Functional Analysis Assignment 6

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Note

Then end of a proof is denoted by $\smile$.

Contents

1 Question 1

Let H be a Hilbert space and $T \in \mathcal{B}(H)$. Show that $T^*T \geqslant 0$.

Proof. First, we need to show that T^*T is self-adjoint. Observe that

$$\begin{aligned} (T^*T)^* &= T^* (T^*)^* & (ST)^* &= T^*S^* \\ &= T^*T. & (T^*)^* &= T \end{aligned}$$

To show that $T^*T \geq 0$, we need to show that $\langle T^*Tx, x\rangle \geq 0$ for each $x \in H$. To this end, let $x \in H$. Then we have

$$\begin{aligned} \langle T^*Tx, x\rangle &= \langle Tx, (T^*)^* x\rangle & &\text{by definition} \\ &= \langle Tx, Tx\rangle \geq 0 & &\text{by definition of inner product} \end{aligned}$$

This completes the proof. $\ddot\smile$

2 Question 2

Let H be a Hilbert space and $T \in \mathcal{B}(H)$. Suppose $A \in \mathcal{B}(H)$ and $A \geqslant 0$. Then show that $T^*AT \geqslant 0$.

Proof. Let $A, T \in \mathcal{B}\left(H\right)$ and suppose that $A \geq 0$. Then we need to show that $T^*AT \geq 0$. First, we need to show that T^*AT is self adjoint. To this end, consider the following:

$$\begin{aligned}\left(T^*AT\right)^* &= T^*A^*(T^*)^* \\ &= T^*AT\end{aligned}$$

Since $A \geq 0$, we have that A is self-adjoint by definition. Now, let us show that $\langle T^*ATx, x\rangle \geq 0$ for each $x \in H$. To this end, let $x \in H$ and consider the following:

$$\langle T^*ATx, x\rangle = \langle ATx, Tx\rangle \geq 0 \text{ since } Tx \in H \text{ and } A \geq 0.$$

This completes the proof. $\ddot\smile$

3 Question 3

Let H be a Hilbert space and $T, S \in \mathcal{B}(H)$, satisfying $T \geqslant 0$ and $S \geqslant 0$. Show that $T + S \geqslant 0$.

Proof. The fact that $T + S$ is selfadjoint follows from the fact that the adjoint map is antilinear. To complete the rest, let $x \in H$ and consider the following:

$$\begin{aligned} \langle (S+T)x, x\rangle &= \langle Sx, x\rangle + \langle Tx, x\rangle \\ &\geq 0. \end{aligned}$$

This completes the proof. $\smile$

4 Question 4

Let H be a Hilbert space and $P_1, P_2 \in \mathcal{B}(H)$, are two orthogonal projections,that is, $P_j^2 = P_j$ and $P_j^* = P_j$ for each $j = 1, 2$. Show that P_1P_2 is an orthogonal projection if and only if $P_1P_2 = P_2P_1$. In this case $\operatorname{ran}(P_1P_2) = \operatorname{ran}(P_1) \cap \operatorname{ran}(P_2)$.

Proof. Let $P_1, P_2 \in \mathcal{B}(H)$ be two orthogonal projections. We show that P_1P_2 is an orthogonal projection iff $P_1P_2 = P_2P_1$.

($\Longrightarrow$) Suppose that P_1P_2 is an orthogonal projection. We wish to show that $P_1P_2 = P_2P_1$. To do so, consider the following:

$$\begin{aligned} P_1P_2 &= (P_1P_2)^* && P_1P_2 \text{ being orthogonal projection} \\ &= P_2^*P_1^* && \text{property of the adjoint} \\ &= P_2P_1 && P_1, P_2. \text{ being orthogonal projection} \end{aligned}$$

($\Longleftarrow$) Suppose that $P_1P_2 = P_2P_1$. We wish to show that P_1P_2 is an orthogonal projection.

P_1P_2 **is self adjoint:** Consider the following:

$$\begin{aligned} (P_1P_2)^* &= P_2^*P_1^* \\ &= P_2P_1 \\ &= P_1P_2. && \text{by assumption} \end{aligned}$$

P_1P_2 **is idempotent:** Consider the following:

$$\begin{aligned} (P_1P_2)^2 &= P_1P_2P_1P_2 \\ &= P_1P_1P_2P_2 && \text{since } P_1P_2 = P_2P_1 \\ &= P_1^2P_2^2 \\ &= P_1P_2 && \text{since } P_1, P_2 \text{ are orthogonal projection} \end{aligned}$$

This shows that the both directions of the iff. Let us assume that $P_1P_2 = P_2P_1$. Now, we proceed to show that im $(P_1P_2) =$ im $P_1 \cap$ im P_2.

($\subset$) **:** Let $y \in \text{im } P_1P_2$. Then we have that $y = P_1P_2x$ for some $x \in H$. Since $P_2x \in H$, we have that $y = P_1x$. Also, note that $y = P_2P_1x$ since P_1 and P_2 commute. Hence, we have that $y \in \text{im } P_2$. This shows that $\text{im } P_1P_2 \subset \text{im } P_1 \cap \text{im } P_2$.

($\supset$) **:** Let $y \in \text{im } P_1 \cap \text{im } P_2$. Then we have $y = P_1x_1 = P_2x_2$ for some $x_1 \in H$ and some $x_2 \in H$. It follows that

$$P_1y = P_1^2x_1 \rightsquigarrow P_1y = P_1x_1 \rightsquigarrow y = P_1y$$

and

$$P_2y = P_2^2x_2 \rightsquigarrow P_2y = P_2x_2 \rightsquigarrow y = P_2y.$$

The above two equalities show that $y \in \text{im } P_1 \cap \text{im } P_2$.

