_{k=1}^{n} e^{-i \arg y_k} e_k\right) \\ &= \sum_{k=1}^{n} e^{-i \arg y_k} f_y(e_k) \\ &= \sum_{k=1}^{n} e^{-i \arg y_k} y_k \\ &= \sum_{k=1}^{n} |y_k|. \end{aligned}$$

[2] It can be shown that if $\sum_{i=1}^{\infty} v_n$ converges then $\lim \|v_n\| = 0$.

Thus, we have that $\|f_y\|_{c_0^*} \geq \sum_{k=1}^{n} |y_k|$. Letting $n \to \infty$, we have that $\|f_y\|_{c_0^*} \geq \sum_{k=1}^{\infty} |y_k| = \|y\|_{\ell^1}$. This shows that T is an isometry.

Let $f \in c_0^*$. We wish to show that there is some $y \in \ell^1$ such that $f_y = f$. Let $y = (y_n)_{n\in\mathbb{N}}$ be the sequence defined by

$$y_n = f\left(e_n\right) \text{ for all } n \in \mathbb{N}.$$

Note the same argument as before shows that $y \in \ell^1$. Do you want me to be more explicit? There you go: Now, define for each $n \in \mathbb{N}$,

$$\psi_n := \sum_{k=1}^{n} e^{-i \arg y_k} e_k.$$

It is easy to see that $\psi_n \in c_0$ for each $n \in \mathbb{N}$ and $\|\psi_n\|_{c_0} = 1$. Now observe that for each $n \in \mathbb{N}$, we have that

$$\begin{aligned} f\left(\psi_n\right) &= f\left(\sum_{k=1}^{n} e^{-i \arg y_k} e_k\right) \\ &= \sum_{k=1}^{n} e^{-i \arg y_k} f(e_k) \\ &= \sum_{k=1}^{n} e^{-i \arg y_k} y_k \\ &= \sum_{k=1}^{n} \left|y_k\right|. \end{aligned}$$

Thus, we have that $\|f\|_{c_0^*} \geq \sum_{k=1}^{n} |y_k|$. Letting $n \to \infty$, we have that $\|f\|_{c_0^*} \geq \sum_{k=1}^{\infty} |y_k| = \|y\|_{\ell^1}$. Didn't I tell you the same argument works?

Now, observe that $f(e_i) = f_y(e_i)$ for each $i \in \mathbb{N}$ and since they agree on a dense subset, namely c_{00}, we have that $f = f_y$ on c_0.

(e) The same proof as above works *mutatis mutandis.*

(f) We first show that closed unit ball of c_0 contains no extreme point. Equivalently, we need to show that every point of c_0 can be written as a convex combination of two distinct points in the closed unit ball of c_0.

Let (x_n) be a sequence in the closed unit ball of c_0. Then select $N \in \mathbb{N}$ such that $|x_N| < \frac{1}{2}$. Now, we define two sequences $(y_n)_{n\in\mathbb{N}}$ and $(z_n)_{n\in\mathbb{N}}$:

$$y_n = \begin{cases} x_n & n \neq N \\ x_N + \frac{1}{2} & n = N \end{cases}$$

and

$$y_n = \begin{cases} x_n & n \neq N \\ x_N - \frac{1}{2} & n = N \end{cases}$$

It is easy to see that both (y_n) and (z_n) are in c_0 and both are distinct. Also note that $(x_n) = \frac{1}{2}(y_n) + \frac{1}{2}(z_n)$. So, no point of the closed unit ball of c_0 is an extreme point.

On the other hand, we show that the closed unit ball of c contains a extreme point. Consider the point $x = (1, 1, 1, \ldots)$. Clearly, x is in the closed unit ball of c. We now show that that x is an extreme point.

Assume the contrary that x is not an extreme point. Then there exists two distinct sequences $y = (y_n)_{n\in\mathbb{N}}$ and $z = (z_n)_{n\in\mathbb{N}}$ such that

$$x = \lambda y + (1-\lambda) z$$

for some $\lambda \in [0, 1]$.

Since y and z are distinct, select a $n_0 \in N$ such $y_{n_0} \neq z_{n_0}$. We may assume that $y_{n_0} < z_{n_0}$ without loss of generality.

We now claim that $z_{n_0} = 1$. If not, let $z_{n_0} < 1$. [3] Then we have that

$$\begin{aligned} 1 = x_{n_0} &= \lambda y_{n_0} + (1-\lambda) z_{n_0} \\ &< \lambda z_{n_0} + (1-\lambda) z_{n_0} \\ &= z_{n_0}. \end{aligned}$$

This contradicts the fact that $z_{n_0} < 1$. Hence $z_{n_0} = 1$.

From here, we can conclude that

$$1 = \lambda y_{n_0} + (1-\lambda) z_{n_0} = \lambda y_{n_0} + (1-\lambda) 1 \rightsquigarrow \lambda = \lambda y_{n_0}.$$

If $\lambda = 0$ then $y = z$ which is a contradiction. Otherwise if $\lambda \neq 0$ then $y_{n_0} = 1$ which contradict the fact that $y_{n_0} \neq z_{n_0}$. This completes the proof.

□

[3] Note that z_{n_0} cannot be bigger than 1 because z lies in the closed unit ball of c.