$\smile$

5 Question 5

Let H be a Hilbert space and $P_1, P_2 \in \mathcal{B}(H)$, are two orthogonal projections in $\mathcal{B}(H)$. Show that $P_1 + P_2$ is an orthogonal projection if and only if $ran(P_1)$ is orthogonal to $ran(P_2)$, that $\langle P_1 x, P_2 y\rangle = 0$ for every $x, y \in H$. In this case $ran(P_1 + P_2) = ran(P_1) + ran(P_2)$.

Proof. Let H be a Hilbert space and let P_1, P_2 be two orthogonal projections in $\mathcal{B}(H)$. We proceed to show that $P_1 + P_2$ is an orthogonal projection iff im P_1 is orthogonal to im P_2.

($\Longrightarrow$) Suppose that $P_1 + P_2$ is an orthogonal projection. Let $y_1 \in$ im P_1 and $y_2 \in$ im P_2. Then we have that $y_1 = P_1 x_1$ and $y_2 = P_2 x_2$ for some $x_1 \in H$ and some $x_2 \in H$.

First, we show that $P_1 P_2 = -P_1 P_2$. Note that

$$\begin{aligned} P_1 + P_2 &= (P_1 + P_2)^2 \\ &= P_1 + P_1 P_2 + P_2 P_1 + P_2 \end{aligned}$$

Hence, this implies that

$$P_1 P_2 = -P_2 P_1. \tag{5.1}$$

Now, multiplying the previous equation by P_2, we have that $P_2 P_1 = -P_2 P_1 P_2$. $P_2 P_1$ is then self-adjoint as

$$(P_2 P_1)^* = (-P_2 P_1 P_2)^* = -P_2^* P_1^* P_2^* = -P_2 P_1 P_2 = P_2 P_1.$$

Taking adjoint in both sides of Equation 5.1, we have that

$$\begin{aligned} P_2 P_1 &= (P_2 P_1)^* \\ &= (-P_1 P_2)^* \\ &= -P_2 P_1 \end{aligned}$$

and consequently,

$$P_2 P_1 = P_1 P_2 = 0. \tag{5.2}$$

Now, we proceed to show that im P_1 is orthogonal to im P_2. To this end, let $x_1, x_2 \in H$. Then we have that

$$\begin{aligned}\langle P_1x_1, P_2x_2\rangle &= \langle P_2P_1x_1, x_2\rangle \\ &= \langle 0x_1, x_2\rangle \\ &= 0 && \text{(See 5.2)}\end{aligned}$$

This shows that im P_1 is orthogonal to im P_2.

($\Longleftarrow$) Suppose that im P_1 is orthogonal to im P_2. We wish to show that $P_1 + P_2$ is an orthogonal projection. The fact that $P_1 + P_2$ is selfadjoint is immediate. It remains to show that $P_1 + P_2$ is idempotent.

But before that, we show that $P_1P_2 = 0$. Let $f \in H$. Observe that

$$\begin{aligned}\|P_1P_2f\|^2 &= \langle P_1P_2f, P_1P_2f\rangle \\ &= \langle P_1^*P_1P_2f, P_2f\rangle \\ &= \left\langle P_1^2P_2f, P_2f\right\rangle \\ &= \langle P_1P_2f, P_2f\rangle \\ &= 0.\end{aligned}$$

The first equality is by definition, the second is by definition of adjoint operator and the third is because P_1 is an orthogonal projection and hence adjoint, the the third is because P_1 is selfadjoint and the last is due to our assumption that im P_1 is orthogonal to im P_2. This shows that $P_1P_2 = 0$.

Similarly, it can be shown that $P_2P_1 = 0$. Now, we proceed to $(P_1+P_2)^2 = P_1 + P_2$. Observe that

$$\begin{aligned}(P_1 + P_2)^2 &= P_1^2 + P_1P_2 + P_2P_1 + P_2^2 \\ &= P_1 + P_2.\end{aligned}$$

The first equality is obvious and in the second one, we are using the fact that P_1, P_2 are idempotent and $P_1P_2 = P_2P_1 = 0$.

Now, we proceed to show that im $(P_1 + P_2) =$ im $P_1 +$ im P_2. Consider the following:

($\subset$): Let $y \in$ im $(P_1 + P_2)$. Then $y = (P_1 + P_2)\,x$ for some $x \in H$. Then we have that

$$y = \underbrace{P_1x}_{\in \text{im } P_1} + \underbrace{P_2x}_{\in \text{im } P_2} \in \text{im } P_1 + \text{im } P_2.$$

($\supset$): Let $y \in \text{im } P_1 + \text{im } P_2$. Then $y = P_1 x_1 + P_2 x_2$ for some $x_1 \in H$ and some $x_2 \in H$. Since we have that $P_1 + P_2$ is an orthogonal projection, we have that $P_1 P_2 = -P_2 P_1$ as in ($\Longrightarrow$) in the first part of the proof. Thus, we have that

$$P_2 y = P_2 P_1 x_1 + P_2^2 x_2 = P_2 x_2$$

and similarly we have that

$$P_1 y = P_1 x_1.$$

Hence we have that

$$y = P_1 y + P_2 y = (P_1 + P_2)\, y$$

and consequently $y \in \text{im } (P_1 + P_2)$.

This completes the proof. $\smile$

6 Question 6

Let H be a Hilbert space and M be a closed subspace of H. Let P_M be the orthogonal projection onto M. Suppose $T \in \mathcal{B}(H)$. Show that M is an invariant subspace for T if and only if $TP_M = P_M T P_M$. Show that M is a reducing subspace for T if and only if $P_M T = T P_M$.

Proof. Let P_M be the orthogonal projection onto the closed subspace M and let $T \in \mathcal{B}(H)$. First, we show that if $x \in M$ then $Px = x$.

We wish to show that M is an invariant subspace of T iff $TP_M = P_M T P_M$.

($\Longrightarrow$) Suppose that M is an invariant subspace of T, that is, $TM \subset M$. We wish to show that $TP_M = P_M T P_M$.

First, we show that if $x \in M$ then $P_M x = x$. To show this, let $x \in M$. Then by the decomposition of Hilbert space by a closed subspace, we have that

$$x = P_M x + P_{M^\perp} x$$

but then $x - P_M x \in M \cap M^\perp = \{0\}$. Thus, we must have that $x = P_M x$.

Now, let $x \in M$ then we have that $TP_M x \in M$ because $P_M x \in M$ and therefore, we have that $TP_M x \in M$. By what we just proved we have that $TP_M x = P_M T P_M x$. Since $x \in H$ was arbitrary, we have that $TP_M = P_M T P_M$.

($\Longleftarrow$) Suppose that $TP_M = P_M T P_M$. Let $x \in M$. Then we have that $P_M x = x$ by what we proved earlier. Thus, we have that $Tx = P_M T P_M x \in M$.

This completes the proof. ⌣̈

7 Question 7

Let H be a Hilbert space and $T \in \mathcal{B}(H)$, satisfying $T^k = 0$ for some $k \in \mathbb{N}$. Show that $\sigma(T) = \{0\}$.

Proof. By the Spectral Mapping Theorem, we have that

$$\begin{aligned} (\sigma(T))^k &= \sigma\left(T^k\right) \\ &= \sigma\left(0\right) \\ &= \{0\} \end{aligned}$$

This shows that $\sigma\left(T\right) \subset \{0\}$. Since the spectrum is always nonempty, we have that $\sigma\left(T\right) = \{0\}$. $\ddot\smile$

8 Question 8

Let H be a Hilbert space and M be a closed subspace of H. Let P_M be the orthogonal projection onto M. Compute the spectrum $\sigma(P_M)$.

Proof. Let P_M be the orthogonal projection onto M. Define a polynomial $p(z) \in \mathbb{C}[z]$ given by $p(z) = z^2 - z$. It is easy to see that $p(P_M) = 0$.

It follows that by the Spectral Mapping Theorem that

$$p(\sigma(P_M)) = \sigma(p(P_M)) = \sigma(0) = \{0\}.$$

Hence, if $\alpha \in \sigma(P_M)$, we have that $\alpha^2 - \alpha = 0$. Thus, $\alpha \in \{0, 1\}$. This shows that $\sigma(P_M) \subset \{0, 1\}$.

Since the spectrum is nonempty, we must have that exactly one of the following holds: $\sigma(P_M) = \{0\}$ or $\sigma(P_M) = \{1\}$ or $\sigma(P_M) = \{0, 1\}$. Since P_M is a projection, we have that $H = \ker P_M \oplus \operatorname{im} P_M$. This shows that this corresponds to $P_M = I$, $P_M = 0$ and "something in between"! $\ddot\smile$

9 Question 9

Let H be a Hilbert space and $T \in \mathcal{B}(H)$. Suppose X is an invertible operator in $\mathcal{B}(H)$. Then show that $\sigma(X^{-1}TX) = \sigma(T)$. (In other words similar operators have same spectrum).

Proof. Let X be an invertible operator in $\mathcal{B}(H)$. Consider the following equivalence:

$$\begin{aligned} \alpha \notin \sigma(T) &\iff T - \alpha I \text{ is invertible} \\ &\iff X^{-1}T - \alpha X^{-1} \text{ is invertible} \\ &\iff X^{-1}TX - \alpha I \text{ is invertible} \\ &\iff \alpha \notin \sigma\left(X^{-1}TX\right). \end{aligned}$$

This shows that $\sigma\left(X^{-1}TX\right) = \sigma(T)$. $\ddot\smile$

10 Question 10

Let H be a Hilbert space and $T \in \mathcal{B}(H)$. Suppose X is a unitary operator in $\mathcal{B}(H)$. Then show that

(i) $\|X^{-1}TX\| = \|T\|$.

(ii) T is normal if and only if $X^{-1}TX$ is normal.

(iii) T is self adjoint if and only if $X^{-1}TX$ is self adjoint.

(iv) $T \geqslant 0$ if and only if $X^{-1}TX \geqslant 0$.

(v) T is an orthogonal projection if and only if $X^{-1}TX$ is orthogonal projection.

Proof. Let H be a Hilbert space and $T \in \mathcal{B}(H)$. Let X be a unitary operator in $\mathcal{B}(H)$.

(i) We intend to show that $\left\|X^{-1}TX\right\| = \|T\|$. Since X is a unitary operator, we have that X is an isometric isomorphism. To this end, let $f \in H$ and consider the following:

$$\begin{aligned}
\left\|X^{-1}TXf\right\|^2 &= \|X^*TXf\|^2 \\
&= \langle X^*TXf, X^*TXf\rangle \\
&= \langle TXf, XX^*TXf\rangle \\
&= \langle TXf, TXf\rangle \\
&= \|TXf\|^2 .
\end{aligned}$$

Thus, we have that

$$\left\|X^{-1}TXf\right\| = \|TXf\| \tag{10.1}$$

for each $f \in H$. Now, we claim that

$$\{\|TXf\| : \|f\| = 1\} = \{\|Tf\| : \|f\| = 1\} . \tag{10.2}$$

To prove this:

$(\subset)$: Let $y = \|TXf\|$ for some $f \in H$ with $\|f\| = 1$. Since X is a unitary operator, we have that $\|Xf\| = \|f\| = 1$, hence, y is in the set of the right side of the equality of the aforementioned claim.

$(\supset)$: Let $y = \|Tg\|$ for some $g \in H$ with $\|g\| = 1$. Since X is unitary, it is invertible and hence $Xf = g$ for some $f \in H$. Since X is unitary, we have that $\|f\| = \|Xf\| = \|g\| = 1$. Thus, y is in the left side of the set in the aforementioned equality.

Thus, we have

$$\begin{aligned} \|X^*TX\| &= \sup_{\|f\|=1} \|X^*TXf\| && \text{(by definition)} \\ &= \sup_{\|f\|=1} \|TXf\| && \text{(see 10.1)} \\ &= \sup_{\|f\|=1} \|Tf\| && \text{(see 10.2)} \\ &= \|T\| . && \text{(by definition)} \end{aligned}$$

This completes the proof.

(ii) $(\Longrightarrow)$ Assume that T is normal. Consider the following:

$$\begin{aligned} (X^*TX)(X^*TX)^* &= X^*TXX^*T^*X \\ &= X^*TT^*X \end{aligned}$$

and on the other hand, we have

$$\begin{aligned} (X^*TX)^*(X^*TX) &= X^*TXX^*TX \\ &= X^*TT^*X \end{aligned}$$

Note that the two computations above show that X^*TX is normal by assuming T is normal.

$(\Longleftarrow)$ The computation in the other direction, normality of X^*TX and the invertibility of X imply that T is normal.

This completes the proof.

(iii) $(\Longrightarrow)$**:** Suppose that T is adjoint. To show that X^*TX is self-adjoint, consider the following:

$$\begin{aligned} (X^*TX)^* &= X^*T^*X \\ &= X^*TX && \text{(using the fact that } T\text{is self-adjoint)} \end{aligned}$$

($\Longrightarrow$) Suppose that X^*TX is self-adjoint. To show T is adjoint, consider the following:

$$\begin{aligned} T^* &= XX^*T^*XX^* \\ &= X\left(X^*TX\right)^* X^* \\ &= XX^*TXX^* && (T \text{ is self-adjoint}) \\ &= T. \end{aligned}$$

(iv) ($\Longrightarrow$) Suppose that $T \geq 0$. The fact that X^*TX is self-adjoint follows from item (iii). Now, let $x \in H$. Consider the following:

$$\begin{aligned} \langle X^*TXx, x\rangle &= \langle TXx, Xx\rangle \\ &\geq 0 && (\text{since } T \text{ is positive}) \end{aligned}$$

($\Longleftarrow$) Suppose that $X^*TX \geq 0$. The fact that T is selfadjoint follows from item (iii). To complete the rest, let $f \in H$ and consider the following:

$$\begin{aligned} \langle Tx, x\rangle &= \langle XX^*TXX^*x, x\rangle \\ &= \langle X^*TX(X^*x), X^*x\rangle \\ &\geq 0 && (\text{since } X^*TX \geq 0) \end{aligned}$$

(v) First of all, note that item (iii) does half of the job.

($\Longrightarrow$) Suppose that T is idempotent. Then we have that

$$\begin{aligned} \left(X^*TX\right)^2 &= X^*TXX^*TX \\ &= X^*T^2X \\ &= X^*TX && (T \text{ is idempotent}) \end{aligned}$$

($\Longleftarrow$) Suppose that X^*TX is idempotent. We wish to show that T is idempotent. Observe that

$$\begin{aligned} T^2 &= TT \\ &= XX^*TXX^*XX^* \\ &= X\left(X^*TX\right)^2 X^* \\ &= XX^*TXX^* && (X^*TX \text{ is idempootent}) \\ &= T. \end{aligned}$$

This completes the entire proof. $\ddot\smile$

11 Question 11

Let H be a Hilbert space and $T \in \mathcal{B}(H)$. Show that the following statements are equivalent :

(i) T is an isometry, that is, $T^*T = I$.

(ii) $\|Tx\| = \|x\|$ for every $x \in H$.

(iii) $\langle Tx, Ty \rangle = \langle x, y \rangle$ for every $x, y \in H$.

Proof. **(i) $\implies$ (ii):** Suppose that $T^*T = I$. Let $x \in H$. Then

$$\begin{aligned} \|Tx\|^2 &= \langle Tx, Tx \rangle \\ &= \langle T^*Tx, x \rangle \\ &= \langle x, x \rangle \\ &= \|x\|^2 . \end{aligned}$$

(ii) $\implies$ (iii) Suppose that $\|Tx\| = \|x\|$ for each $x \in H$. See Axler's Theorem 7.42.

(iii) $\implies$ (i) Suppose that $\langle Tx, Ty \rangle = \|x, y\|$ for each $x, y \in H$. Let $x, y \in H$. Then we have

$$\begin{aligned} \langle (T^*T - I)x, y \rangle &= \langle T^*Tx - x, y \rangle \\ &= \langle T^*Tx, y \rangle - \langle T^*Tx, y \rangle \\ &= \langle Tx, Ty \rangle - \langle x, y \rangle \\ &= 0 \end{aligned}$$

This shows that $T^*T = I$.

$\smile$

12 Question 12

Let H be a Hilbert space and $T \in \mathcal{B}(H)$. Show that the following statements are equivalent:

(i) T is left invertible, that is, there exists $S \in \mathcal{B}(H)$ such that $ST = I$.

(ii) T is bounded below, that is, there exists $c > 0$ such that $\|Tx\| \geq c\|x\|$ for each $x \in H$.

(iii) $\ker T = \{0\}$ and $\operatorname{im} T$ is closed.

Proof. Let $T \in \mathcal{B}(H)$.

(i) $\implies$ (ii): Suppose that T is left invertible. Define $c := \|S\|$ and let $x \in H$. Then we have that

$$\begin{aligned} \|x\| &= \|STx\| && (ST = I) \\ &\leq \|S\|\,\|Tx\| && (S \in \mathcal{B}(H)) \\ &\leq c\,\|Tx\| \end{aligned}$$

Since x was arbitrary, we are done.

(ii) $\implies$ (iii): Suppose that T is bounded below. To see that $\ker T = \{0\}$, let $x \in H$ and suppose $Tx = 0$.Then we have that $c\|x\| \geq \|Tx\| = 0$ which implies $\|x\| = 0$. Hence $x = 0$. To see that $\operatorname{im} T$ is closed, let (y_n) be a sequence in $\operatorname{im} T$ converging to some $y \in H$. Then for each $n \in \mathbb{N}$, we have that $y_n = Tx_n$ for some $x_n \in H$.

Since (y_n) is convergent, it is Cauchy. We show that (x_n) is Cauchy. To this end, let $\varepsilon > 0$ be given. Then there exist some $N \in \mathbb{N}$ such that

$$\|Tx_n - Tx_m\| < \varepsilon/c \text{ for every } n, m \geq N.$$

Now, let $n, m \geq N$. Then we have that

$$\begin{aligned} \|x_n - x_m\| &< \frac{1}{c}\|Tx_n - Tx_m\| \\ &< c\left(\frac{\varepsilon}{c}\right) = \varepsilon. \end{aligned}$$

This shows that (x_n) is Cauchy.

Since H is a Hilbert space, we have that $x_n \to x$ for some $x \in H$. Using the continuity of T, we have that $Tx_n \to Tx$. By the uniqueness of limits, we have that $Tx = y$. Thus, $y \in \text{im } T$. This shows that im T is closed.

(iii) ⇒ (i): Suppose that $\ker T = \{0\}$ and im T is closed. Consider the map $T^{-1} : \text{im } T \to H$ which is given by $T^{-1}(Tx) = x$ for each $x \in H$. This map is well defined because T is injective. Since im T is closed, im T is a Hilbert space in its own right. The fact that T is bijective is clear. Thus, by the Banach Isomorphism Theorem, we have that T^{-1} is invertible, that is, $T^{-1} \in \mathcal{B}(\text{im } T, H)$.

Now, we define $S : H \to H$ in the following fashion:

$$Sx = \begin{cases} T^{-1}x & x \in \text{im } T \\ 0 & \text{otherwise.} \end{cases}$$

It is easy to see that $ST = I$ but it remains to show that $S \in \mathcal{B}(H)$. To this end, consider the following:

$$\begin{aligned} \|S\| &= \sup_{x \in H, \|x\| \leq 1} \|Sx\| \\ &\leq \sup_{x \in \text{im } T, \|x\| \leq 1} \left\|T^{-1}x\right\| \\ &= \left\|T^{-1}\right\|. \end{aligned}$$

This completes the proof. ◡̈

13 Question 13

Let H be a Hilbert space and $T \in \mathcal{B}(H)$. Show that the following statements are equivalent:

(i) $ran(T)$ closed.

(ii) T is bounded below on $(\ker T)^{\perp}$, that is, there exist a $c > 0$ such that $\|Tx\| \geqslant c|x\|$ for every $x \in (\ker T)^{\perp}$.

(iii) There exist a $S, Y \in \mathcal{B}(H)$ such that $ST = I - P_{\ker T}$ and $TY = P_{ran(T)}$.

(iv) $ran(T^*)$ closed.

Proof. Let H be a Hilbert space and $T \in \mathcal{B}(H)$. We proceed to show the equivalences:

(i) $\Rightarrow$ (ii): Assume that im (T) is closed. Then im T is a Hilbert space with the restricted inner product. Since $\ker T$ is closed, we can consider $X/\ker T$ which again is a Banach space with the quotient norm. Consider the linear isomorphism given by first isomorphism theorem:

$$
\begin{aligned}
X/\ker T &\xrightarrow{\tilde{T}} \text{im } T \\
[x] &\mapsto \tilde{T}([x]) = T(x)
\end{aligned}
$$

We aim to show that $\tilde{T}$ is continuous and apply the Banach isomorphism theorem to show that $\tilde{T}^{-1}$ is continuous for it is bijective. Before that, we prove a result about Hilbert spaces:

Lemma. Let H be a Hilbert space, M be a closed subspace of H. Then

$$\left\| [x]_{X/M} \right\| = \|P_{M^{\perp}} x\|_X$$

for every $x \in X$.

Proof of Lemma. Let $x \in X$. Since M is a closed subspace of X, we have that

$$x = P_M x + P_{M^\perp} x$$

where P_M, $P_{M^\perp}$ are projections into the subspaces M and $M^\perp$ respectively.

Let $y \in M$. Then

$$\|x - y\|^2 = \|y - P_M x\|^2 + \|P_{M^\perp} x\|^2 .$$

Taking infimum over $y \in M$, we have what we wanted. ⌣̈

Now, we get back to what we were trying to prove. let $x \in H$. Then we have

$$\begin{aligned}\left\|\tilde{T}\left[x\right]\right\| &= \left\|\tilde{T}\left[P_{(\ker T)^\perp} x\right]\right\| \\ &= \left\|T P_{(\ker T)^\perp} x\right\| \\ &\leq \|T\| \left\|P_{(\ker T)^\perp} x\right\| \\ &= \|T\| \, \|[x]\| \qquad \text{(by Lemma that we just proved).}\end{aligned}$$

This shows that $\tilde{T}$ is continuous.

Now, let $f \in (\ker T)^\perp$. Then we have that

$$\begin{aligned}\|f\| &= \left\|\tilde{T}^{-1}\left(\tilde{T} f\right)\right\| \\ &\leq \left\|\tilde{T}^{-1}\right\| \left\|\tilde{T} f\right\| \\ &\leq \left\|\tilde{T}^{-1}\right\| \|T f\|\end{aligned}$$

This shows that T is bounded below on $(\ker T)^\perp$.

(ii) $\Rightarrow$ (iii):

(iii) $\Rightarrow$ (iv):

(iv) $\Rightarrow$ (i): We showed that (i) $\Rightarrow$ (iv) from the previous arguments, that is, im (T) is closed implies im (T^*) for any bounded operator T on a Hilbert space. Now, if T is bounded, we have that T^* is bounded and furthermore if im (T^*) is closed, we must have that im $((T^*)^*) = $ im (T) must be closed. This completes the proof.

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14 Question 14

Let H be a Hilbert space. Suppose that $T \in \mathcal{B}(H)$ and T is invertible. Then show that

$$\sigma(T^{-1}) = \left\{ \frac{1}{\lambda} : \lambda \in \sigma(T) \right\}.$$

Proof. Let $T \in \mathcal{B}(H)$ and suppose that T is invertible, hence, T^{-1} is invertible. Then $0 \notin \sigma(T^{-1})$. Therefore, whatever we are trying to prove makes sense.

Let $\alpha \in \sigma(T^{-1})$. We may assume $\alpha \neq 0$ because we have shown that $0 \notin \sigma\left(T^{-1}\right)$. Then $T^{-1} - \alpha I$ is not invertible. We claim that $T - \frac{1}{\alpha} I$ is not invertible. If our claim was not true then $T - \frac{1}{\alpha} I$ is invertible and αT^{-1} is invertible. Hence $\alpha T^{-1}(T - \alpha^{-1} I) = \alpha - T^{-1}$ would be invertible which would contradict our assumption. Thus, we have that $\frac{1}{\alpha} \in \sigma(T)$. Hence $\alpha = \frac{1}{\underbrace{1/\alpha}_{\in \sigma(T)}}$.

To show the converse, let $\lambda \in \sigma(T)$. Note that $\lambda \neq 0$ because T is invertible. Now, we wish to show that $\frac{1}{\lambda} \in \sigma(T^{-1})$, that is, $T^{-1} - \frac{1}{\lambda} I$ is not invertible. Suppose that $T^{-1} - \frac{1}{\lambda}$ were invertible. Then we would have that

$$\lambda - T = (\lambda T) \left(T^{-1} - \frac{1}{\lambda} \right)$$

would be invertible which would contradict our assumption that $\lambda \in \sigma(T)$. ⌣̈

15 Question 15

For $\varphi \in L^\infty[0,1]$, consider the operator $M_\varphi : L^2[0,1] \to L^2[0,1]$, defined by $M_\varphi f = \varphi f$, $f \in L^2[0,1]$. The essential range of φ (w.r.t the Lebesgue measure) denoted as ess ran (φ) is defined as follows:

A point $p \in \mathbb{C}$ is said to be not in the ess ran (φ) if there exist a $\delta > 0$ such that the Lebesgue measure $m(\varphi^{-1}(B(p,\delta))) = 0$.

Prove that $\sigma(M_\varphi) = \text{ess ran}\,(\varphi)$.

Proof. We first show that α is an eigenvalue of M_φ iff $m\left(\{t \in [0,1] : \varphi\left(t\right) = \alpha\}\right) > 0$.

($\Longrightarrow$) Let α be an eigenvalue of M_φ. Then there exists $f \neq 0$ such that $M_\varphi f = \alpha f$, that is, $\varphi f = \alpha f$.

Assume for the sake of contradiction that $m\left(\{\varphi = \alpha\}\right) = 0$.

Observe that $\{f \neq 0\} \subset \{\varphi = \alpha\} \cup \{\varphi f \neq \alpha \varphi\}$. This is easy to verify: let $t \in [0,1]$ be such that $f(x) \neq 0$. If $\varphi(x) = \alpha$, we are done because then $x \in \{\varphi = \alpha\}$. So suppose not, that is, $\varphi(x) \neq \alpha$. Then $\varphi(x)f(x) \neq \alpha f(x)$ because we are multiplying by a nonzero number.

Now, $m\left(f \neq 0\right) \leq m\left(\{\varphi = \alpha\}\right) + m\left(\{\varphi f \neq \alpha\varphi\}\right) = 0$. Hence, $m\left(f \neq 0\right) = 0$ as $\varphi(x)f(x) = \alpha f(x)$ for almost all $x \in [0,1]$.

($\Longleftarrow$) Let $\alpha \in \mathbb{C}$ such that $m\left(\{t \in [0,1] : \varphi(t) = \alpha\}\right) > 0$. We wish to show that α is an eigenvalue of M_φ. Note that $\chi_{\{\varphi=\alpha\}} \neq 0$ and also we have that $\varphi\chi_{\{\varphi=\alpha\}} = \alpha\chi_{\{\varphi=\alpha\}}$. This shows that α is an eigenvalue of M_φ as $\chi_{\{\varphi=\alpha\}} \in L^2\left[0,1\right]$.

This describes the point spectrum of M_φ.

Now, we show that $\sigma\left(M_\varphi\right) = \operatorname{essran}\varphi$. This is equivalent to showing that $\alpha \notin \sigma\left(M_\varphi\right)$ iff $m\left(\{|h - \alpha| < \varepsilon\}\right) = 0$ for some $\varepsilon > 0$. Before we prove this, we prove a preliminary lemma:

Lemma 1. Let $(X, \mathscr{A}, \mu)$ be a finite measure space. Let $h \in \mathcal{L}^\infty\left(d\mu\right)$, $\varepsilon > 0$ and suppose that $|h| \geq \varepsilon$ for μ-almost everywhere. Then we have that there is a $\eta \in \mathcal{L}^\infty\left(\mu\right)$ such that $h(x)\eta(x) = \eta(x)h(x) = 1$ for μ-almost all $x \in X$.

Proof of Lemma 1. Let h be as in the hypothesis of the lemma. We can define a function. Define a function $\eta : X \to \mathbb{C}$ given by

$$\eta(x) = \begin{cases} \frac{1}{h(x)} & h(x) \neq 0 \\ 0 & x = 0 \end{cases}$$

Note that $\{x \in X : h(x) = 0\}$ is a set of measure zero because it is contained in $\{x \in X : |h(x)| < \varepsilon\}$ which is a set of measure zero by assumption. Thus, $\eta(x) = \frac{1}{h(x)}$ for μ-almost all $x \in X$. Hence, we have that $h(x)\eta(x) = \eta(x)h(x) = 1$ for μ almost all $x \in X$. Since $|\eta(x)| \leq \frac{1}{\varepsilon}$ for μ-almost all $x \in X$. We have that $\|\eta\|_\infty \leq \frac{1}{\varepsilon}$. This completes the proof of the lemma. ☺

($\Longrightarrow$) Let $\alpha \in \mathbb{C}$ and suppose that there exist $\varepsilon > 0$ such that $m(\{|\varphi - \alpha| < \varepsilon\} = 0)$. Since $\varphi \in \mathcal{L}^\infty[0,1]$, we have that $\varphi - \alpha \in \mathcal{L}^\infty[0,1]$. By hypothesis, we have that $|\varphi - \alpha| \geq \varepsilon$ for almost all $x \in [0,1]$. Hence by the previous lemma, there exists a function (which we call) $(\varphi - \alpha)^{-1} \in \mathcal{L}^\infty[0,1]$ such that $(\varphi - \alpha)(x) \cdot (\varphi - \alpha)^{-1}(x) = (\varphi - \alpha)^{-1}(x) \cdot (\varphi - \alpha)(x) = 1$ for almost all $x \in [0,1]$. It is easy to see now that $M_{\varphi-\alpha}M_{(\varphi-\alpha)^{-1}} = I$ and $M_{(\varphi-\alpha)^{-1}}M_{\varphi-\alpha} = I$ and hence $\alpha \notin \sigma(M_\varphi)$.

($\Longleftarrow$) Suppose that $\alpha \notin \sigma(M_\varphi)$. Thus, we have that $M_{\varphi-\alpha}$ is invertible. Hence, by Question 12, we have that there exists $\varepsilon > 0$ such that

$$\|M_{\varphi-\alpha}x\| \geq \varepsilon \|x\|$$

for each $x \in X$. As a consequence we have that

$$\|M_{\varphi-\alpha}\|_{(L^2[0,1])^*} \geq \varepsilon.$$

Also, it can be easily shown that

$$\|M_{\varphi-\alpha}\| \leq \|\varphi - \alpha\|_\infty.$$

From the above two lines, we have that

$$\|\varphi - \alpha\|_\infty \geq \varepsilon.$$

Thus by definition of esssup, we have that

$$|\varphi(x) - \alpha| \geq \varepsilon$$

for almost all $x \in X$. Hence, we have that

$$m\left(\{|\varphi - \alpha| < \varepsilon\} = 0\right).$$

This completes the proof.

16 Question 16

Let H be a Hilbert space and $A, B \in \mathcal{B}(H)$.

(i) Suppose $(I - AB)$ is invertible with $X = (I - AB)^{-1}$. Show that

$$(I - BA)(I + BXA) = I = (I + BXA)(I - BA).$$

Hence $(I - BA)$ is invertible.

(ii) Show that $\sigma(AB) \cup \{0\} = \sigma(BA) \cup \{0\}$.

(iii) Assume that H is of finite dimension. Then show that $\sigma(AB) = \sigma(BA)$.

Proof. (i) Observe that

$$\begin{aligned}(I - BA)(I + BXA) &= (I - BA)\left(B(I - AB)^{-1} A + I\right) \\ &= B\left((I - AB)^{-1}\right) A + I - BAB(I - AB)^{-1} A - BA \\ &= B\left((I - AB)^{-1} - AB(I - AB)^{-1}\right) A - BA + I \\ &= B\left((I - AB)(I - AB)^{-1}\right) A - BA + I \\ &= BA + I - BA \\ &= I\end{aligned}$$

and it can be also verified easily that

$$(I + BXA)(I - BA) = I.$$

(ii) We proceed to show that $\sigma(AB) \cup \{0\} \subset \sigma(BA) \cup \{0\}$. To this end, let $\lambda \in \sigma(AB) \cup \{0\}$. If $\lambda = 0$ then there is nothing to prove as 0 is a member of the right set. Now, suppose $\lambda \neq 0$. Then that forces $\lambda \in \sigma(AB)$. Thus, $\lambda - AB$ is not invertible. We claim that $\lambda - BA$ is not invertible. If it were invertible then we would have that $I - (\lambda^{-1}B)A$ is invertible as $\lambda \neq 0$. By item (i), we have that $I - A\left(\lambda^{-1}B\right)$ is invertible. Hence $\lambda - BA$ is invertible. That is a contradiction! This shows that $\sigma(AB) \cup \{0\} \subset \sigma(BA) \cup \{0\}$. Interchanging the roles of A and B, we have that $\sigma(BA) \cup \{0\} \subset \sigma(AB) \cup \{0\}$. This completes the proof.

(iii) For finite dimensional vector spaces, we have that the spectrum is the same as the point spectrum. Therefore, we aim to show that $\sigma_p(AB) = \sigma_p(BA)$. Let $\lambda \in \sigma_p(AB)$. First, let us show that $\sigma_p(AB) \subset \sigma_p(BA)$.

Let $\lambda \in \sigma_p(AB)$. Therefore, $\lambda - AB$ is not injective. Thus, there is some nonzero $f \in H$ such that $(\lambda - AB)f = 0$. Now, we wish to show that $\lambda - BA$ is not injective. Consider Bf. If $Bf = 0$ then $ABf = \lambda f = 0$ which would imply that $f = 0$. Hence $Bf \neq 0$.

Now, observe that

$$\begin{aligned}(\lambda - BA)\, Bf &= (\lambda B - BAB)\, f \\ &= B\,(\lambda - AB)\, f \\ &= 0.\end{aligned}$$

This shows that $\lambda \in \sigma_p(BA)$. Hence, $\sigma_p(AB) \subset \sigma_p(BA)$.

Interchanging the roles of A and B, we have $\sigma_p(BA) \subset \sigma_p(AB)$.

$\ddot\smile$

17 Question 17

Let $\lambda = \{\lambda_j\}_{j\in\mathbb{N}}$ be sequence in $\ell^\infty(\mathbb{N})$. Consider the unilateral weighted shift operator W_λ on $\ell^2(\mathbb{N})$ defined by

$$W_\lambda(e_j) = \lambda_j e_{j+1}, \quad j \in \mathbb{N},$$

and extend linearly, that is,

$$W_\lambda(c_1, c_2, c_3, \ldots) = (0, c_1\lambda_1, c_2\lambda_2, c_3\lambda_3, \ldots)$$

Prove that the adjoint of W_λ on $\ell^2(\mathbb{N})$ is given by

$$W_\lambda^*(c_1, c_2, c_3, \ldots) = (c_2\overline{\lambda_1}, c_3\overline{\lambda_2}, c_4\overline{\lambda_3}, \ldots).$$

Moreover show that $\|W_\lambda\| = \|\lambda\|_\infty = \sup\{|\lambda_j| : j \in \mathbb{N}\}$.

Proof. Let $(c_1, c_2, \ldots)$ and $(d_1, d_2, \ldots) \in \ell^2(\mathbb{N})$. Then consider the following:

$$\begin{aligned}
\langle (c_1, c_2, \ldots), W_\lambda^*(d_1, d_2, \ldots)\rangle &= \langle W_\lambda(c_1, c_2, \ldots), (d_1, d_2, \ldots)\rangle \\
&= \langle (0, c_1\lambda_1, c_2\lambda_2, \ldots), (d_1, d_2, d_3, \ldots))\rangle \\
&= c_1\lambda_1\bar{d_2} + c_2\lambda_2\bar{d_3} + \ldots \\
&= \left\langle (c_1, c_2, \ldots), \left(\bar{\lambda_1}d_2, \bar{\lambda_2}d_3, \ldots\right)\right\rangle.
\end{aligned}$$

Since (c_i) and (d_i) were arbitrary, we have that

$$W_\lambda^*(d_1, d_2, \ldots) = \left(\bar{\lambda_1}d_2, \bar{\lambda_2}d_3, \ldots\right)$$

for each $(d_1, d_2, \ldots) \in \ell^2(\mathbb{N})$.

Now, observe that $W_\lambda(e_i) = \lambda_i$ for each $i \in \mathbb{N}$. Thus, we have that

$$\|\lambda\|_\infty = \sup_{j\in\mathbb{N}} |\lambda_j| \leq \|W_\lambda\|.$$

To show the reverse inequality, let $(c_1, c_2, \ldots) \in \ell^2(\mathbb{N})$. Then we have that

$$\begin{aligned}
\|W_\lambda(c_1, c_2, \ldots)\|_2^2 &= \sum_{i=1}^\infty |\lambda_i c_i|^2 \\
&\leq \sup_{i\in\mathbb{N}} |\lambda_i|^2 \sum_{i=1}^\infty |c_i|^2 \\
&= \|\lambda\|_\infty^2 \|(c_1, c_2, \ldots)\|_2
\end{aligned}$$

This implies that $\|W_\lambda\| \leq \|\lambda\|_\infty$. This completes the proof. ⌣̈